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Equalizing solutions for bankruptcy problems revisited

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Abstract
When solving bankruptcy problems through equalizing solutions, agents with small claims prefer to distribute the estate according to the Constrained Equal Awards solution, while the adoption of the Constrained Equal Losses solution is preferred by agents with high claims. Therefore, the determination of which is the central claimant, as a reference to distinguish the agents with a high claim from those with a low claim, is a relevant question when designing hybrid solutions, or new methods to distribute the available estate in a bankruptcy problem. We explore the relationship between the equal awards parameter λ and the equal losses parameter μ that characterize the two solutions. We show that the central claimant is fully determined by these parameters. In addition, we explore how to compute these parameters and present optimization problems that provide the Constrained Equal Awards and the Constrained Equal Losses solutions.

Keywords Bankruptcy problem · Constrained Equal Awards · Constrained equal losses · Relative degree of conflict

JEL Classification C79 · D63 · D74

1 Introduction

Bankruptcy problems describe situations where a given amount of a perfectly divisible good, say money, has to be distributed among some agents according to their demands.

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The key issue of these problems is that there is not enough money to satisfy the demands by all the agents. Even though solutions for bankruptcy problems have been suggested throughout the ages, the seminal paper by O'Neill (1982) can be seen as the starting point of a large literature formally focusing on how bankruptcy problems should be solved when some fairness criteria have to be preserved.

Among the many different solutions proposed along this literature, three of them have captured the attention of several academics. These solutions share a common feature, namely all of them try to equalize the agents' perception of how they are treated. The proportional solution, attributed to Aristotle, is designed to equalize the agents' relative award. The Constrained Equal Awards solution (Maimonides 1180) is designed to equalize the agents' absolute award. Lastly, the Constrained Equal Losses solution (Maimonides 1180) is built to equalize the agents' absolute unfulfilled demand. For the sake of completeness, we also mention that the proportional solution equalizes the agents' relative unfulfilled demand.

The two (Constrained Equal) solutions proposed by Maimonides (1180) have been very relevant to build some alternative 'hybrid' solutions as the *Talmud* (Aumann and Maschler 1985), the *Piniles*' (Piniles 1861), or the $\alpha_{\min}$ (Giménez-Gómez and Peris 2014) solutions, among others.[1]

Our aim in this paper is to delve into the knowledge of how the dual proposals by Maimonides work. To this matter, we propose an expression that lightens the computations needed to get these solutions. Our approach is also useful to better understanding the conflict of interest between agents with a low demand (that prefer to equalize awards) and those with a large demand (that prefer to equalize unfulfilled demands). To be precise, for any given problem, when exploring the opinion of each agent about two different solutions, we can consider two groups of agents: those that prefer the proposal of the first solution and the ones preferring the distribution associated with the second solution. When the two solutions are dual (Aumann and Maschler 1985), this analysis becomes simpler because there is an agent, which we identify as the 'central claimant,' exhibiting the following property: agents whose demand does not exceed that of the central claimant prefer one of the solutions, while the alternative solution is (weakly) preferred by agents whose demand exceeds that of the central claimant.

The Constrained Equal Awards solution is characterized by a 'common award' parameter, usually denoted by λ, that each agent receives unless his demand is lower than this amount, in which case he receives his claim in full. Similarly, the Constrained Equal Losses solution is characterized by a 'common loss' parameter, usually denoted by μ, that each agent discounts from his demand, unless it is lower than μ, in which case he receives no amount. We explore the relationship between these two equalizing parameters, and we find that for each given problem, it is mainly conditioned by the *relative degree of conflict* (Alcalde and Peris 2022): the ratio between the aggregate unfulfilled demand and the aggregate demand.

Our results clarify the relationship between the characteristic values λ and μ defining these solutions, and the agents associated with these parameters: the first agent

[1] Alcalde and Peris (2022) provide a complete description of the main hybrid solutions obtained by combining the Constrained Equal Awards and the Constrained Equal Losses solutions.

obtaining exactly the uniform award λ, or the first agent exactly incurring in the uniform loss μ. These results are useful to better understanding these classical solutions and also to define new solutions combining them. It is also remarkable that the central claimant is precisely the agent with the lowest demand exceeding the value $\lambda + \mu$.

The remaining of the paper is organized as follows: Section 2 introduces the main definitions. Section 3 contains our main results. Section 4 analyzes who is the central claimant in a bankruptcy problem. Finally, Sect. 5 concludes by introducing a new solution, the Consensus-Weighted solution, that combines the two constrained equal solutions according to their 'relative popularity.'

2 Preliminaries

We consider a group of agents $\mathcal{N} = \{1, \ldots, i, \ldots, n\}$, to be named the creditors, that have to distribute among them a certain amount $E \geq 0$ of a perfectly divisible good, called the estate. Each creditor exhibits a claim $c_i \geq 0$ on the estate. A bankruptcy occurs when E is not enough to cover all the creditors' claims:[2] $E \leq C \equiv \sum_{j=1}^{n} c_j$.

Throughout this paper, we assume that the set of creditors $\mathcal{N}$ is fixed. This allows to describe a bankruptcy problem as a pair $(E; c) \in \mathbb{R}_+ \times \mathbb{R}_+^n$ such that $E \leq \sum_{j=1}^{n} c_j$. Without loss of generality, we will concentrate on problems in which creditors are labeled according to their claims, that is, we assume that $c_i \leq c_j$ whenever $i < j$. Let $\mathcal{B}$ denote the set of such bankruptcy problems.

A solution for bankruptcy problems, or simply a solution, is a single-valued function $\varphi : \mathcal{B} \rightarrow \mathbb{R}_+^n$ such that for each given bankruptcy problem $(E; c) \in \mathcal{B}$,

(a) $0 \leq \varphi_i (E; c) \leq c_i$ for each creditor i; and
(b) $\sum_{j=1}^{n} \varphi_j (E; c) = E$.

Aumann and Maschler (1985) introduced the notion of duality for bankruptcy solutions capturing the following idea. Given a problem $(E; c)$, solved according to solution φ, creditor i incurs in a loss of $\ell_i = c_i - \varphi_i (E; c)$. Provided that $E \leq C = \sum_{j=1}^{n} c_j$, the pair $(L; c) = (C - E; c)$ has also the structure of a bankruptcy problem, and thus $(L; c) \in \mathcal{B}$. In this framework, we say that:

Definition 1 Solutions φ and φ^d are **dual** whenever for each problem $(E; c) \in \mathcal{B}$

$$\varphi (E; c) + \varphi^d (C - E; c) = c. \tag{1}$$

This description allows to define self-dual solutions as those whose dual solution is the solution itself.

Definition 2 A solution φ is **self-dual** whenever for each problem $(E; c) \in \mathcal{B}$

$$\varphi (C - E; c) = c - \varphi (E; c).$$

[2] A bankruptcy situation is genuinely declared when $0 < E < C$. That is, the inequalities above become strict and the amount to be distributed is strictly positive. Nevertheless, to avoid some technical problems, we include here the degenerate cases where either there is nothing to be distributed, $E = 0$, or the creditors' claims can be exactly reimbursed, $E = C$.

It is well-known that the dual solution of the Constrained Equal Awards solution is the Constrained Equal Losses solution, and vice versa. Therefore, none of these two solutions is self-dual. Instead, the Proportional solution is self-dual. The next subsections provide a formal definition for the Constrained Equal Awards and the Constrained Equal Losses solutions, and describe intuitive algorithms to obtain these solutions.

2.1 The Constrained Equal Awards solution

The Constrained Equal Awards solution assigns equal amounts to all creditors under the restriction that none of them is awarded more than his claim.

Definition 3 The **Constrained Equal Awards** solution is the function $\varphi^{CEA} : \mathcal{B} \to \mathbb{R}^n_+$ such that for each bankruptcy problem $(E; c) \in \mathcal{B}$ and any creditor $i \in \mathcal{N}$, $\varphi_i^{CEA}(E; c) = \min\{\lambda; c_i\}$, where λ is the unique solution to

$$\sum_{j=1}^{n} \min\left\{\lambda; c_j\right\} = E. \tag{2}$$

Equation (2) can be solved through a simple algorithm that we call the CEA algorithm. Consider a given problem $(E; c)$ and proceed as follows:

Step 1. If $E \leq nc_1$, assign each creditor i the (common) award $\varphi_i^{CEA}(E; c) = E/n$, and stop. Otherwise, assign creditor 1 his claim, that is, $\varphi_1^{CEA}(E; c) = c_1$, and proceed to step 2.

...

Step k. If $E - \sum_{j<k} c_j \leq (n-k+1)\, c_k$, assign each creditor $i \geq k$ the amount

$$\varphi_i^{CEA}(E; c) = \frac{E - \sum_{j=1}^{k-1} c_j}{n-k+1},$$

and stop. Otherwise, assign creditor k his claim c_k and proceed to step $k+1$.

Note that since $(E; c)$ is a bankruptcy problem, there must be a step at which the algorithm above stops. For a given problem $(E; c)$, we denote by $i(\lambda)$ the step at which the CEA algorithm stops.

The solution λ of Eq. (2) is precisely the common award obtained at this final step. That is, for a given problem $(E; c)$,[3]

$$\lambda = \frac{E - \sum_{j=1}^{i(\lambda)-1} c_j}{n - i(\lambda) + 1}. \tag{3}$$

[3] From now on we adopt the convention that $\lambda = E/n$ whenever $i(\lambda) = 1$.

Remark 1 An alternative way to compute the solution of Eq. (2) comes from the following formula. For a given problem $(E; c)$, let us compute for each creditor $i \in \mathcal{N}$

$$\lambda_i = \frac{E - \sum_{j=1}^{i-1} c_j}{n - i + 1}. \quad (4)$$

The coefficient λ_i indicates the common award received by creditors after creditor i, if feasible: $\lambda_i \leq c_i$. The first elements in the sequence are:

$$\lambda_1 = \frac{E}{n} \qquad \lambda_2 = \frac{E - c_1}{n-1} \qquad \lambda_3 = \frac{E - c_1 - c_2}{n-2} \qquad \ldots .$$

Then,

$$\lambda_{i+1} - \lambda_i = \frac{1}{(n-i)(n+1-i)} \left(E - \sum_{j=1}^{i} c_j - (n-i)c_i \right)$$

Observe that from the definition of λ and $i(\lambda)$

$$E - \sum_{j=1}^{i} c_j - (n-i)c_i \geq 0 \quad \Leftrightarrow \quad i \leq i(\lambda)$$

and $\{\lambda_i\}$ is an increasing sequence until it stabilizes at λ (when all creditors receive the same amount) and then decreases. Hence, the connection between Eqs. (3) and (4) comes from the fact that

$$\lambda = \max_{i \in \mathcal{N}} \lambda_i .$$

□

Although the stage $i\ (\lambda)$ has been defined within the algorithm, we use to obtain the Constrained Equal Awards solution, this stage does not depend on such algorithm. Note that $i\ (\lambda)$ denotes the first creditor such that, according to the Constrained Equal Awards solution, receives an amount below his claim.[4] Then, there is a configuration like

creditor	1	2	...	$i\ (\lambda) - 1$	$i\ (\lambda)$	$i\ (\lambda) + 1$	...	n
claim	c_1	c_2	...	$c_{i(\lambda)-1}$	$c_{i(\lambda)}$	$c_{i(\lambda)+1}$	...	c_n
φ_i^{CEA}	c_1	c_2	...	$c_{i(\lambda)-1}$	λ	λ	...	λ

[4] Since it can be the case that $\varphi_{i(\lambda)}^{CEA}(E; c) = \lambda = c_{i(\lambda)}$, roughly speaking, $i\ (\lambda)$ is the first creditor such that for each $E' < E$, $\varphi_i^{CEA}(E'; c) < c_i$.

2.1.1 An optimization approach (1)

Inspired by a result by Schummer and Thomson (1997) establishing that under the Constrained Equal Awards solution, the variance of the amounts received by all the creditors is minimal among all the possible solutions, and we present an alternative way to obtain the Constrained Equal Awards solution, by solving an optimization problem. For a given problem $(E; c)$, let $\mathcal{D}(E; c)$ denote the set of its solutions, that is

$$\mathcal{D}(E; c) = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^{n} x_j = E,\ 0 \leq x_j \leq c_j \ \forall\, j \in \mathcal{N} \right\},$$

then, from Schummer and Thomson (1997), $\varphi^{CEA}(E; c)$ minimizes the variance in $\mathcal{D}(E; c)$; that is, it minimizes

$$\frac{1}{n}\sum_{j=1}^{n} x_i^2 - \bar{x}^2 = \frac{1}{n}\sum_{j=1}^{n} x_i^2 - \left(\frac{\sum_{j=1}^{n} x_i}{n}\right)^2 = \frac{1}{n}\sum_{j=1}^{n} x_i^2 - \left(\frac{E}{n}\right)^2.$$

Since the last term of the previous expression is constant, it can be removed from the optimization problem, so we get that the Constrained Equal Awards solution can be obtained as[5]

$$\left.\begin{array}{ll} \varphi^{CEA}(E; c) = \underset{x}{\operatorname{argmin}} & ||x|| \\ \text{s.t.} & \sum_{j=1}^{n} x_j = E \\ & 0 \leq x_j \leq c_j \ \forall\, j \in \mathcal{N} \end{array}\right\}.$$

□

2.2 The constrained equal losses solution

The Constrained Equal Losses solution φ^{CEL} is also inspired by an egalitarian criterion, but is based on what each creditor fails to recover. This solution aims to equalize the (absolute) level of dissatisfaction among creditors, under the assumption that none of them is awarded a negative amount.

Definition 4 The **Constrained Equal Losses** solution is the function $\varphi^{CEL} : \mathcal{B} \to \mathbb{R}_+^n$ such that for each bankruptcy problem $(E; c) \in \mathcal{B}$ and any creditor $i \in \mathcal{N}$, $\varphi_i^{CEL}(E; c) = \max\{c_i - \mu; 0\}$, where μ is the unique solution to

$$\sum_{j=1}^{n} \max\left\{c_j - \mu; 0\right\} = E. \tag{5}$$

5 For a given vector $x \in \mathbb{R}^n$, $\|x\|$ stands for its length, $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$.

For any given problem $(E; c)$, parameter μ in Eq. (5) can be easily computed through the CEL algorithm below.

Step 1. Denote $L_1 = \sum_{j=1}^{n} c_j - E$. If $L_1 \leq nc_1$, assign each creditor i the award $\varphi_i^{CEL}(E; c) = c_i - L_1/n$, and stop. Otherwise, award creditor 1 the amount $\varphi_1^{CEL}(E; c) = 0$, and proceed to step 2.

...

Step k. Denote $L_k = \sum_{j=k}^{n} c_j - E$. If $L_k \leq (n-k+1)\, c_k$, assign each creditor $i \geq k$ the award

$$\varphi_i^{CEL}(E; c) = c_i - \frac{L_k}{n-k+1}.$$

and stop. Otherwise, award creditor k the amount $\varphi_k^{CEL}(E; c) = 0$, and proceed to step $k+1$.

Note that since $(E; c)$ is a bankruptcy problem there must be a step at which the algorithm above stops. We denote by $i(\mu)$ the step at which the CEL algorithm stops.

The solution μ of Eq. (5) is precisely the common loss obtained at this final step. That is, for a given problem $(E; c)$,

$$\mu = \frac{\sum_{j=i(\mu)}^{n} c_j - E}{n - i(\mu) + 1}. \tag{6}$$

Remark 2 As illustrated in Remark 1, we can provide an alternative way to compute the solution of Eq. (5) that comes from the following formula. For a given problem $(E; c)$, let compute for each creditor $i \in \mathcal{N}$

$$\mu_i = \frac{\sum_{j=i}^{n} c_j - E}{n - i + 1}. \tag{7}$$

The coefficient μ_i indicates the common loss incurred by creditors after creditor i, if feasible: $\mu_i \leq c_i$. The first elements in the sequence are:

$$\mu_1 = \frac{\sum_{j=1}^{n} c_j - E}{n} \qquad \mu_2 = \frac{\sum_{j=2}^{n} c_j - E}{n-1} \qquad \mu_3 = \frac{\sum_{j=3}^{n} c_j - E}{n-2} \qquad \ldots.$$

Then,

$$\mu_{i+1} - \mu_i = \frac{1}{(n-i)}\left(c_i - \frac{E - \sum_{j=1}^{i} c_j}{n+1-i}\right)$$

Observe that from the definition of μ and $i(\mu)$

$$c_i - \frac{E - \sum_{j=1}^{i} c_j}{n+1-i} \geq 0 \quad \Leftrightarrow \quad i \leq i(\mu)$$

and $\{\mu_i\}$ is an increasing sequence until it stabilizes at μ (when all creditors lose the same amount) and then decreases. Then, the connection between Eqs. (5) and (7) comes from the fact that

$$\mu = \max_{i \in \mathcal{N}} \mu_i .$$

□

As in the case of the Constrained Equal Awards, the stage $i\ (\mu)$ we use to obtain the Constrained Equal Losses solution does not depend on such algorithm. Note that $i\ (\mu)$ denotes the first creditor receiving some positive amount.[6] Therefore, agents with a label lower than $i\ (\mu)$ do not obtain a positive award.

creditor	1	2	...	$i\ (\mu) - 1$	$i\ (\mu)$	$i\ (\mu) + 1$	...	n
claim	c_1	c_2	...	$c_{i(\mu)-1}$	$c_{i(\mu)}$	$c_{i(\mu)+1}$	...	c_n
φ_i^{CEL}	0	0	...	0	$c_{i(\mu)} - \mu$	$c_{i(\mu)+1} - \mu$	...	$c_n - \mu$

The relationship between the position of stages $i\ (\lambda)$ and $i\ (\mu)$ defining both solutions is analyzed in Sect. 3.

2.2.1 An optimization approach (2)

As we illustrated for the Constrained Equal Awards solution, there is an alternative way to compute the Constrained Equal Losses solution through solving an optimization problem. In this case,

$$\left.\begin{array}{rl} \varphi^{CEL}(E; c) = \underset{y}{\operatorname{argmin}} & \|c - y\| \\ \text{s.t.} & \sum_{j=1}^{n} y_j = E \\ & 0 \leq y_j \leq c_j \ \forall\, j \in \mathcal{N} \end{array}\right\} .$$

□

3 A comparison of the constrained equal solutions

The aim of this section is to obtain, for a given problem $(E; c)$, a relationship between the parameters λ and μ, as well as between the stages of the CEA and CEL algorithms where λ and μ are determined. Before dealing with this objective, it is useful to define the relative degree of conflict of $(E; c)$, introduced by Alcalde and Peris (2022).

[6] As we mentioned in footnote 4 for the case of $i\ (\lambda)$, $i\ (\mu)$ is, roughly speaking, the first creditor such that for each $E' > E$, $\varphi_i^{CEA}\left(E'; c\right) > 0$.

Definition 5 Given a problem $(E; c) \in \mathcal{B}$, its **relative degree of conflict** is the ratio between the excess demand and the aggregate claim:

$$\gamma(E; c) = \frac{\sum_{j=1}^{n} c_j - E}{\sum_{j=1}^{n} c_j} = \frac{L}{C}.$$

Remark 3 Note that for each bankruptcy problem $(E; c)$, $\gamma(E; c) \in [0, 1]$. Moreover, the parameter $\gamma(E; c)$ measures how strong the deficit associated with this problem is. The smaller $\gamma(E; c)$, the 'lighter' the problem associated with this bankrupt, in the sense that the aggregate claim can be satisfied almost completely. Note that in the extreme (degenerate) case where $\gamma(E; c) = 0$, each creditor would receive his own claim and thus, strictly speaking, no bankruptcy problem occurs.

As we see below, when comparing the parameters λ and μ associated with the Constrained Equal Awards and Constrained Equal Losses solutions for a given problem $(E; c)$, it is very important to know whether the problem exhibits a high or a low degree of conflict.

Theorem 1 *For any given bankruptcy problem* $(E; c)$

$$\mu \geq \lambda \quad \text{if and only if} \quad \gamma(E; c) \geq \frac{1}{2}.$$

Proof For a given problem $(E; c)$, let $\lambda(E; c)$ and $\mu(E; c)$ denote the parameters obtained when running the CEA and CEL algorithms for such a problem, respectively.

Assume that $\gamma(E; c) < 0.5$. Note that $\gamma(E; c) < 0.5$ if and only if $2E > \sum_{j=1}^{n} c_j$ and then $E > L$, where $L = \sum_{j=1}^{n} c_j - E$ is the aggregate loss in which creditors incur when facing the problem $(E; c)$.

As φ^{CEA} is order preserving and resource monotonic[7] (see, for instance, Thomson 2019) we have that for each creditor $i \in \mathcal{N}$, $\varphi_i^{CEA}(E; c) \geq \varphi_i^{CEA}(L; c)$, with the inequality strict for creditor n. Therefore, $\lambda(E; c) = \varphi_n^{CEA}(E; c) > \varphi_n^{CEA}(L; c) = \lambda(L; c) = \mu(E; c)$ where the last equality comes from the fact that φ^{CEA} and φ^{CEL} are dual solutions.

To conclude, let us observe that $\gamma(E; c) \geq 0.5$ if and only if $2E \geq \sum_{j=1}^{n} c_j$ and then $E \geq L$. Therefore, the arguments above can be replicated for the cases where $\gamma(E; c) = 0.5$, or $\gamma(E; c) > 0.5$. □

As an immediate consequence, we obtain the following result.

Corollary 1 *Let* $(E; c)$ *be a bankruptcy problem with* $\gamma(E; c) < 0.5$ *(respectively,* $\gamma(E; c) > 0.5$*;* $\gamma(E; c) = 0.5$*). Then, the CEL algorithm stops not later (not before; at the same time, respectively) than the CEA algorithm stops. That is,* $i(\lambda) \geq i(\mu)$ *(respectively,* $i(\lambda) \leq i(\mu)$*;* $i(\lambda) = i(\mu)$*).*

[7] A bankruptcy solution φ is **order preserving** if for each given problem $(E; c)$ and any two creditors $i, j \in \mathcal{N}$, $\varphi_i(E; c) \leq \varphi_j(E; c)$ whenever $c_i \leq c_j$. Solution φ is **resource monotonic** if for each problem $(E; c)$ and any E', $0 \leq E' \leq E$, $\varphi_i(E'; c) \leq \varphi_i(E; c)$ for each creditor $i \in \mathcal{N}$.

Proof For $(E; c)$ given, let $\lambda = \lambda(E; c)$ and $\mu = \mu(E; c)$ be the solutions for Eqs. (2) and (5), respectively. Note that $i(\lambda) = \max\left\{j \in \mathcal{N} : c_j < \lambda\right\} + 1$. Similarly, we note that $i(\mu) = \max\left\{j \in \mathcal{N} : c_j < \mu\right\} + 1$. Assume that $\gamma(E; c) < 0.5$. Then, by Theorem 1, $\lambda > \mu$ and thus $i(\lambda) \geq i(\mu)$. A similar argument applies when either $\gamma(E; c) > 0.5$ or $\gamma(E; c) = 0.5$. □

The following result establishes that the sum of both solutions is greater than the claim for all creditors, or it is lower than the claim for all creditors, depending on the value of the relative degree of conflict $\gamma(E; c)$.

Theorem 2 *Let $(E; c)$ a bankruptcy problem. Then, either*

(1) $\varphi_i^{CEA}(E; c) + \varphi_i^{CEL}(E; c) \geq c_i$ for each creditor $i \in \mathcal{N}$; or
(2) $\varphi_i^{CEA}(E; c) + \varphi_i^{CEL}(E; c) \leq c_i$ for each creditor $i \in \mathcal{N}$.

Moreover, $\varphi_i^{CEA}(E; c) + \varphi_i^{CEL}(E; c) = c_i$, for each creditor if and only if $\gamma(E; c) = 0.5$.

Proof It is well-known that the Constrained Equal Awards and the Constrained Equal Losses are dual solutions; that is, for each bankruptcy problem $(E; c)$, it holds that $\varphi^{CEA}(E; c) = c - \varphi^{CEL}\left(\sum_{j=1}^n c_j - E; c\right) = c - \varphi^{CEL}(L; c)$. Then,

$$\varphi^{CEA}(E; c) + \varphi^{CEL}(L; c) = c.$$

On the other hand,

$$\gamma(E; c) \geq \frac{1}{2} \quad \Leftrightarrow \quad \frac{C - E}{C} \geq \frac{1}{2} \quad \Leftrightarrow \quad 2C - 2E \geq C \quad \Leftrightarrow \quad C \geq 2E \quad \Leftrightarrow \quad L \geq E,$$

and the result follows straightforwardly, since both solutions fulfill resource monotonicity (see, for instance, Thomson 2019). □

It is important to note that the sum of the Constrained Equal Awards and the Constrained Equal Losses solutions cannot be greater for some agents and lower for some other agents. Then, if we consider the average of these solutions:

$$\varphi^{Av}(E; c) = \frac{1}{2}\varphi^{CEA}(E; c) + \frac{1}{2}\varphi^{CEL}(E; c)$$

it holds that all agents receive at least half of his claim (whenever $\gamma(E; c) \leq 0.5$), or all agents receive at most half of his claim (whenever $\gamma(E; c) \geq 0.5$).

Corollary 2 *For each bankruptcy problem $(E; c)$, and agent $i \in \mathcal{N}$,*

(a) if $\gamma(E; c) \geq 0.5$, then $\varphi_i^{Av}(E; c) \leq \frac{1}{2}c_i$, and
(b) if $\gamma(E; c) \leq 0.5$, then $\varphi_i^{Av}(E; c) \geq \frac{1}{2}c_i$.

This property, that can be understood as a kind of *half-claim reference*, is shared by some well known solutions, as the *Proportional* solution, that associates with each problem $(E;c)$ the share of the estate $\varphi^{P}(E;c)=(1-\gamma(E;c))\,c$; or the *Talmud* solution (Aumann and Maschler 1985), that fits the expression

$$\varphi^{T}(E;c)=\begin{cases}\varphi^{CEA}\left(E;\frac{1}{2}c\right) & \text{if } \gamma(E;c)\geq 0.5\\ \frac{1}{2}c+\varphi^{CEL}\left(E-\frac{1}{2}C;\frac{1}{2}c\right) & \text{if } \gamma(E;c)\leq 0.5\end{cases}$$

4 Finding the central claimant

An interesting question about the Constrained Equal Awards and the Constrained Equal Losses solutions is to which extent we can characterize the problems such that more agents *prefer* the first solution. To this matter, for a given problem $(E;c)$, we define its *central claimant* as the agent z such that agents with a large claim prefer the Constrained Equal Losses solution. Note that ties in claims are allowed, so to *count* how many agents prefer each of these solutions, we select as the central claimant the last one who does not prefer the Constrained Equal Losses (in the order we initially prefixed, $\mathcal{N}=\{1,\ldots,i,\ldots,n\}$, such that $c_i\leq c_j$ whenever $i<j$).

Definition 6 Given a bankruptcy problem $(E;c)\in\mathcal{B}$ the **central claimant** is the agent $z\in\mathcal{N}$ such that:

(a) $\varphi_i^{CEA}(E;c)\geq\varphi_i^{CEL}(E;c)$ for each $i\leq z$, and
(b) $\varphi_i^{CEA}(E;c)<\varphi_i^{CEL}(E;c)$ for each $i>z$.

Remark 4 Related to Condition (a), note that for any problem $(E;c)$, and creditor i such that $c_i=0$, it trivially follows that $\varphi_i^{CEL}(E;c)=\varphi_i^{CEA}(E;c)=0$. Condition (b) is imposed to determine which creditor is named the central claimant, namely the last creditor, with positive claim, weakly preferring the distribution by the Constrained Equal Awards solution to that determined by the Constrained Equal Losses solution. Finally, from now on we consider problems $(E;c)$ such that the vector of (ordered) claims satisfies $c_1<c_n$. Note that, otherwise, for each agent i, $\varphi_i^{CEL}(E;c)=\varphi_i^{CEA}(E;c)=E/n$ which trivially implies that n is the central claimant.

From the configuration of both solutions (see tables in Sect. 2), we observe that if for some individual $i\in\mathcal{N}$, $\varphi_i^{CEA}(E;c)\geq\varphi_i^{CEL}(E;c)$, then for all $j\in\mathcal{N}$ such that $c_j\leq c_i$, it is also true that $\varphi_j^{CEA}(E;c)\geq\varphi_j^{CEL}(E;c)$. Therefore, the central claimant $z\in\mathcal{N}$ is defined by the equation:[8]

$$c_z-\mu(E;c)\leq\lambda(E;c)<c_{z+1}-\mu(E;c),$$

which is equivalent to

$$c_z\leq\lambda(E;c)+\mu(E;c)<c_{z+1}. \tag{8}$$

[8] When the central claimant is $z=n$, this inequality becomes $c_n-\mu(E;c)\leq\lambda(E;c)$. Note that this only holds when either $E=0$ or $E=C$.

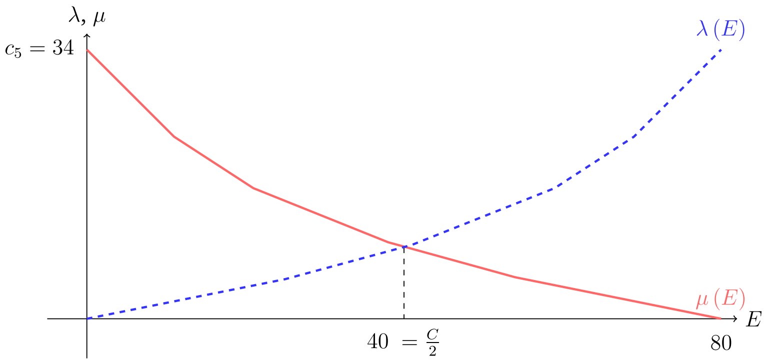

Fig. 1 Graphs of λ (in dashed blue) and μ (in red) as a function of the estate (colour figure online)

That is, in a given problem $(E; c)$, the sum of the common award and the common loss, $\lambda(E; c) + \mu(E; c)$, determines the agent dividing those individuals that prefer the Constrained Equal Awards from those preferring the Constrained Equal Losses.

To illustrate the above assertions, let us consider the following example.

Example 1 Let $\mathcal{B}_5$ the family of five-creditor problems $(E; c)$ where the claims vector is fixed as $c = (5, 8, 13, 20, 34)$ and $0 \leq E \leq C = 80$. Figure 1 illustrates how λ and μ vary with the estate E.

Observe that according to Eq. (8) that characterizes the central claimant, the claim of creditors 1 to 3 is always strictly lower than $\lambda (E; c) + \mu (E; c)$, so these creditors always obtain a higher reimbursement when the estate is distributed according to the Constrained Equal Awards rather than the Constrained Equal Losses solution, no matter the amount of estate to be distributed. On the contrary, $c_5 \geq \lambda (E; c) + \mu (E; c)$ for all E, and thus $\varphi_5^{CEL} (E; c) \geq \varphi_5^{CEA} (E; c)$. Creditor 4 deserves a deeper analysis. For $E \in [0, 70/3] \cup [170/3, 80]$ creditor 4 (weakly) prefers the distribution proposed by the Constrained Equal Awards to that proposed by the Constrained Equal Losses. However, this preference is reversed, that is, $\varphi_4^{CEL} \left(E'; c\right) > \varphi_4^{CEA} \left(E'; c\right)$ whenever the estate E' is in the interval $(70/3, 170/3)$. Therefore for $c = (5, 8, 13, 20, 34)$, the central claimant for each estate, namely $z (E; c)$, is

$$z (E; c) = \begin{cases} 3 \text{ for } E \in (70/3, 170/3) \\ 4 \text{ for } E \in (0, 70/3] \cup [170/3, 80) \\ 5 \text{ for } E = 0 \text{ or } E = 80 \end{cases}$$

Figure 2 illustrates the above reasoning.

The above example invites to describe a simple procedure to find the central claimant for each problem. To proceed, we consider a given vector of claims c. We first identify which creditors can be the central claimant, for an appropriate estate. We then compute the ranks on the estate at which each of these creditors are the central claimant. Next, we formalize this process.

For a claims vector c, and thus aggregate claim $C = \sum_{i=1}^{n} c_i$, compute the equalizing parameters $\lambda (C/2; c) = \mu (C/2; c)$. Let $z (0)$ denote the creditor i such that

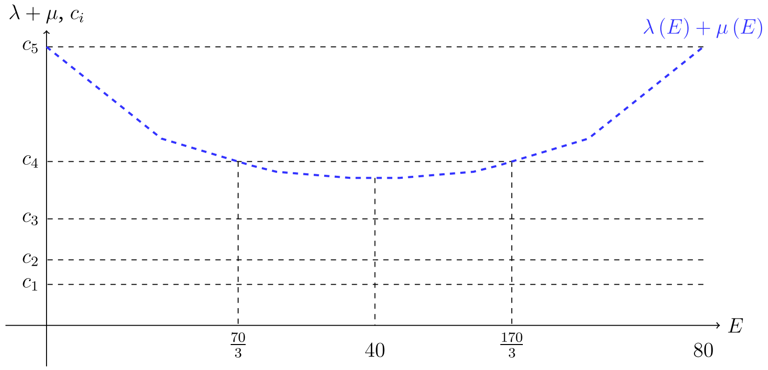

Fig. 2 Finding the central claimant

(a) $c_i \le 2\lambda(C/2; c)$, and
(b) for each $j > i$, $c_j > 2\lambda(C/2; c)$.

Define the sequence of creditors $\{z(k)\}_{k=1}^{t}$, $t \le n$, such that for each $k < t$, $z(k)$ is such that $c_{z(k-1)} < c_{z(k)} < c_{z(k+1)}$; and $z(t) = n$, the last creditor.[9] Once these creditors have been identified, we compute for each of them the two solutions for the equation

$$c_{z(k)} = \lambda\left(E_{z(k)}; c\right) + \mu\left(E_{z(k)}; c\right), \tag{9}$$

and denote them by $E^-_{z(k)}$ and $E^+_{z(k)}$, with $E^-_{z(k)} < E^+_{z(k)}$. Note that in particular, for the last creditor $z(t) = n$, we have $E^-_{z(t)} = 0$ while $E^+_{z(t)} = C$. Moreover, for each k, $E^-_{z(k)} + E^+_{z(k)} = C$. Therefore, when confronting the solutions for all the equations, it follows that

$$0 = E^-_{z(t)} < E^-_{z(t-1)} < \cdots < E^-_{z(1)} < \frac{C}{2} < E^+_{z(1)} < \cdots < E^+_{z(t-1)} < E^+_{z(t)} = C. \tag{10}$$

This allows to identify the central claimant for each problem $(E; c)$ as follows. Given the vector of claims c, once obtained the sequence of claimants $z(0), z(1), \ldots, z(k), \ldots, z(t) = n$, the central claimant is:

$$z(E; c) = \begin{cases} z(0) \text{ for } E \in \left(E^-_{z(1)}, E^+_{z(1)}\right) \\ z(1) \text{ for } E \in \left(E^-_{z(2)}, E^-_{z(1)}\right] \cup \left[E^+_{z(1)}, E^+_{z(2)}\right) \\ \qquad \cdots \\ z(k) \text{ for } E \in \left(E^-_{z(k+1)}, E^-_{z(k)}\right] \cup \left[E^+_{z(k)}, E^+_{z(k+1)}\right) \ \forall k = 2, \ldots, t-1 \\ z(t) \text{ for } E = 0 \text{ or } E = C \end{cases}$$

[9] We are only interested in the claim of each creditor, since agents with equal claims will receive the same amount. In case of a tie in the claims, we select the last creditor with this claim.

As a consequence, we obtain the following result.

Proposition 1 *For each bankruptcy problem with claims vector c, agent i* $\in \mathcal{N}$ *weakly prefers the Constrained Equal Awards solution independently of the value of the estate* $0 \leq E \leq C = \sum_{i=1}^{n} c_i$ *if and only if* $c_i \leq c_{z(0)}$.

4.1 Additional features of $(\lambda + \mu)$ function

Consider, as illustrated by Example 1, the family of problems $(E; c)$ associated with a given vector of claims c, while the estate E varies from 0 to $C = \sum_{i=1}^{n} c_i$. It can be observed that the graphs shown in Figs. 1 and 2 are piece-wise linear. The step points (where the slope changes) can be determined in general terms by the following expressions:

(a) For $\lambda(E)$, the $(n-1)$ step points follow the expression

$$x_i = \sum_{j=1}^{i-1} c_j + (n - i + 1)\, c_i;$$

(b) For $\mu(E)$, the $(n-1)$ step points follow the expression

$$y_i = \sum_{j=n-i}^{n} c_j - (n - i)\, c_i.$$

Observe that for each $k = 1, 2, \ldots, n-1$,

$$x_k + y_{n-k} = \sum_{j=1}^{n} c_j = C. \tag{11}$$

This simplifies the computations needed to obtain all the step points since we only need to compute the λ step points. If we order the $2n-2$ step points from the smallest to largest, say $\mathcal{P} = \{p_1, p_2, \ldots, p_{2n-2}\}$, we obtain a partition of the possible values of $E \in (0, C)$. From (11), exactly half of these values are lower than or equal to $C/2$, so

$$p_{n-1} \leq \frac{C}{2} \leq p_n.$$

Moreover, as the slopes of λ and μ with respect to E are

$$s(\lambda) = \frac{1}{n - i(\lambda) + 1} \qquad \text{and} \qquad s(\mu) = \frac{-1}{n - i(\mu) + 1}$$

the step points occur when the variation in E gives that the CEA or the CEL algorithms need an additional step to stop.

Then, the slope in each interval determined from the above points is

$$s\left(\lambda(E)+\mu(E)\right)=\frac{1}{n-i(\lambda)+1}-\frac{1}{n-i(\mu)+1}.$$

If $E \leq C/2$, then $\gamma(E;c) \geq 1/2$ and $i(\lambda) \leq i(\mu)$, so $s\left(\lambda(E)+\mu(E)\right)$ is negative. Analogously, if $E \geq C/2$, this slope is positive. So, by continuity of both $\lambda\left(\cdot\right)$ and $\mu\left(\cdot\right)$ with respect to E, the minimum value of $\lambda(E)+\mu(E)$ is reached at $E=C/2$. Moreover, $\lambda(E)+\mu(E)$ is symmetric regarding the central value $E=C/2$ and it is constant in the central interval $\left[p_{n-1},p_n\right]$, since $i(\lambda)=i(\mu)$ and then for all $E\in\left[p_{n-1},p_n\right]$

$$\min\left\{\lambda(E)+\mu(E)\colon E\in[0,C]\right\}=\frac{\sum_{j=i(\lambda)}^{n}c_j-\sum_{j=1}^{i(\lambda)-1}c_j}{n-i(\lambda)+1}.$$

With the data in Example 1, $c=(5,8,13,20,34)$, the obtained step points and intervals are:

$$\begin{array}{llllllll} x_1=25 & x_2=37 & x_3=52 & x_4=66 \\ y_1=14 & y_2=28 & y_3=43 & y_4=55 \\ p_1=14 & p_2=25 & p_3=28 & p_4=37 & p_5=43 & p_6=52 & p_7=55 & p_8=66 \end{array}$$

The central interval $[37,43]$ provides the minimum value for $\lambda(E;c)+\mu(E;c)$. In this interval, $\lambda(E;c)+\mu(E;c)$ is constant and then coincides with the value obtained for the estate $E=C/2=40$, that is, $\min_E\left\{\lambda(E;c)+\mu(E;c)\right\}=2\lambda(C/2;c)=18$. Therefore, creditors 1, 2 and 3, with claims $c_i<18$, always prefer the Constrained Equal Awards solution regardless of the value of the estate E.

5 Forgiving debts in bankruptcy solutions

To conclude the paper, we deal with a convex combination of the two ‘constrained equal’ solutions according to the popular support that each one receives. To describe this solution, we introduce some additional notation. For a given problem $(E;c)\in\mathcal{B}$, we denote

$$\mathcal{N}^A\left(E;c\right)=\left\{i\in\mathcal{N}\colon \varphi_i^{CEA}\left(E;c\right)>\varphi_i^{CEL}\left(E;c\right)\right\};\ \text{and}$$
$$\mathcal{N}^L\left(E;c\right)=\left\{j\in\mathcal{N}\colon \varphi_j^{CEL}\left(E;c\right)>\varphi_j^{CEA}\left(E;c\right)\right\},$$

the agents preferring to distribute the estate according to the Constrained Equal Awards or the Constrained Equal Losses solutions, respectively. Their cardinalities are denoted by $n^A\left(E;c\right)$ and $n^L\left(E;c\right)$. Note that $0\leq n^A\left(E;c\right)+n^L\left(E;c\right)\leq n$. For the sake of completeness, we denote by $\mathcal{N}^I\left(E;c\right)$, with cardinality $n^I\left(E;c\right)=n-n^A\left(E;c\right)-$

$n^L(E;c)$, the set of creditors being indifferent between the Constrained Equal Awards and the Constrained Equal Losses solutions at problem $(E;c)$.

In what follows, and just for interpretative purposes, we consider bankruptcy problems where no creditor exhibits a null claim, and the estate is strictly positive. Note that, since claims are increasingly ordered, the central claimant is precisely creditor $z = n - n^L(E;c)$.

Using the proportions of creditors supporting each solution, we can define the Consensus-Weighted solution, a convex combination of the φ^{CEA} and φ^{CEL} solutions, so that the coefficient of each solution is the relative popular support of such a constrained equal solution.

Definition 7 We define the **Consensus-Weighted solution** for bankruptcy problems as the function $\varphi^{CW}: \mathcal{B} \to \mathbb{R}^n_+$ associating each problem $(E;c)$ the share of the estate

$$\varphi^{CW}(E;c) = \frac{n^A(E;c)}{n^A(E;c)+n^L(E;c)}\varphi^{CEA}(E;c) + \frac{n^L(E;c)}{n^A(E;c)+n^L(E;c)}\varphi^{CEL}(E;c), \tag{12}$$

when $c_1 < c_n$; and $\varphi^{CW}(E;c) = \varphi^{CEA}(E;c) = \varphi^{CEL}(E;c)$ when $c_1 = c_n$.

Remark 5 Note that for a problem $(E;c) \in \mathcal{B}$, $\mathcal{N}^A(E;c) = \mathcal{N}^L(E;c) = \emptyset$ if and only if $c_1 = c_n$. In this case, since $n^A(E;c) + n^L(E;c) = 0$, the expression in Eq. (12) is not properly defined.

It is remarkable that from an equity perspective, the Consensus-Weighted solution exhibits a nice behavior. In particular, it is order-preserving because both the Constrained Awards and the Constrained Equal Losses solutions satisfy this property. Nevertheless, it fails to be continuous since the number of agents is finite.

Associated with this discontinuity, we find that some agents might have incentives to hide information about their true claim.[10] The reason is that the Consensus-Weighted solution fails to be claims monotonic. Just to illustrate this, and also to find a rationale of such a behavior, consider a three-claimant instance where $c = (100, 200, 302)$, and $E = 300$. In this case we have that $\varphi^{CEA}(E;c) = (100, 100, 100)$, while $\varphi^{CEL}(E;c) = (0, 99, 201)$. Therefore, $\mathcal{N}^A = \{1, 2\}$ and $\mathcal{N}^L = \{3\}$, and thus $\varphi^{CW}(E;c) = (200/3, 299/3, 401/3)$. Now consider the situation where the claim of creditor 3 diminishes from the (original) 302 to $c'_3 = 298$. Then, creditor 3's award raises to 166. The reason is that, even though both the Constrained Awards and the Constrained Equal Losses solutions are claims monotonic, creditor 3, by reducing his claim, forces the set of high claimants $\mathcal{N}^L$ to become larger. That is $n^L(E;c) < n^L\left(E;c'\right)$. What is relevant for creditor 3 is that the benefit from increasing the popularity of the Constrained Equal Losses solution exceeds the losses from reducing his claim.

The arguments above suggest the interest of studying the following situation. Consider a given bankruptcy problem $(E;c)$. Assume that creditors might 'forgive' (part of) their debt, so that i's effective claim becomes $0 \le x_i < c_i$. Once

[10] Note that in a bankruptcy situation, a creditor might declare a claim x_i lower than her true claim c_i just by 'hiding' some invoices. Nevertheless, no agent can justify a claim exceeding her credit.

each creditor has declared his effective claim, the debtor should face an outlay of $O = \min\left\{E; \sum_{i=1}^{n} x_i\right\}$. Under these circumstances each creditor is awarded x_i when $O = \sum_{j=1}^{n} x_j < E$. Otherwise, $(E; x)$ describes a bankruptcy problem[11] and agents are rewarded according to the Consensus-Weighted solution. In this framework, it can be relevant to explore which creditors select a reduced effective claim $x_i < c_i$. That is, to analyze at which extent some agent has incentives to show a 'forgiving' behavior when the estate is distributed according to the Consensus-Weighted solution. It is clear that if creditor i is trying to manipulate, he will select some $x_i \geq E - \sum_{k \neq i} c_k = \widehat{x}_i$. In other case, his manipulation strategy might be tweaked by selecting $\widehat{x}_i$ and thus reaching a higher award. Therefore, we have no loss of generality in our analysis when assuming that the manipulating strategy x_i by creditor i fits the boundary above.

To formalize our results, we introduce some simplifying additional notation. For a given problem $(E; c)$, agent i and effective claim $x_i < c_i$ for this agent, $(E; (x_i, c_{-i}))$ stands for the situation where i's claim has been reduced from c_i to $x_i \geq 0$ and other claims remain unchanged. We abuse notation and describe, for each $(E; (x_i, c_{-i}))$, and agent j

$$\varphi_j^{CW}(E; (x_i, c_{-i})) = \begin{cases} c_j & \text{if } j \neq i \\ x_i & \text{if } j = i \end{cases}$$

when $x_i < E - \sum_{k \neq i} c_k$. Otherwise, $\varphi_j^{CW}(E; (x_i, c_{-i}))$ fits expression (12), in Definition 7.

Definition 8 We say that creditor i manipulates (or profits from manipulating) φ^{CW} at $(E; c)$ via x_i, $x_i < c_i$, whenever $\varphi_i^{CW}(E; (x_i, c_{-i})) > \varphi_i^{CW}(E; c)$.

For $(E; c)$ given, λ and μ denote its equal awards and equal losses parameters, respectively. For creditor i, and reduced claim x_i, $\lambda(x_i)$ and $\mu(x_i)$ denote the equal awards and equal losses parameters for problem $(E; (x_i, c_{-i}))$ respectively. In addition, since $(E; c)$ is given, from now on, we denote $n^A = n^A(E; c)$, $n^L = n^L(E; c)$, and $n^I = n^I(E; c)$; for agent i and alternative claim $x_i < c_i$, $n^A(x_i) = n^A(E; (x_i, c_{-i}))$, $n^L(x_i) = n^L(E; (x_i, c_{-i}))$, and $n^I(x_i) = n^I(E; (x_i, c_{-i}))$. Similar notation is employed hereafter for sets $\mathcal{N}^A$, $\mathcal{N}^L$ and $\mathcal{N}^I$.

To understand the logic behind the manipulability of φ^{CW}, we introduce some comparative statics insights. Consider a given problem $(E; c)$ and assume that creditor i, by following a forgiving behavior, declares a claim $x_i < c_i$. Since the two constrained equal solutions are claims monotonic, creditor i, by selecting a forgiving strategy $x_i < c_i$ incurs in a 'loss effect' because $\varphi_i^{CEA}(E; (x_i, c_{-i})) \leq \varphi_i^{CEA}(E; c)$, and $\varphi_i^{CEL}(E; (x_i, c_{-i})) \leq \varphi_i^{CEL}(E; c)$, and at least one of the inequalities is strict. For interpretative purposes, we split this global loss effect as the addition of the CEA loss effect, namely $\varphi_i^{CEA}(E; c) - \varphi_i^{CEA}(E; (x_i, c_{-i}))$, and the CEL loss effect, to be described as $\varphi_i^{CEL}(E; c) - \varphi_i^{CEL}(E; (x_i, c_{-i}))$.

To definitively manipulate, creditor i needs to compensate the 'loss effect' by producing a 'popularity effect' in which the solution preferred by i at problem $(E; c)$

[11] We assume that the components of x are increasingly ordered. Otherwise, relabel the indices.

reaches a higher popularity by the remaining creditors. This popularity effect translates into the following. Assume that the manipulating creditor i belongs to $\mathcal{N}^A$. By the loss effect, i is also in $\mathcal{N}^A(x_i)$. To be successful in his manipulation strategy x_i, a necessary (but not sufficient) condition is that $n^A(x_i) > n^A$. Similar justification applies for $i \in \mathcal{N}^L$, where the necessary condition becomes $n^L(x_i) > n^L$. As a consequence of the above facts, we conclude that no creditor being indifferent between the two constrained equal solutions is able to manipulate the Consensus-Weighted solution.

Proposition 2 *Assume that creditor i profits from manipulating φ^{CW} at $(E; c)$ via x_i. Then, $\varphi_i^{CEA}(E; c) \neq \varphi_i^{CEL}(E; c)$.*

Proof Assume that i profits from manipulating φ^{CW} at $(E; c)$ via x_i and $\varphi_i^{CEA}(E; c) = \varphi_i^{CEL}(E; c)$. Note that the Consensus-Weighted solution point-wise describes a convex combination of the two constrained equal solutions,

$$\varphi^{CW}(E; c) = \alpha(E; c)\,\varphi^{CEA}(E; c) + (1 - \alpha(E; c))\,\varphi^{CEL}(E; c),$$

where $\alpha(E; c) = n^A / \left(n^A + n^L\right)$ captures the popularity of the Constrained Equal Awards solution. This implies that $\varphi^{CW}(E; c) = \varphi^{CEA}(E; c)$. Now, consider problem $(E; (x_i, c_{-i}))$.

$$\varphi_i^{CEA}(E; (x_i, c_{-i})) \leq \varphi_i^{CEL}(E; c) \qquad \varphi_i^{CEL}(E; (x_i, c_{-i})) \leq \varphi_i^{CEA}(E; c)$$

so, for any $\alpha \in [0, 1]$

$$\varphi_i^{CW}(E; (x_i, c_{-i})) \leq \alpha\varphi_i^{CEA}(E; c) + (1 - \alpha)\varphi_i^{CEL}(E; c) = \varphi_i^{CW}(E; c),$$

a contradiction. □

We now describe the comparative statics associated with a reduction in i's claim from c_i to $x_i \geq \widehat{x}_i$. This is useful to explore at which extent creditor i is able to manipulate φ^{CW} at $(E; c)$ via x_i.

Consider a given problem $(E; c)$, a manipulating creditor i, and some forgiving strategy $x_i \geq \widehat{x}_i$. Then,

$$\varphi_i^{CEA}(E; (x_i, c_{-i})) = \min\left\{x_i, \varphi_i^{CEA}(E; c)\right\}. \tag{13}$$

The CEA loss effect for i, if any, is uniformly distributed among the creditors $k \neq i$ truthfully constrained by the Constrained Equal Awards solution at $(E; c)$.

For the case of the CEL loss effect, it is null for agents with a low claim. In particular, for the manipulating creditor i,

$$\varphi_i^{CEL}(E; (x_i, c_{-i})) = \varphi_i^{CEL}(E; c) = 0.$$

if and only if $\mu \geq c_i$. Nevertheless, for $\mu < c_i$, the CEL loss effect is positive. That is, in the latter case, $\varphi_i^{CEL}(E; (x_i, c_{-i})) < \varphi_i^{CEL}(E; c)$. Moreover, when the CEL loss effect is positive, $\varphi_k^{CEL}(E;(x_i,c_{-i})) > \varphi_k^{CEL}(E;c)$ for each creditor $k \neq i$ such that $c_k \geq \mu$.

In particular, the above yields the following result.

Proposition 3 *Assume that creditor i profits from manipulating φ^{CW} at $(E;c)$ via x_i. If $i \leq z$, then $x_i < \lambda$.*

Proof Assume that $i \leq z$ is a manipulating creditor. By Proposition 2, $i \notin \mathcal{N}^I$ which implies that $i \in \mathcal{N}^A$. If i manipulates by selecting a forgiving strategy $x_i \geq \lambda$, Eq. (13) implies that for each creditor $k \in \mathcal{N}$, $\varphi_k^{CEA}(E;(x_i,c_{-i})) = \varphi_k^{CEA}(E;c)$. This also applies for the manipulating creditor i.

Moreover, by the arguments above related to the CEL loss effect, for each $k > z$, $\varphi_k^{CEL}(E;(x_i,c_{-i})) > \varphi_k^{CEL}(E;c)$ which implies that $n^A(x_i) \leq n^A$ and $n^L(x_i) \geq n^L$, which contradicts that i is a manipulating creditor. □

In the following proposition, we show that when there is a creditor who is indifferent between both constrained equal solutions, then the Consensus-Weighted solution is always manipulable (except for the trivial case in which all creditors have the same claim).

Proposition 4 *Assume $n^I > 0$. Then, there is a creditor i who profits from manipulating φ^{CW} at $(E;c)$ via $x_i < c_i$ if and only if $n^I < n$.*

Proof Assume $n^I = n$. This implies that $c_1 = c_n$, and thus, for each creditor k,

$$\varphi_k^{CEA}(E;c) = \varphi_k^{CEL}(E;c) = \varphi_k^{CW}(E;c) = \frac{E}{n}.$$

So, for any $\alpha \in [0,1]$

$$\varphi_i^{CW}(E;(x_i,c_{-i})) \leq \alpha\varphi_i^{CEA}(E;c) + (1-\alpha)\varphi_i^{CEL}(E;c) = \frac{E}{n} = \varphi_i^{CW}(E;c),$$

a contradiction.

Now, assume that $n^I < n$. Since $n^I > 0$, the three sets $\mathcal{N}^A, \mathcal{N}^L$, and $\mathcal{N}^I$ are non-empty. Denote by i the $\left(n - n^L + 1\right)$-th creditor. That is, i is such that $\varphi_i^{CEA}(E;c) < \varphi_i^{CEL}(E;c)$, and $\varphi_k^{CEA}(E;c) \geq \varphi_k^{CEL}(E;c)$ for each $k < i$. Since $0 < n^I$, for the central claimant z, we have that $\varphi_z^{CEA}(E;c) = \varphi_z^{CEL}(E;c)$. This implies that $i = z+1$, and also that, for each $k \geq i$, $\varphi_k^{CEA}(E;c) = \lambda < c_k$. Moreover, for each creditor k, and any x_i, $c_k \leq x_i < c_i$, $\varphi_k^{CEA}(E;(x_i,c_{-i})) = \varphi_k^{CEA}(E;c)$.

We now explore the dynamics of the Constrained Equal Losses solution when creditor i exhibits a forgiving behavior. For $(E;c)$ given, denote by t the last creditor such that $c_t < \mu$.[12] By construction, for $x_i \geq (n-t)(c_t - \mu) + c_i$,

$$\varphi_k^{CEL}(E;(x_i, c_{-i})) = \begin{cases} \varphi_k^{CEL}(E;c) & \text{if } k \leq t \\ \varphi_k^{CEL}(E;c) + \dfrac{1}{n-t}(c_i - x_i) & \text{if } k > t, k \neq i \\ \varphi_k^{CEL}(E;c) - \dfrac{n-t-1}{n-t}(c_i - x_i) & \text{if } k = i \end{cases}$$

Since $c_z \geq \max\{\lambda, (n-t)(c_t-\mu)+c_i\}$ we have that, for each x_i, $c_z \leq x_i < c_i$,

(a) $\varphi_z^{CEA}(E;(x_i, c_{-i})) = \varphi_z^{CEA}(E;c)$; and
(b) $\varphi_z^{CEL}(E;(x_i, c_{-i})) > \varphi_z^{CEL}(E;c)$.

Since $\varphi_z^{CEA}(E;c) = \varphi_z^{CEL}(E;c)$, taking into account that the two constrained equal solutions are order preserving it follows that $\mathcal{N}^L(x_i) \supseteq \mathcal{N}^L \cup \mathcal{N}^I$, and thus $n^L(x_i) \geq n^L + n^I$.

Taking into account that $i \in \mathcal{N}^L \cap \mathcal{N}^L(x_i)$,

$$\begin{aligned} &\varphi_i^{CW}(E;(x_i, c_{-i})) \\ &= \frac{n^A(x_i)}{n-n^I(x_i)}\varphi_i^{CEA}(E;(x_i,c_{-i})) + \frac{n^L(x_i)}{n-n^I(x_i)}\varphi_i^{CEL}(E;(x_i,c_{-i})) \\ &\geq \frac{n^A}{n}\varphi_i^{CEA}(E;c) + \frac{n-n^A}{n}\varphi_i^{CEL}(E;c) - \frac{n-n^A}{n}\frac{n-t-1}{n-t}(c_i - x_i). \end{aligned} \tag{14}$$

Since $i \in \mathcal{N}^A$, and $n^I \neq 0$, it follows that

$$\frac{n^A}{n}\varphi_i^{CEA}(E;c) + \frac{n-n^A}{n}\varphi_i^{CEL}(E;c) > \varphi_i^{CW}(E;c).$$

Define the function $F: \mathbb{R} \to \mathbb{R}$ where

$$\begin{aligned} F(x_i) = &\frac{n^A}{n}\varphi_i^{CEA}(E;c) + \frac{n-n^A}{n}\varphi_i^{CEL}(E;c) - \varphi_i^{CW}(E;c) \\ &- \frac{n-n^A}{n}\frac{n-t-1}{n-t}(c_i - x_i) \end{aligned}$$

Since F is continuous and $F(c_i) > 0$ choose some $\epsilon > 0$ such that for all $x_i \in B_\epsilon(c_i)$, $F(x_i) > 0$. Select $\bar{x}_i > \max\{c_z, c_i - \epsilon\}$. Then, by Eq. (14), $\varphi_i^{CW}(E;(\bar{x}_i, c_{-i})) > \varphi_i^{CW}(E;c)$. □

If there are no indifferent creditors, $\mathcal{N}^I = \emptyset$, the existence of a creditor that can profit from manipulating the Consensus-Weighted solution is not guaranteed

[12] That is, t is such that $c_{t+1} \geq \mu$, whereas $c_k < \mu$ for each $k < t$. For completeness, we set $t = 0$ and $c_0 = 0$ when $c_k \geq \mu$ for each $k \in \mathcal{N}$.

and depends, roughly speaking, on the difference between the demand of the central claimant and the next claim, $c_{z+1} - c_z$. The following example illustrates this situation.

Example 2 Consider a three-claimant instance where $c = (100, 150, 240)$, and $E = 180$. In this case, we have that $\varphi^{CEA}(E; c) = (60, 60, 60)$, while $\varphi^{CEL}(E; c) = (0, 45, 135)$. Therefore, $\mathcal{N}^A = \{1, 2\}$ and $\mathcal{N}^L = \{3\}$, and thus $\varphi^{CW}(E; c) = (40, 55, 85)$. If creditor 3 sets $x_3 = 199$, then $\varphi^{CEA}(E; (x_3, c_{-3})) = (60, 60, 60)$, while $\varphi^{CEL}(E; (x_3, c_{-3})) = (10.33, 60.33, 109.33)$. Therefore, $\mathcal{N}^A(x_3) = \{1\}$ and $\mathcal{N}^L(x_3) = \{2, 3\}$, and thus $\varphi^{CW}(E; c) = (26.88, 60.22, 92.88)$, so creditor 3 can profit from manipulating since he gains by reducing some of his claim.

Consider now $c = (100, 150, 480)$, and $E = 180$. In this case, we have that $\varphi^{CEA}(E; c) = (60, 60, 60)$, while $\varphi^{CEL}(E; c) = (0, 0, 180)$. Therefore, $\mathcal{N}^A = \{1, 2\}$ and $\mathcal{N}^L = \{3\}$, and thus $\varphi^{CW}(E; c) = (40, 40, 100)$. In this case, no creditor can profit from manipulating.

A deeper analysis of the Consensus-Weighted solution, which is the aim of future research, can describe the family of problems under which there is no forgiving behavior benefiting a manipulating creditor.

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Declarations

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References

Alcalde, J., Peris, J.E.: Mixing solutions for claims problems. Math. Soc. Sci. **115**, 78–87 (2022)

Aumann, R.J., Maschler, M.: Game theoretic analysis of a bankruptcy from the Talmud. J. Econ. Theory **36**, 195–213 (1985)

Giménez-Gómez, J.M., Peris, J.E.: A proportional approach to claims problems with a guaranteed minimum. Eur. J. Oper. Res. **232**, 109–116 (2014)

Maimonides, M.: Mishneh Torah. The laws of lending and borrowing. Chapter 20, Section 4. English translation by Eliyahu Touger (1180). https://www.chabad.org/library/article_cdo/aid/1161176/jewish/Chapter-20.htm. Accessed: 25 Jan 2022

O'Neill, B.: A problem of rights arbitration from the Talmud. Math. Soc. Sci. **2**, 345–371 (1982)

Piniles, H.: Darkah Shel Torah. Forester, Vienna (1861)

Schummer, J., Thomson, W.: Two derivations of the uniform rule and an application to bankruptcy. Econ. Lett. **55**, 333–337 (1997)

Thomson, W.: How to Divide When there isn't Enough: From Aristotle, the Talmud, and Maimonides to the Axiomatics of Resource Allocation. Econometric Society Monographs. Cambridge University Press, Cambridge (2019)