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TEEMU HEIKKI AUGUST PEKKARINEN

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UNDER ASYMMETRIC INFORMATION

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Abstract

This dissertation consists of an introduction and three independent studies in pricing under asymmetric information. The introduction gives a broad motivation and a brief literature review for this dissertation.

In the first study, we consider an economy where many sellers sell identical goods to many buyers. Each seller has a unit supply and each buyer has a unit demand. The only possible information flow about prices is through costly advertising. We show that in equilibrium the sellers use mixed strategies in pricing which leads to price and advertisement distributions. With convex advertising costs, each seller sends only one advertisement in the market. We also delineate a class of advertising costs which ensures that sellers may send multiple advertisements in equilibrium. Higher prices are advertised more than lower prices.

In the second study, we consider a principal-agent model in which the principal can monitor and punish the agent with a fine if the agent is caught being untruthful. To reduce the probability of being verified, the agent can engage in costly avoidance. We design the optimal regulatory policies with and without avoidance. The optimal mechanism with enforcement allocates the object more often than the optimal mechanism without enforcement. Moreover, enforcement increases the expected transfers to the principal. Avoidance has two implications to the optimal regulatory mechanism: (i) the expected optimal transfers to the principal decrease and (ii) the principal allocates the object to a smaller share of types. If the latter effect dominates the former, it is possible that the agent's capability to engage in avoidance is disadvantageous not only for the principal, but also for the agent *ex ante*.

In the third study, we study a market for 'lemons' from the perspective of mechanism design in a bilateral trade setup. The closed-form solution for the seller-optimal safe mechanism under one-sided private information is provided. We show that a seller can disclose the quality of the goods by controlling the supply of her goods; high-quality sellers want their goods to be scarce and expensive and low-quality sellers abundant and cheap. In this way, sellers can differentiate their products from each other and maximize their payoffs. We extend this model to two-sided private information and give a novel

characterization of the seller-optimal safe mechanism in this setup. It turns out that if there is two-sided asymmetric information, then the seller finds it optimal to engage in price signalling instead of quantity signaling. This is the least-cost way for the seller to signal her private information to the buyer.

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I met Professor Juuso Välimäki in the beginning of my doctoral studies when he was teaching a course on the economics of information. From the very first lecture, I found the topic vastly intriguing. After bombarding Juuso with endless questions on contract theory he accepted to be my supervisor. Since that day, I have been fortunate of having such an inspiring and talented researcher next to me while exploring the world of auctions and mechanism design. I owe my greatest thanks to him. Juuso's door has been open to me anytime I have needed it. No matter if I have had problems in my research or personal life, he has always guided and listened to me.

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I started my economics studies in 2014 together with Petteri Palonen. Since that year he has not been only my helpful colleague but also one of my closest friends. I am ever so happy that we still play with equations and go through all silly models. This keeps my hopes up that some day we will find something big out from this world. I am grateful to Petteri for teaching me the power of storytelling and showing me that imagination is more important than knowledge. I hope that we will have many inventive and colorful years ahead as coauthors and, par excellence, as friends.

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Teemu Pekkarinen
Helsinki, August 2022

Contents

Chapter 1

Introduction

This dissertation is a collection of three independent studies contributing to the theory of *information economics*, which focuses on strategic situations where different agents do not know each others' objectives. Each study suggests a simplified, but coherent, perspective on pricing under asymmetric information.

In principle, studying economic behavior might be extremely complex and nebulous; there is always an ever-changing human being behind our research subjects. This necessitates that also our understanding about economic behavior must be constantly updated and in transition with the subject. Therefore, giving unambiguous and comprehensive explanations for a particular economic phenomenon is, I dare to say, impracticable. This however does not signify that we could not say or learn anything about the complicated economic systems.

Economic theory provides analytical tools for processing and understanding the particular aspects of human behavior such as incentives and mechanisms under asymmetric information. Nowadays economic theories are presented almost without exception in the language of mathematics and the interpretations and implications are more philosophical. This makes economic theory, on the one hand, transparent and consistent (as long as the calculations are correct), but, on the other hand, limited and illegible for a reader who is uninitiated with mathematics. Consequently, communicating new findings in economic theory are always affected by our capability to translate mathematics into the common language, whereas new ideas and underlying assumptions on our research questions are squeezed into a mathematical (or statistical) model. The knowledge produced by economic theories hence rests on our ability to internalize our models and externalize the findings of the models. A theory simplifies presumptions not because the world is simple, but because we, relatively limited and incomplete human beings, want to grasp something about the nearly unattainable reality. In this matter, economics does not stick out from the rest; the necessity of simplifying our ways of thinking runs through all kind of theorizing in (social) sciences regardless whether the presumptions and analysis are formalized or presented in the common language.

In this introduction, we provide a broad motivation of this dissertation. We start by briefly reviewing some central ideas and findings of the economics of pricing. After delving into the classic theories in pricing, we end this chapter by summarizing the upcoming studies. Each study provides its own targeted literature survey of the topic which guides the reader from the trailblazing studies closer to the (current) frontier of research.

1.1 Background

The role of information in pricing decisions has been of great interest in microeconomic theory for centuries. The literature is vast since there is a great number of variations of how information is spread around the people in different kinds of environments. People do not only make decisions based on their prior information but take also into account what kind of information the others possess and whether their own actions reveal relevant information to the others. For instance, imagine an entrepreneur who is willing to sell her company to a buyer. If the entrepreneur has private information about the profitability of the business, the offering price of a share of the firm may disclose relevant information about the profitability to the buyer. The revealed information naturally alters the buyer's willingness to pay which further affects the entrepreneur's pricing strategy.

In order to clearly understand the role of *asymmetric* information in pricing, we need to briefly review how an allocation is determined under *symmetric* and *perfect* information.

1.1.1 Pricing under Symmetric Information

Consider a simple market consisting a single buyer and a single seller who has some goods for sale. If the seller knows the buyer's valuation of the goods, she can set the price and the quantity according to the buyer's preferences and extract the whole surplus from the buyer. For instance, if there is a single object for a sale or if the buyer has a unit demand, then it is optimal for the seller to ask a price equal to the buyer's valuation of the object. And if the seller herself does not value the object more than the buyer or if the production costs are not above this price, then it is optimal for the seller to trade. This principle generalizes to a case in which there are more objects for sale and the buyer is willing to buy more than one object or there are more than one potential buyer interested in the objects. First, if there are more than one buyer with unit demand, then the monopolist seller finds it optimal to sell a good to each buyer at a different price (first-degree price differentiation). Second, if there is a variety of objects for sale, then the seller can use menu pricing and differentiate the price of the objects according to the valuations of each object among the buyers (second-degree price differentiation). This is a classic price differentiation story which ends in favor of the seller (this can be found from many standard economics textbooks or, e.g., from a more recent book "Pricing and Revenue Optimization" by Phillips (2021)),

If the seller cannot engage in the first-degree price differentiation, the optimal price and quantity are determined by the monopoly profit maximization problem given by the buyers' demand and the seller's supply (see, e.g., Mas-Colell et al. (1995) or Varian (2014)). As it is well-known, the monopolist sells all the objects for which the marginal revenue (weakly) exceeds the marginal costs. Naturally, this optimal monopoly pricing scheme applies to all varieties of objects that the seller can provide (second-degree price differentiation).

Price differentiation and monopoly pricing are concepts that have been in the core of economics for centuries. Hence, it is difficult to give credit for some particular author in these contributions. However, to the best of my knowledge, analyzing the market equilibrium under competition was first studied by Antoine Cournot in 1838. Cournot considers a duopoly model in which two companies compete on the amount of output rather than prices. Cournot finds that if there is competition, the equilibrium prices are lower and the amount of supply greater than in the monopoly case. A couple of decades later, in 1883, Joseph Bertrand reviewed Cournot's book "Recherches sur les Principes Mathématiques de la Théorie des Richesses", in which the theory of competition was presented, and formulated his competing theory. In contrast to Cournot's earlier work, Bertrand assumed that the duopoly firms choose prices and the buyers decide quantities. Bertrand's theory suggests that two competitive sellers set equilibrium prices at the marginal cost level and hence the equilibrium of the market is an outcome equivalent to that prevails under perfect competition of many sellers.[1]

Around the time, Leon Walras, Vilfredo Pareto, Francis Edgeworth, and Arthur Bowley continued from and formalized John Stuart Mill's early ideas on general equilibrium theory and the theory of an exchange economy. In an exchange economy there are many agents who possess several divisible goods and are willing to trade the goods with each other. General equilibrium theory considers a large economy which consists of several or many interacting (exchange) economies. The theory focus on the behavior of supply, demand, and prices which result in an equilibrium of each market of the economy. That is to say, in general equilibrium theory the prices are (exogenously) determined such that

[1] Francis Edgeworth formalized Bertrand's model in 1889 in his study "The pure theory of monopoly".

the markets clear so that there is no excess supply or demand. The last completion of the general equilibrium theory was by Kenneth Arrow and Gerard Debreu in the 1950s. This branch of economics under perfect information culminated in two fundamental theorems of welfare economics: 1) the general equilibrium is Pareto Optimal, and 2) Pareto efficiency can be achieved with any redistribution of initial wealth (there are prices that clear the markets).

We end our short journey to the history of pricing under symmetric (and perfect) information. There are many important concepts and models that were not considered here such as the Stackelberg leadership model in which one of the duopoly firms is the leader who decides its pricing strategy first and the follower firm responds to this by setting its own price. However, since this dissertation is about pricing under *asymmetric* information, this amount of background information on the benchmark literature is sufficient to highlight the differences.

1.1.2 Pricing under Asymmetric Information

In the 1960s and 1970s two stimulating articles greatly accelerated economists' interests towards information asymmetries, namely, "The Economics of Information" by George Stigler in 1961 and "The Market for 'Lemons': Quality Uncertainty and the Market Mechanism" by George Akerlof in 1970. We consider the observations of these papers and their implications next, respectively.

Stigler (1961) begins his study with a paragraph:

> *"One should hardly have to tell academicians that information is a valuable resource: knowledge is power. And yet it occupies a slum dwelling in the town of economics. Mostly it is ignored: the best technology is assumed to be known; the relationship of commodities to consumer preferences is a datum. And one of the information-producing industries, advertising, is treated with hostility that economists normally reserve for tariffs or monopolists."*

Stigler considers a market where there are many buyers and sellers. The distribution of prices that sellers use is given and known by the buyers. However, the buyers do not know where the sellers are located and which prices they are asking. Hence, they need to search the goods. The core of the study is to argue that due to the imperfect information in the market, there may be many prices for homogeneous goods. This however is only one side of the story: What rationalizes the sellers' pricing behavior? This question was addressed by Butters in 1977.

Butters generalizes Stigler's analysis and let the sellers to choose the prices in addition to advertising strategies (Stigler (1961) also analyzes advertising but not together with competitive pricing). Butters is the first to show that a price dispersion of homogeneous goods is rationalizable by the sellers' equilibrium behavior; the sellers use a mixed strategy in pricing in equilibrium. The first study of this dissertation continues Butters's work by generalizing the advertising costs and assuming capacity constraints for the sellers. We show that the sellers mix in pricing and with convex advertising costs each seller sends only one advertisement in equilibrium, whereas if the advertising costs are sufficiently concave, higher prices are advertised more than lower prices.

In 1970, Akerlof shows how quality heterogeneity together with asymmetric information about the quality may result in an equilibrium where only low quality products are traded. The model consists of many sellers and buyers such that only the sellers are known whether the good that they are selling is of high ("peach") or low quality ("lemon"). The buyers' willingness to pay then is determined by the average quality of the goods which in turn drives out all the supply of goods that are above average in quality. Asymmetric information can thus even lead to a market *collapse*. This was a striking finding that shed light on markets with asymmetric information such as markets for used cars or labor markets. The third paper of this dissertation continues Akerlof's work and shows that the market collapse is not inevitable if the seller can *credibly* signal the quality of the good, and the signaling is not possible only through prices (as seen in Akerlof (1970)), but altering the supply of the good. Moreover, we show that this is the optimal strategy for the seller among *all* possible selling mechanism. This latter result stands on the wide shoulders of mechanism design which we consider next.

Let us approach mechanism design by considering a simple setup of a single seller and a single buyer. The seller has a good for sale and the buyer's valuation of the good is the private information of the buyer. That is to say, there is *asymmetric* information between the seller and the buyer about the buyer's willingness to pay. Consequently, the seller cannot engage in price differentiation and ask the buyer to just pay her valuation. What is the optimal strategy or mechanism for the seller to sell the good?

In order to have any rationality in pricing strategies, the seller must form beliefs about the buyer's valuation. These beliefs give the seller expectation of how likely is that the buyer accepts a certain price offer. The seller can thus choose a price that maximizes the probability of sale (the probability that the buyer's valuation is greater than or equal to the asked price) multiplied by the price she is asking. This optimization problem gives the seller the optimal take-it-or-leave-it offer that the seller can propose for the buyer. In comparison with the symmetric information counterpart in which the seller can engage in price differentiation, the information asymmetry has the following impacts on the outcome: (i) the good is not always traded and therefore there is inefficiency in the market, (ii) if the trade occurs, the seller does not receive the whole surplus, and (iii) the buyer is given so-called information rent which refers to as the difference between the price paid and the valuation of the good. However, is a take-it-or-leave-it offer the optimal procedure to sell the good? This is not a straightforward question to answer since there are (infinitely) many potential mechanisms how to sell the good such as bargaining, price quotes, randomization, auctions (with multiple buyers), and so on.

In the 1970s, economists like Roger Myerson, Bengt Holmström, Allan Gibbard, Partha Dasgupta, Peter Hammond, and Eric Maskin found that without going through all possible mechanisms, it is without loss of generality to focus on mechanisms in which the buyer reports truthfully her valuation to the seller as long as the seller does not give any incentives for the buyer to be dishonest (the most general form of this result was given by Myerson (1982)). This result, called the Revelation Principle, revolutionized many primarily complex allocation problems under asymmetric information. Soon after this finding Mussa and Rosen (1978) showed that the aforementioned take-it-or-leave-it offer is, indeed, the optimal mechanism to sell the good in our simple bilateral setup above. A couple of years later Myerson (1981) generalized this analysis to a multiple buyer setup and showed that the optimal selling mechanism can be implemented by a second-price auction with a reserve price. This vaunted result highlighted the importance of development of mechanism design and auction theory which was earlier pioneered by Vickrey (1961). In the early 1980s, Maskin and Riley (1984) showed that a monopoly seller with multiple goods finds it optimal to price the goods with quantity discount (if the buyer has a decreasing marginal utility from the good).

One notable result that applies and extends many models of mechanism design and information economics was presented by Harsanyi in 1967 and 1968. Harsanyi shows that a game with incomplete information can be converted into a standard game of imperfect information with an initial move by nature if and only if the players share a common prior over payoffs in some state space. That is to say, if some players (possibly all) lack full information about the basic (mathematical) structure of the game such as payoff functions, endowments, resources, or reciprocal information about the other players' information, and so on and so forth, this incompleteness of information can be reduced to the case where the players have less than full information about each other's payoff functions or so-called types that determined the utilities of the players (higher order belief types). This finding highlights the importance of the common prior assumption when we are considering interactive models with asymmetric information.[2]

In the 1970s and 1980s, mechanism design was developed and used in a wide range of closely related topics to this dissertation. These include, for instance, the theory of

[2]See Morris (1995) for more detailed discussion on the common prior assumption in economic theory.

optimal income taxation (Mirrlees (1971)), the monopoly pricing of insurance (Stiglitz (1977)), and the regulation of a monopolist with private information (Baron and Myerson (1982) and Lewis and Sappington (1988a and 1988b)). The mechanism design approach provides general tools for many applications and has enabled us to derive the somewhat robust results of monopoly pricing under asymmetric information. However, much less is known for the case of a duopoly or a finite number of sellers. There are a couple of difficulties that the literature of competing mechanisms has faced. First, if there are two or more competing mechanisms, there is no clear analog of the Revelation Principle to a standard single seller case; the strategy by buyers select between mechanisms is determined only in equilibrium. Therefore, a mechanism designer's revenue depends on the number of buyers who participate in her mechanisms which results in a fixed point optimization problem. Second, in general it cannot be ruled out that the competing mechanisms depend on each other. This leads to an infinite regress and it is not clear whether an equilibrium (a fixed point) exists in this case. Moreover, even if there is such a fixed point of mechanisms, it is not evident how to analyze that setup.[3]

In order to dodge these difficulties, Burguet and Sákovics (1999) study a setup in which two sellers compete simultaneously setting reserve prices for their second-price sealed bid auctions. The potential buyers have private information about their valuations of the homogeneous goods that the sellers are offering. In stark contrast with Bertrand's earlier observations, Burguet and Sákovics show that this game has at least one equilibrium and that all equilibria reserve prices are not driven to zero cost causing inefficiency. This result supports the earlier finding of Spulber (1995) who shows that in the Bertrand-Nash equilibrium when sellers' costs are unknown, the sellers make positive profits by pricing above marginal costs.

Another way to tackle the problems of the mechanism design approach is to assume that there are infinitely many sellers competing on buyers with private information about their valuations. In this case, a single seller cannot affect the rest of the market and hence the construction of an equilibrium is accessible. McAfee (1993) shows that in this setup there exists an equilibrium where sellers arrange identical auctions and buyers randomize their participation into the auctions. In equilibrium, the sellers set reserve prices equal to their valuations of the goods resulting in an efficient outcome. However, McAfee finds that this equilibrium fails in all finite economies where the number of sellers is finite. In this regard, the results of McAfee (1993) can be viewed as the mechanism design analog to the theory of *perfect* competition. Peters and Severinov (1997) endogenize the buyers' entry strategies in this setup and show that the reason why sellers set reserve prices equal

[3]See, e.g., Peters (2001) and Pavan and Calzolari (2010) for identifying the full set of feasible mechanisms sellers can offer. These impediments are more carefully addressed by Pai (2010).

to their valuations is due to the fact that sellers' profits are discontinuous in the seller's reserve price in a way that resembles the discontinuity in a Bertrand pricing game.

Greenwald and Stiglitz (1986) provide a general framework for analyzing externalities in economies with incomplete markets and imperfect information. They identify the pecuniary effects of these externalities and show that (general) equilibria under imperfect information are rarely constrained Pareto optima. This implies that the aforementioned fundamental welfare theorems do not hold if there are any incomplete markets or imperfect information among the agents. Moreover, Greenwald and Stiglitz show that the pecuniary effects are not due to the finite number of agents of the model; imperfect or incomplete information matters also in large economies.

In this section, we have stressed that information asymmetries might have significant effects on strategical behavior of agents. Sometimes asymmetric information leads to an equilibrium which is more disadvantageous to the mechanism designer than that under perfect information. In this case, the designer would benefit if she was able to disclose the private information of the agents and implement the perfect information outcome. This kind of environment, where the private information of the agents is (at least partially) verifiable, was first studied by Townsend (1979), Diamond (1984), and Gale and Hellwig (1985). Townsend studies a model in which agents are asymmetrically informed on the actual state of nature and, at some cost, this information can be transmitted to other agents. This kind of information structure is present especially in financial contracting problems where there is an entrepreneur with an investment project and an investor who is willing to capitalize the project. The entrepreneur has private information about realized cash flows from the project which can be verified to the investor. Townsend (1979) shows that the optimal financing mechanism is a standard debt contract such that in case of default the entrepreneur fully discloses her cash flows and if the debt is honored, there is no verification. There is a growing literature in mechanism design with verification pioneered by Green and Laffont (1986). Green and Laffont show that the Revelation Principle does not hold in general if the information is only partially verifiable by the designer. Since the literature is somewhat recent, it is covered in the second study of this dissertation. The paper itself studies monopoly pricing with costly state verification. We show that if the principal can verify the agent's private valuation and impose a fine on the agent if she is caught being non-compliant, the principal can extract some of the agent's information rent and get more surplus. In other words, the verification mechanism eliminates some of the asymmetric information between the seller and the buyer leading to an equilibrium which lies somewhere between the first-degree price differentiation and the optimal take-it-or-leave-it offer for a privately informed buyer. These findings throw some light especially on regulation problems in which the regulator can monitor agents' compliance and impose a punishment on non-compliant agents.

For the sake of compactness, we end the literature review at this point. We thus omit excessively many important topics in information economics such as moral hazard (see, e.g., Arrow (1978) and Holmström (1979 & 1982)), signaling (see, e.g., Spence (1978) and Crawford and Sobel (1982)), or mechanism design by an informed principal (see, e.g., Myerson (1983) and Maskin and Tirole (1990 & 1992)). Some of the observations of signaling models and the literature of mechanism design by an informed designer are covered in the third study of this dissertation. The vast and versatile literature of information economics is an undeniable piece of evidence of the diversity of the topic; the gamut of information asymmetries and imperfections is extensive.

1.2 Summary of Studies

In this section we summarize the upcoming chapters. As discussed in the previous section, the studies of this dissertation continue from the distinguished work of Butters (1977) (the first study), Baron and Myerson (1982) and Lewis and Sappington (1988a and 1988b) (the second study), and Akerlof (1970) and Myerson (1983) (the third study). The papers offer new insights into these classic frameworks by relaxing some previous assumptions (the first study), introducing something new into the model (the second study), or bringing two models and solution concepts together (the third study). Some of the results of the studies may appear self-evident once they are stated, but the knowledge is not only in the compact claims themselves; the proofs are the journey from the assumptions into the conclusions. The understanding arises from the presumptions via analysis which provides the machinery to elaborate how different matters are connected with each other.

1.2.1 Equilibrium Pricing and Advertising Distributions

In the first study, we examine a relationship between prices and advertising. We consider an economy where many sellers sell identical goods to many buyers. Each seller has a unit supply and each buyer has a unit demand. The only possible information flow about prices is through costly advertising. Moreover, the sellers do not know which buyers they capture by sending advertisements and whether they have received other price offers from the other sellers as well.

The empirical advertising literature suggests that heavily advertised brands are more expensive than are less-advertised goods within the same class of goods. This phenomenon has usually been explained by persuasive advertising that alters consumers' tastes and brand loyalty. We, however, apply a version of Butters's (1977) model, and show that this positive relationship between prices and advertising is a natural feature of *informative* advertising, too. That is to say, the sellers who advertise more aggressively can ask high prices since with positive probability they capture buyers who lack information about lower prices in the market.

In equilibrium the sellers use mixed strategies in pricing which leads to price and advertisement distributions. With convex advertising costs each seller sends only one advertisement in the market. We delineate a class of advertising costs which ensures that sellers may send multiple advertisements in equilibrium. That is, if the advertising costs belong to this class of concave cost functions, higher prices are advertised more than lower prices.

1.2.2 Optimal Regulation with Costly Verification

In the second study, we consider a regulator who can decide whether to allocate the right to conduct a business to a firm who has private information about its profitability, emissions, or some other verifiable and payoff-relevant parameter. This setup can be modeled as a classical mechanism design problem in which the firm is asked to report its private information to the regulator who then decides the allocation. The optimal allocation and regulation in different kinds of environments are pioneered by Mussa and Rosen (1978), Myerson (1981), Baron and Myerson (1982), and Lewis and Sappington (1988a and 1988b), with the well-known results. However, these results are founded on the assumption that misreporting is not a crime. We diverge from this approach. We assume that the regulator has the power to punish the firm with a fine if the firm is caught being untruthful (e.g. tax evasion or accounting fraud). In order to verify the firm's compliance, the regulator has to invest in costly monitoring. Moreover, following Malik (1990) we suppose that a firm can weaken the regulator's monitoring efforts by covering up its misreporting by engaging in costly 'avoidance' (e.g. by falsification of accounts, corruption, or bribing).

We find that the expected optimal transfers are greater than that in the optimal *standard mechanism* in which the principal can use only a physical allocation and transfers. While a take-it-or-leave-it offer is the optimal standard mechanism, *non-linear* pricing is the optimal mechanism with enforcement. The rationale for this result is that the principal is able to extract a proportion of the agent's information rent by monitoring and fines. However, the agent's ability to engage in avoidance makes this proportion smaller. Avoidance has no direct effects on the equilibrium transfers since with a truthful report the fines are zero and hence the agent has no incentive to invest in costly avoidance. However, avoidance makes the incentive compatibility constraint more rigid (engaging in avoidance may be profitable with off-equilibrium reports) and, consequently, it is optimal for the principal to monitor a smaller proportion of reports than without avoidance. This has two implications: (i) the expected optimal transfers to the principal decrease and (ii) the principal allocates the object to a smaller share of types. So, although avoidance results in greater information rent for the agent, it may also *hurt* the agent by making the mechanism to allocate to a smaller share of types. If the latter effect dominates the former, then it is possible that the agent's capability to engage in avoidance hurts not only the principal, but also the agent *ex ante*.

It turns out that if there is no avoidance and monitoring is costless, then with sufficiently large fines the principal allocates for all types and gets the whole surplus even if the verification is noisy. However, when the agent can engage in avoidance, the full information rent extraction is not possible for all types even if the monitoring was costless.

1.2.3 Bilateral Trade with Interdependent Values

In the third and last study, we derive the seller's utility maximizing selling *safe* mechanism in bilateral trade with interdependent values. In a standard mechanism design problem it is typically assumed that the mechanism designer (principal) does not possess any payoff-relevant information for the agents. Relaxing this assumption may, however, be essential for many applications as dissolving partnerships, agency contracts, trading with externalities, allocating mineral rights, or other comparable contracting problems in which the principal's choice of a mechanism possibly reveals substantive information to the agents (see Myerson (1983)).

One conventional circumstance in which the contract is designed by an informed principal is bilateral trade where the seller, who determines the selling procedure, has private information about the quality of the object which affects the buyer's valuation of the object (market for lemons; see Akerlof (1970)). In this study, the closed-form solution for the seller-optimal safe mechanism under one-sided private information is provided. We show that a seller can disclose the quality of the goods by controlling the supply of her goods; high-quality sellers want their goods to be scarce and expensive and low-quality sellers abundant and cheap. In this way, sellers can differentiate their products from each other and maximize their payoffs. We extend this model to two-sided private information and give a novel characterization of the seller-optimal safe mechanism in this setup. It turns out that if there is two-sided asymmetric information, then the seller finds it optimal to engage in price signalling instead of quantity signaling. This is the least-cost way for the seller to signal her private information to the buyer.

All three studies open several new doors for further research. For instance, how is the equilibrium of the first study altered if the sellers are not capacity constrained? Or what is the optimal allocation procedure for the right to do business for many potential buyers in the model of the second study? If there are more than a single buyer in the model of the third study, is it optimal to share the allocation between several buyers? Finding the solution to the first question would develop the advertising and directed search literature further. The research considering the last two questions would contribute to the literature of auctions in which one of the main interests is to know whether one can implement optimal mechanisms by well-known selling procedures like first or second-price auctions.

Chapter 2

Equilibrium Price and Advertisement Distributions

Abstract

We consider an economy where many sellers sell identical goods to many buyers. Each seller has a unit supply and each buyer has a unit demand. The only possible information flow about prices is through costly advertising. We show that in equilibrium the sellers use mixed strategies in pricing which leads to price and advertisement distributions. With convex advertising costs each seller sends only one advertisement in the market. We also delineate a class of advertising costs which ensures that sellers may send multiple advertisements in equilibrium. Higher prices are advertised more than lower prices.[1]

Keywords: Advertising, Price Distributions.

JEL: D41, D47.

2.1 Introduction

Butters's (1977) article on informative advertising is seminal in at least two respects. First, it is an equilibrium analysis of firms that compete both by prices and advertising. Secondly, the urn-ball meeting technology, which has become widely used in many fields in economics (in particular directed search models), is introduced. In the model there are multiple firms that produce a homogeneous good at a constant marginal cost. There are

[1]This study is written with Klaus Kultti. For helpful comments we would like to thank Daniel Hauser, Pauli Murto, Juuso Toikka, Geert Van Moer, Juuso Välimäki, Takuro Yamashita, and numerous seminar audiences at the Helsinki Graduate School of Economics and the Congress of European Economic Association. Financial support from the Finnish Cultural Foundation and the University of Helsinki is gratefully acknowledged.

many consumers each with a unit demand and identical valuations. The consumers do not know where the goods are available, nor at which prices, unless they receive advertisements (ads, hereafter) from the firms. The firms send multiple ads at a constant unit cost, and the ads are randomly allocated amongst the consumers. Consumers who do not receive any ads cannot consume (there is no search by uninformed consumers). If a consumer receives multiple ads, she contacts the firm with the lowest price. In equilibrium the firms mix over prices which leads to price dispersion. All the firms send the same number of ads.

In the empirical advertising literature there are many papers which suggest that heavily advertised brands are more expensive than are less-advertised goods within the same class of goods (see Bagwell (2007) for a comprehensive review of the literature). This phenomenon has been usually explained by persuasive advertising that alters consumers' tastes and brand loyalty. We apply a version of Butters's model, and show that this positive relationship between prices and advertising is a natural feature of informative advertising, too.[2] To achieve this result we deviate from Butters's model in two respects.

First, we assume that the firms are capacity constrained each firm possessing just one unit of an indivisible good. In Butters (1977) the firms have unlimited capacity which is in stark contrast with the more recent directed search literature: sellers have just one unit for sale (e.g. Burdett, Shi, and Wright, 2001), or firms have just one vacancy (e.g. Pissarides, 2000 and Shimer, 2005).

Second, we generalise the advertising cost scheme by considering a large class of cost functions which plays a crucial role in our set-up.

If the cost function is convex, as in Butters (1977), we show that each firm sends only one ad in equilibrium. Pricing is in mixed strategies, and the equilibrium price distribution of our model coincides with Butters (1977) once the parametrisation between the papers is harmonised (the number of consumers and their valuations of the good are normalised to unity and the cost of production to zero). This is a surprising result as the capacity constraint seems to play no role in pricing. The explanation hinges on the linear cost function in Butters (1977): sending k more ads is equivalent to adding k more firms who send one ad each. Hence, the equilibrium of Butters (1977) can be interpreted as a case in which each firm sends a single ad and there is a free entry.[3]

[2]We point out that interpreting the low prices that result from the mixed pricing strategy as sale-prices is misleading. The concept of a sale would require a multiperiod model and it does not make sense in our static model.

[3]Butters (1977) studies the limit case in which the number of firms is taken to infinity which makes each firm's profits zero. This limit case resembles free entry of firms. Moreover, as the number of the firms goes to infinity, there are some firms who do not send any ads, which can be interpreted as free exit of firms.

Our main contribution is to delineate a class of cost functions such that in equilibrium multiple ads are sent. Pricing is still in mixed strategies. In equilibrium the support of the mixed strategy is divided into intervals such that in each interval the firms send the same number of ads, and *the number of ads increases with the price*. To the best of our knowledge, this is a somewhat novel equilibrium in the theoretical advertising literature.

In a multiple-ad equilibrium the advertising costs must be sufficiently concave. The positive relationship between the prices and the number of ads arises as the firms who price low do not face much competition, while those who price high are likely to be undercut if they send only one ad. Sending more ads increases the probability of a sale and the expected revenue. If the increase in revenue is greater than the increase in advertising costs, then the firm can also ask a higher price (in equilibrium these two effects must be equal). Since in our model the firms have a limited capacity and there is competition for the potential consumers, advertising has diminishing marginal revenue. Hence, price-increasing advertising necessitates, indeed, that the advertising expenditure per ad must fall as more ads are sent in a multiple-ad equilibrium.

On the other hand, the advertising costs cannot grow too slowly. The construction of the equilibrium presupposes that the consumers always contact the firm with the lowest price. This is obvious if there are no capacity constraints (i.e., in Butters (1977)); a consumer who contacts a firm always gets an object. However, if the firms are capacity constrained, not every consumer who receives an ad gets an object. This implies that a consumer who receives multiple ads may find it profitable to choose a higher price offer if it is more probable that she gets the object. To guarantee that the consumers contact the firm with the lowest price there is a minimum speed at which the costs have to increase. It is somewhat surprising that making the lowest priced good the most desirable for the consumers restricts the possible cost functions of the advertisers; this emphasises that the logic of the model with capacity constrained firms is different from that of unlimited capacity.

We delineate a class of cost functions that supports an equilibrium with multiple ads by using *functional equations*. The functional equations determine the upper and lower bounds for the changes in advertising costs. In this class, the equilibrium can be determined simply by examining the successive differences of the cost function. Moreover, the cost of the first ad (which can also include the entry or capacity costs) immediately fixes the highest possible number of ads sent by a single firm in equilibrium. It turns out that this maximum is decreasing in the cost of the first ad.

The theoretical contribution of our model stems from highlighting the issues that arise once we give up the assumption of unlimited capacity. In particular, constructing an equilibrium where the consumers regard low prices as more attractive than high prices turns out to restrict the growth of the cost function from below, while, more expectedly, the firms are willing to send multiple ads only if the growth rate is restricted from above. One would expect the same issues to arise if capacity were allowed to be at any finite level.

The positive association between the price and the number of ads requires the capacity constraint. We elaborate this in Section 5.

This paper is organised as follows: In Section 2 we relate our analysis to the literature. In Section 3 we list the set of assumptions and build the model. In Section 4 we define and construct so-called *configurations* for different amounts of ads sent in the market. After that we study which conditions are needed for a configuration to be an equilibrium. In Section 5 we discuss our findings, and in Section 6 we conclude. We relegate all the proofs to the Appendix to improve readability.

2.2 Related Literature

Our analysis contributes to two different fields. The first consists of directed search models originated by Peters (1991) and Montgomery (1991). A typical application consists of buyers and sellers, the latter ones posting prices. These models aim to depict markets with frictions. The frictions are of coordination type, and they arise as each seller has only one good but in equilibrium she may be contacted by several buyers, or no buyers at all. The frictions, however, arise in a symmetric equilibrium; depending on the details of the model there may be asymmetric equilibria which do not give rise to frictions. Instead of price posting the sellers in our model send ads, and only those who receive the ads get informed about the offers in the market. In this set-up there is a unique equilibrium that gives rise to frictions.

Our results, in particular the price distribution, is reminiscent of what happens in models of noisy search. In these models there are features of directed search but the agents have only partial or noisy information about some aspects of the environment. For instance, in Shi (2018) the buyers enter a submarket based on the maximum price the sellers commit not to exceed. In the submarket the sellers contact the buyers making offers without knowing how many other sellers contact the same buyer. The optimal behaviour in the price offer subgame is mixing, and this results in a price distribution. In a similar vein Bethune et al. (2020) in a model of money and credit, and Acemoglu and Shimer (2000) in a model of labour market, assume that the contacting parties choose how much information they acquire about the deals available. In equilibrium they have only partial information which leads the parties who offer the deals to use mixed strategies as the buyers' partial information gives the offerers some monopoly power but at the same time exposes them to some competitive pressures. In our set-up the buyers have only partial information, albeit endogenously determined, about the available deals, while the sellers still face some competition as a buyer may get ads from several sellers. It is worth noticing that unlike in Bethune et al. (2020) and Acemoglu and Shimer (2000) it is the party that offers the deals, i.e., the sellers, who are responsible for the noisy environment.

The second field naturally deals with the economics of advertising. This is a very large area covered for instance in Bagwell (2007). We only mention a couple of models that are directly related to Butters (1977).

Robert and Stahl (1993) allow the consumers who remain uninformed to search. The model still exhibits price dispersion but a mass of sellers charge the highest price that is paid only by the searchers. Roberts and Stahl assume strictly convex advertising costs, and find that firms advertise lower prices more intensively which is just the opposite of our result. Convex costs and uninformed searchers imply that firms advertise "sale" prices more than high prices in equilibrium.[4]

In McAfee (1994) the firms choose a continuous advertising intensity instead of physical ads as in Butters (1977). McAfee shows that when the firms first choose the intensity and only after that the price, there is one high-intensity high-price firm in equilibrium, while the other firms advertise at lower intensity and mix in prices.

Gomis-Porqueras, Julien, and Wang (2017) study a model which differs from Butters (1977) in two respects. Firms are capacity constrained, and advertising takes place by choosing intensity continuously. The cost of intensity is assumed convex and increasing, while we study a broader class of advertising cost schemes. Moreover, in our model each firm sends a finite number of ads, and there is a discrete jump in the cost between zero and one ads. This means that in our set-up it is natural to assume free entry and exit. In Gomis-Porqueras et al. (2017) the intensity of advertising can be continuously adjusted, and the market tightness is taken as a parameter. They study trading by both posted prices and auction, and in both cases find a unique equilibrium in pure strategies. As the number of firms is fixed the advertising intensity is non-monotonic in the number of buyers. If there are relatively many buyers there is little competition, and a low intensity in advertising results in trade with high probability. If there are relatively few buyers then there is a lot of competition, and the marginal pay-off from advertising is low. Consequently, the equilibrium advertising intensity is low. Between these extremes there is some competition and a need to make sure that advertising reaches the buyers. As a result there is more advertising than at the extremes. If, in our model, the number

[4]Search makes the marginal benefit of sending an offer with a high price smaller than with a low price. In equilibrium the marginal benefit must be equal to the marginal cost, and then the convexity of advertising costs implies that low prices are advertised more than high prices.

of firms were fixed we would expect same kind of non-monotonicity; with relatively few buyers some firms would not advertise at all.

2.3 Model

In the spirit of Butters (1977), we assume the following:

(i) There is a large economy with S sellers and B buyers. Denote the ratio of sellers to buyers by $\theta = \frac{S}{B}$. Since this ratio is the only relevant magnitude in the sequel, we normalise $B = 1$. Then the number of sellers is $S = \theta$.

(ii) All the sellers are risk neutral and have a unit supply of an identical good. They value the good at zero. Also the buyers are risk neutral and have a unit demand. The buyers value the good at unity.

(iii) There is free entry and exit of sellers.

(iv) The sellers can sell their goods only via sending ads. An ad contains the location and price of the good. Buyers who do not receive any ads cannot shop at all.

(v) The price and the number of ads are the choice variables of a seller.

(vi) The cost of sending k ads is given by function $c(k)$ where $c : \mathbb{N}_0 \to \mathbb{R}_+$ such that $c(0) = 0$ and for all $k \in \mathbb{N}_0$, $\Delta c(k+1) \equiv c(k+1) - c(k) > 0$.

(vii) Buyers receive ads randomly and independently of all other ads and each buyer has an equal probability of receiving each offer. Receiving an ad and sending orders (i.e. contacting a seller) are costless for a buyer. Buyers can contact exactly one seller.

The timing of the static game is as follows. First, sellers set prices and send ads. Second, each buyer who has received an ad (or ads) makes an order. Lastly, the orders are executed by sellers. If a seller receives many orders, she chooses randomly with equal probabilities one buyer with whom to trade. There is no discounting between stages.

Since we consider a large economy where there is an infinite number of buyers and sellers, the random process by which the ads are allocated follows the Poisson distribution.[5]

[5] The idea is basically based on the following argument. Assume that there are N discrete buyers, and each buyer has an equal probability of getting an ad – that is, $\frac{1}{N}$. Then for any total amount of ads θN, each buyer receives zero ads with probability $\left(1 - \frac{1}{N}\right)^{\theta N}$. This converges to $e^{-\theta}$ as N goes to infinity. This is the Poisson probability with parameter $\theta > 0$.

In equilibrium, the sellers use mixed strategies in pricing. This can be seen by assuming the opposite. Suppose that all the sellers ask the same price and send a single ad. Then lowering the price a little leads to a discrete increase in the selling probability when a potential buyer receives ads from multiple sellers. This further increases the profits which is a contradiction. The same logic shows that there are no mass points or gaps which means that the support of the mixed strategy is some interval $[l, U] \subset \mathbb{R}$. The highest price is the value of buyers, $U = 1$, by two reasons. First, it is clear that it is never profitable to ask price greater than the value of the buyers. Second, if U were less than 1, then it would be profitable to increase the price since it does not change the probability of sale (a buyer contacts a seller with the highest price only if she does not receive ads from any other seller). For the seller who asks the lowest price in the support, l, it is optimal to send only one ad; an ad always reaches a buyer and the probability of a sale with the lowest price is 1. The free entry and exit assumption implies that we must have $l = c(1)$.

A seller who asks the lowest price need not send more than one ad since this always leads to a sale. If each buyer chooses the lowest price offer she receives, then a seller who asks the highest price only sells if a buyer who receives his ad does not receive any other ads. In this light, we construct the equilibrium of the following type. Depending on the price, sellers send different numbers of ads such that the higher the price, the higher the number of ads sent. Denote a partition of a unit interval by $\mathcal{P}_n = \{p_i\}_{i=-1}^{n}$ where $p_{i-1} < p_i$ for all $i \in \{0, 1, \dots, n\}$, $p_{-1} = 0$, and $p_n = 1$. A seller who advertises a price $p \in [p_{i-1}, p_i)$ sends i ads, and a seller with the highest price, p_n, sends n ads. The partition of the unit interval with a maximum of n ads is illustrated in Figure 2.1.

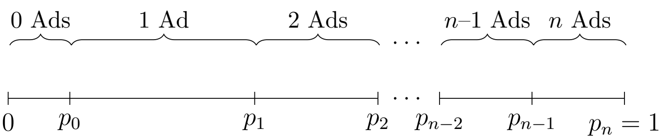

Figure 2.1: The partition of the unit interval with a maximum of n ads.

The equilibrium mixed strategy F is a probability distribution over $[p_0, p_n]$ defined piecewise for each subinterval of the partition. The corresponding probability density function of F is denoted by f and called a *price distribution*.[6] In equilibrium, each seller makes zero profits (due to free entry). Here we assume that the buyers who receive multiple ads always contact the seller with the lowest price; we return to this point in Section 2.4.2

[6]Note that by a price distribution we refer to a density function, not the distribution function.

2.4 Results

In this section we define configurations related to the partition of the unit interval; these are used to construct an equilibrium. In a 1-configuration all the sellers send exactly one ad, in a 2-configuration some sellers (low pricing ones) send one ad, and the others two ads, and in an n-configuration the sellers send different numbers of ads between one and n as depicted in Figure 2.1. In other words, configurations are indexed by their maximum number of ads that are sent. If no one wants to send more ads in a configuration and each buyer chooses the lowest price offer that she receives, then the configuration constitutes an equilibrium. That is why we start the analysis with careful derivation of configurations.

After defining configurations, we determine the conditions on the advertising costs that guarantee that a configuration constitutes an equilibrium. In particular, we determine a class of cost functions under which an n-configuration constitutes an equilibrium.

2.4.1 Configurations

For the formal definition of a configuration we need the following components:

1. The partition of the unit interval $\mathcal{P}_n = \{p_i\}_{i=-1}^{n}$ which assigns prices to the number of ads such that sellers with prices in $[p_{i-1}, p_i)$ sends i ads for all $i \in \{0, 1, \ldots, n\}$ and a seller with the highest price p_n sends n ads.

2. The mixed strategy F_n over $[p_0, p_n]$ defined piecewise for each subinterval of the partition:

$$F_n(p) = \begin{cases} F_n^{(1)}(p) & \text{for } p \in [p_0, p_1) \\ F_n^{(2)}(p) & \text{for } p \in [p_1, p_2) \\ \vdots & \\ F_n^{(n)}(p) & \text{for } p \in [p_{n-1}, p_n]. \end{cases}$$

   Since there are no gaps or mass points in the support, we have $F_n^{(i)}(p_i) = F_n^{(i+1)}(p_i)$ for all $i \in \{1, 2, \ldots, n-1\}$.

3. The number of sellers θ_n.

Since the partition, the mixed strategy, and the number of sellers vary with configurations, we use the subscripts in each component to refer to the index of a configuration.

Using these components we define the expected number of ads with price less than $p \in [p_{i-1}, p_i]$ as follows:

$$\lambda_n(p) = \sum_{j=1}^{i-1} j \cdot \left(F_n^{(j)}(p_j) - F_n^{(j)}(p_{j-1})\right) \cdot \theta_n + i \cdot \left(F_n^{(i)}(p) - F_n^{(i)}(p_{i-1})\right) \cdot \theta_n, \tag{2.1}$$

where term $j \cdot \left(F_n^{(j)}(p_j) - F_n^{(j)}(p_{j-1})\right) \cdot \theta_n$ is the number of ads times the expected number of sellers who send j ads. In particular, the total (expected) number of ads is

$$\lambda_n(p_n) = \sum_{i=1}^{n} i \cdot \left(F_n^{(i)}(p_i) - F_n^{(i)}(p_{i-1})\right) \cdot \theta_n.$$

Next, consider a seller who sends k ads with price $p \in [p_0, p_n]$, and a buyer who receives her ad. Assume that the buyer chooses the lowest price offer that she receives.[7] The number of ads with a price lower than p is distributed as $Poisson(\lambda_n(p))$.[8] Hence, the buyer who receives the seller's ad contacts the seller with probability $e^{-\lambda_n(p)}$, which is the probability that the buyer receives zero ads from price range of $[p_0, p)$. The probability that the buyer does *not* contact the seller is $1 - e^{-\lambda_n(p)}$. Consequently, the seller's expected profit by sending k ads at price p is given by

$$\pi_n(p, k) = \left(1 - \left(1 - e^{-\lambda_n(p)}\right)^k\right) p - c(k), \tag{2.3}$$

where $\left(1 - \left(1 - e^{-\lambda_n(p)}\right)^k\right)$ is the probability that a seller who sends k ads at price p is contacted by at least one buyer.

Using this notation we can give the formal definition of an n-configuration.

Definition 1. *An n-configuration is a triplet $(\mathcal{P}_n, F_n, \theta_n)$ which solves the following system of equations for $i \in \{1, 2, \ldots, n\}$:*

$$\pi_n(p, i) = 0 \qquad \textit{for all } p \in [p_{i-1}, p_i] \tag{ZP_i}$$

$$\pi_n(p_{i-1}, i-1) = \pi_n(p_{i-1}, i). \tag{I_i}$$

[7]In Section 2.4.2 we determine conditions when this is, indeed, optimal behaviour.

[8]This is due to the properties of the Poisson distribution. Buyers receive $n_T \sim Poisson(\lambda_n(1))$ ads in total. The probability of each of these ads has a price less than $p \in [p_{i-1}, p_i]$ is $\frac{\lambda_n(p)}{\lambda_n(1)}$. The probability for a buyer to receive n_L ads with price offer less than p therefore equals

$$\sum_{n_T=n_L}^{\infty} e^{-\lambda_n(1)} \frac{\lambda_n(1)^{n_T}}{n_T!} \binom{n_T}{n_L} \left(\frac{\lambda_n(p)}{\lambda_n(1)}\right)^{n_L} \left(1 - \frac{\lambda_n(p)}{\lambda_n(1)}\right)^{n_T - n_L} = e^{-\lambda_n(p)} \frac{\lambda_n(p)^{n_L}}{n_L!}. \tag{2.2}$$

That is, $n_L \sim Poisson(\lambda_n(p))$ (see, e.g., Lester et al. (2015)). We thank the referee for suggesting this clarification.

Condition (ZP_i) is the zero profit condition which says that each seller has to make zero profits by setting any price $p \in [p_{i-1}, p_i]$ and sending i ads. Conditions (I_i) for all $i \in \{1, 2, \ldots, n\}$ are indifference conditions which require that a seller who sets price p_{i-1} must be indifferent between sending $i-1$ and i ads for all $i \in \{1, 2, \ldots, n\}$. Note that a configuration is not necessarily an equilibrium, but an equilibrium is a configuration. Before we go in more detail to this, we prove that if we find a partition for a configuration, then it is unique. Then given partition $\mathcal{P}_n$, it is always possible to uniquely determine mixed strategies F_n and the number of sellers θ_n.

Proposition 1. *If an n-configuration exists, then it is unique.*

Proposition 1 is a technical result which shows that if an n-configuration exists, it has a unique partition $\mathcal{P}_n$, and given that partition, the total number of ads with price less than $p \in [p_{i-1}, p_i]$ is given by

$$\lambda_n(p) = -\log\left(1 - \sqrt[i]{1 - \frac{c(i)}{p}}\right), \tag{2.4}$$

the mixed strategy over $[p_{i-1}, p_i]$ for the ith subinterval of the partition by

$$F_n^{(i)}(p) = \frac{1}{\theta_n}\left[\frac{\lambda_n(p)}{i} + \sum_{j=1}^{i-1} \frac{\lambda_n(p_j)}{j(j+1)}\right], \tag{2.5}$$

and the number of sellers by

$$\theta_n = \left[\frac{\lambda_n(p_n)}{n} + \sum_{j=1}^{n-1} \frac{\lambda_n(p_j)}{j(j+1)}\right]. \tag{2.6}$$

In words, a configuration is completely pinned down by the maximum number of ads and advertising costs $c(\cdot)$.

Notice that an $(n-1)$-configuration and an n-configuration satisfy the same zero profit and indifference conditions up until price p_{n-2}. Consequently, the partitions in both configurations are the same except that the last subinterval is divided in two in the n-configuration. The number of sellers changes, but for any price $p \le p_{n-2}$ in any subinterval the number of ads remains the same. The mixed strategy in each subinterval has a logarithmic form, and the price distribution is decreasing and convex in each subinterval of the partition.

Next we give an example of a 1-configuration where each seller sends a single ad. In the Appendix we derive 2- and 3-configurations (Examples 2 and 3). Solving the 1- and 2-configurations is pretty simple, but determining the partition of the unit interval for the 3-configuration requires solving of a cubic equation and gets somewhat arduous. Constructing higher-indexed configurations is probably possible only numerically.

Example 1. *Consider a market in which each seller sends only one ad at maximum. A seller who sets the lowest price,* p_0*, sells her good for sure and earns* $p_0 - c(1)$*. Since free entry implies zero profits, we must have* $p_0 = c(1)$*, and the partition of the unit interval becomes* $\mathcal{P}_1 = \{0, c(1), 1\}$*.*

Since in the 1*-configuration each seller sends a single ad, the total number of ads is the same as the number of sellers* θ*. A seller who asks the highest price,* 1*, sells only if the buyer who receives her ad does not get any other ads; this happens with probability* $e^{-\theta}$*. The zero profit condition requires* $e^{-\theta} - c(1) = 0$*, which implies that the number of sellers in the market is* $\theta = -\log c(1)$*.*

We know that all the sellers have to get the same revenue from sending a single ad and setting a price according to the mixed strategy, F*. Consider a seller who sends an ad with price* $p \in (c(1), 1)$*. Her expected revenue is* $e^{-F(p)\theta}p - c(1)$*, where* $e^{-F(p)\theta}$ *is the probability that a buyer who receives the seller's ad does not receive any other ads with price less than* p*. Then we can use the zero profit condition and substitute* $\theta = -\log c(1)$ *into this and obtain*

$$F(p) = 1 - \frac{\log p}{\log c(1)}.$$

We have thus found a unique 1*-configuration that consists of the following three elements: (i) partition of the unit interval* $\mathcal{P}_1 = \{0, c(1), 1\}$*, (ii) mixed strategy* $F(p) = 1 - \frac{\log p}{\log c(1)}$*, for* $p \in [c(1), 1]$*, and (iii) number of sellers* $\theta = -\log c(1)$*.*

The partition of the unit interval and an example of a price distribution with $c(1) = \frac{1}{2}$ *are given in Figures 2.2 and 2.3.*

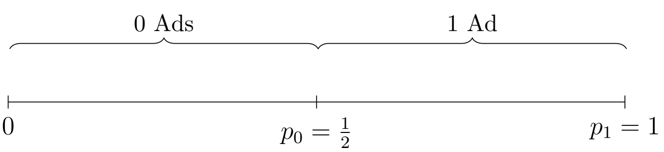

Figure 2.2: The partition of the unit interval in a 1-configuration.

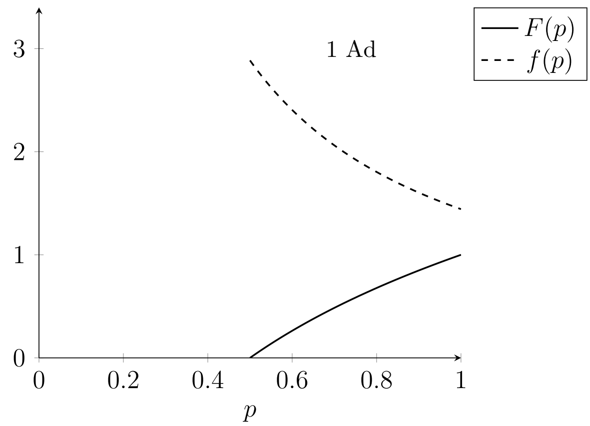

Figure 2.3: The price distribution, $f(p)$, and the mixed strategy, $F(p)$, of the 1-configuration with $c(1) = \frac{1}{2}$.

The 1-configuration constitutes an equilibrium if no seller finds it profitable to send more than 1 ad. It turns out that if the advertising cost function is convex, then each seller sends exactly one ad in equilibrium. This is the case in Butters (1977) where the advertising costs are linear. We postpone the proof of this for later analysis where we have the sufficient tools and notation.

2.4.2 Equilibrium

In this section we construct an equilibrium. There are two things that could go wrong with an n-configuration to be an equilibrium. The first one is that some of the sellers might want to deviate and send more than n ads. The second one is that we have implicitly assumed so far that each buyer always chooses the lowest price offer that she receives. Basically, these two problems occur if the advertising costs are not increasing fast enough. On the other hand, if the costs are increasing too fast, then it is not possible to construct an n-configuration. To tackle these issues we need to determine a class of cost functions under which no seller wants to deviate and all the buyers choose the lowest price offer that they receive.

We start by proving two lemmas. In the first lemma we derive a class of advertising costs under which an optimal behaviour for buyers is to choose the lowest price offer. In the second lemma we show that a seller who asks the highest price, has the highest incentive to send more ads. This result eases the construction of an equilibrium; once we have found a configuration, we only need to check that a seller with the highest price does not find it profitable to send more ads.

First, let us define the following class of advertising cost functions.

Definition 2. *For $n \in \mathbb{N} \backslash \{1\}$ and $\gamma \in (0,1)$, let $\underline{\mathcal{C}}_n(\gamma)$ be the set of advertising costs defined on $\mathbb{N}_0$ with the following two properties:*

1. *Any $c \in \underline{\mathcal{C}}_n(\gamma)$ is strictly increasing on $\mathbb{N}_0$ such that $c(0) = 0$ and $c(1) = \gamma$.*
2. *Let $\underline{c}(k) = \frac{k\gamma}{1+(k-1)\gamma}$. Any $c \in \underline{\mathcal{C}}_n(\gamma)$ satisfies $\Delta \underline{c}(k) < \Delta c(k)$ for all $k \in \{2, 3, \ldots, n\}$.*[9]

In equilibrium the buyers know the sellers' pricing and advertising strategies, and they must best-respond to them. In particular, choosing the lowest price offer received has to be optimal. It turns out that this is the case when the advertising costs belong to the class $\underline{\mathcal{C}}_n(\gamma)$ given in Definition 2, and the construction of an n-configuration is correct.

Lemma 1. *If $c \in \underline{\mathcal{C}}_n(\gamma)$ for any $\gamma \in (0,1)$, then in an n-configuration a buyer always chooses the lowest price offer that she receives.*

The intuition of Lemma 1 is the following. If the advertising costs do not increase fast enough, the proportions of sellers who send multiple ads are relatively large. This means that a buyer who receives an ad with a low price from a seller who has sent many ads is in competition with the other buyers who have received this seller's ads. For all these buyers, the seller's offer is likely to be the lowest one. But then contacting the lowest price compromises the probability of getting a good.

Next consider the ith subinterval of the unit partition and the sellers who send i ads and ask prices between p_{i-1} and p_i. By construction, any seller with price $p < p_i$ makes negative profit by sending $i+1$ ads, while a seller with price $p = p_i$ is just indifferent; both i and $i+1$ ads generate zero profits.

Lemma 2. *Assume $c \in \underline{\mathcal{C}}_n(\gamma)$. In an n-configuration for any $i \in \{1, 2, \ldots, n\}$ a seller who asks the highest price $p_i \in [p_{i-1}, p_i]$ has the highest incentive to send more than i ads. Furthermore, a seller who asks the lowest price $p_{i-1} \in [p_{i-1}, p_i]$ has the highest incentive to send fewer than i ads.*

Using Lemma 1 and 2 we get the following proposition.

Proposition 2. *Assume $c \in \underline{\mathcal{C}}_n(\gamma)$. In an n-configuration, a seller who sets price $p \in [p_{i-1}, p_i]$ cannot increase her profits by sending $k \in \{1, \ldots, i-1, i+1, \ldots, n\}$.*

[9] Recall that $\Delta c(k+1) \equiv c(k+1) - c(k) > 0$.

Although Proposition 2 is not surprising, it provides an easy test for an equilibrium: if in an n-configuration a seller with the highest price does not want to deviate and send more than n ads, then the n-configuration constitutes an equilibrium.[10]

Corollary 1. *Assume $c \in \underline{\mathcal{C}}_n(\gamma)$. An n-configuration constitutes an equilibrium if the seller who asks the highest price does not find it profitable to send more than n ads.*

By these results, the Butters's (1977) model with capacity constrained sellers features each seller sending just one ad in equilibrium as the cost function is linear. This follows because the marginal return of the second ad is always lower than that of the first ad; the second ad is useless if the first ad leads to a sale. Therefore, for linear advertising costs, if a seller finds it profitable to send a second ad, it gets surplus from the first one, which violates the zero profit condition. Consequently, if the second ad is sufficiently more expensive than the first ad, sellers send only one ad in a free entry equilibrium. We state the result as follows.

Proposition 3. *If the advertising cost function is convex, then each seller sends exactly one ad in equilibrium.*

The characterisation of the single-ad equilibrium is given in Example 1. The equilibrium price distribution coincides with Butters (1977) by setting the cost of production to zero, normalising the number of buyers to unity, and assuming that each buyer values the good at unity in the Butters's model.[11] This is due to the free entry and exit assumption and convex advertising costs.

2.4.3 Multi-Advertisement Equilibria

In this section we study a class of advertising costs under which an n-configuration constitutes an equilibrium. It turns out that even the concavity of advertising costs is not enough to guarantee that some sellers send more than 1 ad in equilibrium. Next we construct a class of cost functions that allows an n-configuration to constitute an equilibrium for some $n > 1$. The idea is to determine an upper bound for advertising costs such that if the advertising costs increase faster than the upper bound after $k + n$ ads

[10]We impose that each seller sends all the ads with the same price in equilibrium. Butters (1977) does not use this premise. However, as the proof of Lemma 2 indicates, we can relax the assumption and allow sellers to post different prices in each ad. In equilibrium, sellers who choose a price in interval $[p_{i-1}, p_i]$ use a mixed strategy $F_n^{(i)}$ and send i ads. They could equally well choose i different prices by making i independent draws from $F_n^{(i)}$; they would still make zero profits. If a seller deviates and chooses a price $p' \notin [p_{i-1}, p_i]$ and sends i ads, she makes losses. Assuming that each seller advertises just one price simplifies the analysis; otherwise there would be buyers with different price offers approaching the seller.

[11]Note that Butters's advertising price density $a(p)$ equals $f_1(p)\theta_1$ in our case.

($k \in \{1, 2, \dots\}$), then a seller with the highest price does not find it profitable to send more than n ads. Then, if the advertising costs belong to the intersection of the class of costs given in Definition 2 and the class defined by the upper bound, an n-configuration constitutes an equilibrium.

Consider an n-configuration and a seller who sets price at 1. She does not want to send more than n ads if $\pi_n(1, n+k) \leq \pi_n(1, n)$ for all $k > 1$ – that is,

$$\left(1 - \left(1 - e^{-\lambda_n(1)}\right)^{n+k}\right) - c(n+k) \leq \left(1 - \left(1 - e^{-\lambda_n(1)}\right)^{n}\right) - c(n). \tag{2.7}$$

From the zero profit condition ZP_n we get that $1 - e^{-\lambda_n(1)} = (1 - c(n))^{\frac{1}{n}}$ and so (2.7) becomes

$$c(n) \leq 1 - (1 - c(n+k))^{\frac{n}{n+k}}, \tag{2.8}$$

which gives us the upper bound for the advertising costs. As with the proof of Lemma 1, let us treat the upper bound in (2.8) as a functional equation and denote it as $\bar{c}(x) = 1 - (1 - \bar{c}(x+k))^{\frac{x}{x+k}}$ such that $\bar{c} : \mathbb{R}_+ \to \mathbb{R}$. This functional equation has a solution of $\bar{c}(x) = 1 - \phi(x)^x$ such that $\phi(x) = \phi(x+1)$ for all $x \in \mathbb{R}$. Since the advertising costs are assumed to be increasing, we must have $\phi(x) = \phi \in (0, 1)$ for all x, which makes $\bar{c}$ an increasing concave function. Furthermore, from Example 1 we know that the upper bound for $\Delta c(2)$ is $c(1)(1-c(1))$. This is the initial value for the upper bound from which we can solve $\phi = 1 - c(1)$. The upper bound becomes

$$\bar{c}(x) = 1 - \phi^x \text{ for all } x \in \mathbb{R}_+,$$

where $\phi = 1 - c(1)$. Simple algebra shows that the upper bound is greater than the lower bound in Definition 2 – that is, $\bar{c}(x) > \underline{c}(x)$ for all $x > 1$.[12] If the costs increase as fast as the upper bound, i.e. if $\Delta c(n+k) = \Delta \bar{c}(n+k)$, then a seller who asks the highest price is indifferent between sending n and $n + k > n$ ads.

Using the upper bound we define the following class of advertising costs.

Definition 3. *For $n \in \mathbb{N} \backslash \{1\}$ and $\gamma \in (0, 1)$, let $\bar{\mathcal{C}}_n(\gamma)$ be the set of advertising costs defined on $\mathbb{N}_0$ with the following two properties:*

1. *Any $c \in \bar{\mathcal{C}}_n(\gamma)$ is strictly increasing on $\mathbb{N}_0$ such that $c(0) = 0$ and $c(1) = \gamma$.*
2. *Let $\bar{c}(k) = 1 - (1 - \gamma)^k$. Any $c \in \bar{\mathcal{C}}_n(\gamma)$ satisfies $\Delta c(k) < \Delta \bar{c}(k)$ for all $k \in \{2, 3, \dots, n-1\}$ and $\Delta c(k) \geq \Delta \bar{c}(k)$ for all $k \in \{n, n+1, \dots\}$.*

[12] See the proof of Lemma 4.

Our aim is to determine when an n-configuration constitutes an equilibrium. To that end, let the intersection of the classes of advertising costs in Definition 2 and Definition 3 be denoted by $\mathcal{C}_n(\gamma) = \underline{\mathcal{C}}_n(\gamma) \cap \bar{\mathcal{C}}_n(\gamma)$. The class of advertising costs we study is then defined as $\mathcal{C}(\gamma) = \bigcup_{i=2}^{\infty} \mathcal{C}_i(\gamma)$.[13] Then let the advertising costs be $c(k) = \underline{c}(k) + a_k = \frac{k\gamma}{1+(k-1)\gamma} + a_k$ for all $k \geq 1$.

We still need to check under which conditions $\mathcal{C}_n(\gamma)$ is not an empty set to guarantee than an n-configuration exists and is a equilibrium for $c \in \mathcal{C}_n(\gamma)$. This result is given by the following proposition.

Proposition 4. *There exists a unique $n(\gamma) \in \mathbb{N}$ which gives the highest possible configuration under costs $c \in \mathcal{C}(\gamma)$. Moreover, $n(\gamma)$ is decreasing in γ and $\mathcal{C}_n(\gamma) \neq \emptyset$ for all $n \leq n(\gamma)$.*

In Figure 2.4 we illustrate the relationship between n, $n(\gamma)$, and $x^*(\gamma)$, which are used in the proof of Proposition 4.

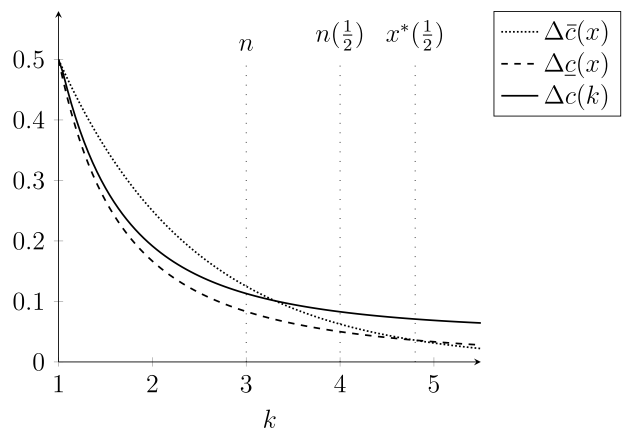

Figure 2.4: An example of advertising costs $c \in \mathcal{C}_3(\frac{1}{2})$.

Next we give our last result which is a direct implication of the construction of $\mathcal{C}(\gamma)$, Proposition 1, Corollary 1, and Proposition 4.

Proposition 5. *Assume that $c \in \mathcal{C}_n(\gamma)$ such that $n \leq n(\gamma)$. Then there exists a unique n-configuration which constitutes an equilibrium.*

If the advertising costs coincide with the upper bound, then a 1-configuration is an equilibrium. This can be easily seen by considering $p_1 = \frac{c(1)^2}{2c(1)-c(2)}$ derived in Example 1. If we substitute $\bar{c}(k)$ into this formula we get that $p_1 = 1$ for all $\gamma \in (0,1)$.

[13] An example of an advertising cost function that belongs to class $\mathcal{C}_n(\gamma)$ can be constructed by using the lower bound. Consider a sequence (a_k) that increases up to a_n sufficiently slowly, and after that sufficiently fast with $a_1 = 0$.

Proposition 5 gives us a simple test to find an equilibrium: if the advertising costs belong to class $\mathcal{C}(\gamma)$, find an $n \leq n(\gamma)$ such that $c \in \mathcal{C}_n(\gamma)$. Then the n-configuration is an equilibrium.

2.5 Discussion

In this section we point out, on the one hand, some limitations of our analysis, and, on the other hand, possible applications and interpretations of the model.

Comparative statics in a multiple-ad equilibrium is complicated. For instance, changing advertising costs affects not only the equilibrium strategies, but also the number of sellers (free entry and exit). It is just hard to keep track of both effects.

Nevertheless, some comparative statics can still be conducted. First, the equilibrium configuration with the highest index is determined by the cost of the first ad. The higher it is, the smaller the maximum index (Proposition 4). This has the following economic intuition. Let us interpret the cost of the first ad as the sum of an entry cost and the marginal cost of the first ad. Then the higher the entry cost, the less there can be potential entrants. This implies that the probability of a sale is greater or competition is less severe. Therefore fewer ads are sent with a higher entry cost in equilibrium.

Furthermore, comparative statics can be done *within* the equilibrium configuration. Suppose that an n-configuration forms the equilibrium. If we decrease the advertisements costs such that the same configuration is still an equilibrium, we have the following effects: (i) the total number of ads sent is greater (see equation (2.4)), and (ii) the number of sellers is higher (a consequence of the first effect).

Capacity constrained sellers is a crucial feature of our model. It guarantees the positive association of prices and the number of ads.

A simple example demonstrates this. Assume that the sellers have unlimited capacity, and assume a cost function such that each seller sends a finite number of ads. By the standard arguments pricing is in mixed strategies on some interval $[p_0, 1]$.

Consider a seller with price p_0, and assume that she sends k ads. Her pay-off is given by $kp_0 - c(k)$. Then consider a seller with price unity, and assume provisionally that she sends k ads, too. The number of buyers she attracts is given by a binomial distribution with success probability $e^{-\theta k}$, where θ is the number of sellers and θk the total number of ads sent assuming that each seller sends k ads. Consequently, the seller's pay-off is given by $ke^{-\theta k} - c(k)$. Under mixed strategy the pay-offs have to be equal, and this condition allows solving $p_0 = e^{-\theta k}$. This implies that the pay-offs of these two sellers are identical, and hence the optimality condition for the sellers is $c(k+h) - c(k) > hp_0$ for $h \in \{-k, -(k-1), \ldots, -1, 0, 1, \ldots\}$ (no profitable deviations to send $k+h$ ads). This shows that in equilibrium all the sellers send the same number of ads.

This leaves open the possibility that there are equilibria where the sellers send different numbers of ads. Let us consider this next. Assume temporarily that low pricing sellers with $p \in [p_0, p_1)$ send k ads, and high pricing sellers with $p \in [p_1, 1]$ send $k+1$ ads. The pay-off of the lowest pricing seller is given by $kp_0 - c(k) = 0$ by free entry. We can thus solve $p_0 = \frac{c(k)}{k}$. Denote the total number of ads with price lower than p by $\lambda(p)$. The pay-off of the highest pricing seller is given by $(k+1)e^{-\lambda(1)} - c(k+1) = 0$, and we can solve $e^{-\lambda(1)} = \frac{c(k+1)}{k+1}$.

In equilibrium the lowest pricing seller does not find it profitable to send $k+1$ ads, and the highest pricing seller does not find it profitable to send k ads, or

$$kp_0 - c(k) > (k+1)p_0 - c(k+1) \tag{2.9}$$

$$(k+1)e^{-\lambda(1)} - c(k+1) > ke^{-\lambda(1)} - c(k). \tag{2.10}$$

Substituting $p_0 = \frac{c(k)}{k}$ in the first condition, and $e^{-\lambda(1)} = \frac{c(k+1)}{k+1}$ in the latter condition, and manipulating a little yields

$$c(k+1) > \frac{k+1}{k} c(k) \tag{2.11}$$

and

$$c(k+1) < \frac{k+1}{k} c(k) \tag{2.12}$$

which is a contradiction. This demonstrates that we lose the positive association with prices and the number of ads in general, if we allow unlimited capacity.

We assume unit capacity, but relaxing this to some finite capacity $k > 1$ does not affect the basic message of our model. In equilibrium pricing is still in mixed strategies, and the seller with the lowest price sends k ads, while higher pricing sellers send more than k ads if the cost function is concave enough.

Although our model is highly stylised in the sense that the capacity constrained sellers are assumed to possess just one unit of a good, it may be applicable to some settings where capacity constraints are salient. For instance, suppose that each seller has a room for rent, and the rooms are more or less equal in quality (distance, ratings, etc.). The sellers use internet platforms to advertise their items. Posting the offer onto a platform is costly, but it is reasonable to argue that the costs are marginally decreasing in the number of platforms chosen since the first offer involves costs, such as taking pictures and composing the ad, that are not incurred for the succeeding offers. Based on this, our model suggests that the sellers that use multiple platforms should ask higher prices.

An alternative interpretation of our theoretical framework is as follows. There is a large number of agents divided into two different types. To produce a unit surplus a member of both types has to form a pair. One party can commit to the division of the surplus in the sense that it sends take-it-or-leave-it offers to the other party. The senders are, however, subject to competition by other senders as the receiving party accepts the best offer. Sending offers is costly, and in equilibrium the senders mix over the offers and number of offers sent. This is the typical setting of a decentralised model of a job market. Low wage offers would then be advertised more as they correspond to high prices of goods offered for sale.

2.6 Conclusion

We study a version of Butters's seminal model of informative advertising with a large number of buyers and sellers, assuming that the sellers are capacity constrained each with one unit of a good. In order to trade a buyer has to receive an ad. Pricing is in mixed strategies, and we establish an equilibrium where high pricing sellers send more ads than low pricing sellers, i.e., a positive relationship between prices and the number of ads. In equilibrium, the buyers who received multiple ads contact the seller with lowest price.

The key ingredient in our analysis is the cost of advertising. If the cost function is convex, then each seller sends exactly one ad in equilibrium. The reason is diminishing marginal returns of advertising that stem from the capacity constraint. If the marginal cost of advertising is decreasing there may be equilibria where multiple ads are sent.

We delineate a class of advertising costs which permits an equilibrium where sellers send multiple ads, and each buyer contacts the seller who offers the lowest price. The first property requires that the marginal cost of advertising is decreasing, and we determine an upper bound for the advertising costs. The second property requires that the cost is increasing sufficiently fast; otherwise there could be too many sellers who price high, and send many ads, such that the buyers could profitably trade-off low price and the probability of getting a good. We determine a lower bound for the advertising costs.

Appendix

Proof of Proposition 1

Proof. We start by showing that an n-configuration has a unique partition. Assume that there is an n-configuration with two different partitions $\mathcal{P}_n$ and $\mathcal{P}'_n$ such that the elements of the partitions are denoted by $p_i \in \mathcal{P}_n$ and $p'_i \in \mathcal{P}'_n$. Let index k be the first one where $p_k \neq p'_k$ and let $p_k < p'_k$. The numbers of ads less than p_k and p'_k with different partitions are denoted by $\lambda_n(p_k)$ and $\lambda'_n(p'_k)$, respectively. Up until price p_k we necessarily have $\lambda_n(p_k) = \lambda'_n(p_k)$ since in both cases the sellers make zero profits.

Now we have the following two equalities

$$\left(1 - \left(1 - e^{-\lambda_n(p_k)}\right)^k\right) p_k - c(k) = \left(1 - \left(1 - e^{-\lambda'_n(p'_k)}\right)^k\right) p'_k - c(k)$$
$$\left(1 - \left(1 - e^{-\lambda_n(p_k)}\right)^{k-1}\right) p_k - c(k-1) = \left(1 - \left(1 - e^{-\lambda'_n(p'_k)}\right)^{k-1}\right) p'_k - c(k-1).$$

This system of equations can be rewritten as

$$\frac{1 - \left(1 - e^{-\lambda'_n(p'_k)}\right)^{k-1}}{1 - \left(1 - e^{-\lambda'_n(p'_k)}\right)^k} = \frac{1 - \left(1 - e^{-\lambda_n(p_k)}\right)^{k-1}}{1 - \left(1 - e^{-\lambda_n(p_k)}\right)^k}. \tag{2.13}$$

Both sides have the same functional form of $f(z) = \frac{1-z^{k-1}}{1-z^k}$ such that

$$\frac{\partial}{\partial z} f(z) = \frac{z^{k-2}(1-z)\left(k - \frac{1-z^k}{1-z}\right)}{\left(1-z^k\right)^2} > 0 \tag{2.14}$$

since $z \in (0,1)$ and $k - \frac{1-z^k}{1-z} = k - \sum_{i=1}^{k} z^{i-1} > 0$ by the sum of a geometric progression where each element $z^{i-1} \in (0,1)$. In other words, f is a strictly increasing function and so $f(x) = f(y)$ only if $x = y$. Hence, the system of equations in (2.13) has a solution of $\lambda_n(p_k) = \lambda'_n(p'_k)$. This is a contradiction and therefore the partition must be unique.

Next we show that given partition $\mathcal{P}_n$, there is a unique mixed strategy F_n that solves the zero profit conditions. Consider a seller who sets price $p \in [p_{i-1}, p_i]$ and sends i ads. From zero profit condition (ZP_i) we get

$$\lambda_n(p) = -\log\left(1 - \sqrt[i]{1 - \frac{c(i)}{p}}\right). \tag{2.15}$$

On the other hand, from expression (2.1) we know the total number of ads with a price less than p is

$$\lambda_n(p) = \lambda_n(p_{i-1}) + i\left[F_n^{(i)}(p) - F_n^{(i-1)}(p_{i-1})\right]\theta_n.$$

Combining these two we have the following expression for mixed strategy $F_n^{(i)}(p)$ for prices $p \in [p_{i-1}, p_i]$:

$$F_n^{(i)}(p) = F_n^{(i-1)}(p_{i-1}) + \frac{1}{i\theta_n}\left[\lambda_n(p) - \lambda_n(p_{i-1})\right].$$

This recursive formula can be rewritten as follows by substituting in the mixed strategies from the earlier subintervals:

$$F_n^{(i)}(p) = \frac{1}{\theta_n}\left[\frac{\lambda_n(p)}{i} + \sum_{j=1}^{i-1}\frac{\lambda_n(p_j)}{j(j+1)}\right], \tag{2.16}$$

where $\lambda_n(\cdot)$ is given in Equation (2.15).

Finally, using the fact that $F_n^{(n)}(1) = 1$ we can solve the number of sellers:

$$\theta_n = \left[\frac{\lambda_n(p_n)}{n} + \sum_{j=1}^{n-1}\frac{\lambda_n(p_j)}{j(j+1)}\right]. \tag{2.17}$$

Equations (2.4), (2.5), and (2.6) uniquely determine the mixed strategies F_n and the number of the sellers in the market θ_n. □

Proof of Lemma 1

Proof. Let us state the obvious case first: if a buyer receives an ad at price $p \in [p_0, p_1)$, she knows that the seller has sent only a single ad and therefore by contacting the seller she always gets the good.

Then, consider a buyer who receives an ad at price $p \in [p_{i-1}, p_i)$ for some $i > 1$. She knows that the seller who has sent this offer has sent i ads. If the buyer contacts this seller, the probability that she gets the object is

$$Q_i(p) \equiv \sum_{k=0}^{i-1}\frac{1}{k+1}\binom{i-1}{k}\left(e^{-\lambda_n(p)}\right)^k\left(1-e^{-\lambda_n(p)}\right)^{i-1-k}. \tag{2.18}$$

This can be written as

$$\frac{e^{\lambda_n(p)}}{i}\sum_{k=1}^{i}\binom{i}{k}\left(e^{-\lambda_n(p)}\right)^k\left(1-e^{-\lambda_n(p)}\right)^{i-k}$$

and using the binomial theorem it becomes

$$Q_i(p) = \frac{e^{\lambda_n(p)}}{i}\left(1 - \left(1 - e^{-\lambda_n(p)}\right)^i\right).$$

Further, from zero profit condition (ZP_i) we get

$$e^{-\lambda_n(p)} = 1 - \left(1 - \frac{c(i)}{p}\right)^{\frac{1}{i}}.$$

Substituting this into the formula of $Q_i(p)$ we get

$$Q_i(p) = \frac{c(i)}{ip\left(1 - \sqrt[i]{1 - \frac{c(i)}{p}}\right)}. \tag{2.19}$$

A buyer's utility of getting an object at price p is $1-p$. Then, the expected utility of a buyer who contacts a seller with price $p \in [p_{i-1}, p_i)$ is $U(p) = Q_i(p)(1-p)$. The derivative of this with respect to p is

$$U'(p) = -\frac{1}{p}\frac{c(i)}{ip\left(1 - \sqrt[i]{1 - \frac{c(i)}{p}}\right)} + \frac{1-p}{p}\frac{c(i)^2\left(1 - \frac{c(i)}{p}\right)^{\frac{1}{i}-1}}{\left(ip\left(1 - \sqrt[i]{1 - \frac{c(i)}{p}}\right)\right)^2}. \tag{2.20}$$

After some multiplications and rearrangements that retain the sign, this expression becomes

$$-1 + U(p)\left(1 - \frac{c(i)}{p}\right)^{\frac{1}{i}-1}. \tag{2.21}$$

First we prove that (2.21) is strictly negative at $p = p_{i-1}$. After that we show that $U(p)$ is an inverted-U-shaped function ($\cap$-shaped) and thus if $U'(p_{i-1}) < 0$, then also $U'(p) < 0$ for all $p \in [p_{i-1}, p_i)$.

Expression (2.21) is strictly negative at $p = p_{i-1}$ if

$$(1 - p_{i-1})\frac{c(i)}{ip_{i-1}} < \left(1 - \frac{c(i)}{p_{i-1}}\right)^{1-\frac{1}{i}} - 1 + \frac{c(i)}{p_{i-1}}. \tag{2.22}$$

We know that a seller who asks price p_{i-1} is indifferent between sending i and $i-1$ ads. We hence get from indifference condition (I_i) and zero profit condition (ZP_i) that

$$\left(1 - \frac{c(i)}{p_{i-1}}\right)^{1-\frac{1}{i}} = 1 - \frac{c(i-1)}{p_{i-1}}.$$

By substituting this into (2.22) and rearranging the terms we get

$$p_{i-1} > 1 - i\frac{\Delta c(i)}{c(i)}. \tag{2.23}$$

We know that $p_{i-1} > c(i-1)$. So, if $c(i-1) > 1 - i\frac{\Delta c(i)}{c(i)}$, then also (2.23) is satisfied. This is equivalent to

$$c(i) \geq \frac{ic(i-1)}{i-1+c(i-1)}. \tag{2.24}$$

From this we get lower bound $\underline{c} : \mathbb{R}_+ \to \mathbb{R}_+$ for the differences of advertising costs under which each buyer chooses the lowest price offer she receives. The lower bound $\underline{c}$ must satisfy

$$\underline{c}(x-1) = 1 - x\frac{\Delta \underline{c}(x)}{\underline{c}(x)}, \tag{2.25}$$

for all $x \in \mathbb{R}_+$. This is a functional equation which has the following increasing solution:

$$\underline{c}(x) = \frac{x}{x+a}$$

for some $a > 0$. The initial value for advertising costs is given by $\underline{c}(1) = c(1)$ from where we can solve $a = \frac{1-c(1)}{c(1)}$. The lower bound for the advertising cost function thus is

$$\underline{c}(x) = \frac{x}{x + \frac{1-c(1)}{c(1)}} = \frac{xc(1)}{1+(x-1)c(1)},$$

which is exactly the function given in Definition (2) with $\gamma = c(1)$. So we know that if $\Delta c(i) \geq \Delta \underline{c}(i)$ then $U'(p_{i-1}) < 0$.

Let us consider the left-hand side and the right-hand side of expression (2.22) as functions of p instead of p_{i-1}. The left-hand side is a strictly convex strictly decreasing function of p for all $p \in [p_{i-1}, p_i)$. The right-hand side is a strictly concave and strictly increasing function of p for all $p \in [p_{i-1}, p_i)$. Moreover, the left-hand side is strictly positive at $p = c(i)$, whereas the right-hand side is 0. If $p = 1$, then the left-hand side is 0 and the right-hand side is strictly positive. Hence, there is a unique intersection which implies that $U(p)$ is a $\cap$-shaped function of p. Since $U'(p_{i-1}) < 0$, then also $U'(p) < 0$ for all $p \in [p_{i-1}, p_i]$.

Finally, we need to show that a buyer wants to contact a seller who has sent an ad at price $p \in [p_{i-1}, p_i)$ rather than a seller who has sent an ad at price $p' \in [p_{k-1}, p_k)$ for some $k > i$. We know that $Q_i(p)(1-p) \geq Q_i(p_i)(1-p_i)$. By the similar arguments, we know that $Q_{i+1}(p_i)(1-p_i) \geq Q_{i+1}(p')(1-p')$ for all $p' \in [p_{i-1}, p_i)$. It is easy to verify that $Q_i(p_i) \geq Q_{i+1}(p_i)$, and hence we have $Q_i(p)(1-p) \geq Q_{i+1}(p')(1-p')$ for all $p' \in [p_i, p_{i+1})$. This implies that $Q_i(p)(1-p) \geq Q_k(p')(1-p')$ for all $k \geq i$ and $p' \geq p$. □

Proof of Lemma 2

Proof. The derivative of $\pi_n(p,k)$ with respect to $p \in [p_{i-1}, p_i]$ and for any $k \in \mathbb{N}_0$ is

$$\pi_n'(p,k) = 1 - \left(1 - \frac{c(i)}{p}\right)^{k/i} - \frac{k}{i}\left(1 - \frac{c(i)}{p}\right)^{\frac{k}{i}-1} \frac{c(i)}{p}. \tag{2.26}$$

Let us simplify the notation and denote $x = \frac{c(i)}{p} \in (0,1)$ and $z = \frac{k}{i}$. By rearranging terms we get

$$\pi_n'(x,z) = 1 - (1-x)^{z-1}\left(1 - (1-z)x\right). \tag{2.27}$$

This is non-negative if

$$(1-x)^{1-z} \geq 1 - (1-z)x. \tag{2.28}$$

and negative if

$$(1-x)^{1-z} < 1 - (1-z)x. \tag{2.29}$$

The derivative in (2.27) is zero if $z = 0$ or $z = 1$. Moreover, since $(1-x)^{1-z}$ is a strictly increasing convex function of z and $1-(1-z)x$ is a strictly increasing linear function of z, we have that $\pi_n'(p,k) \geq 0$ for all $z \geq 1$ and $\pi_n'(x,z) \leq 0$ for all $z \in [0,1]$. In other words, we have that for all $p \in [p_{i-1}, p_i]$, $\pi_n'(p,k) = 0$ if $k = i$, $\pi_n'(p,k) < 0$ if $k \in \{1, 2, \ldots, i-1\}$, and $\pi_n'(p,k) > 0$ if $k > i$. □

Proof of Proposition 2

Proof. Lemma 2 shows that $\pi_n(p,k) \leq \pi_n(p',k)$ for some $p \leq p'$ and $k \in \{i+1, \ldots, n\}$. So, if a seller sends $k \in \{i+1, \ldots, n\}$ ads at price $p \in [p_{i-1}, p_i]$, her profits are $\pi_n(p,k) \leq \pi_n(p_k, k) = 0$ for $p_k \geq p$.

Analogously, if a seller sets price at $p \in [p_{i-1}, p_i]$ and sends $k \in \{1, \ldots, i-1\}$ ads her profits are $\pi_n(p,k) \leq \pi_n(p_k, k) = 0$ for $p_k \leq p$ by Lemma 2. □

Proof of Corollary 1

Proof. Assume that the cost function is linear $c(k) = k\alpha$, for some $\alpha \in (0,1)$. Then consider a 1-configuration and a seller who asks price 1, but deviates and sends k ads. Then her expected profits $\pi_1(k,1)$ are

$$\left(1-\left(1-e^{-\lambda_1(1)}\right)^k\right)-c(k)=\left(1-(1-c(1))^k\right)-c(k) \tag{2.30}$$

since $\lambda_1(1) = -\log c(1)$. This is decreasing for all $k \geq 0$ or

$$\left(1-(1-\alpha)^k\right)-k\alpha \geq \left(1-(1-\alpha)^{k+1}\right)-(k+1)\alpha,$$

which can be simplified to

$$1 \geq (1-\alpha)^k. \tag{2.31}$$

So, in a 1-configuration it is not profitable to send more than one ad if the advertising costs are linear. This clearly holds good also for costs that increase faster, i.e. for convex cost functions. □

Proof of Proposition 4

Proof. The goal of this proof is to show that there exists a unique and decreasing $n(\gamma) \in \mathbb{N}$ such that $\Delta\bar{c}(k) > \Delta\underline{c}(k)$ for all $k = 2, 3, \ldots, n(\gamma)$ and $\Delta\bar{c}(k) \leq \Delta\underline{c}(k)$ for all $k > n(\gamma)$. Then the set $\mathcal{C}_n(\gamma) = \underline{\mathcal{C}}_n(\gamma) \cap \bar{\mathcal{C}}_n(\gamma)$ is non-empty for all $n \leq n(\gamma)$. In order to do that, we first prove that there exists a unique $x^*(\gamma) \in (1,\infty)$ such that $\underline{c}'(x^*(\gamma)) = \bar{c}'(x^*(\gamma))$ and $\frac{\partial}{\partial\gamma}x^*(\gamma) < 0$. Once we have shown this, we can show that this applies to integer values as well.

One can show by induction that $\underline{c}(x) < \bar{c}(x)$ holds for all $x \in \mathbb{N}\backslash\{1\}$, and since the functions are concave and continuous it holds for all real numbers $x > 1$. Moreover, we have that $\underline{c}(1) = \bar{c}(1) = \gamma$ and $\lim_{x\to\infty}\underline{c}(x) = \lim_{x\to\infty}\bar{c}(x) = 1$. The derivatives of the upper and lower bounds are $\bar{c}'(x) = -(1-\gamma)^x\log(1-\gamma)$ and $\underline{c}'(x) = \frac{\gamma(1-\gamma)}{(1+\gamma(x-1))^2}$. These are equal if

$$(1-\gamma)^{x-1} = \frac{-\gamma}{\log(1-\gamma)(1+\gamma(x-1))^2}. \tag{2.32}$$

From here we can see that the left-hand side equals 1 when $x = 1$ and the right-hand side is less unity since $-\log(1-\gamma) = \gamma + \frac{\gamma^2}{2} + \cdots > \gamma$ for all $\gamma \in (0,1)$. Both sides are strictly decreasing functions of x and they both converge to zero as x goes to infinity. However, since the left-hand side decreases exponentially and the right-hand slower, there exists a unique $x^*(\gamma) \in (0, \infty)$ such that $\underline{c}'(x^*(\gamma)) = \bar{c}'(x^*(\gamma))$. In other words, $(1-\gamma)^{x-1} \geq \frac{-\gamma}{\log(1-\gamma)(1+\gamma(x-1))^2}$ for all $x \leq x^*(\gamma)$ and $(1-\gamma)^{x-1} < \frac{-\gamma}{\log(1-\gamma)(1+\gamma(x-1))^2}$ for all $x > x^*(\gamma)$. This is depicted in Figure 2.5.

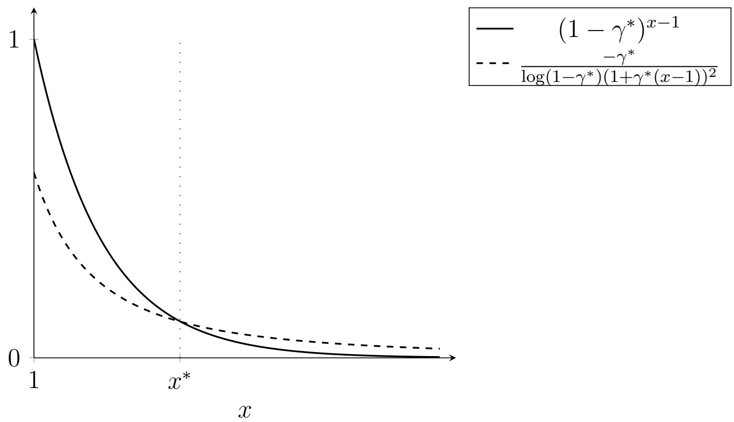

Figure 2.5: Determination of x^*.

Next we show that $\frac{\partial}{\partial\gamma}x^*(\gamma) < 0$. However, it turns out that the proof is not straightforward and we must do it inversely. We show that $\gamma^*(x)$ which solves (2.32) is strictly decreasing in x. Then its inverse $x^*(\gamma)$ is decreasing in γ.

Let us denote $h(x,\gamma) = (1-\gamma)^{x-1}(1+\gamma(x-1))^2$ and $g(\gamma) = \frac{\gamma}{-\log(1-\gamma)}$. Clearly, $h(x,0) = 1$ and $h(x,1) = 0$, whereas $\lim_{\gamma\to 0} g(\gamma) = 1$ and $\lim_{\gamma\to 1} g(\gamma) = 0$. One can show that g is strictly decreasing in γ while

$$\frac{\partial}{\partial\gamma}h(x,\gamma) = (x-1)(1-\gamma)^{x-2}(1+\gamma(x-1))(1-\gamma(x+1)), \tag{2.33}$$

which is non-negative for all $\gamma \leq \frac{1}{x+1}$ and negative for all $\gamma < \frac{1}{x+1}$. Moreover, when γ goes to 0, the derivative in (2.33) approaches $x - 1 \geq 0$. Thus, $h(x,\gamma)$ is a single-peaked function of γ with a global maximum at $\frac{1}{x+1}$. We know that $g(\gamma)$ and $h(x,\gamma)$ intersect once at some $\gamma^* \in (0,1)$, and hence the intersection must be on the decreasing part of $h(x,\gamma)$. This is depicted in Figure 2.6. Next we show that increasing x shifts $h(x,\gamma)$ to the left and, consequently, γ under which $h(x,\gamma) = g(\gamma)$ decreases.

Let us fix arbitrary γ^* and x^* such that equation (2.32) is satisfied or $h(x^*,\gamma^*) = g(\gamma^*)$. By the uniqueness of x^* we know that for all $x > x^*$ the expression in (2.32) holds as inequality:

$$(1-\gamma^*)^{x-1} < \frac{-\gamma^*}{\log(1-\gamma^*)(1+\gamma^*(x-1))^2}. \tag{2.34}$$

This is equivalent to

$$(1-\gamma^*)^{x-1}(1+\gamma^*(x-1))^2 < \frac{\gamma^*}{-\log(1-\gamma^*)}. \tag{2.35}$$

On the left-hand side we have now $h(x,\gamma^*)$ and on the right-hand side $g(\gamma^*)$. However, since $g(\gamma^*) = h(x^*,\gamma^*)$ we have that $h(x,\gamma^*) < h(x^*,\gamma^*)$ for all $x > x^*$. This implies that if $h(x,\gamma) = g(\gamma)$ and $x > x^*$ then $\gamma < \gamma^*$. This is depicted in Figure 2.6.

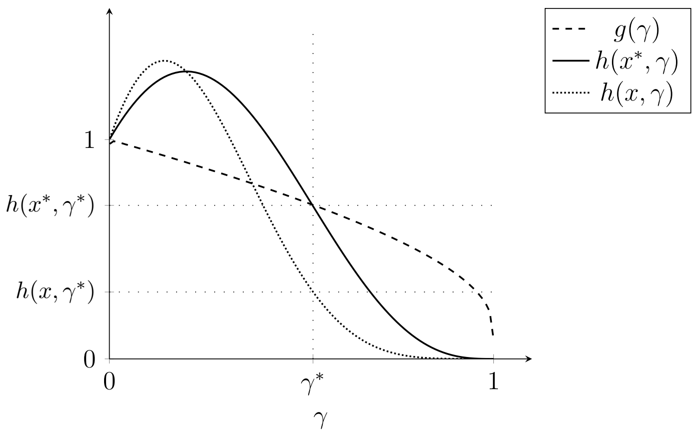

Figure 2.6: Determination of γ^*.

We have thus shown that γ^* which solves (2.32) exists and is unique for all $x > 1$. It is also strictly decreasing in x and therefore its inverse $x^*(\gamma)$ is strictly decreasing in γ for all $\gamma \in (0,1)$.

Finally, let $n(\gamma) = \lceil x^*(\gamma) \rceil - 1$ (where $\lceil\cdot\rceil$ is the ceiling function). By the properties of $x^*(\gamma)$ it directly implies that $\Delta\bar{c}(k) > \Delta\underline{c}(k)$ for all $k \in \{2,3,\ldots,n(\gamma)\}$ and $\Delta\bar{c}(k) \leq \Delta\underline{c}(k)$ for all $k > n(\gamma)$. We have thus shown that $n(\gamma)$ gives us the highest possible configuration under costs $c \in \mathcal{C}(\gamma)$ such that $\mathcal{C}_n(\gamma) \neq \emptyset$ for all $n \leq n(\gamma)$, and $n(\gamma)$ is decreasing in γ. $\square$

Example 2. *(2-configuration) From the example of a* 1*-configuration we get that* $p_0 = c(0)$ *and thus the partition in a* 2*-configuration is* $\mathcal{P}_2 = \{0, c(0), p_1, 1\}$*. So we are left with solving* p_1*.*

Consider a seller who sets a price at p_1. She must make zero profits, and she must be indifferent between sending 1 and 2 ads. More precisely, it means that the following two conditions must hold

$$\begin{cases} e^{-\lambda_2(p_1)}p_1 - c(1) & = 0 \\ e^{-\lambda_2(p_1)}p_1 - c(1) & = \left(1 - \left(1 - e^{-\lambda_2(p_1)}\right)^2\right) p_1 - c(2). \end{cases}$$

From this set of equations we solve that $p_1 = \frac{c(1)}{1-\frac{\Delta c(2)}{c(1)}}$. Since we must have $p_1 \in (p_0, 1)$, it requires that $\Delta c(2) < c(1)(1 - c(1))$ which necessitates the strict concavity of advertising costs.

The mixed strategies are solved by using Equation (2.5). For $p \in [p_0, p_1]$ we have

$$F_2^{(1)}(p) = \frac{\log p - \log c(1)}{\theta_2},$$

and for $p \in [p_1, 1]$

$$F_2^{(2)}(p) = \frac{1}{2\theta_2}\left[\lambda_2(p_1) + \lambda_2(p)\right],$$

where $\lambda_2(p_1) = \log \frac{p_1}{c(1)}$ and

$$\lambda_2(p) = -\log\left(1 - \sqrt[2]{1 - \frac{c(2)}{p}}\right).$$

The number of sellers in the market is given by Equation (2.6):

$$\theta_2 = \frac{1}{2}\log\left[\frac{c(1)}{(2c(1) - c(2))\left(1 - \sqrt{1 - c(2)}\right)}\right].$$

We have thus solved the unique 2-configuration $(\mathcal{P}_2, F_2, \theta_2)$, where $\mathcal{P}_2 = \{0, c(1), \frac{c(1)}{1-\frac{\Delta c(2)}{c(1)}}, 1\}$,

$$F_2(p) = \begin{cases} \frac{1}{\theta_2}\left[\log p - \log c(1)\right] & p \in [p_0, p_1) \\ \frac{1}{2\theta_2}\log\left[\frac{c(1)}{(2c(1)-c(2))\left(1-\sqrt{1-\frac{c(2)}{p}}\right)}\right] & p \in [p_1, 1] \end{cases}$$

and

$$\theta_2 = \frac{1}{2}\log\left[\frac{c(1)}{(2c(1) - c(2))\left(1 - \sqrt{1 - c(2)}\right)}\right].$$

With $c(1) = \frac{1}{2}$ and $c(2) = \frac{5}{8}$ we have $\theta_2 \approx 0.617$, which is less than the number of sellers in the 1-configuration $\theta_1 = -\log(2) \approx 0.69$ with the same $c(1) = \frac{1}{2}$. The partition is $\mathcal{P}_2 = \{0, \frac{1}{2}, \frac{2}{3}, 1\}$. This and the price distribution are given in Figures 2.7 and 2.8.

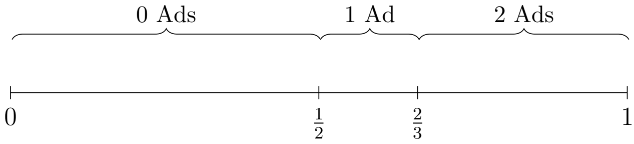

Figure 2.7: The partition of the unit interval in the 2-configuration with $c(1) = \frac{1}{2}$ and $c(2) = \frac{5}{8}$.

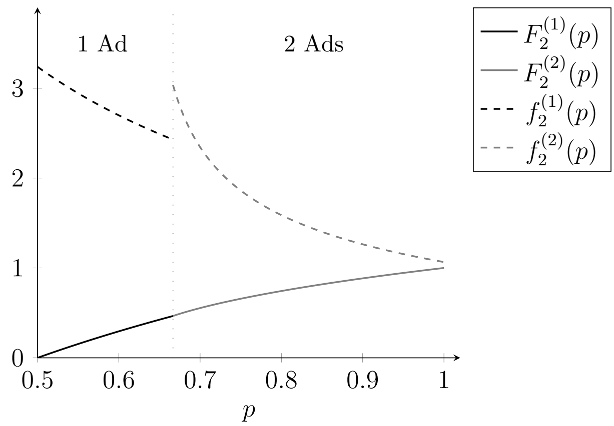

Figure 2.8: The price distribution of the 2-configuration with $c(1) = \frac{1}{2}$ and $c(2) = \frac{5}{8}$.

Example 3. *(3-configuration) From the 2-configuration we know that $p_0 = c(1)$ and $p_1 = \frac{c(1)}{1 - \frac{\Delta c(2)}{c(1)}}$. Combining indifference condition I_2 and zero profit condition (ZP_2) we get*

$$\left(1 - \frac{c(3)}{p_2}\right)^2 = \left(1 - \frac{c(2)}{p_2}\right)^3.$$

There is a unique solution to this cubic equation which satisfies $p_2 > c(3)$ and the advertising costs remain concave:

$$p_2 = \frac{c(3)^2 - 3c(2)^2 + (c(3) - c(2))^{\frac{3}{2}} \sqrt{c(3) + 3c(2)}}{2(2c(3) - 3c(2))}.$$

The mixed strategies are derived by the similar steps as in the 2-configuration:

$$F_3(p) = \begin{cases} \frac{1}{\theta_3}\lambda_3(p), & p \in [p_0, p_1) \\ \frac{1}{2\theta_3}\left(\lambda_3(p_1) + \lambda_3(p)\right), & p \in [p_1, p_2) \\ \frac{1}{\theta_3}\left(\frac{1}{2}\lambda_3(p_1) + \frac{1}{6}\lambda_3(p_2) + \frac{1}{3}\lambda_3(p)\right), & p \in [p_2, 1] \end{cases}$$

where $\lambda_3(\cdot)$ is given in (2.4). θ_3 is given by (2.6):

$$\theta_3 = \frac{1}{2}\lambda_3(p_1) + \frac{1}{6}\lambda_3(p_2) + \frac{1}{3}\lambda_3(1)$$

With $c(1) = \frac{1}{2}, c(2) = \frac{5}{8}$, and $c(3) = \frac{3}{4}$ we have $\theta_3 \approx 0.611$ which is less than the number of sellers in the 2-configuration with the same advertising costs ($\theta_2 \approx 0.617$). The unit partition is $\mathcal{P}_3 = \{0, \frac{1}{2}, \frac{2}{3}, \frac{39+\sqrt{21}}{48}, 1\}$. This and the price distribution are given in Figures 2.9 and 2.10.

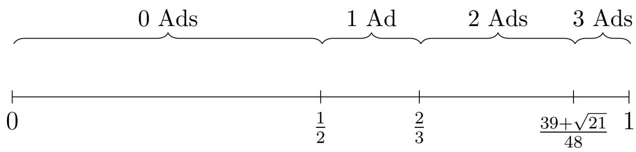

Figure 2.9: The partition of the unit interval in the 3-configuration with $c(1) = \frac{1}{2}, c(2) = \frac{5}{8}$, and $c(3) = \frac{3}{4}$.

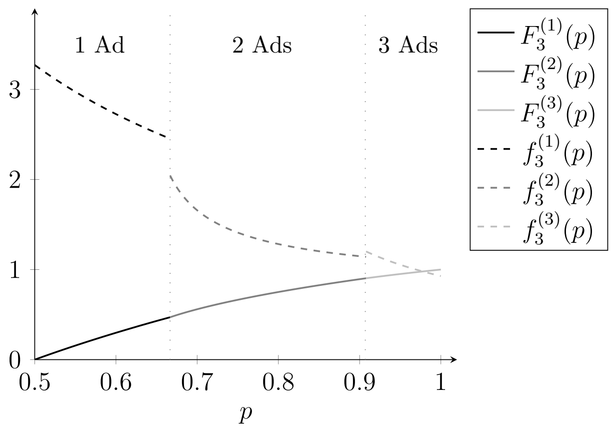

Figure 2.10: The price distribution of the 3-configuration with $p_2 = \frac{39+\sqrt{21}}{48}$ and $c(1) = \frac{1}{2}$, $c(2) = \frac{5}{8}$, and $c(3) = \frac{6}{8}$.

Chapter 3

Optimal Regulation with Costly Verification

Abstract

We consider a principal-agent model in which the principal can monitor and punish the agent with a fine if the agent is caught being untruthful. To reduce the probability of being verified, the agent can engage in costly avoidance. We design the optimal regulatory policies with and without avoidance. The optimal mechanism with enforcement allocates the object more often than the optimal mechanism without enforcement. Moreover, enforcement increases the expected transfers to the principal. Avoidance has two implications to the optimal regulatory mechanism: (i) the expected optimal transfers to the principal decrease and (ii) the principal allocates the object to a smaller share of types. If the latter effect dominates the former, it is possible that the agent's capability to engage in avoidance is disadvantageous not only for the principal, but also for the agent *ex ante*.[1]

Keywords: Mechanism Design, Verification, Enforcement, Monitoring, Avoidance, Fines.

JEL: D82, D86, L51.

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3.1 Introduction

Imagine a regulator who can decide whether to allocate the right to conduct a business to a firm who has private information about its profitability, emissions, or some other verifiable and payoff-relevant parameter. The optimal allocation and regulation in different kind of environments are pioneered by Mussa and Rosen (1978), Myerson (1981), Baron and Myerson (1982), and Lewis and Sappington (1988a and 1988b), with the well-known results. However, these results are founded on the assumption that the regulator does not have (law) enforcement in her tool box. We diverge from this approach. We assume that the regulator has the power to punish the firm with a fine if the firm is caught choosing a contract that was not designed to her (e.g. tax evasion). In order to verify the firm's compliance, the regulator has to invest in costly monitoring. That is to say, in addition to a physical allocation and transfers, the regulator has also the tools of enforcement (monitoring and punishments) at her disposal to achieve desirable outcomes. How does the principal's enforcement affect the firm's information rent? What is the optimal regulatory policy?

Consider next the firm's perspective. We have deviated from the classical allocation narrative by introducing enforcement. Now we must ask: how does enforcement change the firm's behavior? Following Malik (1990) we suppose that a firm can weaken the regulator's monitoring efforts by covering up its non-compliance by engaging in costly 'avoidance' (e.g. by falsification of accounts, corruption, or bribing). Now the firm has not only private information, but can also make a private action, both unobservable to the regulator. If the regulator confronts a firm who can decrease the probability of being caught from non-compliance, how does this affect the optimal regulation mechanism?

We approach these questions by considering a principal-agent model with costly verification. We focus on direct mechanisms, where the agent is asked to report her private information (type) to the principal. We show that it is without loss of optimality to focus on truthful equilibria among all direct mechanisms (Proposition 6).[2]

The principal (regulator) has an exogenous verification technology for the agent's (firm's) private information. The principal can improve the accuracy of the verification by investing in costly monitoring, whereas the agent can decrease the verification probability by engaging in costly avoidance. This battle between avoidance and monitoring makes

[2] In the taxation literature a truthful equilibrium is often referred as horizontal equity (see, e.g., Stiglitz (1982) and Mookherjee and Png (1989)). It means that agents with identical expected incomes and identical reports face the same ex-post taxation scheme. Two popular surveys on optimal taxation and tax evasion are by Andreoni et al. (1998) and Slemrod and Yitzhaki (2002). To the best of our knowledge, our model has the closest connection to the tax evasion models by Border and Sobel (1987) and Mookherjee and Png (1989). See Mookherjee and Png (1990) how the Revelation Principle (for indirect mechanisms) applies in this context.

the verification imperfect.

Once the agent is successfully verified, the principal learns the agent's private information and can fine or rebate the agent. The punishments are determined by the agent's true type and the reported type such that the punishments are zero with the truthful report.[3] If the verification is unsuccessful, the principal learns that the agent's type was the reported one which also leads to zero punishments. The punishment function is bounded and exogenously given to the principal. In this context avoidance can be interpreted as bribing the person who is conducting the monitoring.[4]

The equilibrium payoffs and transfers are derived by using the Envelope Theorem (Propositions 7 and 8). Theorems 1 and 2 give the optimal regulatory policies without and with avoidance, respectively. We find that the principal allocates the object to a larger share of agents and the expected optimal transfers are greater than in the optimal *standard mechanism* in which the principal can design only an allocation consisting of a physical allocation and transfers (e.g. in Mussa and Rosen (1978) and Myerson (1981)). While a take-it-or-leave-it offer is the optimal standard mechanism, *non-linear* pricing is the optimal mechanism with enforcement. The rationale for this result is that the principal is able to extract a proportion of the agent's information rent by monitoring and fines. The agent's ability to engage in avoidance makes this proportion smaller. Avoidance has no direct effects on the equilibrium transfers since with a truthful report the fines are zero and hence the agent has no incentive to invest in costly avoidance. However, avoidance makes the incentive compatibility constraint more rigid (engaging in avoidance may be profitable with off-equilibrium reports) and, consequently, it is optimal for the principal to monitor a smaller proportion of reports than without avoidance. This has two implications: (i) the expected optimal transfers to the principal decrease and (ii) the principal allocates the object to a smaller share of types. So, although avoidance results in greater information rent for the agent, it also *hurts* the agent by making the optimal mechanism to allocate to a smaller share of types. If the latter effect dominates the former, then it is possible that the agent's capability to engage in avoidance is disadvantageous not only for the principal, but also for the agent *ex ante* (Proposition 9).

[3]For instance, fines may be positively dependent on the offender's revenue or income.

[4]This setup is closely related to Mylovanov and Zapechelnyuk (2017) and Li (2020) who also consider limited punishments. Using limited punishments has several reasons. For instance, in the real life, enforcement mechanisms are not perfectly implemented and therefore there exists a risk that compliant agents are erroneously sentenced. Consequently, societies do not impose maximal punishments in order to prevent severe consequences of false positive errors. Moreover, even if the enforcement mechanisms were perfect, societies do not want to discipline amoral agents too harshly. And lastly, if the regulator was able to design the whole punishment function, the enforcement would be trivial: infinite punishments for a non-compliant agent and zero punishments for a compliant agent. This would guarantee that any outcome is incentive compatible.

It turns out that if there is no avoidance and monitoring is costless, then with sufficiently large (bounded) fines, the principal allocates for all types and gets the whole surplus even if the verification is noisy, i.e. the verification probability is less than unity (Corollary 2). However, if monitoring is costly, enforcement weak, or the agent can engage in avoidance, the full information rent extraction is too costly or impossible for the principal.

3.1.1 Related Literature

Our paper contributes to two strands of literature: (i) enforcement and regulation, and (ii) mechanism design with verification. The confluence of these two literature is rewarding. On the one hand, mechanism design offers a useful approach to lay out enforcement tools and regulatory policies, and on the other hand, the enforcement environment introduces new tools, monitoring and fines, for the principal when designing contracts. Since both branches of literature are vast, we comment only a few closest papers to our research leaving out many papers that would have deserved to be mentioned.

The economic literature on enforcement originates from Becker's (1968) seminal article. One can interpret Becker's analysis loosely such that it is optimal for the regulator to increase fines as high as possible and decrease monitoring as low as possible: monitoring has a cost to the society, whereas imposing a fine is costless. Malik (1990) is the first to argue that the optimal fine *might* not be as high as possible when risk-neutral agents can engage in avoidance activities. However, our *comparative statics* on fines do not coincide with Malik's. In our model, the principal can increase the information rent extraction by investing more in monitoring. However, the higher the exogenously given marginal fine, the less monitoring the principal needs for achieving incentive compatibility. Consequently, increasing the marginal fine parameter eventually drives the optimal monitoring closer and closer to zero. Our model thus provides unsurprising results on fines, and hence the focus of our analysis of enforcement is on monitoring and avoidance. For further readings on avoidance and punishments see, for instance, Langlais (2008) and Tabbach (2010).[5]

One of the applications of our model is a monopoly regulation setup. The pioneering papers on this branch of economic theory literature are Baron and Myerson (1982), and Lewis and Sappington (1988a and 1988b). Baron and Myerson assume that the pro-

[5]Siegel and Strulovici (2018) study judicial mechanism design. Perez-Richet and Skreta (2018) study optimal test design in a sender-receiver game. The sender and the receiver can both manipulate the accuracy of a signal, which can be interpreted as a duel of monitoring and avoidance. Finkle and Shin (2020) consider a principal-agent model in which the principal's monitoring can be obstructive to the agent.

duction costs of the monopoly are private information of the monopolist, while Lewis and Sappington (1988a) assume that the demand is private information. Lewis and Sappington (1988b) extend private information on both parameters, production costs and demand.

There are roughly three branches of literature on verification: (i) costly, (ii) partial, and (iii) probabilistic verification.

Costly verification was pioneered by Townsend (1979), Diamond (1984), and Gale and Hellwig (1985). In their models, the verification technology is perfect and it reveals the hidden information once implemented. Border and Sobel (1987) and Mookherjee and Png (1989) study costly verification with noisy verification technology. Costly verification without transfers are studied by Ben-Porath et al. (2014), Mylovanov and Zapechelnyuk (2017), Erlanson and Kleiner (2019), Halac and Yared (2020), Li (2020), and Kattwinkel and Knoepfle (2022).[6]

In partial verification models the principal can restrict the message space of each type and so *partially* verify the agents' private information. This topic was first explored by Green and Laffont (1986), who show that the Revelation Principle does not hold in this setup.[7] Lipman and Seppi (1995) and Bull and Watson (2004) study (partial) verification as hard evidence. Partial verification precludes the use of the envelope theorem in designing equilibrium transfers because the verification probability jumps discontinuously from 0 to 1 at the truthful report.

In probabilistic verification models the authentication probability is conditional on the agent's report and type. Whenever the agent reports truthfully, her true type is revealed with probability 1, otherwise the authentication rate can be anything between 0 and 1. This is not either amenable to the first-order approach *if* there is a non-differentiability at the truthful report. Ball and Kattwinkel (2019) solve this problem and are the first to use the first-order approach in mechanism design with verification. Moreover, by separating communication from testing, they recover a version of the Revelation Principle.[8] Ball and Kattwinkel set the fine such that it normalizes the utility of the agent to zero when getting caught in a misreport (maximal punishments). In contrast to Ball and Kattwinkel (2019), we use an exogenous and limited punishment function. Our approach has two advantages.

[6] Holmström (1979) considers a principal-agent relationship subject to moral hazard in which the principal can acquire imperfect information about the agent's actions by monitoring. Halac and Yared (2019) study a fiscal policy model with limited enforcement.

[7] About implementation without the revelation principle see Nisan and Ronen (2001), Singh and Wittman (2001), Fotakis and Zampetakis (2015), Auletta et al. (2011), Yu (2011), Rochet (1987), and Vohra (2011).

[8] In computer science, Caragiannis et al. (2012) and Ferraioli and Ventre (2018) consider direct probabilistic verification mechanisms. Dziuda and Salas (2018) and Balbuzanov (2019) study binary probabilistic verification models (authentication rate is either 1 or $p \in (0, 1)$) without commitment.

First, we can make simple assumptions in order to utilize the Envelope Theorem. Second, the verification probability can be modeled as a function of monitoring and avoidance; an exogenous punishment function allows *mechanisms* to be independent of the agent's true type, which is not the case in probabilistic verification models. These assumptions in our setting provide an accessible environment which can be used to study the effects of avoidance on the optimal enforcement policies.

3.1.2 Roadmap

This paper is organized as follows. In Section 3.2 we lay out the preliminaries for the analysis. In Section 3.3 we study the requirements for incentive and avoidance compatibility and derive the optimal mechanisms without and with avoidance, respectively. In Section 3.4 we discuss the findings. All the results given in the analysis are proved in Appendix 3.4 to improve readability.

3.2 Preliminaries

Consider a standard principal-agent model where the principal is a regulator and the agent is a firm. The agent has private information about her profitability or the valuation of getting the right to do business $\theta \in \Theta := \left[\underline{\theta}, \bar{\theta}\right] \subset \mathbb{R}_+$ for some $\underline{\theta} < \bar{\theta}$. Let the principal's beliefs about the agent's type θ be given by the distribution function F on Θ with the density function f with full support on Θ. We assume that F is regular, i.e. $\frac{1-F(\theta)}{f(\theta)}$ is decreasing in θ.

The principal designs a direct mechanism $\Gamma = (\Theta, (x, m))$ which consists of a message space Θ, an allocation rule $x = (r, t) : \Theta \to [0,1] \times \mathbb{R}$, where $r : \Theta \to [0,1]$ is the probability that the principal permits the agent to do business, $t : \Theta \to \mathbb{R}$ is the transfer from the agent to the principal (a tax or a price of the right), and monitoring $m : \Theta \to \mathcal{M} := [0,1]$. Next we introduce how these mechanisms determine the outcomes of the game.

Let the agent's type be $\theta \in \Theta$ and let $\theta' \in \Theta$ be an arbitrary report in a mechanism $\Gamma = (\Theta, (x, m))$. The payoff of the agent from reporting θ' and getting an allocation $x(\theta') = (r(\theta'), t(\theta'))$ is given by a function $u : \mathcal{X} \times \Theta \to \mathbb{R}$, where $\mathcal{X} := [0,1] \times \mathbb{R}_+$. For simplicity, we abstract ourselves from the regulation of the price or quantity of output which is allowed in Baron and Myerson (1982) and Lewis and Sappington (1988a and 1988b) and assume that $u(x(\theta'), \theta) = v(r(\theta'), \theta) - t(\theta') = \theta \cdot r(\theta') - t(\theta')$.

We assume that the agent's private information θ is verifiable. The verification probability is given by a function $p : \mathcal{M} \times \mathcal{A} \to \mathbb{R}$ such that $p(m, a) = m(1-a) \in [0,1]$ for all (m, a) on $\mathcal{M} \times \mathcal{A}$, where m is the monitoring designed by the principal and $a \in \mathcal{A} := [0,1]$

is a private action taken by the firm. We call this action *avoidance* (following Malik (1990)). To be more precise with the terminology, by verification we refer to the probability p that the principal learns the agent's true type $\theta \in \Theta$ and to the probability $1-p$ that the principal learns that the agent's type is the reported one, $\theta' \in \Theta$. This simple verification model is similar to that used in the costly verification literature, except now there is avoidance as a counter-force.[9]

We use linear monitoring costs which are given by a function $\kappa : \mathcal{M} \to \mathbb{R}_+$ such that $\kappa(m') = K \cdot m$ for $K > 0$ and m on $\mathcal{M}$. The agent's avoidance costs are also given by a linear function $c : \mathcal{A} \to \mathbb{R}_+$ such that $c(a) = C \cdot a$ for some $C > 0$.

If the agent is verified, the regulator can punish or reward the agent with a fine or rebate. The punishments are given by a function $\Phi : \Theta \times \Theta \to \mathbb{R}$ such that the amount of fine, $\Phi(\theta', \theta)$, is given by the agent's type $\theta \in \Theta$ and report $\theta' \in \Theta$. To keep the model tractable, we use a simple linear punishment function $\Phi(\theta', \theta) = \varphi(\theta - \theta')$ for some constant $\varphi > 0$. This is a standard punishment function in tax evasion literature where $\theta - \theta'$ is interpreted as the difference between a taxpayer's income and report, i.e. the amount of tax evasion (see, e.g., Allingham and Sandmo (1972) and Kleven et al. (2011)). With a report $\theta' > \theta$ it can be considered as a tax refund. We assume that the agent has linear preferences in fines and avoidance costs.

The game proceeds as follows. First, the type θ is drawn for the agent from distribution F. Second, the principal posts and commits to a mechanism Γ. Third, the agent decides whether to participate or not the mechanism. If the agent does not participate, the game ends, whereas if she participates, the game proceeds and the agent sends a message $\theta' \in \Theta$. After receiving the message, the principal makes decision $x(\theta')$. Then the agent chooses avoidance a and the verification is successfully executed with probability $p(m(\theta'), a)$. If the agent is verified, the true type θ is revealed and punishment $\Phi(\theta', \theta)$ is imposed. If the principal's verification fails, fines $\Phi(\theta', \theta') = 0$ are imposed. After that the game ends. The timeline is illustrated in Figure 3.1.[10]

[9]Avoidance can be interpreted, for instance, as falsification of accounts or corruption in monitoring (bribing). If $\mathcal{A}$ was $\{0, 1\}$, then avoidance could be thought as the agent's action to evade the verification mechanism (go into tax exile). In this case avoidance costs could be interpreted as transition or removal expenses.

[10]In many applications it is natural to assume that verification follows the allocation decision, which could be due to monitoring being a slow process. For instance, if the messages are firms' tax reports, auditing is usually enforced after the payment of taxes and it necessitates going through the firms' books.

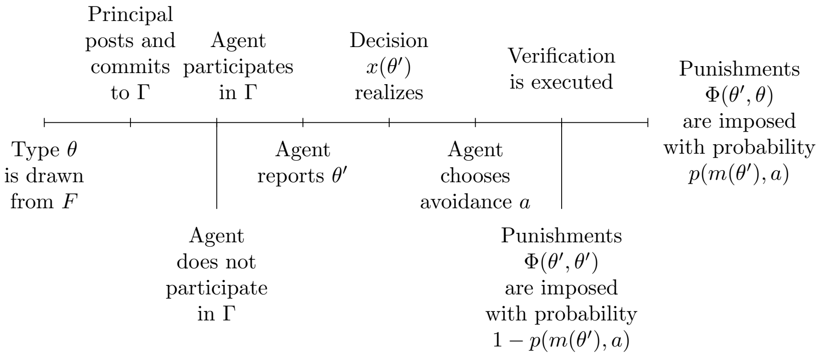


Figure 3.1: Timeline of the game.

Since the principal is fully committed to the mechanism and cannot observe avoidance, the game is static and equivalent to one in which the agent chooses her report and avoidance simultaneously. Hence, given direct mechanism Γ, an agent with type $\theta \in \Theta$ chooses $(\theta', a) \in \Theta \times \mathcal{A}$ to maximize her expected utility[11]

$$\begin{aligned}
&U_A(x(\theta'), m(\theta'), a, \theta) \\
&\quad = p\left(m(\theta'), a\right)\left[u(x(\theta'), \theta) - \Phi(\theta', \theta)\right] \\
&\qquad + \left(1 - p\left(m(\theta'), a\right)\right)\left[u(x(\theta'), \theta) - \underbrace{\Phi(\theta', \theta')}_{=0}\right] - c(a) \qquad (3.1) \\
&\quad = \theta r(\theta') - t(\theta') - m\left(\theta'\right)(1-a)\,\varphi(\theta - \theta') - C \cdot a. \qquad (3.2)
\end{aligned}$$

The principal's ex-post payoff (before verification) is

$$\begin{aligned}
&U_P(x(\theta'), m(\theta,), a, \theta) \\
&\quad = t(\theta') + p(m(\theta'), a)\Phi(\theta', \theta) + \left(1 - p(m(\theta'), a)\right)\underbrace{\Phi(\theta', \theta')}_{=0} \\
&\qquad - \kappa\left(m(\theta')\right) + \alpha U_A(x(\theta'), m(\theta'), a, \theta) \qquad (3.3) \\
&\quad = t(\theta') + m(\theta')(1-a)\varphi(\theta - \theta') - Km(\theta') \\
&\qquad + \alpha U_A(x(\theta'), m(\theta'), a, \theta) \qquad (3.4)
\end{aligned}$$

[11] Ball and Kattwinkel (2019) use a model in which $u(x(\theta'), \theta) - \Phi(\theta', \theta) = 0$ whenever the verification is successful. This requires that the punishments are given by an endogenous function of a report and a type. In our model the functional form of the punishments is exogenous. Moreover, instead of taking the verification probability function as given, Ball and Kattwinkel (2019) design an authentication rate $\alpha : \Theta \times \Theta \to [0, 1]$ and allocation $x : \Theta \to \mathcal{X}$ such that $\alpha(\theta|\theta) = 1$ for all $\theta \in \Theta$. The agent's payoff is given by $\alpha(\theta'|\theta)u(x(\theta'), \theta)$ which is $u(x(\theta), \theta)$ at the truthful report.

for some weight parameter $\alpha \in [0,1]$.[12] That is, the principal designs a direct mechanism Γ to maximize weighted sum of the expected transfers and fines net of monitoring costs and the agent's payoff, i.e. $\mathbb{E}_\theta\left[U_P(x(\theta'), m(\theta,), a, \theta)\right]$.

We say that a direct mechanism is *incentive and avoidance compatible* if the agent finds it optimal to report truthfully not to engage in avoidance.

Definition 4. *A direct mechanism* $\Gamma = (\Theta, (x, m))$ *is incentive and avoidance compatible if and only if for all* $(\theta, \theta') \in \Theta^2$ *and all* $a \in \mathcal{A}$

$$U_A(x(\theta), m(\theta), 0, \theta) \geq U_A(x(\theta'), m(\theta'), a, \theta).$$

We assume that the agent's outside option from not participating the mechanism is zero. Consequently, a mechanism is individually rational if and only if the agent receives non-negative payoff in equilibrium when participating into an incentive and avoidance compatible direct mechanism.

Definition 5. *An incentive and avoidance compatible direct mechanism* Γ *is individually rational if and only if* $U_A(x(\theta), m(\theta), 0, \theta) \geq 0$.

Lastly, by the following proposition we can restrict our attention to equilibria where the agent truthfully reports her type to the principal and does not engage in avoidance.

Proposition 6. *It is without loss of optimality to focus on incentive and avoidance compatible mechanisms among all possible direct mechanisms.*

This result has the following intuitive interpretation and reasoning. Consider an arbitrary direct mechanism. If there is avoidance in equilibrium, the principal's information rent extraction is weaker than in equilibrium, where avoidance is zero. Hence, the principal prefers the outcomes where there is no avoidance. By the linearities of the players' utilities, the principal can reassign the expected punishments into the equilibrium transfers. Moreover, the punishments are assumed to be additively separable and thus the agent's report does not affect the marginal fines that determines the effectiveness of the principal's information rent extraction. Consequently, the principal's maximum payoff over direct mechanisms is attained by some incentive and avoidance compatible mechanism.

[12] The weight parameter does not play a crucial role in our analysis but takes our model slightly closer to the models of Baron and Myerson (1982) and Lewis and Sappington (1988a and 1988b).

3.3 Optimal Regulatory Policy

In this section we design the regulator's payoff maximizing mechanisms. We first derive the optimal mechanism *without* avoidance and then compare it with the optimal mechanism *with* avoidance.

3.3.1 Optimal Mechanism without Avoidance

Assume that the agent cannot engage in avoidance and the verification probability is simply given by monitoring $m(\theta') \in [0,1]$ for any report $\theta' \in \Theta$. Given direct mechanism Γ, we can write the agent's value function as follows:

$$\begin{aligned} V(\theta) &= \max_{\theta' \in \Theta} \Big(\theta r(\theta') - t(\theta') - m(\theta')\varphi(\theta - \theta') \Big) \\ &= \max_{\theta' \in \Theta} \Big(\theta \mathcal{I}(\theta') - \tau(\theta') \Big), \end{aligned} \tag{3.5}$$

where $\mathcal{I}(\theta') := r(\theta') - m(\theta')\varphi$ and $\tau(\theta') := t(\theta') - m(\theta')\varphi\theta'$. These both, $\mathcal{I}$ and τ, are functions of report θ' and independent of true type θ. In other words, the mechanism designer's problem without avoidance reduces to a standard mechanism design problem where the 'physical allocation' is $\mathcal{I}(\theta')$ and 'transfers' are $\tau(\theta')$. We thus know from the well-known results of Myerson (1981) that the mechanism Γ is incentive compatible if and only if $\mathcal{I}(\cdot)$ is non-decreasing and τ is given by the Envelope Theorem:

$$\tau(\theta) = \theta \mathcal{I}(\theta) - V(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} \mathcal{I}(s) ds \tag{3.6}$$

which can be rewritten as

$$t(\theta) = \theta r(\theta) - V(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} \left(r(s) - m(s)\varphi \right) ds. \tag{3.7}$$

Based on this we get the following result.

Proposition 7. *Assume an environment without avoidance. A direct mechanism* $\Gamma = (\Theta, (x, m))$ *is incentive compatible if and only if*

$$t(\theta) = \theta r(\theta) - V(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} \mathcal{I}(s) ds, \tag{3.8}$$

and $\mathcal{I}(\cdot) := r(\cdot) - m(\cdot)\varphi$ *is non-decreasing.*

The equilibrium transfers given in Proposition 7 can be interpreted as follows. The integral in (3.8) has two components: (i) the agent's information rent $\int_{\underline{\theta}}^{\theta} r(s)ds$ and (ii) the amount that the principal can extract the agent's information rent $\int_{\underline{\theta}}^{\theta} m(s)\varphi ds$. The higher the monitoring or the marginal fines, the greater the principal's rent extraction. We call the difference between these two components, $\mathcal{I}$, the agent's *net information rent.* The higher the net information rent, the more the agent gets payoff in equilibrium.

In comparison with the standard monotonicity results, we observe from Proposition 7 that in an incentive compatible mechanism we do not necessarily have non-decreasing physical allocation r. It is possible that the monitoring guarantees that the net information rent $\mathcal{I}(\cdot)$ is non-decreasing even though the physical allocation is non-monotonic. The following short example illustrates this phenomenon.

Example 4. *(Emission Trading System.) Consider a firm which production causes $\theta \in [0, \bar{\theta}]$ amount of emissions. The pollution level is privately known by the firm. The regulator asks the firm to report its emissions in order to allocate the emissions allowance r at price t. The regulator can verify the firm's report by monitoring and punish or rebate the firm if the report is not accurate. For simplicity, let monitoring be costless and the marginal fine satisfy $\varphi \geq 1$. Suppose that the emissions cause harm for the environment. Let the harm be a function $H : \Theta \times [0,1] \to \mathbb{R}$ such that $H(\theta, r(\theta)) = 0$ if $r(\theta) = 0$. The regulator maximizes $\mathbb{E}_{\theta}[t(\theta) - H(\theta, r(\theta))]$ subject to incentive compatibility and participation constraints.*

Since monitoring is costless, it is optimal for the regulator to set $m(\theta) = r(\theta)\varphi^{-1} \in [0,1]$ in order to maximize the information rent extraction (see Proposition 7). This implies that the net information rent $\mathcal{I}(\theta) = r(\theta) - m(\theta)\varphi = 0$ for all $\theta \in \Theta$. That is, the agent does not receive any information rent and the regulator's problem reduces to designing $r(\cdot)$ such that it maximizes $\mathbb{E}[\theta r(\theta) - V(\underline{\theta}) - H(\theta, r(\theta))]$. This has no requirements for the monotonicity of r. For instance, we can assume that a firm with large emissions causes more harm to the society than it yields tax revenue. In this case $\theta < H(\theta, 1)$ for large θ and the optimal r is decreasing in θ.

Recall that the principal maximizes the expected weighted sum of the agent's payoff and the transfers net of monitoring costs subject to the incentive compatibility and the participation constraints. By the similar arguments as in a standard mechanism design problem, we know that if a mechanism Γ maximizes the principal's transfers net of monitoring costs, then at the optimum we have $V(\underline{\theta}) = 0$. This further implies that in the mechanism Γ the individual rationality is satisfied if and only if $\mathcal{I}(\theta) \geq 0$ for all $\theta \in \Theta$. Thus, there is a natural bound for the information rent extraction: the net information rent must be non-negative and so the principal cannot extract more information rent than there is available (see also Lemma 3 in Section 3.3.2). By Proposition 7 we know that

$\mathcal{I}(\cdot)$ is non-decreasing and hence the principal solves

$$\max_{(x;m)} \mathbb{E}\left[t(\theta) - K \cdot m(\theta) + \alpha V(\theta)\right] \tag{MAX}$$

subject to

$$t(\theta) = \theta r(\theta) - \int_{\underline{\theta}}^{\theta} \mathcal{I}(s)ds \tag{TAX}$$

$$\mathcal{I}(\cdot) \qquad \text{is non-decreasing} \tag{IC}$$

$$\mathcal{I}(\underline{\theta}) \geq 0 \tag{IR}$$

for all $\theta \in \Theta$.

By substituting the equilibrium transfers (TAX) into the principal's objective function in (MAX) and using Fubini's theorem for interchanging the order of integration, the principal's objective function becomes

$$\mathbb{E}\left[\psi_r(\theta)r(\theta) + \psi_m(\theta)m(\theta)\right], \tag{3.9}$$

where

$$\psi_r(\theta) := \theta - (1-\alpha)\frac{1-F(\theta)}{f(\theta)} \qquad \text{and} \qquad \psi_m(\theta) := (1-\alpha)\varphi\frac{1-F(\theta)}{f(\theta)} - K.$$

From (3.9) we observe that the principal's objective function is linear in r and m and consequently the optimum is achieved at some extremes of (r, m). By the agent's individual rationality constraint we know that the net information rent must be non-negative and hence all the feasible extreme values are $\{0, \bar{r}, 1\}$ and $\{0, \bar{m}\}$ for r and m, respectively, where $\bar{r} := \min\{1, \varphi\}$ and $\bar{m} := \min\{1, \varphi^{-1}\}$ (see Figure 3.6 in Appendix 3.4).

By regularity of F we know that $\psi_r(\theta)$ is increasing and $\psi_m(\theta)$ is decreasing in θ. This implies that also $\psi_r(\theta)\bar{r} + \psi_m(\theta)\bar{m}$ is increasing in θ since $1 \geq \varphi\bar{m}$. In other words, the regulator's objective function (3.9) is non-decreasing in θ for all possible combinations of feasible extremes of (r, m) and consequently the optimal mechanism is given by the upper envelope of these combinations (see Figure 3.7 in Appendix 3.4). This reasoning is the core of the proof of our first theorem. The complete proof is given in Appendix 3.4.

Before giving our first theorem, let us introduce the following cutoff types:

$$\theta_r = \inf\left\{\theta \in \Theta : \psi_r(\theta) \geq 0\right\} \tag{3.10}$$

$$\theta_m = \inf\left\{\theta \in \Theta : \psi_r(\theta)\bar{r} + \psi_m(\theta)\bar{m} \geq 0\right\} \tag{3.11}$$

$$\theta^m = \sup \{\theta \in \Theta : \psi_m(\theta) \geq 0\}. \tag{3.12}$$

By the regularity of F, the cutoff types θ_r, θ_m, and θ^m are uniquely determined. The cutoff θ_r is the lowest type for whom the principal would allocate without verification (i.e. θ_r is the standard take-it-or-leave-it offer). As we show in the proof of Theorem 1, the interval $[\theta_m, \theta^m)$ with $\theta_m < \theta^m$ gives us the types that the principal finds it optimal to monitor.

Theorem 1. *Assume an environment without avoidance. If the monitoring is relatively costly ($\theta_m \geq \theta^m$), then the optimal mechanism is a standard take-it-or-leave-it offer θ_r with no monitoring.*

If the monitoring is relatively inexpensive ($\theta_m < \theta^m$), then the optimal mechanism is given by the physical allocation

$$r(\theta) = \begin{cases} 0, & \theta \in [\underline{\theta}, \theta_m) \\ \bar{r}, & \theta \in [\theta_m, \theta_r) \\ 1, & \theta \in [\theta_r, \bar{\theta}], \end{cases} \tag{3.13}$$

the monitoring

$$m(\theta) = \begin{cases} \bar{m}, & \theta \in [\theta_m, \theta^m) \\ 0, & \textit{otherwise}, \end{cases} \tag{3.14}$$

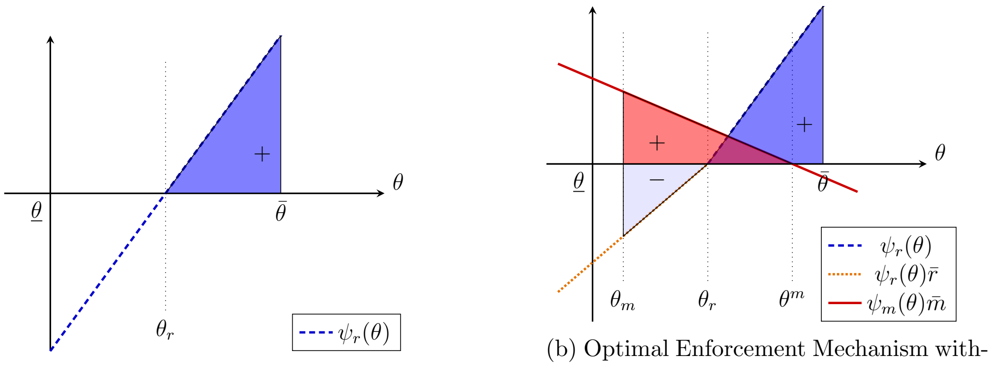
and the transfers

$$t(\theta) = \begin{cases} 0, & \theta \in [\underline{\theta}, \theta_m) \\ \theta\varphi\bar{m}, & \theta \in [\theta_m, \theta_r) \\ \theta_r + \bar{m}\varphi(\theta - \theta_r), & \theta \in [\theta_r, \theta^m) \\ \theta_r + \bar{m}\varphi(\theta^m - \theta_r), & \theta \in [\theta^m, \bar{\theta}]. \end{cases} \tag{3.15}$$

Let us compare this mechanism with a take-it-or-leave-it offer θ_r, i.e. with the optimal mechanism without enforcement (see Mussa and Rosen (1978) or Myerson (1981)). From Theorem 1 we can observe that, whenever the monitoring is relatively cheap ($\theta_m < \theta^m$), then the principal benefits from enforcement tools: (i) the principal allocates the object more often than the optimal standard mechanism: $\theta_m \leq \theta_r$, and (ii) the principal expects to receive a higher transfers net of monitoring costs with enforcement than without enforcement: $\mathbb{E}[t(\theta) - Km(\theta)] \geq \mathbb{E}[\theta_r \mathbf{1}\{\theta \geq \theta_r\}]$, where $\mathbf{1}$ is the indicator function. That is to say, agents with type $\theta \in [\theta_m, \theta_r]$ who did not receive the allocation without enforcement, are allocated the object with enforcement with probability $\bar{r}$. However, even thought agents with type $\theta \in [\theta_m, \theta_r]$ get the object in the optimal enforcement mechanism, they do not receive any surplus from the allocation. This result holds for any marginal fines φ. If the fines were low, $\varphi < 1$, then the regulator boosts the information extraction by decreasing the allocation probability to φ for types $\theta \in [\theta_m, \theta_r]$.[13] And if the marginal fine is large, $\varphi \geq 1$, the regulator's enforcement is so effective that the principal allocates for certainty but extract all the information rent not only from all agents with type $\theta \in [\theta_m, \theta_r)$, but also from all types in $[\theta_r, \theta^m)$.

The principal's expected payoffs with the take-it-or-leave-offer θ_r and with the optimal enforcement mechanism without avoidance are illustrated in Figure 3.2. The dark blue areas in the figures depict the principal's expected payoff in the standard mechanism without monitoring and punishments. The red area illustrates the payoff that the principal can get by extracting extra information rent from the agent by enforcement. The light blue area is the loss that the principal receives from allocating to low types. From these figures we observe that it would be optimal for the principal to allocate the object for all types in $[\theta_r, \overline{\theta}]$ and monitor all types in $[\underline{\theta}, \theta^m]$. However, this mechanism is not incentive compatible or individually rational for the agent because there cannot be monitoring without allocation and the net information rent must be non-decreasing. Hence, if the principal wants to monitor some types in $[\underline{\theta}, \theta_r]$, she faces the loss from allocation of the monitored types in the light blue area. In order to maximize the red area net of the light blue subject to the incentive constraints, the principal finds it optimal to decrease the allocation probability for types $[\theta_m, \theta_r]$.

[13] An alternative interpretation for decreasing r is the following. Suppose the regulator is selling a share r of assets she is possessing, which can be related to for instance an ownership of the business or emission allowances. Then offering a low r can be interpreted as a joint ownership of the business or limited supply of emission allowances. See more discussion from Chapter 4, where the optimal decreased allocation probability is due to the informed principal.



(a) Optimal Standard Mechanism.

(b) Optimal Enforcement Mechanism without Avoidance.

Figure 3.2: The principal's expected revenue.

Due to the enforcement the regulator's optimal pricing is now non-linear for types $\theta \in [\theta_m, \theta^m]$. The regulator can incentivize the agent to be truthful even thought the regulator is asking greater transfers with enforcement than in a standard mechanism. By the linearity of the agent's payoff (risk neutrality), the regulator can shift the punishments into the transfers and hence give all the uncertainty of the verification to the agent. The optimal transfers without avoidance are depicted in Figure 3.3. The jumps in the transfers at θ_m and θ_r are due to the jumps in the allocation probability from 0 to $\bar{r}$ and from $\bar{r}$ to 1, respectively. The linearly increasing parts of transfers are due to the linear punishment function. The slope is given by product of the maximal monitoring and marginal fines, i.e. $\min\{1, \varphi\}$. Since the marginal benefits from monitoring are less than the costs of monitoring for the high types, the principal does not monitor types greater than θ^m and consequently the transfers become flat.

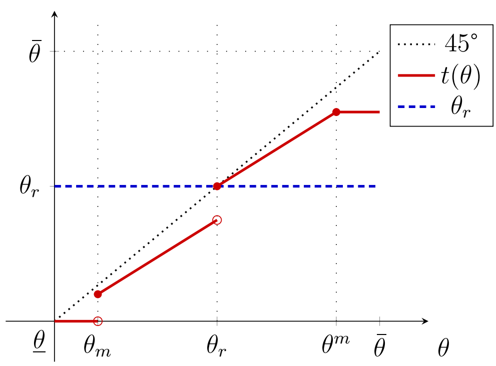


Figure 3.3: Optimal take-it-or-leave-it-offer (blue dashed line) and optimal transfers with enforcement mechanism when $\varphi < 1$ (red solid line).

From Theorem 1 we observe that the more the principal values the agent's utility — that is, the greater α, the more the principal values the allocation and the less the monitoring; θ_r and θ^m are both decreasing in α. If $\alpha = 1$, the principal always allocates the object to the agent for free and does not invest in monitoring. At the other extreme, $\alpha = 0$, the principal wants to extract as much information rent as possible from the agent by monitoring aggressively. One can verify that there exists $\alpha^* \in (0, 1)$ such that $\theta_m \geq \theta^m$ for $\alpha \geq \alpha^*$. That is, for a relatively high α, the principal give up on monitoring and leave the agent more information rent; investing in costly enforcement is no longer profitable if the principal receives high utility from the agent's utility.

If the fines are large and there are no monitoring costs, then the optimal mechanism always allocates the object and extracts the whole information rent from all types. To be more precise, with $K = 0$, the bounds become $\theta_m = \underline{\theta}$ and $\theta^m = \bar{\theta}$, and consequently $r(\theta) = 1$, $m(\theta) = \varphi^{-1}$, and $t(\theta) = \theta$ with $\varphi \geq 1$ for all $\theta \in \Theta$.

Corollary 2. *Assume an environment without avoidance. Then with zero monitoring costs and large marginal fines, $\varphi \geq 1$, the principal extracts the whole surplus from the agent.*

That is, even with *limited* punishments ($\varphi \geq 1$) and *noisy* verification ($m = \varphi^{-1}$), the regulator can leave the agent without information rent. However, if the marginal punishment is small, $\varphi < 1$, the full surplus extraction is not possible.

There is a couple of reasons to assume that the marginal fines are small. First, with $\varphi \in (0, 1)$ it is without loss of generality to assume $m : \Theta \to [0, 1]$ instead of $\hat{m} : \Theta \to [0, \bar{M}]$ for some $\bar{M} < 1$. To see this, we can always write the marginal fine such that $\varphi = \hat{\varphi}\bar{M}$ for some constant $\hat{\varphi}$. This would exactly represent the case in which the monitoring is imperfect and bounded from above by $\bar{M}$. In this case we have $m(\theta')\varphi(\theta - \theta') = m(\theta')\bar{M}\hat{\varphi}(\theta - \theta') = \hat{m}(\theta')\hat{\varphi}(\theta - \theta')$ with $\hat{\varphi} \geq 1$ even though $\varphi = \hat{\varphi}\bar{M} < 1$. Another reason arises if we assume that the monitoring is slow and the fines are realized later than the allocation. Then the punishments are discounted. Letting the discount factor to be $\delta \leq 1$ we can write the marginal fine as $\varphi := \delta\bar{\varphi}$. In this case we can have $\varphi = \delta\hat{\varphi} < 1$ even though $\hat{\varphi} \geq 1$.

If the agent can engage in avoidance the result in Corollary 2 does not hold anymore. Next we show this by deriving the optimal mechanism *with* avoidance.

3.3.2 Optimal Mechanism with Avoidance

Let us now suppose that the agent can engage in costly avoidance. Recall that the verification probability is then given by $p(m, a) = m(1 - a)$ for $(m, a) \in [0, 1]^2$.

Given mechanism Γ, let the agent's value function be

$$V^a(\theta) = \sup_{\theta' \in \Theta, a \in \mathcal{A}} \Big(\theta \cdot r(\theta') - t(\theta') - m(\theta')(1-a)\varphi(\theta - \theta') - C \cdot a\Big). \tag{3.16}$$

For any $(\theta', \theta) \in \Theta^2$, the agent's optimal avoidance is given by

$$a^*(\theta'|\theta) \in \arg\max_{a \in \mathcal{A}} \Big(\theta r(\theta') - t(\theta') - m(\theta')(1-a)\varphi(\theta - \theta') - C \cdot a\Big), \tag{3.17}$$

which has the following solution

$$a^*(\theta'|\theta) = \begin{cases} 1, & m(\theta')\varphi(\theta - \theta') > C \\ 0, & \text{otherwise.} \end{cases} \tag{3.18}$$

In other words, under positive monitoring the agent engages in costly avoidance only with a sufficient under report. For over reports the agent does not engage in costly avoidance since it would decrease the probability of getting the refund.

If monitoring is decreasing in report, then the more the agent under reports her type, the more she invests in avoidance since under reporting leads to increased monitoring and fines. The agent's reaction to enforcement is thus aggressive which makes it harder for the principal to achieve truthful reports.

From (3.18) we get two immediate results. First, if the optimal monitoring is zero, then also the optimal avoidance is zero and we end up in the first part of Theorem 1 ($\theta_m \geq \theta^m$). Second, if the marginal cost of avoidance is large, $C \geq \varphi(\bar{\theta} - \underline{\theta})$, then the agent never engages in avoidance. In this case the mechanism is *avoidance free*, and the results in Theorem 1 and Corollary 2 follow
The following lemma shows that the agent's net information rent is non-decreasing in the agent's type *also* in an incentive and avoidance compatible mechanism.

Lemma 3. *In any incentive and avoidance compatible direct mechanism Γ, the net information rent $\mathcal{I}(\theta|\theta) := r(\theta) - m(\theta)\varphi$ is non-decreasing in θ.*

Based on the agent's optimal choice of avoidance (3.18) we also know that in the truthful equilibrium there is no avoidance. This implies that the equilibrium transfers in an incentive and avoidance compatible mechanism are the same as given in Proposition 7. However, avoidance can be positive with off-equilibrium reports which affects the incentive and avoidance compatibility constraints (recall Definition 4).

Proposition 8. *A direct mechanism $\Gamma = (\Theta, (x, m))$ is incentive and avoidance compat-*

ible if and only if for all $\theta \in \Theta$

$$t(\theta) = \theta r(\theta) - V(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} \mathcal{I}(s|s)ds \tag{3.19}$$

and for all $\theta, \theta' \in \Theta$

$$(\theta - \theta')\left[\mathcal{I}(\theta|\theta) - \mathcal{I}(\theta'|\theta)\right] \geq -c(a^*(\theta'|\theta)), \tag{3.20}$$

or

$$(\theta - \theta')\left[\mathcal{I}(\theta|\theta) - \mathcal{I}(\theta'|\theta')\right] \geq a^*(\theta'|\theta)\left[m(\theta')\varphi(\theta - \theta') - C\right], \tag{3.21}$$

where $\mathcal{I}(\theta'|\theta) := r(\theta') - m(\theta')(1 - a^*(\theta'|\theta))\varphi$ and the optimal avoidance a^* is given by (3.18).

The inequality in (3.21) states that the net information rent must increase sufficiently fast relative to the gains from avoidance. Or, conversely from the inequality in (3.20), the avoidance costs must be greater than the information rent benefits from misreporting. In comparison with an environment without avoidance (Proposition 7), the monotonicity of $\mathcal{I}(\cdot|\cdot)$ is a necessary but not sufficient condition for the incentive and avoidance compatibility.

By Proposition 8 and Lemma 3, the principal's expected revenue is maximized by setting $V(\underline{\theta}) = \underline{\theta} \cdot r(\underline{\theta}) - t(\underline{\theta}) = 0$. If this was not the case, the principal could increase $t(\underline{\theta})$ keeping $r(\underline{\theta})$ unchanged. This implies that the individual rationality constraint becomes $V(\theta) = \int_{\underline{\theta}}^{\theta} \mathcal{I}(s|s) \geq 0$ for all $\theta \in \Theta$. Since $\mathcal{I}(\theta|\theta)$ is non-decreasing in θ, then an incentive and avoidance compatible mechanism Γ is individually rational if and only if $\mathcal{I}(\underline{\theta}|\underline{\theta}) \geq 0$ since the net information rent $\mathcal{I}(\cdot|\cdot)$ is non-decreasing. Consequently, the principal's problem is to solve

$$\max_{(r;m)} \mathbb{E}\left[\psi_r(\theta)r(\theta) + \psi_m(\theta)m(\theta)\right], \tag{MAX}$$

subject to

$$(\theta - \theta')\left[\mathcal{I}(\theta|\theta) - \mathcal{I}(\theta'|\theta)\right] \geq -c(a^*(\theta'|\theta)), \tag{IAC}$$

$$\mathcal{I}(\underline{\theta}|\underline{\theta}) \geq 0 \tag{IR}$$

for all $\theta, \theta' \in \Theta$. That is, the only difference to the maximization problem without avoidance is that (IC) is replaced by (IAC).

Let us start analyzing the optimal mechanism under avoidance by asking when the mechanism given in Theorem 1 is incentive *and* avoidance compatible. We know that in an environment without avoidance the net information rent $\mathcal{I}(\theta|\theta)$ is non-decreasing in θ in the optimal mechanism (see Theorem 1). Therefore, for any over report, (IAC) is satisfied and we need to only check when (IAC) holds for all $\theta' \leq \theta$. To that end, let us go through mechanically all the possible deviations in mechanism Γ given in Theorem 1 and check which types have profitable deviations under avoidance.

First, clearly, if an agent with any type $\theta \in \Theta$ reports $\theta' \in [\underline{\theta}, \theta_m)$ she receives zero surplus. This cannot be profitable for any agent since the mechanism in Theorem 1 is individually rational. We can thus consider under reports to interval $[\theta_m, \bar{\theta}]$ from hereafter.

Consider an agent with a type $\theta \in [\theta^m, \bar{\theta}]$. She receives the maximal information rent with a truthful report: $\mathcal{I}(\theta|\theta) = 1$. Therefore, all under-reports yield her something worse than the truthful report no matter whether the agent engage in avoidance or not. Consequently the mechanism is incentive and avoidance compatible for all agents with type $\theta \in [\theta^m, \bar{\theta}]$.

Let us then consider the second highest type interval $[\theta_r, \theta^m)$. An agent with a type in this interval receives net information rent $1 - \varphi\bar{m}$. If the agent under reports $\theta' \in [\theta_r, \theta^m)$ and *engages* in avoidance, her net information rent is 1. Consequently, (IAC) is violated and the mechanism in Theorem 1 does not satisfy (IAC). Let us then suppose that $C \geq \bar{m}\varphi(\theta^m - \theta_r)$ and hence an agent with type $\theta \in [\theta_r, \theta^m)$ does not find it optimal to engage in avoidance with report $\theta' \in [\theta_r, \theta^m)$. However, if she reports $\theta' \in [\theta_m, \theta_r)$ and invests in avoidance, she receives net information rent of $\bar{r}$. This violates (IAC) unless we have $C \geq (2\bar{r} - 1)(\theta^m - \theta_m)$. That is, if $C \geq \max\{\bar{m}\varphi(\theta^m - \theta_r), (2\bar{r} - 1)(\theta^m - \theta_m)\}$, then the mechanism in Theorem 1 is incentive and avoidance compatible for an agent with type $\theta \in [\theta_r, \theta^m)$.

Lastly, consider an agent with type $\theta \in [\theta_m, \theta_r)$. With a truthful report she receives no information rent and hence (IAC) is violated only if the agent finds it optimal to engage in avoidance with a report $\theta' \in [\theta_m, \theta_r)$. We thus need to have $C \geq \bar{m}\varphi(\theta_r - \theta_m)$ for the mechanism in Theorem 1 to be incentive and avoidance compatible for an agent with type $\theta \in [\theta_m, \theta_r)$.

To sum up our brute force deductions, we get the following corollary.

Corollary 3. *The optimal mechanism given by Theorem 1 is incentive and avoidance compatible, and hence optimal, if the marginal cost of avoidance satisfies:*

$$C \geq \bar{r} \max\{\theta^m - \theta_r, \theta_r - \theta_m, (2 - \bar{r}^{-1})(\theta^m - \theta_m)\}, \tag{3.22}$$

for which the sufficient condition is

$$C \geq \bar{m}\varphi(\theta^m - \theta_m). \tag{3.23}$$

The sufficient condition $C \geq \bar{m}\varphi(\theta^m - \theta_m)$ plays a crucial role in our analysis later on since it states that if the agent does not find it optimal to engage in avoidance with types and reports in the monitoring interval $[\theta_m, \theta^m)$, then the mechanism is incentive and avoidance compatible.

If the avoidance costs are small and the condition is Corollary (3) is not satisfied, then the regulator is forced to give up on some of the monitoring: when C approaches zero, the agent always finds it optimal to engage in avoidance in order to prevent the principal from extracting information rent by monitoring. This drives the level of monitoring or the length of monitoring interval to zero and we end up to the standard take-it-or-leave-it offer without verification (Theorem 1).

For the rest of the paper we focus on the optimal mechanism design with low avoidance and monitoring costs.

Assumption 1. $C < \bar{r}\max\{\theta^m - \theta_r, \theta_r - \theta_m, (2 - \bar{r}^{-1})(\theta^m - \theta_m)\}$ *and* $\theta_m < \theta^m$.

Unfortunately these are the only closed-form results that we can provide without making any other simplifying assumptions; the degree of freedom of choosing functions $(r, m) : \Theta \to [0, 1]^2$ is large when the individual rationality and incentive and avoidance compatibility constraints are binding. For instance, one could choose the level of monitoring so low that it guarantees that the optimal avoidance remains zero for any off-equilibrium report. Or, on the other hand, the regulator could shrink the monitoring interval so small, that the agent would never engage in avoidance within this interval. Together with these two possibilities and all their combinations, the principal could also decrease the allocation probability in order to boost the information rent extraction for small monitoring and marginal punishment levels. That is, the regulator can try to tackle the avoidance by many different ways, which makes the finding of the solution to the problem given in (MAX) relatively demanding.

Without going to the numerical solutions, we take here another path and assume for simplicity that the regulator can only decide whether to allocate or not and whether to monitor or not. In other words, we assume for the rest of this section that $r : \Theta \to \{0, 1\}$ and $m : \Theta \to \{0, \bar{m}\}$ with some constant $\bar{m} \leq \min\{1, \varphi^{-1}\}$ which guarantees that the net information rent remains non-negative under monitoring. In other words, the regulator is prohibited to use stochastic allocation and she can only decide whether or not to monitor the agent with the success probability $\bar{m}$.

Assumption 2. *$(r, m) : \Theta \to \{0, 1\} \times \{0, \bar{m}\}$, where $\bar{m} \le \min\{1, \varphi^{-1}\}$.*

This simplification implies that the regulator can tackle the avoidance only by changing the intervals of types which are monitored and allocated the object. Before going to the optimal mechanism under this assumption, let us introduce a proper benchmark mechanism in an environment without avoidance. This result is a direct implication of Theorem 1.

Corollary 4. *Assume an environment without avoidance or suppose that Assumption 1 does not hold. Under Assumption 2 and relatively inexpensive monitoring ($\theta_m < \theta^m$), the optimal mechanism is given by the physical allocation*

$$r(\theta) = \begin{cases} 0, & \theta \in [\underline{\theta}, \theta_m) \\ 1, & \theta \in [\theta_m, \bar{\theta}], \end{cases} \tag{3.24}$$

by the monitoring

$$m(\theta) = \begin{cases} \bar{m}, & \theta \in [\theta_m, \theta^m) \\ 0, & \textit{otherwise}, \end{cases} \tag{3.25}$$

and by the transfers

$$t(\theta) = \begin{cases} 0, & \theta \in [\underline{\theta}, \theta_m) \\ \theta_m + \bar{m}\varphi(\theta - \theta_m), & \theta \in [\theta_m, \theta^m) \\ \theta_m + \bar{m}\varphi(\theta^m - \theta_m), & \theta \in [\theta^m, \bar{\theta}], \end{cases} \tag{3.26}$$

where $\bar{m} \le \min\{1, \varphi^{-1}\}$ and θ_m, θ_r, and θ^m are the same as in Theorem 1.

In comparison with the optimal mechanism given by Theorem 1, the optimal mechanism under Assumption 2 gives more information rent for the agent. Now an agent with type $\theta \in [\theta_m, \theta_r)$ receive positive equilibrium payoff also if $\bar{m}\varphi < 1$; the regulator cannot decrease the allocation probability in order to get the whole surplus. This implies that now there is no jump in the equilibrium transfers at θ_r. That is to say, under Assumption 2 the agent is offered higher allocation probability r and lower transfers t than in the optimal mechanism without Assumption 2.

Next we introduce the optimal mechanism under avoidance.

Theorem 2. *Under Assumptions 1 and 2, the optimal mechanism Γ^a is given by the physical allocation*

$$r^a(\theta) = \begin{cases} 1, & \theta_a \leq \theta \leq \bar{\theta} \\ 0, & \textit{otherwise}, \end{cases} \tag{3.27}$$

by the monitoring

$$m^a(\theta) = \begin{cases} \bar{m}, & \theta_a \leq \theta < \theta^a \\ 0, & \textit{otherwise}, \end{cases} \tag{3.28}$$

and by the transfers

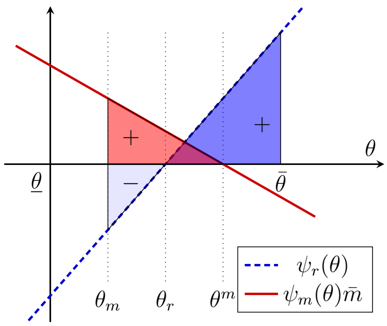
$$t^a(\theta) = \begin{cases} 0, & \theta \in [\underline{\theta}, \theta_m) \\ \theta_a + \bar{m}\varphi(\theta - \theta_a), & \theta \in [\theta_a, \theta^a) \\ \theta_a + \bar{m}\varphi(\theta^a - \theta_a), & \theta \in [\theta^a, \bar{\theta}], \end{cases} \tag{3.29}$$

where

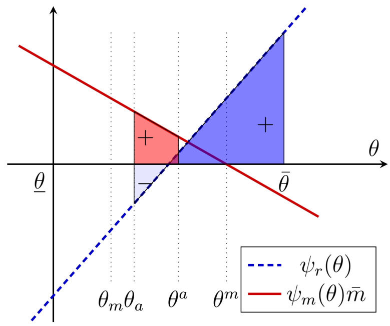
$$\theta_a = \arg \max_{\theta'_a \in \Theta :\, \theta'_a + \frac{C}{\varphi \bar{m}} \leq \bar{\theta}} \left(\int_{\theta'_a}^{\bar{\theta}} \psi_r(\theta) dF(\theta) + \int_{\theta'_a}^{\theta'_a + \frac{C}{\varphi \bar{m}}} \psi_m(\theta) \bar{m} dF(\theta) \right). \tag{3.30}$$

and $\theta^a = \theta_a + \frac{C}{\varphi \bar{m}}$ such that $[\theta_a, \theta^a] \subseteq [\theta_m, \theta^m]$ for $\theta_m < \theta^m$.

Since there is avoidance only if there is monitoring, the optimal mechanism *with* avoidance is the mechanism given by Corollary 4 except the bounds for the physical allocation and monitoring are adjusted such that the agent never finds it optimal to engage in avoidance for types and reports in interval $[\theta_a, \theta^a)$. The determination of interval $[\theta_a, \theta^a]$ is illustrated by Figure 3.4b: the regulator chooses θ_a in order to maximize the sum of the red and dark blue area minus the light blue area.



(a) Without avoidance.



(b) With avoidance.

Figure 3.4: The principal's expected profits using an optimal enforcement mechanism without avoidance (left) and using an optimal enforcement mechanism with avoidance (right).

In comparison with the optimal mechanism in an environment without avoidance given by Corollary 4, we observe that avoidance has two implications. First, the optimal monitoring interval is more narrow than without avoidance. This gives the agent more information rent which is profitable for the agent since it decreases the expected transfers. Second, the share of the types for whom the regulator allocates is smaller than without avoidance. This is harmful for agents with types $\theta \in [\theta_m, \theta_a)$ who do not get the allocation. If the latter negative effect dominates the former positive effect ex ante, the agent's expected equilibrium payoff is lower in the optimal mechanism with avoidance than without avoidance. In words, *avoidance may hurt also the agent* by making the mechanism more inefficient ex ante.

Proposition 9. *Under Assumption 2 and with weak enforcement, i.e.* $\bar{m}\varphi < 1$*, the agent's ability to engage in avoidance can make both, the principal and the agent worse-off ex ante.*

This result holds only with a relatively small marginal fine or monitoring $\bar{m}\varphi < 1$. If the enforcement was strong, i.e. $\bar{m}\varphi = 1$, the principal could extract all information rent from the types that are monitored. In this case, avoidance has only positive effects for the agent ex ante since types $\theta \in [\theta^a, \overline{\theta}]$ receive positive utility by paying transfers $\theta^a \leq \theta^m$, whereas without avoidance agents $\theta \in [\theta^m, \overline{\theta}]$ pay $\theta^m \geq \theta^a$. This can be seen by setting $\bar{m}\varphi = 1$, and observing that $t^a(\theta) = t(\theta) = \theta$ for all $\theta \in [\underline{\theta}, \theta^a]$ and $\theta^a(\theta) = \theta^a < t(\theta)$ for all $\theta \in [\theta^a, \overline{\theta}]$, where t^a given by Theorem 2 and t given by Corollary 4.

Although avoidance decreases the expected transfers, it is not true ex post if the enforcement is weak. Since with avoidance the principal allocates now only for all types $\theta \geq \theta_a \geq \theta_m$ she can exclude agents with low type $\theta < \theta_a$ by asking high prices. Hence, avoidance actually increases the ex-post transfers for some agents. To be more precise, for t^a given by Theorem 2 and t given by Corollary 4 we have $t^a(\theta) > t(\theta)$ for all $\theta \in [\theta_a, \hat{\theta}^a]$, where $\hat{\theta}^a = \theta^a + \frac{1-\bar{m}\varphi}{\bar{m}\varphi}(\theta_a - \theta_m) > \theta^a$ and $\bar{m}\varphi < 1$. That is to say, all types $\theta \leq \hat{\theta}^a$ are worse-off with avoidance than without avoidance.

The equilibrium ex-post transfers are illustrated in Figure 3.5. The blue dashed line is the optimal take-it-or-leave-it-offer, the red solid line is the optimal transfers without avoidance, and the black dotted line is the optimal transfers with avoidance. From here we observe that if F assigns relatively much mass for types $\theta \leq \hat{\theta}^a$ than for types $\theta > \hat{\theta}^a$, then the negative effect of avoidance dominates the positive effect and we would result in the outcome of Proposition 9.

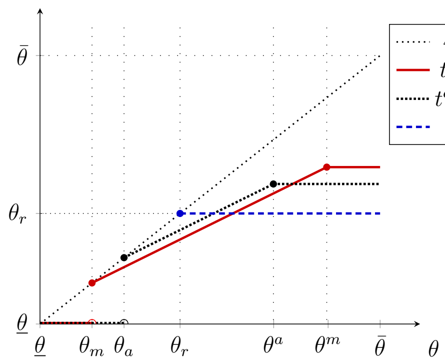


Figure 3.5: Optimal take-it-or-leave-it-offer (blue dashed line), optimal transfers without avoidance (red solid line), and optimal transfers with avoidance (black dotted line).

Finally, we give an example that ratifies Proposition 9. In the example we derive the necessary cutoff types for the closed-form solutions to the optimal mechanism with and without avoidance under Assumptions 1 and 2 such that the type distribution is assumed to be uniform.

Example 5. *(Optimal Mechanisms with a Uniform Distribution.) Let us suppose that Assumption 2 holds and $F(\theta) = \theta$ for all $\theta \in \Theta = [0, 1]$ — that is, the agent's type is uniformly distributed on the unit interval. For simplicity, assume that the monitoring is costless, $\bar{m} = 1$, $\alpha = 0$, and $C, \varphi < 1$.*

After some algebra, one can show that the boundaries are given by $\theta_r = \frac{1}{2}$, $\theta_m = \frac{1-\varphi}{2-\varphi}$, $\theta^m = 1$, $\theta_a = \frac{1-C}{2}$, and $\theta^a = \frac{1-C}{2} + \frac{C}{\varphi}$ when $C \leq \frac{\varphi}{2-\varphi}$, which ensures that $[\theta_a, \theta^a] \subset [\theta_m, \theta^m]$.

The agent's expected equilibrium payoff in the optimal enforcement mechanism without *avoidance (Proposition 1) is given by*

$$\int_0^1 V(\theta)d\theta = \int_{\theta_m}^1 \int_{\theta_m}^{\theta} (1-\varphi)\, dsd\theta = \frac{1-\varphi}{2(2-\varphi)^2}. \tag{3.31}$$

The agent's expected equilibrium payoff in the optimal enforcement mechanism with avoidance (Theorem 2) is given by

$$\int_0^1 V^a(\theta)d\theta = \int_{\theta_a}^{\theta^a} \int_{\theta_a}^{\theta} (1-\varphi)\, dsd\theta + \int_{\theta^a}^1 \left(\int_{\theta_a}^{\theta^a} (1-\varphi)ds + \int_{\theta^a}^{\theta} ds \right) d\theta \tag{3.32}$$

$$= \frac{\varphi + C^2(4-3\varphi) - 2C\varphi}{8\varphi}. \tag{3.33}$$

Lastly, the agent's expected equilibrium payoff in the optimal standard mechanism, i.e. with a take-it-or-leave-it-offer $\theta_r = 1/2$ is $\frac{1}{8}$ (can be also derived from (3.31) by setting $\varphi = 0$).

It is easy to verify that with a small marginal fine and some marginal cost of avoidance, say $\varphi = 1/3$ and $C = 1/10$, the expected payoff for the agent in the optimal enforcement mechanism with avoidance is lower than that in the optimal enforcement mechanism without avoidance. Hence, the agent's ability to engage in avoidance hurts not only the principal, but also the agent in this case.

3.4 Discussion

Generally speaking, in this paper we study a contracting problem in which the principal can use monitoring and punishments to enforce the agent to choose a certain contract. We approach this problem from the perspective of mechanism design with costly verification. We derive the optimal regulatory policies with and without avoidance, respectively.

One of most serviceable application of our model is an emission regulation problem similar to that, for instance, in Bontems and Bourgeon (2005). In this case r is defined as the amount of abatement that the regulator wants a firm to decrease its pollution, θ as a privately known abatement cost by the firm, t as the subsidy from the regulator to the firm, $m(1-a)$ as the probability that the regulator learns the firm's abatement costs, and $\varphi(\theta' - \theta)$ as the fines that are imposed when the firm reports costs θ' under true costs θ. That is, all the elements of the emission regulation model are oppositely beneficial to the players in comparison with the model used in the analysis. This inversion is analogous to that in procurement auctions.

Theorem 1 implies that without avoidance the regulator offers contracts from the following four categories:

1. (No abatement, no subsidies, no monitoring);
2. (Partial abatement, small cost-based subsidies, monitoring);
3. (Full abatement, intermediate cost-based subsidies, monitoring);
4. (Full abatement, a high fixed subsidy, and no monitoring).

The contracts in these categories are designed such that firms with high costs are excluded from the enforcement by the first category. Firms with intermediate abatement costs choose either partial of full abatement contracts from the second or third category accordingly to their costs. Firms with low abatement costs choose a full abatement with a high fixed subsidy and receive the greatest benefits from the regulation.

If the regulator's enforcement is strong, there is no need of offering partial abatement; with efficient monitoring and sufficiently high penalties all intermediate firms are also enforced to the full abatement. This further decreases the regulator's monitoring costs and leads to the higher social welfare than with weak enforcement.

If the agent can engage in avoidance — that is, the agent can decrease the probability of getting caught from taking a wrong contract, contracting becomes less favorable for the regulator. First, abatements are enforced less frequently, and secondly, the regulator expects to pay higher subsidies than without avoidance. These results emerge from the fact that avoidance makes monitoring less effective and the regulator finds it optimal to decrease the number of contracts in the intermediate category. Consequently, there are more contracts in Category 1 which results in that the less firms are asked to abate their pollution. Moreover, since avoidance decreases the number of monitored contracts, deviations are more desirable for the firms with low abatement costs. As a result, the regulator must increase the subsidy in Category 4 in comparison with the optimal regulation contracts without avoidance.

In the language of mechanism design, the firm's ability to engage in avoidance (i) increases the firm's information rent, and (ii) makes the optimal mechanism less efficient ex ante (abatement are enforced less often). The former effect is good for the firm and the latter is not (since then there is no subsidies). Surprisingly, if the inefficiency effect decreases the firm's ex-ante payoffs more than the information rent effect increases it, then avoidance is disadvantageous not only to the regulator, but also to the firm *ex ante* (see Proposition 9 and Example 5).

In real-life we observe many phenomena of monitoring and avoidance. We discover individuals and firms who hide their income to offshore banks and then misreport their wealth to tax officials (tax evasion); we have automobile manufactures who install softwares into their vehicles in order to conceal nitrogen oxide (NOx) emissions in emission tests (see United States Environmental Protection Agency (EPA) (2015));[14] we detect misreported customs declarations and many cunning ways of trying to cover up the true contents of packages or containers from customs (tariff evasion); we see firms allocating jobs to persons with fake certificates; and the list goes on.

All of these examples have a common thread: the principal wants the agents to be compliant, but the agents are capable of deceiving the principal's verification mechanism in order to advantage themselves. As we have shown in this paper, if the regulation mechanisms are designed without taking avoidance into account, compliance (and incentive compatibility) may fail.

In this paper, we design the optimal mechanisms in a linear and more or less stylized setup. How does the optimal mechanism change if we assumed different verification probability or punishment functions? What if avoidance costs are not known to the principal? Furthermore, in many instances avoidance is also a crime. What happens if we assumed that the punishments are conditional on avoidance as well? These questions are left for further studies.

Appendix: Omitted Proofs

In some of the proofs in this section we use the more general notation which was introduced in Section 3.2. In this way we want to highlight that many of our results apply also to a more general class of models. In addition, using the more general notation helps us to stress which assumptions are pivotal for the results.

The Proof of Proposition 6

Let an arbitrary choice set be denoted by $\mathcal{Z} \subseteq [0,1] \times \mathbb{R} \times \mathcal{M} \times \Theta \times \mathcal{A} =: \mathcal{Y}$ such that $z = (r, t, m, \sigma, a) \in \mathcal{Z}$, where $\sigma \in \Theta$ is the agent's report and $a \in \mathcal{A}$ the agent's choice of avoidance. In other words, the principal designs a menu $\Gamma = \{r(\sigma), t(\sigma), m(\sigma)\}_{\sigma \in \Theta}$ from where the agent chooses a contract by reporting σ and engages in a at the same time.

The agent's parameterized objective function $U_A : \mathcal{Y} \times \Theta \to \mathbb{R}$ is given by

$$U_A(z, \theta) = v(r, \theta) - t - p(m, a)\Phi(\sigma, \theta) - c(a) \tag{3.34}$$

$$= \theta r - t - m(1-a)\varphi(\theta - \sigma) - Ca. \tag{3.35}$$

[14] Volkswagen sold around 500,000 cars fitted with so-called 'defeat devices' that are designed to reduce emissions of nitrogen oxide (NOx) under test conditions. When they got caught, they had to pay a fine of $4.3bn.

Let the value function be given by

$$V(\theta) = \sup_{z \in \mathcal{Z}} U_A(z, \theta) \tag{3.36}$$

and the optimal choice correspondence by

$$Z^*(\theta) = \{z \in \mathcal{Z} : U_A(z, \theta) = V(\theta)\}. \tag{3.37}$$

The value function, V, is absolutely continuous and differentiable almost everywhere, and so by the Fundamental Theorem of Calculus: $V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} V'(s)ds$. Moreover, since $U_A(z, \theta)$ is differentiable in θ, we have by the Envelope Theorem that

$$\begin{aligned} V'(\theta) &= v_\theta(r(\sigma^*(\theta)), \theta) - p(m(\sigma^*(\theta)), a^*(\theta))\Phi_\theta(\sigma^*(\theta), \theta) && (3.38) \\ &= r(\sigma^*(\theta)) - m(\sigma^*(\theta))(1 - a^*(\theta))\varphi && (3.39) \end{aligned}$$

almost everywhere and thus the equilibrium payoffs are

$$\begin{aligned} V(\theta) &= V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \left[v_\theta(r(\sigma^*(s)), s) - p(m(\sigma^*(s)), a^*(s))\Phi_\theta(\sigma^*(s), s)\right] ds && (3.40) \\ &= V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \left[r(\sigma^*(s)) - m(\sigma^*(s))(1 - a^*(s))\varphi\right] ds, && (3.41) \end{aligned}$$

for any $z^*(\theta) = (r(\sigma^*(\theta)), t(\sigma^*(\theta)), m(\sigma^*(\theta)), \sigma^*(\theta), a^*(\theta)) \in Z^*(\theta)$ (see Milgrom and Segal (2002), Theorem 2). Denote $r^*(\theta) := r(\sigma^*(\theta))$, $t^*(\theta) := t(\sigma^*(\theta))$, and $m^*(\theta) := m(\sigma^*(\theta))$. Consequently, the equilibrium transfers are given by

$$\begin{aligned} t^*(\theta) &= v(r^*(\theta), \theta) - p(m^*(\theta), a^*(\theta))\Phi(\sigma^*(\theta), \theta) - c(a^*(\theta)) \\ &\quad - V(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} \left[v_\theta(r^*(s), s) - p(m^*(s), a^*(s))\Phi_\theta(\sigma^*(s), s)\right] ds && (3.42) \\ &= \theta r^*(\theta) - m^*(\theta)(1 - a^*(\theta))\varphi(\theta - \sigma^*(\theta)) - Ca^*(\theta) \\ &\quad - V(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} \left[r^*(s) - m^*(s)(1 - a^*(s))\varphi\right] ds. && (3.43) \end{aligned}$$

Substituting (3.43) into the principal's ex-post equilibrium payoff (before verification) we get

$$\begin{aligned} U_P(z^*(\theta), \theta) &= t^*(\theta) + p(m^*(\theta), a^*(\theta))\Phi(\sigma^*(\theta), \theta) - \kappa(m^*(\theta)) + \alpha V(\theta) && (3.44) \\ &= v(r^*(\theta), \theta) - (1 - \alpha)V(\underline{\theta}) - \kappa(m^*(\theta)) - c(a^*(\theta)) \end{aligned}$$

$$- (1-\alpha)\int_{\underline{\theta}}^{\theta} \left[v_\theta(r^*(s),s) - p(m^*(s),a^*(s))\Phi_\theta(\sigma^*(s),s)\right] ds \tag{3.45}$$

$$= \theta r^*(\theta) - (1-\alpha)V(\underline{\theta}) - Km^*(\theta) - Ca^*(\theta)$$

$$- (1-\alpha)\int_{\underline{\theta}}^{\theta} \left[r^*(s) - m^*(s)(1-a^*(s))\varphi\right] ds \tag{3.46}$$

$$\leq \theta r^*(\theta) - (1-\alpha)V(\underline{\theta}) - Km^*(\theta)$$

$$- (1-\alpha)\int_{\underline{\theta}}^{\theta} \left[r^*(s) - m^*(s)\varphi\right] ds. \tag{3.47}$$

From here we observe that the principal prefers contracts in which $a^*(\theta) = 0$ since the marginal fines are positive with respect to the type ($\Phi_\theta(\sigma,\theta) \geq 0$); if the agent engage in avoidance, the principal's information rent extraction by monitoring decreases. Moreover, since the principal and the agent both have quasi-linear utilities in monetary variables (t and Φ) and the punishment function is additively separable ($\Phi_\theta(\sigma,\theta)$ is independent of σ), equilibrium payoffs of both players are independent of the agent's report σ. This last observation is crucial since it allows us to consider only truthful equilibria by the following rationale.

In any mechanisms in which the avoidance is zero in equilibrium, the equilibrium payoffs are

$$U_A(z(\sigma^*(\theta)),\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \left[r(\sigma^*(s)) - m(\sigma^*(s))\varphi\right] ds, \tag{3.48}$$

$$U_P(z(\sigma^*(\theta)),\theta) = \theta r(\sigma^*(\theta)) - (1-\alpha)V(\underline{\theta}) - Km(\sigma^*(\theta)) \tag{3.49}$$

$$- (1-\alpha)\int_{\underline{\theta}}^{\theta} \left[r(\sigma^*(s)) - m(\sigma^*(s))\varphi\right] ds, \tag{3.50}$$

for the agent and the principal, respectively, all $\theta \in \Theta$. Then by implementing a direct mechanism $(\hat{r}(\cdot),\hat{t}(\cdot),\hat{m}(\cdot)) = (r(\sigma^*(\cdot)),t(\sigma^*(\cdot)),m(\sigma^*(\cdot)))$ we induce the same outcomes and payoffs for both players. This direct mechanism is incentive and avoidance compatible by the optimality of σ and by the fact that the optimal avoidance with truthful report is zero. To see this, assume the opposite: there is a $\theta' \neq \theta$ such that

$$U_i(\hat{z}(\theta'),\theta) > U_i(\hat{z}(\theta),\theta), \tag{3.51}$$

for either $i \in \{A,P\}$. However, since $\hat{z}(\theta') = z(\sigma^*(\theta'))$ we get from (3.51) that $U_i(z(\sigma^*(\theta')),\theta) > U_i(z(\sigma^*(\theta)),\theta)$, which contradicts the optimality of σ^*. Hence, the direct mechanism given by $\hat{r}$ must be incentive and avoidance compatible.

We have thus argued that the optimal class of direct mechanisms for the principal is given by mechanisms in which avoidance is zero in equilibrium. Furthermore, all the outcomes of any mechanism in this class can be implemented by a truthful mechanism which also belongs to this class. That is to say, it is *without loss of optimality* to restrict our attention to *truthful* direct mechanisms among all direct mechanisms.[15]

Proof of Theorem 1

The principal's objective function (3.9) is a continuous and linear function in $(r; m)$ on a compact and convex vector space (see Figure 3.6). This implies that the optimal mechanism must occur at an extreme point.

The individual rationality requires that $\mathcal{I}(\theta|\theta) \geq 0$ for all $\theta \in \Theta$ (see Lemma 3). This is satisfied if $(r(\theta), m(\theta)) \in \mathcal{E}$ for all $\theta \in \Theta$, where

$$\mathcal{E} = \{(0,0), (1,0), (1,\bar{m}), (\bar{r}, \bar{m})\} \tag{3.52}$$

is the set of individual rational extreme points, where $\bar{m} := \min\{1, \varphi^{-1}\}$ and $\bar{r} := \min\{1, \varphi\}$.

With the mechanism at the first extreme point, $(0,0)$, the principal does not allocate or monitor; at the second extreme point $(1,0)$, the principal uses only the physical allocation and not monitoring; at the third extreme point $(1,\bar{m})$ the principal allocates the object with probability 1 and adjust the monitoring to level $\bar{m}$ which equals 1 if $\varphi \leq 1$ and φ^{-1} if $\varphi > 1$ (noisy monitoring); and at the fourth extreme point $(\bar{r}, \bar{m})$, the mechanism is the same as the third one if $\varphi > 1$ and $(\varphi, 1)$ if $\varphi \leq 1$. In the latter case the principal chooses maximal monitoring and decreases the probability of physical allocation as low as possible but still satisfy the individual rationality constraint. In conclusion, the principal may find it optimal to use stochastic physical allocation or stochastic monitoring in some instances.

The extreme points $\mathcal{E}$ are depicted in Figure 3.6. If the marginal fine $\varphi > 1$, then $(r; m)$ belongs to the blue area which is the convex hull of points $\{(0,0), (0,1), (1, \varphi^{-1})\}$. As for $\varphi' \leq 1$ a physical allocation and monitoring pair $(r; m)$ belongs to the union of the blue and the red area which is the convex hull of extreme points $\{(0,0), (1,0), (1,1), (\varphi', 1)\}$.

[15] This result generalizes to indirect mechanisms in the same fashion as in Mookherjee and Png (1990). It, however, necessitates that the punishment function can be transformed in terms of arbitrary reports and there are no punishments or rewards in equilibrium (in either indirect or direct mechanisms).

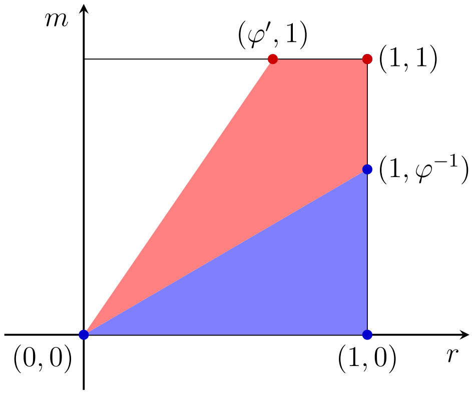


Figure 3.6: Individually rational extreme points with $\varphi > 1$ and $\varphi' \leq 1$.

The finding of optimal mechanism is now significantly easier: we need to choose the extreme points that maximize the principal's objective function point-wise for θ and check whether this satisfies the incentive compatibility constraint.[16]

Let the agent's *virtual valuation* with given mechanism (r, m) be denoted by

$$\Psi(\theta|r, m) := \psi_r(\theta)r(\theta) + \psi_m(\theta)m(\theta). \tag{3.53}$$

That is, Ψ is a function that measures the surplus that can be extracted from the agent minus the monitoring costs. Let us next consider (3.53) at each of our four extreme points $\mathcal{E}$. The first extreme point yields

$$\Psi(\theta|0, 0) = 0$$

and the second

$$\Psi(\theta|1, 0) = \theta - (1 - \alpha)\frac{1 - F(\theta)}{f(\theta)},$$

which is the virtual valuation in the standard mechanism. At the third extreme point we have

$$\Psi(\theta|1, \bar{m}) = \theta - (1 - \varphi\bar{m})(1 - \alpha)\frac{1 - F(\theta)}{f(\theta)} - K\bar{m},$$

and at the last one we get

$$\Psi(\theta|\bar{r}, \bar{m}) = \theta\bar{r} - K\bar{m}.$$

[16] Note that

$$\max_{(r;m)} \mathbb{E}\left[\psi_r(\theta)r(\theta) + \psi_m(\theta)m(\theta)\right] \leq \mathbb{E}\left[\max_{r(\theta), m(\theta)} (\psi_r(\theta)r(\theta) + \psi_m(\theta)m(\theta))\right].$$

Hence, if we can maximize the objective function point-wise, we get also a solution that maximizes the expected value.

These last two ones are the virtual valuations in which the regulator can use monitoring to extract some extra surplus from the agent in comparison with the first two one. The difference between $\Psi(\theta|1, \bar{m})$ and $\Psi(\theta|\bar{r}, \bar{m})$ is that in the latter one the principal decreases the probability of the allocation in order to extract the full information rent from the agent. This is however a costly strategy for the regulator because then she also makes the allocation uncertain.

Next we compare $\Psi(\theta|0, 0)$, $\Psi(\theta|1, 0)$, $\Psi(\theta|1, \bar{m})$, and $\Psi(\theta|\bar{r}, \bar{m})$ at each θ and choose the mechanism that gives the maximal value and check whether the incentive compatibility constraint is satisfied. That is, the optimal mechanism candidate is given by the upper envelope of $\Psi(\theta|r(\theta), m(\theta))$ which we denote by $\bar{\Psi}(\theta)$. The regularity of F implies that $\Psi(\theta|r(\theta), m(\theta))$ is non-decreasing for any choice $(r(\theta), m(\theta)) \in \mathcal{E}$ since $\alpha \in [0, 1]$ and $1 - \varphi\bar{m} \in [0, 1]$. This makes $\bar{\Psi}(\theta)$ convex. The upper envelope of functions $\Psi(\theta|\bar{r}, \bar{m})$ is illustrated in Figure 3.7.

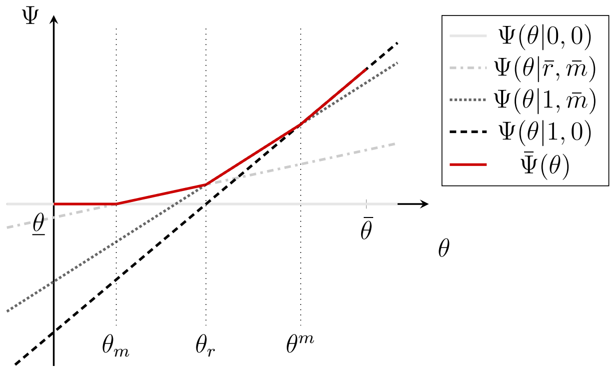


Figure 3.7: Principal's objective with different extreme values.

Let us define

$$\theta_m := \inf\{\theta \in \Theta : \Psi(\theta|0, 0) \le \Psi(\theta|\bar{r}, \bar{m})\} \tag{3.54}$$

$$= \inf\left\{\theta \in \Theta : \theta \ge \frac{K}{\varphi}\right\} \tag{3.55}$$

$$\theta_r := \inf\{\theta \in \Theta : \Psi(\theta|0, 0) \le \Psi(\theta|1, 0)\} \tag{3.56}$$

$$= \inf\left\{\theta \in \Theta : \theta \ge (1 - \alpha)\frac{1 - F(\theta)}{f(\theta)}\right\} \tag{3.57}$$

$$\theta^m := \inf\{\theta \in \Theta : \Psi(\theta|1, \bar{m}) \le \Psi(\theta|1, 0)\} \tag{3.58}$$

$$= \inf\left\{\theta \in \Theta : \frac{K}{\varphi} \ge (1 - \alpha)\frac{1 - F(\theta)}{f(\theta)}\right\}. \tag{3.59}$$

It is also straightforward to show that $\Psi(\theta|\bar{r},\bar{m}) \geq \Psi(\theta|1,\bar{m})$ for $\theta \leq \theta_r$ and $\Psi(\theta|\bar{r},\bar{m}) \leq \Psi(\theta|1,\bar{m})$ for $\theta \geq \theta_r$ (note that if $\varphi > 1$, then $\Psi(\theta|\bar{r},\bar{m}) = \Psi(\theta|1,\bar{m}) = \Psi(\theta|1,\varphi^{-1})$). Therefore, our candidate for the optimal mechanism as the upper envelope of $\Psi(\theta|r(\theta),m(\theta))$ can be determined by choosing $(r;m)$ in the following way

$$(r(\theta),m(\theta)) = \begin{cases} (0,0), & \theta \in [\underline{\theta},\theta_m) \\ (\bar{r},\bar{m}), & \theta \in [\theta_m,\theta_r) \\ (1,\bar{m}), & \theta \in [\theta_r,\theta^m) \\ (1,0), & \theta \in [\theta^m,\bar{\theta}]. \end{cases} \tag{3.60}$$

for $\underline{\theta} \leq \theta_m \leq \theta_r \leq \theta^m \leq \bar{\theta}$. It is easy to confirm that under this mechanism $r(\cdot) - m(\cdot)\varphi$ is non-negative and non-decreasing. Hence, whenever we have $\underline{\theta} \leq \theta_m \leq \theta_r \leq \theta^m \leq \bar{\theta}$, the mechanism given in (3.60) is incentive compatible and individually rational and *thus* optimal among all mechanisms.

Lastly, it might be the case that $\theta_m \geq \theta^m$. This occurs if either (i) monitoring is costly or (ii) fines are small. If (i) is true, then it is profitable for the principal to use the take-it-or-leave-it offer θ_r rather than to invest in expensive monitoring. If (ii) is true, then the principal's information extraction is ineffective and thus an increase in transfers by monitoring is smaller than its costs (again monitoring is relatively costly). Consequently, in both cases the optimal mechanism is the standard take-it-or-leave-it-offer θ_r without any additional enforcement. This can be seen by observing that $\max\{\Psi(\theta|0,0),\Psi(\theta|1,0)\} \geq \max\{\Psi(\theta|\bar{r},\bar{m}),\Psi(\theta|1,\bar{m})\}$ whenever $\theta_m \geq \theta^m$ for all $\theta \in \Theta$.

Proof of Lemma 3

Without loss of generality, let $\theta \geq \theta'$. Then the incentive and avoidance compatibility requires that

$$\theta r(\theta) - t(\theta) \geq \theta r(\theta') - t(\theta') - m(\theta')(1 - a^*(\theta'|\theta))\varphi(\theta - \theta') - Ca^*(\theta'|\theta) \tag{3.61}$$

$$\theta' r(\theta') - t(\theta') \geq \theta' r(\theta) - t(\theta) - m(\theta)(1 - a^*(\theta|\theta'))\varphi(\theta' - \theta) - Ca^*(\theta|\theta'), \tag{3.62}$$

where the optimal avoidance a^* is given by (3.18). Since $\theta \geq \theta'$, the optimal avoidance for type θ with report θ' is zero: $a^*(\theta|\theta') = 0$. Then, by summing up the inequalities we get that

$$(\theta - \theta')\left(\mathcal{I}(\theta|\theta) - \mathcal{I}(\theta'|\theta)\right) \geq -Ca^*(\theta'|\theta), \tag{3.63}$$

or

$$(\theta - \theta')\left(\mathcal{I}(\theta|\theta) - \mathcal{I}(\theta'|\theta')\right) \geq a^*(\theta'|\theta)\left(m(\theta')\varphi(\theta - \theta') - C\right), \tag{3.64}$$

where $\mathcal{I}(\theta'|\theta) = r(\theta') - m(\theta')(1 - a^*(\theta'|\theta))\varphi$. Since $a^*(\theta'|\theta) = 1$ iff $m(\theta')\varphi(\theta - \theta') \geq C$ and zero otherwise, the right-hand side of inequality (3.64) is non-negative. This implies that $\mathcal{I}(\cdot|\cdot)$ is non-decreasing.

Proof of Proposition 8

By definition, the mechanism Γ is incentive and avoidance compatible if and only if for all $\theta, \theta' \in \Theta$ we have

$$u(x(\theta), \theta) \geq u(x(\theta'), \theta) - p(m(\theta'), a^*(\theta'|\theta))\Phi(\theta', \theta) - c(a^*(\theta'|\theta)). \tag{3.65}$$

We know from the proof of Proposition 6 that the equilibrium transfers in a truthful mechanism are

$$t(\theta) = v(r(\theta), \theta) - V(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} \left[v_\theta(r(s), s) - p(m(s), 0)\Phi_\theta(s, s)\right] ds. \tag{3.66}$$

By substituting these into the inequality in (3.65) we observe

$$\begin{aligned} &\int_{\theta'}^{\theta} \left[v_\theta(r(s), s) - p(m(s), 0)\Phi_\theta(s, s)\right] ds \\ &\quad \geq \int_{\theta'}^{\theta} \left[v_\theta(r(\theta'), s) - p(m^*(\theta'), a^*(\theta'|\theta))\Phi_\theta(\theta', \theta)\right] ds - c(a^*(\theta'|\theta)). \end{aligned} \tag{3.67}$$

Let $\mathcal{I}(\theta'|\theta) := v_\theta(r(\theta'), \theta) - p(m(\theta'), a^*(\theta'|\theta))\Phi_\theta(\theta', \theta)$ and rewrite (3.67) as follows

$$\int_{\theta'}^{\theta} \left(\mathcal{I}(s|s) - \mathcal{I}(\theta'|\theta)\right) ds \geq -c(a^*(\theta'|\theta)). \tag{3.68}$$

Obviously, if $(\theta - \theta')\left[\mathcal{I}(\theta|\theta) - \mathcal{I}(\theta'|\theta)\right] \geq -c(a^*(\theta'|\theta))$ for all $\theta', \theta \in \Theta$, then (3.68) is satisfied. Conversely, we know by the proof of Lemma 3, that in every incentive and avoidance compatible mechanism we have

$$(\theta - \theta')\left[\mathcal{I}(\theta|\theta) - \mathcal{I}(\theta'|\theta)\right] \geq -c(a^*(\theta'|\theta)) \tag{3.69}$$

for all $\theta', \theta \in \Theta$, which completes the proof.

Proof of Theorem 2

By Lemma 3 we know that in all incentive and avoidance compatible mechanism the net information rent is non-decreasing. Hence, our candidate for the optimal mechanism is

$$(r^a(\theta), m^a(\theta)) = \begin{cases} (0,0), & \theta \in [\underline{\theta}, \theta_a) \\ (1,\bar{m}), & \theta \in [\theta_a, \theta^a) \\ (1,0), & \theta \in [\theta^a, \bar{\theta}], \end{cases} \tag{3.70}$$

for some $\underline{\theta} \leq \theta_a \leq \theta^a \leq \bar{\theta}$. By the similar structure as the optimal mechanism in Corollary 4, we know that the incentive and avoidance compatibility is satisfied for all other type-report pairs but $\theta, \theta' \in [\theta_a, \theta^a]$ such that $\theta' \leq \theta$ and $a^*(\theta'|\theta) = 1$ (see the proof of Theorem 1). Moreover, the individual rationality is satisfied for all $\theta \in \Theta$. We can thus proceed by finding optimal interval $[\theta_a, \theta^a]$ under which (IAC) is satisfied.

Optimal interval $[\theta_a, \theta^a]$**.** Let us assume that the agent finds it optimal to engage in avoidance with some $\theta, \theta' \in [\theta_a, \theta^a)$. Then (IAC) condition

$$(\theta - \theta')(r(\theta) - m(\theta)\varphi - r(\theta') + m(\theta')(1 - a^*(\theta'|\theta))\varphi) \geq -Ca^*(\theta'|\theta) \tag{3.71}$$

is satisfied only if

$$C \geq \bar{m}\varphi(\theta - \theta'). \tag{3.72}$$

However, since the optimal avoidance is one iff $\bar{m}\varphi(\theta - \theta') > C$, we have a contradiction. In other words, there cannot be avoidance for $\theta, \theta' \in [\theta_a, \theta^a)$.

Since the regulator prefers monitoring for all reports $\theta' \in [\theta_a, \theta^a] \subset [\theta_m, \theta^m]$, she wants the optimal interval $[\theta_a, \theta^a]$ to be as broad as possible, but still keeping the avoidance zero. Therefore, **the optimal upper bound** can be determined by setting $C \geq \varphi\bar{m}(\theta^a - \theta_a)$ to hold as equality:

$$\theta^a = \theta_a + \frac{C}{\varphi\bar{m}}. \tag{3.73}$$

Now the agent never engages in avoidance with any $\theta, \theta' \in [\theta_a, \theta^a) = \left[\theta_a, \theta_a + \frac{C}{\varphi\bar{m}}\right)$ and the mechanism given in (3.70) satisfies (IAC).

Lastly, we need to choose the lower bound θ_a such that the principal's objective is maximized. To that end, **the optimal lower bound** is given by

$$\theta_a = \arg \max_{\theta'_a \in \Theta \colon \theta'_a + \frac{C}{\varphi \bar{m}} \leq \bar{\theta}} \left(\int_{\theta'_a}^{\bar{\theta}} \psi_r(\theta) dF(\theta) + \int_{\theta'_a}^{\theta'_a + \frac{C}{\varphi \bar{m}}} \psi_m(\theta) \bar{m} dF(\theta) \right). \tag{3.74}$$

Chapter 4

Bilateral Trade with Interdependent Values

Abstract

We study a market for 'lemons' from the perspective of mechanism design in a bilateral trade setup. The closed-form solution for the seller-optimal safe mechanism under one-sided private information is provided. We show that a seller can disclose the quality of the goods by controlling the supply of her goods; high-quality sellers want their goods to be scarce and expensive and low-quality sellers abundant and cheap. In this way, sellers can differentiate their products from each other and maximize their payoffs. We extend this model to two-sided private information and give a novel characterization of the seller-optimal safe mechanism in this setup. It turns out that if there is two-sided asymmetric information, then the seller finds it optimal to engage in price signalling instead of quantity signaling. This is the least-cost way for the seller to signal her private information to the buyer.[1]

Keywords: Informed Principal, Bilateral Trade, Mechanism Design, Interdependent Values.

JEL: D42, D82.

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4.1 Introduction

Mechanism design by an informed principal studies contracting problems where the principal has some private information about the object that she is allocating to agents. This kind of asymmetric information structure is present in many economic circumstances; practically almost all firms or sellers have some relevant information about their products that their customers or buyers do not know. For example, if a car owner is selling her used car, she may want to conceal the information about the car's quality from potential buyers. However, the seller's choice of how to sell the car may still signal something about her private information to the buyers. Thus, the seller faces the following dilemma: How to optimally choose a selling mechanism that may reveal some substantive information to buyers?

In this paper, we study a market for 'lemons' from the perspective of mechanism design. We assume that there are a single seller and a single buyer. The seller maximizes her payoffs by designing a selling mechanism to allocate her goods to the buyer. The valuations of the goods for the seller and the buyer are interdependent. That is, the seller has some payoff-relevant information for the buyer. This approach generalizes Akerlof's (1970) famous model by considering a large class of selling mechanisms.

We focus on seller's utility-maximizing mechanisms that are incentive compatible and individually rational (incentive feasible hereafter) even though the seller's type was common knowledge. These mechanisms are called optimal *safe* mechanisms by Myerson (1983). Seller-optimal safe allocations form an important class of mechanisms, since they correspond to the least-cost separating equilibria of the mechanism selection game.[2]

Theorem 3 provides the closed-form solution for the seller-optimal safe mechanism under one-sided private information. It turns out that sellers can disclose their private information by controlling the supply of goods; high-quality sellers want their goods to be *scarce* and *expensive* and low-quality sellers *abundant* and *cheap*. In this way, different types of sellers can differentiate their products from each other and maximize their payoffs. However, since the whole capacity of the goods is not traded, the seller's private information leads to a *deadweight loss*. These findings support the early observations of Akerlof in the markets for lemons; among all safe mechanisms, the monopolist finds it optimal to strictly decrease the quantity of high-quality products. However, the analysis in this paper reveals that the market never collapses for a high-quality seller because of

[2] Maskin and Tirole (1992) show that under discrete type spaces, certain sorting assumptions, and quasi-linear utilities, the optimal safe mechanism gives weakly higher interim utility for the seller than any other incentive feasible mechanism. Nishimura (2022) characterizes the set of prior beliefs for which the seller-optimal safe mechanism is undominated by any other incentive compatible and individually rational mechanism. This result provides a necessary and sufficient condition for the existence of a strong solution (undominated safe mechanism) introduced by Myerson (1983).

the possibility of credible signaling.

We extend our analysis to a two-sided private information model where also the buyer has some private information. Theorem 4 provides a novel characterization of the seller-optimal safe mechanism in this setup. From this characterization we can elucidate the signaling costs of the seller and their connection to the surplus extraction. It turns out that the seller finds it optimal to increase the equilibrium prices from the second-best counterpart where the seller's type is common knowledge. In other words, instead of engaging in quantity-signaling, the seller signals her private information to the buyer buy asking higher prices than it would be optimal if she was not informed. In this way, the seller makes the signaling credible: By decreasing the expected probability of trade, she gives up some of her payoffs.

Finally, we show that if there was a trustworthy mediator who can verify the seller's type to the buyer, then it would always be profitable for the seller to outsource the trade for the mediator. This is due to the fact that the seller-optimal safe mechanism under one-sided private information gives the seller always greater payoff than that under two-sided private information. Therefore, a mediator who can verify the seller's type can benefit from the payoff difference between these two seller-optimal safe mechanisms. The mediator's solution allocates always all the goods, and hence there is no inefficiency by the private information of the seller. Moreover, this mediator-agreement is profitable also for the seller if the price paid by the mediator to the seller minus the brokerage paid by the seller to the mediator is greater than the expected profit that the seller would receive by organizing the trade by herself.

This paper is organized as follows. First, in Sections 4.2 and 4.3, we build the model and introduce the results, respectively. After that, in Section 4.4, we compare our results to the mediator's solution, which determines the seller's valuation for the full disclosure of information or perfect verification. We elaborate the connection of our results to the earlier literature in Section 4.5. Our findings provide some novel rationales for different types of economic behavior, such as joint ownership agreements, part-time employment contracts, or unused production capacities. Section 4.6 discusses these applications in more detail. Appendix 4.6 is devoted to the proofs of our results.

4.2 Preliminaries

We focus on the following principal-agent model. The principal is a seller, S, who has an object for sale. The agent is a buyer, B, who is willing to buy the object. Player $i \in \{S, B\}$ has privately known type $t_i \in T_i := [\underline{t_i}, \overline{t_i}] \subset \mathbb{R}_+$. We denote a type profile by $t = (t_S, t_B) \in T_S \times T_B =: T$. The seller's beliefs about the type of buyer, t_B, are given by distribution F_B with full support on T_B. The buyer's prior beliefs about the seller's type, t_S, are given by a distribution F_S with full support on T_S. We assume that t_S and t_B are independently distributed and that F_B is regular, i.e. $\frac{1-F_B(t_B)}{f_B(t_B)}$ is *strictly* decreasing in t_B.

The types of the players form the valuations of the object for each player in the following fashion. The buyer's valuation of the object is given by function $v_B(t) = t_B + \alpha_B t_S$ and the seller's valuation by $v_S(t) = \alpha_S t_S$ where $\alpha_i \geq 0$ for $i \in \{B, S\}$. That is, α_i determines the importance of the seller's type for Player i. For simplicity, we assume that the seller's valuation is independent of the buyer's type.

Player i's ex-post utility is given by the function $u_i : X \times T \to \mathbb{R}$ for $i \in \{B, S\}$ such that

$$u_B(x;t) = v_B(t)q - p \tag{4.1}$$

$$u_S(x;t) = p - v_S(t)q, \tag{4.2}$$

where $x = (q,p)$ and $X := [0,1] \times \mathbb{R}$. The quantity of goods is denoted by q, and the seller's capacity of the goods is normalized to unity.[3] The price paid by the buyer to the seller is given by p.

By the Revelation Principle, we can focus on direct revelation mechanisms (see, e.g., Myerson (1981, 1982) or Sugaya and Wolitzky (2021)). That is, an equilibrium of any indirect mechanism with given beliefs corresponds to a truthful equilibrium of a direct revelation mechanism (DRM).

Let $\mathcal{G}$ be the set of all DRMs and $\Gamma = (T, x) \in \mathcal{G}$ be an arbitrary direct mechanism, where $x = (q,p)$ is given by the functions $(q,p) : T \to X$.

The timing of the game is as follows. First, the valuations $t \in T$ are drawn independently according to the distributions F_S and F_B. After learning her type, the seller designs a mechanism and commits to it. The buyer updates her beliefs about the seller's type based on the observed mechanism and decides whether or not to participate in the mechanism. If the buyer decides not to participate, the game ends and both players receive their outside options, which are normalized to zero. If the buyer participates in the mechanism, both players report their types to the mechanism and the outcomes of the designed game realize. The timing of the game is similar to that in Myerson (1983) and Maskin and Tirole (1990, 1992).

By Myerson (1983) it is without loss of generality to focus on mechanisms that will not convey information to the buyer; all types of the seller choose the same mechanism. This

[3]Alternatively we can consider a single divisible good. We could also use a model where the seller's ex-post utility is given by $p - v_S(t)(1-q)$ without affecting our results. However, by sticking to the canonical model, we can also interpret q as the probability of trade.

property is called the *Inscrutability Principle* of Myerson (1983). The justification of this claim is based on the fact that the seller does not need to communicate any information to the buyer by her choice of mechanism, because she can always build such communication into the process of the mechanism itself.

We focus on *safe* mechanisms that are incentive compatible and individually rational for the buyer and the seller even thought the seller's type was common knowledge. Hence, by the Revelation and Inscrutability Principles, the seller's optimization problem can be written as follows:

$$\text{Program I: } \max_{\Gamma\in\mathcal{G}} \mathbb{E}_{t_B}\left(u_S(x(t);t)\right) \text{ for all } t_S \in T_S \tag{OBJ}$$

subject to

$$\mathbb{E}_{t_B}\left(u_S(x(t);t)\right) \geq \mathbb{E}_{t_B}\left(u_S(x(t_B,t'_S);t)\right) \qquad \text{for all } t_S, t'_S \in T_S \tag{S-IC}$$

$$\mathbb{E}_{t_B}\left(u_S(x(t);t)\right) \geq 0 \qquad \text{for all } t_S \in T_S \tag{S-IR}$$

$$u_B(x(t);t) \geq u_B(x(t'_B,t_S);t) \qquad \text{for all } t_B, t'_B \in T_B \tag{B-EPIC}$$

$$u_B(x(t);t) \geq 0 \qquad \text{for all } t_B \in T_B \tag{B-EPIR}$$

with given off-equilibrium beliefs of the buyer. Constraints (S-IC) and (S-IR) are the seller's interim incentive compatibility and participation constraints, respectively, whereas constraints (B-EPIC) and (B-EPIR) are the buyer's ex-post incentive compatibility and participation constraints, respectively. In other words, the program above gives us a *safe* mechanism $\Gamma \in \mathcal{G}$ that maximizes each seller's interim payoffs subject to the incentive compatibility and individual rationality constraints. Since the same mechanism is optimal for each seller type, all sellers choose this mechanism, and thus the choice of the mechanism does not reveal any information about the sellers' types.

Let the buyer's ex-post utility with a given DRM $\Gamma = (T, x)$ be given by the following value function

$$V_B(t) = \max_{t'_B\in T_B}\left(v_B(t)q(t'_B,t_S) - p(t'_B,t_S)\right), \tag{4.3}$$

for all $t \in T$. Similarly, let the seller's interim utility be given by the following value function

$$V_S(t_S) = \max_{t'_S\in T_S} \mathbb{E}_{t_B}\left(p(t_B,t'_S) - v_S(t)q(t_B,t'_S)\right), \tag{4.4}$$

for all $t_S \in T_S$.

Program I can equivalently be written as follows.

Lemma 4. *The seller-optimal safe mechanism is given by the following maximization problem:*

$$\textit{Program II:} \max_{q:T\to[0,1]} \mathbb{E}_{t_B}\big(J(t)q(t) - V_B(\underline{t_B}, t_S)\big) \textit{ for all } t_S \in T_S \tag{4.5}$$

subject to

$$\mathbb{E}_{t_B}\left(J(t_B, t'_S)q(t_B, t'_S) - \alpha_S \int_{t'_S}^{\overline{t_S}} q(t_B, s)ds\right) = V_S(\overline{t_S}) + V_B(\underline{t_B}, t'_S) \tag{4.6}$$

$$\mathbb{E}_{t_B}\left(q(t_B, \cdot)\right) \textit{ nonincreasing}, \tag{4.7}$$

$$q(\cdot, t'_S) \textit{ nondecreasing}, \tag{4.8}$$

$$p(t_B, t'_S) = v_B(t_B, t'_S)q(t_B, t'_S) - V_B(\underline{t_B}, t'_S) - \int_{\underline{t_B}}^{t_B} q(s, t'_S)ds \tag{4.9}$$

for all $t'_S \in T_S$*, where*

$$J(t) = v_B(t) - \frac{1 - F_B(t_B)}{f_B(t_B)} - v_S(t), \tag{4.10}$$

for all $t \in T$.

4.3 Optimal Safe Mechanisms

We divide this section in two. First, we analyze the model where only the seller has private information. After that, we focus on models with two-sided asymmetric information where both players have privately known types.

4.3.1 One-Sided Private Information

Let us start our analysis by stating the obvious: When both types t_B and t_S are common knowledge, the seller-optimal safe mechanism is the following.

Remark 1. *Assume that* t *is common knowledge, i.e.* $T = \{t_B\} \times \{t_S\}$*. The unique seller's utility maximizing safe mechanism* $\Gamma^F = \big(T, x^F\big) \in \mathcal{G}$ *is given by the allocation rule*

$$q^F(t) = \begin{cases} 1, & S(t) \geq 0 \\ 0, & \textit{otherwise}, \end{cases} \tag{4.11}$$

where $S(t) = v_B(t) - v_S(t)$, and by the transfer rule

$$p^F(t) = v_B(t) q^F(t), \tag{4.12}$$

for all $t \in T$.

That is, when the types are common knowledge and the trade is profitable for the seller, the seller receives the whole surplus $S(t)$ and the whole capacity of the goods are traded: $q^F(t) = 1$.

Remark 1 gives us an important benchmark for our next-presented main result. When the buyer's type t_B is common knowledge, but the type t_S is private information of the seller, the seller's utility maximizing safe mechanism is given by the following theorem.

Theorem 3. *Assume that t_B is common knowledge, i.e. $T_B = \{t_B\}$. The unique seller's utility maximizing safe mechanism $\Gamma^{FI} = \left(T, x^{FI}\right) \in \mathcal{G}$ is given by the allocation rule*

$$q^{FI}(t) = \begin{cases} \left(\frac{S(t)}{S(t_B, \underline{t}_S)}\right)^{\lambda}, & \textit{if } S(t) \geq 0 \textit{ and } S(t_B, \underline{t}_S) > 0 \\ 0, & \textit{otherwise,} \end{cases} \tag{4.13}$$

where $S(t) = v_B(t) - v_S(t)$ and $\lambda := \frac{\alpha_B}{\alpha_S - \alpha_B}$, and by the pricing rule is

$$p^{FI}(t) = v_B(t) q^{FI}(t), \tag{4.14}$$

for all $t \in T$. The mechanism Γ is unique only if $\alpha_S > 0$.

Remark 2. *If $\alpha_B = \alpha_S = \alpha$, then the allocation rule given by Theorem 3 becomes*

$$q^{FI}(t) = \begin{cases} \exp\left\{\frac{-\alpha(t_S - \underline{t}_S)}{t_B}\right\}, & \textit{if } S(t) \geq 0 \textit{ and } t_B > 0 \\ 0, & \textit{otherwise,} \end{cases} \tag{4.15}$$

for all $t \in T$.

Theorem 3 states that whenever the seller's type is unknown to the buyer, the optimal mechanism no longer allocates the whole capacity to the buyer as in Remark 1. The allocation rule given in (4.13) is strictly lower than unity for all types of seller $t_S > \underline{t}_S$ when $\alpha_B > 0$. Only a seller with the lowest possible type, $\underline{t}_S$, allocates all goods to the buyer and receives her first-best outcome. In other words, *the seller's private information results in a deadweight loss.* Moreover, the allocation rule (4.13) is strictly decreasing in the seller's type t_S for all $\alpha_B > 0$. That is, the higher the type of the seller, the lower the quantity offered, and the greater is the inefficiency. Furthermore, the optimal allocation rule (4.13) is increasing in the buyer's type: the more the buyer values the allocation, the greater the quantity allocated to the buyer.

To see these results, observe that the optimal allocation rule (4.13) is a function of the *total surplus*

$$S(t) = v_B(t) - v_S(t),$$

where $v_B(t)$ is the buyer's gain from the trade and $v_S(t)$ is the seller's (opportunity) cost of allocating the goods to the buyer. When $\alpha_S > \alpha_B$, the total surplus S decreases in t_S and the exponent $\lambda > 0$. As for $\alpha_S < \alpha_B$, the surplus S is increasing in t_S, but the exponent λ is negative. Hence, the weighted surplus ratio $\left(\frac{S(t)}{S(t_B,\underline{t_S})}\right)^{\lambda}$ is always below unity and decreases in t_S.

If the seller's type is not payoff relevant for the buyer — that is, $\alpha_B = 0$, then the mechanism given in Theorem 3 reduces back to full-information allocation where the seller gets the whole surplus and all goods are traded. In other words, if the information content of a proposed mechanism is irrelevant to the buyer, then there is no need to take this into account when designing the mechanism.

Remark 3. *If the buyer's valuation of the goods is independent of the seller's type, that is, $\alpha_B = 0$, then the seller receives her first-best outcome and the allocation is efficient.*

If the seller's type is relevant only for the buyer, that is, $\alpha_S = 0$, then the seller-optimal safe mechanism yields the same utility for all seller types. This is the only way to keep the mechanism incentive compatible for the seller.

Remark 4. *Assume that the seller values the goods at zero, i.e. $\alpha_S = 0$, then the seller-optimal safe mechanism given by Theorem 3 becomes*

$$q^{FI}(t) = \frac{v_B(t_B,\underline{t_S})}{v_B(t)} \tag{4.16}$$

with the price rule $p^{FI}(t) = v_B(t)q^{FI}(t) = v_B(t_B,\underline{t_S})$ for all $t \in T$. That is, all seller-types receive the same payoff $v_B(t_B,\underline{t_S})$ in equilibrium.

This seller-optimal safe mechanism is not unique. A mechanism $q(t) = 1$ and $p(t) = v_B(t_B,\underline{t_S})$ is clearly incentive feasible and yields the seller the same payoff as the mechanism given in Remark 4.16. However, it gives the buyer strictly positive payoff if $t_S > \underline{t_S}$. Hence, the seller-optimal safe mechanism given in Theorem 3 is unique only if $\alpha_S > 0$.

The optimal mechanism given in Theorem 3 is *ex post* individually rational for the buyer. This is an important feature of the optimal mechanism since this allows us to make the following interpretation. A seller of type t_S offers a contract $(q(t),p(t))$, where the quantity q^{FI} and the price p^{FI} are given by Theorem 3. Since this allocation is unique for each seller type $t_S \in T_S$, it perfectly reveals the seller's type to the buyer. Because this mechanism is incentive compatible and individually rational even though the seller's type was known (safe), it is feasible, and the buyer accepts the proposed allocation. In other words, high-type sellers can signal their private information to the buyer by decreasing their supply and asking high prices. Decreasing the quantity of goods leads to an inefficient allocation. However, in this way, the seller's signaling is credible and the buyer perfectly learns the seller's type. We can thus call the seller-optimal safe mechanism the least-cost separating equilibrium whenever $S(t) \geq 0$ for all $t \in T$.[4]

The equilibrium transfers given by (4.14) are increasing in t_B, and hence in q^{FI} (recall that q^{FI} is increasing in t_B). Since the seller does not always sell the whole capacity of the goods, the relevant magnitude for pricing is the price per quantity, i.e., the unit price for a single good. One can show by a straightforward, albeit arduous, calculus that the price per quantity $\frac{p^{FI}(t)}{q^{FI}(t)} = v_B(t)$ is nondecreasing in both t_B and t_S for $t \in T$ such that $S(t) \geq 0$. This further implies that $p^{FI}(t)/q^{FI}(t)$ is increasing in q^{FI} meaning that the seller is selling the goods with a *premium*; the more the buyer wants to buy the goods, the higher the price per quantity that the buyer needs to pay.

We know that $S(t)$ is increasing in α_B, which gives us a clear implication of how α_B affects the cutoff types when to allocate at all. This still does not give us the complete effect of α_B on q^{FI} (and so on p^{FI}) since $\left(\frac{S(t)}{S(t_B,\underline{t}_S)}\right)^\lambda$ is not decreasing in α_B for all $t \in T$. We can only conclude that as α_B reaches zero, the offered quantity converges point-wise to an indicator function $q^{FI}(t)|_{\alpha_B=0}$ (see Figure 4.1). That is, once α_B gets small and so the seller's private information less relevant to the buyer, the closer the optimal mechanism goes to a take-it-or-leave-it-offer.

For the seller's interdependence parameter α_S the connection is clear: $S(t)$ and $\left(\frac{S(t)}{S(t_B,\underline{t}_S)}\right)^\lambda$ both decrease in α_S and, therefore, q^{FI} and p^{FI} decrease in α_S. This is intuitive, since the more the seller values the goods, the less she wants to trade them to the buyer.

Figures 4.1 and 4.2 illustrate the comparative statics of the optimal mechanism of an informed seller. In Figure 4.1, the seller's type, t_S, and the interdependence parameter, α_S, are fixed, and the optimal allocation rule given by Theorem 3 is plotted as a function of the buyer's type, t_B. In Figure 4.2, the same plot is made in terms of t_S with fixed t_B and α_S. In both figures, the allocation rule q is depicted with six different interdependence parameters of the buyer: $\alpha_B \in \{0, 0.1, 0.3, 0.5, 0.7, 0.9\}$.

[4] Decreasing the quantity of the goods is similar kind of signaling as in Crawford and Sobel (1982) where the cost of signaling is created endogenously to achieve equilibria with partial sorting.

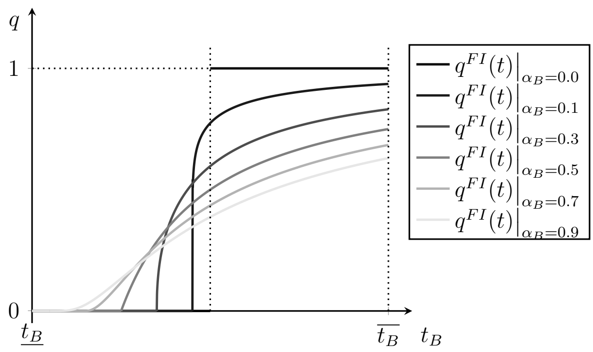

Figure 4.1: Optimal allocation rule with a fixed $t_S \in T_S$.

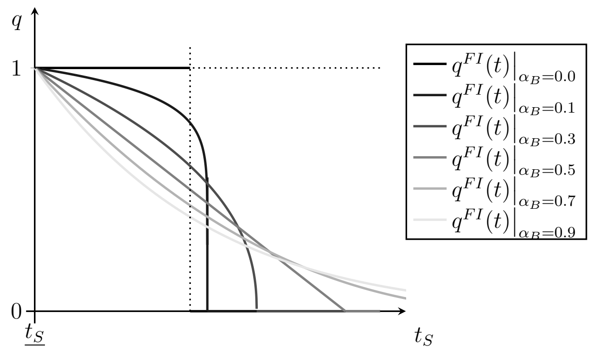

Figure 4.2: Optimal allocation rule with a fixed $t_B \in T_B$.

4.3.2 Two-Sided Asymmetric Information

Assume next that both players have private types. For concreteness, in this section we make the following assumption.

Assumption 3. $v_B(\overline{t_B}, t_S) \geq \alpha_S t_S \geq v_B(\underline{t_B}, t_S) - \frac{1}{f_B(\underline{t_B})}$ *for all* $t_S \in T_S$.

The first inequality in Assumption 3 guarantees that for all types of sellers, it is profitable to trade *per se*; there are some buyers who are willing to buy from any seller. The latter inequality says that the virtual valuation of the lowest-type buyer is smaller than the seller's own valuation, and hence it is never optimal to allocate for the lowest-type buyer. For example, if $\underline{t_B} = 0$, $\alpha_S \geq \alpha_B$, and $\overline{t_B} \geq (\alpha_S - \alpha_B)\overline{t_S}$, Assumption 3 is satisfied since $f_B(t_B) > 0$ for all $t_B \in T_B$.

Let us first state the well-known benchmark: the seller-optimal safe mechanism when the seller's type is known but the buyer is informed. This mechanism can be implemented by the canonical take-it-or-leave-it-offer (see Myerson (1981) and Mussa and Rosen (1978)).

Corollary 5. *Assume that t_S is common knowledge, i.e. $T_S = \{t_S\}$. Under Assumption 3, the seller-optimal safe mechanism $\Gamma^S = \left(T, x^S\right)$ is given by the allocation rule*

$$q^S(t) = \begin{cases} 1, & \text{if } t_B \geq b^S(t_S), \\ 0, & \text{otherwise} \end{cases} \tag{4.17}$$

and the price rule $p^S(t) = v_B(b^S(t_S), t_S)q^S(t)$, where $b^S : T_S \to T_B$ solves

$$v_B(b^S(t_S), t_S) - \frac{1 - F_B(b^S(t_S))}{f_B(b^S(t_S))} - \alpha_S t_S = 0 \tag{4.18}$$

for all $t_S \in T_S$ such that $b^S(\cdot)$ is non-decreasing.

When both players are informed, the seller's expected utility-maximizing safe mechanism is characterized by Theorem 4.

Theorem 4. *Under Assumptions 3, the seller-optimal safe mechanism $\Gamma^{SI} = \left(T, x^{SI}\right)$ is given by the allocation rule*

$$q^{SI}(t) = \begin{cases} 1, & \text{if } t_B \geq b^{SI}(t_S), \\ 0, & \text{otherwise} \end{cases} \tag{4.19}$$

and the price rule $p^{SI}(t) = v_B(b^{SI}(t_S), t_S)q^{SI}(t)$, where $b^{SI} : T_S \to T_B$ is strictly increasing in t_S and solves the seller's incentive feasibility conditions (4.6) and (4.7). Moreover, if $\alpha_i > 0$ for both $i \in \{B, S\}$, then b^{IS} is always greater than b^S which given by Corollary 6 except at $\underline{t_S}$ when they are equal.

If b^{SI} is differentiable almost everywhere, then it is given by the following differential equation

$$v_B(b^{SI}(t_S), t_S) - \frac{1 - F_B(b^{SI}(t_S))}{f_B(b^{SI}(t_S))} \left[1 + c(t_S)\right] - \alpha_S t_S = 0 \tag{4.20}$$

for all $t_S \in T_S$ and $b^{SI}(\underline{t_S}) = b^S(\underline{t_S})$ where

$$c(t_S) = \frac{\alpha_B}{\frac{\partial}{\partial t_S} b^{SI}(t_S)} \tag{4.21}$$

is the signaling cost such that $c(\underline{t_S}) = 0$.

This characterization helps us to understand the role of the seller's private information: when b^{SI} is differentiable, the only difference from the optimal mechanism given by Corollary 6 is the term $c(t_S)$ that we call *signalling costs*. The function c is the ratio between the buyer's interdependence parameter α_B and the derivative of the cutoff type b^{SI}. That is, the more relevant is the seller's private information for the buyer, the more the seller needs to invest in credible signalling. The signaling costs are strictly positive for all $t_S > \underline{t_S}$ when $\alpha_B > 0$.

Equation (4.20) represents the net virtual valuation which measures the surplus that can be extracted from that buyer. In equation (4.20), the buyer's information rent term $\frac{1-F_B(t_B)}{f_B(t_B)}$ is multiplied by the signaling costs. This signifies the fact that the seller's private information comes with a cost; the seller needs to give up on some of her surplus extraction in order to signal her private information to the buyer. This has the following interpretation. Due to the strictly positive signaling costs, it is straightforward to see that $b^{SI}(t_S) > b^S(t_S)$ for all $t_S > \underline{t_S}$ when $\alpha_B > 0$. That is, the seller finds it optimal to increase the prices p^{SI} to signal her type to the buyer. This is in the stark contrast with Theorem 3: If there are two-sided private information, then it is optimal for the seller to engage in price-signaling instead of quantity-signaling.

If the buyer's utility is independent of the seller private information, then the seller does no need to engage in signaling and the signaling costs are zero.

Remark 5. *If $\alpha_B = 0$, the signaling costs are zero and all sellers receive their second-best allocations given by Corollary 5.*

The signaling costs for the lowest-type seller are zero. The rationale for this observation is the following. It is never optimal for a seller to try to mimic the lowest-type seller because this would lead to the lowest transfers. Indeed, all seller have incentives to report upwards and therefore the lowest-type seller does not need to invest in costly signaling.

Remark 6. *Property $b^{SI}(\underline{t_S}) = b^S(\underline{t_S})$ guarantees that the lowest-type seller receives her second-best allocation given by Corollary 6.*

If the seller does not value the goods at all, we observe the similar effects as in Remark 4: the only mechanism that is incentive compatible for the seller is the one that gives all the seller equal payoffs.

Corollary 6. *Assume that the seller values the goods at zero, that is, $\alpha_S = 0$. Then all the seller types receive the same payoff as the lowest-type seller who gets her second best allocation. That is, the optimal cutoff type for the buyer $b(t_S)$ solves*

$$\int_{b^{SI}(t_S)}^{\overline{t_B}} J(t) dF_B(t_B) = V_S(\underline{t_S}). \tag{4.22}$$

Again this mechanism can be implemented in a couple of ways: (1) by solving b^{SI} from (4.22) or from (4.20), which gives as the least-cost separating equilibrium or (2) by making a take-it-or-leave-it-offer $p^{SI}(t) = b^S(\underline{t_S}) + \alpha_B \underline{t_S}$ to the buyer, which gives us a pooling equilibrium. Both mechanisms yield the each seller type the same expected payoff, but the latter gives the buyer always a weakly greater payoff than the first one. We illustrate this by an example.

Example 6. *Assume that $F_i(t_i) = t_i$ on $[0, 1]$ for $i \in \{B, S\}$ and $\alpha_B = 1$ and $\alpha_S = 0$. Now the net virtual valuation becomes*

$$J(t) = v_B(t) - \frac{1 - F_B(t_B)}{f_B(t_B)} - v_S(t) = 2t_B - 1 + t_S,$$

for all $t \in T$.

The optimal cutoff can be solved from differential equation (4.20):

$$b(t_S) + t_S - (1 - b(t_S))(1 + 1/b'(t_S)) = 0. \tag{4.23}$$

This has a unique increasing solution

$$b^{SI}(t_S) = \frac{1}{2}\left(1 - t_S + \sqrt{c_1 + t_S(t_S + 2)}\right) \tag{4.24}$$

where c_1 is a constant. We know that initially $b^{SI}(0) = b^S(0) = \frac{1}{2}$ (which can be solved from Corollary 6). Hence, we must have $c_1 = 0$.

Alternatively, we can solve b^{SI} from

$$\int_{b^{SI}(t_S)}^{1} (2t_B - 1 + t_S)dt_B = \frac{1}{4}, \tag{4.25}$$

from where we can solve the same cutoff type as above:

$$b^{SI}(t_S) = \frac{1}{2}\left(1 - t_S + \sqrt{t_S(t_S + 2)}\right) \tag{4.26}$$

for all $t_S \in [0, 1]$.

The equilibrium payoffs become:

$$V_S(t_S) = \frac{1}{4} \tag{4.27}$$

$$V_B(t) = t_B - b^{SI}(t_S). \tag{4.28}$$

However, if all the seller just pooled to the lowest-type seller's mechanism, then they would *receive payoffs:*

$$V_S(t_S) = \frac{1}{4} \tag{4.29}$$

$$V_B(t) = t_B - b^{SI}(0), \tag{4.30}$$

where $b^{SI}(0) = \frac{1}{2} < b^{SI}(t_S)$ *for all* $t_S > 0$.

Let us give a short example that summarizes and illustrates our findings under different kinds of information structure.

Example 7. *Consider a symmetric model:* $F_i(t_i) = t_i$ *on* $[0,1]$ *and* $\alpha_i = 1$ *for* $i \in \{B, S\}$. *Now the net virtual valuation becomes*

$$J(t) = v_B(t) - \frac{1 - F_B(t_B)}{f_B(t_B)} - v_S(t) = 2t_B - 1$$

for all $t \in T$. *This is independent of* t_S *and non-negative for for all* $t_B \geq \frac{1}{2}$.

Let us derive the seller-optimal safe mechanisms in the following four information structures: (1) full-information, (2) informed buyer, (3) informed seller, and (4) informed buyer and seller, respectively.

CASE 1. *(Full Information.) Suppose that both* t_B *and* t_S *are common knowledge. Then the seller-optimal mechanism is to sell the whole capacity of the goods at a price* $v_B(t)$.

CASE 2. *(Informed Buyer.) Assume that* t_B *is private information of the buyer who knows the seller's type* t_S. *By Corollary 5 we know that it is optimal for the seller to ask price* $v_B(2, t_S) = 1/2 + t_S$. *That is, all buyer's that have type* t_B *greater than* t_B *accept the offer.*

CASE 3. *(Informed Seller.) Let us then focus on the case where* t_B *is common knowledge and* t_S *is private information of the seller. By Theorem 3 and Remark 2 the seller-optimal safe mechanism is given by the allocation rule*

$$q(t) = \exp\{-\frac{t_S}{t_B}\} \tag{4.31}$$

and the price per quantity $p(t)/q(t) = t_B + t_S$ *for all* $t \in T$. *From this we observe that the higher the seller's type, the less she offers the goods to the buyer. The trade occurs with probability 1 since the buyer's valuation is always greater than the seller's valuation, but the seller's private information rent results in inefficiency: the whole capacity is not traded.*

A seller of the lowest type sells the entire capacity at a price t_B. *The highest-type seller offers quantity* $q(t_B, 1) = \exp\{-\frac{1}{t_B}\} \in [0, 1/e]$ *at unit prices* $p(t_B, 1)/q(t_B, 1) = t_B + 1 \in [1, 2]$.

CASE 4. *(Informed Buyer and Seller.) Assume that* t_i *is private information of Player* i. *From Theorem 4 we observe that* b^{SI} *is given by the following differential equation:*

$$b^{SI}(t_S) - (1 - b^{SI}(t_S))\left(1 + \frac{1}{\frac{\partial}{\partial t_S} b^{SI}(t_S)}\right) = 0 \tag{4.32}$$

or

$$\frac{\partial}{\partial t_S} b^{SI}(t_S) = \frac{1 - b^{SI}(t_S)}{2b^{SI}(t_S) - 1} \tag{4.33}$$

for all $t_S \in (0, 1)$. *Since this is a continuous first-order nonlinear differential equation with an initial value* $b^{SI}(0) = b^S(0) = \frac{1}{2}$, *it has the following unique solution:*

$$b^{SI}(t_S) = 1 + \frac{1}{2} W\left(2\exp\{c_1 - t_S\}\right), \tag{4.34}$$

where W *is the Lambert W function. We know that* $W(-1/e) = -1$ *and hence our cutoff becomes*

$$b^{SI}(t_S) = 1 + \frac{1}{2} W\left(-\exp\{-1 - t_S\}\right), \tag{4.35}$$

for all $t_S \in T_S$ *by the initial condition. Note that*

$$\frac{\partial}{\partial t_S} b^{SI}(t_S) = \frac{-W\left(-\exp\{-1 - t_S\}\right)}{2(1 + W\left(-\exp\{-1 - t_S\}\right))} \longrightarrow +\infty \tag{4.36}$$

as $t_S \to 0$ *since* $W(-1/e) = -1$. *That is,* $\lim_{t_S \to 0} c(t_S) = 0$ *and hence* $b^{SI}(0) = b^S(0) = \frac{1}{2}$.

The cutoff functions b^S *and* b^{SI} *are illustrated in Figure 4.3. From here we observe that* b^{SI} *increases first concavely indicating that the signaling costs are also concavely increasing in* t_S.[5]

[5] Note that

$$\frac{\partial^2}{(\partial t_S)^2} b^{SI}(t_S) = \frac{W\left(-\exp\{-1 - t_S\}\right)}{2(1 + W\left(-\exp\{-1 - t_S\}\right))^3} \leq 0$$

for all $t_S \in T_S$.

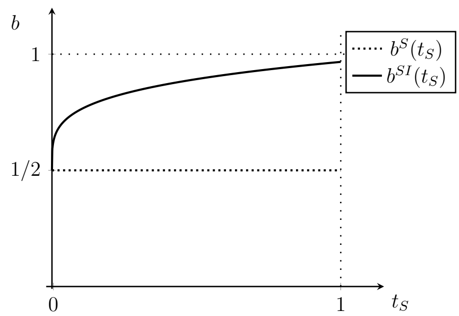

Figure 4.3: Optimal cutoff functions.

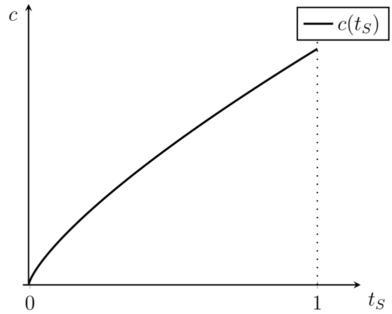

Figure 4.4: Signalling costs.

4.4 Trading with a Mediator

Suppose that there is a trustworthy mediator (a broker) who arranges the trade. Assume that the mediator is able to verify the seller's type without any extra costs and always reveals it honestly to the buyer. For concreteness, let the buyer be uninformed. The following arguments are generalized to the two-sided private information model.

The mediator must design a mechanism that gives the seller weakly better payoff than the mechanisms given in Theorem 3, or otherwise the seller arranges the trade by herself. However, the highest expected payoff that the mediator can receive from the trade is given by the mechanisms given in Remark 1. Therefore, the mediator can make the buyer a take-it-or-leave-it offer $v_B(t) = t_B + \alpha_B t_S$ and pay the seller $r(t) + \alpha_B t_S$, where $r(t)$ satisfies

$$r(t) + \alpha_B t_S - \alpha_S t_S \geq S(t) q^{FI}(t) \tag{4.37}$$

where q^{FI} is given by Theorem 3. That is, in the mediated mechanism the whole capacity of goods is traded and there is no deadweight loss.

When receiving $r(t_S) + \alpha_B t_S$ from the mediator, the seller receives at least the same utility that she would have received by settling the trade herself. When the buyer receives the offer from the mediator who reliably verifies the seller's type, she accepts the offer. The mediator profits the spread between the price she receives from the buyer and the price she pays to the seller: $t_B - r(t) \geq 0$.[6]

The same reasoning applies to the case where the buyer has a private type. In this case, the mediator makes a take-it-or-leave-it offer $b^S(t_S) + \alpha_B t_S$ to the buyer and pays the seller a price that is at least as high as $b^{SI}(t_S) + \alpha_B t_S$. This is a profitable deal for all parties.

Proposition 10. *Outsourcing the trade to a third party is a weakly dominant strategy for the seller ex ante. The optimal mechanism designed by a trustworthy mediator who can verify the seller's type is a take-it-or-leave-it offer given by Corollary 6 (or Remark 1 under one-sided asymmetric information) to the buyer with a brokerage that is smaller than the difference between the seller's payoffs in mechanisms given by Corollary 6 and Theorem 4 (or between Remark 1 and Theorem 3 under one-sided private information).*

That is, if the seller has the ability to perfectly disclose her private information via the mediator to the buyer at some cost, then it is profitable for the seller to do so if the brokerage is sufficiently low. If the verification is costly, then both the mediator and the seller accept the mediated mechanism in which the expected brokerage is greater than the expected verification costs but smaller than the expected difference between the seller's payoffs in mechanisms given by Corollary 6 and Theorem 4 (or between Remark 1 and Theorem 3 under one-sided private information).

4.5 Related Literature

One of the closest articles to ours is Koessler and Skreta (2016). Koessler and Skreta consider a bilateral trade setup in which the buyer's valuation of the object is a function of the buyer's and seller's types. The types are private information of the players, which leads to an informed seller problem similar to our setup. Koessler and Skreta assume that the value of the object for the seller is type-independent and hence does not affect the seller's incentives (in our model $\alpha_S = 0$). Koessler and Skreta (2016) show that in an

[6] Since the mediator can verify the seller's type, the incentive compatibility for the seller can be enforced by giving the seller 0 payment if $t'_S \neq t_S$ and $r(t_S) + \alpha_B t_S$ if $t'_S = t_S$.

ex-ante revenue-maximizing equilibrium, the seller benefits from her private information.[7] This result holds for a larger set of equilibrium allocations than strong solutions, since the ex-ante optimal allocation is not safe in general.

Koessler and Skreta (2019) study an informed-principal problem where the valuations of the seller and the buyer are interdependent similarly to our setup. They show that when the seller's information can be certified, there is an ex ante profit-maximizing selling procedure that is an equilibrium of the mechanism proposal game. Certifiability has two effects: on the one hand, it relaxes the seller's incentive constraint for certifiable reports and, on the other hand, makes deviations from noncertifiable reports more effective: a seller with high type has strong incentives to reveal that her product is valuable for the buyer. Due to the costless certifiability, the seller can always achieve at least as good outcome as in the case where her type was common knowledge.

Nishimura (2022) extends some of the results by Maskin and Tirole (1992) for bilateral asymmetric information. Nishimura's model is slightly more general than ours but coincides in two respects: (i) the utility functions are quasilinear in transfers, and (ii) the valuation functions of the players are additively separable in types. He characterizes the *RSW* (Rotchild-Stiglitz-Wilson) allocations for arbitrary posterior beliefs and provides conditions under which the *RSW* allocations are undominated. In other words, Theorem 2 in Nishimura (2022) states a necessary and sufficient condition under which a strong solution exists. Moreover, Nishimura shows that the equilibrium allocations passing the *intuitive criterion* (see Cho and Kreps (1987)) are interim-payoff-equivalent to the *RSW* allocations.

Mechanism design with an informed principal has been first studied by Myerson (1983) on whose shoulders we are standing in this paper. Myerson introduces several pivotal solution concepts for our analysis, such as safe mechanisms, strong solutions, and inscrutability of the principal. While Myerson analyzes a general model of multiple agents, Maskin and Tirole (1990, 1992) study a single agent setup as in this paper. Maskin and Tirole (1990) assume private values and Maskin and Tirole (1992) common values. Maskin and Tirole (1990) shows that in the quasilinear case with *private* values, the principal neither gains nor loses if her type is revealed to the agent before the reporting stage (Proposition 11), whereas Maskin and Tirole (1992) argue that this does no longer hold with common values.

Mylovanov and Tröger (2012) generalize the approach of Maskin and Tirole (1990) in a private values setup and establish the existence of equilibria. Mylovanov and Tröger (2014) derive the equilibrium allocation characterization by imposing an additional structure of

[7] In the language of Myerson (1983), the set of ex-ante optimal allocations coincides with the set of *core* mechanisms.

transferable utility to Mylovanov and Tröger (2012).[8]

In an informed-principal problem, the contract (mechanism) is designed after the principal privately learns her type or signal (interim contracts). Natural benchmarks to interim contracts used in the literature are ex-ante and ex-post contracts — that is, contracts designed before the principal learns her type or signal and contracts designed before the agents learn the principal's private information, respectively. The equivalence between the outcomes of these three contracts (ex-ante, interim, and ex-post) is shown in many *private values* environments: first in Maskin and Tirole (1990) with risk-neutral players and transferable utilities and later, for example, in Tan (1996), Yilankaya (1999), Skreta (2011), and Mylovanov and Tröger (2014). However, Fleckinger (2007) and Mylovanov and Tröger (2014) show that the equivalence does not generally hold in private-value models.[9] One of the main contributions of Maskin and Tirole (1992) is that they show that the equilibrium set of a mechanism selection game coincides with the set of allocations that weakly dominate the *RSW* allocations. That is, if a *RSW* allocation is dominated by some other mechanism, then the *RSW* mechanism gives the seller the worst outcome of the mechanism selection game.

Cella (2008) studies mechanism selection by an informed principal and a single agent with correlated types. Cella shows that the principal can extract extra information rent from the agent in comparison with the ex-post contracts. Skreta (2011) considers optimal information disclosure by an informed principal who maximizes her expected revenue after observing a vector of signals correlated with the agents' valuations. Skreta shows that under general allocation environments under *agents with interdependent values* it is optimal for the principal to disclose no information. This is in stark contrast to our results which say that under a slight interdependence in *the principal's and the agent's* values, it is optimal for the principal to fully disclose her private information. Based on our interpretation, that arises from the assumption that the seller's valuation for the object is zero in Skreta (2011), whereas in our model this is not the case.

Severinov (2008) provides conditions under which an ex-post efficient solution exists in an informed principal problem under interdependent values (among all players). Balkenborg and Makris (2015) studies undominated mechanisms designed by an informed principal who has common values with the agent.

Jullien and Mariotti (2006) consider a second-price auction with an informed seller who announces an informative reserve price in advance. Jullien and Mariotti characterize

[8]Utility is transferable if one player can transfer part of its utility to another player without any additional cost.

[9]Mylovanov and Tröger (2012, 2014) provide a comprehensive survey of the informed principal literature with private values.

the equilibria of the game (see Cai et al. (2007) and Tsuchihashi (2020) for reserve price signaling). Zhao (2018) studies optimal auction design by an informed seller and observes that reserve price signaling is optimal for the seller. Both, Jullien and Mariotti (2006) and Zhao (2018), find that the optimal reserve price is higher than that in the full-information case where the seller's type is common knowledge. In a multi-agent model, Zhao (2018) shows that there exists a solution to the optimization problem in (4.5) such that the seller's interim allocation probability is strictly decreasing in t_S and strictly less than that in the optimal mechanism when the seller's type t_S is common knowledge except for the lowest-type seller (Zhao (2018), Theorem 1 and Proposition 4).

Informed principal problems in moral-hazard environments are studied, for instance, by Beaudry (1994), Jost (1996), Benabou and Tirole (2003), Kaya (2010), and Wagner et al. (2015). For more recent studies in this field, see Mekonnen (2021) and Clark (2022a and 2022b). Particularly, Clark (2022b) shows that there can be equilibria that do not principal-payoff-dominate the optimal safe outcomes when moral hazard is present.

There is extensive literature on auctions and mechanism design with interdependent values by an *uninformed* seller. The revenue rankings of auctions with interdependent values between buyers are pioneered by Milgrom and Weber (1982). Crémer and McLean (1985, 1988) show that the seller is capable of extracting the full surplus from the buyers if the valuations of the buyers are interdependent (1985) or the signals of the buyers' valuations are correlated (1988).[10] This mechanism is usually referred to as the generalized Vickrey-Clarke-Groves mechanism, which is later studied, for example, by Ausubel (1999), Dasgupta and Maskin (2000), and Perry and Reny (2002). Their focus is on efficient design under interdependent values as in Holmström and Myerson (1983), Jehiel and Moldovanu (2001), Fieseler et al. (2003), Mezzetti (2004), and Li (2017). Revenue-maximizing mechanisms with interdependent values are studied, e.g., in Myerson (1981) and Roughgarden and Talgam-Cohen (2016). For bargaining with interdependent values, one can see, for instance, Deneckere and Liang (2006) and Fuchs and Skrzypacz (2013). The crucial feature in these bargaining models is typically that the uninformed party makes all the offers.

[10] McAfee et al. (1989) assume that the signals are continuously distributed and that the buyers assign the same value to the object. They show that almost all surplus can be extracted also in this case. Myerson (1981) was the first to point out that the full surplus extraction may be possible if the signals of the buyers' valuations are correlated.

4.6 Discussion

In this paper we have analyzed bilateral trade with an indormed seller. We observe that if a seller has payoff-relevant information for a buyer, then the seller finds it optimal to decrease the offered quantity in order to credibly signal her type to the buyer. If the seller's type is common knowledge, then the optimal mechanism is a take-it-or-leave-it offer for the whole capacity.

The model analyzed here is adjustable for many applications. Next, we exemplify how our results generalize to explain some well-known economic phenomena.

Market for Lemons. In the spirit of Akerlof (1970), consider a market for goods of unknown quality t_S. There are buyers who are willing to buy a unit quantity of the goods at a maximum price $v_B(t) = t_B + \alpha_B t_S$. That is, the ex-post valuation of the buyers is a combination of the buyers' private type t_B and the quality of the goods t_S, which is the private information of a seller. The seller's production costs are given by $v_S(t)q = t_S q$, and the production capacity is limited to unity (by normalization). Based on Theorem 3, only a seller who has the lowest quality, $\underline{t_S}$, supplies the entire capacity and sellers with higher qualities engage in production shortage; the supplier signals the quality of the goods to buyers by not producing the entire capacity. This signaling behavior gives one rationale for the deficiency of new high-quality or luxury goods.[11]

An alternative lemon story can be found in advertising models. Suppose that a monopolist chooses a share q of demand that she can capture by investing in advertising. The production costs of quality t_S are given by $v_S(t)q = t_S q$. According to Theorem 3, the most efficient way for a high-quality monopolist to signal its type is to engage a low level of demand-enhancing advertising. This behavior predicts a negative relationship between advertising and product quality (see Figure 4.2).[12]

In both examples, low-quality products are the most widely supplied. This indicates similar market outcomes as in Akerlof (1970): only lemons are traded, and the market for the highest quality goods is marginal. In this paper, we derive the optimal selling mechanism among all possible selling mechanisms (by the Revelation Principle). Therefore, our findings suggest that there is less trade with a high-quality seller in any kind of circumstances (no matter what the selling strategy is), and this is the best she can get even with the full monopoly power.

[11]For quality signaling via product scarcity see Stock and Balachander (2005). Stock and Balachander show that a high-quality monopoly firm that signals quality by inducing shortage can make more profit than using price alone.

[12]To some extent this prediction is supported by advertising literature. Many of these studies can be found in Bagwell (2007), which provides comprehensive empirical and theoretical surveys on advertising and quality literature.

Joint-Ownership. In the partnership models, there are two or more parties who initially own some shares of the company. The partners have private information about their valuations of the ownership and they want to renegotiate the shares of the company in order to achieve a profitable balance of ownership for all parties (see, for instance, Cramton et al. (1987), Jehiel and Pauzner (2006), and Loertscher and Wasser (2019)). Our model can be interpreted as the origin of this story: how the joint ownership was formed in the first place. Consider an entrepreneur who initially owns all shares of her company and is willing to sell a share q to a buyer at price p. The seller has private information about the profitability of the firm. The buyer's valuation of full ownership is given by a weighted sum of her own profitability t_B and the seller's private information about the firm's current profitability t_S. In this case, the seller wants to signal that the firm has high profitability by offering a partial ownership to the buyer; owning a share of the company, the seller can credibly reveal the profitability of the company to the buyer.[13]

Part-time Employment. Consider a worker (seller) who offers to work a share q of her working hours in a company (buyer). The productivity of the worker is partly known by the firm and partly known by the worker herself. The worker proposes a part-time contract of employment in order to signal that she is competent and profitable in her current position and, thus, also for the firm. In this way, the worker chooses the optimal allocation of her work load: she works share q of her working hours in the new company and share $1 - q$ in her current position. It is profitable for the worker to negotiate such a contract since there are positive externalities in the firm's production (recall that by interdependence $v_B(t) = t_B + \alpha_B t_S$) and so the company is willing to pay a high salary for the worker.

Trading with Externalities. Consider next a trade between two firms, $\{B, S\}$. The firms' profits are negatively interacted (e.g. due to competition) such that firm i's profit is given by $\hat{\pi}_i = r_i - \alpha_i \hat{\pi}_j$, where r_i is firm i's gains from the trade and $\hat{\pi}_j$ is the profit of the rival firm multiplied by an externality parameter α_i. By presuming that $\alpha_i \in [0, 1)$ we can solve the reduced-form profits of the trade: $\pi_i := (1 - \alpha_i \alpha_j)\hat{\pi}_i = r_i - \alpha_i r_j$ for both

[13] This example generalizes to applications in which there is a seller who owns an asset (or unity mass of assets), which profitability is private information of the seller. The seller determines the asset pricing scheme and decides the share of assets that she retains. The finance literature, e.g. the papers that consider Ininitial Public Offerings (IPOs), have documented signaling motives of entrepreneurs in these kinds of setup. Leland and Pyle (1977) were the first who theoretically showed that the retention of firm ownership by the entrepreneur can signal the firm's characteristics. Later, the empirical results by Downes and Heinkel (1982) show that firms in which entrepreneurs retain high fractional ownership do indeed have a higher value. These findings support the hypotheses of Leland and Pyle (1977) and our paper.

$i \in \{B, S\}$.[14]

Assume next that firm S is selling goods to firm B. The profits of the trade are given by $r_B(x;t) = qt_B - p$ and $r_S(x;t) = p - t_S q$, where an allocation $x = (q,p) \in X = [0,1] \times \mathbb{R}$ is given by the quantity $q \in [0,1]$ and the price of the goods $p \in \mathbb{R}$. The buyer's valuation of the goods, t_B, and the seller's marginal cost of producing the goods, t_S, are private information of the firms. The ex-post profits of the firms can be written as

$$u_B(x;t) = (t_B + \alpha_B t_S)\, q - (1 + \alpha_B)p =: v_B(t)q - p_B \tag{4.38}$$

$$u_S(x;t) = (1 + \alpha_S)p - (t_S + \alpha_S t_B)\, q =: p_S - v_S(t)q. \tag{4.39}$$

This is indeed the model we consider in this paper, except that the prices are given by $p_i = (1 + \alpha_i)p$ for both $i \in \{B, S\}$. Therefore, Theorem 3 can be generalized into this example: If the buying firm, B, receives negative externalities from the trade, the selling firm, S, finds it profitable to decrease the quantity offered to the firm B. In this way, the firm S can efficiently signal its production costs to the firm B.

Bargaining. Consider a continuous-time bargaining problem with transfers. Let the common discount factor be $\delta \in (0,1)$. Then reformulate the optimal mechanism given by Theorem 3 as follows: $(q(t), p(t)) = \left(\hat{q}(t; T(t))e^{-\delta T(t)}, \hat{p}(t; T(t))e^{-\delta T(t)}\right)$, where $\hat{q}(t;T)e^{-\delta T(t)}$ and $\hat{p}(t;T(t))e^{-\delta T(t)}$ are the discounted allocation probability and price in period $T(t)$ given by the report t, respectively. By setting $\hat{q}(t;T(t)) = 1$ whenever $q(t)$ is positive, we get $T(t) = -\log q(t)/\delta$. This gives us the following interpretation of the optimal mechanism: the seller rejects to sell the object until time $T(t) = -\log q(t)/\delta$ has passed. The price of the object in period $T(t)$ is given by $\hat{p}(t;T(t)) = p(t)/q(t)$. By postponing the trade, the seller can signal her private information for the buyer. This is credible because waiting is also costly for the seller by the discount factor.

Mineral Rights. Lastly, consider a government that owns the mineral rights to exploit an area for the minerals it harbors. There is a single potential firm whose value of the rights is determined by its own productivity t_B and the amount of minerals in the area, t_S, which is private information of the government. If the firm cannot verify the amount of minerals in the area prior to contracting, the government finds it optimal to signal this information to the firm by offering a joint ownership of the minerals. However, it is profitable for both parties to let the firm acquire information about the amount of

[14] This is essentially a model of socially interacted agents. The model can be found from Becker's (1974) seminal article "A Theory of Social Interactions", where Becker introduces an economic theory under social interactions. The theory incorporates a general treatment of interactions in the theory of consumer demand in order to explain, for instance, intrafamily relations, charity, merit goods, and envy and hatred.

minerals, and the government is even willing to pay for the acquisition of information if it is relatively inexpensive for her (Proposition 10).[15]

All the examples given above offer a simplistic and shallow standpoint to the applications, which are certainly more intrinsic than we have illustrated here. Our intention is not to give an unequivocal explanation to these phenomena but rather to highlight some possible implications of an informed principal to some of the well-known frameworks.

Appendix: Proofs

Proof of Lemma 4

Let the buyer's ex-post utility with a given DRM $\Gamma = (T, x)$ be given by the following value function

$$V_B(t) = \max_{t'_B \in T_B} \left(v_B(t) q(t'_B, t_S) - p(t'_B, t_S) \right). \tag{4.40}$$

By the standard envelope theorem argument (see Milgrom and Segal (2002)) we know that a DRM $\Gamma = (T, x)$ is ex-post incentive compatible for the buyer iff the equilibrium transfers are given by

$$p(t) = v_B(t) q(t) - V_B(\underline{t_B}, t_S) - \int_{\underline{t_B}}^{t_B} q(s, t_S) ds \tag{4.41}$$

and $q(\cdot, t_S)$ is nondecreasing for all $t_S \in T_S$.

The seller's interim utility is given by the following value function

$$V_S(t_S) = \max_{t'_S \in T_S} \mathbb{E}_{t_B} \left(p(t_B, t'_S) - v_S(t) q(t_B, t'_S) \right). \tag{4.42}$$

By the envelope theorem for the seller's value function, we know that $V'_S(t_S) = -\mathbb{E}_{t_B} \left(q(t_B, t_S) \right)$ almost everywhere (a.e.) at the truthful equilibrium. Then, by the fundamental theorem of calculus, the seller's interim equilibrium utility becomes the following

$$V_S(t_S) = V_S(\overline{t_S}) + \alpha_S \int_{t_S}^{\overline{t_S}} \mathbb{E}_{t_B} \left(q(t_B, s) \right) ds. \tag{4.43}$$

[15] On information acquisition in mechanism design and auctions with interdependent values see, for instance, Bergemann and Välimäki (2002, 2005) and Bergemann et al. (2009).

It is straightforward to show that a DRM $\Gamma = (T, x)$ is interim incentive compatible for the seller iff the seller's interim utility is given by (4.43), and $\mathbb{E}_{t_B}(q(t_B, \cdot))$ is nonincreasing.

By substituting the transfers (4.41) into the seller's value function (4.42), and interchanging the order of integration, we observe that the seller's equilibrium utility can be written as

$$V_S(t_S) = \mathbb{E}_{t_B}\left(J(t)q(t) - V_B(\underline{t_B}, t_S)\right), \tag{4.44}$$

where $J(t) = v_B(t) - \frac{1-F_B(t_B)}{f_B(t_B)} - v_S(t)$ is the *net virtual valuation* with given $t \in T$. This finishes the proof.

Proof of Theorem 3

Let us assume that t_B is commonly known — that is, $T_B = \{t_B\}$. Then the equilibrium transfers are given by (4.43):

$$p(t) = v_S(t)q(t) + V_S(\overline{t_S}) + \alpha_S \int_{t_S}^{\overline{t_S}} q(t_B, s)ds \tag{4.45}$$

for all $t \in T$. It is straightforward to show that a mechanism $\Gamma = (T, x)$ is incentive compatible for the seller iff the equilibrium transfers satisfy (4.45) and $\mathbb{E}_{t_B}[q(t_B, \cdot)]$ is non-increasing. Consequently, the seller's problem is to design a mechanism $\Gamma \in \mathcal{G}$ to maximize $p(t) - v_S(t)q(t)$ subject to (4.45), $q(t_B, \cdot)$ non-increasing, and the buyer's *ex-post* individual rationality constraint $v_B(t)q(t) - p(t) \geq 0$ for all $t \in T$. By substituting (4.45) into the buyer's participation constraint, we observe

$$\left(v_B(t) - v_S(t)\right)q(t) \geq V_S(\overline{t_S}) + \int_{t_S}^{\overline{t_S}} q(t_B, s)ds, \tag{4.46}$$

must hold for all $t \in T$. This implies that we must have $q(t) = 0$ for all $t \in T$ such that $v_B(t) < v_S(t)$. For $t \in T$ such that $v_B(t) \geq v_S(t)$, it is optimal for the seller to choose a mechanism such that (4.46) holds at equality; otherwise the seller could increase her payoffs by increasing the prices (see Zhao (2018) and Maskin and Tirole (1992)). The binding individual rationality constraint forms the following integral equation:

$$S(t)q(t) = V_S(\overline{t_S}) + \int_{t_S}^{\overline{t_S}} q(t_B, s)ds, \tag{4.47}$$

where $S(t) = v_B(t) - v_S(t)$ for all $t \in \{t' \in T : S(t') \geq 0\}$.

The equation in (4.47) is a linear Volterra's integral equation of the second kind for t_S

which has a *unique* continuous solution.[16] It is straightforward to verify that the following function is the solution to (4.47):

$$q(t) = \beta(t_B)S(t)^{\lambda}, \tag{4.48}$$

for some $\beta(t_B) \in \mathbb{R}$, where $\lambda := \frac{\alpha_B}{\alpha_S - \alpha_B}$ for $\alpha_S \neq \alpha_B$.

It is optimal for the seller to set $q(t)$ as high as possible, and so we must have $q(t_B, \underline{t_S}) = 1$ for $t_B \geq t_B^*(\underline{t_S}) := \inf\{t_B' \in T_B : S(t_B', t_S) \geq 0\}$ at the optimum. That is, the lowest-type seller receives the same payoff as in the case where her type was common knowledge (see, e.g., Maskin and Tirole (1992), Zhao (2018), and Nishimura (2022)). This gives us the boundary condition from which we can solve $\beta(t_B) = \frac{1}{J(t_B, \underline{t_S})^{\lambda}}$.

Consequently, the optimal safe mechanism with an uninformed buyer is given by

$$q(t) = \begin{cases} \left(\frac{S(t)}{S(t_B, \underline{t_S})}\right)^{\lambda}, & \text{if } J(t) \geq 0 \\ 0, & \text{otherwise,} \end{cases} \tag{4.49}$$

where $S(t) = v_B(t) - v_S(t)$ and $\lambda := \frac{\alpha_B}{\alpha_S - \alpha_B}$.

Lastly, write

$$\left(\frac{S(t)}{S(t_B, \underline{t_S})}\right)^{\lambda} = \left(1 + \frac{\frac{\alpha_B(\underline{t_S} - t_S)}{t_B - (\alpha_S - \alpha_B)\underline{t_S}}}{\lambda}\right)^{\lambda}. \tag{4.50}$$

We know that $\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x$, and so the expression in (4.50) converges to

$$\exp\left(\frac{-\alpha(t_S - \underline{t_S})}{t_B}\right) \tag{4.51}$$

when $\alpha_B \to \alpha_S = \alpha$ for $t_B > 0$.

Proof of Corollary 5

From the optimization problem given by Lemma 4, we can conclude that whenever t_S is common knowledge, i.e. $T_S = \{t_S\}$, the constraints (4.6) and (4.7) are redundant. Then the optimal mechanism can be solved point-wise from the optimization problem since the point-wise solution solution satisfies the monotonicity constraint (4.8).

[16] To be more precise, this can be a linear Volterra's integral equation of the third kind since if $t_S = t_S^*(t_B) := \sup\{t_S' \in T_S : S(t_B, t_S') \geq 0\}$ we may have $S(t) = 0$, which causes discontinuity to the kernel of the integral equation. However, since this is only a single point and both sides of (4.47) are zero at $t_S = t_S^*(t_B)$, this means that any q^* satisfies (4.47) at that point. See the original paper of Evans (1911) for a more detailed analysis of Volterra's integral equations of the second kind with discontinuous kernels.

Proof of Theorem 4

By Zhao (2018), Corollary 1 and Proposition 4, the seller-optimal mechanism is a cutoff-mechanism:

$$q(t) = \begin{cases} 1, & \text{if } t_B \geq b^{SI}(t_S), \\ 0, & \text{otherwise} \end{cases} \tag{4.52}$$

for all $t \in T$ and some continuous and stictly increasing function $b^{SI} : T_S \to T_B$. The pricing rule is given by (4.9). Note that by Assumption 3 the expected probability of trade is never unity for any seller type, i.e., $b^{SI} > \underline{t_B}$.

To see this property of the seller-optimal safe mechanism, consider the optimization problem given by Lemma 4 for an arbitrary $t_S \in T_S$. The objective function is linear and hence concave in q. Let $\mathcal{F}$ be the set of all functions $q : T \to [0,1]$ that satisfies conditions (4.6) and (4.8). Let $\mathcal{F}$ has the L^1 norm and endow it with the metric induced by this norm. The set $\mathcal{F}$ is convex since both these constraints are convex in q; the convex combination of two non-decreasing functions is non-decreasing, and the convexity of (4.6) is given by the fact that it is continuous and linear in q. The set $\mathcal{F}$ is also compact by Helly's Selection Theorem and the bounded convergence theorem of Lebesgue integration.

Hence, by the Extreme Point Theorem, a function $q \in \mathcal{F}$ takes the form of (4.52) for almost all $t \in T$. The fact that b^{SI} is continuous and stictly increasing comes from the fact that $\alpha_B \geq 0$. For more detailed arguments and proof of this result, see Zhao (2018).

By the standard optimality argument, we must have $V_B(\underline{t_B}, t_S) = 0$ for all $t_S \in T_S$; otherwise the seller could increase the equilibrium transfers. Then, by the assumption that b^{SI} is differentiable, the seller's incentive compatibility constraint is satisfied iff

$$\frac{\partial}{\partial m} \int_{b^{SI}(t_S+m)}^{\overline{t_B}} \left(v_B(t_B, t_S + m) - \frac{1 - F_B(t_B)}{f_B(t_B)} - \alpha_S t_S \right) dF_B(t_B) \Big|_{m=0} = 0 \tag{4.53}$$

or

$$v_B(b^{SI}(t_S), t_S) - \frac{1 - F_B(b^{SI}(t_S))}{f_B(b^{SI}(t_S))} \left[1 + \frac{\alpha_B}{\frac{\partial}{\partial t_S} b^{SI}(t_S)} \right] - \alpha_S t_S = 0 \tag{4.54}$$

for all $t_S \in T_S$.

Since $\frac{1-F_B(t_B)}{f_B(t_B)}$ strictly decreasing and non-negative, we must have $\frac{1-F_B(b^{SI}(\underline{t_S}))}{f_B(b^{SI}(\underline{t_S}))} > 0$. Moreover, by Zhao (2018) the seller-optimal safe mechanism satisfies $b^{SI}(\underline{t_S}) = b^S(\underline{t_S})$, where b^S is given by Corollary 6. These two conditions imply that we must have $\lim_{t_S \to \underline{t_S}} \frac{\partial}{\partial t_S} b^{SI}(t_S) = +\infty$ when $\alpha_B > 0$.

Lastly, by Assumptions 3 and $\frac{1-F_B(t_B)}{f_B(t_B)}$ strictly decreasing in t_B, we know that $b^{SI}(t_S) \in T_B$ for all $t_S \in T_S$. Hence, condition (4.54) is well-defined.

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