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2022-06

Hagger , R , Liu , C , Taskinen , J & Virtanen , J A 2022 , ' Toeplitz operators on the unit ball with locally integrable symbols ' , Integral Equations and Operator Theory , vol. 94 , no. 2 , 17 . <https://doi.org/10.1007/s00020-022-02695-3>

<http://hdl.handle.net/10138/347451>

<https://doi.org/10.1007/s00020-022-02695-3>

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Toeplitz Operators on the Unit Ball with Locally Integrable Symbols

R. Hagger, C. Liu, J. Taskinen and J. A. Virtanen

Abstract. We study the boundedness of Toeplitz operators T_ψ with locally integrable symbols on weighted harmonic Bergman spaces over the unit ball of \mathbb{R}^n . Generalizing earlier results for analytic function spaces, we derive a general sufficient condition for the boundedness of T_ψ in terms of suitable averages of its symbol. We also obtain a similar “vanishing” condition for compactness. Finally, we show how these results can be transferred to the setting of the standard weighted Bergman spaces of analytic functions.

Mathematics Subject Classification. Primary 47B35.

Keywords. Toeplitz operator, Harmonic Bergman space, Bergman projection, Boundedness, Compactness.

1. Introduction

Denote by dV the normalized n -dimensional Lebesgue measure on the unit ball \mathbb{B}_n of \mathbb{R}^n with $n \geq 2$. For $\lambda > -1$ and $p \geq 1$, let $b_\lambda^p = b_\lambda^p(\mathbb{B}_n)$ be the harmonic Bergman space consisting of all harmonic complex-valued functions in $L_\lambda^p = L^p(\mathbb{B}_n, dV_\lambda)$, where $dV_\lambda = c(n, \lambda)(1 - |x|^2)^\lambda dV$ and $c(n, \lambda)$ is a normalization constant (see (3.2)). We further write b^p for b_0^p and L^p for L_0^p .

The Toeplitz operator T_ψ with symbol ψ is defined by

$$T_\psi f = P_\lambda(\psi f), \quad (1.1)$$

where P_λ is the orthogonal projection of L_λ^2 onto b_λ^2 (see (3.3)), ψ is a measurable function on \mathbb{B}_n , and f is harmonic in \mathbb{B}_n . It is known that P_λ can be extended to a bounded projection from L_λ^p onto b_λ^p for $1 < p < \infty$ (see [3]). It follows that T_ψ is well defined and bounded on b_λ^p whenever ψ is bounded.

The first author was supported by the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Grant Agreement No 844451. The fourth author was supported in part by Engineering and Physical Sciences Research Council (EPSRC) Grant EP/T008636/1.

For $f \in L^1_\lambda$, $P_\lambda f$ is still well defined as a function (see (3.3) below), but we may have $P_\lambda f \notin L^1_\lambda$ in general.

The study of T_ψ on b^p goes back to [8], where it was proven for non-negative symbols $\psi \in L^1$ that T_ψ is bounded (compact) on b^p if and only if the averaging function

$$x \mapsto \frac{1}{|E_r(x)|} \int_{E_r(x)} \psi \, dV \tag{1.2}$$

is bounded on \mathbb{B}_n (vanishes as $|x| \rightarrow 1$), where $E_r(x) = \{y \in \mathbb{B}_n : |y - x| < r(1 - |x|)\}$ with $r \in (0, 1)$ and $|E_r(x)|$ denotes its volume. Further, it was shown that for symbols ψ continuous on $\overline{\mathbb{B}_n}$, T_ψ is compact on b^p if and only if $\psi = 0$ on $\partial\mathbb{B}_n$, which was generalized to b^2_λ in [8]. It is easy to see that these conditions for boundedness (compactness) can also be formulated in terms of boundedness (vanishing) of the Berezin transform of ψ or variants of the averaging function (1.2). Moreover, applying (1.2) to the positive and negative parts of $\operatorname{Re} \psi$ and $\operatorname{Im} \psi$ shows that if

$$\sup_{x \in \mathbb{B}_n} \frac{1}{|E_r(x)|} \int_{E_r(x)} |\psi| \, dV < \infty, \tag{1.3}$$

then T_ψ is bounded. However, the modulus in the integrand makes (1.3) far from being necessary if the symbol is not positive. In this paper we propose averages over different sets, certain spherical boxes, but also with the modulus outside of the integral; see (2.3). In particular, we give a new, weaker sufficient condition for the boundedness and compactness of Toeplitz operators on the weighted harmonic Bergman spaces b^p_λ in Theorems 2.3 and 2.5. In Corollary 2.6 we present the corresponding results in the case of weighted Bergman spaces A^p_λ of analytic functions on the unit ball of \mathbb{C}^n .

Most results about Toeplitz operators on b^p_λ are generalizations from the setting of analytic Bergman spaces A^p_λ using similar but also additional ideas related to non-analyticity. As in b^p_λ , characterizing bounded Toeplitz operators is an open problem in $A^p(\mathbb{D})$ (the unweighted Bergman space of the unit disk \mathbb{D} of the complex plane \mathbb{C}) even in the case $p = 2$. Compared with (1.3), much more general and weaker sufficient conditions were found for Toeplitz operators on $A^p(\mathbb{D})$ in [9] with further improvements in [10]. Roughly speaking, the conditions resemble (1.3), but the modulus appears outside the integral. This means that the modulus of a (wildly oscillating) symbol may be very large, but it may still induce a bounded operator due to cancellation in the integral; for an illustrative example, see [10]. Recently, the same idea was used to introduce weak BMO and VMO type conditions in [11]. Here, we prove similar results for the space b^p_λ of the unit ball with standard weights for the sake of maximal generality, which carries a number of technical challenges. We note that recently in [12] it was shown that the mentioned sufficient conditions in A^p are not necessary for the boundedness. We also refer to [12] for a concise account of the study of boundedness of Toeplitz operators on Bergman spaces.

As for the contents of this paper, the main results are formulated in Sect. 2 and their proofs are prepared in Sects. 3 and 4, which contain some

more necessary notation, definitions and preliminary lemmas. The proofs of the main results are completed in Sect. 5, and in Sect. 6 we construct examples of radial, oscillating symbols, the modulus of which may grow arbitrarily fast, but which still induce bounded Toeplitz operators.

2. The Main Results

In what follows we generalize the results of [9, 10] from the case of analytic functions in \mathbb{D} to the case of harmonic functions in \mathbb{B}_n and also consider weighted norms. Our analogous sufficient condition is a rather weak requirement of the boundedness of certain averages of ψ over spherical boxes (see Theorem 2.3). We present the main results, Theorems 2.3–2.5 and Corollary 2.6, with a minimal amount of notation.

It seems likely that analogous results hold for little Hankel operators on b_{λ}^p , too, but we do not consider this question here; cf. [10].

Let $\mathbb{Q}_n := [0, 1] \times [0, \pi]^{n-2} \times [0, 2\pi]$ and $\sigma : \mathbb{Q}_n \rightarrow \mathbb{B}_n$ be defined by

$$\begin{aligned} \sigma(r, \theta_2, \dots, \theta_{n-1}, \theta_n) &= (r \cos \theta_2, r \sin \theta_2 \cos \theta_3, r \sin \theta_2 \sin \theta_3 \cos \theta_4, \dots, \\ &\quad r \sin \theta_2 \cdots \sin \theta_{n-1} \cos \theta_n, r \sin \theta_2 \cdots \sin \theta_{n-1} \sin \theta_n). \end{aligned}$$

We note that σ is surjective, and injective almost everywhere (i.e. for almost every $x \in \mathbb{Q}_n$, $\sigma(x) = \sigma(y)$ implies $x = y$). For $x, y \in \mathbb{Q}_n$ we define the boxes

$$Q(x, y) := \{z \in \mathbb{Q}_n : z_j \in [\min\{x_j, y_j\}, \max\{x_j, y_j\}]\}$$

and $B(x, y) := \sigma(Q(x, y))$. In order to distinguish between $B(x, y) \subset \mathbb{B}_n$ and $Q(x, y) \subset \mathbb{Q}_n$, we call $B(x, y)$ a spherical box. For the following, we need to single out certain spherical boxes. Let

$$\mathcal{K} := \left\{ (m, k_2, k_3, \dots, k_n) \in \mathbb{N}_0^n : 0 \leq k_n \leq k_{n-1} \leq \dots \leq k_2 \leq 2^m - 1 \right\}.$$

For $k \in \mathcal{K}$ we define

$$\begin{aligned} C_k &:= [1 - 2^{-m}, 1 - 2^{-m-1}] \times [\frac{\pi}{2} k_2 2^{-m}, \frac{\pi}{2} (k_2 + 1) 2^{-m}] \times [\frac{\pi}{2} \frac{k_3}{k_2+1}, \frac{\pi}{2} \frac{k_3+1}{k_2+1}] \\ &\quad \times \cdots \times [\frac{\pi}{2} \frac{k_{n-1}}{k_{n-2}+1}, \frac{\pi}{2} \frac{k_{n-1}+1}{k_{n-2}+1}] \times [2\pi \frac{k_n}{k_{n-1}+1}, 2\pi \frac{k_n+1}{k_{n-1}+1}]. \end{aligned}$$

Note that $\bigcup_{k \in \mathcal{K}} C_k = [0, 1] \times [0, \frac{\pi}{2}]^{n-2} \times [0, 2\pi]$. If we add all the sets that can be obtained from the sets C_k by repeatedly using the reflections $\theta_2 \mapsto \pi - \theta_2, \dots, \theta_{n-1} \mapsto \pi - \theta_{n-1}$, we obtain a cover of \mathbb{Q}_n . Let \mathcal{Q} denote the collection of these sets. Enumerate the elements of \mathcal{Q} by Q_1, Q_2, \dots and define $B_j := \sigma(Q_j)$ for $j \in \mathbb{N}$. Then $\bigcup_{j \in \mathbb{N}} B_j$ covers \mathbb{B}_n and $B_j \cap B_k$ is a null set whenever $j \neq k$. If $|x| \in [1 - 2^{-m}, 1 - 2^{-m-1}]$ for all $x \in B_j$, then B_j is called a dyadic box of generation m . For a dyadic box B_j of generation m , we denote $B_j^* := B_j + 2^{-m-2} \mathbb{B}_n$. The collection of these spherical boxes B_j has the property that they are all of size comparable to 2^{-mn} and there is a constant $N \in \mathbb{N}$ such that every $x \in \mathbb{B}_n$ is contained in at most N of the sets B_j^* . We will prove these facts in Lemma 4.3 below.

We remark that it may be tempting and more natural to extend the decomposition of the unit disk in [9] to higher dimensions using boxes of the form

$$C_k = [1 - 2^{-m}, 1 - 2^{-m-1}] \times \prod_{j=2}^n [2^{-m}k_j\pi, 2^{-m}(k_j + 1)\pi] \subset \mathbb{Q}_n,$$

where $k_j = 1, \dots, 2^m - 1$ for $j = 2, \dots, n - 1$ and $k_n = 1, \dots, 2^{m+1} - 1$. After mapping them to \mathbb{B}_n and some reordering, these boxes also provide a countable, pairwise essentially disjoint cover $\{B_j : j \in \mathbb{N}\}$ of \mathbb{B}_n . However, it turns out that the size of these boxes is not always comparable to 2^{-mn} . This is due to the fact that the Jacobian of σ is singular if any of $\theta_2, \dots, \theta_{n-1}$ is equal to 0 or π . It also means that too many of the sets B_j^* overlap close to the singular points. As these two properties are crucial for our analysis, we had to be more careful in the definition of the sets B_j . Having said that, it is highly expected that many other decompositions also lead to the same result. Our choice is not special in that regard, but it is somewhat natural as it leads to a definition of generalized Toeplitz operators that is independent of the decomposition; cf. Theorem 2.4.

The set of locally integrable functions on \mathbb{B}_n will be denoted by L^1_{loc} . Note that L^1_{loc} does not depend on λ . The characteristic function of a measurable set $E \subset \mathbb{B}_n$ will be denoted by χ_E .

Definition 2.1. Let $\psi \in L^1_{loc}$, $1 < p < \infty$ and assume that the series

$$T_\psi f(x) := \sum_{j=1}^\infty T_\psi(\chi_{B_j} f)(x) = \sum_{j=1}^\infty P_\lambda(\psi \chi_{B_j} f)(x) \tag{2.1}$$

converges for almost every $x \in \mathbb{B}_n$ and all $f \in b^p_\lambda$. Then T_ψ is called a *generalized Toeplitz operator*.

As P_λ is bounded on L^p_λ , it is clear that $T_\psi f = P_\lambda(\psi f)$ whenever $\psi f \in L^p_\lambda$. In particular, if ψ is bounded, then T_ψ is just the usual Toeplitz operator as defined in the introduction. However, without any further assumptions it is a priori not even clear if $T_\psi f \in b^p_\lambda$. Our first main result is that T_ψ is a well-defined bounded linear operator under the “weak” Carleson-type condition (2.4).

For any measurable set $B \subset \mathbb{B}_n$ we denote the weighted volume by $|B|_\lambda := \int_B dV_\lambda$. Moreover, $|B| := |B|_0$.

Let $x^{(j)}$ and $y^{(j)}$ denote the smallest, respectively largest, element of Q_j with respect to the partial order

$$\begin{aligned} x \leq y \iff & \quad x_1 \leq y_1, \left| \frac{\pi}{2} - x_2 \right| \geq \left| \frac{\pi}{2} - y_2 \right|, \dots, \left| \frac{\pi}{2} - x_{n-1} \right| \geq \left| \frac{\pi}{2} - y_{n-1} \right|, \\ & \quad x_n \leq y_n \end{aligned} \tag{2.2}$$

imposed on \mathbb{Q}_n .

Remark 2.2. Note that for $x, y \in [0, 1] \times [0, \frac{\pi}{2}]^{n-2} \times [0, 2\pi]$ this is just the usual partial order of points in \mathbb{R}^n , which is then mirrored to all of \mathbb{Q}_n to account for the construction of the sets Q_j and B_j . In particular, the $x^{(j)}$ and $y^{(j)}$ are two opposite corners of Q_j and we have $B_j = B(x^{(j)}, y^{(j)})$. In the

following, we will only work in the subset $[0, 1) \times [0, \frac{\pi}{2}]^{n-2} \times [0, 2\pi]$ and then use that everything is symmetric around $\theta_2 \mapsto \pi - \theta_2, \dots, \theta_{n-1} \mapsto \pi - \theta_{n-1}$. This simplifies the notation significantly and also means that we do not have to worry about the slightly unusual order (2.2).

Given a function $\psi \in L^1_{\text{loc}}$, we define

$$\widehat{\psi}_j := \sup_{y \in B_j} \left| \int_{B(x^{(j)}, y)} \psi \, dV_\lambda \right|. \tag{2.3}$$

Theorem 2.3. *Let $\psi \in L^1_{\text{loc}}$, $1 < p < \infty$ and the family $(B_j)_{j \in \mathbb{N}}$ be as above. If there exists a constant $C_\psi \geq 0$ such that*

$$\widehat{\psi}_j \leq C_\psi |B_j|_\lambda \tag{2.4}$$

for all $j \in \mathbb{N}$, then the series (2.1) converges almost everywhere and in L^p_λ . Moreover, T_ψ defines a bounded linear operator on b^p_λ and there is a constant $C \geq 0$ independent of ψ such that $\|T_\psi\| \leq CC_\psi$.

For $\psi \in L^1_{\text{loc}}$ and $0 < \rho < 1$, we define $\psi_\rho(z) = \psi(z)$ for $|z| \leq \rho$ and $\psi_\rho(z) = 0$ for $\rho < |z| < 1$. Note that T_{ψ_ρ} is bounded on b^p_λ by the previous theorem (see also Lemma 3.2 below). Our next theorem gives an alternative definition of Toeplitz operators with L^1_{loc} -symbol that does not depend on the decomposition $(B_j)_{j \in \mathbb{N}}$.

Theorem 2.4. *Let $1 < p < \infty$ and $1/p + 1/q = 1$, and suppose that $\psi \in L^1_{\text{loc}}$ satisfies (2.4). Then*

$$T_\psi f = \lim_{\rho \rightarrow 1} T_{\psi_\rho} f$$

for all $f \in b^p_\lambda$ and $T_\psi^* : b^q_\lambda \rightarrow b^q_\lambda$ can be expressed as

$$T_\psi^* f = \lim_{\rho \rightarrow 1} T_{\overline{\psi}_\rho} f$$

for $f \in b^q_\lambda$.

Here, T_ψ^* denotes the adjoint of the operator T_ψ with respect to the standard duality of b^p_λ -spaces.

Concerning the compactness of the Toeplitz operator, we formulate the result in terms of the corresponding vanishing weak Carleson condition.

Theorem 2.5. *Let the symbol ψ satisfy the assumptions of Theorem 2.3 and in addition the condition*

$$\lim_{j \rightarrow \infty} \frac{\widehat{\psi}_j}{|B_j|_\lambda} = 0. \tag{2.5}$$

Then $T_\psi : b^p_\lambda \rightarrow b^p_\lambda$ is compact for all $1 < p < \infty$.

We emphasize that these sufficient conditions do not concern the modulus of the symbol ψ , only that of the integral.

Finally, we consider Toeplitz operators, denoted by T_ψ^{an} , in the case of the weighted Bergman spaces $A^p_\lambda = A^p_\lambda(\mathbf{B}_n)$ of the open unit ball \mathbf{B}_n of \mathbb{C}^n . Here, \mathbb{C}^n is identified in the canonical way with \mathbb{R}^{2n} , hence $\mathbf{B}_n = \mathbb{B}_{2n}$, and the

Bergman space A_λ^p is the closed subspace of $L_\lambda^p(\mathbb{B}_{2n})$ consisting of analytic functions of n complex variables in the domain \mathbf{B}_n . The Toeplitz operator T_ψ^{an} with symbol $\psi \in L_{\text{loc}}^1(\mathbb{B}_{2n})$ is defined as in (1.1) by replacing the projection P_λ by the orthogonal projection P_λ^{an} from $L_\lambda^p(\mathbb{B}_{2n})$ onto $A_\lambda^p(\mathbf{B}_n)$, and the generalized Toeplitz operator corresponding to (2.1) is defined analogously.

Corollary 2.6. *Let the assumptions of Theorem 2.3 (respectively, Theorem 2.5) be valid for the symbol ψ in the domain \mathbb{B}_{2n} , $n \in \mathbb{N}$. Then, the Toeplitz operator $T_\psi^{\text{an}} : A_\lambda^p(\mathbf{B}_n) \rightarrow A_\lambda^p(\mathbf{B}_n)$ is bounded (resp. compact). The statement corresponding to Theorem 2.4 is also valid in the space $A_\lambda^p(\mathbf{B}_n)$.*

The proofs of the main results will be given in Sect. 5.

3. Further Notation and Definitions

By C, c, C' etc. we mean positive constants which may vary from place to place. If the constant depends on some parameter, say n , this is shown as $C(n)$. For positive valued expressions f and g depending on some variables or parameters, the notation $f \approx g$ (respectively, $f \lesssim g$) means the existence of constants c, C such that $cf \leq g \leq Cf$ (resp. $f \leq Cg$) for all values of the variables or parameters. By $E(a, r) \subset \mathbb{R}^n$ we denote the open Euclidean ball with center $a \in \mathbb{R}^n$ and radius $r > 0$.

On \mathbb{B}_n we define the standard weight function

$$w(x) = 1 - |x|^2. \tag{3.1}$$

Given $\lambda > -1$ and $1 \leq p < \infty$, the norm of the weighted space L_λ^p and harmonic Bergman space $b_\lambda^p \subset L_\lambda^p$ is defined by

$$\|f\|_{p,\lambda} := \left(\int_{\mathbb{B}_n} |f|^p dV_\lambda \right)^{1/p},$$

where

$$dV_\lambda := c(n, \lambda)w^\lambda dV := \frac{2}{n} \cdot \frac{\Gamma(n/2 + \lambda + 1)}{\Gamma(n/2)\Gamma(\lambda + 1)}w^\lambda dV. \tag{3.2}$$

The orthogonal projection P_λ from L_λ^2 onto b_λ^2 (the harmonic Bergman projection) can be expressed as an integral operator

$$P_\lambda f(x) = \int_{\mathbb{B}_n} f(y)R_\lambda(x, y) dV_\lambda(y), \tag{3.3}$$

where, for $x, y \in \mathbb{B}_n$,

$$R_\lambda(x, y) = \frac{\Gamma(n/2)}{\Gamma(n/2 + \lambda + 1)} \sum_{k=0}^\infty \frac{\Gamma(k + n/2 + \lambda + 1)}{\Gamma(k + n/2)} Z_k(x, y)$$

and $Z_k(x, y)$ denote the extended zonal harmonics of order k . We refer to [1, Chapter 5] for the definition of these functions and related facts. This series converges absolutely and uniformly on $K \times \mathbb{B}_n$ for every compact set $K \subset \mathbb{B}_n$; see [7, Proposition 2.6]. In particular, R_λ is a smooth bounded function on

$K \times \mathbb{B}_n$ and also on $\mathbb{B}_n \times K$ by symmetry. For $1 < p < \infty$ the boundedness of $P_\lambda : L_\lambda^p \rightarrow b_\lambda^p$, defined as in (3.3), is proven in [3, Theorem 3.1].

The following pointwise estimate follows directly from the mean value property of harmonic functions.

Lemma 3.1. *Let $\lambda > -1$, $1 \leq p < \infty$, and $f \in b_\lambda^p$. Then*

$$|f(x)| \lesssim \frac{\|f\|_{p,\lambda}}{w(x)^{(n+\lambda)/p}}$$

for all $x \in \mathbb{B}_n$.

For compactly supported symbols we have the following result, which is well known to experts; however, we include its proof for completeness because we do not know a reference.

Lemma 3.2. *Let $\psi \in L_\lambda^1$ have compact support, that is, $\text{supp } \psi \subseteq r\overline{\mathbb{B}_n}$ for some $r < 1$. Then, $P_\lambda\psi$ is harmonic and there is a constant $C = C(r)$ such that*

$$\|P_\lambda\psi\|_{p,\lambda} \leq C\|\psi\|_{1,\lambda}$$

for all $1 < p < \infty$. In particular, T_ψ is well-defined and bounded on b_λ^p .

Proof. We have

$$|P_\lambda\psi(x)| \leq \int_{r\overline{\mathbb{B}_n}} |\psi(y)| |R_\lambda(x, y)| dV_\lambda(y) \leq C \int_{r\overline{\mathbb{B}_n}} |\psi(y)| dV_\lambda(y) = C\|\psi\|_{1,\lambda}$$

for all $x \in \mathbb{B}_n$ because $R_\lambda(x, y)$ is bounded on $\mathbb{B}_n \times r\overline{\mathbb{B}_n}$ as noted above. This implies that $P_\lambda\psi$ is harmonic and $\|P_\lambda\psi\|_{p,\lambda} \leq C\|\psi\|_{1,\lambda}$ for all p .

By Lemma 3.1, $|f(y)| \leq \tilde{C}(r)\|f\|_{p,\lambda}$ for $|y| \leq r$. The same estimate as above thus yields the boundedness of T_ψ . \square

The maximal harmonic Bergman projection P_λ^M is the nonlinear operator

$$P_\lambda^M f(x) = \int_{\mathbb{B}_n} |f(y)| |R_\lambda(x, y)| dV_\lambda(y).$$

For $1 < p < \infty$ this is a well-defined mapping $L_\lambda^p \rightarrow L_\lambda^p$, and it is also bounded in the sense that, for some constant

$$\|P_\lambda^M f\|_{p,\lambda} \leq C\|f\|_{p,\lambda} \tag{3.4}$$

for $f \in L_\lambda^p$; see again [3].

Recall that for each $a \in \mathbb{B}_n$, the Möbius transformation $\varphi_a : \mathbb{B}_n \rightarrow \mathbb{B}_n$ is defined by the formula

$$\varphi_a(x) = \frac{|x - a|^2 a - (1 - |a|^2)(x - a)}{[x, a]^2}, \tag{3.5}$$

where

$$[x, a] := (1 - 2x \cdot a + |x|^2|a|^2)^{1/2}. \tag{3.6}$$

As is well known, φ_a is an automorphism (analytic bijection) of \mathbb{B}_n onto itself, which maps the point a to the origin, and it is also an involution, i.e., $\varphi_a \circ \varphi_a(z) = z$ for all $z \in \mathbb{B}_n$.

The technical challenge of our paper arises from the decomposition of the unit ball into spherical boxes and a tricky integration by parts argument. Accordingly, it is important to introduce suitable combinatorial notation. First, we will use standard multi-index notation so that for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, we denote $|\alpha| := \alpha_1 + \dots + \alpha_n$, and if $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$, then $\beta \leq \alpha$ means that $\beta_k \leq \alpha_k$ for all k . Given a multi-index α , the corresponding partial differential operator acting on functions with n real variables is defined by

$$D^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n},$$

where $\partial_k = \partial/\partial x_k$ for all k . Differentiation will be performed both in Cartesian and spherical coordinates in the following.

Moreover, in order to perform the integration by parts, we need some notation for certain parts of the boundary. For $x, y \in \mathbb{Q}_n$ and multi-indices $\alpha \in \{0, 1\}^n$, let

$$Q_\alpha(x, y) = \prod_{\substack{k=1, \dots, n \\ \alpha_k=1}} [x_k, y_k] \times \prod_{\substack{k=1, \dots, n \\ \alpha_k=0}} \{y_k\}. \tag{3.7}$$

For $z \in Q(x, y)$ we denote by z_α the element in $Q(x, y)$ with coordinates

$$(z_\alpha)_k = \begin{cases} z_k & \text{if } \alpha_k = 1, \\ y_k & \text{if } \alpha_k = 0. \end{cases} \tag{3.8}$$

4. Preparatory Lemmas

4.1. Remarks Concerning the Hyperbolic Metric

The following facts about Möbius transforms φ_a are well known and easy to check from the definitions, cf. (3.1), (3.5), (3.6).

Lemma 4.1. *Let $a \in \mathbb{B}_n$. The identities*

$$1 - |\varphi_a(x)|^2 = \frac{w(a)w(x)}{[x, a]^2} \quad \text{and} \quad |\varphi'_a(x)| = \frac{w(a)}{[x, a]^2} \tag{4.1}$$

hold for every $x \in \mathbb{B}_n$. Moreover, for all $x, y \in \mathbb{B}_n$,

$$[\varphi_a(x), \varphi_a(y)]^2 = |\varphi'_a(x)| |\varphi'_a(y)| [x, y]^2. \tag{4.2}$$

The distance of two points $a, b \in \mathbb{B}_n$ in the hyperbolic (or Poincaré) metric is given by

$$d(a, b) = \log \left(\frac{1 + |\varphi_a(b)|}{1 - |\varphi_a(b)|} \right),$$

and, in particular, $d(0, a) = \log((1 + |a|)/(1 - |a|))$.

Lemma 4.2. *Let $a, b \in \mathbb{B}_n$. Then*

$$e^{-d(a,b)} \leq \frac{[x, a]}{[x, b]} \leq e^{d(a,b)}$$

for all $x \in \mathbb{B}_n$.

Proof. We write $x = \varphi_a(y)$ and $b = \varphi_a(c)$. By (4.2), we have

$$\frac{[x, a]^2}{[x, b]^2} = \frac{[\varphi_a(y), \varphi_a(0)]^2}{[\varphi_a(y), \varphi_a(c)]^2} = \frac{|\varphi'_a(y)||\varphi'_a(0)||[y, 0]^2}{|\varphi'_a(y)||\varphi'_a(c)||[y, c]^2} = \frac{[a, c]^2}{[y, c]^2}$$

using (4.1) for $|\varphi'_a(0)|$ and $|\varphi'_a(c)|$. Since $c = \varphi_a(b)$, we have

$$\frac{[x, a]}{[x, b]} = \frac{[a, c]}{[y, c]} \leq \frac{1 + |c|}{1 - |c|} = \frac{1 + |\varphi_a(b)|}{1 - |\varphi_a(b)|} = e^{d(a,b)}.$$

The other inequality follows from interchanging the roles of a and b . \square

Next we prove some crucial properties of the dyadic boxes B_j and their enlargements B_j^* . Recall that the B_j are essentially disjoint and $B_j^* = B_j + 2^{-m-2}\mathbb{B}_n$.

Lemma 4.3. *Let B_j be a dyadic box of generation m . Then*

- (i) $|B_j| \approx 2^{-mn}$,
- (ii) $\text{diam}(B_j) \approx 2^{-m}$,
- (iii) $|B_j^*| \approx 2^{-mn}$,
- (iv) *there is a constant $N \in \mathbb{N}$ such that every $x \in \mathbb{B}_n$ is contained in at most N of the sets B_j^* ,*
- (v) $1 - |x| \approx 1 - |x|^2 \approx 2^{-m}$ whenever $x \in B_j^*$,
- (vi) $[x, a] \approx [x, b]$ for all $x \in \mathbb{B}_n$ and $a, b \in B_j^*$.

Proof. Without loss of generality, we may always assume that $B_j = \sigma(C_k)$ for some $k \in \mathcal{K}$.

(i) A direct computation gives

$$\begin{aligned} |B_j| &= \int_{1-2^{-m}}^{1-2^{-m-1}} \int_{\frac{\pi}{2}k_2 2^{-m}}^{\frac{\pi}{2}(k_2+1)2^{-m}} \cdots \int_{\frac{\pi}{2}\frac{k_{n-1}}{k_{n-2}+1}}^{\frac{\pi}{2}\frac{k_{n-1}+1}{k_{n-2}+1}} \int_{2\pi\frac{k_n}{k_{n-1}+1}}^{2\pi\frac{k_n+1}{k_{n-1}+1}} \\ &\quad r^{n-1} \sin^{n-2} \theta_2 \dots \sin \theta_{n-1} d\theta_n \dots d\theta_2 dr \\ &\approx \int_{1-2^{-m}}^{1-2^{-m-1}} \int_{\frac{\pi}{2}k_2 2^{-m}}^{\frac{\pi}{2}(k_2+1)2^{-m}} \cdots \int_{\frac{\pi}{2}\frac{k_{n-1}}{k_{n-2}+1}}^{\frac{\pi}{2}\frac{k_{n-1}+1}{k_{n-2}+1}} \int_{2\pi\frac{k_n}{k_{n-1}+1}}^{2\pi\frac{k_n+1}{k_{n-1}+1}} \\ &\quad r^{n-1} \theta_2^{n-2} \dots \theta_{n-1} d\theta_n \dots d\theta_2 dr \\ &\approx ((1 - 2^{-m-1})^n - (1 - 2^{-m})^n) (((k_2 + 1)2^{-m})^{n-1} - (k_2 2^{-m})^{n-1}) \\ &\quad \times \prod_{l=3}^n \left(\left(\frac{k_l + 1}{k_{l-1} + 1} \right)^{n-l+1} - \left(\frac{k_l}{k_{l-1} + 1} \right)^{n-l+1} \right) \\ &\approx 2^{-m} ((k_2 + 1)^{n-1} - k_2^{n-1}) 2^{-m(n-1)} \\ &\quad \times \prod_{l=3}^n \frac{1}{(k_{l-1} + 1)^{n-l+1}} ((k_l + 1)^{n-l+1} - k_l^{n-l+1}) \\ &\approx 2^{-mn} \prod_{l=2}^{n-1} \frac{(k_l + 1)^{n-l+1} - k_l^{n-l+1}}{(k_l + 1)^{n-l}} \\ &\approx 2^{-mn}. \end{aligned}$$

- (ii) The edges of the dyadic boxes are coordinate lines γ_l of the coordinate transform σ . Those are of the form

$$\gamma_l(t) = \sigma(r, \theta_2, \dots, \underset{\substack{\uparrow \\ \theta_l - \text{coordinate}}}{t}, \dots, \theta_n).$$

The length ℓ of the edges can therefore be computed directly:

$$\begin{aligned} \ell(\gamma_1) &= \int_{1-2^{-m}}^{1-2^{-m-1}} \|\gamma_1'(t)\| dt = \int_{1-2^{-m}}^{1-2^{-m-1}} 1 dt = 2^{-m-1}, \\ \ell(\gamma_2) &= \int_{\frac{\pi}{2}k_2 2^{-m}}^{\frac{\pi}{2}(k_2+1)2^{-m}} \|\gamma_2'(t)\| dt = \int_{\frac{\pi}{2}k_2 2^{-m}}^{\frac{\pi}{2}(k_2+1)2^{-m}} r dt \lesssim 2^{-m}, \\ &\vdots \\ \ell(\gamma_l) &= \int_{\frac{\pi}{2} \frac{k_l}{k_{l-1}+1}}^{\frac{\pi}{2} \frac{k_l+1}{k_{l-1}+1}} \|\gamma_l'(t)\| dt = \int_{\frac{\pi}{2} \frac{k_l}{k_{l-1}+1}}^{\frac{\pi}{2} \frac{k_l+1}{k_{l-1}+1}} r \sin \theta_2 \dots \sin \theta_{l-1} dt \\ &\lesssim (k_2 + 1)2^{-m} \cdot \frac{k_3 + 1}{k_2 + 1} \cdot \dots \cdot \frac{k_{l-1} + 1}{k_{l-2} + 1} \cdot \frac{1}{k_{l-1} + 1} = 2^{-m}, \\ &\vdots \\ \ell(\gamma_n) &= \int_{\frac{\pi}{2\pi} \frac{k_n}{k_{n-1}+1}}^{\frac{\pi}{2} \frac{k_n+1}{k_{n-1}+1}} \|\gamma_n'(t)\| dt = \int_{\frac{\pi}{2} \frac{k_n}{k_{n-1}+1}}^{\frac{\pi}{2} \frac{k_n+1}{k_{n-1}+1}} r \sin \theta_2 \dots \sin \theta_{n-1} dt \\ &\lesssim (k_2 + 1)2^{-m} \cdot \frac{k_3 + 1}{k_2 + 1} \cdot \dots \cdot \frac{k_{n-1} + 1}{k_{n-2} + 1} \cdot \frac{1}{k_{n-1} + 1} = 2^{-m}. \end{aligned}$$

It follows $\text{diam}(B_j) \approx 2^{-m}$.

- (iii) We only need to show $|B_j^*| \lesssim 2^{-mn}$. Clearly, B_j^* is contained in a ball of diameter at most $\text{diam}(B_j) + 2^{-m-1} \approx 2^{-m}$. Hence $|B_j^*| \lesssim 2^{-mn}$.
- (iv) Let $x \in \mathbb{B}_n$ with $|x| \in [1 - 2^{-m}, 1 - 2^{-m-1}]$ and denote the set of indices j such that $B_j \cap E(x, 2^{-m-2}) \neq \emptyset$ by \mathcal{J}_x . It suffices to show that the cardinality of \mathcal{J}_x is bounded independently of x . Clearly, B_j is of generation $m - 1$, m or $m + 1$ for every $j \in \mathcal{J}_x$. Therefore, by (i) and (ii), $|B_j| \approx 2^{-nm}$ and $\text{diam}(B_j) \approx 2^{-m}$ for every $j \in \mathcal{J}_x$. It follows that there is a constant C (only depending on n) such that $B_j \subset E(x, C2^{-m})$ for every $j \in \mathcal{J}_x$. Since $|E(x, C2^{-m})| \approx 2^{-mn}$, $E(x, C2^{-m})$ can only contain finitely many essentially disjoint sets of volume $\approx 2^{-mn}$. This finite number is the constant N we are looking for; it only depends on n . This finishes the proof.
- (v) As $1 - 2^{-m} \leq |x| \leq 1 - 2^{-m-1}$ for x in a dyadic box of generation m , this is clear.
- (vi) In view of (ii), there is a constant C such that $|a - b| \leq C2^{-m}$ for all $a, b \in B_j^*$. We also have $1 - |a|^2 \geq 2^{-m-2}$ and $1 - |b|^2 \geq 2^{-m-2}$. Hence,

by (3.6) and (4.1),

$$\begin{aligned} \frac{1}{1 - |\varphi_a(b)|^2} &= \frac{[b, a]^2}{w(a)w(b)} = \frac{w(a)w(b) + |a|^2 + |b|^2 - 2a \cdot b}{w(a)w(b)} \\ &= 1 + \frac{|a - b|^2}{w(a)w(b)} \leq 1 + 16C^2, \end{aligned}$$

and therefore,

$$d(a, b) = \log \left(\frac{(1 + |\varphi_a(b)|)^2}{1 - |\varphi_a(b)|^2} \right) \leq \log \left(\frac{4}{1 - |\varphi_a(b)|^2} \right) \leq \log(4 + 64C^2).$$

The assertion (vi) now follows from Lemma 4.2. □

4.2. Basic Facts About the Space b_λ^p

We will need a number of results on the spaces b_λ^p . The following was proven in [3, Lemma 2.8].

Lemma 4.4. *For all multi-indices $\alpha \in \mathbb{N}_0^n$ and all $\lambda > -1$,*

$$|D_x^\alpha R_\lambda(x, y)| \lesssim [x, y]^{-n-\lambda-|\alpha|}, \quad x, y \in \mathbb{B}_n.$$

To prove the next lemma, we use the Forelli–Rudin type estimate

$$\int_{\mathbb{B}_n} \frac{(1 - |y|^2)^t}{[x, y]^{n+s+t}} dV(y) \approx (1 - |x|^2)^{-s}, \tag{4.3}$$

where $s > 0$ and $t > -1$ (see [4, Proposition 2.2]).

Lemma 4.5. *Let $\lambda > -1$ and $1 < p < \infty$. Then the integral operator defined by*

$$\Lambda_\lambda f(x) := \int_{\mathbb{B}_n} \frac{f(y)}{[x, y]^{n+\lambda}} dV_\lambda(y), \quad x \in \mathbb{B}_n,$$

is bounded on L_λ^p .

Proof. Define $h(x) = (1 - |x|^2)^\alpha$ and choose $\frac{-1-\lambda}{\max\{p,q\}} < \alpha < 0$. Then, by (4.3), when $1/p + 1/q = 1$, we get

$$\int_{\mathbb{B}_n} \frac{h(y)^q}{[x, y]^{n+\lambda}} dV_\lambda(y) = \int_{\mathbb{B}_n} \frac{(1 - |y|^2)^{\alpha q + \lambda}}{[x, y]^{n+\lambda}} dV(y) \lesssim h(x)^q$$

and similarly $\int_{\mathbb{B}_n} \frac{h(x)^p}{[x, y]^{n+\lambda}} dV_\lambda(x) \lesssim h(y)^p$. Now Schur’s test completes the proof. □

Lemma 4.6. *Let $\lambda > -1$, $1 < p < \infty$ and $k \in \mathbb{N}$. If α is a multi-index with $|\alpha| = k$ and $f \in b_\lambda^p$, then $w^k D^\alpha f \in L_\lambda^p$, and*

$$\|w^k D^\alpha f\|_{p,\lambda} \lesssim \|f\|_{p,\lambda}.$$

Proof. In view of Lemma 3.1, $b_\lambda^p \subset b_\gamma^2$ for sufficiently large γ , and hence

$$f(x) = \int_{\mathbb{B}_n} R_\gamma(x, y) f(y) dV_\gamma(y).$$

Differentiating under the integral sign, we obtain

$$w(x)^{|\alpha|} D^\alpha f(x) = \int_{\mathbb{B}_n} K(x, y) f(y) dV_\lambda(y), \tag{4.4}$$

where

$$K(x, y) := w(x)^{|\alpha|} w(y)^{\gamma-\lambda} D_x^\alpha R_\gamma(x, y).$$

By Lemma 4.4,

$$|K(x, y)| \lesssim \frac{w(x)^{|\alpha|} w(y)^{\gamma-\lambda}}{[x, y]^{n+\gamma+|\alpha|}} \lesssim \frac{1}{[x, y]^{n+\lambda}},$$

where the last inequality follows from

$$[x, y] \geq 1 - |x||y| \geq \max\{1 - |x|, 1 - |y|\} \geq \frac{1}{2} \max\{w(x), w(y)\}.$$

Using (4.4) and Lemma 4.5, we get

$$\left\| w^{|\alpha|} D^\alpha f \right\|_{p,\lambda} \lesssim \|f\|_{p,\lambda},$$

which completes the proof. □

4.3. Integration by Parts in Spherical Boxes

The results in [9] were based on a tricky integration by parts. We have to exploit this procedure in even higher generality, hence, it is useful to expose the corresponding general integration-by-parts-formula. The set of n -times continuously differentiable functions on \mathbb{B}_n will be denoted by $C^n(\mathbb{B}_n)$. In the following, the Jacobian determinant of the coordinate transform σ will be denoted by J_σ .

Lemma 4.7. *Let $f \in L^1_{loc}$, $g \in C^n(\mathbb{B}_n)$ and $x, y \in \mathbb{Q}_n$. Then, with $F = f \circ \sigma$ and $G = g \circ \sigma$,*

$$\begin{aligned} \int_{B(x,y)} fg dV_\lambda &= \int_{Q(x,y)} F(\gamma)G(\gamma)J_\sigma(\gamma)w(\sigma(\gamma))^\lambda d\gamma \\ &= \sum_{\alpha \in \{0,1\}^n} s_\alpha \int_{Q_\alpha(x,y)} \left(\int_{Q(x,\gamma_\alpha)} F(\tau)J_\sigma(\tau)w(\sigma(\tau))^\lambda d\tau \right) \\ &\quad \times D^\alpha G(\gamma_\alpha) d\gamma_\alpha, \end{aligned} \tag{4.5}$$

where $s_\alpha \in \{\pm 1\}$ for all $\alpha \in \{0, 1\}^n$.

Recall that $Q_\alpha(x, y) = \prod_{\substack{k=1, \dots, n \\ \alpha_k=1}} [x_k, y_k] \times \prod_{\substack{k=1, \dots, n \\ \alpha_k=0}} \{y_k\}$, which means that the integration $\int_{Q_\alpha(x,y)} d\gamma_\alpha$ is performed only in those variables γ_k where $\alpha_k = 1$. This is in concordance with the notation (3.8) since $(\gamma_\alpha)_k = y_k$ if $\alpha_k = 0$.

Proof. Let $\mathbb{1} := (1, \dots, 1)$. We will use the following well-known formula

$$(D^{\mathbb{1}}u) \cdot v = \sum_{\alpha \in \{0,1\}^n} (-1)^{|\alpha|} D^{\mathbb{1}-\alpha}(u \cdot D^{\alpha}v), \tag{4.6}$$

which is easily proven by induction. Now choose

$$u(\gamma) := \int_{Q(x,\gamma)} F(\tau) J_{\sigma}(\tau) w(\sigma(\tau))^{\lambda} d\tau, \quad \text{and} \quad v(\gamma) := G(\gamma),$$

and observe that $u(\gamma) = 0$ if $x_k = \gamma_k$ for some k . Integrating the formula (4.6) over $Q(x, y)$ and using the fundamental theorem of calculus multiple times yields the result. \square

5. Proofs of the Main Results

Before giving the proofs of our main results, we still need to consider four lemmas. The first one is used to fix a small flaw in [9]: in the reference, the inequality (3.8) is not true as such, but the integration domain has to be replaced by a larger set. This, however, is not difficult, and we use here the enlarged dyadic boxes B_j^* to this end.

Lemma 5.1. *Let f be a harmonic function on \mathbb{B}_n and let B_j be one of our dyadic boxes. Then for each j ,*

$$|f(x)| \lesssim \frac{1}{|B_j|_{\lambda}} \int_{B_j^*} |f| dV_{\lambda} \tag{5.1}$$

for every $x \in B_j$.

Proof. Suppose that B_j is of generation m . By definition of the sets B_j^* , we have $E(x, \delta 2^{-m-2}) \subset B_j^*$ for every $x \in B_j$. The mean value property of harmonic functions yields

$$f(x) = \frac{1}{|E(x, 2^{-m-2})|} \int_{E(x, 2^{-m-2})} f(y) dV(y).$$

Thus,

$$\begin{aligned} |f(x)| &\leq \frac{1}{|E(x, 2^{-m-2})|} \int_{E(x, 2^{-m-2})} |f(y)| dV(y) \\ &\lesssim 2^{nm} \int_{B_j^*} |f(y)| dV(y) \\ &\lesssim 2^{(n+\lambda)m} \int_{B_j^*} |f(y)| w(y)^{\lambda} dV(y), \end{aligned}$$

where the last inequality follows from the fact that $w(y) \approx 2^{-m}$ for $y \in B_j^*$; see Lemma 4.3. Since $|B_j|_{\lambda} \approx 2^{-(n+\lambda)m}$, this proves (5.1). \square

In order to apply Lemma 4.7, we need to transfer partial derivatives from spherical coordinates to Cartesian coordinates. For this we need the following lemma. We note that there is an explicit formula for iterated partial derivatives, the Faà di Bruno formula, but since we do not need it in full generality, we decided to give a proof in simple terms.

Lemma 5.2. *Let $g \in C^n(\mathbb{B}_n)$ and $\alpha \in \{0, 1\}^n$, $|\alpha| \geq 1$. Then for every $\beta \in \mathbb{N}_0^n$ with $1 \leq |\beta| \leq |\alpha|$ there is a $d_\beta \in C^\infty(\mathbb{Q}_n)$ such that*

$$D_\gamma^\alpha g(\sigma(\gamma)) = \sum_{1 \leq |\beta| \leq |\alpha|} c_\beta(\gamma) d_\beta(\gamma) (D_x^\beta g)(x),$$

where

$$c_\beta(\gamma) = \prod_{j=2}^{n-1} (\sin \theta_j)^{\max\{0, |\beta| - \sum_{i=1}^j \alpha_i\}},$$

$\gamma = (r, \theta_2, \dots, \theta_n) \in \mathbb{Q}_n$ and $x = \sigma(\gamma)$.

Proof. We only prove that one can factor out $(\sin \theta_2)^{|\beta| - \alpha_1 - \alpha_2}$. As will be clear from the proof, factoring out the other terms in the product can then be done in the same way. We also note that the power of each $\sin \theta_j$ in the expansion of $D_\gamma^\alpha (g \circ \sigma)$ can never get negative. It can therefore be assumed that $|\beta| - \alpha_1 - \alpha_2 \geq 1$; otherwise there is nothing to prove.

We prove the claim by induction over $|\alpha|$. First assume that $|\alpha| = 1$. Then

$$D_\gamma^\alpha (g \circ \sigma) = \frac{\partial (g \circ \sigma)}{\partial \gamma_k} = \sum_{l=1}^n \frac{\partial x_l}{\partial \gamma_k} \frac{\partial g}{\partial x_l}$$

for some $k \in \{1, \dots, n\}$. Now observe that $\sin \theta_2$ is necessarily a factor of $\frac{\partial x_l}{\partial \gamma_k}$ unless $k \in \{1, 2\}$. This completes the proof for $|\alpha| = 1$.

To complete the induction process, we compute

$$\frac{\partial}{\partial \gamma_k} \sum_{1 \leq |\beta| \leq |\alpha|} (\sin \theta_2)^{|\beta| - \alpha_1 - \alpha_2} d_\beta D_x^\beta g$$

for each $k \in \{1, \dots, n\}$ with $\alpha_k = 0$ and show that we can factor out sufficiently many $\sin \theta_2$. First assume $k \neq 2$. Then

$$\begin{aligned} & \frac{\partial}{\partial \gamma_k} \sum_{1 \leq |\beta| \leq |\alpha|} (\sin \theta_2)^{|\beta| - \alpha_1 - \alpha_2} d_\beta D_x^\beta g \\ &= \sum_{1 \leq |\beta| \leq |\alpha|} (\sin \theta_2)^{|\beta| - \alpha_1 - \alpha_2} \left(\frac{\partial d_\beta}{\partial \gamma_k} D_x^\beta g + d_\beta \sum_{l=1}^n \frac{\partial x_l}{\partial \gamma_k} \frac{\partial}{\partial x_l} D_x^\beta g \right). \end{aligned}$$

Let α' be equal to α except that the k -th entry is flipped from 0 to 1. If $k \neq 1$, we can factor out another $\sin \theta_2$ from $\frac{\partial x_l}{\partial \gamma_k}$ as above and therefore the power of $\sin \theta_2$ again matches the order of the derivative in each term. If $k = 1$, then $\frac{\partial x_1}{\partial \gamma_1} = \cos \theta_2$, so there is no additional $\sin \theta_2$ to factor out. However, since

$\alpha'_1 = \alpha_1 + 1$, we still have the correct power of $\sin \theta_2$ in the expansion. Now let $k = 2$. In this case

$$\begin{aligned} & \frac{\partial}{\partial \gamma_k} \sum_{1 \leq |\beta| \leq |\alpha|} (\sin \theta_2)^{|\beta| - \alpha_1 - \alpha_2} d_\beta D_x^\beta g \\ &= \sum_{1 \leq |\beta| \leq |\alpha|} (\sin \theta_2)^{|\beta| - \alpha_1 - \alpha_2} \left(\frac{\partial d_\beta}{\partial \gamma_k} D_x^\beta g + d_\beta \sum_{l=1}^n \frac{\partial x_l}{\partial \gamma_k} \frac{\partial}{\partial x_l} D_x^\beta g \right) \\ & \quad + (\sin \theta_2)^{|\beta| - \alpha_1 - \alpha_2 - 1} \cos \theta_2 d_\beta D_x^\beta g. \end{aligned}$$

As $\alpha'_2 = \alpha_2 + 1$, we are in the same situation as for $k = 1$. We again have the correct power of $\sin \theta_2$ in each term. This completes the proof. \square

The functions c_β have the following important property.

Lemma 5.3. *Let $\alpha \in \{0, 1\}^n$, $|\alpha| \geq 1$ and $c_\beta(\gamma)$ be as above. Then*

$$\int_{Q_\alpha(x^{(j)}, y^{(j)})} c_\beta(\gamma) d\gamma_\alpha \lesssim 2^{-m|\beta|}$$

for all $j \in \mathbb{N}$.

Proof. As usual, we may assume that $Q(x^{(j)}, y^{(j)}) = C_k$ for some $k \in \mathcal{K}$. Choose $h \in \mathbb{N}$ such that $|\beta| = \sum_{i=1}^h \alpha_i$. If $h = 1$, then $|\beta| = \alpha_1 = 1$ as well as $c_\beta(\gamma) = 1$. Therefore,

$$\int_{Q_\alpha(x^{(j)}, y^{(j)})} c_\beta(\gamma) d\gamma_\alpha \lesssim 2^{-m} = 2^{-m|\beta|}.$$

If $h \geq 2$, another direct computation shows

$$\begin{aligned} \int_{Q_\alpha(x^{(j)}, y^{(j)})} c_\beta(\gamma) d\gamma_\alpha & \lesssim 2^{-m\alpha_1} (k_2 + 1)^{|\beta| - \alpha_1 - \alpha_2} 2^{-m(|\beta| - \alpha_1)} \\ & \quad \times \frac{(k_3 + 1)^{|\beta| - \alpha_1 - \alpha_2 - \alpha_3}}{(k_2 + 1)^{|\beta| - \alpha_1 - \alpha_2}} \cdots \frac{(k_{n-1} + 1)^{|\beta| - \sum_{l=1}^h \alpha_l}}{(k_{n-2} + 1)^{|\beta| - \sum_{l=1}^{h-1} \alpha_l}} \\ & = 2^{-m|\beta|}. \end{aligned} \quad \square$$

The following lemma is the most important technical step in the proof of our theorems.

Lemma 5.4. *Let $\psi \in L^1_{\text{loc}}$. For $z \in \mathbb{B}_n$ and every dyadic box B_j , we have*

$$|T_\psi(\chi_{B_j} f)(z)| \lesssim \frac{\widehat{\psi}_j}{|B_j|^\lambda} \sum_{|\beta| \leq n} \int_{B_j^*} \frac{w(y)^{|\beta|} |D^\beta f(y)|}{[z, y]^{n+\lambda}} dV_\lambda(y).$$

Proof. Suppose that the generation of B_j is m . We may further assume that $B_j = \sigma(C_k)$ for some $k \in \mathcal{K}$. Let $\gamma = (r, \theta_2, \dots, \theta_n) \in \mathbb{Q}_n$ be such that $y = \sigma(\gamma) \in B_j$ and $\alpha \in \{0, 1\}^n$, $|\alpha| \geq 1$. By Lemma 4.4, we have

$$|D_y^\alpha R_\lambda(z, y)| \lesssim [z, y]^{-n-\lambda-|\alpha|} \tag{5.2}$$

for every $z \in \mathbb{B}_n$. Using Lemma 5.2, we obtain

$$\begin{aligned} |D_\gamma^\alpha(R_\lambda(z, \sigma(\gamma))f(\sigma(\gamma)))| &\leq \sum_{1 \leq |\beta| \leq |\alpha|} c_\beta(\gamma) |d_\beta(\gamma)| |D_y^\beta(R_\lambda(z, y)f(y))| \\ &\lesssim \sum_{1 \leq |\beta| \leq |\alpha|} c_\beta(\gamma) \sum_{\tilde{\beta} \leq \beta} |D_y^{\beta - \tilde{\beta}} R_\lambda(z, y)| |D_y^{\tilde{\beta}} f(y)| \\ &\lesssim \sum_{1 \leq |\beta| \leq |\alpha|} c_\beta(\gamma) \sum_{\tilde{\beta} \leq \beta} 2^{m(|\beta| - |\tilde{\beta}|)} [z, y]^{-n-\lambda} |D_y^{\tilde{\beta}} f(y)| \end{aligned}$$

where the last inequality follows from (5.2) and $[z, \sigma(\gamma)] \geq 1 - |\sigma(\gamma)| \approx 2^{-m}$ by Lemma 4.3. We now apply Lemma 5.1 to the functions $D^{\tilde{\beta}} f$ and use (v) and (vi) of Lemma 4.3. This yields

$$\begin{aligned} &|D_\gamma^\alpha(R_\lambda(z, \sigma(\gamma))f(\sigma(\gamma)))| \\ &\lesssim \sum_{1 \leq |\beta| \leq |\alpha|} c_\beta(\gamma) \sum_{\tilde{\beta} \leq \beta} 2^{m(|\beta| - |\tilde{\beta}|)} [z, y]^{-n-\lambda} \frac{1}{|B_j|_\lambda} \int_{B_j^*} |D^{\tilde{\beta}} f| dV_\lambda \\ &\lesssim \sum_{1 \leq |\beta| \leq |\alpha|} c_\beta(\gamma) \sum_{\tilde{\beta} \leq \beta} 2^{m|\beta|} \frac{1}{|B_j|_\lambda} \int_{B_j^*} \frac{w(y)^{|\tilde{\beta}|} |D^{\tilde{\beta}} f(y)|}{[z, y]^{n+\lambda}} dV_\lambda(y) \\ &\lesssim \sum_{1 \leq |\beta| \leq |\alpha|} c_\beta(\gamma) 2^{m|\beta|} \frac{1}{|B_j|_\lambda} \int_{B_j^*} \frac{w(y)^{|\beta|} |D^\beta f(y)|}{[z, y]^{n+\lambda}} dV_\lambda(y) \end{aligned} \tag{5.3}$$

for all $z \in \mathbb{B}_n$. We are now going to apply the integration by parts lemma, Lemma 4.7. Recall that $B_j = B(x^{(j)}, y^{(j)}) = \sigma(Q(x^{(j)}, y^{(j)}))$. Integrating (5.3) over $Q_\alpha(x^{(j)}, y^{(j)})$ and using Lemma 5.3 yields

$$\begin{aligned} &\int_{Q_\alpha(x^{(j)}, y^{(j)})} |D_\gamma^\alpha(R_\lambda(z, \sigma(\gamma))f(\sigma(\gamma)))| d\gamma_\alpha \\ &\lesssim \sum_{1 \leq |\beta| \leq |\alpha|} \frac{1}{|B_j|_\lambda} \int_{B_j^*} \frac{w(y)^{|\beta|} |D^\beta f(y)|}{[z, y]^{n+\lambda}} dV_\lambda(y). \end{aligned} \tag{5.4}$$

We now apply (4.5) to the integral

$$T_\psi(\chi_{B_j} f)(z) = \int_{B_j} \psi(y) f(y) R_\lambda(z, y) dV_\lambda(y). \tag{5.5}$$

For the factor f in (4.5) we take the function ψ and for g the function $R_\lambda(z, \cdot)$ with a fixed $z \in \mathbb{B}_n$. This yields (see the remark just after this proof)

$$\begin{aligned} &\left| \int_{B_j} \psi(y) f(y) R_\lambda(z, y) dV_\lambda(y) \right| \\ &\leq \sum_{\alpha \in \{0,1\}^n} \int_{Q_\alpha(x^{(j)}, y^{(j)})} \left| \int_{Q(x^{(j)}, \gamma_\alpha)} \psi(\sigma(\tau)) w(\sigma(\tau))^\lambda J_\sigma(\tau) d\tau \right| \\ &\quad \times \left| D_\gamma^\alpha(R_\lambda(z, \sigma(\gamma_\alpha))f(\sigma(\gamma_\alpha))) \right| d\gamma_\alpha. \end{aligned} \tag{5.6}$$

The change of the variables formula turns the integral over the set $Q(x^{(j)}, \gamma_\alpha)$ into

$$\left| \int_{B(x^{(j)}, y)} \psi \, dV_\lambda \right| \leq \widehat{\psi}_j \tag{5.7}$$

for $y = \sigma(\gamma_\alpha) \in B_j$, see (2.3). Applying (5.4), we thus see that (5.6) is bounded by a constant times

$$\frac{\widehat{\psi}_j}{|B_j|^\lambda} \sum_{|\beta| \leq n} \int_{B_j^*} \frac{w(y)^{|\beta|} |D^\beta f(y)|}{[z, y]^{n+\lambda}} \, dV_\lambda(y), \tag{5.8}$$

since $|\alpha| \leq n$. So the lemma follows by combining (5.5)–(5.8). □

We emphasize that the deduction (5.6) is in the core of our result: By using integration by parts, it is possible to make estimates only in terms of the modulus of the integral of ψ , and a direct bound involving the modulus of ψ can be avoided.

We are now in the position to prove our main results:

Proof of Theorem 2.3. Recall that the Toeplitz operator T_ψ is defined with the help of the series

$$T_\psi f(x) = \sum_{j=1}^\infty T_\psi(\chi_{B_j} f)(x),$$

cf. (2.1). We show that the series converges absolutely for almost every $x \in \mathbb{B}_n$ and that the resulting operator is bounded. Indeed, by the previous lemma, assumption (2.4) and Lemma 4.3, we have

$$\begin{aligned} \sum_{j=1}^\infty |T_\psi(\chi_{B_j} f)(x)| &\lesssim C_\psi \sum_{j=1}^\infty \sum_{|\beta| \leq n} \int_{B_j^*} \frac{w(y)^{|\beta|} |D^\beta f(y)|}{[x, y]^{n+\lambda}} \, dV_\lambda(y) \\ &\lesssim C_\psi \sum_{|\beta| \leq n} \int_{\mathbb{B}_n} \frac{w(y)^{|\beta|} |D^\beta f(y)|}{[x, y]^{n+\lambda}} \, dV_\lambda(y) \end{aligned} \tag{5.9}$$

for all $x \in \mathbb{B}_n$. By Lemmas 4.5 and 4.6, we see that for all $|\beta| \leq n$,

$$\left\| \int_{\mathbb{B}_n} \frac{w(y)^{|\beta|} |D^\beta f(y)|}{[x, y]^{n+\lambda}} \, dV_\lambda(y) \right\|_{p, \lambda} \lesssim \left\| w^{|\beta|} D^\beta f \right\|_{p, \lambda} \lesssim \|f\|_{p, \lambda}.$$

This implies that the series $\sum_{j=0}^\infty |T_\psi(\chi_{B_j} f)(x)|$ is pointwise bounded by an L_λ^p -function, and thus it converges for almost all $x \in \mathbb{B}_n$. Moreover, the above argument implies

$$\left\| \sum_{j=1}^\infty T_\psi(\chi_{B_j} f) \right\|_{p, \lambda} \lesssim C_\psi \|f\|_{p, \lambda}. \tag{5.10}$$

By dominated convergence, the series also converges in L_λ^p . In particular, $T_\psi f \in b_\lambda^p$ for all $f \in b_\lambda^p$ and $\|T_\psi\| \lesssim C_\psi$. □

Proof of Theorem 2.4. We first show that $T_{\psi_\rho} f \rightarrow T_\psi f$ for every $f \in b_\lambda^\rho$. As $(\widehat{\psi}_\rho)_j \leq \widehat{\psi}_j$, the estimate (5.9) is uniform in ρ . That is, for almost every $x \in \mathbb{B}_n$, the series $\sum_{j \in \mathbb{N}} T_{\psi_\rho}(\chi_{B_j} f)(x)$ converges absolutely and uniformly in ρ .

In particular,

$$\lim_{\rho \rightarrow 1} \sum_{j=1}^\infty T_{\psi_\rho}(\chi_{B_j} f)(x) = \sum_{j=1}^\infty \lim_{\rho \rightarrow 1} T_{\psi_\rho}(\chi_{B_j} f)(x) = \sum_{j=1}^\infty T_\psi(\chi_{B_j} f)(x).$$

Moreover, $\|T_{\psi_\rho} f\|_{p,\lambda} \lesssim C_\psi \|f\|$ by (5.10). By dominated convergence, we get $T_{\psi_\rho} f \rightarrow T_\psi f$ as $\rho \rightarrow 1$. For the adjoint observe

$$\langle T_\psi f, g \rangle = \sum_{j=1}^\infty \langle \psi \chi_{B_j} f, g \rangle = \sum_{j=1}^\infty \langle f, \bar{\psi} \chi_{B_j} g \rangle = \langle f, T_{\bar{\psi}} g \rangle$$

and thus $T_{\bar{\psi}_\rho} g \rightarrow T_{\bar{\psi}} g$ follows analogously. □

Proof of Theorem 2.5. A routine normal families argument shows that the unit ball of b_λ^ρ is compact in the topology of uniform convergence on compact subsets of \mathbb{B}_n . It therefore suffices to show that $\|T_\psi f_k\|_{p,\lambda} \rightarrow 0$ for all norm bounded sequences $(f_k)_{k=1}^\infty \subset b_\lambda^\rho$ which converge to zero uniformly on compacta of \mathbb{B}_n .

So, we fix such a sequence $(f_k)_{k=1}^\infty$ with $\|f_k\|_{p,\lambda} \leq 1$ for all k . Let $\varepsilon > 0$ be arbitrary. For all $j, k \in \mathbb{N}$ Lemma 5.4 implies

$$|T_\psi(\chi_{B_j} f_k)(x)| \lesssim \frac{\widehat{\psi}_j}{|B_j|_\lambda} \sum_{|\beta| \leq n_{B_j^*}} \int \frac{w(y)^{|\beta|} |D^\beta f_k(y)|}{[x, y]^{n+\lambda}} dV_\lambda(y),$$

and the weak vanishing Carleson condition (2.5) implies

$$\lim_{j \rightarrow \infty} \frac{\widehat{\psi}_j}{|B_j|_\lambda} = 0.$$

We choose $N \in \mathbb{N}$ such that $\frac{\widehat{\psi}_j}{|B_j|_\lambda} < \varepsilon$ for $j > N$.

As the sets B_j^* are bounded, there exists a constant $C > 0$ such that

$$\sum_{j=1}^N \sum_{|\beta| \leq n_{B_j^*}} \int \frac{w(y)^{|\beta|}}{[x, y]^{n+\lambda}} dV_\lambda(y) \leq C$$

for all $x \in \mathbb{B}_n$. Since the sequence $(f_k)_{k=1}^\infty$, as well as the sequences of the derivatives of f_k , converge to 0 uniformly on compact subsets, we may choose $M \in \mathbb{N}$ such that

$$|D^\beta f_k(y)| \leq \varepsilon$$

for all $k \geq M$, $|\beta| \leq n$, $y \in B_j^*$ and $j \leq N$.

For $k \geq M$ we get

$$\begin{aligned} \sum_{j=1}^{\infty} |T_{\psi}(\chi_{B_j} f_k)(x)| &\lesssim C_{\psi} \sum_{j=1}^N \sum_{|\beta| \leq n} \int_{B_j^*} \frac{w(y)^{|\beta|} |D^{\beta} f_k(y)|}{[x, y]^{n+\lambda}} dV_{\lambda}(y) \\ &\quad + \varepsilon \sum_{j=N+1}^{\infty} \sum_{|\beta| \leq n} \int_{B_j^*} \frac{w(y)^{|\beta|} |D^{\beta} f_k(y)|}{[x, y]^{n+\lambda}} dV_{\lambda}(y) \\ &\lesssim C_{\psi} \varepsilon + \varepsilon \sum_{|\beta| \leq n} \int_{\mathbb{B}_n} \frac{w(y)^{|\beta|} |D^{\beta} f_k(y)|}{[x, y]^{n+\lambda}} dV_{\lambda}(y). \end{aligned}$$

Arguing as in the proof of Theorem 2.3 we find that $\|T_{\psi} f_k\|_{p, \lambda} \lesssim \varepsilon$ for sufficiently large k . □

Proof of Corollary 2.6. As P_{λ}^{an} and P_{λ} are orthogonal projections and obviously $A_{\lambda}^p(\mathbf{B}_n) \subset b_{\lambda}^p(\mathbb{B}_{2n})$, we have $P_{\lambda}^{\text{an}} = P_{\lambda}^{\text{an}} P_{\lambda}$ on $L_{\lambda}^2 = L_{\lambda}^2(\mathbb{B}_{2n})$. Let $f \in L_{\lambda}^1$ have compact support. Choose a sequence $(f_n)_{n \in \mathbb{N}}$ in L_{λ}^2 with the same support as f such that $\|f - f_n\|_{1, \lambda} \rightarrow 0$ as $n \rightarrow \infty$. Then

$$\|P_{\lambda}(f - f_n)\|_{p, \lambda} \lesssim \|f - f_n\|_{1, \lambda} \quad \text{and} \quad \|P_{\lambda}^{\text{an}}(f - f_n)\|_{p, \lambda} \lesssim \|f - f_n\|_{1, \lambda} \tag{5.11}$$

by Lemma 3.2 and the respective result for the analytic case, which can be proven verbatim. In particular,

$$P_{\lambda}^{\text{an}} P_{\lambda} f = P_{\lambda}^{\text{an}} P_{\lambda}(f - f_n) + P_{\lambda}^{\text{an}} P_{\lambda} f_n = P_{\lambda}^{\text{an}} P_{\lambda}(f - f_n) + P_{\lambda}^{\text{an}} f_n.$$

By (5.11), the former converges to 0 and the latter converges to $P_{\lambda}^{\text{an}} f$ as $n \rightarrow \infty$. Hence $P_{\lambda}^{\text{an}} f = P_{\lambda}^{\text{an}} P_{\lambda} f$ for $f \in L_{\lambda}^1$ with compact support. Now let $\psi \in L_{\text{loc}}^1$ and $f \in A_{\lambda}^p$. Then

$$\sum_{j=1}^{\infty} T_{\psi}^{\text{an}}(\chi_{B_j} f) = \sum_{j=1}^{\infty} P_{\lambda}^{\text{an}}(\psi \chi_{B_j} f) = \sum_{j=1}^{\infty} P_{\lambda}^{\text{an}} P_{\lambda}(\psi \chi_{B_j} f).$$

As the series $\sum_{j=1}^{\infty} P_{\lambda}(\psi \chi_{B_j} f)$ converges in L_{λ}^p by Theorem 2.3, and P_{λ}^{an} is continuous, we obtain

$$\sum_{j=1}^{\infty} P_{\lambda}^{\text{an}} P_{\lambda}(\psi \chi_{B_j} f) = P_{\lambda}^{\text{an}} \left(\sum_{j=1}^{\infty} P_{\lambda}(\psi \chi_{B_j} f) \right) = P_{\lambda}^{\text{an}} T_{\psi} f.$$

In short, $T_{\psi}^{\text{an}} = P_{\lambda}^{\text{an}} T_{\psi}|_{A_{\lambda}^p}$. Properties such as boundedness and compactness therefore transfer from the harmonic to the analytic setting. □

6. An Example

Sufficient conditions for the boundedness of Toeplitz operators with radial symbols in Bergman spaces $A_{\lambda}^p(\mathbf{B}_n)$ have been given for example in [2], where $p = 2$, or in [5], where $n = 1$. All such results concern symbols ψ belonging at least to $L^1(\mathbf{B}_n)$. Modifying an example first presented in [9] we show here that there are radial symbols with arbitrarily fast growing modulus, still

satisfying the sufficient condition (2.4) of Theorem 2.3 and thus inducing bounded Toeplitz operators.

If $\lambda > -1$ and $\psi = \psi(r)$ is a radial symbol on the ball \mathbb{B}_n , then our condition (2.4) for boundedness simplifies to

$$\sup_{m \in \mathbb{N}_0} 2^{m(1+\lambda)} \sup_{\rho \in [1-2^{-m}, 1-2^{-m-1}]} \left| \int_{1-2^{-m}}^{\rho} r^{n-1} \psi(r) (1-r^2)^\lambda dr \right| < \infty. \tag{6.1}$$

Let $f : [1, \infty) \rightarrow \mathbb{R}$ be any continuous function with $\inf_{x \in [1, \infty)} f(x)x^\lambda > 0$.

Define

$$g : [1, \infty) \rightarrow [0, \infty), \quad g(x) = \int_1^x \frac{f(y)}{y^{1-\lambda}} dy$$

and note that $g'(x) = \frac{f(x)}{x^{1-\lambda}} > 0$ and $g(x) \rightarrow \infty$ as $x \rightarrow \infty$. In particular, g is invertible. Define

$$\psi(r) = r^{-n+1} (1-r^2)^{-\lambda} f((1-r)^{-1}) \exp(i\pi g((1-r)^{-1})).$$

It is plain that the modulus of ψ can be made to grow arbitrarily fast as $r \rightarrow 1$, nevertheless, we claim that ψ satisfies (6.1) and thus by Theorem 2.3, $T_\psi : b_\lambda^p \rightarrow b_\lambda^p$ is bounded for all $1 < p < \infty$.

With the substitution $s := g((1-r)^{-1})$, i.e., $1-r = 1/g^{-1}(s)$, we get for the modulus of imaginary part (the real part is treated in the same way)

$$\begin{aligned} & \left| \operatorname{Im} \left(\int_{1-2^{-m}}^{\rho} r^{n-1} \psi(r) (1-r^2)^\lambda dr \right) \right| \\ &= \left| \int_{1-2^{-m}}^{\rho} (1-r)^{-1+\lambda} g'((1-r)^{-1}) \sin(\pi g((1-r)^{-1})) dr \right| \\ &= \left| \int_{g(2^m)}^{g((1-\rho)^{-1})} (g^{-1}(s))^{-1-\lambda} \sin(\pi s) ds \right|. \end{aligned}$$

Let a be the smallest integer such that $a > g(2^m)$ and b the largest integer such that $g((1-\rho)^{-1}) > b$. Then, as $\frac{1}{g^{-1}(s)}$ is decreasing, we get

$$\begin{aligned} & \left| \int_{g(2^m)}^{g((1-\rho)^{-1})} (g^{-1}(s))^{-1-\lambda} \sin(\pi s) ds \right| \\ & \leq \left| \int_{g(2^m)}^a (g^{-1}(s))^{-1-\lambda} \sin(\pi s) ds \right| \\ & \quad + \left| \sum_{k=a}^{b-1} (-1)^k \int_k^{k+1} (g^{-1}(s))^{-1-\lambda} |\sin(\pi s)| ds \right| \\ & \quad + \left| \int_b^{g((1-\rho)^{-1})} (g^{-1}(s))^{-1-\lambda} \sin(\pi s) ds \right| \\ & \leq 3 \cdot 2^{-m(1+\lambda)}, \end{aligned}$$

since we have the lower estimate $g^{-1}(s) \geq 2^m$ on all integration intervals and the series in the middle term is alternating with decreasing absolute values

of the terms. This shows that T_ψ is bounded. Similarly, replacing g by

$$g : [1, \infty) \rightarrow [0, \infty), \quad g(x) = \int_1^x f(y)y^\lambda dy$$

and choosing ψ as above, yields a compact Toeplitz operator T_ψ . These examples show that symbols of arbitrary growth can induce bounded or even compact Toeplitz operators.

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Received: November 9, 2020.

Revised: February 8, 2021.

Accepted: February 9, 2021.