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Extrapolation of compactness on weighted Morrey spaces

by

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Abstract. In a previous work, “compact versions” of Rubio de Francia’s weighted extrapolation theorem were proved, which allow one to extrapolate the compactness of a linear operator from just one space to the full range of weighted Lebesgue spaces where this operator is bounded. In this paper, we extend these results to the setting of weighted Morrey spaces. As applications, we easily obtain new results on the weighted compactness of commutators of Calderón–Zygmund singular integrals, rough singular integrals and Bochner–Riesz multipliers.

1. Introduction.

We refer to a locally integrable positive almost everywhere function w on $\mathbb{R}^d$ as a *weight* and we define the weighted Lebesgue and Morrey spaces as follows:

Definition 1.1. Let $1 \le p < \infty$ and w be a weight. Then a *weighted Lebesgue space* is defined by

$$L^p(w) := \Big\{ f : \mathbb{R}^d \to \mathbb{C} \text{ measurable} : \|f\|_{L^p(w)} := \Big(\int_{\mathbb{R}^d} |f|^p w \Big)^{1/p} < \infty \Big\}.$$

Definition 1.2 ([34]). Let $1 \le p < \infty$, $0 < \lambda < d$ and w be a weight. Then the *Samko type weighted Morrey space* is defined by

$$\mathcal{L}^{p,\lambda}(w) := \Big\{ f \in L^p_{\mathrm{loc}}(w) : \|f\|_{\mathcal{L}^{p,\lambda}(w)} := \sup_Q |Q|^{-\frac{\lambda}{dp}} \Big(\int_Q |f|^p w \Big)^{1/p} < \infty \Big\},$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^d$.

Remark 1.3. Alternatively, we could define the Samko type weighted Morrey spaces with balls instead of cubes. If $w \equiv 1$, then $\mathcal{L}^{p,\lambda}(w) = \mathcal{L}^{p,\lambda}(\mathbb{R}^d)$, where $\mathcal{L}^{p,\lambda}(\mathbb{R}^d)$ is the classical Morrey space (see [31]).

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As we will work with Muckenhoupt weight characteristics, we recall the following definitions:

Definition 1.4 ([22, 32]). A weight $w \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is called a *Muckenhoupt $A_p(\mathbb{R}^d)$ weight* (or $w \in A_p(\mathbb{R}^d)$) if

$$[w]_{A_p} := \sup_Q \langle w\rangle_Q \langle w^{-\frac{1}{p-1}}\rangle_Q^{p-1} < \infty, \qquad 1 < p < \infty,$$
$$[w]_{A_1} := \sup_Q \langle w\rangle_Q \|w^{-1}\|_{L^\infty(Q)} < \infty, \qquad p = 1,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^d$ and $\langle w\rangle_Q := |Q|^{-1}\int_Q w$.

A weight w is said to belong to the *reverse Hölder class* $\mathrm{RH}_\sigma(\mathbb{R}^d)$ (or $w \in \mathrm{RH}_\sigma(\mathbb{R}^d)$) if

$$[w]_{\mathrm{RH}_\sigma} := \sup_Q \langle w^\sigma\rangle_Q^{1/\sigma} \langle w\rangle_Q^{-1} < \infty, \qquad 1 < \sigma < \infty,$$
$$[w]_{\mathrm{RH}_\infty} := \sup_Q \|w\|_{L^\infty(Q)} \langle w\rangle_Q^{-1} < \infty, \qquad \sigma = \infty.$$

The classes $\mathrm{RH}_\sigma(\mathbb{R}^d)$ and $A_p(\mathbb{R}^d)$ were introduced to study the *L^p-integrability* of the partial derivatives of a quasiconformal mapping and the weighted norm inequalities for *Hardy–Littlewood maximal function*, respectively; see [22, 32].

The following theorem of Rubio de Francia [33] (see also the work of Auscher–Martell [3]) on the extrapolation of *boundedness* on weighted spaces is one of the most important tools of modern harmonic analysis:

Theorem 1.5 ([3, Theorem 4.9] and [33]). *Let $1 \le p_- < p_+ \le \infty$, and T be a linear operator simultaneously defined and bounded on $L^{p_1}(\tilde{w})$ for* **some** *$p_1 \in [p_-, p_+]$ and* **all** *$\tilde{w} \in A_{p_1/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p_1)'}(\mathbb{R}^d)$, with $p' := p/(p-1)$. Then T is also defined and bounded on $L^p(w)$ for all $p \in (p_-, p_+)$ and all $w \in A_{p/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p)'}(\mathbb{R}^d)$.*

Remark 1.6. We do not seem to have an analogue of Rubio de Francia's extrapolation theorem on weighted Morrey spaces.

In [14, Theorem 3.31], a version of Theorem 1.5 is stated in terms of non-negative measurable pairs of functions (f, g). This means that one does not need to work with specific operators since nothing about the operators themselves is used (like linearity or sublinearity) and they play no role. However, we work with the pairs $(|f|, |Tf|)$, where T is a linear operator, since the abstract compactness results that we will use in order to prove Theorem 1.8 below hold for linear operators (see Theorem 2.3 of Cwikel–Kalton and Theorem 2.4 of Cwikel–Rochberg).

In the recent paper [25], the authors provided the following version for extrapolation of *compactness* (see also [8, 26] for extensions to multilinear operators):

Theorem 1.7 ([25, Theorems 1.3 and 2.4]). *In the setting of Theorem 1.5, suppose in addition that T is compact on $L^{p_1}(w_1)$ for* **some** $w_1 \in A_{p_1/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p_1)'}(\mathbb{R}^d)$. *Then T is compact on $L^p(w)$ for all $p \in (p_-, p_+)$ and all $w \in A_{p/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p)'}(\mathbb{R}^d)$.*

In this paper, we extend Theorem 1.7 to the setting of weighted Morrey space. In particular, we obtain the following:

Theorem 1.8. *Let $0 < \lambda < d$, $1 \le p_- < p_+ \le \infty$, and T be a linear operator simultaneously defined and bounded on $\mathcal{L}^{p,\lambda}(w)$ for* **all** $p \in (p_-, p_+)$ *and* **all** $w \in A_{p/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p)'}(\mathbb{R}^d)$. *Suppose in addition that T is compact on $\mathcal{L}^{p_1,\lambda}(w_1)$ for* **some** $p_1 \in [p_-, p_+]$ *and* **some** $w_1 \in A_{p_1/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p_1)'}(\mathbb{R}^d)$. *Then T is compact on $\mathcal{L}^{p,\lambda}(w)$ for all $p \in (p_-, p_+)$ and all $w \in A_{p/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p)'}(\mathbb{R}^d)$.*

Remark 1.9. Theorem 1.8 remains true in the case $p_+ = \infty$, provided that $p_1 < \infty$. Thus the reverse Hölder conditions on w, w_1 are vacuous. Due to Theorem 2.8 below, it seems that our results do not apply to other types of weighted Morrey spaces such as the Komori–Shirai type weighted Morrey space considered in [28]. In addition, notice that if we let $\lambda \to 0$ in Definition 1.2, then $\mathcal{L}^{p,0}(w) \equiv L^p(w)$ and hence Theorem 1.8 formally recovers Theorem 1.7.

When $w_1 \equiv 1$, Theorem 1.8 says that we can obtain weighted compactness if the weighted boundedness and unweighted compactness are already known. This case is relevant to all our applications in Sections 4–7.

The paper is organized as follows. In Section 2 we collect some previously known results from which the proof of Theorem 1.8 will be worked out in Section 3. In Sections 4–7 we provide several applications of our main result. An example of these applications to commutators of *Calderón–Zygmund operators* is the following (we refer to Sections 4 and 5 for the notions of $\mathrm{CMO}(\mathbb{R}^d)$ and Calderón–Zygmund operators, respectively):

Theorem 1.10. *Let $b \in \mathrm{CMO}(\mathbb{R}^d)$, and T be a Calderón–Zygmund operator that extends boundedly to $L^2(\mathbb{R}^d)$ and satisfies for all $f \in C_c^\infty(\mathbb{R}^d)$ the condition $Tf(x) = \lim_{\varepsilon \to 0} \int_{|x-y| \ge \varepsilon} K(x,y) f(y)\, dy$ for a.e. $x \in \mathbb{R}^d$. Then the commutator $[b, T]$ is compact on $\mathcal{L}^{p,\lambda}(w)$ for all $w \in A_s(\mathbb{R}^d) \cap \mathrm{RH}_t(\mathbb{R}^d)$ and all p, λ, s, t such that*

$$p \in (1, \infty), \quad 0 < \lambda < d, \quad s \in \left[1, \min\left\{p, \frac{d}{\lambda}\right\}\right], \quad t \in \left(\left(\frac{d}{s\lambda}\right)', \infty\right).$$

See also Theorems 6.4 and 7.4 for similar results on *rough singular integrals* and *Bochner–Riesz multipliers*. Although compactness of such operators on the unweighted Morrey spaces has been considered in the literature, obtaining compactness results on weighted Morrey spaces appears to be entirely new altogether.

Notation. Throughout the paper, we denote by C a positive constant which is independent of the main parameters but may change at each occurrence, and we write $f \lesssim g$ if $f \leq Cg$. The term cube will always refer to a cube $Q \subset \mathbb{R}^d$ and $|Q|$ will denote its Lebesgue measure. Recall from Definition 1.4 that $\langle w\rangle_Q$ denotes $|Q|^{-1}\int_Q w$, the average of w over Q. We write p' for the conjugate exponent to p, that is, $p' := p/(p-1)$.

2. Preliminaries. We collect the results from which the proof of Theorem 1.8 will be deduced in Section 3.

Let $S := \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$ and $\bar{S} := \{z \in \mathbb{C} : 0 \leq \mathrm{Re}(z) \leq 1\}$. Following [7, 24], we recall the following definitions of two complex interpolation functors:

Definition 2.1 (Calderón's first complex interpolation space). Let (X_0, X_1) be a compatible couple of Banach spaces.

(1) Define $\mathcal{F}(X_0, X_1)$ as the set of all functions $F : \bar{S} \to X_0 + X_1$ such that
   (a) F is continuous on $\bar{S}$ and $\sup_{z\in\bar{S}} \|F(z)\|_{X_0+X_1} < \infty$,
   (b) F is holomorphic on S,
   (c) the functions $t \in \mathbb{R} \mapsto F(j + it) \in X_j$ are bounded and continous on $\mathbb{R}$ for $j = 0, 1$.

   The space $\mathcal{F}(X_0, X_1)$ is equipped with the norm

$$\|F\|_{\mathcal{F}(X_0,X_1)} := \max\Big\{\sup_{t\in\mathbb{R}} \|F(it)\|_{X_0}, \sup_{t\in\mathbb{R}} \|F(1 + it)\|_{X_1}\Big\}.$$

(2) Let $\theta \in (0, 1)$. Define the complex interpolation space $[X_0, X_1]_\theta$ with respect to (X_0, X_1) as the set of all functions $x \in X_0 + X_1$ such that $x = F(\theta)$ for some $F \in \mathcal{F}(X_0, X_1)$. The norm on $[X_0, X_1]_\theta$ is defined by

$$\|x\|_{[X_0,X_1]_\theta} := \inf\{\|F\|_{\mathcal{F}(X_0,X_1)} : x = F(\theta) \text{ for some } F \in \mathcal{F}(X_0, X_1)\}.$$

Let Y be a Banach space. Set

$$\mathrm{Lip}(\mathbb{R}, Y) := \{f : \mathbb{R} \to Y \text{ continuous} : \|f\|_{\mathrm{Lip}(\mathbb{R},Y)} < \infty\},$$

where

$$\mathrm{Lip}(\mathbb{R}, Y) := \sup_{-\infty<s<t<\infty} \frac{\|f(t) - f(s)\|_Y}{|t - s|}.$$

Definition 2.2 (Calderón's second complex interpolation space). Suppose that $\bar{X} = (X_0, X_1)$ is a compatible couple of Banach spaces.

(1) Define $\mathcal{G}(X_0, X_1)$ as the set of all functions $F : \bar{S} :\to X_0 + X_1$ such that
   (a) F is continuous on $\bar{S}$ and $\sup_{z\in\bar{S}} \big\|\frac{F(z)}{1+|z|}\big\|_{X_0+X_1} < \infty$,

(b) F is holomorphic on S,
(c) the functions $t \in \mathbb{R} \mapsto F(j+it) \in X_j$ are Lipschitz continuous on $\mathbb{R}$ for $j = 0, 1$.

The space $\mathcal{G}(X_0, X_1)$ is equipped with the norm

$$\|F\|_{\mathcal{G}(X_0,X_1)} := \max\{\|F(i\,\cdot)\|_{\mathrm{Lip}(\mathbb{R},X_0)}, \|F(1+i\,\cdot)\|_{\mathrm{Lip}(\mathbb{R},X_1)}\}.$$

(2) Let $\theta \in (0,1)$. Define the complex interpolation space $[X_0, X_1]^\theta$ with respect to (X_0, X_1) to be the set of all functions $x \in X_0 + X_1$ such that $x = F'(\theta)$ for some $F \in \mathcal{G}(X_0, X_1)$. The norm on $[X_0, X_1]^\theta$ is defined by

$$\|x\|_{[X_0,X_1]^\theta} := \inf\{\|F\|_{\mathcal{G}(X_0,X_1)} : x = F'(\theta) \text{ for some } F \in \mathcal{G}(X_0, X_1)\}.$$

Our main abstract tools are the following theorems of Cwikel–Kalton [15] and Cwikel–Rochberg [16]:

THEOREM 2.3 ([15]). *Let (X_0, X_1) and (Y_0, Y_1) be Banach couples and T be a linear operator such that $T : X_0 + X_1 \to Y_0 + Y_1$ and $T : X_j \to Y_j$ boundedly for $j = 0, 1$. Suppose moreover that $T : X_1 \to Y_1$ is compact. Then also $T : [X_0, X_1]_\theta \to [Y_0, Y_1]_\theta$ is compact for $\theta \in (0,1)$ under* **any** *of the following four side conditions:*

(1) *X_1 has the UMD (unconditional martingale differences) property,*
(2) *X_1 is reflexive, and $X_1 = [X_0, E]_\alpha$ for some Banach space E and $\alpha \in (0,1)$,*
(3) *$Y_1 = [Y_0, F]_\beta$ for some Banach space F and $\beta \in (0,1)$,*
(4) *X_0 and X_1 are both complexified Banach lattices of measurable functions on a common measure space.*

We have swapped the roles of the indices 0 and 1 in comparison to [15]. For the UMD property, see [27, Ch. 4].

THEOREM 2.4 ([16, Theorem 2.2]). *Let (X_0, X_1) and (Y_0, Y_1) be arbitrary Banach couples. Let T be a bounded linear operator such that $T : X_0 + X_1 \to Y_0 + Y_1$. Suppose moreover that $T : [X_0, X_1]_\theta \to [Y_0, Y_1]_\theta$ is compact for some $\theta \in (0,1)$. Then also $T : [X_0, X_1]^\theta \to [Y_0, Y_1]_\theta$ is compact for the same $\theta \in (0,1)$.*

We consider a measurable function f, weights v, v_0, v_1 and numbers $p_0, p, p_1, R > 0$ fixed. The following objects depend on these quantities, but we do not always indicate this in the notation. We define

$$E_{R,0} := \left\{x \in \mathbb{R}^d : |f(x)|^{p_0-p}\frac{v_0(x)}{v(x)} \geq R\right\},$$

$$E_{R,1} := \left\{x \in \mathbb{R}^d : |f(x)|^{p_1-p}\frac{v_1(x)}{v(x)} \geq R\right\},$$

$$E_R := E_{R,0} \cup E_{R,1}.$$

Define further

$$f_R := f(1 - \chi_{E_R})$$

for $f \in \mathcal{L}^{q,\lambda}(v)$ and consider the condition

$$f = \lim_{R\to\infty} f_R \quad \text{in } \mathcal{L}^{q,\lambda}(v). \tag{2.1}$$

We will use Theorems 2.3 and 2.4 in the following special setting:

PROPOSITION 2.5. *Let* $0 < \lambda < d$, $1 \le p_- < p_+ \le \infty$, $q_1 \in [p_-, p_+]$, $q \in (p_-, p_+)$, *and*

$$v \in A_{q/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/q)'}(\mathbb{R}^d), \qquad v_1 \in A_{q_1/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/q_1)'}(\mathbb{R}^d).$$

Then

$$[\mathcal{L}^{q_0,\lambda}(v_0), \mathcal{L}^{q_1,\lambda}(v_1)]^\gamma = \mathcal{L}^{q,\lambda}(v),$$
$$[\mathcal{L}^{q_0,\lambda}(v_0), \mathcal{L}^{q_1,\lambda}(v_1)]_\gamma = \{f \in \mathcal{L}^{q,\lambda}(v) : (2.1) \textit{ holds}\}$$

for some $q_0 \in (p_-, p_+)$, $v_0 \in A_{q_0/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/q_0)'}(\mathbb{R}^d)$, *and* $\gamma \in (0,1)$.

We postpone the proof of Proposition 2.5 to the next section.

LEMMA 2.6. *If* $0 < \lambda < d$, $1 \le p_j < \infty$ *and* w_j *are weights, then the spaces* $X_j = \mathcal{L}^{p_j,\lambda}(w_j)$ *satisfy the condition 4 of Theorem* 2.3.

Proof. It is easy to verify that $X_j = \mathcal{L}^{p_j,\lambda}(w_j)$ are complexified Banach lattices of measurable functions on the common measure space $\mathbb{R}^d$ (see also [36]). ∎

REMARK 2.7. We observe that Morrey spaces do not satisfy any of the conditions (1), (2) and (3) of Theorem 2.3. For more details, see [27, Theorem 4.3.3] and [36].

We quote the following results from which the proof of Proposition 2.5 will follow:

THEOREM 2.8 ([24, Theorem 2.3]). *If* $0 < \lambda < d$, $q_0, q_1 \in [1, \infty)$ *and* w_0, w_1 *are two weights, then for all* $\theta \in (0,1)$,

$$[\mathcal{L}^{q_0,\lambda}(w_0), \mathcal{L}^{q_1,\lambda}(w_1)]^\theta = \mathcal{L}^{q,\lambda}(w),$$
$$[\mathcal{L}^{q_0,\lambda}(w_0), \mathcal{L}^{q_1,\lambda}(w_1)]_\theta = \{f \in \mathcal{L}^{q,\lambda}(w) : (2.1) \textit{ holds}\},$$

where

$$\frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \qquad w^{\frac{1}{q}} := w_0^{\frac{1-\theta}{q_0}} w_1^{\frac{\theta}{q_1}}. \tag{2.2}$$

REMARK 2.9. The authors of [24] proved Theorem 2.8 in the framework of generalized weighted Morrey spaces. As explained in [24, Example 1.2], if one takes $\varphi(x,r) = |B(x,r)|^{\frac{1}{q} - \frac{\lambda}{dq}}$ and $v = 1$ in [24, Definition 1.1], where $B = B(x,r)$ is the ball with center x and radius r and v is a weight, then the Samko type weighted Morrey space is an example of the generalized weighted Morrey space.

In order to connect Theorem 2.8 with the $A_{q/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/q)'}(\mathbb{R}^d)$ weights, we need:

LEMMA 2.10 ([25, Lemma 4.9]). *Let* $1 \le p_- < p_+ \le \infty$, $q_1 \in [p_-, p_+]$, $q \in (p_-, p_+)$, *and*

$$w_1 \in A_{q_1/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/q_1)'}(\mathbb{R}^d), \qquad w \in A_{q/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/q)'}(\mathbb{R}^d).$$

Then there exist $q_0 \in (p_-, p_+)$, $w_0 \in A_{q_0/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/q_0)'}(\mathbb{R}^d)$, *and* $\theta \in (0,1)$ *such that* (2.2) *holds.*

REMARK 2.11. Lemma 2.10 remains true in the case $p_+ = \infty$, provided that $q_1 < \infty$. In this case the reverse Hölder conditions on w, w_0, w_1 are vacuous and the proof is given in [25, Lemma 4.4].

3. The proof of the key Proposition 2.5 and Theorem 1.8. To complete the proof of Theorem 1.8 it remains to verify Proposition 2.5:

Proof of Proposition 2.5. We consider the case $p_+ < \infty$. The case $p_+ = \infty$ is proved in a similar way. With some $0 < \lambda < d$, we are given $1 \le p_- < p_+ < \infty$, $q_1 \in [p_-, p_+]$, $q \in (p_-, p_+)$ and weights $v \in A_{q/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/q)'}(\mathbb{R}^d)$, $v_1 \in A_{q_1/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/q_1)'}(\mathbb{R}^d)$. By Lemma 2.10, there is some $q_0 \in (p_-, p_+)$, a weight $v_0 \in A_{q_0/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/q_0)'}(\mathbb{R}^d)$, and $\theta \in (0,1)$ such that

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \qquad v^{\frac{1}{q}} = v_0^{\frac{1-\theta}{q_0}} v_1^{\frac{\theta}{q_1}}.$$

By Theorem 2.8, we then have

$$[\mathcal{L}^{q_0,\lambda}(v_0), \mathcal{L}^{q_1,\lambda}(v_1)]^\theta = \mathcal{L}^{q,\lambda}(v),$$
$$[\mathcal{L}^{q_0,\lambda}(v_0), \mathcal{L}^{q_1,\lambda}(v_1)]_\theta = \{f \in \mathcal{L}^{q,\lambda}(v) : (2.1) \text{ holds}\},$$

as we claimed. ■

We can now give the proof of our main result:

Proof of Theorem 1.8. Let $p_+ < \infty$ and recall that the assumptions of Theorem 1.8 are in force. The case $p_+ = \infty$ is proved in a similar way. In particular, T is a bounded linear operator on $\mathcal{L}^{p,\lambda}(w)$ for all $p \in (p_-, p_+)$ and all $w \in A_{p/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p)'}(\mathbb{R}^d)$. In addition, it is assumed that T is a compact operator on $\mathcal{L}^{p_1,\lambda}(w_1)$ for some $p_1 \in [p_-, p_+]$ and some $w_1 \in A_{p_1/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p_1)'}(\mathbb{R}^d)$. We need to prove that T is actually compact on $\mathcal{L}^{p,\lambda}(w)$ for all $p \in (p_-, p_+)$ and all $w \in A_{p/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p)'}(\mathbb{R}^d)$. Now, fix some $p \in (p_-, p_+)$ and $w \in A_{p/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p)'}(\mathbb{R}^d)$. By Proposition 2.5, we have

$$[\mathcal{L}^{p_0,\lambda}(w_0), \mathcal{L}^{p_1,\lambda}(w_1)]^\theta = \mathcal{L}^{p,\lambda}(w),$$
$$[\mathcal{L}^{p_0,\lambda}(w_0), \mathcal{L}^{p_1,\lambda}(w_1)]_\theta = \{f \in \mathcal{L}^{p,\lambda}(w) : (2.1) \text{ holds}\}$$

for some $p_0 \in (p_-, p_+)$, some $w_0 \in A_{p_0/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p_0)'}(\mathbb{R}^d)$ and some $\theta \in (0,1)$. Writing $X_j = Y_j = \mathcal{L}^{p_j,\lambda}(w_j)$, we know that $T : X_0+X_1 \to Y_0+Y_1$, that $T : X_0 \to Y_0$ is bounded (since T is bounded on all $\mathcal{L}^{q,\lambda}(w)$ with $q \in (p_-, p_+)$ and $w \in A_{q/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/q)'}(\mathbb{R}^d)$), and that $T : X_1 \to Y_1$ is compact (since this was assumed). By Lemma 2.6, the last condition 4 of Theorem 2.3 is also satisfied by these spaces $X_j = \mathcal{L}^{p_j,\lambda}(w_j)$. By Theorem 2.3, it follows that T is also compact on $[X_0, X_1]_\theta = [Y_0, Y_1]_\theta = \{f \in \mathcal{L}^{p,\lambda}(w) : (2.1) \text{ holds}\}$. Hence, by Theorem 2.4, we conclude that T is also compact from $[X_0, X_1]^\theta = \mathcal{L}^{p,\lambda}(w)$ to $[Y_0, Y_1]_\theta = \{f \in \mathcal{L}^{p,\lambda}(w) : (2.1) \text{ holds}\}$. In particular, this implies that T is also compact on $\mathcal{L}^{p,\lambda}(w)$. ∎

4. Commutators with functions of bounded mean oscillation. We indicate several applications of Theorem 1.8 which deal with commutators of the form

$$[b, T] : f \mapsto bT(f) - T(bf),$$

where the pointwise multiplier b belongs to the space

$$\mathrm{BMO}(\mathbb{R}^d) := \Big\{f : \mathbb{R}^d \to \mathbb{C} : \|f\|_{\mathrm{BMO}} := \sup_Q \langle |f - \langle f\rangle_Q|\rangle_Q < \infty\Big\}$$

of functions of bounded mean oscillation, or its subspace

$$\mathrm{CMO}(\mathbb{R}^d) := \overline{C_c^\infty(\mathbb{R}^d)}^{\mathrm{BMO}(\mathbb{R}^d)},$$

where the closure is in the BMO norm. We will need the following results of Duoandikoetxea–Rosenthal [20] (see also [19, 21]) on the extrapolation of boundedness on the corresponding weighted Morrey spaces from the assumption of weighted estimates on $L^p(w)$ spaces:

Theorem 4.1 ([20, Theorem 1.1]). *Let $1 \le \kappa \le p_1 < \infty$, and T be an operator defined and bounded on $L^{p_1}(w)$ for all $w \in A_{p_1/\kappa}(\mathbb{R}^d)$. Then T is bounded on $\mathcal{L}^{p,\lambda}(w)$ for all $p \in (\kappa, \infty)$ (and also $p = \kappa$ if $p_1 = \kappa$), all $0 < \lambda < d - d/\sigma$ and all $w \in A_{p/\kappa}(\mathbb{R}^d) \cap \mathrm{RH}_\sigma(\mathbb{R}^d)$.*

Theorem 4.2 ([20, Corollary 3.2]). *Let $1 \le p_- \le p_1 \le p_+ < \infty$, and T be an operator defined and bounded on $L^{p_1}(w)$ for all $w \in A_{p_1/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p_1)'}(\mathbb{R}^d)$. Then for all $p \in (p_-, p_+)$ (and also $p = p_-$ if $p_1 = p_-$) and $\sigma > (p_+/p)'$, T is bounded on $\mathcal{L}^{p,\lambda}(w)$ for all $0 < \lambda < d(1 - p/p_+ - 1/\sigma)$ and all $w \in A_{p/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_\sigma(\mathbb{R}^d)$.*

Remark 4.3. In [20], Theorems 4.1 and 4.2 are stated in terms of non-negative measurable pairs of functions (f, g). This has the advantage of providing immediately several different versions. In our applications below, we will apply these with $[b,T]$ in place of T.

In [1, 4], the following general results on weighted boundedness about commutators were obtained:

THEOREM 4.4 ([1]). *Let $1 \le \kappa < p_1 < \infty$, and T be a linear operator defined and bounded on $L^{p_1}(\tilde{w})$ for all $\tilde{w} \in A_{p_1/\kappa}(\mathbb{R}^d)$, with the operator norm dominated by some increasing function of $[\tilde{w}]_{A_{p_1/\kappa}}$. Suppose moreover that $b \in \mathrm{BMO}(\mathbb{R}^d)$. Then also $[b,T]$ extends to a bounded linear operator on $L^{p_1}(\tilde{w})$ for all $\tilde{w} \in A_{p_1/\kappa}(\mathbb{R}^d)$, and its operator norm is dominated by another increasing function of $[\tilde{w}]_{A_{p_1/\kappa}}$.*

The statement in [1, Theorem 2.13] is somewhat more general, but the above particular case is easily seen to be contained in it.

THEOREM 4.5 ([4, Corollary 5.3]). *Let $1 \le p_- < p_1 < p_+ \le \infty$, and T be a linear operator bounded on $L^{p_1}(\tilde{w})$ for all $\tilde{w} \in A_{p_1/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p_1)'}(\mathbb{R}^d)$. If $b \in \mathrm{BMO}(\mathbb{R}^d)$, then $[b,T]$ is bounded on $L^{p_1}(\tilde{w})$ for all $\tilde{w} \in A_{p_1/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p_1)'}(\mathbb{R}^d)$.*

A combination of Theorems 1.8, 4.1, 4.2, 4.4 and 4.5 immediately gives the following two corollaries:

COROLLARY 4.6. *Let $1 \le \kappa < p_+ < \infty$, $1 < \sigma_1 < \infty$,*

$$p_- = \max\{\kappa, p_+(1 - 1/\sigma_1)\},$$

and $0 < \lambda_1 < d - d/\sigma_1$. Suppose moreover that

(1) *T is a linear operator defined and bounded on $L^{\tilde{p}_1}(\tilde{w})$ for some $\tilde{p}_1 \in (\kappa,\infty)$ and all $\tilde{w} \in A_{\tilde{p}_1/\kappa}(\mathbb{R}^d)$, with the operator norm dominated by some increasing function of $[\tilde{w}]_{A_{\tilde{p}_1/\kappa}}$,*
(2) *the commutator $[b,T]$ is compact on $\mathcal{L}^{p_1,\lambda_1}(w_1)$ for some $b \in \mathrm{BMO}(\mathbb{R}^d)$, some $p_1 \in [p_-, p_+]$ and some $w_1 \in A_{p_1/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p_1)'}(\mathbb{R}^d)$.*

Then $[b,T]$ is compact on $\mathcal{L}^{p,\lambda_1}(w)$ for the same b, for all $p \in (p_-, p_+)$ and all $w \in A_{p/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p)'}(\mathbb{R}^d)$.

Proof. We verify the assumptions of Theorem 1.8 for the fixed numbers $\lambda_1, p_1, \sigma_1, \kappa, p_+, d$ and the weight w_1 appearing in the statement of the corollary, and the operator $[b,T]$ in place of T. By Theorem 4.4, $[b,T]$ is bounded on $L^{\tilde{p}_1}(\tilde{w})$ for the same $\tilde{p}_1 \in (\kappa,\infty)$ and all $\tilde{w} \in A_{p_1/\kappa}(\mathbb{R}^d)$. Then, by Theorem 4.1 with $[b,T]$ in place of T, $[b,T]$ is bounded on $\mathcal{L}^{p,\lambda}(\tilde{w})$ for all $p \in (\kappa,\infty)$, all $0 < \lambda < d - d/\sigma$ and all $\tilde{w} \in A_{p/\kappa}(\mathbb{R}^d) \cap \mathrm{RH}_\sigma(\mathbb{R}^d)$. By choosing $\lambda = \lambda_1$ and $\sigma = \sigma_1$ to be the fixed numbers appearing in the statement of the corollary, $[b,T]$ is bounded on $\mathcal{L}^{p,\lambda_1}(\tilde{w})$ for all $p \in (\kappa,\infty)$ and all $\tilde{w} \in A_{p/\kappa}(\mathbb{R}^d) \cap \mathrm{RH}_{\sigma_1}(\mathbb{R}^d)$. In particular, $[b,T]$ is bounded on $\mathcal{L}^{p,\lambda_1}(\tilde{w})$

for all $p \in (p_-, p_+)$ and all $\tilde{w} \in A_{p/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p)'}(\mathbb{R}^d)$ where we recall that $p_- = \max\{\kappa, p_+(1 - 1/\sigma_1)\}$ and $1 \le p_- < p_+ < \infty$. By assumption, $[b, T]$ is compact on $\mathcal{L}^{p_1,\lambda_1}(w_1)$ for some $p_1 \in [p_-, p_+]$ and some $w_1 \in A_{p_1/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p_1)'}(\mathbb{R}^d)$. Thus the assumptions of Theorem 1.8 hold for the operator $[b, T]$ in place of T. Hence, the conclusion of Theorem 1.8 gives the compactness of $[b, T]$ on $\mathcal{L}^{p,\lambda_1}(w)$ for all $p \in (p_-, p_+)$ and all $w \in A_{p/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p)'}(\mathbb{R}^d)$. ■

Corollary 4.7. *Let $1 \le p_- < p_+ < \infty$, $1 < \sigma_1 < \infty$, $0 < \lambda_1 < d - d/\sigma_1$, and*

$$q_+ = p_+\left(1 - \frac{1}{\sigma_1} - \frac{\lambda_1}{d}\right), \qquad q_- = \max\left\{q_+\left(1 - \frac{1}{\sigma_1}\right), p_-\right\}.$$

Suppose moreover that

1. *T is a linear operator defined and bounded on $L^{\tilde{p}_1}(\tilde{w})$ for some $\tilde{p}_1 \in (p_-, p_+)$ and all $\tilde{w} \in A_{\tilde{p}_1/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/\tilde{p}_1)'}(\mathbb{R}^d)$,*
2. *the commutator $[b, T]$ is compact on $\mathcal{L}^{p_1,\lambda_1}(w_1)$ for some $b \in \mathrm{BMO}(\mathbb{R}^d)$, some $p_1 \in [q_-, q_+]$ and some $w_1 \in A_{p_1/q_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(q_+/p_1)'}(\mathbb{R}^d)$.*

Then $[b, T]$ is compact on $\mathcal{L}^{p,\lambda_1}(w)$ for the same b, for all $p \in (q_-, q_+)$ and all $w \in A_{p/q_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(q_+/p)'}(\mathbb{R}^d)$.

Proof. We verify the assumptions of Theorem 1.8 for the fixed numbers $\lambda_1, p_1, \sigma_1, p_-, p_+, d$ and the weight w_1 appearing in the statement of the corollary, and the operator $[b, T]$ in place of T. By Theorem 4.5, $[b, T]$ is bounded on $L^{\tilde{p}_1}(\tilde{w})$ for the same $\tilde{p}_1 \in (p_-, p_+)$ and all $\tilde{w} \in A_{\tilde{p}_1/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/\tilde{p}_1)'}(\mathbb{R}^d)$. Then, by Theorem 4.2 with $[b, T]$ in place of T, for all $p \in (p_-, p_+)$ and $\sigma > (p_+/p)'$, $[b, T]$ is bounded on $\mathcal{L}^{p,\lambda}(\tilde{w})$ for all $0 < \lambda < d(1 - p/p_+ - 1/\sigma)$ and all $\tilde{w} \in A_{p/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_\sigma(\mathbb{R}^d)$. Equivalently, by expressing the range of parameter p in terms of λ, $[b, T]$ is bounded on $\mathcal{L}^{p,\lambda}(\tilde{w})$ for all $p \in (p_-, p_+(1 - 1/\sigma - \lambda/\sigma))$ with $0 < \lambda < d - d/\sigma$ and all $\tilde{w} \in A_{p/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_\sigma(\mathbb{R}^d)$. By choosing $\lambda = \lambda_1$ and $\sigma = \sigma_1$ to be the fixed numbers appearing in the statement of the corollary, $[b, T]$ is bounded on $\mathcal{L}^{p,\lambda_1}(\tilde{w})$ for all $p \in (p_-, q_+)$ and all $\tilde{w} \in A_{p/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{\sigma_1}(\mathbb{R}^d)$ where we recall that $q_+ = p_+(1 - 1/\sigma_1 - \lambda_1/d)$ and $1 \le p_- < q_+ < \infty$. In particular, $[b, T]$ is bounded on $\mathcal{L}^{p,\lambda_1}(\tilde{w})$ for all $p \in (q_-, q_+)$ and all $\tilde{w} \in A_{p/q_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(q_+/p)'}(\mathbb{R}^d)$ where we recall that $q_- = \max\{q_+(1 - 1/\sigma_1), p_-\}$ and $1 \le q_- < q_+ < \infty$. By assumption, $[b, T]$ is compact on $\mathcal{L}^{p_1,\lambda_1}(w_1)$ for some $p_1 \in [q_-, q_+]$ and some $w_1 \in A_{p_1/q_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(q_+/p_1)'}(\mathbb{R}^d)$. Thus the assumptions of Theorem 1.8 hold for the operator $[b, T]$ in place of T. Hence, the conclusion of Theorem 1.8 gives the compactness of $[b, T]$ on $\mathcal{L}^{p,\lambda_1}(w)$ for all $p \in (q_-, q_+)$ and all $w \in A_{p/q_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(q_+/p)'}(\mathbb{R}^d)$. ■

5. Commutators of Calderón–Zygmund singular integrals. In our application below, we consider *Calderón–Zygmund singular integral operators* T which are as follows: T is a linear operator defined on a suitable class of test functions on $\mathbb{R}^d$, and it has the representation

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)\,dy, \qquad x \notin \operatorname{supp} f,$$

where the kernel K satisfies the *size condition*

$$|K(x,y)| \lesssim \frac{1}{|x-y|^d}$$

and, for some $\delta \in (0,1]$, the *smoothness condition*

$$|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \lesssim \frac{|x-z|^\delta}{|x-y|^{d+\delta}}$$

for all $x, z, y \in \mathbb{R}^d$ such that $|x-y| > \frac{1}{2}|x-z|$. We will need the following classical result of Coifman–Fefferman [11]:

Theorem 5.1 ([11]). *Let T be a Calderón–Zygmund operator that extends to a bounded operator on $L^2(\mathbb{R}^d)$. Then T extends to a bounded operator on $L^p(w)$ for all $p \in (1,\infty)$ and all $w \in A_p(\mathbb{R}^d)$.*

The following result of Arai–Nakai [2], based on Sawano and Shirai's method [35], provides a concrete condition to verify the assumptions of Corollary 4.6:

Theorem 5.2 ([2]). *Let $0 < \lambda < d$, $1 < p < \infty$ and T be a Calderón–Zygmund operator that extends to a bounded operator on $L^2(\mathbb{R}^d)$. Assume also that for all $f \in C_c^\infty(\mathbb{R}^d)$ we have $Tf(x) = \lim_{\varepsilon\to 0} \int_{|x-y|\ge\varepsilon} K(x,y)f(y)\,dy$ for a.e. $x \in \mathbb{R}^d$. If $b \in \mathrm{CMO}(\mathbb{R}^d)$, then $[b,T]$ is compact on the unweighted $\mathcal{L}^{p,\lambda}(\mathbb{R}^d)$.*

Remark 5.3. In [2], Theorem 5.2 is stated and proved in the setting of the generalized Morrey space $L^{(p,\varphi)}(\mathbb{R}^d)$, where $\varphi : \mathbb{R}^d \times (0,\infty) \to (0,\infty)$ is a variable growth function. This space coincides with $\mathcal{L}^{p,\lambda}(\mathbb{R}^d)$ by choosing $\varphi(x,r) = |B(x,r)|^{\lambda/d-1}$, where $B = B(x,r)$ is the ball with center x and radius r.

By combining Corollary 4.6 and Theorems 5.1 and 5.2 we obtain the following result which appears to be new:

Lemma 5.4. *Let $1 < p_+ < \infty$, $1 < \sigma < \infty$,*

$$p_- = \max\{1, p_+(1-1/\sigma)\},$$

and $0 < \lambda < d - d/\sigma$. Suppose moreover that $b \in \mathrm{CMO}(\mathbb{R}^d)$, T is a Calderón–Zygmund operator that extends boundedly to $L^2(\mathbb{R}^d)$ and satisfies for all $f \in C_c^\infty(\mathbb{R}^d)$ the condition $Tf(x) = \lim_{\varepsilon\to 0}\int_{|x-y|\ge\varepsilon} K(x,y)f(y)\,dy$ for a.e. $x \in \mathbb{R}^d$.

Then the commutator $[b,T]$ *is compact on* $\mathcal{L}^{p,\lambda}(w)$ *for all* $p\in(p_-,p_+)$ *and all* $w\in A_{p/p_-}(\mathbb{R}^d)\cap \mathrm{RH}_{(p_+/p)'}(\mathbb{R}^d)$.

Proof. Let us fix some $\sigma_1\in(1,\infty)$, $\lambda_1\in(0,d-d/\sigma_1)$, $p_1\in[p_-,p_+]$ and $\tilde p_1\in(1,\infty)$ for which we verify the assumptions of Corollary 4.6 with $\kappa=1$. By Theorem 5.1, T extends to a bounded operator on $L^{\tilde p_1}(\tilde w)$ for all $\tilde w\in A_{\tilde p_1}(\mathbb{R}^d)$. By Theorem 5.2, $[b,T]$ is a compact operator on $\mathcal{L}^{p_1,\lambda_1}(\mathbb{R}^d)=\mathcal{L}^{p_1,\lambda_1}(w_1)$ with $w_1\equiv 1\in A_{p_1/p_-}(\mathbb{R}^d)\cap\mathrm{RH}_{(p_+/p_1)'}(\mathbb{R}^d)$. Thus Corollary 4.6 applies to give the compactness of $[b,T]$ on $\mathcal{L}^{p,\lambda_1}(w)$ for all $p\in(p_-,p_+)$ and all $w\in A_{p/p_-}(\mathbb{R}^d)\cap\mathrm{RH}_{(p_+/p)'}(\mathbb{R}^d)$. ∎

Now, we check the conditions of Lemma 5.4 in such a way that we obtain the following:

Proof of Theorem 1.10. Under the conditions appearing in the assumptions and conclusion of the theorem, we check that we can find parameters σ and p_+,p_- as in Lemma 5.4. In particular, by choosing $\sigma\in((d/\lambda)',(t's)']$ $p_+\in[t'p,\sigma'p/s]$, and $p_-=\max\{1,p_+(1-1/\sigma)\}$, Lemma 5.4 applies to give the compactness of $[b,T]$ on $\mathcal{L}^{p,\lambda}(w)$ for all $p\in(p_-,p_+)$ and all $w\in A_{p/p_-}(\mathbb{R}^d)\cap\mathrm{RH}_{(p_+/p)'}(\mathbb{R}^d)$. It remains to check that this covers all $w\in A_s(\mathbb{R}^d)\cap\mathrm{RH}_t(\mathbb{R}^d)$ as in the statement of the theorem. Since $p/p_-\ge s$ and $p_+/p\ge t'$, the monotonicity of the $A_s(\mathbb{R}^d)$ and $\mathrm{RH}_t(\mathbb{R}^d)$ classes implies that $[b,T]$ is compact on $\mathcal{L}^{p,\lambda}(w)$ for all $w\in A_s(\mathbb{R}^d)\cap\mathrm{RH}_t(\mathbb{R}^d)$ and all p,λ,s,t such that

$$p\in(1,\infty),\qquad 0<\lambda<d,\qquad s\in\left[1,\min\left\{p,\frac{d}{\lambda}\right\}\right],\qquad t\in\left(\left(\frac{d}{s\lambda}\right)',\infty\right). \ \blacksquare$$

6. Commutators of rough singular integrals. Let Ω be homogeneous of degree zero in $\mathbb{R}^d$, integrable, and have mean value zero on the unit sphere S^{d-1}. Define the singular integral operator T_Ω by

$$T_\Omega f(x):=\lim_{\varepsilon\to 0}\int\limits_{|x-y|>\varepsilon}\frac{\Omega(x-y)}{|x-y|^d}f(y)\,dy.$$

Duoandikoetxea [17] and Watson [37] considered the following weighted estimates for T_Ω:

THEOREM 6.1 ([17, 37]). *Let* $r\in(1,\infty)$ *and* $\Omega\in L^r(S^{d-1})$ *be homogeneous of order zero with vanishing mean on* S^{d-1}. *Then* T_Ω *extends to a bounded operator on* $L^p(w)$ *for all* $p\in(r',\infty)$ *and all* $w\in A_{p/r'}(\mathbb{R}^d)$.

The unweighted compactness result about the commutator $[b,T_\Omega]$ is due to Guo–Hu [23]:

THEOREM 6.2 ([23, Theorem 1.8]). *Let* $0<\lambda<d$, $r\in(1,\infty)$ *and* $\Omega\in L^r(S^{d-1})$ *be homogeneous of order zero with vanishing mean on* S^{d-1}.

Let $b \in \mathrm{CMO}(\mathbb{R}^d)$*. Then the commutator* $[b, T_\Omega]$ *is compact on* $\mathcal{L}^{p,\lambda}(\mathbb{R}^d)$ *for all* $p \in (r', \infty)$*.*

A combination of Corollary 4.6 and Theorem 6.1, together with Theorem 6.2, gives the following new result:

Lemma 6.3. *Let* $1 \le r' < p_+ < \infty$*,* $1 < \sigma < \infty$*,*

$$p_- = \max\{r', p_+(1 - 1/\sigma)\},$$

$0 < \lambda < d - d/\sigma$*, and* $r \in (1, \infty)$*, and let* $\Omega \in L^r(S^{d-1})$ *be homogeneous of order zero with vanishing mean on* S^{d-1}*. Let* $b \in \mathrm{CMO}(\mathbb{R}^d)$*. Then the commutator* $[b, T_\Omega]$ *is compact on* $\mathcal{L}^{p,\lambda}(w)$ *for all* $p \in (p_-, p_+)$ *and all* $w \in A_{p/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p)'}(\mathbb{R}^d)$*.*

Proof. We verify the assumptions of Corollary 4.6 with $\kappa = r'$, an arbitrary $\sigma_1 \in (1, \infty)$, $\lambda_1 \in (0, d - d/\sigma_1)$, $p_1 \in [p_-, p_+]$, $\tilde{p}_1 \in (r', \infty)$ and $w_1 \equiv 1 \in A_{p_1/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p_1)'}(\mathbb{R}^d)$. Theorem 6.2 guarantees that $[b, T_\Omega]$ is compact on $\mathcal{L}^{p_1,\lambda_1}(w_1)$ for the exponent $p_1 \in [p_-, p_+]$ and weight $w_1 \equiv 1 \in A_{p_1/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p_1)'}(\mathbb{R}^d)$. On the other hand, a direct application of Theorem 6.1 shows that T_Ω is bounded on $L^{\tilde{p}_1}(\tilde{w})$ for all $\tilde{w} \in A_{\tilde{p}_1/r'}(\mathbb{R}^d)$. Thus Corollary 4.6 applies to give the compactness of $[b, T_\Omega]$ on $\mathcal{L}^{p,\lambda_1}(w)$ for all $p \in (p_-, p_+)$ and all $w \in A_{p/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p)'}(\mathbb{R}^d)$. ∎

Now, we check the conditions of Lemma 6.3 in such a way that we obtain the following:

Theorem 6.4. *Let* $r \in (1, \infty)$ *and* $\Omega \in L^r(S^{d-1})$ *be homogeneous of order zero with vanishing mean on* S^{d-1}*. Let* $b \in \mathrm{CMO}(\mathbb{R}^d)$*. Then the commutator* $[b, T_\Omega]$ *is compact on* $\mathcal{L}^{p,\lambda}(w)$ *for all* $w \in A_s(\mathbb{R}^d) \cap \mathrm{RH}_t(\mathbb{R}^d)$ *and all* p, λ, s, t *such that*

$$p \in (r', \infty), \qquad 0 < \lambda < d, \qquad s \in \left[1, \min\left\{\frac{p}{r'}, \frac{d}{\lambda}\right\}\right], \qquad t \in \left(\left(\frac{d}{s\lambda}\right)', \infty\right).$$

Proof. Under the conditions appearing in the assumptions and conclusion of the theorem, we check that we can find parameters σ and p_+, p_- as in Lemma 6.3. In particular, by choosing $\sigma \in ((d/\lambda)', (t's)']$, $p_+ \in [t'p, \sigma' p/s]$ and $p_- = \max\{r', p_+(1 - 1/\sigma)\}$, Lemma 6.3 applies to give the compactness of $[b, T_\Omega]$ on $\mathcal{L}^{p,\lambda}(w)$ for all $p \in (p_-, p_+)$ and all $w \in A_{p/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/p)'}(\mathbb{R}^d)$. It remains to check that this covers all $w \in A_s(\mathbb{R}^d) \cap \mathrm{RH}_t(\mathbb{R}^d)$ as in the statement of the theorem. Since $p/p_- \ge s$ and $p_+/p \ge t'$, the monotonicity of the $A_s(\mathbb{R}^d)$ and $\mathrm{RH}_t(\mathbb{R}^d)$ classes implies that $[b, T_\Omega]$ is compact on $\mathcal{L}^{p,\lambda}(w)$ for all $w \in A_s(\mathbb{R}^d) \cap \mathrm{RH}_t(\mathbb{R}^d)$ and all p, λ, s, t such that

$$p \in (r', \infty), \qquad 0 < \lambda < d, \qquad s \in \left[1, \min\left\{\frac{p}{r'}, \frac{d}{\lambda}\right\}\right], \qquad t \in \left(\left(\frac{d}{s\lambda}\right)', \infty\right). \ ∎$$

7. Commutators of Bochner–Riesz multipliers. In this section we will apply Theorem 1.8 to the commutators of Bochner–Riesz multipliers in dimensions $d \geq 2$. Following [25, 29], we recall that a *Bochner–Riesz multiplier* is a Fourier multiplier B^κ with the symbol $(1-|\xi|^2)_+^\kappa$, where $\kappa > 0$ and $t_+ = \max(t,0)$. That is, the Bochner–Riesz operator is defined, on the class $\mathcal{S}(\mathbb{R}^d)$ of Schwartz functions, by

$$\widehat{B^\kappa f}(\xi) = (1-|\xi|^2)_+^\kappa \widehat{f}(\xi),$$

where $\widehat{f}$ denotes the Fourier transform of f.

The following Bochner–Riesz conjecture is well-known (see also the works [10, 13] in two dimensions and [5, 30] in the case $d \geq 3$):

Conjecture 7.1 (Bochner–Riesz conjecture). *For $0 < \kappa < \frac{d-1}{2}$, we have $B^\kappa : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ if*

$$p \in \left(\frac{2d}{d+1+2\kappa}, \frac{2d}{d-1-2\kappa}\right).$$

In [29], an equivalent form of the Bochner–Riesz Conjecture 7.1 is stated as follows (see also [9, 12, 13] and [18, Section 8.5] for the connection between the Bochner–Riesz and the S_τ Fourier multipliers):

Conjecture 7.2. *Let $\mathbf{1}_{[-1/4,1/4]} \leq \chi \leq \mathbf{1}_{[-1/2,1/2]}$ be a Schwartz function and denote by the S_τ Fourier multiplier with symbol $\chi((|\xi|-1)/\tau)$. If $\frac{2d}{d+1} < p < \frac{2d}{d-1}$, then*

$$\|S_\tau\|_{L^p(\mathbb{R}^d)\mapsto L^p(\mathbb{R}^d)} \leq C_\epsilon \tau^{-\epsilon}, \tag{7.3}$$

where $0 < \tau < 1$ and C_ϵ is a constant that depends on $0 < \epsilon < 1$.

The following weighted estimates for B^κ were obtained in [25]:

Theorem 7.1 ([25, Corollary 10.5]). *If $d = 2$, $0 < \kappa < 1/2$ and*

$$p \in \left(\frac{4}{1+6\kappa}, \frac{4}{1-2\kappa}\right),$$

then B^κ is bounded on $L^p(w)$ for all

$$w \in A_{\frac{p(1+6\kappa)}{4}}(\mathbb{R}^2) \cap RH_{\left(\frac{4}{p(1-2\kappa)}\right)'}(\mathbb{R}^2).$$

Moreover, if $d \geq 3$, $0 < \kappa < \frac{d-1}{2}$, $1 < p_0 < 2$ is such that the estimate (7.3) of Conjecture 7.2 holds, and

$$p \in \left(\frac{p_0(d-1)}{d-1+2\kappa(p_0-1)}, \frac{p_0(d-1)}{d-1-2\kappa}\right),$$

then B^κ is bounded on $L^p(w)$ for all

$$w \in A_{\frac{p(d-1+2\kappa(p_0-1))}{p_0(d-1)}}(\mathbb{R}^d) \cap \mathrm{RH}_{\left(\frac{p_0(d-1)}{p(d-1-2\kappa)}\right)'}(\mathbb{R}^d).$$

In [6], Bu–Chen–Hu proved the following unweighted compactness result:

THEOREM 7.2 ([6, Theorems 1.4 and 1.5]). *If $d=2$, $0<\kappa<1/2$,*

$$p \in \left(\frac{4}{3+2\kappa}, \frac{4}{1-2\kappa}\right),$$

and

$$\lambda \in (0, 2\kappa\theta_p/(2\kappa\theta_p + 1 - 2\kappa)) \quad \text{with}$$
$$\theta_p = \frac{1}{1+2\kappa} \min\{4/p - (1-2\kappa), 3 + 2\kappa - 4/p\},$$

then for $b \in \mathrm{CMO}(\mathbb{R}^2)$, the commutator $[b, B^\kappa]$ is compact on $\mathcal{L}^{p,\lambda}(\mathbb{R}^2)$. Moreover, if $d \geq 3$, $\frac{d-1}{2d+2} < \kappa < \frac{d-1}{2}$,

$$p \in \left(\frac{2d}{d+1+2\kappa}, \frac{2d}{d-1-2\kappa}\right),$$

and

$$\lambda \in (0, 2\kappa\theta_p/(2\kappa\theta_p + d - 1 - 2\kappa)) \quad \text{with}$$
$$\theta_p = \frac{1}{1+2\kappa} \min\{2d/p - (d-1-2\kappa), d + 1 + 2\kappa - 2d/p\},$$

then for $b \in \mathrm{CMO}(\mathbb{R}^d)$, the commutator $[b, B^\kappa]$ is compact on $\mathcal{L}^{p,\lambda}(\mathbb{R}^d)$.

By combining Corollary 4.7 and Theorem 7.1, together with Theorem 7.2, we obtain the following new weighted compactness result:

LEMMA 7.3. *If $d=2$, $0<\kappa<1/2$, $1<\sigma<\infty$, $0<\lambda<d-d/\sigma$, and*

$$q_+ = \frac{4}{1-2\kappa}\left(1 - \frac{1}{\sigma} - \frac{\lambda}{d}\right), \qquad q_- = \max\left\{q_+\left(1-\frac{1}{\sigma}\right), \frac{4}{1+6\kappa}\right\},$$

then for $b \in \mathrm{CMO}(\mathbb{R}^2)$, the commutator $[b, B^\kappa]$ is compact on $\mathcal{L}^{p,\lambda}(w)$ for all $p \in (q_-, q_+)$ and all

$$w \in A_{p/q_-}(\mathbb{R}^2) \cap \mathrm{RH}_{(q_+/p)'}(\mathbb{R}^2).$$

Moreover, if $d \geq 3$, $\frac{d-1}{2d+2} < \kappa < \frac{d-1}{2}$, $1<\sigma<\infty$, $0<\lambda<d-\frac{d}{\sigma}$, $1<p_0<2$ is such that the estimate (7.3) of Conjecture 7.2 holds and

$$q_+ = \frac{p_0(d-1)}{d-1-2\kappa}\left(1-\frac{1}{\sigma}-\frac{\lambda}{d}\right),$$
$$q_- = \max\left\{q_+\left(1-\frac{1}{\sigma}\right), \frac{p_0(d-1)}{d-1+2\kappa(p_0-1)}\right\},$$

then for $b \in \mathrm{CMO}(\mathbb{R}^d)$, the commutator $[b, B^\kappa]$ is compact on $\mathcal{L}^{p,\lambda}(w)$ for all $p \in (q_-, q_+)$ and all

$$w \in A_{p/q_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(q_+/p)'}(\mathbb{R}^d).$$

Proof. Let $d \geq 3$, $\frac{d-1}{2d+2} < \kappa < \frac{d-1}{2}$ and p_0 be as in the assumptions. We verify the assumptions of Corollary 4.7 for the fixed exponents $\sigma_1 \in (1, \infty)$, $\lambda_1 \in (0, d - d/\sigma_1)$, $p_1 \in [q_-, q_+]$ and $\tilde{p}_1 \in (p_-, p_+)$ with $p_- = \frac{p_0(d-1)}{d-1+2\kappa(p_0-1)}$, $p_+ = \frac{p_0(d-1)}{d-1-2\kappa}$ and $1 < p_- < p_+ < \infty$. By Theorem 7.1, B^κ is a bounded operator on $L^{\tilde{p}_1}(\tilde{w})$ for all

$$\tilde{w} \in A_{\tilde{p}_1/p_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(p_+/\tilde{p}_1)'}(\mathbb{R}^d).$$

By Theorem 7.2, $[b, B^\kappa]$ is a compact operator on $\mathcal{L}^{p_1,\lambda_1}(\mathbb{R}^d) = \mathcal{L}^{p_1,\lambda_1}(w_1)$ with

$$w_1 \equiv 1 \in A_{p_1/q_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(q_+/p_1)'}(\mathbb{R}^d).$$

Thus Corollary 4.7 applies to give the compactness of $[b, B^\kappa]$ on $\mathcal{L}^{p,\lambda_1}(w)$ for all $p \in (q_-, q_+)$ and all

$$w \in A_{p/q_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(q_+/p)'}(\mathbb{R}^d).$$

The case $d = 2$ follows in a similar way. ∎

Now, we check the conditions of Lemma 7.3 in such a way that we obtain the following:

THEOREM 7.4.

(1) *Let $d = 2$, $0 < \kappa < 1/2$ and denote $p_- = \frac{4}{1+6\kappa}$. Then for $b \in \mathrm{CMO}(\mathbb{R}^2)$, the commutator $[b, B^\kappa]$ is compact on $\mathcal{L}^{p,\lambda}(w)$ for all $w \in A_s(\mathbb{R}^2) \cap \mathrm{RH}_t(\mathbb{R}^2)$ and all p, λ, s, t such that*

$$p \in (p_-, \infty), \quad 0 < \lambda < d, \quad s \in \left[1, \min\left\{\frac{p}{p_-}, \frac{d}{\lambda}\right\}\right], \quad t \in \left(\left(\frac{d}{s\lambda}\right)', \infty\right).$$

(2) *Let $d \geq 3$, $\frac{d-1}{2d+2} < \kappa < \frac{d-1}{2}$, $1 < p_0 < 2$ be such that the estimate (7.3) of Conjecture 7.2 holds and denote $p_- = \frac{p_0(d-1)}{d-1+2\kappa(p_0-1)}$. Then for $b \in \mathrm{CMO}(\mathbb{R}^d)$, the commutator $[b, B^\kappa]$ is compact on $\mathcal{L}^{p,\lambda}(w)$ for all $w \in A_u(\mathbb{R}^d) \cap \mathrm{RH}_v(\mathbb{R}^d)$ and all p, λ, u, v such that*

$$p \in (p_-, \infty), \quad 0 < \lambda < d, \quad u \in \left[1, \min\left\{\frac{p}{p_-}, \frac{d}{\lambda}\right\}\right], \quad v \in \left(\left(\frac{d}{u\lambda}\right)', \infty\right).$$

Proof. We prove the theorem in the case (2). Under these assumptions and the conditions appearing in the conclusion of theorem, we check that we can find parameters σ and q_+, q_- as in the Lemma 7.3. In particular, by choosing $\sigma \in ((d/\lambda)', (v'u)']$, $q_+ \in [v'p, \sigma' p/u]$, $q_- = \max\{q_+(1 - 1/\sigma), p_-\}$, Lemma 7.3 applies to give the compactness of $[b, B^\kappa]$ on $\mathcal{L}^{p,\lambda}(w)$ for all $p \in (q_-, q_+)$ and all

$$w \in A_{p/q_-}(\mathbb{R}^d) \cap \mathrm{RH}_{(q_+/p)'}(\mathbb{R}^d).$$

It remains to check that this covers all $w \in A_u(\mathbb{R}^d) \cap \mathrm{RH}_v(\mathbb{R}^d)$ as in the statement of the theorem. Since $p/q_- \geq u$ and $q_+/p \geq v'$, the monotonicity of the $A_u(\mathbb{R}^d)$ and $\mathrm{RH}_v(\mathbb{R}^d)$ classes implies that $[b, B^\kappa]$ is compact on $\mathcal{L}^{p,\lambda}(w)$ for all $w \in A_u(\mathbb{R}^d) \cap \mathrm{RH}_v(\mathbb{R}^d)$ and all p, λ, u, v such that

$$p \in (p_-, \infty), \qquad 0 < \lambda < d, \qquad u \in \left[1, \min\left\{\frac{p}{p_-}, \frac{d}{\lambda}\right\}\right], \qquad v \in \left(\left(\frac{d}{u\lambda}\right)', \infty\right).$$

The case (1) of the theorem follows in a similar way. ∎

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