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Coagulation equations for aerosol dynamics

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Abstract

Binary coagulation is an important process in aerosol dynamics by which two particles merge to form a larger one. The distribution of particle sizes over time may be described by the so-called Smoluchowski's coagulation equation. This integrodifferential equation exhibits complex non-local behaviour that strongly depends on the coagulation rate considered. We first discuss well-posedness results for the Smoluchowski's equation for a large class of coagulation kernels as well as the existence and nonexistence of stationary solutions in the presence of a source of small particles. The existence result uses Schauder fixed point theorem, and the nonexistence result relies on a flux formulation of the problem and on power law estimates for the decay of stationary solutions with a constant flux. We then consider a more general setting. We consider that particles may be constituted by different chemicals, which leads to multi-component equations describing the distribution of particle compositions. We obtain explicit solutions in the simplest case where the coagulation kernel is constant by using a generating function. Using an approximation of the solution we observe that the mass localizes along a straight line in the size space for large times and large sizes.

Contents

1 Introduction

We consider particle systems where moving particles undergo binary coagulation, forming larger particles. This simple system can be used to study the dynamics of aerosols in the atmosphere [17] as well as raindrop formation, smoke, sprays and galaxies [1, 9].

When the number of particles is very large it becomes more relevant to study the average behaviour of the particles rather than individual particle behaviour. This motivates a statistical description of the system. In 1916 Smoluschowski [33] proposed an equation to describe the particle size distribution over time, assuming that the system is homogeneous in space.

Let $f(x,t)$ be the density of particles of size $x > 0$ at time $t > 0$. The Smoluchowski's coagulation equation, or simply coagulation equation, is the following mean-field equation for the evolution of f

$$\partial_t f(x,t) = \frac{1}{2}\int_0^x K(x-y,y)f(x-y)f(x)dy - \int_0^\infty K(x,y)f(x)f(y) \tag{1.1}$$

where $K(x,y)$ is the coagulation rate between particles of size x and y. The first term on the right hand-side is the gain term due to the coagulation between particles of size $x-y$ and particles of size x to create a particle of size x. The second term is the loss term due to the coagulation of particles of size x with any other particle in the system. Equation (1.1) is an integrodifferential equation belonging to the class of kinetic equations as well as General dynamic equations.

If the size is an integer, for example, if the size is described by the number of molecules that constitute a particle, then the density of particles of size $\alpha \in \mathbb{N}_+$ at time t, $n_\alpha(t)$, satisfies the discrete Smoluchowski's coagulation,

$$\partial_t n_\alpha = \frac{1}{2}\sum_{\beta<\alpha} K_{\alpha-\beta,\beta} n_{\alpha-\beta} n_\beta - n_\alpha \sum_{\beta>0} K_{\alpha,\beta} n_\beta. \tag{1.2}$$

Complementary research lines have emerged over the last decades on experimental, numerical and theoretical issues. Algorithms to simulate Smoluschowski's coagulation equation have been developed to test hypotheses drawn from atmospheric data [25, 35] (see [26] for a survey on numerical methods). On the other hand, theoretical results have clarified issues such as existence and uniqueness of solutions for general classes of kernels [31, 16] or the behaviour of the solutions for explicitly solvable kernels [30] as well as general kernels [4, 5, 12, 15].

A particle may be characterized not only by its size but also by its composition, leading to *multicomponent coagulation equations* where the size is described by a vector $x \in \mathbb{R}_+^d$ (or $\alpha \in \mathbb{N}^d$) representing the size of each of the chemical components of a particle. An application of multicomponent equations to aerosol dynamics is described in Section 2.2.

In this paper we review analytic results related to the continuous onecomponent equation (1.1) and to the discrete multicomponent equation (1.2), with $\alpha, \beta \in \mathbb{N}^d$. We start with a preliminary Section 2 where we give a short overview on various topics related to properties of the solutions, applications, derivation from particle systems and notation. In Section 3

we study well-posedness and existence of stationary solutions for one-component equation of (1.1) for a class of kernels satisfying some growth bounds. More specifically, in Section 3.2 we present the main steps of the proofs of a well-posedness result for unbounded coagulation kernels obtained in 1999 by Norris [31] and in Sections 3.3 and 3.4 we review the proofs for existence and non-existence of stationary solutions to coagulation equation with source, respectively, obtained recently in [13]. In Section 4 we consider the discrete multicomponent equation with constant kernel. Following the computations presented in [21], we compute in Section 4.1 explicit time-dependent solutions to the discrete multicomponent coagulation equation with constant kernel $K(x, y) = 1$ and in Section 4.2 stationary solutions when an additional source at the monomers is present. We also obtain approximations of both solutions that show explicitly that mass localizes along a straight line in the multidimensional size space for large times and large sizes. Finally in Section 5 we mention some more recent results in the literature and open questions.

2 Preliminaries

2.1 Conservation of mass and continuity equation

By multiplying (1.1) by x and integrating in x from 0 to ∞ one obtains formally an equation for the mass $M_1(t) = \int_0^\infty x f(x,t)dx$ given by $\frac{d}{dt}M_1(t) = 0$. This shows that the mass is conserved, provided that the integrals are well-defined. Associated to the mass-conservation, one may write a continuity equation that shows that the mass is transported continuously along the size space:

$$\partial_t(xf(x,t)) = \partial_x J(x,t) \tag{2.1}$$

where the flux of mass from small to large clusters is given by

$$J(x,t) = \int_0^x \int_{x-y}^\infty y f(y,t) f(z,t) K(y,z) dz dy \tag{2.2}$$

As we will see in Section 3.3 a (non-equilibrium) stationary solution has a constant flux of mass at large sizes, i.e., $J(x)$ is constant for all $x > L$, for some positive L. Moreover this flux plays an important role in the proof of non-existence of stationary solutions in Section 3.4.

Interestingly, if the coagulation rate is sufficiently large, the mass-conservation is lost. Such phenomenon, called gelation, corresponds to the formation of infinitely large clusters that are not seen any more by the equations and it may be interpreted as a change in state from gas to gel. Mathematically gelation poses interesting challenges [11], however since it has not been observed in atmospheric aerosols we do not elaborate about it in this text.

We note that, contrarily to the Boltzmann equation, the coagulation equation does not preserve number of particles nor energy.

2.2 Aerosol dynamics in the atmosphere

Atmospheric aerosols are suspensions of small particles in the air, whose diameter ranges approximately between 1 nanometre, in the case of molecular particles, to 100 micrometres, in the case of cloud droplets and dust particles [17]. Aerosols influence sunlight scattering by reflecting and absorbing radiation, and they constitute the seeds that originate the clouds. Therefore, they play an important role in weather and climate forecast [6].

Aerosols are subject to complex processes that influence their size distribution over time. One important process is the coagulation of particles to produce larger ones. Other processes include the formation of new small particles, or monomers, due to certain physical and chemical processes, the removal of particles due to gravity or diffusion, and the growth/shrinkage due to condensation/evaporation [25]. Atmospheric aerosols are typically constituted by different chemicals, leading to multicomponent systems, which may alter the rate of the processes mentioned before and consequently, the particle size distribution [35].

We consider the regime in which the particles are uniformly distributed in space. Moreover, we assume that removal and growth of particles is not important, which in practice may correspond roughly to sizes between 10 nanometers and 10 micrometers [17]. We are then led to the study of multicomponent systems where particles undergo binary coagulation in the presence of a source of small particles.

Coagulation kernels K have been derived for atmospheric aerosols using kinetic theory under several assumptions on the shape and movement of the particles [17]. Aerosol particles are commonly assumed to be spherical and to undergo elastic collisions with background air particles. The number of such collision events is assumed to be much larger than the number of collisions between two coalescing particles, which drives the system towards an equilibrium where the particle velocities follow a Maxwell-Boltzmann distribution.

Moreover, any collision between coalescing particles yields a coalescing particle. Two different coagulation kernels have been derived under the previous conditions for two different particle size regimes, based on the relation between the particle size and the mean free path in air, ie, the average distance travelled by a particle between two collisions. Under normal pressure and temperature conditions, the mean free path in air, ℓ, is of the order of $\ell \sim 10$ nanometres. If the size (diameter) of a spherical particle, d_p, is much smaller than the mean free path $d_p \ll \ell$, the particle is more more likely to travel in straight lines before meeting another coalescing particle. In this case the rate of coagulation has been estimated by the free molecular coagulation kernel:

$$K_{\alpha,\beta} = (\alpha^{1/3} + \beta^{1/3})^2(\alpha^{-1} + \beta^{-1})^{1/2}. \tag{2.3}$$

Otherwise, if the size of a particle is much larger than the mean free path, $d_p \gg \ell$, the coalescing particle will meet many background air particles before meeting another coalescing particle. In this case, the air behaves like a fluid and the coalescing particle is more likely to diffuse. The coagulation rate has been estimated by the diffusive coagulation kernel:

$$K_{\alpha,\beta} = (\alpha^{-1/3} + \beta^{-1/3})(\alpha^{1/3} + \beta^{1/3}). \tag{2.4}$$

This kernel has been first derived in the original work by Smoluchowski [33]. Other kernels have been derived under different assumptions on the underlying background gas and particles features, such as particles moving in a laminar shear or turbulent flow [17], and particles having electric charges [34, 35].

The behaviour and even the existence of solutions to (1.2) strongly depends on the coagulation kernel $K_{\alpha,\beta}$. In Sections 3.3 and 3.4 we review existence of stationary solutions for a large class of kernels which includes in particular the free molecular (2.3) and the diffusive kernels (2.4).

2.3 Derivation of Smoluchowski coagulation equation from particle model

The Smoluchowski coagulation equation has been rigorously derived using different approaches that consider different types of particle systems. In one approach, a purely stochastic particle system is considered, where pairs of particles are randomly picked to originate a new particle. The associated stochastic process is usually called Markov-Lushnikov process. A different approach considers spatial particle systems, where particles move in an Euclidean domain in a deterministic manner and when they collide they merge with a certain probability.

The first approach is inspired in Kac-models for the derivation of the Boltzmann equation [14]. A common strategy is to start from an infinite stochastic particle system where particles of size x and y coalesce at a rate $K(x, y)$ and to prove that the number density, after conveniently rescaled in such a way that the mean free time is constant, converges, as the unit volume tends to infinity, to a measure that solves the Smoluchowski coagulation equation with kernel K. This has been obtained for the additive kernel, product kernel as well as for a class of sub-multiplicative kernels using combinatorial techniques and random graphs. See [2] (Chapter 5.2) for an accessible exposition and [1] for a review on existing results and open problems.

In the second approach, there are fewer rigorous results. The first result to the best of our knowledge is due to Lang and Xanh [22]. They consider Brownian particles moving in the three-dimensional Euclidean space according to Brownian motion with a diffusion coefficient D. The particles are assumed to move independently on each other provided they are at a distance greater than the sum of their radius $2R$. Once they come closer than $2R$ they coalesce with probability $1/2$, forming one Brownian particle with the same radius R and the same diffusion coefficient D. In the limit when the number of particles N goes to infinity and the radius R goes to zero, such that RN remains constant, the authors prove that the particles remain independent on each other (propagation of chaos) and that the density function converges in probability to the solution to the Smoluchowski's coagulation equation with constant coagulation kernel. The limit where RN remains constant is the so-called Boltzmann-Grad limit and is the limit of constant mean free time. A more general case of coalescing Brownian particles with different diffusion coefficients where the diffusion coefficient changes after coalescence, but not the size R, has been treated in [20]. More recently, the change in size after coalescence has been considered in [32] in the case of a one tracer particle moving in a straight line and coalescing with randomly distributed fixed particles. In this case, a linear coagulation equation with a simple shear kernel was derived in the kinetic limit where the volume fraction filled by the background of particles tends to zero.

2.4 Notation

We use the notation $\mathbb{R}_* := (0, \infty)$ and $\mathbb{R}_+ := [0, \infty)$. We denote by $\mathscr{M}(I)$ the space of signed Radon measures on $I \subset \mathbb{R}_+$, i.e., the non-negative measures having finite total variation on any compact subset of I, and by $\|\cdot\|$ the total variation norm. We denote by $\mathscr{M}_+(I)$ the space of measures on $\mathscr{M}(I)$ that are nonnegative. The measures from $\mathscr{M}_+(I)$ that are also bounded are denoted by $\mathscr{M}_{+,b}(I) := \{\mu \in \mathscr{M}_+(I) \,|\, \mu(I) < \infty\}$. The space $\mathscr{M}_{+,b}(I)$ equipped with the norm $\|\cdot\|$ is a Banach space. The notation $f_t(x)$ will sometimes be used to denote $f(t, x)$. We denote by $C_c(I)$ or $C_b(I)$ the spaces of continuous functions on I that are compactly supported or bounded, respectively. For simplicity, we use a generic constant $C > 0$ which

may change from line to line.

3 One component equation

3.1 Main results

We consider a large class of kernels $K : (0,\infty)^2 \to [0,\infty)$ satisfying

$$K \text{ is continuous,} \tag{3.1}$$

and for each $(x,y) \in (0,\infty)^2$, there exist constants $c_1,\ c_2, \lambda$ and γ such that

$$K(x,y) = K(y,x),\ K(x,y) \geq 0, \tag{3.2}$$
$$K(x,y) \geq c_1[x^{\lambda+\gamma}y^{-\lambda} + y^{\lambda+\gamma}x^{-\lambda}] \tag{3.3}$$
$$K(x,y) \leq c_2[x^{\lambda+\gamma}y^{-\lambda} + y^{\lambda+\gamma}x^{-\lambda}], \tag{3.4}$$
$$0 < c_1 \leq c_2 < \infty, \tag{3.5}$$
$$\text{and } \lambda,\ \gamma \in \mathbb{R}. \tag{3.6}$$

Note that condition (3.1) implies that

$$K \text{ is measurable.} \tag{3.7}$$

This class includes in particular the physical kernels (2.3) and (2.4). The parameter γ represents the homogeneity of the kernel, while λ represents the "off-diagonal" rate. The parameter γ yields the behaviour under the scaling of the particle size, while λ measures the importance of collisions between particles of different sizes. Note that the bounds in (3.3) and (3.4) are homogeneous, i.e., they satisfy for any $k > 0$, $h(kx,ky) = k^\gamma h(x,y)$, but the kernels are not necessarily homogeneous.

We assume the following conditions on the source $\eta \in \mathscr{M}_+(\mathbb{R}_*)$

$$\operatorname{supp}\eta \in [1,L], \quad \text{for some } L > 1 \tag{3.8}$$

Note that then the source is bounded, i.e., $\eta(\mathbb{R}_*) < \infty$.

We consider the following definition of time-dependent solution [31].

Definition 3.1 *Assume that K is a measurable function satisfying (3.2) and (3.4). Let $\eta \equiv 0$. We will say that the map $t \mapsto f_t : [0,T) \to \mathscr{M}_+(\mathbb{R}_*)$, where $T \in (0,\infty]$ is a local solution if it satisfies*

1. *for all compact sets $B \subset \mathbb{R}_*$, $t \mapsto f_t(B) : [0,T) \to [0,\infty)$ is measurable*
2. *for all $t < T$ and all compact sets $B \subset \mathbb{R}_*$*

$$\int_0^t \int_{B\times\mathbb{R}_*} K(x,y) f_s(dx) f_s(dy) ds < \infty,$$

3. for all bounded measurable functions φ of compact support and $t < T$ it holds

$$\langle \varphi, f_t \rangle = \langle \varphi, f_0 \rangle + \int_0^t \langle \varphi, L(f_s) \rangle ds \tag{3.9}$$

where $L(f)$ is defined by

$$\langle \varphi, L(f) \rangle = \frac{1}{2} \int_{\mathbb{R}_*} \int_{\mathbb{R}_*} K(x,y)[\varphi(x+y) - \varphi(x) - \varphi(y)] f(dx) f(dy),$$

4. $\int_{\mathbb{R}_*} x \mathbb{1}_{x \leq 1} f_0(dx) < \infty$ and (3.9) holds with $\varphi(x) = x \mathbb{1}_{x \leq 1}$.

If $T = \infty$ we call time-dependent solution.

One can easily check that condition 2 is the minimal one to have well-defined integrals. Condition 3 is the weak formulation commonly used in the literature and it is obtained by formally multiplying by a test function and integrating in x. Condition 4 is a boundary condition imposing that no mass enters at 0.

Theorem 3.2 *Let K a measurable function satisfying (3.2) and (3.4) with $\lambda = 0$ and $\gamma < 1$. Let $\eta = 0$. If $\langle x^2, f_0 \rangle < \infty$, then there exists a unique time-dependent solution $(f_t)_{t>0}$.*

We consider now a source $\eta \neq 0$ of small particles entering into the system at a constant rate and we study the existence of stationary injection solutions, i.e., solutions that satisfy $f(t,x) = f(0,x)$ for all $t > 0$ [13].

Definition 3.3 *Assume that $K : \mathbb{R}_*^2 \to \mathbb{R}_+$ is a continuous function satisfying (3.2) and the upper bound (3.4). Assume further that $\eta \in \mathscr{M}_+(\mathbb{R}_*)$ satisfies (3.8). We will say that $f \in \mathscr{M}_+(\mathbb{R}_*)$, satisfying $f((0,1)) = 0$ and*

$$\int_{\mathbb{R}_*} x^{\gamma+\lambda} f(dx) + \int_{\mathbb{R}_*} x^{-\lambda} f(dx) < \infty, \tag{3.10}$$

is a stationary injection solution of (1.1) if the following identity holds for any test function $\varphi \in C_c(\mathbb{R}_*)$:

$$\frac{1}{2} \int_{\mathbb{R}_*} \int_{\mathbb{R}_*} K(x,y) [\varphi(x+y) - \varphi(x) - \varphi(y)] f(dx) f(dy) + \int_{\mathbb{R}_*} \varphi(x) \eta(dx) = 0. \tag{3.11}$$

Condition (3.10) is the minimal one for the integrals in (3.11) be well-defined. Stationary injection solutions have a constant in time flux of mass from small to large sizes, due to the source, therefore they are non-equilibrium solutions. Note that to be able to be stationary, the volume of particles entering the system has to balance the volume of particles leaving the system. Interestingly, there is an implicit removal of particles from the system at infinite sizes that allows the existence of these solutions. As we see in the next two Theorems, for some class of coagulation rates, including the diffusive kernel (2.4), such balance exists, while for other class of kernels, including the free molecular kernel (2.3), such balance does not exist.

Theorem 3.4 *Assume that K satisfies (3.2)–(3.6) and $|\gamma + 2\lambda| < 1$. Let $\eta \neq 0$ satisfy (3.8). Then, there exists a stationary injection solution $f \in \mathscr{M}_+(\mathbb{R}_*)$, $f \neq 0$, to (1.1) in the sense of Definition 3.3.*

Theorem 3.5 *Suppose that $K(x,y)$ satisfies (3.2)–(3.6) as well as $|\gamma + 2\lambda| \geq 1$. Let us assume also that $\eta \neq 0$ satisfies (3.8). Then, there is not any solution of (1.1) in the sense of the Definition 3.3.*

Note that the diffusive kernel (2.4) satisfies the growth conditions (3.3)-(3.4) with $\gamma = 0$ and $\lambda = 1/3$, while the free molecular kernel (2.3) satisfies the growth conditions with $\gamma = 1/6$ and $\lambda = 1/2$. Therefore there exists a stationary solution for the diffusive but not for the free molecular kernel.

The mass flux from small to large sizes associated to a stationary injection solutions is given in the next Lemma.

Lemma 3.6 *Suppose that the assumptions of Theorem 3.4 hold. Let f be a stationary injection solution in the sense of Definition 3.3. Then f satisfies for any $R > 0$*

$$J(R) = \int_{(0,R]} x\eta(dx) \tag{3.12}$$

with

$$J(R) = \int_{(0,R]} \int_{(R-x,\infty)} K(x,y) x f(dx) f(dy)$$

Remark 3.7 If $R \geq L_\eta$, the right-hand side of (3.12) is constant equal to $J_\eta = \int_{[1,L_\eta]} x\eta(dx) > 0$. Therefore, $J(R) = J_\eta$ for $R > L_\eta$, i.e., the mass flux is constant in the regions that include large sizes.

Proof: [Idea of the proof] For for each $\varepsilon > 0$, define a test function $\varphi(x) = x\chi_\varepsilon(x) \in C_c(\mathbb{R}_*)$ where $\chi_\varepsilon \in C_c^\infty(\mathbb{R}_*)$ is such that $0 \leq \chi_\varepsilon \leq 1$, $\chi_\varepsilon(x) = 1$, for $1 \leq x \leq R$, and $\chi_\varepsilon(x) = 0$, for $x \geq R + \varepsilon$. Then using the test function in (3.11) the result follows after some computations and after letting $\varepsilon \to 0$. □

3.2 Well-posedness

We describe the main ideas of the proofs obtained in [31] (Section 2).

The first step is to prove well-posedness for a truncated problem. The second step is to compute estimates that allow us to remove the truncation and to obtain well-posedness for the original problem.

Let $B \subset \mathbb{R}_*$ be a compact set. We consider the space $\mathscr{M}(B)$ of finite signed measures supported on B. Note that all measures in $\mathscr{M}(B)$ are bounded. Note that from the hypotheses of Theorem 3.2 on the kernel we have that

$$0 \leq K(x,y) \leq w(x) + w(y), \quad \text{with } w(x) := x^\gamma \text{ and } \gamma < 1. \tag{3.13}$$

The truncated operator $L^B : \mathscr{M}(B) \times \mathbb{R} \to \mathscr{M}(B) \times \mathbb{R}$ is defined by

$$\langle (\varphi, a), L^B(f,\lambda) \rangle :=$$
$$\frac{1}{2} \int_{\mathbb{R}_* \times \mathbb{R}_*} \{\varphi(x+y)\mathbb{1}_{\{x+y \in B\}} + aw(x+y)\mathbb{1}_{\{x+y \notin B\}} - \varphi(x) - \varphi(y)\} \times K(x,y) f(dx) f(dy)$$
$$+\lambda \int_{\mathbb{R}_*} \{aw(x) - \varphi(x)\} w(x) f(dx)$$

for all bounded measurable functions φ on $\mathbb{R}_*$ and all $a \in \mathbb{R}$, where $\langle(\varphi, a), (f, \lambda)\rangle$ denotes $\langle \varphi, f\rangle + a\lambda$. The truncated equation reads

$$\langle(\varphi, a), (f_t, \lambda_t)\rangle = \langle(\varphi, a), (f_0, \lambda_0)\rangle + \int_0^t \langle(\varphi, a), L^B(f, \lambda)\rangle ds. \tag{3.14}$$

An interpretation of the operator L^B is the following (see [31] for more details). Particles of size x and y merge at a rate $K(x, y)$ and they produce a new particle of size $x + y$. If the merging particle has size outside B, we add $w(x + y)$ to λ.

A solution to (3.14) is defined next.

Definition 3.8 *Let $T \in (0, \infty)$. We will say that $(f_t, \lambda_t)_{t\in[0,T]}$ is a local solution to* (3.14) *if $t \mapsto (f_t, \lambda_t) : [0, T] \to \mathscr{M}(B) \times \mathbb{R}$ is a continuous map satisfying* (3.14) *for all $t \in [0, T]$. Additionally, $(f_t)_{t\in[0,T]}$ is called a solution when $[0, T]$ is replaced by $[0, \infty)$.*

Proposition 3.9 *Suppose that $f_0 \in \mathscr{M}(B)$ with $f_0 \geq 0$ and $\lambda_0 \in [0, \infty)$. The equation* (3.14) *has a unique solution $(f_t, \lambda_t)_{t\geq 0}$ starting from (f_0, λ_0). Moreover $f_t \geq 0$ and $\lambda_t \geq 0$ for all $t \geq 0$.*

Proof:[idea of the proof] The first step is to show that there is a constant $T > 0$ depending only on γ and B such that there exists a unique local solution $(f_t, \lambda_t)_{t\in[0,T]}$ to (3.14) starting from (f_0, λ_0). This is obtained by using an iterative scheme of continuous maps $(f_t^n, \lambda_t^n) : [0, \infty) \mapsto \mathscr{M}(B) \times \mathbb{R}$ defined by

$$\begin{aligned} (f_t^0, \lambda_t^0) &= (f_0, \lambda_0) \\ (f_t^n, \lambda_t^n) &= (f_0, \lambda_0) + \int_0^t L^B(f_t^{n-1}, \lambda_t^{n-1}) \end{aligned}$$

and proving that there exists a $T > 0$ such that (f^n, λ^n) converges in $\mathscr{M}(B) \times \mathbb{R}$ uniformly in $t \leq T$ to the desired local solution, which is also unique.

The second step is to prove that $f_t \geq 0$, $t \in [0, T]$, which is obtained using again an iterative argument similar to the one used in the first step.

Finally, the third step is to show that the solution exists for all times $t \in [0, \infty)$. Choosing $\varphi = w$ and $a = 1$ we obtain that

$$\frac{d}{dt}(\langle w, f_t\rangle + \lambda_t) = \frac{1}{2}\int_{\mathbb{R}_*\times\mathbb{R}_*} \{w(x + y) - w(x) - w(y)\}K(x, y)f_t(dx)f_t(dy) \leq 0,$$

which implies that

$$\|f_T\| + |\lambda_T| \leq \langle w, f_T\rangle + \lambda_T \leq \langle w, f_0\rangle + \lambda_0,$$

using a scaling argument, we may assume without loss of generality that $\langle w, f_0\rangle + \lambda_0 \leq 1$, consequently $\|f_T\| + |\lambda_T| \leq 1$. We can start again from (f_T, λ_T) at time T to extend the solution to $[0, 2T]$ and so on. Moreover, choosing $\varphi = 0$ and $a = 1$ in (3.14), we obtain

$$\frac{d}{dt}\lambda_t = \frac{1}{2}\int_{\mathbb{R}_*\times\mathbb{R}_*} \{w(x + y)\mathbb{1}_{\bar{B}}(x + y)\}K(x, y)f(dx)f(dy) + \lambda_t \int_{\mathbb{R}_*} w^2(x)f(dx),$$

which implies that $\lambda_t \geq 0$, for all $t \geq 0$, due to $f_t \geq 0$, which ends the proof of the proposition. □

Proof:[Theorem 3.2] Fix $f_0 \in \mathscr{M}_+$, such that $\langle w, f_0\rangle < \infty$. For each compact set $B \subset \mathbb{R}_*$ define $f_0^B = \mathbb{1}_B f_0$ and $\lambda_0^B = \int_B w(x) f_0(dx)$. From Proposition 3.9 there is a unique solution $(f_0^B, \lambda_t^B)_{t\geq 0}$ to (3.14) starting from (f_0^B, λ_0^B). We now set $f_t = \lim_{B\to\mathbb{R}_*} f_t^B$ and $\lambda_t = \lim_{B\to\mathbb{R}_*} \lambda_t^B$. Using (3.13), we obtain by dominated convergence,

$$\frac{d}{dt}\langle \varphi, f_t\rangle = \frac{1}{2}\int_{\mathbb{R}_*\times\mathbb{R}_*} \{\varphi(x+y) - \varphi(x) - \varphi(y)\} K(x,y) f_t(dx) f_t(dy) - \lambda_t \langle \varphi w, f_t\rangle,$$

for all bounded measurable functions φ. One can prove that for all $t < T$ and for any local solution $(g_t)_{t<T}$,

$$f_t \leq g_t, \quad \langle w, f_t\rangle + \lambda_t \geq \langle w, g_t\rangle. \tag{3.15}$$

By hypothesis $\langle w^2, f_0\rangle < \infty$. Since $K(x,y) \leq w(x)+w(y)$ it holds $\langle w^2, f_t\rangle \leq \langle w^2, f_0\rangle \exp(2\langle w, f_0\rangle t)$. Therefore $\langle w^2, f_t\rangle < \infty$ for all $t > 0$, which allows to pass to the limit as $B \to \mathbb{R}_*$ in (3.14) and to deduce that

$$\lambda_t = 0, \quad t > 0. \tag{3.16}$$

Then (3.15) and (3.16) imply that $(f_t)_{t\geq 0}$ is a solution and moreover, it is the only solution. □

3.3 Existence of stationary solutions

We present the main ideas of the proofs obtained in [13].

The general strategy to prove existence of stationary solutions is similar to the one used in the proof of well-posedness presented in the previous Section. First, we prove existence of a stationary solution for a truncated problem and second, we obtain estimates that allow to remove the truncation and to obtain the existence result for the original problem. Unfortunately the method used to prove existence does not give uniqueness, that problem needs a separate treatment (see [23] for a simple explanation of the available techniques).

Let $\varepsilon > 0$ and $R_* \geq L_\eta$, where L_η is the upper bound of the support of the source η defined in (3.8). We will eventually make $\varepsilon \to 0$ and $R_* \to \infty$. We consider kernels K_{ε,R_*} that are continuous, bounded and have compact support, such that

$$K_{\varepsilon,R_*}(x,y) \leq a_2(\varepsilon), \qquad (x,y) \in \mathbb{R}_+^2 \tag{3.17}$$

$$K_{\varepsilon,R_*}(x,y) \in [a_1(\varepsilon), a_2(\varepsilon)], \qquad (x,y) \in [1, 2R_*] \tag{3.18}$$

$$K_{\varepsilon,R_*}(x,y) = 0, \qquad x \geq 4R_* \text{ or } y \geq 4R_*, \tag{3.19}$$

and

$$\lim_{R_*\to\infty} K_{\varepsilon,R_*}(x,y) = K_\varepsilon(x,y) \tag{3.20}$$

where K_ε is continuous and satisfies $K_\varepsilon(x,y) \in [a_1(\varepsilon), a_2(\varepsilon)]$, for all $(x,y) \in \mathbb{R}_+^2$ and

$$\lim_{\varepsilon\to 0} K_\varepsilon(x,y) = K(x,y). \tag{3.21}$$

Additionally, in the evolution equation, we consider a cut-off of the gain term due to the coagulation that ensures that the measure solutions are supported in $[1, 2R_*]$ and bounded at all times. To this end, we choose $\zeta_{R_*} \in C(\mathbb{R}_*)$ such that $0 \leq \zeta_{R_*} \leq 1$, $\zeta_{R_*}(x) = 1$ for

$0 \leq x \leq R_*$, and $\zeta_{R_*}(x) = 0$ for $x \geq 2R_*$. The regularized time evolution equation then reads as

$$\partial_t f(x,t) = \frac{\zeta_{R_*}(x)}{2} \int_{(0,x]} K_{\varepsilon,R_*}(x-y,y) f(x-y,t) f(y,t) dy - \int_{\mathbb{R}_*} K_{\varepsilon,R_*}(x,y) f(x,t) f(y,t) dy + \eta(x) \,. \tag{3.22}$$

Definition 3.10 *Let $\varepsilon > 0$ and $R_* \geq L_\eta$. Suppose that K_{ε,R_*} satisfies* (3.17)-(3.19) *and $\eta \in \mathscr{M}_+(\mathbb{R}_+)$ satisfies* (3.8)*. Consider some initial data $f_0 \in \mathscr{M}_+(\mathbb{R}_*)$ for which $f_0((0,1) \cup (2R_*, \infty)) = 0$. Then $f_0 \in \mathscr{M}_{+,b}(\mathbb{R}_*)$. We will say that $f \in C^1([0,T], \mathscr{M}_{+,b}(\mathbb{R}_*))$ satisfying $f(\cdot,0) = f_0(\cdot)$ is a time-dependent solution of (3.22) if the following identity holds for any test function $\varphi \in C^1([0,T], C_c(\mathbb{R}_*))$ and all $0 < t < T$,*

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}_*} \varphi(x,t) f(dx,t) - \int_{\mathbb{R}_*} \dot{\varphi}(x,t) f(dx,t) \\ = \; & \frac{1}{2} \int_{\mathbb{R}_*} \int_{\mathbb{R}_*} K_{\varepsilon,R_*}(x,y) \left[\varphi(x+y,t) \zeta_{R_*}(x+y) - \varphi(x,t) - \varphi(y,t) \right] f(dx,t) f(dy,t) \\ & + \int_{\mathbb{R}_*} \varphi(x,t) \eta(dx) \,, \end{aligned} \tag{3.23}$$

where $\dot{\varphi}$ denotes the Fréchet time-derivative of φ.

Proposition 3.11 *Let $\varepsilon > 0$ and $R_* \geq L_\eta$. Suppose that K_{ε,R_*} satisfies* (3.17)-(3.19) *and $\eta \in \mathscr{M}_+(\mathbb{R}_+)$ satisfies* (3.8)*. Then, for any initial condition f_0 satisfying $f_0 \in \mathscr{M}_+(\mathbb{R}_*)$, $f_0((0,1) \cup (2R_*, \infty)) = 0$ there exists a unique time-dependent solution $f \in C^1([0,T], \mathscr{M}_{+,b}(\mathbb{R}_*))$ to (3.22) which solves it in the classical sense. Moreover, f is a Weak solution of (3.22) in the sense of Definition 3.10 such that*

$$f((0,1) \cup (2R_*, \infty), t) = 0 \,, \quad \text{for } 0 \leq t \leq T \,,$$

and the following estimate holds

$$\int_{\mathbb{R}_*} f(dx,t) \leq \int_{\mathbb{R}_*} f_0(dx) + Ct \,, \quad t \geq 0 \,,$$

for $C = \int_{\mathbb{R}_*} \eta(dx) \geq 0$ which is independent of f_0, t, and T.

Proof:[Idea of the proof] Since the kernel is bounded, the result may be obtained using Banach fixed-point theorem. □

Definition 3.12 *Let $\varepsilon > 0$ and $R_* \geq L_\eta$. Suppose that K_{ε,R_*} satisfies* (3.17)-(3.19) *and $\eta \in \mathscr{M}_+(\mathbb{R}_+)$ satisfies* (3.8)*. We will say that $f \in \mathscr{M}_+(\mathbb{R}_*)$, satisfying $f((0,1) \cup (2R_*, \infty)) = 0$ is a stationary injection solution of (3.22) if the following identity holds for any test function $\varphi \in C_c(\mathbb{R}_*)$:*

$$\begin{aligned} 0 \; = \; & \frac{1}{2} \int_{\mathbb{R}_*} \int_{\mathbb{R}_*} K_{\varepsilon,R_*}(x,y) \left[\varphi(x+y) \zeta_{R_*}(x+y) - \varphi(x) - \varphi(y) \right] f(dx) f(dy) \\ & + \int_{\mathbb{R}_*} \varphi(x) \eta(dx) \,. \end{aligned}$$

We denote by $S(t)$ the semigroup defined by the time-dependent solution f obtained in Proposition 3.11,

$$S(t)f_0 = f(\cdot, t)$$

that satisfies the semigroup property

$$S(t+s)f = S(t)S(s)f,\ t, s \in \mathbb{R}_+.$$

The operators $S(t)$ define mappings

$$S(t) : \mathscr{X}_{R_*} \to \mathscr{X}_{R_*}, quad \text{ for each } t_1, t_2 \in \mathbb{R}_+$$

with $\mathscr{X}_{R_*} = \{f \in \mathscr{M}_+(\mathbb{R}_*) : f((0,1) \cup (2R_*, \infty)) = 0\}$.

Proposition 3.13 *Under the assumptions of Proposition 3.11, there exists a stationary injection solution $\hat{f} \in \mathscr{M}_+(\mathbb{R}_*)$ to (3.22) as defined in Definition 3.12.*

Proof: [Idea of the proof] The key point of the proof is to use Schauder fixed point theorem. The first step is to obtain the existence of an invariant region for the evolution problem (3.23). To that end, we choose a time independent test function $\varphi(x) = 1$ for $x \in [1, 2R_*]$. Using the lower bound for the kernel (3.18) and that $f(\cdot, t)$ has support in $[1, 2R_*]$ we obtain the following estimate

$$\frac{d}{dt}\int_{[1,2R_*]} f(dx,t) \leq -\frac{a_1}{2}\left(\int_{[1,2R_*]} f(dx,t)\right)^2 + c_0$$

where $c_0 = \int_{\mathbb{R}_*} \eta(dx)$. This implies that for a large enough $M > 0$, the set

$$\mathscr{U}_M = \left\{ f \in \mathscr{X}_{R_*} : \int_{[1,2R_*]} f(dx) \leq M \right\}.$$

is invariant under the time evolution (3.22). Moreover, $\mathscr{U}_M$ is compact in the $*-$weak topology due to Banach-Alaoglu's Theorem (cf.[3]), since it is an intersection of a $*-$weak closed set $\mathscr{X}_{R_*}$ and the closed ball $\|f\| \leq M$.

The second step is to prove that for each $t > 0$, both maps $S(t) : \mathscr{U}_M \to \mathscr{U}_M$ and $t \mapsto S(t)f_0$ are continuous in the $*-$weak topology.

Finally, the third step of the proof reads as follows. Since for each t, the operator $S(t)$ is continuous and $\mathscr{U}_M$ is compact and convex when endowed with the $*-$weak topology, we can apply Schauder fixed point theorem to conclude that for all $\delta > 0$ there is a fixed point f_δ of $S(\delta)$ in $\mathscr{U}_M$. Moreover, since $\mathscr{U}_M$ is metrizable and hence sequentially compact, there is a convergent sequence $\{f_{\delta_n}\}_{n\in\mathbb{N}}$, i.e., there exists $\hat{f} \in \mathscr{U}_M$ such that $f_{\delta_n} \to \hat{f}$ when $\delta_n \to 0$ in the $*-$weak topology. For each t we choose $\delta_n = t/n$. Using the semigroup property we obtain that $S(t)f_{\delta_n} = S(n\delta_n)f_{\delta_n} = S(\delta_n)f_{\delta_n}$. Using the continuity of $t \mapsto S(t)f_0$ and the fact that $S(0)\hat{f} = \hat{f}$, we obtain $S(\delta_n)f_{\delta_n} \to \hat{f}$. On the other hand using the continuity of $S(t)$ we obtain that $S(t)f_{\delta_n} \to S(t)\hat{f}$. Therefore $S(t)\hat{f} = \hat{f}$ and thus $\hat{f}$ is a stationary solution to (3.22), which concludes the proof. □

The next Lemma provides uniform estimates for integrals.

Lemma 3.14 *Let $a > 0$, $R \geq a$ and $b \in (0,1)$ be such that $bR > a$. Suppose $f \in \mathscr{M}_+(\mathbb{R}_*)$, $\varphi \in C(\mathbb{R}_*)$, $g \in L^1(\mathbb{R}_*)$, and $g, \varphi \geq 0$. If*

$$\frac{1}{z}\int_{[bz,z]} \varphi(x) f(dx) \leq g(z)\,, \quad \textit{for } z \in [a,R]\,,$$

then

$$\int_{[a,R]} \varphi(x) f(dx) \leq \frac{\int_{[a,\infty)} g(z)dz}{\ln(b^{-1})} + Rg(R)\,.$$

We now extend the previous existence result to general unbounded kernels K supported in $\mathbb{R}^2_+$ and satisfying the conditions of the theorem 3.4.

Proof:[idea of the proof of Theorem 3.4] Let f_{ε,R_*} be a stationary injection solution to (3.22) as defined in Definition 3.12 provided by Proposition 3.13. The idea is to obtain estimates that are independent on both ε and R_* that allow to pass to the limit as $\varepsilon \to 0$ and $R_* \to \infty$ and to obtain the existence of a stationary injection solution to the original problem as defined in (3.3).

First we obtain an estimate uniform in R_*:

$$\int_{[0,2R_*/3]} f_{\varepsilon,R_*}(dx) \leq \bar{C}_\varepsilon, \quad R_* > 0,$$

where $\bar{C}_\varepsilon$ is a constant independent on R_*. This estimate implies, that taking a subsequence if needed, there exists $f_\varepsilon \in \mathscr{M}_+(\mathbb{R}_+)$ such that $f_\varepsilon([0,1)) = 0$ and:

$$f_{\varepsilon,R^n_*} \rightharpoonup f_\varepsilon \text{ as } n \to \infty \text{ in the } *\text{−weak topology}$$

with $R^n_* \to \infty$ as $n \to \infty$. For any bounded continuous test function $\varphi : [0,\infty) \to \mathbb{R}$, one proves that f_ε satisfies

$$\frac{1}{2}\int_{[0,\infty)^2} K_\varepsilon(x,y)\left[\varphi(x+y) - \varphi(x) - \varphi(y)\right] f_\varepsilon(dx)\, f_\varepsilon(dy) + \int_{[0,\infty)} \varphi(x)\eta(dx) = 0.$$

where K_ε is defined in (3.20).

Second, we obtain estimates independent on ε:

$$\frac{1}{z}\int_{\left[\frac{2z}{3},z\right]} f_\varepsilon(dx) \leq \frac{\tilde{C}}{z^{\frac{3}{2}}}\left(\frac{1}{\min\left(z^\gamma, \frac{1}{\varepsilon}\right)}\right)^{\frac{1}{2}}, \tag{3.24}$$

and

$$\frac{1}{z}\int_{\left[\frac{2z}{3},z\right]} f_\varepsilon(dx) \leq \frac{\tilde{C}}{z^{\frac{3}{2}}\sqrt{\varepsilon}}.$$

where $\tilde{C}$ is independent on ε. This estimate yields $*$−weak compactness of the family of measures $\{f_\varepsilon\}_{\varepsilon>0}$ in $\mathscr{M}_+(\mathbb{R}_+)$. Therefore, there exists $f \in \mathscr{M}_+(\mathbb{R}_+)$ such that:

$$f_{\varepsilon_n} \rightharpoonup f \text{ as } n \to \infty \text{ in the } *\text{−weak topology}$$

for some subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}\varepsilon_n = 0$. Using (3.21) and Lemma 3.14, one can prove that f_ε satisfies (3.11) for any $\varphi \in C_c(\mathbb{R}_+)$. In particular, $f \neq 0$ due to $\eta \neq 0$.

It only remains to prove (3.10). Taking the limit of (3.24) as $\varepsilon \to 0$ we arrive at:

$$\frac{1}{z}\int_{[2z/3,z]} f(dx) \leq \frac{\widetilde{C}}{z^{3/2+\gamma/2}} \quad \text{for all } z \in (0,\infty),$$

which implies

$$\frac{1}{z}\int_{[2z/3,z]} x^{\mu} f(dx) \leq \widetilde{C}\frac{z^{\mu}}{z^{3/2+\gamma/2}} \quad \text{for all } z \in (0,\infty),$$

for any $\mu \in \mathbb{R}$. From Lemma 3.14 we obtain the boundedness of the moment of order μ:

$$\int_{[0,\infty)} x^{\mu} f(dx) < \infty.$$

for any μ satisfying $\mu < \frac{\gamma+1}{2}$. In particular, since $|\gamma + 2\lambda| < 1$, then the moments $\mu = -\lambda$ and $\mu = \gamma + \lambda$ are bounded, which proves (3.10). □

3.4 Nonexistence of stationary solutions

We present the main ideas of the proof obtained in [13].

The proof is done by contradiction. Let the kernel K satisfy the power law bounds (3.3)-(3.4) with $|\gamma + 2\lambda| \geq 1$. Suppose that $f \in \mathscr{M}_+(\mathbb{R}_*)$ is a stationary injection solution of (1.1) in the sense of Definition 3.3. Then f satisfies the weak formulation (3.11) as well as the condition on the moments (3.10).

The first step is to rewrite (3.11) using the flux formulation. Consider the function $J : \mathbb{R}_* \to \mathbb{R}_+$ defined by

$$J(R) = \iint_{\Sigma_R} K(x,y)\, x f(dx)\, f(dy) \tag{3.25}$$

where

$$\Sigma_R = \{x \geq 1,\ y \geq 1 : x + y > R,\ x \leq R\}.$$

Let $\varepsilon > 0,\ R \geq 1$ and $\chi_\varepsilon \in C^\infty(\mathbb{R}_+)$ satisfy $\chi_\varepsilon(x) = 1,\ x \leq R$ and $\chi_\varepsilon(x) = 0,\ x \geq R + \varepsilon$. Choosing a test function $\varphi(x) = x\chi_\varepsilon(x)$ we obtain from (3.11) the flux formulation

$$J(R) = \int_{[1,R]} x\eta(dx),\ R \geq 1.$$

We note that J describes the flux of particles passing through Σ_R and that this flux is constant for all $R \geq L_\eta$ and equal to $J(L_\eta) = \int_{[1,\infty)} x\eta(dx) > 0$, i.e.,

$$J(R) = J(L_\eta),\ R \geq L_\eta.$$

The second step is to prove that the main contribution to the integral (3.25) as $R \to \infty$ is due to collisions between particles of size close to R and particles of size of order 1. To that end, for a given $\delta > 0$ small, we consider a partition of $\Sigma_R = D_\delta^{(1)} \cap D_\delta^{(2)}$ such that

$$D_\delta^{(1)} = \{x \geq 1,\ y \geq 1 : y \leq \delta x\}\ ,$$
$$D_\delta^{(2)} = \{x \geq 1,\ y \geq 1 : y > \delta x\}$$

and we define

$$J_k(R) = \iint_{\Sigma_R \cap D_\delta^k} K(x,y)\, x f(dx)\, f(dy), \ k = 1, 2.$$

Therefore

$$J(R) = J_1(R) + J_2(R).$$

Using the upper bound for the kernel (3.4), the moment condition (3.10) and the fact that $\Sigma_R \cap D_\delta^2 \subset [1, R] \times [\frac{\delta R}{1+\delta}, \infty)$ one concludes after some computations that the contribution of J_2 vanishes as $R \to \infty$, i.e.,

$$\lim_{R\to\infty} J_2(R) = 0$$

which implies that

$$\lim_{R\to\infty} J_1(R) = \lim_{R\to\infty} J(R) = J(L_\eta).$$

In the remainder of the proof we will use the notation $a := \gamma+\lambda$ and $b := -\lambda$ if $(\gamma+2\lambda) \geq 1$, or $a := \gamma + \lambda$ and $b := -\lambda$ if $(\gamma + 2\lambda) \leq -1$. Then, the assumption (3.10) may be rewritten as

$$\int_{\mathbb{R}_*} x^a f(dx) < \infty. \tag{3.26}$$

The third step of the proof consists in obtaining a lower bound for the fluxes that implies a lower bound for the number of particles in some region of the size space. Using the upper bound for the kernel (3.4) we obtain after some computations

$$\liminf_{R\to\infty} \left(R^{a+1} \iint_{\Sigma_R \cap D_\delta^{(1)}} y^b f(dx)\, f(dy) \right) \geq \frac{J(L_\eta)}{c_3\left(1 + \delta^{|a-b|}\right)}\ . \tag{3.27}$$

For R sufficiently large we have that

$$\Sigma_R \cap D_\delta^{(1)} \subset \{(x,y) : 1 \leq y \leq \delta R,\ R < x + y,\ x \leq R\}$$

whence, (3.27) implies the inequality

$$\int_{[1,\delta R]} y^b f(dy) \int_{(R-y,R]} f(dx) \geq \frac{J(L_\eta)}{2c_3\left(1 + \delta^{|a-b|}\right)} \frac{1}{R^{a+1}} \tag{3.28}$$

for $R \geq R_0$ with R_0 large enough. We now consider two cases separately $a \geq 0$ and $a < 0$. Let first $a \geq 0$. Due to (3.26) we may define

$$F(R) = \int_{(R,\infty)} f(dx) \quad , \quad R \geq 1\ . \tag{3.29}$$

Using (3.29) we can rewrite (3.28) as

$$-\int_{[1,\delta R]} [F(R-y) - F(R)]\, y^b f(dy) \leq -\frac{J(L_\eta)}{2c_3\left(1 + \delta^{|a-b|}\right)} \frac{1}{R^{a+1}} \quad \text{for } R \geq R_0.$$

Then, using a comparison argument (see Lemma 4.1 in [13]), for some constant $B > 0$, it follows that

$$F(R) \geq \frac{B}{R^a} \quad \text{if} \quad R \geq R_0, \quad \text{for } a > 0, \tag{3.30}$$

and

$$F(R) \geq B \log(R) \quad \text{if} \quad R \geq R_0, \quad \text{for } a = 0. \tag{3.31}$$

In the case $a > 0$, (3.30) implies

$$B \leq R^a F(R) \leq \int_{(R,\infty)} x^a f(dx)$$

Taking the limit when $R \to \infty$ and using (3.26) it follows that $B \leq 0$, which leads to a contradiction. In the case $a = 0$, the contradiction follows from (3.31) in a similar way using (3.26).

Let now $a < 0$. We define the function F by

$$F(R) = \int_{[1,R]} f(dx) \quad , \quad R \geq 1. \tag{3.32}$$

Using (3.32) we can rewrite (3.28) as:

$$-\int_{[1,\delta R]} [F(R) - F(R-y)]\, y^b f(dy) \leq -\frac{J(L_\eta)}{2c_3\left(1+\delta^{|a-b|}\right)} \frac{1}{R^{a+1}} \quad \text{for } R \geq R_0.$$

As in the previous case, using a comparison argument (see Lemma 4.2 in [13]), it follows that there is $B > 0$ such that

$$F(R) \geq \frac{B}{R^a} \quad \text{if} \quad R \geq R_0\,.$$

For a small $\varepsilon > 0$ satisfying $\varepsilon < B$ there exists M such that

$$\int_{[M,\infty)} x^a f(dx) = \varepsilon\ .$$

Then for all $R > M$ we have

$$B \leq R^a \int_{[1,R]} f(dx) \leq R^a \int_{[1,M]} f(dx) + \int_{[M,R]} x^a f(dx) \leq R^a \int_{[1,M]} f(dx) + \varepsilon.$$

Since $a < 0$, taking the limit as $R \to \infty$ we obtain $B \leq \varepsilon$, which leads to a contradiction.

4 Multicomponent equation with constant kernel

4.1 Mass localization in time-dependent solution

We consider the multicomponent discrete equation with constant kernel $K_{\alpha,\beta} = 2$,

$$\frac{dn_\alpha(t)}{dt}(t) = \sum_{\beta<\alpha} n_\beta(t) n_{\alpha-\beta}(t) - 2n_\alpha(t) \sum_{\beta>0} n_\beta(t) \tag{4.1}$$

with initial condition

$$n_\alpha(0) = \frac{1}{d} \sum_{|\beta|=1} \delta_{\alpha,\beta} \tag{4.2}$$

where $\beta > \alpha$ denotes $\beta_i \geq \alpha_i$ for all $i = 1, ..., d$ and $\beta > \alpha$, and $\delta_{\alpha,\beta} = 1$ if $\alpha = \beta$ and $\delta_{\alpha,\beta} = 0$ otherwise. Note that the initial condition (4.2) is supported in the monomers and its mass is uniformly distributed by the types of particles. The initial mass of each type of particle is $\frac{1}{d}$. Existence and uniqueness of a solution to (4.1)-(4.2) in the onecomponent case $d = 1$ is proven in [30] for any initial condition satisfying $\sum_{\alpha=1}^{\infty} n_\alpha(0) < \infty$ using Laplace transforms. An explicit solution to the multicomponent problem has been obtained in [27] and [21] using a generating function.

In this Section we review the computations described in [21] to obtain an explicit solution to (4.19). We then study the long-time behaviour using an approximation of the solution for large times and large sizes obtained in [21], in particular, we observe the phenomenon of mass localization along a straight line in the size space.

Multiplying (4.19) by a compactly supported test function ψ_α and summing in α we obtain the weak formulation

$$\partial_t \sum_{\alpha=1}^{\infty} \psi_\alpha n_\alpha(t) = \sum_{\alpha,\beta=1}^{\infty} [\psi_{\alpha+\beta} - \psi_\alpha - \psi_\beta] n_\alpha(t) n_\beta(t) \tag{4.3}$$

The solution may be obtained using the generating function $F : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ defined by

$$F(z,t) = \sum_{\alpha>0} z^\alpha n_\alpha(t) \tag{4.4}$$

where $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} ... z_d^{\alpha_d}$. Using $\psi_\alpha = z^\alpha$ in (4.3) we obtain an equation for F,

$$\partial_t F(z,t) = F(z,t)^2 - 2F(z,t)N(t) \tag{4.5}$$

where $N(t) = F(0,t) = \sum_{\alpha>0} n_\alpha$ is the total number of particles at time t. From (4.2) the initial number of particles is $N(0) = 1$. Using $\psi_\alpha = 1$ in (4.3) we obtain an equation for N,

$$\partial_t N(t) = -N^2(t),\ N(0) = 1 \quad \iff \quad N(t) = \frac{1}{1+t} \tag{4.6}$$

If we subtract equations (4.5) and (4.6) we obtain an equation for $F - N$, $\partial_t(F-N) = (F-N)^2$. Solving this equation and using (4.6) yields an expression for F

$$F(z,t) = \frac{F_0(z)}{(1+t)(1+t-tF_0(z))} \tag{4.7}$$

where $F_0(z) = F(z,0)$ is given by $F_0(z) = \frac{1}{d}\sum_{i=1}^d z_i$ after substituting (4.2) in (4.4). The expression for F will be used in the following to determine the solution to (4.1).

We note that if $\{n_\alpha\}_{\alpha>0}$ is a solution to the multicomponent coagulation equation (4.1), then $\{n_{|\alpha|}\}_{\alpha>0}$, where $|\alpha| = \sum_{i=1}^d \alpha_i$ is the sum variable and $n_{|\alpha|}$ is defined by $n_{|\alpha|} = \sum_{\beta>0} n_\beta \delta_{|\alpha|,|\beta|}$, is a solution to the one component equation with constant kernel $K_{|\alpha|,|\beta|} = 2$ and initial condition $n_{|\alpha|}(0) = \delta_{|\alpha|,1}$. This result may be obtained using the weak formulation (4.3) with a test function of the form $\psi_\alpha = \varphi_{|\alpha|}$. We first solve the one component equation to find an expression for $\{n_{|\alpha|}\}_{\alpha>0}$.

We consider the generating function $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ associated to the one component problem

$$f(z,t) = \sum_{|\alpha|=1}^{\infty} z^{|\alpha|} n_{|\alpha|}(t), \tag{4.8}$$

which may be expressed by (4.7) with $f_0(z,t) = \sum_{|\alpha|=1}^{\infty} z^{|\alpha|} n_{|\alpha|}(0) = z$, i.e.

$$f(z,t) = \frac{z}{(1+t)(1+t-tz)} \tag{4.9}$$

Using the Taylor series, we expand f around $z = 0$ and obtain

$$f(z,t) = \sum_{k=1}^{\infty} z^k \frac{t^{k-1}}{(1+t)^{k+1}}, \quad z < \frac{1+t}{t},\ t > 0. \tag{4.10}$$

Comparing each term of the two series (4.10) and (4.8) we conclude that the solution to the one component equation is

$$n_{|\alpha|}(t) = \frac{t^{|\alpha|-1}}{(1+t)^{|\alpha|+1}}. \tag{4.11}$$

The solution to the multicomponent equation (4.1) can now be computed by expanding (4.7) and comparing with (4.4). Using the Taylor series in several variables we obtain the expansion of (4.7) around 0,

$$f(z,t) = \sum_{k=1}^{\infty} \frac{1}{d^k} \frac{t^{k-1}}{(1+t)^{k+1}} |z|^k, \quad |z| < d\frac{1+t}{t},\ t > 0. \tag{4.12}$$

Comparing with (4.8) and using (4.11) and the fact that $(z_1+...+z_d)^k = \sum\limits_{|\alpha|=k} \frac{k!}{\alpha_1!\alpha_2!...\alpha_d!} z_1^{\alpha_1} z_2^{\alpha_2} ... z_d^{\alpha_d}$ we finally obtain the solution to the multicomponent coagulation equation (4.1) expressed in terms of $n_{|\alpha|}$,

$$n_\alpha(t) = n_{|\alpha|}(t) g(\alpha) \quad \text{with} \quad g(\alpha) = \frac{1}{d^k} \frac{k!}{\alpha_1!\alpha_2!...\alpha_d!}. \tag{4.13}$$

To study the long time behaviour, we use the fact that $\lim(\frac{1+t}{t})^t = e$ to obtain an approximation for $n_{|\alpha|}(t)$ for large $|\alpha|$ and large time t such that $|\alpha| \sim t$

$$n_{|\alpha|}(t) \approx t^{-2} \exp(-\frac{|\alpha|}{t}),\ t > 0 \tag{4.14}$$

Remark 4.1 In [30] it was proven that $n(x,t) = t^{-2} \exp\left(-\frac{x}{t}\right)$ is in fact the limit of a rescaled solution, provided the initial mass is either finite, which includes the case treated in this Section, or its distribution function diverges sufficiently weakly.

We also consider an approximation of the function g

$$g(\alpha) \approx |\alpha|^{-(d-1)/2} \exp(-\frac{|\alpha|_-^2}{2|\alpha|}) \tag{4.15}$$

where $|\alpha|_-^2 = \frac{1}{d} \sum\limits_{i,j=1}^{d} (\alpha_i - \alpha_j)^2$ denotes the generalized mass difference variable. Using (4.14) and (4.15) in (4.13) we obtain for large t and $|\alpha|$ the approximation

$$n_\alpha(t) \approx t^{-2} |\alpha|^{-(d-1)/2} \exp(-\frac{|\alpha|}{t}) \exp(-\frac{|\alpha|_-^2}{2|\alpha|}). \tag{4.16}$$

We observe that besides the mass scale imported from the solution to the one component equation, there is a second mass scale given by $|\alpha|_- \sim \sqrt{t}$. Introducing the variables $\xi = \frac{|\alpha|}{t}$ and $\rho = \frac{|\alpha|_-}{\sqrt{t}}$ we may then write the solution in a scaling form

$$n_\alpha(t) \approx t^{-(d+3)/2}\phi(\xi, \rho) \tag{4.17}$$

where

$$\phi(\xi, \rho) = \xi^{-(d-1)/2} \exp(-\xi) \exp(-\frac{\rho^2}{2\xi}) \tag{4.18}$$

Finally we note from (4.16) that for any fixed time, $n_\alpha(t)$ reaches maximum values when $|\alpha|_-^2 = 0$. This condition defines a straight line in the size space given by $\{\alpha \in \mathbb{N}_+^d \mid \alpha_1 = \alpha_2 = ... = \alpha_d\}$.

4.2 Mass localization in stationary solutions

Mass localization is also observed in stationary solutions to coagulation equations with source by applying a similar study as in the previous Section. We consider the stationary multicomponent coagulation equation with source and constant kernel as before $K_{\alpha,\beta} = 2$,

$$0 = \sum_{\beta<\alpha} n_{\alpha-\beta} n_\beta - 2 \sum_{\beta>0} n_\alpha n_\beta + s_\alpha \tag{4.19}$$

where s_α is the source term. In analogy to the initial conditions in the time-dependent case (4.2), the source term is given by

$$s_\alpha = \frac{h}{d} \sum_{|\beta|=1} \delta_{\alpha,\beta}, \tag{4.20}$$

for some given $h > 0$.

The constant kernel belongs to the class of kernels that are considered in Section 3. In particular, the constant kernel belongs to the subclass of kernels for which there is a stationary injection solution (see Theorem 3.4). Following the computations of [21] we compute in the following an explicit solution to the multicomponent equation (4.19).

For any compactly supported test function ψ_α, the weak formulation is now given by

$$0 = \sum_{\alpha,\beta=1}^{\infty} [\psi_{\alpha+\beta} - \psi_\alpha - \psi_\beta] n_\alpha n_\beta + \frac{h}{d} \sum_{|\alpha|=1} \psi_\alpha. \tag{4.21}$$

The generating function

$$F(z) = \sum_{|\alpha|=1}^{\infty} z^{|\alpha|} n_\alpha \tag{4.22}$$

satisfies

$$F(z)^2 - 2F(z)N + S(z) = 0 \tag{4.23}$$

where $S(z) = \frac{h}{d} \sum_{|\alpha|=1} z^\alpha = \frac{h}{d} \sum_{i=1}^d z_i$ and $N = \sqrt{h}$ is obtained using an appropriate test function in (4.21). The solution to (4.23) reads

$$F(z) = \sqrt{h}[1 - \sqrt{1 - \frac{|z|}{d}}]. \tag{4.24}$$

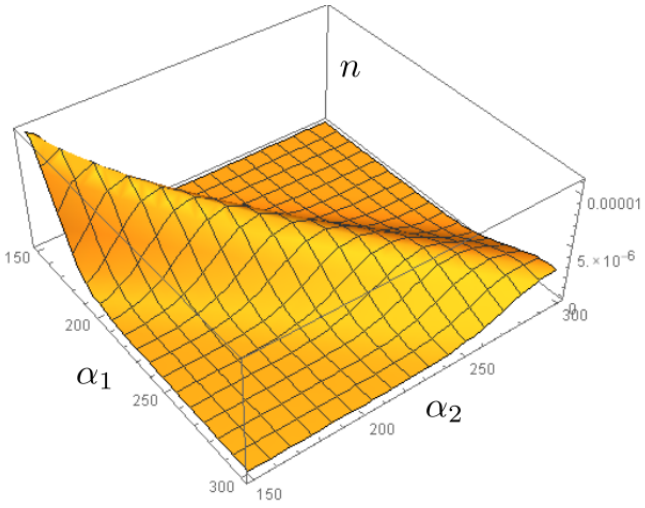

Figure 1: Representation of an approximation of a stationary solution to the two component coagulation equation with source and constant kernel (4.19). We observe a concentration of particles along a straight line.

The solution to (4.19) is obtained by expanding F in powers of the variables z_i and comparing with (4.22), yielding

$$n_\alpha = n_{|\alpha|} g(\alpha) \tag{4.25}$$

where g is defined in (4.13) and

$$n_{|\alpha|} = \frac{\sqrt{h}(2|\alpha|)!}{(2|\alpha|-1)(2^{|\alpha|}|\alpha|!)^2}. \tag{4.26}$$

For large sizes we may approximate $n_{|\alpha|}$ by $\sqrt{h}|\alpha|^{-3/2}$, therefore using also (4.15), we obtain

$$n_\alpha \approx \sqrt{h}|\alpha|^{-(d+2)/2} \exp(-\frac{|\alpha|_-^2}{2|\alpha|}). \tag{4.27}$$

Like in the time-dependent problem, an additional size scale is observed $|\alpha|_- \sim \sqrt{|\alpha|}$. Also here we can see from (4.27) that a stationary solution n_α reaches maximum values at the straight line defined by $\{\alpha \in \mathbb{N}_+^d \mid \alpha_1 = \alpha_2 = ... = \alpha_d\}$. A representation of (4.27) is shown in Figure 4.2.

5 Perspectives and open problems

The existence and uniqueness result of a time-dependent solution have also been established [31] for coagulation kernels satisfying (3.3)-(3.4) with $\gamma + \lambda = -\lambda$ and $\lambda > -1/2$ using a similar reasoning as in the proof of Theorem 3.2. Moreover, in [10] existence is obtained using a different framework where the solution f is a function, for a class of kernels satisfying (3.3)-(3.4) with $c_1 = c_2 = 1$, $\lambda \in [-1, 1]$, $\gamma \in [0, 2]$, $\gamma \leq -2\lambda$, $\gamma + \lambda \in [-1, 1]$ and $(\gamma, \lambda) \neq (-\lambda, -1)$. In [16] uniqueness is proved globally in time for a class of kernels satisfying (3.3)-(3.4) with $\gamma \leq 1$, $\lambda = 0$, and for a different class of kernels with $\gamma \in (1, 2]$, $\lambda = 1$ up to a gelation time T.

In general well-posedness remains an open problem. We refer to the survey [24] for further references.

In the presence of a constant source of monomers, the existence and non-existence results presented in Section 3 are the most recent ones to the best of our knowledge. Previous results [8] have been obtained for particular classes of kernels that are included in the more general setting presented here. In the multicomponent equation with kernel K satisfying

$$c_1 w(x,y) \le K(x,y) \le c_2 w(x,y) \quad \text{with} \quad w(x,y) = \sum_i x^{\gamma_i-\lambda_i} y^{\lambda_i} + y^{\gamma_i-\lambda_i} x^{\lambda_i}, \tag{5.1}$$

we expect the existence and nonexistence results to be still valid for each class of kernels satisfying $|\gamma_i + 2\lambda_i| < 1$ for all i and $|\gamma_i + 2\lambda_i| \ge 1$ for some i, respectively. Similarly to the time-dependent solutions, we also expect that stationary solutions exhibit localization along a straight line provided the bounds for the kernel are invariant under permutations.

Almost nothing is known about rigorous results for the multicomponent coagulation equations with general kernels. However, the well-posedness results for the one-component case, are expected to also hold in the multicomponent case provided the kernel satisfies the bounds (5.1) with γ_i and λ_i satisfy the same conditions for well-posedness in dimension $d = 1$ for all i. Mass localization for large times is expected to hold for a class of kernels that satisfy the same bounds with the additional condition that w is invariant under any permutation of the components. An additive kernel that does not satisfy this invariance has been considered in [36]. We expect that in that case the mass does not localize in a straight line, because each component will have a different rate of coagulation and complex multiscale behaviour is expected to emerge that may break down the nice localization structure.

Mass localization results are very important in the optimization of current algorithms as they allow to focus the computations on the region of size space where the mass is expected to be localized. The computational complexity of the multicomponent problem may in this way be reduced to the complexity of the onecomponent problem.

There are also many open problems for more general coagulation equations with fragmentation and additional sink and growth terms as well as on the derivation of these equations from appropriate particle systems. We refer to [7] for a brief overview on some of these topics.

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