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Independence-friendly logic without Henkin quantification

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Abstract We analyze the expressive resources of IF logic that do not stem from Henkin (partially-ordered) quantification. When one restricts attention to regular IF sentences, this amounts to the study of the fragment of IF logic which is individuated by the game-theoretical property of action recall (AR). We prove that the fragment of prenex AR sentences can express all existential second-order (ESO) properties. We then show that the same can be achieved in the non-prenex fragment of AR, by using "signalling by disjunction" instead of Henkin or signalling patterns.

We also study irregular IF logic (in which requantification of variables is allowed) and analyze its correspondence to regular IF logic. By using new methods, we prove that the game-theoretical property of knowledge memory is a first-order syntactical constraint also for irregular sentences, and we identify another new first-order fragment. Finally we discover that irregular prefixes behave quite differently in finite and infinite models. In particular, we show that, over infinite structures, every irregular

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prefix is equivalent to a regular one; and we present an irregular prefix which is second order on finite models but collapses to a first-order prefix on infinite models.

Keywords Independence-friendly logic · Existential second-order logic · Signalling · Action recall · Knowledge memory · Irregular formulas

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1 Introduction

Independence-friendly logic ([**?**], [**?**]) is one of a number of formalisms that have been developed in order to make various notions of *dependence* and *independence* accessible to the instruments of logical investigation. Independence-friendly (IF) logic and similar formalisms (Dependence-friendly logic, Dependence logic [**?**]), in particular, were developed as more flexible approaches to the logic of *Henkin quantifiers* ([**?**]). The Henkin quantifier H^n_k is a matrix

$$\begin{pmatrix} \forall x^1_1 & \forall x^2_1 & \dots & \forall x^n_1 & \exists y_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \forall x^1_k & \forall x^2_k & \dots & \forall x^n_k & \exists y_k \end{pmatrix}$$

which, differently from a linear sequence of the same quantifiers, is meant to state that each y_i is supposed to be chosen as a function of $x^1_i,\dots,x^n_i$ only. In IF logic, the same is achieved by means of a linear prefix, together with a slashing device. For example, the Henkin quantifier H^1_2, that is,

$$\begin{pmatrix} \forall x^1_1 & \exists y_1 \\ \forall x^1_2 & \exists y_2 \end{pmatrix}$$

is expressed in IF logic by the following linear sequence of quantifiers[1]

$$\forall x^1_1 \exists y_1 \forall x^1_2 (\exists y_2/\{x^1_1, y_1\}).$$

The slashed quantifier $(\exists y_2/\{x^1_1, y_1\})$ expresses the fact that y_2 is *independent* from x^1_1 and y_1.

It has been gradually realized that, in spite of the fact that it stems from the study of Henkin quantifiers, IF logic derives its expressiveness also from other sources. Henkin quantifiers are partial orderings of first-order quantifiers; but in IF logic also intransitive (thus not ordered) dependence sequences are allowed, for example

$$\forall x \exists y (\exists z/\{x\}).$$

Here y depends on x, z depends on y, but z does not depend on x. It is known that such quantifier sequences, also known as *signalling* patterns, can be used to express higher-order concepts ([**?**],[**?**],[**?**]); for example, the IF sentence

$$\exists v \forall x \exists y (\exists z/\{x\})(x = z \land y \neq v)$$

[1] There are (up to renaming of variables and permutation of quantifiers) a few alternative ways for expressing the Henkin quantifier H^1_2 in IF logic, for example $\forall x^1_1 \forall x^1_2 (\exists y_1/\{x^1_2\})(\exists y_2/\{x^1_1, y_1\})$ and $\forall x^1_1 \forall x^1_2 (\exists y_2/\{x^1_1\})(\exists y_1/\{x^1_2, y_2\})$.

is known to characterize the class of all (Dedekind) infinite structures (this idea is attributed, in [?], to Fred Galvin).

By a dichotomy result of Sevenster ([?]), Henkin and signalling patterns are known to exhaust the higher-order expressive power of *prenex regular*[2] IF logic: if a regular sentence is in prenex normal form and does not contain Henkin or signalling patterns, then it is equivalent to some first-order sentence ([?]). Non-prenex, regular IF logic is known to contain further expressive syntactical patterns (involving the interaction of quantifiers and connectives) that are neither of the Henkin nor the signalling type, yet allow describing NP-complete problems such as SAT and SET PARTITIONING ([?]); the problem of a complete classification of such patterns is still open. Less is known of irregular IF logic, which will be addressed here in Sections 7.2, 8 and 9. The aim of the present paper is a better understanding of the resources of IF logic that do *not* stem from Henkin quantification.

A peculiarity of independence-friendly logic is the close link between its syntax and the theory of extensive games of imperfect information. The link is given by the so-called Game-Theoretical Semantics, which we will review in Section 2. Through this connection, game-theoretical concepts throw light on peculiarities of the logic; and vice versa, the study of logical phenomena can cast new light on the foundations of game theory.

It is well known, through the works of Henkin, Hintikka and others, that it is possible to define a notion of truth for first-order languages in terms of certain games of perfect information, which involve two players called Verifier ("Eloise") and Falsifier ("Abelard"), who take it in turns to point out evidence for or against the truth of a given sentence φ in a given structure M. The resulting Game-Theoretical Semantics (GTS) is equivalent to the usual Tarskian one.

When moving from first-order to IF languages, extending the Tarskian semantics is not straightforward[3]; instead, it is quite natural to generalize semantic games by allowing *imperfect information*, in a way that the independence constraints expressed by syntax correspond (roughly speaking) to the fact that a player is forced to make his/her choices in ignorance of the outcomes of some earlier moves ([?]). This generalization allows new complex possibilities. Many IF games are actually games of *imperfect recall*: the players may forget what they knew at earlier stages of the game.

In this paper, we will be particularly interested in a game-theoretical property called *action recall* (AR). Eloise has action recall if she cannot forget her own moves; assuming regularity, an IF sentence has AR (i.e., all its corresponding games have action recall) for Eloise if its sets of slashed variables associated to existential quantifiers contain no existentially quantified variables. Thus e.g. $\forall x(\exists y/\{x\})R(x,y)$ has AR, while $\exists x(\exists y/\{x\})R(x,y)$ does not have it.

The fragment of sentences with action recall for Eloise is particularly important for our purposes because, in it, it is impossible to write the usual IF translations of Henkin prefixes[4], and yet it is a highly expressive fragment. Therefore, it is natural

[2] Regular sentences do not contain requantification of variables.

[3] It can be done, at the cost of defining a notion of satisfaction by sets of assignment, instead of the usual single assignments. See e.g. [?],[?],[?],[?],[?].

[4] This point is exemplified by the IF rendition of the H^1_2 prefix, shown above: its "slash set" $\{x^1_1, y_1\}$ contains an existentially quantified variable, y_1.

to wonder to what degree the IF-definable concepts are expressible under the restriction of action recall. IF logic is known to capture exactly the existential second-order (ESO) definable classes ([?]). We will show that the Henkin prefixes H_2^n are explicitly definable in the *prenex*, *regular* fragment of AR (therefore, by means of signalling). The H_2^n prefixes, taken together, are known to capture all ESO definable concepts ([?]); thus, the prenex, regular fragment of action recall suffices for full IF expressive power. We also present a simpler translation of the H_2^n prefixes into the *non*-prenex fragment of AR *without signalling patterns*; in this case, the relevant source of expressiveness is a form of "signalling by disjunction". Furthermore, we present an alternative proof for this claim by using *irregular* IF sentences instead.

Properties of irregular IF sentences and prefixes have not been studied much and their expressive power is not very well understood. We extend the understanding of them by studying irregular IF logic from different perspectives. Firstly, we present a new kind of "regularization procedure" which can be used for naturally translating irregular sentences into regular ones. Moreover, we observe that when this procedure is applied to prenex sentences, it leaves the quantifier-free part intact. We also present an alternative type of regularization procedure that can be applied to particular fragments of IF logic, among which is the fragment of sentences that have the property of *knowledge memory* (KM)[5]. This procedure can be used to generalize a result of [?], stating that the fragment of knowledge memory is first order, to the case of irregular sentences. The alternative regularization also allows us to prove another first-orderness criterion, orthogonal to the criterion of knowledge memory.

After this we study irregular prefixes by first making observations on how irregular Henkin/signalling prefixes have different properties compared to regular ones. Then we apply our regularization procedure to irregular prefixes to show that their expressive power amounts to a more general question of the expressive power of regular prefixes "relative" to variable sets. Next we demonstrate how irregular prefixes behave quite differently over infinite and finite models. In infinite models, requantified variables can be used "in full potential" for signalling. The proof of this claim also provides a straightforward technique for checking whether an irregular prefix is second order over infinite models. Finally we show how irregular prefixes may gain expressive power in finite models, and present an example of an irregular prefix $\mathbf{Q}^*$ which defines a problem in LOGSPACE which is not first-order definable. However, interestingly $\mathbf{Q}^*$ collapses to a first-order prefix on infinite models.

Structure of the paper.

First, in Section 2, we review preliminary notions about IF syntax, game-theoretical semantics and Skolemizations; we also generalize the notion of expressive power of prefixes relative to variable sets. In Section 3 we review the definitions of Henkin and signalling patterns and extend them to the case of irregular formulae. In Section 4 we define memory properties, Action Recall (AR), Knowledge Memory (KM) and Perfect Recall (PR), and generalize their syntactical characterizations to include irregular

[5] An IF sentence is said to have KM if in the corresponding semantic game the Verifier never forgets what she knew in earlier stages of the game.

sentences. In Section 5 we present several examples of expressing NP-complete problems in the fragment of AR. In Section 6 we show how to capture ESO in the prenex fragment of AR and in Section 7 we do the same in the non-prenex fragment of AR without signalling. In Section 8 we present two regularization methods for irregular sentences and use them to isolate two new first-order fragments, one of which is the irregular fragment of KM. Finally, in Section 9 we study the properties of irregular prefixes in finite and infinite models.

2 Preliminaries

Structures are denoted by capital italic letters. To keep the notation simple, we do not introduce a separate symbol for the domain of a structure; thus if M is a structure, $a \in M$ and $R^M \subseteq M^2$ mean that a is an element of the domain of M and R^M is a binary relation on the domain of M, respectively.

An *assignment* of variables on a structure M is a function $s\colon V \to M$, where the domain V of s is a finite set of variables. We denote the set of all assignments on M with domain V by $\mathrm{As}(V,M)$. Given $s \in \mathrm{As}(V,M)$ and $a \in M$, we write $s(a/v)$ for the assignment with domain $V \cup \{v\}$ such that $s(a/v)(v) = a$ and $s(a/v)(u) = s(u)$ for $u \in V \setminus \{v\}$. If $\bar{x} = (x_1,\ldots,x_n)$ is a tuple of variables, we use the shorthand notation $s(\bar{x})$ for $(s(x_1),\ldots,s(x_n))$.

2.1 Syntax of IF logic

The syntax of IF logic is a restriction of the usual first-order syntax, to which we add quantifiers of the forms $(\exists v/V)$ and $(\forall v/V)$, where V is a finite set of variables, called the *slash set* of the quantifier. When $V = \emptyset$, we use the abbreviation $Qx := (Qx/V)$. The syntax is restricted, with regards to usual first-order languages, in that

- we only allow the connectives $\wedge, \vee$ and $\neg$, and
- for simplicity, we only allow $\neg$ to occur in front of atomic formulae.

The set $\mathrm{Free}(\varphi)$ of free variables of a formula φ is defined as usual, with the proviso that also variables from slash sets can be either free or bound. In detail:

- $\mathrm{Free}(\alpha)$ is the set of all variables occurring in α, in case α is a literal (i.e., an atomic formula or the negation of an atomic formula)
- $\mathrm{Free}(\psi_1 \wedge \psi_2) = \mathrm{Free}(\psi_1 \vee \psi_2) = \mathrm{Free}(\psi_1) \cup \mathrm{Free}(\psi_2)$
- $\mathrm{Free}((\exists v/V)\psi) = \mathrm{Free}((\forall v/V)\psi) = (\mathrm{Free}(\psi) \setminus \{v\}) \cup V$.

For example, in $\forall x(\exists y/\{x,y,z\})\psi$, the occurrence of x in the slash set is bound, while the occurrences of y,z are free. We say that an IF formula φ is a *sentence* if $\mathrm{Free}(\varphi) = \emptyset$.

We will need two different notions of substitution, one for variables and one for terms. Let φ be a formula, x,y variables and t a term.

- Substitution of a variable for a variable: $\mathrm{Subv}(\varphi,x,y)$ denotes the formula obtained by replacing with y all the free occurrences of x in φ.

– Substitution of a term for a variable: $\mathrm{Subt}(\varphi,x,t)$ denotes the formula obtained replacing with t all the free occurrences of x in φ, *except* those occurring in slash sets.

Notice indeed that replacing a complex term within a slash set would not produce a well-formed IF formula. On the other hand, both kinds of substitution allow replacing a variable for another variable, but the result in the two cases may differ; for example, $\mathrm{Subv}((\exists z/\{x\})R(x,z),x,y) = (\exists z/\{y\})R(y,z) \neq (\exists z/\{x\})R(y,z) = \mathrm{Subt}((\exists z/\{x\})R(x,z),x,y)$.

When considering *subformulae* ψ of an IF formula φ, we define (as normally for IF logic) that different *occurrences* of ψ in φ are considered as different subformulae. Hence the set of subformulae, $\mathrm{Sf}(\varphi)$, of φ is actually the set of nodes in the syntactical tree of φ.

A further restriction on the syntax of IF logic that is often assumed in the literature (see, e.g., [**?**] and [**?**]) is that variables are not requantified:

– A sentence φ is *regular* if no quantifier (Qv/V) occurs in the scope of another quantifier $(Q'v/W)$ over the same variable v.

More generally, an IF formula φ is regular if the condition above holds, and furthermore $\mathrm{Free}(\varphi) \cap \mathrm{Bound}(\varphi) = \emptyset$. Note that this definition allows several quantifications of variable v as long as they are on different branches in the syntactical tree of φ.

We denote IF logic with this regularity restriction by IF^{r}; by IF^{p} the set of *prenex* IF sentences; and by $\mathrm{IF}^{\mathrm{p,r}}$ the set of IF sentences that satisfy both regularity and prenex form. We will at first restrict our studies to IF^{r}, but in Sections 4, 7.2, 8 and 9 we will also consider irregular sentences.

2.2 Game-Theoretical Semantics

Game-Theoretical Semantics (GTS) associates a 2-player win-lose extensive game of imperfect information $G(\varphi,M,s)$ to each triple (φ,M,s), where φ is an IF formula, M is a structure, and $s \in \mathrm{As}(V,M)$ for a set V of variables such that $\mathrm{Free}(\varphi) \subseteq V$. In case φ is a sentence and $s = \emptyset$, we simply write $G(\varphi,M)$. The two players, usually called Eloise and Abelard, can be thought of as trying to verify, respectively falsify, the sentence φ on the structure M. Their moves are triggered by the most external logical operator of φ:

– in $G(\psi_1 \vee \psi_2,M,s)$, Eloise chooses a disjunct ψ_i, and then $G(\psi_i,M,s)$ is played;
– in $G(\psi_1 \wedge \psi_2,M,s)$, the same kind of move is performed by Abelard;
– in $G((\exists v/V)\psi,M,s)$, Eloise picks an element $a \in M$ and then $G(\psi,M,s(a/v))$ is played;
– in $G((\forall v/V)\psi,M,s)$ the same kind of move is performed by Abelard;
– in $G(\alpha,M,s)$, with α a literal (i.e., an atomic formula or the negation of an atomic formula), the winner is decided: it is Eloise in case $M,s \models \alpha$ (in the usual first-order sense), and Abelard otherwise.

In the recursive definition above, the semantical game $G(\varphi,M,s)$ defined by using "subgames" $G(\psi,M,s')$, where ψ is a subformula of φ. Provided we treat the

$G(\psi, M, s')$ corresponding to distinct occurrences of ψ in φ as distinct games, we can see these subgames as *positions* within the original game $G(\varphi, M, s)$. We will often use this terminology and write "position (ψ, M, s')" (corresponding to the *occurrence* ψ) instead of "subgame $G(\psi, M, s')$". A *history* of a game $G(\varphi, M, s)$ is any tuple $h := p_1, \ldots, p_n$ of positions[6] starting from the *initial position* (φ, M, s) so that each position p_{i+1} can be reached from p_i by the rules of the game.

Imperfect information manifests itself in that some histories of the game are considered indistinguishable for the player who has the turn to move at the end of them. If two histories h and h' both end with the choice of a subgame associated with the same occurrence of a subformula $(Qv/V)\psi$ and with assignments $s_h, s_{h'} \in \mathrm{As}(W, M)$ such that $s_h(w) = s_{h'}(w)$ for every $w \in W \setminus V$, then we say that h and h' are indistinguishable for the player associated to (Qv/V), and we write $h \sim_V h'$. If instead h, h' end in the same occurrence of $\psi_1 \circ \psi_2$, where $\circ \in \{\vee, \wedge\}$, with assignments $s_h, s_{h'} \in \mathrm{As}(W, M)$ such that $s_h = s_{h'}$ we write $h \sim_\emptyset h'$. (Note that here h and h' may differ on choices for the values of those variables which have been requantified before reaching $\psi_1 \circ \psi_2$; this situation does not arise in regular sentences.)

A *strategy* for Eloise in game $G(\varphi, M, s)$ is a function associating, to each history ending in a subgame $G((\exists v/V)\psi, M, s')$, an element $a \in M$; and, to every history ending in a subgame $G(\psi_1 \vee \psi_2, M, s')$, either ψ_1 or ψ_2. Strategies for Abelard can be defined similarly. A strategy of Eloise is *winning* if, playing according to it, Eloise wins, whatever moves Abelard makes. Winning strategies for Abelard are defined dually. A strategy σ is *uniform* if, whenever two histories h, h' are in its domain and $h \sim_V h'$ (for the only appropriate V), then $\sigma(h) = \sigma(h')$.

Remark 1 As usually done with IF logic, strategies are defined above as mappings on the set of histories of the semantic game. An alternative way would be to define strategies as mappings on the set of *positions* of the game instead. As several histories can lead to the same position, the former approach is more general. However, in the context of IF-logic, these two definitions give rise to the same uniform strategies as the moves given by such strategies are determined by the *last position* (i.e. "current position") of the history. Indeed, for a quantifier (Qv/V), the V-equivalence of histories h and h' is defined in terms of the assignments and formula occurrences which appear in the last positions on h and h'; and likewise for the equivalence of histories ending in a disjunctive formula. So the definition of uniform strategy can be in turn expressed in terms of last positions. In this paper we assume uniform strategies to be defined in this alternative way when it is more convenient.

With this game-theoretical apparatus, it is possible to define the notions of *truth* and *falsity* for IF sentences as the existence of appropriate strategies:

$$M \models \varphi \text{ if Eloise has a uniform winning strategy in } G(\varphi, M)$$
$$M \models^{-} \varphi \text{ if Abelard has a uniform winning strategy in } G(\varphi, M).$$

[6] Note that also all the moves made by the players can be read from the positions (ψ, M, s) in the history as earlier moves for quantifiers are recorded by the assingments s and earlier moves for connectives can be read from the formulae ψ (as they refer to occurrences of formulae).

There is also a third possibility: it may happen that neither player has a uniform winning strategy (consider, e.g., the sentence $\forall x(\exists y/\{x\})x = y$). In that case, the game and the truth value of the sentence on M are said to be *undetermined*. In this paper, we only focus on the truth/nontruth distinction. Accordingly, we say that two IF sentences φ, θ are (truth-)*equivalent* if, for all structures M,

$$M \models \varphi \iff M \models \theta;$$

and we say that a class K of structures is definable in IF logic if there is an IF sentence φ such that, for all structures M,

$$M \in K \iff M \models \varphi.$$

As was already shown in [**?**], IF logic has the same expressive power as existential second-order logic ESO: a class of structures is definable in IF logic if and only if it is definable in ESO.

2.3 Skolemization

Any IF sentence φ can be translated into an equivalent ESO sentence $\mathrm{SK}(\varphi)$ by means of a process of (inside-out) Skolemization which generalizes the Skolemization procedure of first-order logic (see [**?**]). As an intermediate step, we can define the *first-order* Skolemization $\mathrm{Sk}(\varphi)$ of an IF sentence φ, or more generally the first-order Skolemization $\mathrm{Sk}_U(\psi)$ of an IF formula ψ, relativized to a set of variables $U \supseteq \mathrm{Free}(\psi)$.

Given a regular IF formula ψ with occurrences $(\exists v_1/V_1), \ldots, (\exists v_n/V_n)$ of existential quantifiers, let $h_1, \ldots, h_n$ be fresh and distinct function symbols (the correct arities can be deduced from the inductive definition below). Given a set of variables $U \supseteq \mathrm{Free}(\psi)$, define

$$\begin{aligned}
&\mathrm{Sk}_U(\psi) = \psi \text{ (if } \psi \text{ is a literal)}\\
&\mathrm{Sk}_U(\psi \vee \psi') = \mathrm{Sk}_U(\psi) \vee \mathrm{Sk}_U(\psi')\\
&\mathrm{Sk}_U(\psi \wedge \psi') = \mathrm{Sk}_U(\psi) \wedge \mathrm{Sk}_U(\psi')\\
&\mathrm{Sk}_U((\exists v_i/V_i)\psi) = \mathrm{Subt}(\mathrm{Sk}_{U\cup\{v_i\}}(\psi), v_i, h_i(y_1, ..., y_m))\\
&\mathrm{Sk}_U((\forall v/V)\psi) = \forall v\, \mathrm{Sk}_{U\cup\{v\}}(\psi)
\end{aligned}$$

where $y_1, ..., y_m$ is a list of the variables in $U \setminus V_i$. Then, the (unrelativized) Skolemization of a sentence φ is defined as $\mathrm{Sk}(\varphi) := \mathrm{Sk}_\varnothing(\varphi)$. It can be proved that, for any regular IF sentence φ of bound variables $v_1, \ldots, v_n$, and any suitable structure M,

$$M \models \varphi \iff (M, f_1, \ldots f_n) \models \mathrm{Sk}(\varphi) \text{ for some interpretations } f_1, \ldots f_n \text{ of the function symbols } h_1, \ldots h_n.$$

On the right hand side, the symbol $\models$ denotes the usual notion of first-order truth.

The (second-order) Skolemization $\mathrm{SK}(\varphi)$ of a regular IF sentence φ with occurrences $(\exists v_1/V_1), \ldots, (\exists v_n/V_n)$ of existential quantifiers is the existential second-order sentence $\exists h_1 \ldots \exists h_n \mathrm{Sk}(\varphi)$. $\mathrm{SK}(\varphi)$ is equivalent to φ in the sense that, for any M,

$$M \models \varphi \iff M \models \mathrm{SK}(\varphi).$$

In [?] (Theorem 4.13) it is claimed that the procedure just described works correctly also on irregular IF formulas. However, one of the anonymous reviewers of this paper pointed out a counterexample to this claim. It turns out that one can even find counterexamples among irregular *first-order* sentences. Consider the sentence $\forall x \exists y \forall x(x=y)$; it is obviously equivalent to $\exists y \forall x(x=y)$, which is non-true on structures with at least two elements. However, we have:

$$\begin{aligned}\mathrm{Sk}(\forall x \exists y \forall x(x=y)) &= \forall x\, \mathrm{Sk}_{\{x\}}(\exists y \forall x(x=y))\\ &= \forall x\, \mathrm{Subt}(\mathrm{Sk}_{\{x,y\}}(\forall x(x=y)), y, f_{\exists y}(x))\\ &= \forall x\, \mathrm{Subt}(\forall x(x=y), y, f_{\exists y}(x))\\ &= \forall x \forall x(x=f_{\exists y}(x)),\end{aligned}$$

so that we may take $\mathrm{SK}(\forall x \exists y \forall x(x=y))$ to be $\exists f \forall x \forall x(x=f(x))$. This formula is valid (just take f to be the identity function), therefore it is not equivalent to $\forall x \exists y \forall x(x=y)$. We will see how to define Skolemization correctly for irregular sentences in section 8.1 (Remark 6).

2.4 IF prefixes

For clarity, some of the notions that apply to formulas are extended here to quantifier prefixes (i.e. tuples of the form $(Q_1 v_1/V_1)\dots(Q_n v_n/V_n)$, where $Q_i \in \{\exists, \forall\}$). For a quantifier prefix $\mathbf{Q}$, the set of free variables of $\mathbf{Q}$, denoted by $\mathrm{Free}(\mathbf{Q})$, is defined recursively as follows:

- $\mathrm{Free}((\exists v/V)) = \mathrm{Free}((\forall v/V)) = V$
- $\mathrm{Free}((\exists v/V)\mathbf{Q}) = \mathrm{Free}((\forall v/V)\mathbf{Q}) = (\mathrm{Free}(\mathbf{Q}) \setminus \{v\}) \cup V$.

If $\mathrm{Free}(\mathbf{Q}) = \emptyset$, then the prefix $\mathbf{Q}$ is said to be *sentential*. Notice that, if a prefix is sentential, then the obviously defined set $\mathrm{Bound}(\mathbf{Q})$ of bound variables of $\mathbf{Q}$ coincides with the set of all variables occurring in $\mathbf{Q}$.

Let $\mathbf{Q}$ and $\mathbf{Q}'$ be sentential prefixes and let U be a set of variables for which $U \subseteq \mathrm{Bound}(\mathbf{Q}) \cap \mathrm{Bound}(\mathbf{Q}')$. We say that $\mathbf{Q}$ *is equivalent to* $\mathbf{Q}'$ *over* U, if $\mathbf{Q}\psi$ is equivalent to $\mathbf{Q}'\psi$ for any quantifier-free formula ψ for which $\mathrm{Free}(\psi) \subseteq U$. In the special case when $U = \mathrm{Bound}(\mathbf{Q}) = \mathrm{Bound}(\mathbf{Q}')$, if $\mathbf{Q}$ and $\mathbf{Q}'$ are equivalent over U, we simply say that $\mathbf{Q}$ and $\mathbf{Q}'$ are equivalent.

Let $\mathbf{Q}$ be a sentential prefix for which $U \subseteq \mathrm{Bound}\,\mathbf{Q}$. We say that $\mathbf{Q}$ is *first order over* U if $\mathbf{Q}\psi$ is equivalent to a first-order sentence for every quantifier-free formula ψ for which $\mathrm{Free}(\psi) \subseteq U$. And we say that $\mathbf{Q}$ is NP-*hard over* U if there is a quantifier-free formula ψ such that $\mathrm{Free}(\psi) \subseteq U$ and the sentence $\mathbf{Q}\psi$ defines an NP-hard problem. [7] In the special case when $U = \mathrm{Bound}(\mathbf{Q})$ we may simply say that $\mathbf{Q}$ is first order or that $\mathbf{Q}$ is NP-hard. Note here that when $\mathbf{Q}$ is first order, it is not necessarily syntactically a first-order prefix as it may contain nonempty slash sets.

[7] We remark that, since IF sentences are translatable into ESO sentences, by a well-known result of Fagin ([?]) all the problems definable in IF logic are in NP. Therefore, in this context an NP-hard problem is, more exactly, an NP-complete problem.

It is worth noting that in the earlier literature the notions above have only been defined in the special case when $U = \mathrm{Bound}(\mathbf{Q})$. The generalization of these notions "relatively" to a given set U of variables will be useful when we study irregular prefixes and their regular counterparts in Section 9.

3 Signalling and Henkin patterns

We have been talking informally of Henkin and signalling patterns of quantifiers. Exact definitions were given in [**?**] for the prenex regular fragment $\mathrm{IF}^{\mathrm{p,r}}$ and extended to IF^{r} in [**?**]. We further generalize these definitions, so that they may cover also the case of irregular IF formulae.

3.1 Effective scopes and linear patterns

In the definitions of this section we consider occurences of quantifiers in any (fixed) IF formula. This more general definition can naturally be applied to IF sentences and to the quantifier prefixes of IF formulae.

- In an IF formula (resp. in a quantifier prefix), we say that a quantifier (Qv/V) *is in the effective scope* of $(Q'u/U)$, denoted by $(Qv/V) \in \mathrm{Es}(Q'u/U)$, if the following conditions hold:
  - (Qv/V) is in the scope of $(Q'u/U)$,
  - there is no quantifier $(Q''u/W)$ in the scope of $(Q'u/U)$ such that (Qv/V) is in the scope of $(Q''u/W)$.

When the quantifier (Qv/V) is in the effective scope of $(Q'u/U)$, it intuitively means that the value of the variable u, quantified by $(Q'u/U)$, is "available" when the variable v is quantified by (Qv/V).

Example 1 Consider the following IF formula:

$$\delta := \forall x \exists y (\exists y/\{x\}) \exists z\, \psi.$$

Here $\exists z$ is not in the effective scope of $\exists y$ as the variable y is requantified before the quantification of z. However, $(\exists y/\{x\}) \in \mathrm{Es}(\exists y)$ as the "old value" of y is still available when y is about to be requantified. Moreover, $\exists y, (\exists y/\{x\}), \exists z \in \mathrm{Es}(\forall x)$ as the requantification of any other variables in δ does not "block" the effective scope of $\forall x$.

Note that if a formula does not contain any requantification of variables, then the effective scope simply means scope. Hence, in the case of regular formulae, the effective scopes may be replaced with scopes in the following definitions.

- In an IF formula (resp. in a quantifier prefix), we say that a quantifier (Qv/V) *depends* on a quantifier $(Q'u/U)$ if $(Qv/V) \in \mathrm{Es}(Q'u/U)$ and $u \notin V$.

If (Qv/V) is not in the effective scope of $(Q'u/U)$, then the value of v intuitively cannot depend on the value of u (quantified by $(Q'u/U)$) as it is not available. But all the quantifiers (Qv/V) in the effective scope of $(Q'u/U)$ intuitively may depend on the value of u unless this dependence is explicitly forbidden by including u in the slash set V. From the perspective of semantical games, (Qv/V) depends on $(Q'u/U)$ if and only if the value that was chosen by $(Q'u/U)$ "can be seen" by the player who makes a move for (Qv/V).

- Suppose that the quantifiers $(Q_1v_1/V_1),\dots,(Q_nv_n/V_n)$ occur in an IF formula (resp. in a quantifier prefix).
  - If (Q_jv_j/V_j) is in the effective scope of (Q_iv_i/V_i), whenever $i<j$, we say that $(Q_1v_1/V_1),\dots,(Q_nv_n/V_n)$ form a *linear pattern* (in the given order).
  - If the condition above holds for scopes (but not necessarily effective scopes), we say that $(Q_1v_1/V_1),\dots,(Q_nv_n/V_n)$ form a *weakly linear pattern*.

The requirement of the above guarantees that quantifiers forming a (weakly) linear pattern all occur *on the same branch in the syntactical tree*. Also note that quantifiers in a prefix always form a weakly linear pattern, but not necessarily a linear pattern. Also *non-weakly linear* patterns in which quantifiers occur on several different branches have been studied in the literature (cf. Remark 2 after the definition of a Henkin pattern).

Note that the last variable that is quantified in a linear pattern may be identical to some earlier variable in the pattern. For example the quantifiers $\forall x$, $\exists y$, $(\exists y/\{x\})$ form a linear pattern in the formula δ in Example 1. However, it is easy to see that no other variables, except for the last one, can be repeated in a linear pattern. For example the quantifiers $\exists y$, $(\exists y/\{x\})$, $\exists z$ form a weakly linear pattern in δ which is not a linear pattern because $\exists z \notin \mathrm{Es}(\exists y)$. For the same reason $\forall x$, $\exists y$, $\exists z$ is not a linear pattern in δ. Note, however, that the pair $\forall x$, $\exists z$ and also the triple $\forall x$, $(\exists y/\{x\})$, $\exists z$ form a linear pattern in δ.

3.2 Signalling and Henkin patterns as linear patterns

We are now ready to define signalling and Henkin patterns as linear patterns of certain quantifiers that have specific dependencies between each other.

- A *signalling pattern* in an IF formula (resp. in a quantifier prefix) is a linear pattern formed by three quantifiers $(\forall x/X)$, $(\exists y/Y)$, $(\exists z/Z)$ in such a way that the following dependency conditions hold:
  - $(\exists y/Y)$ depends on $(\forall x/X)$;
  - $(\exists z/Z)$ depends on $(\exists y/Y)$, but not on $(\forall x/X)$.

In a signalling pattern the dependency relation between the quantifiers *is not transitive* as the quantifier $(\exists y/Y)$ depends on $(\forall x/X)$ and $(\exists z/Z)$ depends on $(\exists y/Y)$, but $(\exists z/Z)$ does not depend (directly) on $(\forall x/X)$. However, the variable z may depend on x "indirectly" as the value of y may be used for "signalling" information on the value of x.

Example 2 The IF sentence $\chi := \forall x \exists y (\exists z/\{x\})(P(y) \wedge z = x)$ contains the most simple signalling pattern $\forall x \exists y (\exists z/\{x\})$. By observing the Skolemization

$$\exists f_y \exists f_z \forall x (P(f_y(x)) \wedge f_z(f_y(x)) = x)$$

of χ, we see that χ is true in a model M if and only if the interpretation of P in M has the same cardinality as the domain of M. This property is not FO-definable (not even relative to the class of infinite structures; we will use this observation in Section 9).

- A *Henkin pattern* in an IF formula (resp. in a quantifier prefix) is a linear pattern formed by four distinct quantifiers $(\forall x/X)$, $(\exists y/Y)$, $(\forall z/Z)$, $(\exists w/W)$ in such a way that the following dependency conditions hold:
  - $(\exists y/Y)$ depends on $(\forall x/X)$, but neither on $(\forall z/Z)$ nor $(\exists w/W)$;
  - $(\exists w/W)$ depends on $(\forall z/Z)$, but neither on $(\forall x/X)$ nor $(\exists y/Y)$.

In order to see the idea behind this definition, recall the simplest (nontrivial) Henkin quantifier H_2^1:

$$\begin{pmatrix} \forall x & \exists y \\ \forall z & \exists w \end{pmatrix}.$$

Here y depends on x but neither on z nor w, and likewise w depends on z but neither on x nor y. Note that exactly the same dependency requirement is given for the four quantifiers in the definition of a Henkin pattern.

The Henkin quantifier H_2^1 can be translated into an IF prefix in the following three distinct ways:

$$A_1 : \forall x \forall z (\exists y/\{z\})(\exists w/\{x,y\})$$
$$A_2 : \forall x \forall z (\exists w/\{x\})(\exists y/\{z,w\})$$
$$A_3 : \forall x \exists y \forall z (\exists w/\{x,y\}).$$

Further equivalent prefixes can be obtained by renaming the variables or permuting quantifiers; these further prefixes do not differ in any essential way from the three forms listed above.

Remark 2 In this paper, we require Henkin patterns to be linear, but one could also consider "non-weakly linear Henkin patterns" which satisfy the given dependency conditions, but need not be linear. The paper [**?**] considers such patterns in the context of regular IF logic (see also Section 3.3). The signalling patterns on the other hand need to be linear as otherwise the given dependency conditions could not be satisfied.

If a quantifier prefix contains a signalling (resp. Henkin) pattern, we say that the prefix is signalling (resp. Henkin). More generally we say that a syntactical tree is signalling or Henkin if it contains such a pattern. It is known that a regular prefix has second-order expressive power if and only if it signalling or Henkin, as in the following theorem.

Theorem 1 (Sevenster's dichotomy, [?]) *Let* **Q** *be a regular, sentential prefix.*

1. *If* **Q** *is signalling or Henkin, then* **Q** *is* NP*-hard.*
2. *Otherwise,* **Q** *is first order.*

Irregular signalling/Henkin prefixes have quite different properties than the regular ones. We will discuss these differences in Section 9.1 in more detail.

3.3 Signalling by disjunction.

We have used the word “signalling”, until now, to refer to the transmission of information that is forbidden by slash sets by means of an intermediary existential quantifier. However, in the literature (e.g. [**?**], [**?**], [**?**], [**?**]) an alternative form of signalling has been discussed, in which disjunctions are used as a way to encode binary signals. A typical example from [**?**] is as follows. Observe first that the formula $\forall x(\exists y/\{x\})x \neq y$ is undetermined (in particular: not true) over any structure M with at least 2 elements in the domain. However, the seemingly redundant formula

$$\forall x((\exists y/\{x\})x \neq y \lor (\exists y/\{x\})x \neq y)$$

is valid for models where $|M| \geq 2$. Indeed, Eloise can abide to the following winning strategy: pick a fixed element a for y in the left disjunct; pick a distinct element b in the right disjunct; choose the left disjunct if Abelard picked an element $c \neq a$, and the right disjunct otherwise. In this strategy, the disjunction is used to store a bit of information, the answer to the question “is c distinct from a”?

We can use the notation $\forall x((\exists y/\{x\})[\ ] \lor (\exists y/\{x\})[\ ])$ in order to isolate the purely logical part of the syntactical tree of the formula above; the gaps $[\ ]$ should be thought of as placeholders for quantifier-free formulas with appropriate sets of free variables. These syntactical constructs are the natural analogues of quantifier prefixes outside the realm of prenex logic; they have been studied in relation to IF logic in [**?**] under the name of “tree prefixes”.

Despite the interest of the example above, the tree $\forall x((\exists y/\{x\})[\ ] \lor (\exists y/\{x\})[\ ])$ is provably inadequate to express statements beyond first order. Analogously, the tree prefix $\forall x(\exists y[\ ] \lor \forall z(\exists w/\{x\})[\ ])$ can be shown to be first order, as is the case for many tree prefixes using conjunctions, such as $\forall x(\exists y[\ ] \land \forall z(\exists w/\{x\})[\ ])$ and $\forall x\forall z((\exists y/\{z\})[\ ] \land (\exists w/\{x\})[\ ])$. But in [**?**] it was demonstrated that it is possible to strengthen in various ways the “signalling by disjunction” patterns in order to obtain constructs that can express NP-complete problems; one such example is the tree prefix $\forall x\forall z((\exists y/\{z\})[\ ] \lor (\exists w/\{x\})[\ ])$. These stronger forms of signalling by disjunction contain further universal quantifications with respect to our initial example, but they do not contain (linear) signalling or Henkin patterns, as defined above. We will return on this in Section 7.

With some care, it is possible to extend the notions of “being first order” and “being NP-hard” to tree prefixes. We point out that it is an open problem whether there exist (even regular) tree prefixes which are neither first order nor NP-hard.

4 IF fragments induced by memory properties of games

Even though many different games are associated to each single sentence (one game for each sentence-structure pair), some interesting properties of the games are characterized by syntactical properties of the associated sentences; they are invariants of the sentence alone. As a consequence, such game-theoretical properties define associated fragments of IF logic. In particular, in the literature ([**?**],[**?**],[**?**] Sect. 6.4, [**?**]) there has been some interest in properties that limit the “ability” of players to forget. We first describe these properties generally for an arbitrary (not necessarily semantical) game.

- A player has *action recall* (AR) if (s)he remembers all of the moves (s)he has made earlier in the game.
- A player has *knowledge memory* (KM) if (s)he remembers all the information that was available to her/him in earlier positions of the game.
- A player has *perfect recall* (PR) if (s)he remembers the whole history[8] of the game which was visible to her/him.

Generally these properties depend on the formal definition of the given game. In particular, (KM) is very sensitive to the information that is "encoded" into the positions of the game and which part of that information is available to the players.

From now on we will concentrate on these properties in the case of the semantical games of IF logic. We only consider them from Eloise's perspective, but similar considerations can be done dually for Abelard. Under the assumption of regularity, each of these properties has been given a syntactical characterization in the literature (see e.g. [**?**], [**?**]). We will examine these properties in more detail in order to check under which assumptions these earlier syntactical characterizations are correct. Moreover, we generalize these syntactical characterizations also to the case of irregular sentences.

The analysis of memory properties in semantical games of IF logic is based on the implicit assumption that the players are following uniform strategies. As discussed in Remark 1, the moves given by uniform strategies only depend on the last position in the history. Therefore the analysis of the information that is available to Eloise in different phases of the game can be restricted to the information that is encoded in the positions.

We first make some general remarks on semantical games of IF logic – in particular, on the information that is available to the players. A position of the form $(\psi_1 \vee \psi_2, M, s)$ encodes information on an occurrence of the subformula $\psi_1 \vee \psi_2$, on the structure M and on the current assignment s; here Eloise may use all the information that is available in the position when choosing a disjunct.[9] On the contrary, in a position of the form $((\exists x/V)\psi, M, s)$ the move of Eloise cannot depend on the values of the variables in V; hence we may assume that Eloise only "sees" the part of s that is restricted to the variables that are not in V. In the positions where Eloise does not need to make any move, we assume that Eloise does not perceive any of the information in the position (this assumption is relevant for KM and PR).

We now briefly discuss the case of perfect recall. Clearly, if Eloise has PR, she also has AR and KM. In the semantic games of IF logic, apart from Eloise's own moves (covered by AR) and visible information in the earlier positions (covered by KM), the only remaining information in a play of the game are the moves made by

[8] In this paper we have defined a history to be a sequence of earlier positions in a semantical game. Alternatively, also the moves made by the players could be included in histories (although, in the case of *regular* IF sentences, all the previous moves can simply be "read" from the latest position).

[9] In the variant of IF logic where also disjunctions may have slash sets (see [**?**] or [**?**]), the information on s would be restricted as in the case of existential quantifiers. The definitions of AR, KM and PR could naturally be generalized for this variant of IF logic.

Abelard. But if Eloise were also allowed to see those moves, the whole semantical game would become a perfect information game for Eloise and thus it would simply correspond to a semantical game of first-order logic. Hence, we assume that Eloise cannot see the moves of Abelard (unless she is allowed read them from the assignment s in some of her later positions) whence PR simply becomes the combination of AR and KM.

4.1 Action recall

In the semantical games for IF logic Eloise only needs to make a move in positions of the form $((\exists x/U)\psi, M, s)$ or $(\psi_1 \vee \psi_2, M, s)$. We first consider the latter case, where Eloise chooses one of the disjuncts. In the present paper we have assumed (as usual in the earlier literature) that the formulae in the positions of the game are actually *occurrences* of formulae (that is, nodes in the syntactical tree). Therefore the players can indirectly see, from the current position of the game, which choices were made for the disjunctions – and for the conjunctions. Hence, with this approach, both players have action recall "by default" for all the choices that correspond to disjunctions and conjunctions.

Remark 3 If the formulae in the positions of the game were to be just subformulae (and not occurrences of them), then the truth values of some formulae would change. Consider e.g. the sentence $\forall x((\exists y/\{x\})x \neq y \vee (\exists y/\{x\})x \neq y)$ from Section 3.3. This formula is valid for all models that have at least two elements as Eloise can use the disjunction for signalling. However, if Eloise, in the next position, were only able to see what subformula has been reached, and not which specific occurrence, then such signalling would be impossible and therefore Eloise would not have a winning strategy.

By the observations above, supposing that the occurrences of the subformulae are part of the positions, Eloise automatically remembers her choices for disjunctions. Therefore, in order to have action recall, it suffices that Eloise remembers her choices for existential quantifiers. We first consider the case of regular formulae, where the values of existentially quantified variables never change. Suppose that a position of the form $((\exists x/U)\psi, M, s)$ is reached in the game. After Eloise makes her move, the value chosen for x is recorded in the assignment s. In order to access the value of x in her later moves, it suffices that x is not included in any of the slash sets of the existential quantifiers. Syntactically this amounts to the following condition: a regular sentence φ has AR if the following holds

AR_1: If an existential quantifier $(\exists y/V)$ occurs in the effective scope of another existential quantifier $(\exists x/U)$ in φ, then $x \notin V$.

We could have simply referred to the *scope* of $(\exists x/U)$, since scope and effective scope coincide in regular sentences; but the present formulation is more natural in the general case.

Let us then consider the case of irregular sentences. Here the condition AR_1 does not guarantee action recall for Eloise. For example, $\exists x \exists x \exists y(x = y)$ violates action recall: in her third move, Eloise forgets the value chosen in the first move because it has been overwritten by the second move.

Let φ be an irregular IF sentence in which a variable x is quantified two times. If x is first universally quantified, this requantification does not violate action recall for Eloise, since the first value for x is chosen by Abelard. But if x is first existentially quantified, then there is a play of the semantic game for φ in which Eloise chooses the value for x and then forgets that value when x is requantified, and thus she does not have action recall – supposing that she has to perform at least one action in the game after x has been requantified[10].

By the observations above, we obtain the following syntactical characterization: an irregular sentence φ has AR if the condition AR_1 holds and moreover the following holds

AR_2: If some operator O for Eloise (either a disjunction or an existential quantifier) is in the scope of a quantifier $(\exists x/U)$ then it is in its *effective* scope.

Example 3 Let ψ be a quantifier-free sentence such that $\mathrm{Free}(\psi) \subseteq \{x,y\}$. Let us also assume ψ has no occurrences of $\vee$. Eloise does not have AR for the sentence $\exists x \forall x \exists y\, \psi$ since she no longer remembers the first value that she chose for x when choosing a value for y. But she has AR for the sentence $\exists x \forall x \forall y\, \psi$ because she has no move to make after x has been requantified by Abelard. Likewise she has AR for the sentence $\exists x \forall y \exists x\, \psi$ because she still remembers the first value of x when choosing a second value for x.

4.2 Knowledge memory

We first consider KM in the case of regular sentences. Since the values of the quantified variables are here permanently stored in the assignment s, knowledge memory can be violated only if the value of some variable x is available to Eloise at some position p_1 and it becomes unavailable in some later position p_2. Since we assume that Eloise needs to make a move at p_1 and p_2, the formula in these positions has either an existential quantifier or a disjunction as its main operator. Since disjunctions do not have slash sets (in the current paper), KM cannot be violated if the move in position p_2 is for a disjunction; so we may assume that the move in p_2 is for a quantifier $(\exists z/W)$. The value of x is available for Eloise in p_1 if it is quantified before p_1 and the move in p_1 is either for a disjunction, or for an existential quantifier $(\exists y/V)$ such that $x \notin V$. In both cases KM is violated if x is not available at p_2, which is the case when $x \in W$.

By the observations above, we get the following syntactical characterization: a regular sentence φ has KM if the following holds[11]

– Suppose that (Qx/U) and $(\exists z/W)$ occur in φ so that $(\exists z/W) \in \mathrm{Es}(Qx/U)$. Then

[10] We have added this last condition since we have assumed (as in earlier literature) that Eloise can "perceive" the position of the game only when she has a move to make.

[11] In the case of regular sentences the effective scopes could simply be replaced with scopes. However, in order to generalize this definition to irregular sentences, we need to use effective scopes.

KM_1: If there is a quantifier $(\exists y/V)$ such that $(\exists y/V) \in \mathrm{Es}(Qx/U)$ and $(\exists z/W)$ is in the scope of $(\exists y/V)$, then $x \in W$ implies that $x \in V$.

KM_2: If there is a disjunction $\vee$ such that $\vee \in \mathrm{Es}(Qx/U)$ and $(\exists z/W)$ is in the scope of $\vee$, then $x \notin W$.[12]

Note that the condition KM_1 prohibits all signalling patterns and KM_2 essentially prohibits all useful signalling with disjunctions.

Let us then consider the case of irregular sentences. Here KM can be violated also because the value of some variable is erased by a requantification. Suppose that some variable x is first quantified by (Qx/U) and then again by $(Q'x/U')$. In order for Eloise to observe the first value given for x, there needs to be a position p_1, where Eloise needs to make a move, that is "in between" the quantifications of x. Eloise can see x in p_1 if the move there is either a disjunction or a quantifier $(\exists y/V)$ for which $x \notin V$. KM is now violated if there is also a position p_2 where Eloise needs to make a move after the requantification of x.

By the observations above, we obtain the following syntactical characterization: a (possibly irregular) sentence φ has KM if both KM_1 and KM_2 hold and moreover the following holds

– Suppose that (Qx/U) occurs in φ so that some operator O for Eloise is in the scope, but *not in the effective scope* of (Qx/U). Then

  KM_3: There is no quantifier $(\exists y/V)$ such that $x \notin V$, $(\exists y/V) \in \mathrm{Es}(Qx/U)$ and O is in the scope of $(\exists y/V)$.

  KM_4: There is no disjunction $\vee$ such that $\vee \in \mathrm{Es}(Qx/U)$ and O is in the scope of $\vee$.

Example 4 Let ψ be a quantifier-free sentence such that $\mathrm{Free}(\psi) \subseteq \{x,y,z\}$. Eloise does not have KM for $\forall x \exists y \forall x \exists z\, \psi$ since she sees the first value of x when choosing a value for y, but she no longer sees it when choosing a value for z. But she has KM for $\forall x \forall y \forall x \exists z\, \psi$ because she does not see the first value of x at any position of the game. Likewise, provided ψ has no occurrences of $\vee$, Eloise has KM for $\forall x \exists y \forall x \forall z\, \psi$ because she does not have any move to make after x has been requantified by Abelard.

In the case of the semantical games for IF logic, KM does not imply AR. For example Eloise does not have AR for the sentence $\exists x (\exists y/\{x\})(x = y)$, but she has KM (since the value of x is not available to her in any position). Also AR does not imply KM, as seen by the signalling sentence $\forall x \exists y (\exists z/\{x\}) R(x,y,z)$.

4.3 Expressive power of the corresponding syntactical fragments

Let $\mathrm{P} \in \{\exists, \forall\}$ denote Eloise ($\exists$) or Abelard ($\forall$) and let $\mathrm{MP} \in \{\mathrm{AR}, \mathrm{KM}, \mathrm{PR}\}$ be one of the "memory properties" defined above. By $\mathrm{IF}_{\mathrm{MP(P)}}$ we denote the fragment of IF logic consisting of all, including irregular, sentences that satisfy the syntactic condition of

[12] "Being in the effective scope of (an occurrence of) a quantifier" is defined for a connective similarly as it is for a quantifier. Secondly, a logical operator is in the scope of a disjunction if it occurs in a subformula of either of the disjuncts (and similarly for conjunctions).

the property MP for the player P. As before, we may add a superscript “p” or “r” to denote the corresponding prenex or regular subfragments, respectively.

Note that it is impossible to form any Henkin pattern in $\mathrm{IF}^{\mathrm{r}}_{\mathrm{AR}(\exists)}$; therefore the fragment $\mathrm{IF}^{\mathrm{r}}_{\mathrm{AR}(\exists)}$ will be one of the main objects of study in this paper. However, note that the lack of Henkin patterns does not imply action recall for Eloise as e.g. for $\eta := \exists x(\exists y/\{x\})(x=y)$ we have $\eta \notin \mathrm{IF}^{\mathrm{r}}_{\mathrm{AR}(\exists)}$ even though η has no Henkin patterns. We will also be interested in the fragments $\mathrm{IF}^{\mathrm{p,r}}_{\mathrm{AR}(\exists)}$ (Section 6) and $\mathrm{IF}_{\mathrm{KM}(\exists)}$ (Section 8).

The regular fragments of perfect recall and knowledge memory are relatively well-understood; truth, in both of them, can only capture first-order concepts (for the former fragment, the result was anticipated in [**?**], [**?**] and adequately proved in [**?**]; the latter fragment was addressed in [**?**]). We will extend this result to irregular sentences in Section 8.1.

The regular action recall fragment $\mathrm{IF}^{\mathrm{r}}_{\mathrm{AR}(\exists)}$ is by far less understood; some examples in the literature show that it is capable of expressing higher-order concepts, such as infinity over the empty signature, and some NP-complete problems (see Section 5). But a general understanding of its expressive power is lacking, and will be addressed in the present paper.

5 Defining NP-complete problems in $\mathrm{IF}^{\mathrm{p,r}}_{\mathrm{AR}(\exists)}$: examples

The main result that will be proved in Section 6 implies that any ESO concept can be expressed by some regular, prenex action recall formula (therefore, by means of signalling). However, the defining sentences provided by the theorem are often unnecessarily complicated. We give here some examples of NP-complete problems that can be expressed by relatively simple sentences of $\mathrm{IF}^{\mathrm{p,r}}_{\mathrm{AR}(\exists)}$.

Example 5 In [**?**], it was shown that the EXACT COVER BY 3-SETS problem can be defined by an $\mathrm{IF}^{\mathrm{p,r}}_{\mathrm{AR}(\exists)}$ sentence. This problem consists in deciding, given a set U of $3k$ elements and a family C of 3-element subsets of U, whether there is a subfamily of C which is a partition of U. It is defined by the sentence

$$\forall x \exists y (\exists z/\{x\})(U(x) \to (K(y) \wedge E(x,z)))$$

on finite structures M of domain $U \cup C$ (where $U \cap C = \emptyset$), such that $U^M = U$, $\mathrm{Card}(K^M) = k$ and $E^M = \{\langle a,B\rangle \mid a \in U, B \in C, a \in B\}$. We wish to point out that, if we restrict, w.l.o.g., the class of structures by the additional constraint that $K^M \subseteq M \setminus U^M$, then the condition above can be shown to be equivalent to an ESO sentence of prefix $\exists f \forall x$:

$$\varphi^* := \exists f \forall x (U(x) \to (K(f(x)) \wedge E(x, f(f(x))))).$$

In order to prove this, we first apply Skolemization to $\forall x \exists y (\exists z/\{x\})(U(x) \to (K(y) \wedge E(x,z)))$, obtaining an equivalent ESO sentence

$$\varphi := \exists h \exists g \forall x (U(x) \to (K(h(x)) \wedge E(x, g(h(x))))).$$

A proof that this ESO sentence defines the problem EXACT COVER BY 3-SETS on appropriate structures can be found in [**?**]. Instead, we prove here that φ and φ^*

are equivalent relative to the class of adequate structures which satisfy the additional constraint $K^M \subseteq M \setminus U^M$. (Note that this additional constraint does not decrease the generality of the problem.)

In one direction, it is apparent that φ^* logically implies φ. Suppose instead that φ holds in an appropriate structure M. Let $g, h\colon M \to M$ be two functions that satisfy ψ. Define

$$f(x) = \begin{cases} h(x) & \text{if } x \in U^M \\ g(x) & \text{if } x \in M \setminus U^M \end{cases}$$

Then, for all $a \in U^M$, we have $f(a) = h(a)$ and so from $h(a) \in K^M$ we obtain $f(a) \in K^M$; from our assumption that $K^M \subseteq M \setminus U^M$ we get $f(a) \in M \setminus U^M$; so, $g(h(a)) = g(f(a)) = f(f(a))$; then, from the fact that $(a, g(h(a))) \in E^M$ we deduce $(a, f(f(a))) \in E^M$. Therefore $M \models \varphi^*$.

$\exists f \forall x$ is the simplest non-trivial prefix of functional ESO. The fact that it can capture NP-complete problems was shown by Grandjean ([**?**]); he applied this prefix to a conjunction of *twenty-one* clauses to define the HAMILTON PATH problem.

Example 6 We consider another NP-complete problem, DOMINATING SET: the problem of deciding, given an integer k and a graph $G = (V, E^G)$ as input, whether there is a set $D \subseteq V$ of vertices of size at most k such that for every vertex $x \in V$: either $x \in D$ or $(y, x) \in E^G$ for some $y \in D$. Assuming that the intended structures encode k by an interpreted unary predicate P^G of cardinality k, the problem is described by the $\mathrm{IF}^{\mathrm{p,r}}_{\mathrm{AR}(\exists)}$ sentence

$$\forall x \exists z (\exists y / \{x\})((E(y,x) \vee y = x) \wedge P(z)).$$

This description is based on an analogous result for Dependence logic ([**?**]).

By using Skolemization, it suffices to prove our claim for the ESO sentence

$$\zeta := \exists f \exists g \forall x ((E(g(f(x)), x) \vee g(f(x)) = x) \wedge P(f(x))).$$

Fix an integer k. Let $G = (V, E^G, P^G)$ be any structure such that (V, E^G) is a graph, and such that $P^G = \{d_1, \dots, d_k\}$ is a subset of V of cardinality k. Suppose first that G has a dominating set D of cardinality k. Enumerate D as $\{c_1, \dots, c_k\}$. Since D is a dominating set, to each $a \in V$ we can associate a $b_a \in D$ such that either $(b_a, a) \in E^G$ or $b_a = a$. Now, define $f\colon V \to P^G$ as follows: if $b_a = c_i$, then set $f(a) := d_i$. Define $g\colon V \to V$ as follows: $g(d_i) = c_i$; for $a \in V \setminus P^G$, $g(a)$ takes an arbitrary value. Note then that, by the definitions, for every $a \in V$, $g(f(a)) = b_a$. Therefore, $(G, f, g) \models E(g(f(x)), x) \vee g(f(x)) = x$. And the definition of f implies that $(G, f, g) \models P(f(x))$.

Suppose instead that $G \models \zeta$. Then, there are functions $f\colon V \to P^G$ and $g\colon V \to V$ such that, for every $a \in V$, either $(g(f(a)), a) \in E^G$ or $g(f(a)) = a$. Define $D := g[P^G] = \{g(a) \mid a \in P^G\}$. Clearly $\mathrm{Card}(D) \leq \mathrm{Card}(P^G) = k$, and since $g(f(a)) \in D$ for every $a \in V$, D is a dominating set.

Example 7 Also the problem SAT is expressible by means of signalling. SAT is stated as follows: given a propositional formula π in conjunctive normal form, is π satisfiable? The problem can be modeled over structures M of signature $P, N, C, 0, 1$, with $0^M, 1^M$ distinct constants; $C^M \subseteq M$ representing the set of clauses; $P^M, N^M \subseteq$

$(M \setminus C^M) \times C^M$, representing the fact that the first argument (a propositional letter) occurs positively, respectively negatively, in the second argument (a clause). In this class of structures, SAT is described by the following $\mathrm{IF}^{\mathrm{p,r}}_{\mathrm{AR}(\exists)}$ sentence:

$$\forall x \exists y (\exists z/\{x\})(C(x) \to ((P(y,x) \land z = 1) \lor (N(y,x) \land z = 0))).$$

In order to prove that this sentence captures SAT, we apply Skolemization again and prove the claim for the ESO sentence $\xi := \exists f \exists g \psi$, where

$$\psi := \forall x (C(x) \to ((P(f(x),x) \land g(f(x)) = 1) \lor (N(f(x),x) \land g(f(x)) = 0))).$$

Let M be an appropriate structure, and π the propositional formula encoded by it. Suppose first that M is a "yes" instance of SAT; then there is a truth assignment T such that each clause c of π contains a literal α_c for which we have $T(\alpha_c) = 1$. A literal α_c can either be of the form p_c or $\neg p_c$, with p_c a proposition symbol. In the first case, we then have $T(p_c) = 1$, while in the second $T(p_c) = 0$. Let $f\colon M \to M$ be the function that maps c to p_c (define it arbitrarily on elements that are not clauses); let $g\colon M \to M$ be defined by $g(p) := T(p)$ if p is a proposition symbol, and an arbitrary constant otherwise. With these f and g, $(M, f, g) \models \psi$.

Vice versa, suppose $M \models \xi$. Let f, g be two functions that satisfy ψ. Let T be a truth assignment such that $T(p) = g(p)$ for all the proposition symbols in π. Now for any $c \in C^M$, either $(f(c), c) \in P^M$ and $g(f(c)) = 1$, or $(f(c), c) \in N^M$ and $g(f(c)) = 0$. In the former case $f(c)$ is a proposition symbol occurring positively in c, to which T assigns truth value 1. Similarly, in the second case $f(c)$ is a proposition symbol which occurs negatively in c, to which T assigns truth value 0. These remarks show that T satisfies π.

Since this specific form of SAT is known to be NP-complete under quantifier-free reductions ([**?**]), we could give an argument based on standard tools to show that $\mathrm{IF}^{\mathrm{p,r}}_{\mathrm{AR}(\exists)}$ captures NP. The general idea is that, if K is any NP problem, then it has a quantifier-free reduction S to SAT. Given that SAT is definable by the sentence ξ above, it is then possible to define (in the spirit of [**?**], ch. 3.2) a sentence $\hat{S}(\xi)$ in $\mathrm{IF}^{\mathrm{p,r}}_{\mathrm{AR}(\exists)}$ such that

$$S(M) \models \xi \iff M \models \hat{S}(\xi).$$

The sentence $\hat{S}(\xi)$ then characterizes problem K.

In principle, we could extend this argument to show that an "infinite" version of SAT is complete for ESO under quantifier-free reductions, and thus $\mathrm{IF}^{\mathrm{p,r}}_{\mathrm{AR}(\exists)}$ captures ESO. However, we will prove this result in the next section with a more direct argument.

6 Explicit definition of Henkin quantifiers by signalling

In this section we show that the prenex action recall fragment $\mathrm{IF}^{\mathrm{p,r}}_{\mathrm{AR}(\exists)}$ has the same expressive power as the full IF logic. In the proof of this result we exploit the fact that existential second-order logic is captured by Henkin quantifiers with two rows:

Theorem 2 **([?])** *For any* ESO *sentence there is an equivalent sentence of the form*

$$\begin{pmatrix} \forall x_1 \dots \forall x_n \ \exists u \\ \forall y_1 \dots \forall y_n \ \exists v \end{pmatrix} \psi,$$

where ψ is a quantifier-free formula.

By this result, it suffices to prove that, for any n, any sentence that is obtained by applying the Henkin quantifier H_2^n to a quantifier-free formula, is expressible in $\mathrm{IF}^{\mathrm{p,r}}_{\mathrm{AR}(\exists)}$. Since $\mathrm{IF}^{\mathrm{p,r}}_{\mathrm{AR}(\exists)}$ is a fragment of IF, and IF is expressively equivalent to ESO, it follows then that the expressive powers of all the three logics $\mathrm{IF}^{\mathrm{p,r}}_{\mathrm{AR}(\exists)}$, IF and ESO coincide.

Thus, we consider a sentence starting with the Henkin quantifier H_2^n; let

$$\varphi := \begin{pmatrix} \forall x_1 \dots \forall x_n \ \exists u \\ \forall y_1 \dots \forall y_n \ \exists v \end{pmatrix} \psi(x_1,\dots,x_n,u,y_1,\dots,y_n,v),$$

where ψ is a quantifier-free formula. In order to make the argument below more transparent, we formulate the truth condition of φ in a slightly non-standard way ([?]): $M \models \varphi$ if and only if there are relations $F_a, F_b \subseteq M^{n+1}$ such that

(a) $(M,F_a) \models \forall \bar{z} \exists w F_a(\bar{z},w)$,
(b) $(M,F_b) \models \forall \bar{z} \exists w F_b(\bar{z},w)$,
(c) $(M,F_a,F_b) \models \forall \bar{x} \forall u \forall \bar{y} \forall v (\neg F_a(\bar{x},u) \lor \neg F_b(\bar{y},v) \lor \psi(\bar{x},u,\bar{y},v))$.

Here, and in the sequel, $\bar{z}$ denotes a tuple $(z_1,\dots,z_n)$ of distinct variables; similarly, $\bar{x} = (x_1,\dots,x_n)$ and $\bar{y} = (y_1,\dots,y_n)$.

We will now build a sentence θ of $\mathrm{IF}^{\mathrm{p,r}}_{\mathrm{AR}(\exists)}$ that expresses the three conditions above. The idea is to use the variables $\bar{z}$ and w for expressing conditions (a) and (b), and the variables $\bar{x}$, u, $\bar{y}$ and v for expressing (c). In addition we use an "index variable" i that Abelard will use in the game $G(\theta,M)$ to separate the conditions (a), (b) and (c) from each other, and another "index variable" j that Eloise uses either to signal the value of i, or to choose a disjunct of the quantifier-free part in (c).

To simplify the presentation, we assume first that the signature contains three constants, **a**, **b** and **c**, and consider only structures in which they are interpreted by distinct elements. In this case, the sentence θ is defined as follows[13]:

$$\theta := \forall \bar{x} \forall u \forall \bar{y} \forall v \forall i \exists j \forall \bar{z} (\exists w / X)\, \eta,$$

where X is the set $\{x_1,\dots,x_n,u,y_1,\dots,y_n,v,i\}$ and η is the following quantifier-free formula

$$(i = \mathbf{a} \to j = \mathbf{a}) \tag{1}$$
$$\land (i = \mathbf{b} \to j = \mathbf{b}) \tag{2}$$
$$\land (i = \mathbf{c} \land \bar{z} = \bar{x} \land j = \mathbf{a} \to w \neq u) \tag{3}$$
$$\land (i = \mathbf{c} \land \bar{z} = \bar{y} \land j = \mathbf{b} \to w \neq v) \tag{4}$$
$$\land (i = \mathbf{c} \land j = \mathbf{c} \to \psi(\bar{x},u,\bar{y},v)) \tag{5}$$
$$\land (i = \mathbf{c} \to (j = \mathbf{a} \lor j = \mathbf{b} \lor j = \mathbf{c})) \tag{6}$$

[13] The formula θ has been slightly modified compared to the proceedings version of this paper ([?]); now it only has a single slash set.

Lemma 1 *Let φ and θ be as defined above. Let M be a structure such that $a \neq b \neq c \neq a$, where $a = \mathbf{a}^M$, $b = \mathbf{b}^M$ and $c = \mathbf{c}^M$. Then $M \models \varphi \iff M \models \theta$.*

Proof Assume first that $M \models \varphi$. Let F_a and F_b be relations satisfying the conditions (a), (b) and (c) above. Without loss of generality, we can assume that F_a and F_b are actually functions. We describe a winning strategy σ for Eloise in the semantic game $G(\theta, M)$. In the first $2n+3$ moves of the game, Abelard chooses interpretations for the variables $\overline{x}, u, \overline{y}, v$ and i; let s be the assignment formed during these moves. Then Eloise answers by choosing a value d_s for the variable j as follows:

- If $s(i) = a$, then Eloise sets $d_s = a$,
- If $s(i) = b$, then Eloise sets $d_s = b$,
- Assume then that $s(i) = c$. By condition (c), one of the following holds:
  (i) $s(u) \neq F_a(s(\overline{x}))$, (ii) $s(v) \neq F_b(s(\overline{y}))$, or (iii) $M, s \models \psi$.
  In case (i), Eloise sets $d_s = a$; in case (ii), Eloise sets $d_s = b$; in case (iii), Eloise sets $d_s = c$.

Note that we can assume that $s(i) \in \{a,b,c\}$, as otherwise $M, t \models \eta$ for any extension $t \in \mathrm{As}(X \cup \{j, z_1, \ldots, z_n, w\})$ of s, whence Eloise has an easy winning strategy.

In the next n moves, Abelard chooses interpretations for the variables $\overline{z}$. Let t be the corresponding extension of $s(d_s/j)$ by these interpretations. In the last quantifier move, Eloise chooses a value e_t for the variable w. If $d_s \in \{a,b\}$, she sets $e_t = F_{d_s}(t(\overline{z}))$; in case $d_s = c$, she chooses an arbitrary $e_t \in M$.

Note that the choice of e_t is independent of $t(\overline{x}), t(u), t(\overline{y}), t(v)$ and $t(i)$. Thus, the strategy of Eloise described above is uniform. Furthermore, it is straightforward to verify that $M, t' \models \eta$, for $t' = t(e_t/w)$, whence Eloise has a winning strategy in $G(\eta, M, t')$. Thus, we see that $M \models \theta$.

Assume then for the other direction that $M \models \theta$. Then, given any assignment $s \in \mathrm{As}(X, M)$, Eloise can choose a value d_s for the variable j, and given any extension $t \in \mathrm{As}(X \cup \{j, z_1, \ldots, z_n\}, M)$ of $s(d_s/j)$, she can choose a value e_t for w such that e_t does not depend on $t(x)$ for any $x \in X$ (but may depend on $t(j) = d_s$), and Eloise has a winning strategy in the game $G(\eta, M, t(e_s/w))$.

We define now relations $F_a, F_b \subseteq M^{n+1}$ as follows:

$$F_a := \{(t(\overline{z}), e_t) \mid t \in \mathrm{As}(Y, M),\, t(j) = d_s = a\},$$
$$F_b := \{(t(\overline{z}), e_t) \mid t \in \mathrm{As}(Y, M),\, t(j) = d_s = b\},$$

where $Y = X \cup \{j, z_1, \ldots, z_n\}$ and $s = t \upharpoonright X$ (the restriction of t to X). It suffices to show that conditions (a), (b) and (c) hold for these relations. In order to prove (a), let $\overline{m} = (m_1, \ldots, m_n) \in M^n$, and consider an assignment $t \in \mathrm{As}(Y, M)$ such that $t(\overline{z}) = \overline{m}$, $t(i) = a$, and $t(j) = d_s$, where $s = t \upharpoonright X$. Then, $t(j)$ is necessarily a, since otherwise Eloise would lose the game $G(\eta, M, t(e_t/w))$ if Abelard chooses the first conjunct $i = \mathbf{a} \to j = \mathbf{a}$ of η. Thus, by the definition above, $(\overline{m}, e_t) \in F_a$. Condition (b) is proved symmetrically by using the conjunct (2) of η. Note that since the choice of e_t depends only on $t(\overline{z})$ and $t(j)$, the relations F_a and F_b are actually functions $M^n \to M$.

To prove (c), let s_0 be an assignment with domain $X \setminus \{i\}$. We need to show that $F_a(s_0(\overline{x})) \neq s_0(u)$, $F_b(s_0(\overline{y})) \neq s_0(v)$ or $M, s_0 \models \psi$. Let $s = s_0(c/i)$. Then necessarily $d_s \in \{a,b,c\}$, since otherwise Eloise would lose the game $G(\eta, M, t')$, for any extension $t' \in \mathrm{As}(Y \cup \{w\}, M)$ of s, if Abelard chooses the last conjunct (6) of η.

Assume first that $d_s = a$. Consider the assignment $t = s(a/j, s_0(\overline{x})/\overline{z}) \in \mathrm{As}(Y,M)$. Then by the definition of F_a, we have $e_t = F_a(t(\overline{z})) = F_a(s_0(\overline{x}))$. On the other hand, it must be the case that $e_t \neq t(u) = s_0(u)$, since otherwise Eloise would lose the game $G(\eta, M, t(e_t/w))$ if Abelard chooses the conjunct (3) of η. Thus, we see that $F_a(s_0(\overline{x})) \neq s_0(u)$. In the case $d_s = b$, we can prove in the same way that $F_b(s_0(\overline{y})) \neq s_0(v)$, by using the conjunct (4) of η. Assume finally, that $d_s = c$. Then it follows immediately that $M, s_0 \models \psi$. This is because otherwise Eloise would lose the game $G(\eta, M, t(e_t/w))$ if Abelard chooses the conjunct (5) of η. □

We will next eliminate the assumption of three constants with distinct interpretations. On structures with at least two different elements, this is done by replacing the quantifiers $\forall i$ and $\exists j$ in θ by the sequences $\forall i \forall i' \forall i''$ and $\exists j \exists j' \exists j''$, respectively. Furthermore, the subformulae $i = \mathbf{a}$, $i = \mathbf{b}$ and $i = \mathbf{c}$ of η are replaced by $i = i' \wedge i \neq i''$, $i = i'' \wedge i \neq i'$ and $i' = i'' \wedge i \neq i'$, and similarly for the subformulae $j = \mathbf{a}$, $j = \mathbf{b}$ and $j = \mathbf{c}$. Let θ' be the formula obtained from θ by performing these changes. By a straightforward modification of the proof of Lemma 1, we see that $M \models \varphi \Leftrightarrow M \models \theta'$ holds for all structures M with at least two elements.

If M has only one element, then clearly $M \models \varphi \Leftrightarrow M \models \forall \overline{x} \forall u \forall \overline{y} \forall v\, \psi$. Furthermore, the implication $M \models \forall \overline{x} \forall u \forall \overline{y} \forall v\, \psi \Rightarrow M \models \varphi$ holds for all structures. Thus, we see that φ is equivalent to θ^* on all structures, where θ^* is obtained from θ' by adding (in the end of the prefix) the sequence $\forall \overline{x}' \forall u' \forall \overline{y}' \forall v'$ of universal quantifiers and the disjunct $\psi(\overline{x}', u', \overline{y}', v')$ to the quantifier-free part, for some fresh variables $\overline{x}' = (x'_1, \ldots, x'_n), \overline{y}' = (y'_1, \ldots, y'_n), u'$ and v'. This completes the proof of the main result in this section:

Theorem 3 $\mathrm{IF}^{\mathrm{p,r}}_{\mathrm{AR}(\exists)}$ *has the same expressive power as* ESO. *In particular, any class definable in* IF *is already definable in* $\mathrm{IF}^{\mathrm{p,r}}_{\mathrm{AR}(\exists)}$.

Note that the length of the $\mathrm{IF}^{\mathrm{r}}_{\mathrm{AR}(\exists)}$ translation θ^* given in the proof of Theorem 3 is only linear with respect to the length of the original H^n_2 formula φ. Another interesting observation that follows from the proof is that there is no hierarchy of expressive power based on the number of existential quantifiers used in signalling: each signalling pattern in θ^* consists of one universal quantifier from the prefix $\forall \overline{x} \forall u \forall \overline{y} \forall v \forall i$, one of the three existential quantifiers from $\exists j \exists j' \exists j''$ (used as a signal), and the existential quantifier $(\exists w/X)$. Note further that in the semantical sense, the sequence $\exists j \exists j' \exists j''$ forms a single three valued signal for the existential quantifier $(\exists w/X)$. Moreover, the proof of Theorem 3 shows that it suffices to use sentences with only a single nonempty slash set in order to capture the whole ESO.

7 Capturing ESO using only signalling by disjunction

The results of [?] tell us that Henkin and signalling prefixes constitute the totality of sources of second-order expressive power of $\mathrm{IF}^{\mathrm{p,r}}$. We abandon here the requirement of prenex form, and show that the whole ESO can then be captured without using any signalling or Henkin patterns. This can be achieved by using only "signalling by disjunction" instead (c.f. Section 3.3). In Section 7.1 we prove this result for the fragment IF^{r} and in Section 7.2 we present an alternative proof method by using irregular IF sentences.

7.1 Explicit definition of Henkin quantifiers in non-prenex, signalling-free, regular IF logic of action recall

In this section we investigate the expressive resources of IF^{r}. It was shown in [**?**] that there are syntactical constructs in this fragment, based on the idea of signalling by disjunction, that capture NP-complete problems, even though these constructs contain neither (weakly linear) Henkin nor signalling patterns. We show here that it is possible to translate each sentence of the form $\mathrm{H}^n_2\psi$, with ψ quantifier free, into a sentence of $\mathrm{IF}^{\mathrm{r}}_{\mathrm{AR}(\exists)}$ without signalling patterns. It immediately follows from this and Theorem 2 that the fragment $\mathrm{IF}^{\mathrm{r}}_{\mathrm{AR}(\exists)}$ without signalling patterns captures ESO.

Let $\psi(\bar{x},u,\bar{y},v)$ be a quantifier-free formula, with $\bar{x},\bar{y}$ n-tuples of variables, and let φ be the IF sentence

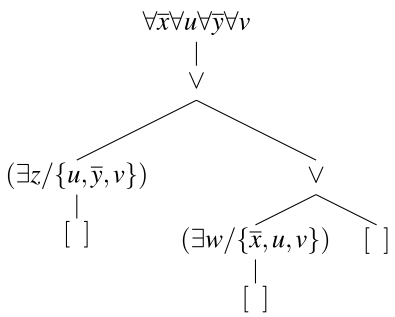
$$\forall\bar{x}\,\exists u\,\forall\bar{y}\,(\exists v/\{\bar{x},u\})\,\psi(\bar{x},u,\bar{y},v)$$

corresponding to H^n_2 quantification applied to ψ.

Let ξ be the IF sentence

$$\forall\bar{x}\forall u\forall\bar{y}\forall v\,((\exists z/\{u,\bar{y},v\})z\neq u\vee((\exists w/\{\bar{x},u,v\})w\neq v\vee\psi(\bar{x},u,\bar{y},v))).$$

It may help to visualize the syntactical structure of ξ as a tree T:



We will show the equivalence of φ and ξ.

Theorem 4 *For any structure M, we have $M\models\varphi \iff M\models\xi$.*

Proof By using Skolemization on ξ, we obtain the equivalent sentence

$$\mathrm{SK}(\xi):=\exists f\,\exists g\,\forall\bar{x}\forall u\forall\bar{y}\forall v\,(f(\bar{x})\neq u\vee(g(\bar{y})\neq v\vee\psi(\bar{x},u,\bar{y},v))).$$

On the other hand, the Skolemized form of the sentence φ is

$$\mathrm{SK}(\varphi):=\exists f\,\exists g\,\forall\bar{x}\forall\bar{y}\,\psi(\bar{x},f(\bar{x}),\bar{y},g(\bar{y})),$$

which is clearly equivalent with $\mathrm{SK}(\xi)$. □

Theorem 4 could also be proven by a similar game-theoretical argument as used for the sentence θ^* in the proof of Theorem 3, but Skolemization makes the proof much shorter. However, the proof of Theorem 3 cannot be shortened by the same method, which is why we decided to give the more illustrative game-theoretical proof instead. By comparing the two translations, one can see an interesting link by observing that the constants **a**, **b** and **c** in θ in Lemma 1 are intuitively used to "simulate" the three disjuncts in the sentence ξ above.

It is interesting to note that the quantifier-free formula ψ in sentence φ is not in the scope of *any existential quantifier* in the sentence ξ equivalent to φ. Signalling by disjunction can be used in this manner to "change the role" of universal quantifiers $\forall u$ and $\forall v$ essentially into existential quantifiers $\exists u$ and $\exists v$.

The syntactical tree of our sentence ξ belongs to a category that was called *generalized Henkin* in [**?**]. In [**?**], a second class of expressive syntactical trees was isolated, under the name of *coordinated* class. These syntactical constructs more markedly differ from Henkin patterns than the generalized Henkin trees do. We remark that our definitory sentence ξ does not contain coordinated patterns.

Coordinated trees are seemingly weaker than the generalized Henkin trees; however, Reijo Jaakkola has shown[14] that an appropriate class of coordinated syntactical trees (devoid of Henkin, signalling and generalized Henkin patterns) is sufficient to simulate the H_2^n quantifiers, and thus to capture ESO.

7.2 Eliminating Henkin and signalling patterns by requantification

Here we present an alternative method for expressing the whole ESO in non-prenex IF logic without using signalling or Henkin patterns. Here we abandon the requirement of regularity and show how both Henkin and signalling patterns can then be eliminated from any IF formula by a very straightforward translation. By using requantification, we can change all second-order quantifier patterns into patterns that use signalling by disjunction.

For creating a Henkin or a signalling pattern, we need to have two existential quantifiers that have certain dependencies to each other. In order to eliminate these patterns, we now attempt to replace existential quantifiers with universal quantifiers that have essentially the same role in the semantical game. This can be done by a simple trick if we allow requantification of variables.

Let φ be a regular IF formula and let y be a fresh variable. Suppose that φ has a subformula of the form $(\exists x/V)\psi$. Now $(\exists x/V)\psi$ is equivalent to the irregular IF formula $(\exists y/V)\forall x(x \neq y \vee \forall y\, \psi)$. The correctness of this equivalence can be seen by the following game-theoretical reasoning[15]: after the quantification for $(\exists y/V)$, Abelard has to choose the same value for x as Eloise chose for y, since else he loses the game (when Eloise chooses the left disjunct). Hence we see that Eloise can indirectly "force" Abelard to choose a value for x in any (V-independent) way she wishes. And since Abelard may then choose a new value for y, Eloise cannot use the older value (as a signal) later in the game.

[14] Private communication.

[15] A precise formal proof can be found in the proceedings version of this paper, [**?**].

We now generalize the idea above to modify all existential quantifications in φ. Let $\{x_1,\dots,x_k\}$ be the set of the occurrences of variables in existential quantifiers of φ. Let $\{y_1,\dots,y_k\}$ be a set of distinct variables that do not occur in φ. We define the formula φ' recursively as follows:

$$\begin{aligned}
\varphi' &= \varphi, \quad \text{if } \varphi \text{ is a literal}\\
(\psi\vee\theta)' &= \psi'\vee\theta',\\
(\psi\wedge\theta)' &= \psi'\wedge\theta',\\
((\forall x/V)\psi)' &= (\forall x/V)\psi',\\
((\exists x_i/V)\psi)' &= (\exists y_i/V)\forall x_i(x_i\neq y_i\vee\forall y_i\psi').
\end{aligned}$$

By the observations above, it is easy to see that φ' is equivalent to φ. Since no existential quantifier in φ' is in the effective scope of any other existential quantifier, φ' has no signalling nor Henkin patterns.

As we observed about the proof of Theorem 4, the role of existential quantifiers can be changed essentially into that of universal quantifiers by using signalling by disjunction. This trick is even more explicit in the translation above, where Abelard is forced to give the same value for x_i as Eloise chose for y_i, or otherwise he will immediately lose the game.

Note that the translation above can be directly applied to any IF formula – including non-prenex and irregular formulae. Furthermore, this translation increases the length of a given formula only by a small constant for each existential quantifier in it. If a sentence φ in prenex form is translated to φ' as above, the prenex form is lost. However, φ' is still "almost prenex" since only disjunctions with a literal as the left disjunct are created within the quantifier prefix. See the following example for explicitly expressing the Henkin prefix H^1_2 without using Henkin or signalling patterns.

Example 8 Consider an IF sentence $\forall x\exists y\forall z(\exists w/\{x,y\})\psi$, where ψ is quantifier free, and suppose that y' does not occur in ψ. By applying our translation procedure to the most external occurrence of an existential quantifier, $\exists y$, we obtain the formula $\chi := \forall x\exists y'\forall y(y\neq y'\vee\forall y'\forall z(\exists w/\{x,y\})\psi)$. Note here that there is no need to apply the translation procedure to the innermost existential quantifier $(\exists w/\{x,y\})$. What happens to the flow of information in χ? In the right disjunct, the variables y and y' carry the same value; as a signal, y is blocked by the slash set of $\exists w$ and, as a signal, y' is blocked by $\forall y'$; but the value of y (equal to the value of y') can still be used within ψ.

Remark 4 We want to analyze more carefully what memory properties are violated in the various translations of Sections 6, 7.1 and 7.2, to pin down what is needed to achieve ESO expressive power. We first note that the sentences θ^* and ξ used in Theorems 3 and 4, respectively, both have action recall (conditions AR_1–AR_2) for Eloise. However, θ^* does not satisfy the condition KM_1 of knowledge memory due to signalling (with existential quantifiers) and similarly ξ does not satisfy KM_2 due to the use of signalling with disjunctions. The translation presented in this section typically violates AR_2 and KM_4, but it cannot violate AR_1 as no existential quantifier is in the effective scope of another existential quantifier.

In the last two sections we will study the properties of irregular IF sentences and prefixes in more detail.

8 Regularization

In this section we study irregular IF logic and demonstrate its correspondence to regular IF logic. In Section 8.1 we present a regularization procedure which can be used for translating irregular sentences into equivalent regular ones in a natural way; it will be very useful for the results in Section 2.4. Furthermore, in Section 8.2 we present an alternative regularization method that can be applied to special fragments of IF logic, and in particular to sentences with knowledge memory. We use it to show that the fragment of knowledge memory (including irregular sentences) is first order and also to identify another first-order fragment of IF logic.

8.1 General regularization procedure

Any IF sentence can be *regularized* by renaming the variables that are requantified and by making additions to certain slash sets. We present here a regularization procedure that is slightly different from the one that has been presented earlier in the literature (cf. Remark 5). Suppose that a quantifier $(Q'x/W)$ occurs in the effective scope of a quantifier (Qx/V) in an IF sentence. In order to eliminate this requantification, we now replace the quantifier (Qx/V) with (Qy/V), where y is a fresh variable. Moreover, we substitute x with y everywhere within the *effective* scope of (Qx/V). Furthermore, since the requantification of x removes the information on the old value of x, we also need to "hide" the information on the value of y within the effective scope of $(Q'x/W)$. This can be done simply by adding y to all slash sets that are in the scope of $(Q'x/W)$.[16] The formal definition of this regularization procedure now follows.

Let φ be an IF sentence, where quantifier (Qx/V) occurs so that there is no quantifier $(Q'x/V')$ for which $(Qx/V) \in \mathrm{Es}(Q'x/V')$; if there are several such quantifiers that quantify x in φ, we fix any one of them. We call such a quantifier an *outermost quantification of x in φ*. Let $(Qx/V)\mu$ be the corresponding subformula of φ and let y be a fresh variable. Let then $\varphi_{y/x}$ be the sentence that is obtained from φ by replacing the subformula $(Qx/V)\mu$ with $(Qy/V)\mu'$, where μ' is obtained by modifying μ as follows:

1. every *free* occurrence of x in μ (including occurrences in slash sets) is substituted by y; and
2. y is added to the slash set of every quantifier in μ which is not in $\mathrm{Es}(Qx/V)$.

[16] Note that in order the hide the information on y completely, y should also be added to the slash sets of all connectives (disjunctions and conjunctions) that are in the effective scope of $(Q'x/W)$. However, we will see that there is no need to add y to the slash sets of any connectives, if they initially had empty slash sets. Intuitively this is because in a position with move for a disjunction (resp. conjunction), Eloise (resp. Abelard) can already see the values of all the free variables occurring in disjuncts (resp. conjuncts) and thus the value of y does not give any new information as a signal.

We next prove that the elimination of the requantification of x, as described above, preserves logical equivalence.

Lemma 2 *Let φ be an* IF *sentence in which x is requantified and let $\varphi_{y/x}$ be the sentence that is obtained by replacing an outermost quantification (Qx/V) of x in φ with (Qy/V) as described above. Now we have $M \models \varphi \iff M \models \varphi_{y/x}$.*

Proof The translation from φ to $\varphi_{y/x}$ only changes certain occurrences of x to y and adds y to certain slash sets. As the syntactical trees of φ and $\varphi_{y/x}$ are otherwise identical, there is a canonical one-to-one correspondence between them; for any $\psi \in \mathrm{Sf}(\varphi)$, let ψ' denote the corresponding subformula in $\varphi_{y/x}$. We illustrate this below:

$$\begin{array}{ccl}
\varphi & \varphi_{y/x} & \\
\vdots & \vdots & \\
(Qx/V)\mu & (Qy/V)\mu' & \\
\vdots & \vdots & \left.\right\}\ \text{free occurrences of } x \text{ replaced with } y \\
(Q'x/W)\eta & (Q'x/W')\eta' & \\
\vdots & \vdots & \left.\right\}\ y \text{ added to all slash sets} \\
\psi & \psi' &
\end{array}$$

We next extend this correspondence for the positions of the semantical games $G(\varphi, M)$ and $G(\varphi_{y/x}, M)$ with respect to subformulas ψ of φ:

(1) If ψ is not in the scope of (Qx/V) and $\psi \neq (Qx/V)\mu$, then any position of the form (ψ, M, s) in $G(\varphi, M)$ corresponds to the position (ψ', M, s) in $G(\varphi_{y/x}, M)$.
(2) If $\psi = (Qx/V)\mu$, then any position of the form (ψ, M, s) in $G(\varphi, M)$ corresponds to the position $((Qy/V)\mu', M, s)$ in $G(\varphi_{y/x}, M)$.
(3) If $\psi \in \mathrm{Es}(Qx/V)$, then any position of the form (ψ, M, s) in $G(\varphi, M)$ corresponds to the position (ψ', M, s') in $G(\varphi_{y/x}, M)$, where s and s' are otherwise identical except for that $y \notin \mathrm{dom}(s)$, $x \notin \mathrm{dom}(s')$ and $s(x) = s'(y)$.
(4) If ψ is in the scope, but not in the effective scope, of (Qx/V), then any position of the form (ψ, M, s) in $G(\varphi, M)$ corresponds to *each* position of the form $(\psi', M, s(a/y))$ in $G(\varphi_{y/x}, M)$, where $a \in M$.

We observe that in cases (1) – (3) this correspondence is indeed bijective. However, in the case (4), a position in $G(\varphi, M)$ corresponds to several positions in $G(\varphi_{y/x}, M)$ – the only difference between them being that the assignment in the position has a different value for y.

As discussed in Remark 1, uniform strategies could be alternatively defined over the set of positions, instead of histories, of the game. For technical convenience, we assume uniform winning strategies to be defined here in this way. The main idea in the proof below is to always follow moves in the *corresponding positions* in the other game where a uniform winning strategy is given by the assumption. Thus the proof is quite straightforward for positions of type (1) – (3), but the case (4) requires more attention as there this correspondence is not bijective.

Suppose first that Eloise has a uniform winning strategy σ in $G(\varphi, M)$. We define a strategy σ' for Eloise in $G(\varphi_{y/x}, M)$ so that, in any position p, the strategy σ' simply assigns the same move as σ does in the corresponding position in $G(\varphi,M)$. Note that here each position p in $G(\varphi_{y/x},M)$ has a unique corresponding position in $G(\varphi,M)$ (even in the case (4)). Now for any literal ψ' in $\varphi_{y/x}$, if a position for ψ' can be reached with σ', then the corresponding position for ψ can be reached with σ. Since for literals $\psi' = \psi$ and y does not occur in ψ, it is thus easy to see that σ' is a winning strategy. The uniformity of σ' also clearly follows from the uniformity of σ.

Suppose then that Eloise has a uniform winning strategy σ' in $(M,\varphi_{y/x})$. For positions of the form (1) – (3) we may again define σ to do the same as in the (unique) corresponding position in $G(\varphi_{y/x},M)$. Consider next a position $p = ((\exists z/U)\psi,M,s)$ of the form (4). Such a position corresponds to several $p_a = ((\exists z/U\cup\{y\})\psi',M,s(a/y))$ in $G(\varphi_{y/x},M)$ – with different values a for y. However, since y is in the slash set of the quantifier for z, the strategy σ' assigns the same move for each p_a. We thus define σ to assign this particular move for p as well. Finally we need to define σ for a position $p = (\psi\vee\theta,M,s)$ of the form (4). This corresponds to each $p_a = (\psi'\vee\theta',M,s(a/y))$ in $G(\varphi_{y/x},M)$, for a ranging in M. If there is at least one such a for which σ' can reach p_a and σ' chooses ψ' in p_a, we define also σ to select ψ in p. Else we define σ to select θ in p.

By the definition of σ, if some literal can be reached by σ, then a corresponding literal can be reached by σ'. Thus, since σ' is a winning strategy, also σ is a winning strategy. The uniformity of σ is clear for all the positions of types (1) – (3), but we need to check the positions of the type (4) more carefully as there the value of y is available as additional information in $G(\varphi_{y/x},M)$. However, as y is in the slash sets of all existential quantifiers in positions of type (4) in $G(\varphi_{y/x},M)$, the uniformity of σ follows from the uniformity of σ'. Moreover, the uniformity of σ is also clear for positions with disjunctions: as we have defined σ on positions of the game (instead of histories), σ prescribes the same move for histories h,h' that differ only in the choices of values for variables that have been requantified. □

If the variable x is requantified in φ, then $\varphi_{y/x}$ has one less requantification of x than φ. Hence, this translation can be iterated until no variable is requantified anymore, and we have a regular sentence that is equivalent to φ. We call the sentence obtained by this procedure the *regularization of* φ and denote it by $\mathrm{reg}(\varphi)$.

Theorem 5 *Any* IF *sentence* φ *is equivalent to its regularization* $\mathrm{reg}(\varphi)$.

Since the regularization procedure acts in the same way on both kinds of quantifiers, it is easy to see that actually φ is *strongly* equivalent to $\mathrm{reg}(\varphi)$ (that is, $M \models \varphi$ iff $M \models \mathrm{reg}(\varphi)$ and also $M \models^- \varphi$ iff $M \models^- \mathrm{reg}(\varphi)$). Moreover, it can be shown that, apart from the names chosen for the new fresh variables y, the sentence $\mathrm{reg}(\varphi)$ is constructed in a unique way.

Example 9 Consider the following sentence:

$$\varphi = \forall x(\exists y/\{x\})\big(R(x,y)\wedge(\forall y/\{y\})(\exists x/\{x,y\})\,R(x,y)\big).$$

By replacing the first quantification of x with a fresh variable v and the first quantification of y with a fresh variable w, we obtain an equivalent regularized sentence $\mathrm{reg}(\varphi) = \forall v(\exists w/\{v\})\big(R(v,w)\wedge(\forall y/\{w\})(\exists x/\{v,y,w\})\,R(x,y)\big)$.

We make the following important observation on the regularization process described above:

– For any IF sentence φ, $\mathrm{reg}(\varphi)$ has the *same quantifier-free part as* φ.

Let $\mathbf{Q}$ be an irregular sentential IF prefix. By the observation above, it makes sense to write $\mathrm{reg}(\mathbf{Q})$ for the regular prefix of any sentence of the form $\mathrm{reg}(\mathbf{Q}\psi)$ – where ψ is any quantifier-free formula with $\mathrm{Free}(\psi) \subseteq \mathrm{Bound}(\mathbf{Q})$.[17] Then, any such sentence $\mathrm{reg}(\mathbf{Q}\psi)$ can be written as $\mathrm{reg}(\mathbf{Q})\psi$. By this observation it could seem that every irregular prefix is equivalent to a regular one. However, this equivalence holds only "relatively" to the set $\mathrm{Bound}(\mathbf{Q})$ which is a proper subset of $\mathrm{Bound}(\mathrm{reg}(\mathbf{Q}))$. We get back to this topic in Section 9.2.

Remark 5 There is a similar regularization procedure described in [**?**], Theorem 9.3. However, our regularization proceeds "outside-in" while the one in [**?**] proceeds "inside-out". For an example, consider the sentence $\varphi := \exists x(\forall x/\{x\})\exists z\,\psi$, where ψ is quantifier free. By our approach

$$\mathrm{reg}(\varphi) = \exists y(\forall x/\{y\})(\exists z/\{y\})\,\psi.$$

In contrast, the approach in [**?**] produces an alternative regularization

$$\mathrm{reg}'(\varphi) = \exists x(\forall y/\{x\})\exists z\,\mathrm{Subv}(\psi, x, y)$$

where x is not added to the slash set of $\exists z$ *only because such set is empty*. An advantage of our approach is that reg does not modify the quantifier-free part of ψ; as we mentioned, this turns out to be particularly useful for the study of irregular quantifier prefixes. On the other hand, our regularization procedure may turn first-order quantifiers into slashed ones, as with the quantifier $\exists z$ above; reg', instead, always preserves the empty slash sets.

Remark 6 As we have already pointed out, the Skolemization procedure described in Section 2.3 works correctly only for regular formulas. We are now in the position to define a correct Skolemization of any IF sentence: just let $\mathrm{SK}(\varphi) := \mathrm{SK}(\mathrm{reg}(\varphi))$. Notice that if φ is regular, we have $\mathrm{reg}(\varphi) = \varphi$, and thus this definition coincides with that given in Section 2.3 for regular formulas. As an example, consider a formula φ of the form $\forall x\exists x\forall z(\exists w/\{x\})\psi(x,z,w)$, with $\psi(x,z,w)$ quantifier-free. We have

$$\begin{aligned}\mathrm{SK}(\varphi) = \mathrm{SK}(\mathrm{reg}(\varphi)) &= \mathrm{SK}(\forall y\exists x(\forall z/\{y\})(\exists w/\{x,y\})\psi(x,z,w)) \\ &= \exists f\exists g\forall y\forall z\psi(f(y),z,g(z)).\end{aligned}$$

The regularization procedure allows us to analyse irregular sentences by examining the corresponding regular sentences instead. Also the semantic games related to an irregular sentence φ and its regularization $\mathrm{reg}(\varphi)$ are very similar to each other. However, regularization does not preserve all memory properties of the game. For example, a sentence of the form $\varphi = \forall x\forall x(\psi \vee \exists z\psi)$, with ψ quantifier-free, has

[17] Note here that $\mathrm{reg}(\mathbf{Q})$ is not uniquely defined as we allow any fresh variables to be used to eliminate requantifications in the regularization process. However, this fact – which fresh variables are chosen to form $\mathrm{reg}(\mathbf{Q})$ – is irrelevant for all of the results in this paper.

knowledge memory for Eloise, while $\mathrm{reg}(\varphi) = \forall y \forall x(\psi \vee (\exists z/\{y\})\psi)$ violates the condition KM_2. Moreover, violations of KM_3 turn into violations of KM_1 as demonstrated by the sentence $\forall x \exists x \exists z \psi$ and its regularization $\forall y \exists x(\exists z/\{y\})\psi$. Similarly, regularization turns any violation of KM_4 into a violation of KM_2. In the next section we will present an alternative regularization procedure which preserves the properties KM_1–KM_2 and use it to identify first order fragments of IF logic.[18]

8.2 Regularization procedure for sentences with knowledge memory

In the regularization procedure defined in the previous section we eliminated the requantification of a variable x by renaming its initial quantification with a fresh variable y; and by also adding y to all the slash sets that were not in the effective scope of the original quantifier. The latter modification was made in order to simulate the *loss of information on the old value of x due to requantification*. We will show in this section that such additions to slash sets are unnecessary if the original sentence has knowledge memory, and thus we may use an alternative regularization process which does not hide the value y anywhere beyond the effective scope of of the initial quantification of x. Intuively this means that, when Eloise has knowledge memory, she would not gain any benefit even if she were able to see the old values of those variables which have been requantified. We also give examples which show that these old values may carry useful information in games where Eloise does not have knowledge memory.

Let φ be an IF sentence satisfying the same assumptions as given in the definition of $\varphi_{y/x}$ in Section 8.1. We define the sentence $\varphi^*_{y/x}$ as the sentence $\varphi_{y/x}$, except that the clause 2 in the definition of μ' is omitted. That is, the introduced fresh variable y is not added to any of the slash sets which are not in $\mathrm{Es}(Qx/V)$. When all the requantifications in φ are eliminated by this translation, we finally obtain a regular sentence that is denoted by $\mathrm{reg}^*(\varphi)$. Note that in $\mathrm{reg}^*(\varphi)$ the renamed variables correspond to those variables which will be requantified in φ – the only essential difference being that they are not hidden via requantification. So the semantical games for $\mathrm{reg}^*(\varphi)$ can, in a sense, be seen as variants of the corresponding games for φ with the alternative rule that the players can see the old values of the variables even after the requantification.

Example 10 Recall $\varphi = \forall x(\exists y/\{x\})\big(R(x,y) \wedge (\forall y/\{y\})(\exists x/\{x,y\})R(x,y)\big)$ from Example 9. We again regularize φ by replacing the first quantification of x with v and the first quantification of y with w:

$$\mathrm{reg}^*(\varphi) = \forall v(\exists w/\{v\})\big(R(v,w) \wedge (\forall y/\{w\})(\exists x/\{v,y\})R(x,y)\big).$$

The only difference between the regularizations $\mathrm{reg}(\varphi)$ and $\mathrm{reg}^*(\varphi)$ is that w is not added to the slash set of the rightmost existential quantifier in $\mathrm{reg}^*(\varphi)$. As φ has knowledge memory for Eloise, it will follow from Theorem 6 that φ and $\mathrm{reg}^*(\varphi)$ are indeed equivalent.

[18] Without going into details, we mention that all of the memory properties could alternatively be maintained in a natural way by adding fresh variables y also to the slash sets of the *connectives* which are in the scope – but not in the effective scope – of the quantifier for (Qx/V) which is replaced by (Qy/V) in $\varphi_{y/x}$.

Next we will show that a sentence φ is equivalent to $\varphi^*_{y/x}$ when certain conditions of knowledge memory hold in φ. We consider separately the case where the variable x, replaced with y, is universally quantified and the case where it is existentially quantified. The necessary assumptions in these two cases are quite different. We begin with the case where the replaced quantification of x is *universal*.

Lemma 3 *Let φ be an* IF *sentence, where $(\forall x/V)$ is an outermost quantification of x. Let $\varphi^*_{y/x}$ be the sentence that is obtained by replacing $(\forall x/V)$ with $(\forall y/V)$ as described above. Moreover, assume that* KM_3 *and* KM_4 *hold for φ. Now we have $M \models \varphi \iff M \models \varphi^*_{y/x}$.*

Proof We first note that there is a natural correspondence between the positions of games $G(\varphi, M)$ and $G(\varphi^*_{y/x}, M)$. This correspondence is defined for cases (1) – (4) here exactly in the same way as in the proof of Lemma 2. (Indeed, the only difference between $G(\varphi_{y/x}, M)$ and $G(\varphi^*_{y/x}, M)$ is that y is not added to the slash sets of quantifiers in positions of type (4).) As in the proof of Lemma 2, we assume the uniform winning strategies to be defined on positions of the games (instead of histories), and also here the main idea in the proof is to always follow moves in the corresponding positions in the other game. The implication from left to right can be proven exactly as in the proof of Lemma 2. (Note here that $G(\varphi^*_{y/x}, M)$ is an easier version of $G(\varphi_{y/x}, M)$ for Eloise since she has a bit more information available in the former game).

Suppose that Eloise has a uniform winning strategy σ' in $(M, \varphi^*_{y/x})$. First for positions of the form (1) – (3) we define σ to do the same move as σ' does in the corresponding position in $G(M, \varphi^*_{y/x})$. Next, for positions of type (4), we first *fix some value $a_0 \in M$* and then define σ at *any position $p(s)$ of type* (4) to use the choice of σ' for $p(s(a_0/y))$, where $p(s)$ denotes the position p with a given assignment s as a parameter.

Before showing that σ is a winning strategy, we make some observations. Let $p^\dagger$ be a position of type (4) in $G(M, \varphi)$ such that the position immediately preceding it is still of type (3). Now $p^\dagger$ is (ψ, M, s) for some ψ preceded by a quantifier $(Q'x/W)$, i.e. $p^\dagger$ is a position that occurs immediately after a requantification of x. We first consider the special case when no operator for Eloise occurs in the scope of $(Q'x/W)$. Then, since Eloise does not have a move to make at any position p reachable from $p^\dagger$, it is clear that if such a position p can be reached with σ, then also some position corresponding to p can be reached with σ'.

Suppose then that some operator for Eloise is in the scope of $(Q'x/W)$. Then, by conditions KM_3 and KM_4, Eloise cannot see the initial value of x (quantified by $(\forall x/V)$) in any position of the type (3) which precedes $p^\dagger$ in $G(M, \varphi)$. Therefore she also cannot see the value of y in any position of type (3) which precedes a position corresponding to $p^\dagger$ in $G(M, \varphi^*_{y/x})$. Since corresponding assignments in corresponding positions differ only on y, then, the uniformity of σ' forces Eloise to pick the same moves independently of what value a Abelard chooses for y. It is thus clear that:

If $p^\dagger(s)$ can be reached with σ, then a corresponding position $p^\dagger(s(a/y))$ can be reached with σ' for each $a \in M$.

By the definition of σ and by the condition above, it is also easy to see that whenever some *later position* $p(s)$ *of type* (4) can be reached with σ, then also the corresponding position $p(s(a_0/y))$ can be reached with σ'.[19]

By the observations above, we conclude that if a position p in $G(M,\varphi)$ can be reached with σ, then a position corresponding to p in $G(M,\varphi^*_{y/x})$ can be reached with σ'. Since y does not occur in any of the literals of type (4) in $G(M,\varphi^*_{y/x})$, it is thus easy to see that σ is a winning strategy.

For checking the uniformity of σ in positions of the form $((\exists z/U)\psi, M, s)$ of type (4), let s_1 and s_2 be assignments that can occur in such position and that only differ for the variables in U. For these assignments σ follows the choices of σ' for $s_1(a_0/y)$ and $s_2(a_0/y)$, respectively. As $s_1(a_0/y)$ and $s_2(a_0/y)$ only differ on the values in U, it follows from the uniformity of σ' that σ assigns the same value for the positions with s_1 and s_2. For all other positions of the game the uniformity of σ can be proven as in the proof of Lemma 2. We thus conclude that σ is a uniform winning strategy. □

The assumptions KM_3 and KM_4 are both necessary for proving the claim of Lemma 3. See the example below.

Example 11 Consider the sentence $\psi = \forall x \exists u(u = x \wedge \forall x(\exists z/\{u\})z = u)$ for which $\psi^*_{y/x} = \forall y \exists u(u = y \wedge \forall x(\exists z/\{u\})z = u)$. We first note that ψ does not satisfy KM_3 as $\exists u$ can see the "first value of x" while it is hidden from $(\exists z/\{u\})$ due to requantification. The sentence ψ is not true in any model with at least two elements as Eloise has to copy the value of x to u and thus she cannot match the value of z with the value of u when she cannot see the value of u nor the "old value of x". However, $\psi^*_{y/x}$ is clearly valid since Eloise wins simply by copying the value of y to both u and z.

Consider then the sentence $\theta = \forall x \forall u(u \neq x \vee \forall x(\exists z/\{u\})z = u)$ for which we have $\theta^*_{y/x} = \forall y \forall u(u \neq y \vee \forall x(\exists z/\{u\})z = u)$. Note that θ does not satisfy KM_4 as $\vee$ can see the first value of x while it is hidden from $(\exists z/\{u\})$. The sentence θ is not true in any model with at least two elements as Eloise has to choose the right disjunct always when u and x have the same value and thus she cannot match the values of z and u when she cannot see the value of u nor the old value of x. However, $\theta^*_{y/x}$ is clearly valid since Eloise wins by choosing the right disjunct whenever u and y have the same value and then copying the value of y to z.

Note that in both $\psi^*_{y/x}$ and $\theta^*_{y/x}$ the universally quantified variable y can signal something useful for $(\exists z/\{u\})$ even though it has already become a dummy variable (by not occurring in any of the literals that can be reached). Hence, in the corresponding games for ψ and θ, it would be useful for Eloise if she could see the *old value of* x when selecting a value for u.

Next we prove a claim that gives necessary conditions for the equivalence of φ and $\varphi^*_{y/x}$ in the case when the replaced quantification of x is *existential*. From the assumptions given below it follows that *all the information that Eloise can see when*

[19] Note however that for later positions $p(s)$ of type (4), the position $p(s(a/y))$ is not necessarily reached with σ' for all elements $a \in \mathrm{dom}(M)$. This is because σ' may indeed choose different moves depending on the value a for y. Nevertheless, since σ' is a winning strategy and y is a dummy variable for positions of type (4), it is irrelevant which fixed value a_0 for y Eloise "follows" when copying σ' to σ.

quantifying y is available to her also later in the game. Therefore, intuitively, any information that the dummy variable y can potentially signal to Eloise is useless for her, since she already knows it.

Lemma 4 *Let φ be an* IF *sentence, where $(\exists x/V)$ is an outermost quantification of x. Let $\varphi^*_{y/x}$ be the sentence that is obtained by replacing $(\exists x/V)$ with $(\exists y/V)$ as described above. Moreover, we assume that the condition* KM_1 *and the following condition $(\star)$ hold for φ:*

$(\star)$ *$(\exists x/V)$ is not in the scope of any quantifier (Qz/U) such that z is requantified after $(\exists x/V)$ in φ.*

Now we have $M \models \varphi \iff M \models \varphi^*_{y/x}$.

Proof We use also here the same correspondence between the games $G(\varphi, M)$ and $G(\varphi^*_{y/x}, M)$ as in the proof of Lemma 2. Moreover, the implication from left to right can be proven as in the proof of Lemma 2.

Suppose that Eloise has a uniform winning strategy σ' in $(M, \varphi^*_{y/x})$. We now define a strategy σ for Eloise in $G(\varphi, M)$ so that, in a position p, the strategy σ assigns the same move as σ' does in some position p' which corresponds to p and *which can be reached with* σ' in $G(\varphi^*_{y/x}, M)$. In case p is of type (4) there can be several positions which correspond to p, but at most one of them can be reached with σ'. This holds because the value of y chosen by σ' is determined by the values of the variables that are stored in the assignment when quantifying y, and none of these variables is requantified after y in $\varphi^*_{y/x}$ by the condition $(\star)$. Moreover, it is not relevant to consider the case when no position corresponding to p can be reached with σ', as such positions p are not reachable with σ either.[20] For the same reasons as in the proof of Lemma 2, it is now easy to see that σ is a winning strategy and that its uniformity condition holds for positions of types (1) – (3) and positions of type (4) with disjunctions.

For checking the uniformity of σ in positions of the form $((\exists z/U)\psi, M, s)$ of type (4), let s_1 and s_2 be assignments that can occur in such position and that only differ for the variables in U. Let s'_1 and s'_2 be the respective assignments in the corresponding positions that are reachable by σ'. We first show that these positions agree on the value of y. Suppose that $s'_1(y) \neq s'_2(y)$. Let s^*_1 and s^*_2 be the assignments preceding s'_1 and s'_2, respectively, in the corresponding histories such that s^*_1 and s^*_2 occur in a position for the quantifier $(\exists y/V)$ in $G(\varphi^*_{y/x}, M)$. Since σ' assigns different values of y for s^*_1 and s^*_2, there must be some other variable w for which $s^*_1(w) \neq s^*_2(w)$. Let $(Q''w/T)$ be the quantifier for which $(\exists y/V) \in \mathrm{Es}(Q''w/T)$. See the order of the relevant quantifiers in $\varphi^*_{x/y}$ below for illustration:

$$(Q''w/T)\ldots(\exists y/V)\ldots(Q'x/W')\ldots(\exists z/U)$$

[20] Towards σ being a winning strategy, it is irrelevant how it is defined for positions that are not reachable by it. It is also obvious that when the uniformity condition is not violated by reachable positions, we can define σ for non-reachable positions in such a way that uniformity still holds. Moreover, the fact that σ does not reach such positions for which no corresponding position is reachable by σ' can be proven by a simple induction argument using the fact that σ is defined by copying the moves of σ'.

Since the choice of σ' for the value of y is independent of V, we may assume that $w \notin V$. By the assumption ($\star$), w is not requantified in $\varphi^*_{x/y}$ after y and hence we must have $s'_1(w) \neq s'_2(w)$ and thus also $s_1(w) \neq s_2(w)$. We next apply the assumption KM_1 for the quantifiers $(Q''w/T)$, $(\exists x/V)$ and $(\exists z/U)$ in φ. Because $w \notin V$, by KM_1, it has to be that $w \notin U$. But this is impossible since s_1 and s_2 were supposed to differ only for the variables in U. Hence $s'_1(y) = s'_2(y)$ and thus, by the uniformity of σ', the strategy σ assigns the same value for positions $((\exists z/U)\psi, M, s_1)$ and $((\exists z/U)\psi, M, s_2)$. We thus conclude that σ is a uniform winning strategy for Eloise. □

The assumptions KM_1 and ($\star$) are both necessary for the claim of Lemma 4. See the example below.

Example 12 Consider the sentence $\psi = \forall z \exists x \forall x (\exists u/\{z\}) u = z$ for which we have $\psi^*_{y/x} = \forall z \exists y \forall x (\exists u/\{z\}) u = z$. We first note that ψ does not satisfy KM_1 as $\exists x$ can see the value of z while it is hidden from $(\exists u/\{z\})$. The sentence ψ is not true in models with at least two elements as Eloise cannot match the value of u with the value of z when she cannot see the value of z nor the first value of x. However, $\psi^*_{y/x}$ is valid since Eloise wins by copying the value of z to y which signals it to u.

Consider then the sentence $\theta = \forall z \exists x \exists w (x = z \wedge w = z \wedge \forall z \forall x (\exists u/\{w\}) u = w)$ for which $\theta^*_{y/x} = \forall z \exists y \exists w (y = z \wedge w = z \wedge \forall z \forall x (\exists u/\{w\}) u = w)$. Now θ satisfies KM_1, but it does not satisfy the condition ($\star$) as $\exists x$ is in the scope of $\forall z$ which is requantified in θ after $\exists x$. The sentence θ is not true in any model with at least two elements as Eloise has to copy the value of z to both x and w, and thus she cannot match the value of u with the value of w when she cannot see the value of w nor the old values of x or z. However, $\theta^*_{y/x}$ is clearly valid since Eloise wins simply by copying the value of z to y and w, and then copying y to u.

In the following theorem we give various sufficient conditions under which the sentence $\mathrm{reg}^*(\varphi)$, given by the alternative regularization process, is equivalent to φ.

Theorem 6 *An* IF *sentence* φ *is equivalent to its alternative regularization* $\mathrm{reg}^*(\varphi)$ *if at least one of the conditions below holds:*

(1) φ *satisfies* KM_3, KM_4 *and all the replaced quantifiers in* $\mathrm{reg}^*(\varphi)$ *are universal.*[21]
(2) φ *satisfies* KM_1 *and all the replaced quantifiers in* $\mathrm{reg}^*(\varphi)$ *are existential.*
(3) φ *satisfies* KM_1, KM_3 *and* KM_4.

Proof The first case is proven by applying Lemma 3 for every requantification in φ. Similarly, the second case is proven by applying Lemma 4 for every requantification in φ – in such an order that the additional assumption ($\star$) is maintained in the translation. Finally the third case is obtained by combining the earlier two translations. □

Theorem 6 isolates three fragments of IF logic on which reg^* is correct, i.e., it produces equivalent sentences. It is worth pointing out that these are all "large" fragments that can define second-order properties, as each of them allows either signalling

[21] Note that this holds if and only if we have: $(Q'x/U) \in \mathrm{Es}(Qx/V) \Rightarrow Q = \forall$ for all quantifiers in φ.

or signalling by disjunction; thus reg* is correct on a significant part of IF logic. By contrast, in the rest of the section we will use reg* to isolate a first-order fragment.

As demonstrated in the end of Section 8.1, our standard regularization reg does not necessarily preserve the conditions KM_1 and KM_2. However the alternative regularization procedure reg* preserves these properties and thus it follows that reg* preserves the condition of knowledge memory.

Lemma 5 *Let φ be an* IF *sentence where (Qx/X) is an outermost quantification of x; let y be a variable that does not occur in φ.*

(1) If φ has KM_1*, then $\varphi^*_{y/x}$ has* KM_1.
(2) If φ has KM_2*, then $\varphi^*_{y/x}$ has* KM_2.

Proof We prove only (1), as the proof of (2) is similar. Suppose for the sake of contradiction that $\varphi^*_{y/x}$ does not have KM_1. We first note that all quantifications, with the exception of (Qx/X) and the corresponding (Qy/X) in $\varphi^*_{y/x}$, are visible to other quantifiers in φ if and only if they are visible to other quantifiers in $\varphi^*_{y/x}$. Hence the reason why KM_1 fails in $\varphi^*_{y/x}$ is that there are quantifiers $(\exists u/U'), (\exists v/V')$ in $\varphi^*_{y/x}$ such that $(\exists u/U'), (\exists v/V') \in \mathrm{Es}(Qy/X)$, $y \notin U'$, $y \in V'$, and $(\exists v/V')$ is in the scope of $(\exists u/U')$. Let $(\exists u/U), (\exists v/V)$ be the corresponding quantifiers in φ. By the definition of the alternative regularization procedure, $y \in V'$ implies that, in φ, $(\exists v/V) \in \mathrm{Es}(Qx/X)$ and $x \in V$. Now since $(\exists v/V) \in \mathrm{Es}(Qx/X)$ and $(\exists v/V)$ is in the scope of $(\exists u/U)$, we must have $(\exists u/U) \in \mathrm{Es}(Qx/X)$. But then again, by the definition of the regularization procedure, from $y \notin U'$ we conclude $x \notin U$. But then, since $(\exists u/U), (\exists v/V) \in \mathrm{Es}(Qx/X)$, $x \in V$ and $x \notin U$, and $(\exists v/V)$ is in the scope of $(\exists u/U)$, we conclude that φ does not have KM_1: a contradiction. □

Since the alternative regularization process reg* is correct for sentences with knowledge memory and reg* preserves knowledge memory for Eloise, we now can generalize a result of [**?**] – stating that the *regular* fragment of knowledge memory is first order – to the case of irregular sentences.

Theorem 7 (Cf. [?]) *If Eloise has knowledge memory for a (possibly irregular)* IF *sentence φ, then φ is equivalent to a first-order sentence.*

Proof Let φ be an IF sentence for which KM_1–KM_4 hold. By Theorem 6(3), φ is equivalent to reg$^*(\varphi)$. By repeated applications of Lemma 5, we see that also reg$^*(\varphi)$ has the properties KM_1–KM_2; being regular, then, reg$^*(\varphi)$ has KM. Thus reg$^*(\varphi)$ is equivalent to a first-order sentence by Theorem 8.10 in [**?**]. □

We can also use reg* to identify a different kind of first order fragment.

Theorem 8 *A (possibly irregular)* IF *sentence φ is equivalent to a first order sentence if φ satisfies the conditions* KM_1 *and* KM_2 *of knowledge memory, and the following condition holds for φ: for every quantifier (Qx/U) which has a quantifier $(Q'x/V)$ in its scope, we have $Q = \exists$.*

Proof By Theorem 6(2), φ is equivalent to $\text{reg}^*(\varphi)$. By similar reasoning as in the proof above, $\text{reg}^*(\varphi)$ also has KM and thus it is equivalent to a first order sentence. □

Note that the theorem above does not follow from Theorem 7 as we do not need to assume the conditions KM_3 and KM_4 of knowledge memory. For example, any sentence $\varphi = \forall x \exists y \exists z \forall y (\exists w/\{z\})\psi$, where ψ is quantifier free, is equivalent to a first-order sentence by the theorem above even though φ does not have knowledge memory for Eloise because it violates the property KM_3.

9 Irregular IF prefixes

In this section we make a number of observations on irregular prefixes. First in Section 9.1 we study properties of irregular Henkin and signalling prefixes. In Section 9.2 we show how the study of irregular prefixes can be reduced to the study of regular prefixes by using the regularization procedure from Section 8.1. Finally in Sections 9.3 and 9.4 we show how irregular prefixes behave quite differently in infinite and finite models.

9.1 Irregular signalling and Henkin prefixes

In this section we compare the irregular signalling and Henkin prefixes to the regular ones. In the case of regular prefixes, a so-called Extension Lemma ([**?**], section 4) ensures that the expressive power of a prefix $\mathbf{Q}$ does not decrease if it is appropriately embedded in a longer prefix $\mathbf{Q}'$ ("appropriately" means, roughly, that the dependencies in $\mathbf{Q}$ are preserved). Consequently, for a regular prefix to have second-order expressive power it suffices that a second-order prefix be embeddable in it; that is, it is sufficient that the prefix *contains* certain specific patterns. The Extension Lemma fails for irregular prefixes. For example, the signalling pattern $\forall x \exists y (\exists z/\{x\})$ is NP-hard, but its extension $\forall x \exists y (\exists z/\{x\}) \forall x \forall y \forall z$ is equivalent to the first-order prefix $\forall x \forall y \forall z$. Requantification may inhibit an expressive quantification pattern.

Remember that by the dichotomy result (Theorem 1) every regular (sentential) IF prefix is either first order or can express NP-complete problems; and the NP-hard prefixes are exactly those that contain a signalling or a Henkin pattern as a subprefix. It is not known whether a similar dichotomy result holds for irregular IF prefixes. We will get back to this question in Sections 9.2, 9.3 and 9.4, where we study irregular prefixes as compared to regular ones.

Recall that our definition of a linear pattern requires that the relation of "being in the effective scope" forms a linear order among the quantifiers in the pattern. Hence the requirement that signalling and Henkin patterns be linear patterns excludes many prefixes that resemble signalling/Henkin prefixes, but have repetitions of variables. For example, the prefix $\forall x \exists x \exists y$ is not signalling because $\exists y$ is not in the effective scope of $\forall x$; and similarly, the prefixes $\forall x \exists x \forall z (\exists w/\{x\})$, $\forall x \exists y \forall x (\exists w/\{y\})$, $\forall x \exists y \forall y (\exists w/\{x\})$ are not classified as Henkin. By observing the Skolemizations of these sentences (see Remark 6), it is easy to show that these specific prefixes are actually first order. For example, consider a sentence $\forall x \exists x \forall z (\exists w/\{x\})\, \psi(x,z,w)$ (where ψ is quantifier-free), which has the first of the prefixes above. We have already seen (Remark 6) that its Skolemization is $\exists f \exists g \forall y \forall z \psi(f(y),z,g(z))$, which is easily proved to be equivalent to $\exists c \exists g \forall z \psi(c,z,g(z))$ as y does not occur in ψ. By "reversing" the Skolemization process (see [?]), this is seen to be equivalent to the first-order sentence $\exists u \forall z \exists w \psi(u,z,w)$.

As observed earlier, our definitions do *not* enforce that the variables in a signalling or Henkin pattern are all distinct. For example, the prefix $\forall x \exists y (\exists y/\{x\})$ is signalling, and the prefixes $\forall x \forall z (\exists y/\{z\})(\exists y/\{x,y\})$ and $\forall x \exists y \forall z (\exists z/\{x\})$ are Henkin. Using the Skolemizations as above, it is easy to show that these specific prefixes are actually first order – in contrast with the case of regular Henkin and signalling prefixes, which can all express second-order properties. However, we will see in Section 9.4 that the signalling prefix $\forall x \exists y (\exists y/\{x\})$ can be extended to a second-order prefix by adding two additional universal quantifiers.

We end the discussion with some notes on signalling in irregular prefixes. We first consider *long signalling sequences* or "chained signalling patterns". The regular prefix $\forall x \exists y (\exists z/\{x\})(\exists w/\{x,y\})$ is an example of a long signalling sequence as the value of x may be signalled via y to z and then via to z to w. In the case of regular prefixes long signalling sequences are not so interesting as any long signalling sequence contains a signalling pattern (essentially of the form $\forall x \exists y (\exists z/\{x\})$ which can already express NP-hard problems ([?]). However, in the irregular case, there are long signalling sequences that are beyond first order, but such that all the signalling patterns occurring as subprefixes are first order. The simplest example is the prefix $\forall x \exists y (\exists y/\{x\})(\exists z/\{x\})$. By observing the Skolemizations, it is easy to show that this prefix is equivalent to the standard (NP-hard) signalling pattern $\forall x \exists y (\exists z/\{x\})$. But, according to our definitions, the only signalling pattern that occurs within this prefix is $\forall x \exists y (\exists y/\{x\})$ which is first order, as stated above.

9.2 Regularizing irregular prefixes

As shown in Section 8.1, irregular IF sentences in prenex form can be regularized by only modifying the quantifier prefix without changing the quantifier-free part. Thus irregular prefixes are equivalent to regular prefixes in a "relative sense" stated by the following theorem. (Recall Section 2.4 for the definition of equivalence of prefixes *over a given set of variables*.)

Theorem 9 *Any irregular prefix* $\mathbf{Q}$ *is equivalent to* $\mathrm{reg}(\mathbf{Q})$ *over* $\mathrm{Bound}(\mathbf{Q})$.

Proof Let ψ be a quantifier-free formula for which $\mathrm{Free}(\psi) \subseteq \mathrm{Bound}(\mathbf{Q})$. As stated in Section 8.1, $\mathbf{Q}\psi$ is equivalent to its regularization $\mathrm{reg}(\mathbf{Q}\psi) = \mathrm{reg}(\mathbf{Q})\psi$. As the sentences $\mathbf{Q}\psi$ and $\mathrm{reg}(\mathbf{Q})\psi$ are equivalent for each such ψ, we have shown that the prefixes $\mathbf{Q}$ and $\mathrm{reg}(\mathbf{Q})$ are equivalent over the set $\mathrm{Bound}(\mathbf{Q})$. □

However, since requantifications in $\mathbf{Q}$ are replaced with fresh variables in $\mathrm{reg}(\mathbf{Q})$, we have $\mathrm{Bound}(\mathbf{Q}) \subsetneq \mathrm{Bound}(\mathrm{reg}(\mathbf{Q}))$. Therefore $\mathbf{Q}$ is not equivalent to $\mathrm{reg}(\mathbf{Q})$ in general. See the following example for illustration.

Example 13 Consider the irregular quantifier prefix $\mathbf{Q} := \forall x \exists z (\exists z/\{x\})$. As we have stated before, this prefix is first order. However, $\text{reg}(\mathbf{Q}) = \forall x \exists y (\exists z/\{x\})$ is NP-hard as a regular signalling prefix. This seeming contradiction is explained by the fact that $\mathbf{Q}\psi \equiv (\text{reg}(\mathbf{Q}))\psi$ for all ψ with $\text{Free}(\psi) \subseteq \{x,z\}$. Hence the prefix $\forall x \exists y (\exists z/\{x\})$ is indeed first order over the set $\{x,z\}$.

By Theorem 9, when studying the expressive power of an irregular prefix $\mathbf{Q}$, we may equivalently study the corresponding regular prefix $\text{reg}(\mathbf{Q})$ applied to quantifier-free formulae ψ for which $\text{Free}(\psi) \subseteq \text{Bound}(\mathbf{Q})$. Since the "dummy variables" in $\text{Bound}(\text{reg}(\mathbf{Q}))$ do not occur in ψ, their only role in the semantical game can be as signals for the other quantifiers[22]. It is thus natural to ask whether we could remove all these dummy quantifiers from $\text{reg}(\mathbf{Q})$ by doing suitable modifications to the slash sets in $\text{reg}(\mathbf{Q})$.

We first show that the dummy *universal* quantifiers $(\forall x/V)$ cannot signal anything useful for Eloise. Indeed, they can be removed from $\text{reg}(\mathbf{Q})$ simply by removing the quantified variable x from all the slash sets as well. This result holds also more generally for non-prenex sentences that have dummy universal quantifications. An intuitive idea for the truth of this claim is that when Abelard can choose any value for the dummy variable x, the quantifier $(\forall x/V)$ does not signal anything useful for Eloise.

Lemma 6 *Let φ be a regular sentence, where x is quantified by $(\forall x/V)$, but x does not occur in any literal in φ. Let $\varphi_{|\forall x}$ denote the sentence which is obtained by*

1. *removing the quantifier $(\forall x/V)$ from φ; and*
2. *removing x from all the slash sets in the scope of $(\forall x/V)$.*

Now we have $M \models \varphi \iff M \models \varphi_{|\forall x}$.

Proof Similarly as in the proof of Lemma 2, there is a natural correspondence between the syntactical trees of φ and $\varphi_{|\forall x}$. Here the subformula $(\forall x/V)\mu$ does not correspond to any subformula in $\varphi_{|\forall x}$, but for any $\psi \in \text{Sf}(\varphi) \setminus \{(\forall x/V)\mu\}$, the corresponding subformula in $\varphi_{|\forall x}$ is denoted by ψ'. This correspondence is extended to positions of semantic games by defining that a position of the form (ψ, M, s), where $\psi \neq (\forall x/V)\mu$, corresponds to the position (ψ', M, s'), where (i) $s' = s$ if ψ is not in the scope of $(\forall x/V)$; and (ii) $s' = s_{-x}$ if ψ is in the scope of $(\forall x/V)$. (The assignment s_{-x} is obtained by removing x from the domain of s.)

As in the proof of Lemma 2, the general idea is to copy the moves made by the winning strategy in the corresponding positions. Again we may assume for technical convenience that the winning strategies are defined over positions of the game instead of histories (cf. Remark 1).

Suppose first that Eloise has a uniform winning strategy σ' in $G(\varphi_{|\forall x}, M)$. Now every position where Eloise needs to make a choice in $G(\varphi, M)$ has a unique corresponding position in $G(\varphi_{|\forall x}, M)$ (note here that in positions for $(\forall x/V)\mu$ Eloise does not need to make a choice). We may thus define a strategy σ for $G(M, \varphi)$ by simply

[22] It is intuitively clear that the dummy variables cannot signal anything useful for disjunctions in ψ as Eloise is allowed to see the values of all relevant variables when choosing a disjunct.

copying σ' in corresponding positions. As x does not occur in any literals in φ, it is clear that σ is a winning strategy. Moreover, the uniformity of σ follows from the uniformity of σ'.

Suppose next that Eloise has a uniform winning strategy σ in $G(\varphi, M)$. As positions for those formulas ψ', for which ψ not in the scope of $(\forall x/V)$, have a unique corresponding position, we may there define σ' to follow σ in the corresponding position. For the other positions we proceed as in the proof of Lemma 3. That is, we fix any value $a_0 \in M$ and define σ' at (ψ', M, s') to follow σ at $(\psi, M, s'(a_0/x))$. As (ψ', M, s') can be reached with σ' only when $(\psi, M, s'(a_0/x))$ can be reached with σ, it is easy to see that σ' is a winning strategy. Moreover, also the uniformity of σ' now follows from the uniformity of σ (cf. the corresponding part in the proof of Lemma 3.) □

The next natural question is whether also the dummy *existential* quantifiers could be removed from sentences by making suitable modifications to slash sets. It turns out that this can indeed be done – when we consider only *infinite models*.

9.3 Irregular prefixes in infinite models

When considering the effectiveness of signalling, there is a big difference between finite and infinite models. Since in an infinite model finite tuples can be injectively mapped to the domain, we can use the value of a single variable as a signal encoding all the information that is produced by any (finite) number of quantifiers. This observation will be the key for showing that, in infinite models, the expressive power of irregular IF prefixes amounts to the expressive power of regular IF prefixes.

Suppose that an IF sentence φ contains a dummy variable x, quantified by an existential quantifier $(\exists x/V)$, such that x does not occur in any of the literals and therefore its role can be relevant only for signalling. We then eliminate the quantification of x from φ and modify the slash sets of other existential quantifiers in such a way that:

- The quantifiers that could "see" the value of x (as a signal) are now allowed to see the values of all the variables that $(\exists x/V)$ can "see".
- The quantifiers that cannot see the value of x do not need any essential modifications. We may simply remove x from their slash set as the value of x is no longer available.

As a result we obtain a sentence $\varphi_{|\exists x}$ in which the value of the dummy variable x can no longer be used as a signal, but all the quantifiers that were allowed to depend on x in φ are now allowed to depend on all the values that x was allowed to depend on. Hence the existential quantifiers in $\varphi_{|\exists x}$ intuitively can use the signal given by the dummy variable x "in its maximal potential". A formal definition of $\varphi_{|\exists x}$ now follows.

Let φ be a regular IF sentence in which the variable x occurs in the quantifier $(\exists x/V)$, but it does not occur in any literal in φ. The set of those variables that are quantified before $(\exists x/V)$ in φ, but which do not occur in V, is denoted by $\overline{V}$. Let then $\varphi_{|\exists x}$ be the sentence that is obtained from φ by replacing the subformula $(\exists x/V)\mu$ with μ' (omitting $(\exists x/V)$), where μ' is obtained by modifying μ so that every existential quantifier $(\exists y/U)$ in μ is replaced with $(\exists y/U')$, where

1. $U' := U \setminus \overline{V}$ if $x \notin U$; and
2. $U' := U \setminus \{x\}$ else.

Note that since φ is a sentence and x does not occur in any literal of φ, also $\varphi_{|\exists x}$ is a sentence. Moreover, $\varphi_{|\exists x}$ has exactly the same quantifier-free part as φ.

Example 14 Let ψ_1, ψ_2, ψ_3 be quantifier-free formulae whose free variables are from the set $\{x_1, x_2, x_3, v_1, v_2, v_3\}$. Consider the following sentence

$$\varphi := \forall x_1 \forall x_2 \forall x_3 (\exists w/\{x_1\}) \\ \big((\exists v_1/\{x_1,x_2\})\psi_1 \vee (\exists v_2/\{x_2,x_3\})\psi_2 \vee (\exists v_3/\{w,x_2,x_3\})\psi_3\big).$$

Now we have

$$\varphi_{|\exists w} = \forall x_1 \forall x_2 \forall x_3 \big((\exists v_1/\{x_1\})\psi_1 \vee \exists v_2 \psi_2 \vee (\exists v_3/\{x_2,x_3\})\psi_3\big).$$

The intuition behind this translation is that because v_1 can see w and w can see x_2, also v_1 becomes able to see x_2. However, v_1 cannot see x_1 since w cannot see x_1 either. Likewise v_2 can see both x_2 and x_3 since w can see them. But v_3 cannot see x_2 nor x_3 since it cannot see w.

The following example demonstrates how $\varphi_{|\exists x}$ becomes equivalent to φ in all infinite models. (The general result will be proven in Lemma 7.)

Example 15 Consider the sentence

$$\eta := \forall x \forall y \exists u (\exists v/\{x,y\})(\exists w/\{x,y\})(v = x \wedge w = y).$$

Here both $(\exists v/\{x,y\})$ and $(\exists w/\{x,y\})$ can only see the value of u. But if the model is infinite, it is possible to map each pair of values for (x,y) to a different value of u and thus signal the value of both x and y to the quantifiers $(\exists v/\{x,y\})$ and $(\exists w/\{x,y\})$. Hence, in all infinite models, η becomes equivalent to the sentence $\eta_{|\exists u} = \forall x \forall y \exists v \exists w (v = x \wedge w = y)$ which is trivially valid.

The idea in the example above can be used to prove the following Lemma which intuitively states that signalling by dummy variables can be used maximally in infinite models.

Lemma 7 *Let φ be a regular* IF *sentence in which x is once existentially quantified by the quantifier $(\exists x/V)$, but x does not occur in any literal. Let $\varphi_{|\exists x}$ be the sentence that is obtained from φ by eliminating all the occurrences of x as described above. Now for any infinite model M we have: $M \models \varphi \iff M \models \varphi_{|\exists x}$.*

Proof As in the proof of Lemma 6, there is a natural correspondence between the subformulas of φ and $\varphi_{|\exists x}$: for any $\psi \in \mathrm{Sf}(\varphi) \setminus \{(\exists x/V)\mu\}$, let ψ' denote the corresponding subformula in $\varphi_{|\exists x}$. This correspondence is again extended to positions of semantic games so that any position (ψ, M, s), where $\psi \neq (\exists x/V)\mu$, corresponds to (ψ', M, s'), where $s' = s$ if ψ is not in the scope of $(\exists x/V)$; and else $s' = s_{-x}$. (The assignment s_{-x} is obtained by removing x from the domain of s.) We assume again that winning strategies are defined over positions of the game.

Suppose first that Eloise has a uniform winning strategy σ in $G(\varphi, M)$. We then formulate a strategy σ' for Eloise so that, in every position p in $G(M, \varphi_{|\exists x})$, the strategy σ' assigns the same move as σ does in some corresponding position that can be reached with σ in $G(\varphi, M)$. Note here that there can be several positions in $G(\varphi, M)$ that correspond to p (with different values for x), but at most one of them can be reached with σ because the value chosen for x by σ is determined by the values of the other variables. We do not need to consider the case when no position corresponding to p can be reached with σ as such positions are not reachable by σ' either. Now, for any literal $\psi \in \mathrm{Sf}(\varphi)$, if a position (ψ', M, s') can be reached with σ' in $G(M, \varphi_{|\exists x})$, then a corresponding position (ψ, M, s) can be reached with σ in $G(M, \varphi)$. Since $\psi' = \psi$ and x does not occur in ψ, the strategy σ' is thus clearly a winning strategy in $G(\varphi_{|\exists x}, M)$.

For checking the uniformity of σ', consider a position $((\exists y/(U \setminus \overline{V}))\psi', M, s')$ which correspond to positions $((\exists y/U)\psi, M, s)$, where $x \notin U$, for such s for which $s_{-x} = s'$. Let s'_1 and s'_2 be assignments that only differ on $U \setminus \overline{V}$. Let s_1 and s_2 be the respective assignments in the corresponding positions that are reachable by σ. Now s_1 and s_2 can only differ on $(U \setminus \overline{V}) \cup \{x\}$. Since s_1 and s_2 agree on all variables $v_i \notin V$ that are quantified before x in φ and since the choice of σ for the value of x is independent of V, it must be that $s_1(x) = s_2(x)$. Thus s_1 and s_2 agree on all variables that are not in U. Since the choice of σ for the value of y is independent of U, the strategy σ must thus assign the same choice of y for positions with s_1 and s_2. Thus σ' assigns the same move for the corresponding positions with s'_1 and s'_2. For all the other positions, the uniformity of σ' clearly follows from the uniformity of σ. We thus conclude that σ' is a uniform winning strategy Eloise in $G(\varphi_{|\exists x}, M)$.

Suppose then that Eloise has a uniform winning strategy σ' in $G(M, \varphi_{|\exists x})$. We define a strategy σ for Eloise in $G(\varphi, M)$ as follows. Consider first a position of the form $((\exists x/V)\psi, M, s)$ and let $\overline{V} = \{v_1, \ldots, v_n\}$. Since M is infinite, there is an injection $f\colon M^n \to M$. We now define σ to assign the value $f(s(v_1), \ldots s(v_n))$ for $((\exists x/V)\psi, M, s)$; note that this choice is clearly independent of V. In any other position p in $G(\varphi, M)$, the strategy σ picks the same move as σ' does in the (unique) corresponding position in $(\varphi_{|\exists x}, M)$. As in the other direction of the proof, it is now easy to see that σ is a winning strategy.

For checking the uniformity of σ, consider first a position $((\exists y/U)\psi, M, s)$ such that $(\exists y/U)$ is in the scope of $(\exists x/V)$ and $x \notin U$. This corresponds to the position $((\exists y/(U \setminus \overline{V}))\psi', M, s_{-x})$ in $G(\varphi_{|\exists x}, M)$. Let s_1 and s_2 be assignments that differ only on U. Suppose first that s_1 and s_2 also agree on all variables in $\overline{V}$. Now $(s_1)_{-x}$ and $(s_2)_{-x}$ only differ on $U \setminus \overline{V}$ and thus σ' must assign the same move for $(s_1)_{-x}$ and $(s_2)_{-x}$; and hence σ assigns the same move for s_1 and s_2. Suppose then that s_1 and s_2 differ for some variable $v_i \in \overline{V}$. Since the value of x is given by the injection f and $v_i \in \mathrm{dom}(f)$, it must now be that $s_1(x) \neq s_2(x)$. But since $x \notin U$, this is impossible as we assumed s_1 and s_2 to only differ on U. Consider then a position $((\exists y/U)\psi, M, s)$ such that $(\exists y/U)$ is in the scope of $(\exists x/V)$ and $x \in U$. This corresponds to the position $((\exists y/(U \setminus \{x\}))\psi', M, s_{-x})$. Even though x is not included in the slash set $U \setminus \{x\}$, the value chosen by σ' is here trivially independent of the value of x as x is not in the domain of s_{-x}. Also for all the other types of positions the uniformity of σ follows directly from the uniformity of σ'. We thus conclude that σ is a uniform winning strategy Eloise in $G(\varphi, M)$. □

By applying Lemmas 6 and 7 in the particular case of the regularizations of IF-sentences in prenex form, we can prove the following theorem.

Theorem 10 *In infinite models every irregular sentential prefix* $\mathbf{Q}$ *is equivalent to a regular prefix* $\mathbf{Q}'$ *such that* $\mathrm{Bound}(\mathbf{Q}) = \mathrm{Bound}(\mathbf{Q}')$.

Proof Let $\mathbf{Q}$ be an irregular prefix and $\mathbf{Q}\psi$ be a sentence, where ψ is quantifier free. Next, by regularizing $\mathbf{Q}\psi$, we obtain the sentence $\mathrm{reg}(\mathbf{Q})\psi$. Note that the prefix $\mathrm{reg}(\mathbf{Q})$ may now contain variables which do not occur in $\mathbf{Q}$ nor in any literal in ψ. By applying Lemmas 6 and 7 repeatedly we can eliminate all such variables from $\mathrm{reg}(\mathbf{Q})\psi$ and obtain a sentence $\mathbf{Q}'\psi$ which is equivalent to $\mathbf{Q}\psi$ in all infinite models. Moreover, $\mathbf{Q}'$ is a regular prefix which contains exactly the same variables as $\mathbf{Q}$. We conclude that $\mathbf{Q}$ and $\mathbf{Q}'$ are equivalent over infinite models. □

We will also show that it is easy to check which irregular prefixes are first order and which second order in infinite models. For this we will also use the following dichotomy result (cf. Theorem 1) for infinite models.

Theorem 11 *A regular sentential quantifier prefix* $\mathbf{Q}$ *is second order over infinite models if and only if it contains either a Henkin pattern or a signalling pattern.*

Proof As observed in Example 2, the most simple signalling prefix $\forall x \exists y (\exists z/\{x\})$ is beyond first order in infinite models. Likewise, the most simple Henkin prefix can express second-order properties on infinite models (for example, the sentence $\forall x \exists y \forall z (\exists w/\{x,y\})(P(y) \wedge (x = z \leftrightarrow y = w))$ expresses equicardinality of a structure M with its subset P^M). Since all (regular, sentential) signalling and Henkin prefixes are extensions of these smallest ones[23], by the Extension lemma (in the form given in [**?**]) all (regular, sentential) signalling and Henkin prefixes can express second-order properties on infinite models. The claim thus follows from Theorem 1. □

In order to check if a given irregular prefix $\mathbf{Q}$ is second order in infinite models, we first translate it to a corresponding regular prefix $\mathbf{Q}'$ as in the proof of Theorem 10 (note that this translation is a straightforward procedure). Then, by Theorem 11, we can simply check whether $\mathbf{Q}'$ contains any Henkin/signalling pattern to determine whether $\mathbf{Q}$ is second order on infinite models. The following example demonstrates how this procedure is applied in practice.

Example 16 Consider the irregular prefix $\mathbf{Q} := \forall x \exists y (\exists z/\{x\})(\exists y/\{x,y\})$. By regularizing $\mathbf{Q}$, we obtain $\mathrm{reg}(\mathbf{Q}) = \forall x \exists w (\exists z/\{x\})(\exists y/\{x,w\})$. Since the new fresh variable w does not occur in $\mathrm{Bound}(\mathbf{Q})$, we may eliminate its quantification by modifying the slash sets to obtain the prefix $\mathbf{Q}' = \forall x \exists z (\exists y/\{x\})$ which is equivalent to $\mathbf{Q}$ in infinite models (cf. the proof of Theorem 10). Since $\mathbf{Q}'$ contains a signalling pattern, it is second order in infinite models. Therefore, also $\mathbf{Q}$ is second order (in particular, it is such on infinite models).

[23] This includes the alternative forms for the Henkin prefixes described in Section 3.2. It is easy to provide similar descriptions of second-order properties using these prefixes.

In finite models the dummy variables cannot always be removed in the same way as in infinite models. The reason for this is that in a finite model there might not be enough elements to "encode" all the information that is available to a dummy quantifier into a single signal. See the following example.

Example 17 Recall the sentence $\eta = \forall x \forall y \exists u (\exists v/\{x,y\})(\exists w/\{x,y\})(v = x \wedge w = y)$ from Example 15 and let M be a finite model over the empty signature. Now the pair (x,y) may get $|M|^2$ different values, but u may get only $|M|$ different values. Hence, by using u, it is impossible to signal all the information about x and y to $(\exists v/\{x,y\})$ and $(\exists w/\{x,y\})$. Thus it is easy to see that η is true in M only when $|M| = 1$.

9.4 Irregular prefixes in finite models

It is natural to ask if there are irregular IF prefixes that are not equivalent to any regular IF prefix. By the results of the previous section (Theorem 10), this nonequivalence can manifest itself only in finite models.

In this section we consider the following irregular signalling prefix, which has some interesting properties:

$$\mathbf{Q}^* := \forall x \forall y \exists u \forall z (\exists u/\{x,y\}).$$

We make two preliminary observations on $\mathbf{Q}^*$:

1. By the proof of Theorem 10, in infinite models, $\mathbf{Q}^*$ is equivalent to the regular prefix $\forall x \forall y \forall z \exists u$ which is first order. Note that, as seen by comparing Theorem 1 and 11, all regular prefixes that are beyond first order are also beyond first order in infinite models.
2. $\mathbf{Q}^*$ contains only a single variable that is existentially quantified (although it is quantified two times). However, all the regular prefixes that are beyond first order contain at least two existentially quantified variables. This follows from the dichotomy result of IF prefixes (Theorem 1) as both Henkin and signalling patterns always contain at least two existentially quantified variables.

These two points suggest that $\mathbf{Q}^*$ should also be first order. However, it turns out to be second order, as shown by Theorem 12 below.

In the prefix $\mathbf{Q}^*$ Eloise can use the first quantification of u to signal: (1) the value of x; (2) the value of y; or (3) some information about the relationship between the values of x and y. The third case, which is the most interesting one, is demonstrated by the proof below, where Eloise uses u to signal the "distance" between x and y.

Lemma 8 *Let R be a binary relation symbol. Let M be a finite model over vocabulary $\{R\}$ such that R^M is the disjoint union of directed cycles that contain all the elements of M. We write*

$$\theta := \forall x \forall y \exists u \forall z (\exists u/\{x,y\})(\psi_1 \wedge \psi_2), \text{ where } \begin{cases} \psi_1 := (x = z \rightarrow y = u) \\ \psi_2 := (R(x,z) \rightarrow R(y,u)). \end{cases}$$

Now Eloise has a winning strategy in the semantic game $G(\theta, M)$ if and only if all the cycles in M have the same length.

Proof Suppose first that M is the disjoint union of cycles $C_0,\dots,C_{m-1}$ and each C_i is of length n. Thus, we can enumerate the elements of M as $\{a_{i,j} \mid i<m, j<n\}$ in such a way that $(a_{i,j},a_{i,k}) \in R^M \iff k = j+1 \pmod{n}$. We define natural summation and subtraction operations on M as follows:

$a_{i,j} + a_{k,l} := a_{p,q}$, where $p := i+k \pmod{m}$ and $q := j+l \pmod{n}$,
$a_{i,j} - a_{k,l} := a_{p,q}$, where $p := i-k \pmod{m}$ and $q := j-l \pmod{n}$.

Observe now that the sum $+$ is associative, and $(b-a)+a = b$ for all $a,b \in M$. Furthermore, denoting $\underline{1} := a_{0,1}$, we have $R^M = \{(a,a+\underline{1}) \mid a \in M\}$.

We formulate a strategy σ for Eloise in $G(\theta,M)$ as follows. Suppose that Abelard chooses a for the value of x and b for the value of y. Then Eloise chooses $b-a$ for the value of u. Assume next that Abelard chooses c for the value of z. Then Eloise answers by choosing the value of (the second) u to be $(b-a)+c$.

Since the new value of u only depends on the old value of u and the value of z, this strategy is uniform. Thus, it suffices to show that $M,s \models \psi_1 \wedge \psi_2$ for the assignment s defined by the values $a,b,c,(b-a)+c$ for x,y,z,u, respectively, chosen by Abelard and Eloise. Assume first that $M,s \models x = z$, i.e., $a = c$. Then

$$s(y) = b = (b-a)+a = (b-a)+c = s(u),$$

whence $M,s \models y = u$ and thus we see that $M,s \models \psi_1$. Assume then that $M,s \models R(x,z)$. As observed above, this means that $c = a+\underline{1}$. Then

$$s(u) = (b-a)+c = (b-a)+(a+\underline{1}) = ((b-a)+a)+\underline{1} = b+\underline{1},$$

whence $M,s \models R(y,u)$, and we conclude that $M,s \models \psi_2$.

Suppose then that Eloise has a winning strategy σ in the game $G(\theta,M)$. Then for each $a \in M$, there is a function $f_a\colon M \to M$ such that for each $b \in M$, $f_a(b)$ is the value for the variable u given by σ after Abelard has chosen the values a and b for the variables x and y. Furthermore, for each $e \in M$, there is a function $g_e\colon M \to M$ such that if e is the old value of u and Abelard chooses the value c for z, then σ gives $g_e(c)$ as the new value for u. Since σ is a winning strategy, in case $e = f_a(b)$ we have $M,s \models \psi_1 \wedge \psi_2$ for the assignment $s = \{(x,a),(y,b),(z,c),(u,g_e(c))\}$.

We make the following observations on the functions f_a and g_e:

(i) If $e = f_a(b)$, then $g_e(a) = b$.

Indeed, if s is the assignment arising in the game $G(\theta,M)$ when Abelard's choices for the variables x,y,z are a,b,a, respectively, and Eloise uses σ, then $M,s \models x = z$ and $M,s \models \psi_1$, whence $M,s \models y = u$ and thus

$$b = s(y) = s(u) = g_e(c) = g_e(s(z)) = g_e(s(x)) = g_e(a).$$

(ii) For any $a \in M$, f_a is a bijection.

Since M is finite, it suffices to show that f_a is an injection. Suppose that $f_a(b) = f_a(b') = e$ for some $b,b' \in M$. By observation (i), we see that $b = g_e(a) = b'$.

(iii) For any $e \in M$, g_e is a homomorphism of M:

$$\forall a, a' \in M : (a, a') \in R^M \Rightarrow (g_e(a), g_e(a')) \in R^M.$$

Assume that $a, a' \in M$, $b = g_e(a)$, $b' = g_e(a')$, and $(a, a') \in R^M$. Since, by observation (ii), f_a is a surjection, there exists $b'' \in M$ such that $e = f_a(b'')$. By observation (i), we have $b'' = g_e(a) = b$, whence $e = f_a(b)$. Let s be the assignment arising in the game $G(\theta, M)$ when Abelard's choices for the variables x, y, z are a, b, a' and Eloise uses σ. Now $M, s \models R(x, z)$ and $M, s \models \psi_2$, whence $M, s \models R(y, u)$. By the above, Eloise picks e as the first choice for u, and thus her second choice of u is made using function g_e. Thus we see that

$$(g_e(a), g_e(a')) = (b, g_e(s(z))) = (s(y), s(u)) \in R^M.$$

To complete the proof, let C and D be any two cycles in M. We need to show that C and D are of the same length. Pick arbitrary elements a from C and b from D, and let $e = f_a(b)$. Then by observation (iii), g_e is a homomorphism of M, and by observation (i), it maps a to b. Let h be the restriction of g_e to the cycle C. We claim that $h(c) \in D$ for every $c \in C$. Otherwise there are $c, c' \in C$ such that $(c, c') \in R^M$, $h(c) \in D$ and $h(c') \notin D$. But then $(h(c), h(c')) \notin R^M$, which contradicts the fact that h is a homomorphism. Thus, h is a homomorphism $C \to D$.

Finally, we show that h is surjective, and consequently C contains at least as many elements as D. By reversing the roles of C and D, we see that the converse is also true, whence we conclude that C and D are of the same length. Assume towards contradiction that h is not surjective. Then there are $d, d' \in D$ such that $(d, d') \in R^M$, $d \in h[C]$ and $d' \notin h[C]$ (here $h[C]$ is the image of C under h). Thus, $d = h(c)$ for some $c \in C$. Let c' be the unique element of C such that $(c, c') \in R^M$. Since h is a homomorphism, we have $(d, h(c')) \in R^M$. On the other hand, d' is the unique element of D such that $(d, d') \in R^M$, whence $d' = h(c')$, in contradiction with the assumption that $d' \notin h[C]$. □

Note that in infinite models the sentence θ is equivalent to $\forall x \forall y \forall z \exists u(\psi_1 \wedge \psi_2)$ (as shown by the proof of Theorem 10). This sentence is clearly true if the model is a disjoint union of directed cycles of length at least two even if some of the cycles are of different length. Thus, the assumption of finiteness was crucial in the proof above. It was indeed used in observation (ii) for proving that f_a is surjective by showing that it is injective.

Theorem 12 *The irregular prefix* $\mathbf{Q}^*$ *can define a property in* LOGSPACE *which is not first-order definable.*

Proof Let $\mathscr{P}$ be the following property of finite $\{R\}$-models M:

> The relation R^M is either empty or a disjoint union of directed cycles of the same length such that all the elements of M are on those cycles.

Note first that R^M is a disjoint union of directed cycles containing all the elements of M if and only if $M \models \forall y \exists u R(y, u) \wedge \forall x \forall y \forall z \psi_0$, where

$$\psi_0 := (R(x, y) \wedge R(x, z) \to y = z) \wedge (R(x, z) \wedge R(y, z) \to x = y).$$

Let θ be as in Lemma 8, and let η be the sentence that is obtained from θ by adding ψ_0 as a conjunct to its quantifier-free part. Our aim is to show that η defines the property $\mathscr{P}$. Before proving this, we make the following two observations:

1. η is equivalent to the conjunction $\forall x \forall y \forall z \psi_0 \wedge \theta$. Indeed if $M \models \eta$, then obviously $M \models \forall x \forall y \forall z \psi_0$, and Eloise wins the game $G(\theta, M)$ by using exactly the same strategy as she uses for winning $G(\eta, M)$. Similarly, if $M \models \forall x \forall y \forall z \psi_0 \wedge \theta$, Eloise can use in $G(\eta, M)$ the same strategy as she uses in $G(\theta, M)$. This guarantees a win for her, since ψ_0 is true irrespective of the the values chosen for x, y and z.
2. If $M \models \theta$ and $R^M \neq \emptyset$, then $M \models \forall y \exists u R(y,u)$. Assume that Abelard chooses in $G(\theta, M)$ values a, b, c for the variables x, y, z in such a way that $(a,c) \in R^M$. Then Eloise has to choose a value e for the (second) u such that $(b,e) \in R^M$, since otherwise $M, s \not\models \psi_2$ for the corresponding assignment s. Clearly this is not possible unless $M \models \forall y \exists u R(y,u)$.

Suppose first that M has property $\mathscr{P}$. If $R^M = \emptyset$, then is it straightforward to verify that $M \models \eta$. If, instead, M is a disjoint union of finitely many cycles of equal length, then $M \models \forall x \forall y \forall z \psi_0$, and by Lemma 8, $M \models \theta$, whence by the first observation above, $M \models \eta$. For the other direction, assume that $R^M \neq \emptyset$, and $M \models \eta$. Then by observation 1, we have $M \models \forall x \forall y \forall z \psi_0 \wedge \theta$. Furthermore, by observation 2, we have $M \models \forall y \exists u R(y,u)$, whence R^M is a disjoint union of cycles containing all the elements of M. It follows now from Lemma 8 that all the cycles are of the same length.

It is easy to see that the property $\mathscr{P}$ is in LOGSPACE.[24] Finally, we can use Ehrenfeucht-Fraïssé games to show that $\mathscr{P}$ is not expressible in first-order logic (see, e.g., [**?**] for techniques proving similar results with Ehrenfeucht-Fraïssé games). □

We leave it open whether the prefix $\mathbf{Q}^*$ is NP-hard or not. Both positive and negative answer to this question would be interesting: if $\mathbf{Q}^*$ were NP-hard, then we would have an example of an NP-hard prefix which is first order in all infinite models and which contains only a single existentially quantified variable. On the other hand, if $\mathbf{Q}^*$ were not NP-hard, then the dichotomy result (Theorem 1), which holds for regular IF prefixes, would fail for irregular IF prefixes.

Remark 7 By Theorem 9, the study of the expressive power of irregular prefixes amounts to the study of the expressive power of regular prefixes relative to variable sets. That is, instead of considering the expressive power of an irregular prefix $\mathbf{Q}$ we may equivalently consider the expressive power of a regular prefix reg($\mathbf{Q}$) over the variable set Bound($\mathbf{Q}$). For example, in the case of $\mathbf{Q}^*$, we can equivalently study its regularization $\forall x \forall y \exists w \forall z (\exists u / \{x,y\})$ over the set $\{x,y,z,u\}$.

However, we conjecture that the expressive power of a regular prefix $\mathbf{Q}$ over a set $U \subseteq$ Bound($\mathbf{Q}$) is a more general problem than the (unrelativized) expressive power of irregular prefixes.[25] There might indeed be regular prefixes $\mathbf{Q}$ and sets $U \subseteq$

[24] To verify that a model M has property $\mathscr{P}$ it suffices to check that the in-degree and out-degree of every element is one, and for each pair $a, b \in M$ chase the paths starting from a and b by moving markers along the R^M-edges in synchronous steps, and check that the markers come back to a and b at the same time. During the chase it suffices to store in memory only the starting points a and b, and the marked points.

[25] We also note that this problem cannot be generalized any further even if we also allow irregular prefixes. This is because the expressive power of an irregular prefix $\mathbf{Q}$ over $U \subseteq$ Bound($\mathbf{Q}$) always amounts to the expressive power of reg($\mathbf{Q}$) over U.

Bound($\mathbf{Q}$) such that no $\mathbf{Q}'$ with Bound($\mathbf{Q}'$) $= U$ is equivalent (relative to U) with $\mathbf{Q}$. This possibility is supported by the observation that, for some choices of $\mathbf{Q}$ and $U \subseteq$ *Bound*($\mathbf{Q}$), there is no $\mathbf{Q}'$ of bound variables U such that $\mathbf{Q} = \text{reg}(\mathbf{Q}')$, as shown in the example below.

Consider a (first-order) regular prefix of the form $\mathbf{Q} = Q_1 x Q_2 y Q_3 z$. We can show that there is no prefix $\mathbf{Q}'$ such that $\text{reg}(\mathbf{Q}') = \mathbf{Q}$ and $\text{Bound}(Q') = \{z\}$. To see this, we try to apply the regularization procedure (Sec. 8.1) backwards, starting from the variable y. We see that $\mathbf{Q} = \text{reg}(Q_1 x Q_2 z Q_3 z)$. However, given any quantifier-free ψ with $\text{Free}(\psi) = \{z\}$, we have $(Q_1 z Q_2 z Q_3 z \psi)_{x/z} = Q_1 x Q_2 z (Q_3 z/\{x\}) \psi \neq Q_1 x Q_2 z Q_3 z \psi$. As a less trivial example, by the same kind of reasoning one can see that a quantifier prefix of the form $Q_1 x Q_2 y Q_3 z (Q_4 u/\{z\}) (Q_5 v/\{y\})$ is not the regularization of any irregular prefix $\mathbf{Q}'$ with $\text{Bound}(\mathbf{Q}') = \{u, v\}$. We leave the deeper analysis of regular prefixes over variable sets for future work.

10 Conclusions

We have shown that full IF (i.e., ESO) expressive power can be achieved, without the use of Henkin prefixes, already within each of the two following fragments of IF logic: (1) prenex, regular IF logic with action recall ($\text{IF}^{\text{r}}_{\text{AR}(\exists)}$), and (2) non-prenex, regular IF logic without Henkin and signalling patterns. The proof of the first result shows that the H^n_2 Henkin prefixes are explicitly definable by means of signalling prefixes with a constant number of signalling variables and with only one existential quantifier with a nonempty slash set. Consequently, there are no hierarchies of expressive power based on the number of existential quantifiers in $\text{IF}^{\text{r}}_{\text{AR}(\exists)}$. As a related approach to (2), we gave a general method for translating IF sentences into *irregular* sentences without Henkin nor signalling patterns.

We deepened the analysis of irregular sentences and prefixes, firstly by identifying syntactical clauses that characterize action recall and knowledge memory without the restriction of regularity; secondly, by extending the definitions of Henkin and signalling patterns to the irregular case; thirdly, by producing two outside-in translation procedures of IF sentences into regular sentences. Both procedures have the advantage, over the usual regularization procedure, that they leave the quantifier-free matrix of prenex sentences unaltered. Because of this our procedures allow us to analyse irregular quantifier prefixes by analysing the corresponding regular ones. The first of these regularization procedures applies to all IF sentences, but it has the disadvantage of failing to preserve the property of knowledge memory. The second of these regularization procedures is correct when applied to sentences that satisfy knowledge memory, and can be extended to three larger (and non-first-order) fragments, which are listed in Theorem 6. While its range of applicability is more limited, this procedure does preserve the property of knowledge memory. This allows us to extend the result that the fragment of KM is first order to irregular sentences. We also used this procedure to prove an alternative first-orderness criterion: a (possibly irregular) IF sentence is first-order if it has the properties KM_1 and KM_2 of knowledge memory, and only existentially quantified variables are requantified in it.

We also showed that, over infinite structures, each irregular prefix can be translated into a regular one, with the same set of bounded variables; this yields a criterion for distinguishing first-order prefixes from second-order prefixes on infinite structures. We then identified a specific irregular prefix $\mathbf{Q}^*$, containing only a single existentially quantified variable, such that $\mathbf{Q}^*$ is first order over infinite structures, but on finite structures it can express a LOGSPACE problem that is not definable in first-order logic. We leave it as an open problem whether this prefix is in LOGSPACE (thus violating the FO/NPC dichotomy that holds for regular prefixes, [**?**]) or whether it can express problems of higher complexity.

These results extend the analysis of the expressive resources of IF logic which was initiated in [**?**] and [**?**], and they raise a number of questions to be further investigated:

- When considering irregular prenex sentences, are there other sources of second-order expressive power, besides Henkin and signalling patterns? What we can say, for now, is that long signalling sequences should be considered as basic second-order patterns.
- Are there other interesting hierarchies of signalling prefixes? For example, do hierarchies based on the length of signalling sequences (recall Section 9.1) arise if the number of universal quantifiers is kept fixed?
- In order to capture ESO within $\mathrm{IF}^{\mathrm{r}}_{\mathrm{AR}(\exists)}$ without making use of Henkin and signalling patterns, we used syntactical constructs which contained two disjunctions. Can the same result be obtained using only one disjunction?
- As discussed in Remark 7, the expressive power of irregular prefixes can be seen as a specific case of a plausibly more general problem: given a regular prefix $\mathbf{Q}$ and a set $U \subseteq \mathrm{Bound}(\mathbf{Q})$, what is the expressive power of $\mathbf{Q}$ over U? Moreover, which complexity classes can be captured by different $\mathbf{Q}$ and U?

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Conflict of interest

The authors declare that they have no conflict of interest.