Surface Tension

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Tiivistelmä – Referat – Abstract

Konveksia funktiota, jonka hessiaani on yksi, määriteltynä konveksissa ja rajoitetussa monikulmiossa kutsutaan pintajännitteeksi. Lisäksi vaaditaan, että annetun monikulmion reunalla funktio on paloittain affiini. Systeemin perusta on dimeerimallien teoriassa, mutta tässä tutkielmassa käsitellään sitä sellaisenaan. Reunaehdosta saadaan erityisinä pisteinä monikulmion kulmat ja niin kutsutut kvasijäätyneet pisteet, joissa reunafunktio ei ole differentioituva.

Yksikköympyrä voidaan sopivalla homeomorfismilla kuvata monikulmioksi. Tässä asetelmassa johdetaan eksplisiittinen lauseke pintajännitteen gradientille. Lisäksi gradientin arvot kulma- ja kvasijäätyneissä pisteissä johdetaan raja-arvoina, josta seurauslauseena tutkitaan pintajännitteen suunnattuja derivaattoja.

Avainsanat – Nyckelord – Keywords
Pintajännite, harmoninen mitta

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Tiivistelmä – Referat – Abstract

A convex function, which Hessian determinant equals to one, defined in a convex and bounded polygon is called a surface tension. Moreover at the boundary of the given polygon it is demanded that the function is piece-wise affine. This setting originates from the theory of dimer models but in this thesis the system is studied as such. The boundary condition gives us points of interest i.e. the corners of the polygon and so called quasi-frozen points, where the boundary function is not differentiable.

With a suitable homeomorphism one can map the unit disc to the polygon in question. In this setting an explicit formula for the gradient of the surface tension is derived. Furthermore the values of the gradient in corner and quasi-frozen points are derived as limits from, which as a corollary the directed derivatives of the surface tension are studied.

Avainsanat – Nyckelord – Keywords

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Contents

1 Introduction

Let N be a compact and convex polygon in the two-dimensional Euclidean space and $L : N \to \mathbb{R}$ a convex function such that it is piecewise affine on the boundary ∂N. We denote the set of corners of the polygon N by $\mathscr{P} = \{p_1, \dots, p_k\}$. The function L might have points on $\partial N \setminus \mathscr{P}$, where it is not differentiable. We call these points quasi-frozen and denote this set by $\mathscr{Q} = \{p_{k+1}, \dots, p_{k+m}\}$.

Definition 1.1. Surface tension is a convex function $\sigma : N \to \mathbb{R}$ satisfying

$$\begin{cases} \det D^2\sigma = 1 & \text{in } N^\circ; \\ \sigma = L & \text{on } \partial N. \end{cases} \tag{$*$}$$

The first equation is called Monge-Ampère equation and the second describes a boundary condition. The equation $(*)$ comes from the theory of dimer models but we restrict ourselves to study $(*)$ just as a differential equation. More precisely the goal of this thesis is to present an explicit formulation for the gradient of the surface tension. This is done in the section 5.1. After this in section 5.2 we study limits of the gradient near the corners and quasi-frozen points. As a corollary we obtain presentations for some directional derivatives of the surface tension. Section 5 is based on [1].

Throughout the paper we work on the complex differentiation by using the so called Wirtinger derivatives which are presented in the section 2.1. In section 2.2 we present some elementary remarks on Monge-Ampère measure and as a main result we derive the comparison principle. In section 2.3 we study the harmonic measure in the unit disc. As main results we derive the Poisson formula and an explicit formula for the harmonic measure of an arc. For each of these preliminary sections we use mainly [2], [3] and [4].

To guarantee that the explicit formulation of the gradient of the surface tension behaves well near $\partial\mathbb{D}$ we need to study the behaviour of univalent, harmonic and orientation-preserving functions near $\partial\mathbb{D}$. For this we mainly rely on [6] and do it in section 3. Transforming the unit disc into the polygon N is done via non-linear Cayley transform which we present in the section 4.

2 Preliminaries

2.1 Wirtinger Derivatives

From [8, p. 52] we recall that a complex function

$$f(x, y) = u(x, y) + iv(x, y) \tag{1}$$

given by real functions u and v is complex differentiable (or equivalently holomorphic or analytic), when it satisfies the Cauchy-Riemann equations

$$\partial_x u = \partial_y v \text{ and } \partial_x v = -\partial_y u.$$

After checking this we know that the derivative $\partial_z f$ exists. But complex differentiation can be done also in respect to the complex conjugate of the variable. To achieve this we'll give the following definition.

Definition 2.1. Let $z = x + iy$ be a complex number. The differential operators

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y) \text{ and } \partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y) \tag{2}$$

are called Wirtinger derivatives.

We notice that the second Cauchy-Riemann equation in (2) can be written as $-i\partial_x v = i\partial_y u$. This allows us to obtain a new definition for the complex differentiability.

Definition 2.2. Function $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable if $\partial_{\overline{z}} f = 0$.

If the complex conjugate of a function is holomorphic, we say the function is antiholomorphic.

Let the function $f : \mathbb{C} \to \mathbb{C}$ be twice continuously differentiable. Then we may calculate

$$\partial_z \partial_{\overline{z}} f(z) = \frac{1}{4}\big(\partial_{xx} f(z) - i\partial_{yx} f(z) + i\partial_{xy} f(z) - i^2 \partial_{yy} f(z)\big) = \frac{1}{4}\Delta f(z).$$

If we change the order of the Wirtinger derivatives we end up with the same result. Hence we obtain a new definition for a function to be harmonic.

Definition 2.3. Complex function $f \in C^2$ is harmonic if $\partial_{\overline{z}z} f = \partial_{z\overline{z}} f = 0$.

From the Definition 2.1 we can derive with ease two properties for Wirtinger derivatives of a function $f : \mathbb{C} \to \mathbb{C}$. Namely

$$\overline{\partial_{\overline{z}} f(z)} = \frac{1}{2}\big(\partial_x u(x, y) - i\partial_y u(x, y) + i\partial_x(-v(x, y)) + \partial_y(-v(x, y))\big) = \partial_z \overline{f(z)}$$

and

$$\overline{\partial_z f(z)} = \frac{1}{2}\big(\partial_x u(x,y) + i\partial_x(-v(x,y)) + i\partial_y u(x,y) - \partial_y(-v(x,y))\big) = \partial_{\overline{z}}\overline{f(z)},$$

which are called the conjugation rules of Wirtinger derivatives. Let us denote $w := g(z)$ for a function $g : \mathbb{C} \to \mathbb{C}$. We recall from [2, p. 10] that the chain rule for Wirtinger derivative of a composition function $h := f \circ g$ in respect to variable z is

$$\partial_z f(g(z)) = \partial_w h(w)\partial_z g(z) + \partial_{\overline{w}} h(w)\partial_z \overline{g(z)}$$

To prove this we just need to write

$$h(w) = f(g(z)) = f_1(g_1(x,y), g_2(x,y)) + if_2(g_1(x,y), g_2(x,y)),$$

where $f = f_1 + if_2$ and $g = g_1 + ig_2$. Let us denote $w_1 := g_1(x,y)$ and $w_2 := g_2(x,y)$. Now we may calculate that

$$\begin{aligned}
\partial_z f(g(z)) &= \frac{1}{2}\big(\partial_x f_1(g(x,y)) + i\partial_x f_2(g(x,y)) - i\partial_y f_1(g(x,y)) + \partial_y f_2(g(x,y))\big) \\
&= \frac{1}{2}(\partial_{w_1} f_1(w_1,w_2)\partial_x g_1(x,y) - i\partial_{w_1} f_1(w_1,w_2)\partial_y g_1(x,y) \\
&+ \partial_{w_2} f_1(w_1,w_2)\partial_y g_1(x,y) + i\partial_{w_2} f_1(w_1,w_2)\partial_x g_2(x,y) \\
&+ i\partial_{w_1} f_2(w_1,w_2)\partial_x g_1(x,y) + \partial_{w_1} f_2(w_1,w_2)\partial_y g_1(x,y) \\
&- i\partial_{w_2} f_1(w_1,w_2)\partial_y g_2(x,y) + \partial_{w_2} f_1(w_1,w_2)\partial_x g_1(x,y)) \\
&= \partial_w h(w)\partial_z g(z) + \partial_{\overline{w}} h(w)\partial_z \overline{g(z)}.
\end{aligned}$$

It is easy to see that for the derivative of the conjugate of composition function it stands that

$$\partial_z \overline{f(g(z))} = \partial_w \overline{h(w)}\partial_z g(z) + \partial_{\overline{w}} \overline{h(w)}\partial_z \overline{g(z)}.$$

Hence with the composition rule above and with conjugation rules we obtain a second chain rule for the Wirtinger derivatives, that is

$$\partial_{\overline{z}} f(g(z)) = \overline{\partial_z \overline{f(g(z))}} = \partial_{\overline{w}} h(w)\partial_{\overline{z}} \overline{g(z)} + \partial_w h(w)\partial_{\overline{z}} g(z).$$

Example 2.1. Let $a \in \mathbb{C}$ be a complex constant. For a future reference and as a short example of Wirtinger derivation we show that the function

$$f(z) := \frac{\overline{a}g(z)}{1 - \overline{a}z},$$

where $g : \mathbb{C} \to \mathbb{C}$ is holomorphic, is holomorphic. For the sake of convenience we assume that $\overline{a}z \neq 1$.

It is easy to show that the product rules for Wirtinger derivatives are as the those of products of real functions, that is

$$\partial_z uv = v\partial_{\overline{z}} u + u\partial_z v \text{ and } \partial_z uv = v\partial_{\overline{z}} u + u\partial_{\overline{z}} v,$$

as stated in [2, p. 8] for functions $u, v : \mathbb{C} \to \mathbb{C}$. Using the product and chain rules we get

$$\begin{aligned}\partial_{\overline{z}} f(z) &= \overline{a}\left(g(z)\partial_{\overline{z}}\frac{1}{1-\overline{a}z} + \frac{1}{1-\overline{a}z}\partial_{\overline{z}} g(z)\right)\\ &= \overline{a}g(z)\left(\partial_{\overline{w}}\frac{1}{w}\partial_{\overline{z}}\overline{(1-\overline{a}z)} + \partial_w\frac{1}{w}\partial_{\overline{z}}(1-\overline{a}z)\right).\end{aligned}$$

Note that from the definition 2.1 it follows that $\partial_{\overline{w}} 1/w = 0$ and $\partial_{\overline{z}} z = 0$. Thus the function f is holomorphic.

2.2 Monge-Ampère Measure

Let $f : U \to \mathbb{R}$ be a function defined in a convex domain $U \subset \mathbb{R}^n$. We say a vector $\xi \in \mathbb{R}^n$ is the subgradient of f on $p_0 \in U$ if

$$f(p) \geq f(p_0) + \langle \xi, p - p_0 \rangle \tag{3}$$

for every $p \in U$. The name is easily undestandable as from [11, p. 29] we recall that the gradient satisfies

$$f(p) = u(p_0) + \langle \nabla f(p_0), p - p_0 \rangle + |p - p_0|\varepsilon(p - p_0),$$

where $\varepsilon(p - p_0) \to 0$ as $p \to p_0$. We call the collection $\partial f(p_0)$ of all subgradients ξ of f on p_0 to be its subdifferential, that is

$$\partial f(p_0) := \{\xi \in \mathbb{R}^n : f(p) \geq f(p_0) + \langle \xi, p - p_0 \rangle \text{ for every } \xi \in U\},$$

and for a set $E \subset \mathbb{R}^n$ we write

$$\partial f(E) := \bigcup_{p \in E} \partial f(p).$$

The geometric interpretation for the mappings $p \mapsto u(p_0) + \langle \xi, p - p_0 \rangle$ is that they define tangential planes for the graph of f i.e. they touch the graph of f from below at $(p_0, f(p_0))$. We call this kind of plane a supporting plane to f at p_0 and it comes as a direct consequence from (3).

Definition 2.4. Let U be a convex set and $E \subset U$ a Borel set. Let $f : U \to \mathbb{R}$ be a function. The Monge-Ampère measure of f is given by

$$\mu_f(E) := \big|\partial f(E)\big|,$$

where $|\cdot|$ is the Lebesgue measure.

Lemma 2.5. *Let f and g be convex functions in $\mathbb{R}^n$ and E be an open and bounded set so that $f = g$ on ∂E and $f \leq g$ in E. Then $\partial f(E) \supset \partial g(E)$ and $\mu_f(E) \geq \mu_g(E)$.*

Proof. Let $\xi \in \partial g(E)$ that is $\xi \in \partial g(p_0)$ for some $p_0 \in E$. Then the mapping

$$p \mapsto g(p_0) + \langle \xi, p - p_0 \rangle$$

defines a supporting plane to g at p_0. Note that we have $f \leq g$ in E. Let us the lower the supporting plane to g at p_0 until it's below the graph of f and then lift it upwards until it touches the graph of f again. Clearly we now have a constant $a \leq g(p_0)$ such that the mapping

$$p \mapsto a + \langle \xi, p - p_0 \rangle$$

defines a supporting plane for f at some point $p' \in \overline{E}$. Moreover we note that the (strictly) convex graphs of f and g are homeomorphic a $n+1$-dimensional hypercones. Hence the supporting plane to g touches the graph of f at the point which maps to the vertex of the cone. Would the functions f or g have multiple minima, we could map the equivalence class of these to the vertex of the cone and then the interpretation would be the same.

At the boundary ∂E we have $f = g$ and therefore $a = g(p_0)$ for a point $p' \in \partial E$. Furthermore we then have that $f(p_0) = g(p_0)$ and the plane is a supporting plane to f at p_0. From this we obtain that $\xi \in \partial f(p_0)$ also and from the additivity of measure it follows that $\mu_f(E) \geq \mu_g(E)$. $\square$

As a main result of this section we provide the following so called comparison principle. It will be used to study the behaviour of the gradient of the surface tension near the border of the polygon N.

Theorem 2.6. *Let f and g be convex functions defined on an open bounded convex set $U \subset \mathbb{R}^n$. If $f \geq g$ on ∂U and in the sense of Monge-Ampère measures*

$$\det D^2 f \leq \det D^2 g$$

in U, then $f \geq g$ in U.

Proof. Let $x_0 \in U$ and define new function $\tilde{g}$ by setting

$$\tilde{g}(x) := g(x) + \varepsilon \left(|x - x_0|^2 - \operatorname{diam}(U)^2 \right).$$

We know that convex functions are positive semi-definite and therefore their determinants are positive or zero. Moreover we know that the Euclidean distance squared is a convex function. Hence

$$\det D^2 \tilde{g} > \det D^2 f$$

and then let $\varepsilon \to 0$. Let us assume that the set $E = \{x \in U : f(x) < \tilde{g}(x)\}$ is not empty. Then Lemma 2.5 gives us

$$\mu_f(E) \geq \mu_{\tilde{g}}(E) \implies \det D^2 f \geq \det D^2 \tilde{g},$$

but this is a contradiction and therefore the set E must be empty. Let $\varepsilon \to 0$ and the claim follows. □

2.3 Harmonic Measure

Let $a, b \in \mathbb{R}$ be real numbers such that $a < b$ and $z \in \mathbb{C}$ be a point on the upper half of the complex plane $\mathbb{H} = \{z \in \mathbb{C} : \mathfrak{Im} z > 0\}$. We recall from [4, p. 1] that the angle at point z of the triangle with endpoints $\{a, z, b\}$ is given by the function

$$\theta(z) := \arg\left(\frac{z-b}{z-a}\right) = \mathfrak{Im} \log\left(\frac{z-b}{z-a}\right). \tag{4}$$

Also from [9, p. 60] we recall that the complex logarithm is defined as

$$\log z = \log|z| + i \arg z,$$

where $z \in \mathbb{C}$. From this we derive that

$$\mathfrak{Im} \log z = -i \log \frac{z}{|z|}.$$

With fixed points $a, b \in \mathbb{R}$ and an angle θ_0, the equation $\theta(z) = \theta_0$ defines a circular arc with end points a and b. We can generalize this by replacing the real numbers a and b with complex numbers $\alpha, \beta \in \mathbb{C}$.

Lemma 2.7. *Let $z, \alpha, \beta \in \mathbb{C}$. The equation*

$$\theta(z) := \arg\left(\frac{z-\alpha}{z-\beta}\right) = \theta_0,$$

where $-\pi < \theta_0 \leq \pi$ is the angle at z, defines a circular arc through z with endpoints α and β.

Proof. Clearly we can draw a line that connects α and β with each other. Then with a suitable rotation ρ we can map this line to the real line after which the circular arc is given by (4) at the point $\rho(z)$. $\square$

Let the set $E = \bigcup_{k=1}^{n} (a_k, b_k)$, where $a_k, b_k \in \mathbb{R}$ for all $k \leq n$, be a finite union of open intervals in the real line so that $b_{k-1} < a_k < b_k$. Then for each interval we set a corresponding function

$$\theta_k(z) := \arg\left(\frac{z - b_k}{z - a_k}\right)$$

and say that the function

$$\omega(z, E, \mathbb{H}) := \frac{1}{\pi} \sum_{k=1}^{n} \theta_k(z)$$

is the harmonic measure of E at point $z \in \mathbb{H}$. To define the harmonic measure on the unit disc we need a conformal map $\phi : \mathbb{H} \to \mathbb{D}$, which maps the upper half-plane to the unit disc. Assume that $I \subset \partial\mathbb{D}$ is a finite set of arcs in the unit circle. We may now define the harmonic measure of I at $z \in \mathbb{D}$ by setting

$$\omega(z, I, \mathbb{D}) = \omega(\phi(z), \phi(I), \mathbb{H}). \tag{5}$$

In $(*)$ we had a boundary condition for the surface tension in the convex polygon N. Similarly we can try to find a function in the unit disc when we have defined what values it should have on the border of the unit disc. Assume the function on the border is continuous and we call this a Dirichlet problem.

Theorem 2.8. *Let $f(e^{i\theta})$ be an integrable function on $\partial\mathbb{D}$ and set*

$$u_f(z) := \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \frac{1 - |z|^2}{\left|e^{i\theta} - z\right|^2} d\theta.$$

Then $u_f(z)$ is harmonic on $\mathbb{D}$. Moreover if f is continuous on $e^{i\theta_0} \in \partial\mathbb{D}$, then $u_f(z) \to f(e^{i\theta_0})$, when $\mathbb{D} \ni z \to e^{i\theta_0}$.

We say that the function $u_f(z)$ is a Poisson integral of f and

$$P(z, \theta) := \frac{1 - |z|^2}{\left|e^{i\theta} - z\right|^2}$$

is called the Poisson kernel.

We recall that a function ϕ is conformal in an open set $U \subset \mathbb{C}$ if it is holomorphic and has a nonzero derivative in U. Let us take a function $\phi(z) := \frac{z-i}{z+i} : \mathbb{D} \to \mathbb{H}$. Now we can calculate $\partial_{\overline{z}}\phi(z) = 0$ and $\partial_z \phi(z) \neq 0$ in the open unit disc. Hence it is conformal. Moreover from (5) we can derive

$$\omega(a+bi, E, \mathbb{D}) = \omega(\phi(a+bi), \phi(E), \mathbb{H}) = \frac{1}{\pi}\int_E \frac{b}{(x-a)^2+b^2}dx \tag{6}$$

as stated in [4, p. 4]. Here the integrand is said to be the Poisson kernel in the upper half-plane, since by the change of variables, given by ϕ, it coincides with $P(z,\theta)$.

Proof of Theorem 2.8. Let $z \in \mathbb{D}$ be a point in the unit disc and $\zeta = e^{i\theta}$, where $0 \leq \theta < 2\pi$, be a point in the boundary. Clearly $z = 2\mathfrak{Re}(z) - \overline{z}$ it follows that

$$\mathfrak{Re}\left(\frac{\zeta+z}{\zeta-z}\right) = \frac{\zeta\overline{\zeta} - z\overline{z}}{(\zeta-z)(\overline{\zeta}-\overline{z})}. \tag{7}$$

In the unit disc we can write $z = re^{i\varphi}$, where $0 \leq r < 1$ and $0 \leq \varphi < 2\pi$, and using this we obtain

$$\mathfrak{Re}\left(\frac{e^{i(\theta-\varphi)}+r}{e^{i(\theta-\varphi)}-r}\right) = \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}. \tag{8}$$

Moreover remembering that $2\cos\theta = e^{i\theta} + e^{-i\theta}$ we realize the equality (8) is in fact the Poisson kernel and that the denominators in the equalities (7) and (8) are the same. Thus also (7) gives a reformulation for the Poisson kernel.

Let $f : U \to \mathbb{C}$ be a holomorphic function in an open domain $U \subset \mathbb{C}$. From [8, p. 120] we recall the Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta, \tag{9}$$

where γ is a contour such that the curve γ and the (interior) set confined by it are contained in the domain of f. Let us now consider the function

$$g(\zeta) := \frac{\overline{z}f(\zeta)}{1-\overline{z}\zeta},$$

where f is holomorphic, defined in the unit disc. In Example 2.1 we showed that it is holomorphic. Let γ be a convex contour and then by Cauchy's theorem, given in [8, p. 107], we have that the integral of g over this contour equals to zero. Clearly we can set the contour to be the boundary of the unit circle. Combining this with (9) we obtain the Poisson formula by using the change of variables and remembering (7) since

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z} + \frac{\overline{z}f(\zeta)}{1-\overline{z}\zeta}d\zeta \\ &= \frac{1}{2\pi i}\int_{\partial\mathbb{D}} f(\zeta)\left(\frac{1-\overline{z}\zeta+\overline{z}(\zeta - z)}{(\zeta - z)(\overline{\zeta}\zeta - \overline{z}\zeta)}\right)d\zeta \\ &= \frac{1}{2\pi i}\int_0^{2\pi} f(e^{i\theta})\left(\frac{1-|z|^2}{(e^{i\theta}-z)(\overline{e^{i\theta}}-\overline{z})e^{i\theta}}\right)\frac{d}{d\theta}e^{i\theta}d\theta. \end{aligned}$$

□

Making a change of variables, given by the inverse mapping of $\phi(z) = \frac{z-i}{z+i}$, to the integral in (6) we obtain

$$\omega(z, E, \mathbb{D}) = \frac{1}{2\pi}\int_E P(z,\theta)d\theta. \tag{10}$$

From this we may then derive the following presentation for the harmonic measure in the unit disc.

Theorem 2.9. *Let $z \in \mathbb{D}$ and $I = (e^{i\theta_1}, e^{i\theta_2})$ be an arc on $\partial\mathbb{D}$. Then the harmonic measure of I is given by*

$$\omega(z, I, \mathbb{D}) = \frac{1}{\pi}\arg\left(\frac{z-e^{i\theta_1}}{z-e^{i\theta_2}}\right) - \frac{\theta_2-\theta_1}{2\pi}.$$

Proof. For the Poisson kernel it stands that

$$\begin{aligned} \mathfrak{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) &= \mathfrak{Re}\left(\frac{2e^{i\theta}}{e^{i\theta}-z} - 1\right) \\ &= -2\frac{d}{d\theta}\mathfrak{Re}\left(i\log\left(e^{i\theta}-z\right)\right) - 1 \\ &= 2\frac{d}{d\theta}\arg\left(e^{i\theta}-z\right) - 1. \end{aligned}$$

Integrating this statement over an arc (θ_1, θ_2) in respect to θ and remembering the relation between the harmonic measure and the Poisson kernel given in (10) we obtain the claim. □

As a last remark let us consider the setting of Theorem 2.9. We can draw lines from the endpoints of the arc $I = (e^{i\theta_1}, e^{i\theta_2})$ to z and extend them so that they intersect the boundary of the unit circle at some points $e^{i\theta_1'}$ and $e^{i\theta_2'}$ forming the arc I' respectively. We also have the well known fact from elementary geometry that

$$\angle AZB = \frac{1}{2}\left(\angle AOB + \angle A'OB'\right),$$

where A and B are the endpoints of I, $Z = z$, A' and B' are the endpoints of I' and O is the center of the circle. In our case $\angle AOB = \theta_2 - \theta_1$ and the angle $\angle AZB$ is given by the argument of $(z - e^{i\theta_1})/(z - e^{i\theta_2})$. Hence we see that the geometric interpretation of the harmonic measure is given by

$$2\pi\omega(z, I, \mathbb{D}) = \angle A'OB'.$$

3 Radial Limits in the Unit Disc

Definition 3.1. Let M and N be connected and oriented smooth manifolds. The diffeomorphism $f : M \to N$ is called orientation preserving if the induced orientation in N is the same as in M.

In the opposite case we would call f orientation reversing. It can be shown that this is equivalent for the Jacobian determinant of f, write J_f, being positive. For the Jacobian determinant of a composition mapping $f \circ g$ at point x we have the identity

$$J_{f\circ g}(x) = J_f(g(x))J_g(x).$$

From this it is easy to see that the composition of orientation preserving mappings is also orientation preserving. In addition the inverse function theorem gives that the inverse of an orientation preserving map preserves orientation.

Moreover a nonsingular linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ is called orientation preserving if it has a nonnegative determinant. For having a negative determinant we would call it an orientation reversing. From the properties of determinant we have again that the composition of two orientation preserving mappings is again an orientation preserving map and that the inverse of an orientation preserving map is also orientation preserving.

Before proving the main result of this section we recall from [4, p. 469] the definitions for cluster sets of function in respect to a point and a set.

Definition 3.2. Let $f : U \to \mathbb{C}$ be an analytic function and $(z_n)_{n=1}^{\infty} \subset U$ a sequence with a limit $z_0 \in \partial U$. We call the collection

$$\mathrm{Cl}(f, z_0) := \{\zeta : \lim_{n\to\infty} f(z_n) = \zeta\}$$

to be the cluster set of f at z_0. Moreover the cluster set of f at $I \subset \partial U$ is

$$\mathrm{Cl}(f, I) := \bigcup_{z_0 \in I} \mathrm{Cl}(f, z_0).$$

Theorem 3.3. *Let D be a bounded and simply-connected domain which has locally connected boundary. Let $f : \mathbb{D} \to D$ be a univalent, harmonic and orientation-preserving mapping which radial limits $\lim_{r\to1} f(re^{i\theta})$ belong to ∂D for almost every θ. Then there exists a countable set $E \subset \partial\mathbb{D}$ such that*

1. *the unrestricted limit $\hat{f}(e^{i\theta}) := \lim_{z\to e^{i\theta}} f(z)$ exists, is continuous and belongs to ∂D for $e^{i\theta} \in \partial\mathbb{D} \setminus E$,*
2. *$\lim_{\theta\to\theta_0^-} \hat{f}(e^{i\theta})$ and $\lim_{\theta\to\theta_0^+} \hat{f}(e^{i\theta})$ exist, are different and belong to D for $e^{i\theta_0} \in E$ and*
3. *the cluster set of the mapping f at $e^{i\theta_0} \in E$ is the straight line segment joining $\lim_{\theta\to\theta_0^-} \hat{f}(e^{i\theta})$ to $\lim_{\theta\to\theta_0^+} \hat{f}(e^{i\theta})$.*

Proof. Let $\phi : \mathbb{D} \to D$ be a univalent, analytic and orientation preserving mapping. It can be shown that such a function has a continuous extension such that $\phi(\partial\mathbb{D}) = \partial D$ as stated in [6, p. 479]. Moreover the function $\phi^{-1} \circ f : \mathbb{D} \to \mathbb{D}$ is an orientation-preserving homeomorphism. Let $h(z) := \lim_{r\to1} \phi^{-1} \circ f(rz)$ be the radial-limit function of $\phi^{-1} \circ f$. Clearly the radial-limit function has modulus one almost everywhere at $\partial\mathbb{D}$.

Through some redefining on a set of measure zero we can write that $h(e^{i\theta}) = e^{i\eta(\theta)}$, where η is a suitable non-descreasing function on the real line for which $\eta(\theta + 2\pi) = \eta(\theta) + 2\pi$. The function η has countably many points of discontinuities $e^{i\theta} \in \partial\mathbb{D}$, set of which we denote by E_0. Thus on $\partial\mathbb{D} \setminus E_0$ the function $\phi \circ h : \mathbb{R} \to \partial D$ becomes continuous and in E_0 and it has one-sided limits belonging to ∂D since the boundary is a closed set. Therefore the function $\phi \circ h$ and the radial limit function of f agree a.e. on ∂D. Thus by the Theorem 2.8 we get that

$$\lim_{r\to1} f(rz) = \frac{1}{2\pi} \int_0^{2\pi} \phi \circ h(e^{i\theta}) P(z, \theta) d\theta$$

almost everywhere. Furthermore we obtain

$$\hat{f}(e^{i\theta}) = \lim_{r\to1} f(re^{i\theta}) = \phi \circ h(e^{i\theta})$$

for all $z \in \partial\mathbb{D} \setminus E_0$. This proves the claim 1. and also 2. up to the difference of the limits.

Let $e^{i\theta_0} \in E_0$ and $-1 < \lambda < 1$. Now the equation

$$\arg\left(\frac{e^{i\theta_0} + z}{e^{i\theta_0} - z}\right) = \frac{\lambda\pi}{2}$$

gives us a circular arc as in Lemma 2.7. Its endpoints are $-z$ and z and it goes through $e^{i\theta_0}$. We then let z approach $e^{i\theta_0}$ on the circular arc. Now $f(z)$ converges to

$$\frac{1}{2}(1-\lambda)\lim_{\theta\to\theta_0^+}\hat{f}(e^{i\theta}) + \frac{1}{2}(1+\lambda)\lim_{\theta\to\theta_0^-}\hat{f}(e^{i\theta}),$$

which is a line-segment connecting the one-sided limit points. It is also the cluster set of f at θ_0, which proves the claim 3. in the set E_0. Let E_1 be the set of points for which the one-sided limits of $\hat{f}(e^{i\theta})$ are equal. Now the cluster set of f at θ_0 becomes a singleton and therefore f has a limit and $\hat{f}$ is continuous there as the one-sided limits coincide. Thus the claim holds for the set $E = E_0 \setminus E_1$. $\square$

4 Non-Linear Cayley Transform

Definition 4.1. Let $\sigma(z)$ be as in $(*)$ and $z \in N^\circ$. Its Lewy transform is given by

$$L_\sigma(z) := z + \nabla\sigma(z).$$

Clearly the Lewy transform L_σ is continuous as the identity mapping and the derivatives of the surface tension are. The surface tension is strictly convex in N°. Hence with the Theorem 1 from [5] we have that the Lewy transform of the surface tension is a homeomorphism.

Definition 4.2. Let $\sigma(z)$ be as in $(*)$. Its non-linear Cayley transform is given by

$$\mathcal{H}_\sigma(w) := (I - \nabla\sigma)(I + \nabla\sigma)^{-1}(\overline{w}),$$

where I is the identity matrix and $\overline{w}$ is chosen so that $\mathcal{H}$ is analytic.

Let $w = z + \nabla\sigma(z)$ be a point in $L_\sigma(N^\circ)$. We have that $(I + \nabla\sigma)^{-1}w = z$ and thus the non-linear Cayley transform is given by

$$\mathcal{H}_\sigma(\overline{w}) = z - \nabla\sigma(z).$$

Note that from the equality

$$\left(2I - (I + \nabla\sigma)\right)(I + \nabla\sigma)^{-1} = 2(I + \nabla\sigma)^{-1} - I$$

it follows that

$$\mathcal{H}_\sigma(\overline{w}) = 2(I + \nabla\sigma)^{-1}(w) - w = 2z - w.$$

Substituting now $z = w - \nabla\sigma(z)$ we get an expression for the gradient of the surface tension that is

$$\nabla\sigma(z) = \frac{1}{2}\left(w - \mathcal{H}(\overline{w})\right). \tag{11}$$

Solving the variable z from this we get the inverse Lewy transform, that is

$$L_\sigma^{-1}(w) = z = \frac{1}{2}\left(w + \mathcal{H}_\sigma(\overline{w})\right). \tag{12}$$

Theorem 4.3. *Let $\sigma(z)$ be a solution to $(*)$. Then the non-linear Cayley transform $\mathcal{H}_\sigma$ is complex analytic in its domain*

$$\operatorname{Dom}\mathcal{H}_\sigma := \{\overline{w} : w = L_\sigma(z), z \in N^\circ\},$$

with the derivative $\left|\mathcal{H}_\sigma'(w)\right| < 1$ for all $w \in \operatorname{Dom}\mathcal{H}_\sigma$.

Proof. Let h and b be continuously differentiable functions and let $a = h \circ b$. The chain rules for Wirtinger derivatives give us then

$$a_z\overline{b_z} - a_{\overline{z}}\overline{b_{\overline{z}}} = (h_\zeta \circ b)b_z\overline{b_z} + (h_{\overline{\zeta}} \circ b)\overline{b}_z\overline{b_z} - (h_\zeta \circ b)b_{\overline{z}}\overline{b_{\overline{z}}} - (h_{\overline{\zeta}} \circ b)\overline{b}_{\overline{z}}\overline{b_{\overline{z}}}.$$

We remember that $b\overline{b} = |b|^2$. and by conjugation rules for Wirtinger derivatives we obtain

$$a_z\overline{b_z} - a_{\overline{z}}\overline{b_{\overline{z}}} = (h_\zeta \circ b)\left(|b_z|^2 - |b_{\overline{z}}|^2\right) \tag{13}$$

and also

$$a_{\overline{z}}b_z - a_z b_{\overline{z}} = (h_{\overline{\zeta}} \circ b)\left(|b_z|^2 - |b_{\overline{z}}|^2\right). \tag{14}$$

Let us now choose our functions to be $h(\zeta) = (I - \nabla\sigma)(I + \nabla\sigma)^{-1}(\zeta)$ and $b(z) = z + 2\sigma_{\overline{z}}$. It follows from the definition of Wirtinger derivatives that $2\sigma_{\overline{z}} = \nabla\sigma(z)$. Hence it stands that $a(z) = z - \nabla\sigma(z)$ and $b(z) = L_\sigma(z)$.

We calculate that

$$a_z = 1 - \frac{1}{2}\left(\sigma_{xx} + \sigma_{yy}\right) = 1 - \frac{1}{2}\Delta\sigma,$$

$$b_z = 1 + \frac{1}{2}\Delta\sigma$$

and

$$a_{\overline{z}} = -\frac{1}{2}\left(\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy}\right) = -b_{\overline{z}}.$$

Substituting these to (13) gives us

$$a_z\overline{b_z} - a_{\overline{z}}\overline{b_{\overline{z}}} = a_z b_z + |b_{\overline{z}}|^2 = 1 - \sigma_{xx}\sigma_{yy} + \sigma_{xy}^2 = 1 - \det D^2\sigma$$

and

$$\begin{aligned}|b_z|^2 - |b_{\overline{z}}|^2 &= \left(1 + \frac{1}{2}\Delta\sigma\right)^2 - \left(\frac{1}{4}\left(\sigma_{xx} - \sigma_{yy}\right)^2 + \sigma_{xy}^2\right)\\ &= 1 + \Delta\sigma + \sigma_{xx}\sigma_{yy} - \sigma_{xy}^2\\ &= 2 + \Delta\sigma,\end{aligned}$$

since from $(*)$ we have $\det D^2\sigma = 1$. In addition we note that

$$\det DL_\sigma = 1 + \sigma_{yy} + \sigma_{xx} + \sigma_{xx}\sigma_{yy} - \sigma_{xy}^2 = 2 + \Delta\sigma.$$

Thus we have

$$1 - \det D^2\sigma = (h_\zeta \circ b)\det DL_\sigma$$

from the equation (13). We note that in N° we have $\det DL_\sigma \geq 4$. Thus $\mathcal{H}_\sigma$ is complex analytic since the limit of the difference quotient becomes now finite. The equation (14) gives us

$$\left|h_{\overline{\zeta}}\right|^2 = \frac{\left|a_{\overline{z}}(b_z + a_z)\right|^2}{\det(DL_\sigma)^2} = \frac{(\sigma_{xx} - \sigma_{yy})^2 + 2\sigma_{xy}^2}{(\Delta\sigma + 2)^2} = \frac{(\Delta\sigma)^2 - 4}{(\Delta\sigma + 2)^2} \leq 1$$

and from definition of derivative it now follows that $\mathcal{H}_\sigma$ is a contraction. Furthermore $\mathcal{H}'_\sigma$ takes its values from the open unit disc and attains its maximum on the boundary due to maximum principle. □

5 Surface Tension

5.1 Explicit Form

Before we can present the explicit formulation for the gradient of the surface tension, derived in the Theorem 5.2 and presented in the Corollary 5.3, we need the following lemma.

Lemma 5.1. *Let σ be the unique bounded and convex function that solves $(*)$ and suppose $J \subset \partial N$ is a closed interval that does not containing any points of $\mathscr{P} \cup \mathscr{Q}$. Then for $p \in N^\circ$ we have*

$$\left|\nabla\sigma(p)\right| \to \infty \text{ as } p \to J.$$

Proof. With using some suitable transforms (rotations and translations) we can assume that $0 \in J \subset \mathbb{R}$ and that the interior of the polygon N lies in the upper half of the complex plane. Let us then take an isosceles triangle T with corners $\{0, x_1, ix_1\}$, where the point $0 < x_1 \in J$ is chosen to be so small that the triangle T is contained in N°. With more transforms (namely by adding linear map and making an affine change of coordinates) we may assume that $x_1 = 1$.

Now we have that $L = \sigma$ in the interval $[0, 1]$ and $L(i) = \sigma(i)$. Moreover the convexity of the surface tension σ means that

$$\sigma\big(tx + (1-t)y\big) \leq t\sigma(x) + (1-t)\sigma(y),$$

where $0 \leq t \leq 1$, for every $x, y \in T$. Thus we get $\sigma(x) \leq L(x)$ in the triangle T by remembering that $0 \leq x, y \leq 1$. This setting is called a lozenge tilling model, described in [1, p.17–18], and here we have a surface tension $\sigma_L := \sigma + L$ for which $\det\big(D^2(\sigma_L)\big) = 1$ as stated in $(*)$. Let $p \in T$ be a point in the triangle T and then Theorem 2.6 gives us

$$\sigma(p) \leq \sigma_L(p). \tag{15}$$

In [10, p. 34-36] an explicit form for the surface tension σ_L in the triangle T is derived to be

$$\sigma_L(s,t) = -\frac{1}{\pi}\left(\mathcal{L}(\pi s) + \mathcal{L}(\pi t) + \mathcal{L}(\pi(1-s-t))\right),$$

where

$$\mathcal{L}(\theta) := -\int_0^\theta \log|2\sin x| dx$$

is the Lobachevsky function. Since $0 \leq s \leq 1$ we may ignore the absolute value in the logarithm. Recalling the Leibnitz integral rule

$$\partial_s \int_{a(s)}^{b(s)} f(s,x)dx = f(s,b(s))\partial_s b(s) - f(s,a(s))\partial_s a(s) + \int_{a(s)}^{b(s)} \partial_s f(s,x)dx$$

we can calculate the gradient of the surface tension. Note that when using the Leibnitz integration rule for the Lobachevsky function the middle and last term vanish. Hence we get

$$\begin{aligned}\partial_s \sigma_L(s,t) &= -\frac{1}{\pi}\partial_s\left(\mathcal{L}(\pi s) + \mathcal{L}(\pi(1-s-t))\right) \\ &= \log\big(2\sin(\pi s)\big) - \log\Big(2\sin\big(\pi(1-s-t)\big)\Big)\end{aligned}$$

and therefore via elementary properties of sine and logarithm we have

$$\nabla\sigma_L(s,t) = \left(\log\frac{\sin(\pi s)}{\sin\bigl(\pi(s+t)\bigr)}, \log\frac{\sin(\pi t)}{\sin\bigl(\pi(s+t)\bigr)}\right),$$

where $(s,t)\in T$. Furthermore we can calculate the second derivatives

$$\partial_{st}\sigma(s,t) = -\frac{1}{\tan\bigl(\pi(s+t)\bigr)}$$

and

$$\partial_{ss}\sigma(s,t) = \frac{\pi\sin(\pi t)}{\sin(\pi s)\sin\bigl(\pi(s+t)\bigr)} \leq -\frac{\pi}{\sin(\pi s)\sin\bigl(\pi(s+t)\bigr)}.$$

From this it is easy to see that for all second derivatives we have that $\partial_2\sigma(s,t)\to-\infty$, when $(s,t)\to[\delta,1-\delta]\subset J$, where $0<\delta<1$. Since the surface tensions σ and σ_L have the same boundary value on the interval $[0,1]$ i.e the function L, with (15) this forces also $\bigl|\nabla\sigma(p)\bigr|\to\infty$ as $p\to[\delta,1-\delta]$. We finish the proof by covering J with such subintervals. $\square$

Theorem 5.2. *Let $N\subset\mathbb{R}^2$ be a convex polygon and $\sigma(z)$ a convex solution to $\det\bigl(D^2\sigma\bigr)=1$ in the interior N°. If $\psi:\mathbb{D}\to L_\sigma(N^\circ)$ is a Riemann map, then the mapping $U(\zeta) := L_\sigma^{-1}\circ\psi(\zeta)$ defines a harmonic homeomorphism $U:\mathbb{D}\to N^\circ$ and it has a representation*

$$U(\zeta) = \sum_{j=1}^{m} p_j\omega(\zeta; I_j), \tag{16}$$

where $\zeta\in\mathbb{D}$, $I_j\subset\partial\mathbb{D}$ are pairwise disjoint open arcs whose closure covers the unit circle and where $\omega(\zeta;I_j)$ is the harmonic measure of an arc I_j in the unit disc.

Proof. Using the inverse Lewy transform given in (12) we get that

$$U(\zeta) = \frac{\psi(\zeta)+\mathcal{H}\left(\overline{\psi(\zeta)}\right)}{2} \tag{17}$$

and also

$$\nabla\sigma\circ U(\zeta) = \frac{\psi(\zeta)-\mathcal{H}\left(\overline{\psi(\zeta)}\right)}{2} \tag{18}$$

as we denote $w = L_\sigma$ in (11) and remember that $U = L_\sigma^{-1}\circ\psi$.

Let us denote $w := \overline{\psi(\zeta)}$. We calculate that

$$\begin{aligned}\partial_{\bar{\zeta}\zeta}U(\zeta) &= \frac{1}{2}\partial_{\bar{\zeta}}\left(\partial_\zeta\psi(\zeta) + i\partial_\zeta\mathcal{H}\left(\overline{\psi(\zeta)}\right)\right)\\ &= \frac{1}{2}\partial_{\bar{\zeta}}\left(\partial_\zeta\psi(\zeta) + i(\partial_w\mathcal{H}(w)\partial_\zeta\overline{\psi(\zeta)}) + \partial_{\overline{w}}\mathcal{H}(w)\partial_\zeta\psi(\zeta))\right).\end{aligned}$$

Lemma 4.3 gives us that $\partial_{\overline{w}}\mathcal{H}(w) = 0$. Also with conjugation rules we get $\partial_\zeta\overline{\psi(\zeta)} = 0$ as the Riemann map ψ is holomorphic. Finally we may change the order of the derivatives for ψ and see that $\partial_{\bar{\zeta}\zeta}U(\zeta)$ equals to zero. Moreover we may calculate that $\partial_{\zeta\bar{\zeta}}U(\zeta)$ is also equal to zero and then the Definition **??** gives that the function U is harmonic.

As a composition of bijective mappings the mapping U is bijective. Moreover its inverse is given by $\psi^{-1} \circ L_\sigma(p)$, where $p \in N^\circ$. Since Lewy transform is a homeomorphism and a Riemann map is holomorphic with holomorphic inverse we know that the mapping U is continuous and so is its inverse. Thus U is homeomorphism.

The unit disc is clearly bounded and simply connected and its boundary is locally connected. Moreover the mapping U has a positive Jacobian and therefore it is orientation-preserving. We may now use Theorem 3.3 and thus for each $\zeta \in \mathbb{D}$ we get unrestricted limits

$$U(e^{ix}) := \lim_{\zeta \to e^{ix}} U(\zeta) \in \partial N,$$

where $e^{ix} \notin E$, and for $e^{ix} \in E$ the cluster set of U is a non-degenerate segment $J \subset \partial N$. Moreover on the complement set of the set E we have that the mapping $e^{ix} \mapsto U(e^{ix})$ is continuous with positive Jacobian.

Let $U(e^{ix_0}) \in \partial N \setminus (\mathscr{P} \cup \mathscr{Q})$ for some point $e^{ix_0} \in \partial\mathbb{D} \setminus E$. Since the function U is continuous we can use Lemma 5.1 to derive that

$$\nabla\sigma \circ U(re^{ix}) \to \infty$$

as $r \to 1$, whenever the distance $|x - x_0|$ is small and $e^{ix} \notin E$. But this isn't possible since the function U is bounded as N° is bounded. Thus it has almost everywhere finite radial boundary values as it results from Theorem 3.3. Moreover all of the unrestricted limits of U at points outside of E must be contained in the set $\mathscr{P} \cup \mathscr{Q}$.

The boundary function U is orientation preserving and therefore piecewise constant. Furthermore Theorem 2.8 and identity (10) gives it a presentation

$$U(\zeta) = \sum_{p_j \in \mathscr{P} \cup \mathscr{Q}} p_{k_j}\omega(\zeta; I_j), \tag{19}$$

where the intervals $I_j \subset \partial\mathbb{D}$ covering the unit circle have pairwise disjoint interiors. On the other hand we have that the cluster set of U at any point in E is a line segment on ∂N. Hence each corner of N appears in the sum (19). We need to show that also every quasifrozen point appears in it and then the proof is finished.

From the Lemma 2.9 we recall that the harmonic measure of the counter clockwise oriented arc $I \subset \partial\mathbb{D}$ between boundary points $\eta_1, \eta_2 \in \partial\mathbb{D}$ is given by

$$\omega(\zeta; I) = \frac{1}{\pi}\mathfrak{Im}\log\left(\frac{\zeta-\eta_2}{\zeta-\eta_1}\right) + c(I) \tag{20}$$

for a point $\zeta \in \mathbb{D}$. Combining this with the equality (19) and remembering that $z - \overline{z} = 2i\mathfrak{Im}z$ we get from (17) that

$$\frac{1}{2}\left(\psi(\zeta) + \mathcal{H}\left(\overline{\psi(\zeta)}\right)\right) = \frac{1}{2i\pi}\sum_{j=0}^{m-1} p_{k_j}\left(\log\left(\frac{\zeta-\eta_{j+1}}{\zeta-\eta_j}\right) - \overline{\log\left(\frac{\zeta-\eta_{j+1}}{\zeta-\eta_j}\right)}\right) + C,$$

where C is some constant. Since the logarithm is holomorphic its complex conjugate is antiholomorphic. Moreover we know that if a function $f(z)$ is holomorphic then $\overline{f(\overline{z})}$ is also. Hence $\mathcal{H}(\overline{w})$ is antiholomorphic. We know that decompositions to holomorphic and antiholomorphic parts as above coincide with the one given by (18). To be more exact the holomorphic parts coincide up to some constant c and antiholomorphic parts up to a constant $-c$. Now with $z + \overline{z} = 2\mathfrak{Re}z$ we get

$$\nabla\sigma \circ U(\zeta) = \frac{1}{i\pi}\sum_{j=0}^{m-1} p_{k_j}\log\left|\frac{\zeta-\eta_{j+1}}{\zeta-\eta_j}\right| + C.$$

Furthermore with $\eta_0 = \eta_m$ we get

$$\begin{aligned}
\nabla\sigma \circ U(\zeta) &= \frac{1}{i\pi}\sum_{j=0}^{m-1} p_{k_j}\Big(\log|\zeta-\eta_{j+1}| - \log|\zeta-\eta_j|\Big) + C \\
&= \frac{p_{k_1}}{i\pi}\log|\zeta-\eta_1| - \frac{p_{k_1}}{i\pi}\log|\zeta-\eta_0| + \frac{p_{k_2}}{i\pi}\log|\zeta-\eta_2| \\
&\quad - \frac{p_{k_2}}{i\pi}\log|\zeta-\eta_1| + \cdots \\
&= \cdots + \frac{p_{k_1}-p_{k_2}}{i\pi}\log|\zeta-\eta_1| + \frac{p_{k_2}-p_{k_3}}{i\pi}\log|\zeta-\eta_2| + \cdots \\
&= \frac{1}{\pi}\sum_{j=0}^{m-1} i\left(p_{k_j} - p_{k_{j+1}}\right)\log\frac{1}{|\zeta-\eta_j|} + C.
\end{aligned}$$

Let us fix an index $j = j_0$. Now the term $i(p_{k_{j0}} - p_{k_{j_0+1}})$ is an outer normal (i.e. the normal pointing outside of the set) for the polygon N on the side $\left[p_{k_{j_0}}, p_{k_{j_0+1}}\right] \subset \partial N$. The tangent on the side $\left[p_{k_{j_0}}, p_{k_{j_0+1}}\right]$ is $T := p_{k_{j_0}} - p_{k_{j_0+1}}$. Moreover the cluster set of the function U on a point $\zeta = \eta_{j_0}$ is the set $\left[p_{k_{j_0}}, p_{k_{j_0+1}}\right]$ since that is the line where the point η_{j_0} resides.

We may now calculate the derivative of the surface tension in the direction of the tangent T. By the definition of directed derivative [11, p. 49] we get

$$\begin{aligned}
(\partial_T \sigma) \circ U(\zeta) &= \langle p_{k_{j_0}} - p_{k_{j_0+1}}, \nabla\sigma \rangle \circ U(\zeta) \\
&= \langle p_{k_{j_0}} - p_{k_{j_0+1}}, \nabla\sigma \circ U(\zeta) \rangle \\
&= \frac{1}{\pi} \sum_{j \neq j_0} \langle p_{kj_0} - p_{kj_0+1}, i(p_{kj_0} - p_{kj_0+1}) \rangle \log \frac{1}{|\zeta - \eta_j|} + C,
\end{aligned}$$

where C is a constant. Notice that the term $j = j_0$ vanishes as the tangent T is perpendicular to the outer normal $i(p_{k_{j0}} - p_{k_{j_0+1}})$. The expression above shows that along the side (p_{kj_0}, p_{kj_0+1}) the tangential derivative of the surface tension σ is continuous since the logarithm is. This means that no quasifrozen point $p \in \mathscr{Q}$ can lie in such an open interval, since $\sigma = L$ is not differentiable on such points. Thus they all must be among the image points p_{k_j} in the sum (19). □

Corollary 5.3. *Suppose N, $\sigma(z)$ and $U(\zeta) := (L_\sigma^{-1} \circ \psi)(\zeta)$ with $\psi : \mathbb{D} \to L_\sigma(N^\circ)$ are as in Theorem 5.2. Then for $\zeta \in \mathbb{D}$ we have*

$$\nabla\sigma \circ U(\zeta) = \frac{1}{\pi} \sum_{j=1}^{m} i(p_j - p_{j+1}) \log \frac{1}{|\zeta - \eta_j|} + c_0,$$

where the arcs $\{I_j\}$ are as in Theorem 5.2, the $\{\eta_j\}$ are their endpoints and c_0 is a constant.

Proof. Appears naturally in the proof of Theorem 5.2. □

5.2 Behaviour Near the Boundary Points

Finally we study the behaviour of the gradient of the surface tension near the the corners and quasi-frozen points in the boundary of the polygon N.

Theorem 5.4. *Let $p_0 \in \mathscr{P} \cup \mathscr{Q}$ and σ a convex solution to $(*)$. Then for any point $p \in N^\circ$, $p \neq p_0$ there exists the finite limit*

$$\hat{\nabla}\sigma(p_0, p - p_0) := \lim_{\tau \to 0^+} \nabla\sigma\left(p_0 + \tau(p - p_0)\right).$$

Proof. Let $p_0 = p_j \in \mathscr{P} \cup \mathscr{Q}$. We're now able to use Theorem 5.2 and Corollary 5.3. Let $U : \mathbb{D} \to N^\circ$ be a harmonic homeomorphism in (16) and combining it with (20) we get

$$U(\eta) = \frac{1}{\pi} \sum_{j=1}^{m} p_j \mathfrak{Im} \log \left(\frac{\eta - \eta_{j+1}}{\eta - \eta_j} \right) + p_j c(I_j),$$

where $\eta \in I_j = (\eta_{j-1}, \eta_j)$. Which outside the endpoints of intervals $\{I_j\}$ extends smoothly on the unit circle as the logarithm there exists. Moreover its radial derivative is given by

$$\partial_r U(\eta) = \frac{1}{\pi} \sum_{j=1}^{m} (p_j - p_{j+1}) \mathfrak{Im} \left(\frac{\eta}{\eta - \eta_j} \right) \tag{21}$$

as stated in [1, p. 31]. From the definition of differentiability we get

$$U(r\eta) = p_j + \partial_r U(\eta)(r - 1) + o_\eta(r - 1) \text{ as } r \to 1$$

and see that U maps all points $\eta \in I_j$ to the point $p_j \in I_j$.

Let $\eta \to \eta_{j-1}$ in the arc $(\eta_{j-1}, \eta) \subset \partial\mathbb{D}$. Then we have that the direction of $\partial_r U(\eta)$ approaches the direction of the tangent $p_j - p_{j-1}$. Also if we let $\eta \to \eta_j$ in the arc (η_{j-1}, η_j), we get that $\partial_r U(\eta)$ goes to the same direction as the tangent $p_j - p_{j+1}$.

We remember that a Riemann map is holomorphic and from Theorem 4.3 that the non-linear Cayley transform is also. Therefore we see easily from (17) and (18) that the pair $(i\nabla\sigma \circ U, U)$ satisfies the Cauchy-Riemann equations. As a homeomorphism the function U is univalent and therefore its derivative in the open unit disc is not zero. Hence we get that

$$\partial_r U(\eta) \neq 0 \text{ for } \eta \in \partial\mathbb{D} \setminus \{\eta_j\}.$$

When we have a given point $p \in N^\circ$ and a direction $p - p_j$, we can find a point $\eta \in (\eta_{j-1}, \eta_j)$ on the arc such that the tangent to the curve $r \mapsto U(r\eta)$ at $r = 1$ has the direction $p - p_j$. Definition of the tangent T of U at point p_j gives us

$$T = U(p_j) + \partial_r U(p_j)(p - p_j) := p_j + \tau(p - p_j).$$

Letting $\tau \to 0^+$ we obtain

$$\lim_{\tau \to 0^+} \nabla\sigma(p_j + \tau(p - p_j)) = \lim_{r \to 1} \nabla\sigma \circ U(r\eta), \tag{22}$$

which by the Corollary 5.3 exists and is finite. $\square$

Corollary 5.5. *Let $p_j, p_{j+1} \in \mathscr{P} \cup \mathscr{Q}$ be neighbouring points in the order induced by the boundary ∂N and $\hat{p} \in (p_j, p_{j+1}) \subset \partial N$. Then for a point $p \in N^\circ$ in the interior the limits*

$$\lim_{p \to \hat{p}} \langle \hat{\nabla}\sigma(p_j, p - p_j), p - p_j \rangle = \sigma(\hat{p}) - \sigma(p_j) = \lim_{p \to \hat{p}} \langle \hat{\nabla}\sigma(p_j, p - p_j), \hat{p} - p_j \rangle$$

exist and are equal as stated.

Proof. From the expression given in the Corollary 5.3 we can disregard the term $j = k$ to obtain a function

$$(\delta_k \sigma)(\zeta) := \frac{1}{\pi} \sum_{j \neq k} i(p_j - p_{j+1}) \log \frac{1}{|\zeta - \eta_j|} + c_0,$$

where $\zeta \in \mathbb{D}$, which has a finite limite as $\zeta \to \eta_j$ in the closed unit disc. Moreover we note that the term $j = k$ is orthogonal to the side $[p_j, p_{j+1}]$ and thus gives inner product zero. The side is also the accumulation set of the function U at η_j and also we remember that the surface tension is affine on the side. Hence the limits

$$\lim_{\zeta \to \eta_j} \langle \nabla\sigma \circ U(\zeta), \hat{p} - p_j \rangle = \lim_{\zeta \to \eta_j} \langle (\delta_j \sigma)(\zeta), \hat{p} - p_j \rangle,$$

where $\hat{p} \in [p_j, p_{j+1}]$, exist. Since the gradient of the surface tension takes every point from the accumulation set $[p_j, p_{j+1}]$, the sum of inner products is the Riemann-Stieltjes integral for which we have that

$$\int_{p_j}^{\hat{p}} \nabla\sigma dU(\zeta) = \sigma(\hat{p}) - \sigma(p_j). \tag{23}$$

Now the equality (22) yields the claim for the right-hand side. For the left-hand side we just need to remember (22) and (23). We can calculate from (21) a quantitative estimate

$$\frac{\partial_r U(\eta)}{|\partial_r U(\eta)|} = \frac{p_j - p_{j+1}}{|p_j - p_{j+1}|} \Big(1 + \mathcal{O}(|\eta - \eta_j|) \Big),$$

when we let $\eta \to \eta_j$ in the arc (η_{j-1}, η_j), as descrided in [1, p. 31]. Then when the point p is near enough to the set $[p_j, p_{j+1}]$ we have

$$\operatorname{dist}(p, [p_j, p_{j+1}]) \leq d(p, \hat{p}) \leq |p_j - p_{j+1}| \leq \mathcal{O}(|\eta - \eta_j|),$$

which finishes the proof. □

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