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# Inverse problems for heat equation and space–time fractional diffusion equation with one measurement

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## Abstract

Given a connected compact Riemannian manifold  $(M, g)$  without boundary,  $\dim M \geq 2$ , we consider a space–time fractional diffusion equation with an interior source that is supported on an open subset  $V$  of the manifold. The time-fractional part of the equation is given by the Caputo derivative of order  $\alpha \in (0, 1]$ , and the space fractional part by  $(-\Delta_g)^\beta$ , where  $\beta \in (0, 1]$  and  $\Delta_g$  is the Laplace–Beltrami operator on the manifold. The case  $\alpha = \beta = 1$ , which corresponds to the standard heat equation on the manifold, is an important special case. We construct a specific source such that measuring the evolution of the corresponding solution on  $V$  determines the manifold up to a Riemannian isometry.

**Keywords:** inverse problem, space–time fractional diffusion equation, regularity, uniqueness

AMS subject classifications: 35R11, 35R30.

## 1 Introduction

### 1.1 Statement of the problem and main results

Throughout this paper,  $(M, g)$  will denote a connected compact smooth Riemannian manifold without boundary, with metric  $g$  and  $\dim M \geq 2$ , and  $V \subset M$  will be a nonempty open subset with smooth boundary. Also,

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$0 < \alpha \leq 1$  and  $0 < \beta \leq 1$  will be fixed parameters. An important special case that we also consider is the heat equation that corresponds to  $\alpha = \beta = 1$ . We note that to the knowledge of the authors, the main results (Theorems 1 and 3, and Proposition 2) are new also in this special case.

We consider the following space–time fractional diffusion equation:

$$\partial_t^\alpha u(x, t) + (-\Delta_g)^\beta u(x, t) = f(x, t), \quad (x, t) \in M \times (0, \infty), \quad (1a)$$

$$u(x, 0) = 0, \quad x \in M. \quad (1b)$$

Here the source term  $f$  is supported on  $V \times (0, T)$ , for some  $T > 0$ , and  $\partial_t^\alpha$  is the Caputo (also known as the Djrbashian–Caputo) fractional-derivative of order  $\alpha$ . For a smooth function  $y$  defined on  $[0, \infty)$ , the Caputo fractional-derivative is defined by  $\partial_t^1 y = y'$  for  $\alpha = 1$ , and

$$\partial_t^\alpha y(t) := \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - \tau)^{-\alpha} y'(\tau) d\tau \quad (t \geq 0, 0 < \alpha < 1), \quad (2)$$

where  $\Gamma$  is the Euler’s gamma function. In the second term of (1a),  $\Delta_g$  is the Laplace–Beltrami operator, and the fractional power is taken in the sense of functional calculus. The precise definition of  $\partial_t^\alpha$  and  $(-\Delta_g)^\beta$  can be found in Section 2.

We show that for a smooth compactly supported source  $f$  there exists a unique so-called strong solution  $u^f$  of (1). The local source-to-solution operator  $L_V$  is then defined as the operator

$$f \mapsto L_V f := u^f|_{V \times [0, \infty)}.$$

In this paper we consider an inverse problem for the space–time fractional diffusion equation (1), namely does  $L_V$  determine the manifold  $(M, g)$  uniquely? Note that the input  $f$  to the local source-to-solution operator  $L_V$ , i.e., the source term in equation (1a), is supported on  $V$ . Also, the value  $L_V f$ , i.e., the evolution of the solution  $u^f$  of (1), is observed only on  $V$ . Hence  $L_V$  is determined by information residing on  $V$  only.

We show that  $L_V$  indeed determines  $(M, g)$  up to an isometry. In fact, we show the stronger result that we do not need to know the operator  $L_V$  completely, but it is enough to know the value of  $L_V h$  on some nonempty time interval  $[0, T)$  with only one source  $h$ , provided the source  $h$  is chosen appropriately. Below,  $\text{cl}(V)$  denotes the closure of the set  $V$ .

**Theorem 1.** *Let  $(M, g)$  be a connected compact smooth Riemannian manifold without boundary, with metric  $g$  and  $\dim M \geq 2$ , let  $V \subset M$  be a nonempty open subset with smooth boundary and let  $T > 0$ . Then it is possible to construct a source  $h \in C_c^\infty((0, T); L^2(V))$  such that the data*

$$(V, u^h|_{V \times [0, T)})$$

determines the manifold  $(M, g)$  up to a Riemannian isometry. More precisely this means the following:

Let  $(\widetilde{M}, \widetilde{g})$  be another smooth, connected and compact Riemannian manifold without boundary, with metric  $\widetilde{g}$ , and let  $\widetilde{V} \subset \widetilde{M}$  be an open and nonempty set with smooth boundary. Then it is possible to construct a source  $h \in C_c^\infty((0, T); L^2(V))$  that has the following property: If there exists a diffeomorphism

$$\theta : \text{cl}(\widetilde{V}) \rightarrow \text{cl}(V)$$

such that the solutions  $u^h$  of (1) with source  $h$  and the solution  $\widetilde{u}^{\theta^*h}$  of the corresponding equation on  $(\widetilde{M}, \widetilde{g})$  with source  $\theta^*h$  satisfy

$$(\theta^*u^h)|_{\widetilde{V} \times [0, T]} = (\widetilde{u}^{\theta^*h})|_{\widetilde{V} \times [0, T]}, \quad (3)$$

then  $(M, g)$  and  $(\widetilde{M}, \widetilde{g})$  are Riemannian isometric.

*Remark 1.* The pullback  $\theta^*$  of the diffeomorphism  $\theta$  acts on an  $L^2(V)$ -valued function  $f$  by

$$(\theta^*f)(t) := \theta^*(f(t)).$$

Above, an open set with smooth boundary refers in local coordinates to definition given in [15, Appendix C.1].

*Remark 2.* An explicit expression for the source  $h$  is given in Definition 14.

In Section 3.2 we show that the local source-to-solution operator  $L_V$  is well-defined as an operator

$$L_V : C_c^2((0, \infty); L^2(V)) \rightarrow C^1([0, \infty); L^2(V)) \cap L^\infty([0, \infty); L^2(V)). \quad (4)$$

As in (4), instead of considering functions depending on both the space variable  $x \in M$  and the time variable  $t \in \mathbb{R}$ , it is convenient to consider them as functions of time  $t \in \mathbb{R}$  taking values in the Hilbert space of square-integrable functions on  $M$ . For the convenience of the reader, we review the necessary definitions and results of calculus of Hilbert space valued functions in the Appendix.

The proof of Theorem 1 consists of two parts. The first part is to show that the function  $L_V h|_{[0, T]} \in C^1([0, T]; L^2(V)) \cap L^\infty([0, T]; L^2(V))$  uniquely determines the operator  $L_V$ . The second part is to show that the operator  $L_V$  determines the manifold  $(M, g)$  up to a Riemannian isometry. These steps are formulated below as two independent results.

In the following, let  $T > 0$  be a constant,  $(M, g)$  and  $(\widetilde{M}, \widetilde{g})$  be Riemannian manifolds,  $V \subset M$  and  $\widetilde{V} \subset \widetilde{M}$  be open sets, and  $\theta : \text{cl}(\widetilde{V}) \rightarrow \text{cl}(V)$  be a diffeomorphism, and assume they all satisfy the assumptions of Theorem 1. Furthermore, let

$$L_{\widetilde{V}} : C_c^2((0, \infty); L^2(\widetilde{V})) \rightarrow C^1([0, \infty); L^2(\widetilde{V})) \cap L^\infty([0, \infty); L^2(\widetilde{V}))$$

be the local source-to-solution operator on the manifold  $(\widetilde{M}, \widetilde{g})$ .

**Proposition 2.** Let  $h \in C_c^\infty((0, T); L^2(V))$  be the source defined in Definition 14. If

$$(\theta^* L_V h)|_{[0, T]} = (L_{\tilde{V}} \theta^* h)|_{[0, T]}, \quad (5)$$

then

$$\theta^* L_V f = L_{\tilde{V}} \theta^* f \quad (6)$$

for every  $f \in C_c^2((0, \infty); L^2(V))$ .

**Theorem 3.** If the equality (6) holds for every  $f \in C_c^2((0, \infty); L^2(V))$ , then the manifolds  $(M, g)$  and  $(\tilde{M}, \tilde{g})$  are Riemannian isometric.

## 1.2 Motivation and literature

Einstein's celebrated paper [14] introduced the classical explanation of Brownian motion as a random walk, in which the dynamics of a particle suspended in a fluid is described by an uncorrelated, Markovian, Gaussian stochastic process. A key result of this theory is that the mean-square displacement of the random walk is proportional to time, i.e.,  $\langle x^2 \rangle \propto t$  for large  $t$ . At the continuum limit, it follows that the concentration of a large number of independent particles is governed by the diffusion equation.

Despite the success of standard diffusion model, there are a number of experimental observations of diffusion processes, where the mean-square displacement does not scale linearly. A random walk interpretation can also be given to such processes: In a standard discrete random walk the step length is a fixed distance and the steps occur at discrete times. In a more general walk (Continuous Time Random Walk, CTRW) a waiting time and step length are sampled from given probability distributions. At the continuum limit, a suitable power law distribution for the waiting time results to *subdiffusive* processes, where  $\langle x^2 \rangle \propto t^\alpha$ ,  $0 < \alpha < 1$ . Analogous to the classical diffusion, the concentration of random particles satisfies a model where the time derivative in the diffusion equation is replaced by a fractional time derivative of order  $\alpha$ . Similarly, a suitable power law step length distribution replaces the Laplacian in the diffusion equation by a fractional power  $(-\Delta)^\beta$ . The variability of these distributions gives rise to the class of fractional PDEs in (1).

Anomalous diffusion processes described by equation (1) appear in spatially disordered systems such as porous media, in turbulent fluids and plasma, biological systems and finance (see, e.g., [2, 34, 21, 56, 49, 5, 11, 8, 9, 67, 13]). Following the random walk analogy, our main result in Theorem 1 can be interpreted as follows: we introduce a rigorous strategy to inject new particles into a diffusion process taking place in an unknown medium so that a single long-term observation of the concentration determines the properties of the medium.

Mathematical work on fractional calculus is extensive. For a general overview, see textbooks [33, 59], reviews [25, 7] and references therein.

Without providing a comprehensive list, we mention that classical properties for the fractional diffusion equations, such as the fundamental solutions, the regularity estimates and the maximum principles are established in [57, 46, 45, 44, 47]. Moreover, numerical analysis for fractional PDEs is considered in [28, 29, 10, 64, 42].

Inverse problems for fractional PDEs have gained major attention in recent years. The review [30] summarizes work on some common fractional inverse problems and collects some open problems. Uniqueness and reconstruction of unknown parameters are considered in [55, 66, 43, 60, 61]. In particular, we mention the article [32] by Y. Kian, L. Oksanen, E. Soccorsi, and M. Yamamoto, where the uniqueness of the Riemannian metric is proved for time-fractional PDE given Dirichlet-to-Neumann map at a fixed time at the boundary of the manifold. For techniques based on Carleman estimates, we refer to [63, 27].

There are a number of other interesting setups for fractional inverse problems: In the static case, the fractional Calderon problems are investigated in [53, 16, 17, 58, 18]. If a more general waiting time probability distribution is considered, then  $\partial_t^\alpha$  may need to be replaced by a weighted mixture of fractional derivatives. This leads to the so-called multi-term time fractional diffusion equations and the distributed order differential equations [54, 39, 37, 40, 38]. Also, there is recent effort to study statistical fractional inverse problems [62, 20, 48, 65, 50].

The work in this paper is connected to geometric inverse problems outside fractional PDEs through many aspects of the observational setup. In wave propagation models with finite speed of propagation single measurement data has been studied in [22, 23]. The setup with multiple measurements is better understood: in such a case geometric version of boundary control method can be used for deriving uniqueness and reconstruction [4, 3, 31, 1, 6]. Finally, let us mention that closed manifolds have been studied also in the framework of inverse spectral problems [35].

### 1.3 Outline of the paper

This paper is organized as follows. In Section 2 we record some preliminary definitions and present some well-known results regarding the Mittag–Leffler function  $E_{a,b}$ , which plays a central role in representing the solution to (1).

We investigate the direct problem for (1) in Section 3, where the existence, uniqueness and representation results of the solution are proved. Also, the source-to-solution operator  $L_V$  is defined, which will be studied in the inverse problem part.

The inverse problem is considered in Section 4. First, we prove that the operator  $L_V$  can be uniquely determined given a single measurement (Proposition 2). Second, we prove that the operator  $L_V$  determines the manifold up to an isometry (Theorem 3). The main result, Theorem 1,

immediately follows from these two results.

## 2 Preliminaries

This section contains some technical tools that are required in understanding equation (1). We recall the definition of the Caputo derivative of fractional order, and we review the Laplace–Beltrami operator  $\Delta_g$ , as well as some basic functional calculus to define its fractional powers. We also give a definition of the strong solution of (1).

### 2.1 The Mittag–Leffler function and fractional derivatives

The (two-parameter) Mittag–Leffler function  $E_{a,b}$  has a role in the fractional differential equations analogous to the role of the exponential function in the case of the integer order differential equations. The function is defined as

$$E_{a,b}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ka + b)} \quad (a > 0, b > 0, z \in \mathbb{C}). \quad (7)$$

In particular,  $E_{1,1}(z) = e^z$ . For a treatise of the Mittag–Leffler function, see [19].

The radius of convergence of the power series (7) is infinite, so  $E_{a,b}$  is an entire function. A recurrence relation for the gamma function together with termwise differentiation of the power series shows that  $E'_{a,1}(z) = a^{-1}E_{a,a}(z)$ . For every  $\lambda \in \mathbb{C}$ , the function

$$G_\lambda(z) := E_{a,1}(-\lambda z^a) \quad (z \in \mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}) \quad (8)$$

is holomorphic on  $\mathbb{C}_+$ , and therefore above reasoning shows that

$$G'_\lambda(z) = -\lambda z^{a-1} E_{a,a}(-\lambda z^a) \quad (z \in \mathbb{C}_+). \quad (9)$$

**Proposition 4.** *For  $0 < a \leq 1$ , the following hold:*

(i) *There exists a constant  $C_a > 0$  (that depends on  $a$ ) such that*

$$|E_{a,a}(z)| \leq C_a \quad (z \in \mathbb{C} \setminus \mathbb{C}_+).$$

(ii) *Let  $\lambda \geq 0$  and define a function  $F_\lambda : (0, \infty) \rightarrow \mathbb{C}$  by*

$$F_\lambda(t) := t^{a-1} E_{a,a}(-\lambda t^a) \quad (t > 0).$$

*Then the Laplace transform  $\mathcal{L}F_\lambda(s)$  of  $F_\lambda$  exists at every point  $s \in \mathbb{C}_+$ , and*

$$\mathcal{L}F_\lambda(s) = \frac{1}{s^a + \lambda} \quad (s \in \mathbb{C}_+). \quad (10)$$

*Proof.* 1. For  $a = 1$  the boundedness is evident, because  $E_{1,1}(z) = e^z$ . For  $0 < a < 1$ , see Theorem 1.6 in [51].

2. For  $s \in \mathbb{C}$  with  $\operatorname{Re} s > \lambda^{1/a}$ , formula (10) is proved in [51] (cf. formula (1.80) there). By the boundedness of  $E_{a,a}$  on  $\mathbb{C} \setminus \mathbb{C}_+$ , the Laplace transform of  $F_\lambda$  exists on the whole half-plane  $\mathbb{C}_+$ . It follows from the uniqueness of analytic continuation that (10) holds for every  $s \in \mathbb{C}_+$ .  $\square$

Recall that  $0 < \alpha \leq 1$  and consider a complex-valued function  $y \in C^1([0, \infty))$ . Here the space is the space of continuously differentiable functions on  $[0, \infty)$ , with the derivative at the left endpoint being the appropriate one-sided derivative. The *Caputo derivative of order  $\alpha$*  of  $y$  at point  $t \in [0, \infty)$ , denoted by  $\partial_t^\alpha y(t)$ , is defined as

$$\partial_t^\alpha y(t) := \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} y'(\tau) d\tau, & 0 < \alpha < 1, \\ y'(t), & \alpha = 1. \end{cases} \quad (11)$$

In particular, if  $\alpha = 1$ , then  $\partial_t^\alpha y$  is just the standard first order derivative of  $y$ .

Another commonly used fractional derivative is the Riemann–Liouville fractional derivative. The *Riemann–Liouville fractional derivative of order  $\alpha$*  of  $y \in C^1([0, \infty))$  at point  $t \in [0, \infty)$ , denoted by  $\partial_{t,\text{RL}}^\alpha y(t)$ , is defined by

$$\partial_{t,\text{RL}}^\alpha y(t) := \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} y(\tau) d\tau, & 0 < \alpha < 1, \\ y'(t), & \alpha = 1. \end{cases} \quad (12)$$

It is clear from (12) that the Riemann–Liouville derivative can be defined for a larger class of functions than the continuously differentiable ones. It can also be shown that

$$\partial_t^\alpha y(t) = \partial_{t,\text{RL}}^\alpha (y(t) - y(0)) \quad (t \geq 0), \quad (13)$$

(see, e.g., Chapter 3 in [12]), and often (13) is in fact taken as the definition of the Caputo derivative, because the right-hand side of (13) is defined for a larger class of functions than (11).

In this paper we mainly consider continuously differentiable functions  $y \in C^1([0, \infty))$  with  $y(0) = 0$ . For such functions (13) shows that the Caputo fractional derivative and the Riemann–Liouville fractional derivative coincide. For consistency of notation, we use the Caputo fractional derivative  $\partial_t^\alpha$  throughout the paper.

For a scalar nonhomogeneous linear fractional differential equation, there are the following existence and uniqueness results (see [12]):



**Proposition 5.** *Let  $0 < \alpha \leq 1$ ,  $\lambda \in \mathbb{R}$ ,  $b \in C_c^1((0, \infty))$ , and consider the fractional differential equation*

$$\partial_t^\alpha y(t) + \lambda y(t) = b(t), \quad (t \geq 0), \quad (14a)$$

$$y(0) = 0. \quad (14b)$$

*There exists a unique function  $y \in C^1([0, \infty))$  for which equations (14a) and (14b) are valid. This function can be represented as*

$$y(t) = \int_0^t (t - \tau)^{\alpha-1} E_{\alpha, \alpha}(-\lambda(t - \tau)^\alpha) b(\tau) d\tau \quad (t \geq 0). \quad (15)$$

*Proof.* By Corollary 6.9 in [12], there exists at most one continuously differentiable function for which (14a) and (14b) are valid. Some standard properties of convolutions and the assumed regularity of  $b$  imply that  $y$  as defined by (15) is continuously differentiable. Theorem 7.2 in [12] states that this function satisfies (14a) and (14b).  $\square$

Of course, in the case  $\alpha = 1$ , the existence and uniqueness of a solution to (14a) and (14b) follow from the standard theory of linear ordinary differential equations, and (15) is just the variation of parameters formula.

## 2.2 The Laplace–Beltrami operator

The Laplace–Beltrami operator  $\Delta_g$  is an unbounded self-adjoint operator on  $L^2(M)$  with domain of definition  $\mathcal{D}(\Delta_g) = H^2(M)$ . The operator is defined in local coordinates by

$$\Delta_g \xi := |g|^{-1/2} \partial_j (|g|^{1/2} g^{jk} \partial_k \xi) \quad (\xi \in H^2(M)),$$

where  $|g|$  is the determinant of the metric  $g$  and  $(g^{jk})$  is the inverse matrix of  $g$ . Here and below, we use Einstein’s summation convention and sum over indexes appearing as sub- and superindexes.

Let  $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$  be the eigenvalues of  $-\Delta_g$ , listed according to their multiplicities, and let  $(\varphi_k)_{k=1}^\infty$  be some complete orthonormal sequence of associated eigenfunctions. The exponent  $(-\Delta_g)^\beta$  of  $(-\Delta_g)$  is then defined by

$$(-\Delta_g)^\beta \xi := \sum_{k=1}^\infty \lambda_k^\beta \langle \xi, \varphi_k \rangle_{L^2(M)} \varphi_k \quad (\xi \in L^2(M)),$$

(see [41]) with domain

$$\mathcal{D}((-\Delta_g)^\beta) := \left\{ \xi \in L^2(M) : \sum_{k=1}^\infty \lambda_k^{2\beta} |\langle \xi, \varphi_k \rangle_{L^2(M)}|^2 < \infty \right\} = H^{2\beta}(M).$$

### 2.3 Fractional derivatives of $L^2(M)$ -valued functions and the strong solution of (1)

Let  $0 < T \leq \infty$  and recall that  $0 < \alpha \leq 1$ . Let  $y \in C^1([0, T]; L^2(M))$ . The *Caputo derivative of order  $\alpha$*  of the  $L^2(M)$ -valued function  $y$  at point  $t \in [0, T)$ , denoted by  $\partial_t^\alpha y(t)$ , is defined analogously to the scalar case:

$$\partial_t^\alpha y(t) := \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} y'(\tau) d\tau, & 0 < \alpha < 1, \\ y'(t), & \alpha = 1. \end{cases} \quad (16)$$

Here the derivative  $y'$  is in the sense of the derivative of an  $L^2(M)$ -valued function of a real variable, and the integral is in the sense of Bochner. Note that the existence of the integral is guaranteed by the assumption of continuous differentiability of  $y$ .

We are now ready to give a definition of a solution of the fractional diffusion equation (1). Let  $f : [0, \infty) \rightarrow L^2(M)$ . We say that a function  $u \in C^1([0, \infty); L^2(M))$  is a *strong solution of the fractional diffusion equation (1)*, if

- (i)  $u(0) = 0$ ,
- (ii)  $u(t) \in \mathcal{D}((-\Delta_g)^\beta)$  for every  $t \geq 0$ , and
- (iii)  $\partial_t^\alpha u(t) + (-\Delta_g)^\beta u(t) = f(t)$  for every  $t \geq 0$ .

## 3 Analysis of the direct problem

Here we prove an existence and uniqueness result for the fractional diffusion equation (1). We also establish a representation of the solution, which will later be used in solving the inverse problem. Furthermore, we define the local source-to-solution operator  $L_V$ .

### 3.1 Uniqueness and existence of a strong solution

Let  $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$  be the eigenvalues of  $-\Delta_g$ , listed according to their multiplicities, and let  $(\varphi_k)_{k=1}^\infty \subset C^\infty(M)$  be some complete orthonormal sequence of corresponding eigenfunctions.

**Proposition 6.** *Suppose that  $f \in C_c^2((0, \infty); L^2(M))$ . Then there exists a unique strong solution  $u \in C^1([0, \infty); L^2(M))$  of the fractional diffusion equation (1). The strong solution can be represented as*

$$u(t) = \sum_{k=1}^{\infty} u_k(t) \varphi_k \quad (t \geq 0), \quad (17)$$

where the series converges in  $L^2(M)$  for every  $t \geq 0$ , and

$$u_k(t) := \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k^\beta(t - \tau)^\alpha) \langle f(\tau), \varphi_k \rangle_{L^2(M)} d\tau \quad (t \geq 0). \quad (18)$$

The proposition is proved by an eigenfunction expansion analogously to [57]. As we use spectral theoretical approach to consider direct and inverse problem for fractional power operator  $(-\Delta_g)^\beta$ , instead of theory of integral operators, we provide the detailed proof for the convenience of the reader.

The proof is split in several steps, starting with uniqueness of the solution, which holds without any assumptions on the source function  $f$ .

**Proposition 7.** *There exists at most one strong solution of the fractional diffusion equation (1).*

*Proof.* Suppose  $u, \tilde{u} : [0, \infty) \rightarrow L^2(M)$  are two strong solutions of the fractional diffusion equation (1), and define  $v := u - \tilde{u}$ . Then  $v$  is a strong solution of (1) with the zero source term.

Fix  $k \in \mathbb{Z}^+ := \{1, 2, 3, \dots\}$  and define a complex-valued function  $v_k : [0, \infty) \rightarrow \mathbb{C}$  by  $v_k(t) := \langle v(t), \varphi_k \rangle_{L^2(M)}$ . It is evident that  $v_k \in C^1([0, \infty))$  and  $v'_k = \langle v', \varphi_k \rangle_{L^2(M)}$ , because  $v \in C^1([0, \infty); L^2(M))$  by the definition of a strong solution. If  $0 < \alpha < 1$ , combining the previous result with Proposition 23 shows that

$$\langle \partial_t^\alpha v(t), \varphi_k \rangle_{L^2(M)} = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - \tau)^{-\alpha} v'_k(\tau) d\tau = \partial_t^\alpha v_k(t) \quad (t \geq 0).$$

Note that  $v(t) \in \mathcal{D}((-\Delta_g)^\beta)$ . Then the definition of  $(-\Delta_g)^\beta$  implies that

$$\langle (-\Delta_g)^\beta v(t), \varphi_k \rangle_{L^2(M)} = \lambda_k^\beta v_k(t) \quad (t \geq 0).$$

Above considerations show that

$$\begin{aligned} \partial_t^\alpha v_k(t) + \lambda_k^\beta v_k(t) &= 0, & (t \geq 0) \\ v_k(0) &= 0. \end{aligned} \quad (19)$$

By Proposition 5 the unique continuous solution of (19) is the zero function. Because  $k \in \mathbb{Z}^+$  is arbitrary,  $v = 0$ , and therefore  $u = \tilde{u}$ .  $\square$

Following lemma provides useful estimates for the component functions  $u_k$  of  $u$ .

**Lemma 8.** *Let  $T > 0$  and  $f \in C_c^2((0, T); L^2(M))$ , and let  $u_k$  be defined by (18). Then the following hold:*

(i) The functions  $u_k$  satisfy  $u_k \in C^1([0, \infty))$ , and there exists a constant  $\epsilon > 0$  such that  $\text{supp } u_k \subset (\epsilon, \infty)$ , for every  $k \in \mathbb{Z}^+$ .

(ii) For every  $k \in \mathbb{Z}^+$ , it holds that

$$\lambda_k^\beta u_k(t) = \int_0^t (1 - E_{\alpha,1}(-\lambda_k^\beta(t-\tau)^\alpha)) \langle f'(\tau), \varphi_k \rangle_{L^2(M)} d\tau \quad (t \geq 0), \quad (20)$$

and

$$\lambda_k^{2\beta} |u_k(t)|^2 \leq \min\{t, T\} \int_0^t |\langle f'(\tau), \varphi_k \rangle_{L^2(M)}|^2 d\tau \quad (t \geq 0). \quad (21)$$

(iii) If in (20) and (21) the functions  $u_k$  and  $f'$  are replaced by  $u'_k$  and  $f''$ , respectively, (20) and (21) remain valid.

*Proof.* (i) If  $\epsilon > 0$  is small enough so that  $f(t) = 0$  for  $0 \leq t \leq 2\epsilon$ , then  $u_k(t) = 0$  for  $0 \leq t \leq 2\epsilon$ . Therefore  $\text{supp } u_k \subset (\epsilon, \infty)$ .

Define

$$F_k(t) := \begin{cases} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k^\beta t^\alpha), & t > 0, \\ 0, & t \leq 0. \end{cases} \quad (22)$$

Then  $F_k$  is locally integrable, and  $u_k$  is the convolution

$$u_k(t) = (F_k * \langle f, \varphi_k \rangle_{L^2(M)})(t) \quad (t \geq 0). \quad (23)$$

It follows that  $u_k \in C^1([0, \infty))$ .

(ii) If  $\lambda_k = 0$ , then both sides of (20) vanish, and (21) holds trivially. Therefore we may assume  $\lambda_k > 0$ .

Note that as  $f(0) = 0$ , (9) and integration by parts show that

$$\begin{aligned} \lambda_k^\beta u_k(t) &= \lim_{\epsilon \rightarrow 0^+} \int_0^{t-\epsilon} \lambda_k^\beta (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k^\beta (t-\tau)^\alpha) \langle f(\tau), \varphi_k \rangle_{L^2(M)} d\tau \\ &= \lim_{\epsilon \rightarrow 0^+} E_{\alpha,1}(-\lambda_k^\beta \epsilon^\alpha) \langle f(t-\epsilon), \varphi_k \rangle_{L^2(M)} \\ &\quad - \int_0^{t-\epsilon} E_{\alpha,1}(-\lambda_k^\beta (t-\tau)^\alpha) \langle f'(\tau), \varphi_k \rangle_{L^2(M)} d\tau \\ &= \langle f(t), \varphi_k \rangle_{L^2(M)} - \int_0^t E_{\alpha,1}(-\lambda_k^\beta (t-\tau)^\alpha) \langle f'(\tau), \varphi_k \rangle_{L^2(M)} d\tau. \end{aligned}$$

By combining above with the fact that

$$\langle f(t), \varphi_k \rangle_{L^2(M)} = \int_0^t \langle f'(\tau), \varphi_k \rangle_{L^2(M)} d\tau, \quad (24)$$

we obtain (20).

Set  $t^* := \min\{t, T\}$ . The Cauchy–Schwarz inequality applied to (20) shows that

$$\lambda_k^{2\beta} |u_k(t)|^2 \leq \int_0^{t^*} |1 - E_{\alpha,1}(-\lambda_k^\beta(t-\tau)^\alpha)|^2 d\tau \int_0^t |\langle f'(\tau), \varphi_k \rangle_{L^2(M)}|^2 d\tau. \quad (25)$$

The function

$$(0, \infty) \ni \tau \mapsto M(\tau) := E_{\alpha,1}(-\lambda_k^\beta \tau^\alpha)$$

is completely monotonic, meaning that  $(-1)^k (\frac{d}{d\tau})^k M(\tau) \geq 0$  for  $k = 0, 1, 2, \dots$  and  $\tau > 0$  (for  $\alpha = 1$  this is immediate from differentiating the exponential function, for  $0 < \alpha < 1$  see Theorem 7.3 in [12]). In particular

$$0 \leq M(\tau) \leq M(0) = 1 \quad (\tau > 0).$$

It follows that the first integrand in (25) has values in  $[0, 1]$ , and (ii) is proved.

- (iii) From (23) and properties of convolution it follows that (18) holds if  $u_k$  and  $f$  are substituted by  $u'_k$  and  $f'$ , respectively. This implies that (20) and (21) also hold under the same substitution.  $\square$

Next two lemmas prepare for the proof of Proposition 6.

**Lemma 9.** *The series (17) converges in  $L^2(M)$  for every  $t \geq 0$ . The limit function  $u$  is in  $C^1([0, \infty); L^2(M))$ , and*

$$u'(t) = \sum_{k=1}^{\infty} u'_k(t) \varphi_k \quad (t \geq 0), \quad (26)$$

where the convergence is pointwise in  $L^2(M)$ . Moreover,  $u(t) \in \mathcal{D}((-\Delta_g)^\beta)$  for every  $t \geq 0$ , and  $\text{supp } u \subset (0, \infty)$ .

*Proof.* Fix  $t \geq 0$ . Inequality (21) implies that

$$\sum_{k=1}^{\infty} \lambda_k^{2\beta} |u_k(t)|^2 \leq t \int_0^t \|f'(\tau)\|_{L^2(M)}^2 d\tau < \infty. \quad (27)$$

Because  $\lambda_k \rightarrow \infty$  as  $k \rightarrow \infty$ , inequality (27) implies that  $\sum_{k=1}^{\infty} |u_k(t)|^2 < \infty$ , and therefore the series (17) converges in  $L^2(M)$ . It also follows that  $u(t) \in \mathcal{D}((-\Delta_g)^\beta)$ .

The first part of Lemma 8 implies that  $\text{supp } u \subset (0, \infty)$ . Therefore on a neighborhood of the origin  $u$  is smooth and (26) holds.

Because  $u_k \in C^1([0, \infty))$ , the partial sums of (17) satisfy  $(\sum_{k=1}^N u_k \varphi_k)' = \sum_{k=1}^N u'_k \varphi_k$ . Hence to prove  $u \in C^1([0, \infty))$  and (26), it is enough to prove

that the series on the right-hand side of (26) converges uniformly on every subinterval  $(0, T) \subset (0, \infty)$ .

For  $N$  large enough so that  $\lambda_N \geq 1$ , (iii) of Lemma 8 yields

$$\sum_{k=N}^{\infty} |u'_k(t)|^2 \leq T \int_0^T \sum_{k=N}^{\infty} |\langle f''(\tau), \varphi_k \rangle_{L^2(M)}|^2 d\tau \quad (0 \leq t < T).$$

The integrand converges to zero pointwise as  $N \rightarrow \infty$ , and it is dominated by the integrable function  $\|f''\|_{L^2(M)}^2$ . By the Lebesgue's dominated convergence theorem, the integral tends to zero as  $N \rightarrow \infty$ . This implies uniform convergence of  $\sum_{k=1}^{\infty} u'_k \varphi_k$  on  $(0, T)$ .  $\square$

**Lemma 10.** *The Caputo derivative of order  $\alpha \in (0, 1]$  of the  $L^2(M)$ -valued function  $u$  defined by (17) and (18) exists on  $[0, \infty)$ , and*

$$\partial_t^\alpha u(t) = \sum_{k=1}^{\infty} [\partial_t^\alpha u_k(t)] \varphi_k = \sum_{k=1}^{\infty} (-\lambda_k^\beta u_k(t) + \langle f(t), \varphi_k \rangle_{L^2(M)}) \varphi_k \quad (t \geq 0). \quad (28)$$

*Proof.* By Lemma 9 we have  $u \in C^1([0, \infty); L^2(M))$ , hence the Caputo derivative of order  $\alpha$  of  $u$  exists at every point  $t \geq 0$ .

If  $\alpha = 1$ , the first equality of (28) is true by Lemma 9. If  $0 < \alpha < 1$ , the first equality follows from an application of (ii) of Proposition 23 and (26) in the definition of  $\partial_t^\alpha$ .

The second equality follows from Proposition 5.  $\square$

Proving the existence and uniqueness of a strong solution of (1) is now straightforward:

*Proof of Proposition 6.* Proposition 7 implies that a strong solution, should it exist, is unique. Lemma 9 proves that the function  $u$  specified by (17) and (18) is a well-defined function with range in  $\mathcal{D}((-\Delta_g)^\beta)$  and  $u(0) = 0$ . By Lemma 10 the Caputo derivative of order  $\alpha$  of  $u$  exists on  $[0, \infty)$ , and

$$\partial_t^\alpha u(t) = -(-\Delta_g)^\beta u(t) + f(t) \quad (t \geq 0).$$

Therefore a strong solution exists, and the solution is given by (17).  $\square$

### 3.2 The local source-to-solution operator $L_V$

Given  $f \in C_c^2((0, \infty); L^2(M))$ , let  $u^f \in C^1([0, \infty); L^2(M))$  denote the strong solution of the fractional diffusion equation (1).

**Proposition 11.** *Let  $T > 0$ . There exists a constant  $C_{T,M,\alpha} > 0$  (that depends on  $T$ ,  $\alpha$ , and the manifold  $(M, g)$ ) such that*

$$\sup_{t \geq 0} \|u^f(t)\|_{L^2(M)} \leq C_{T,M,\alpha} \left( \int_0^T \|f'(\tau)\|_{L^2(M)}^2 d\tau \right)^{1/2}, \quad (29)$$

for every  $f \in C_c^2((0, T); L^2(M))$ .

*Proof.* We have the representation of  $u^f$  given by (17) and (18). Let us first estimate the first term  $u_1$  of the representation.

Since  $\lambda_1 = 0$ , for every  $t \geq 0$  it holds that

$$|u_1(t)| \leq \frac{1}{\Gamma(\alpha)} \sup_{\tau \geq 0} |\langle f(\tau), \varphi_1 \rangle_{L^2(M)}| \int_0^{\min\{t, T\}} (t - \tau)^{\alpha-1} d\tau. \quad (30)$$

The inner product in (30) can be estimated with (24). Applying the Cauchy–Schwarz inequality to (24) and noticing that the integral in (30) as a function of  $t$  obtains its maximum at  $t = T$  show that

$$|u_1(t)|^2 \leq \frac{1}{\Gamma(\alpha)^2} \int_0^T |\langle f'(\tau), \varphi_1 \rangle_{L^2(M)}|^2 d\tau \frac{T^{2\alpha+1}}{\alpha^2}. \quad (31)$$

If  $k > 1$ , then  $\lambda_k \geq \lambda_2 > 0$ . Therefore (ii) of Lemma 8 can be used to estimate  $|u_k(t)|^2$ . These estimates together with (31) readily yield (29).  $\square$

Recall that in the inverse problem we consider sources supported on an open set  $V \subset M$ , and observe the evolution of the corresponding solutions of the fractional diffusion equation (1) on the set  $V$ . From this information we want to recover the manifold  $(M, g)$ .

In what follows, we identify  $L^2(V)$  as a subset of  $L^2(M)$  by identifying functions with their zero extensions. Also, by an abuse of notation,  $u^f|_V$  denotes the function  $[0, \infty) \ni t \mapsto u^f(t)|_V$ . Then Proposition 6 and Proposition 11 imply that if  $f \in C_c^2((0, \infty); L^2(V))$ , then  $u^f|_V \in C^1([0, \infty); L^2(V)) \cap L^\infty([0, \infty); L^2(V))$ .

**Definition 12.** *Let  $V \subset M$  be a nonempty open set with smooth boundary. The local source-to-solution operator on  $V$ , denoted by  $L_V$ , is the operator*

$$L_V : C_c^2((0, \infty); L^2(V)) \rightarrow C^1([0, \infty); L^2(V)) \cap L^\infty([0, \infty); L^2(V))$$

defined by

$$L_V f := u^f|_V.$$

For  $T > 0$ , the truncated local source-to-solution operator on  $V$ , denoted by  $L_{V,T}$ , is the operator

$$L_{V,T} : C_c^2((0, \infty); L^2(V)) \rightarrow C^1([0, T]; L^2(V)) \cap L^\infty([0, T]; L^2(V))$$

defined by  $L_{V,T} f := (L_V f)|_{[0, T]}$ .

Note that as a topological vector space, the space  $L^2(V)$  is independent of the Riemannian metric  $g|_V$ . Therefore the domain and codomain of  $L_V$  and  $L_{V,T}$  do not depend on  $g$ .

If  $(\varphi_k)_{k=1}^\infty$  and  $(\lambda_k)_{k=1}^\infty$  are as in Section 3.1, the local source-to-solution operator can be represented as

$$\mathsf{L}_V f(t) = \sum_{k=1}^{\infty} \left[ \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k^\beta(t-\tau)^\alpha) \langle f(\tau), \varphi_k \rangle_{L^2(V)} d\tau \right] \varphi_k|_V, \quad (32)$$

where the sum converges in  $L^2(V)$ , for every  $t \geq 0$ .

As a consequence of Proposition 11, we obtain the following continuity result for the local source-to-solution operator:

**Proposition 13.** *Let  $T > 0$ . Suppose that*

$$f \in C_c^2((0, T); L^2(V)) \text{ and } (f_k)_{k=1}^\infty \subset C_c^2((0, T); L^2(V))$$

*are such that  $f'_k(t) \rightarrow f'(t)$  in  $L^2(V)$  as  $k \rightarrow \infty$ , for every  $t \in (0, T)$ , and*

$$\sup_{\substack{k \in \mathbb{Z}^+, \\ t \in (0, T)}} \|f'_k(t)\|_{L^2(V)} < \infty.$$

*Then*

$$\mathsf{L}_V f_k(t) \rightarrow \mathsf{L}_V f(t) \text{ in } L^2(V) \text{ as } k \rightarrow \infty,$$

*uniformly in  $t \geq 0$ .*

*Proof.* From the definition of the local source-to-solution operator and inequality (29) of Proposition 11, it follows that for every  $t \geq 0$

$$\begin{aligned} \|\mathsf{L}_V f(t) - \mathsf{L}_V f_k(t)\|_{L^2(V)} &\leq \|u^{f-f_k}(t)\|_{L^2(M)} \\ &\leq C_{T,M,\alpha} \left( \int_0^T \|(f-f_k)'(\tau)\|_{L^2(M)}^2 d\tau \right)^{1/2}. \end{aligned} \quad (33)$$

By assumption  $f'_k(\tau) \rightarrow f'(\tau)$  in  $L^2(V)$  as  $k \rightarrow \infty$ , for every  $\tau \in (0, T)$ , and the same holds for their zero extensions in  $L^2(M)$ .

Because the integrand is uniformly bounded with respect to  $\tau \in (0, T)$  and  $k \in \mathbb{Z}^+$ , the Lebesgue's dominated convergence theorem can be applied. This concludes the proof.  $\square$

## 4 Analysis of the inverse problem

We begin by showing that the local source-to-solution operator can be determined with a single measurement, provided the source is chosen appropriately. After that we show that the manifold is determined up to a Riemannian isometry by this operator.



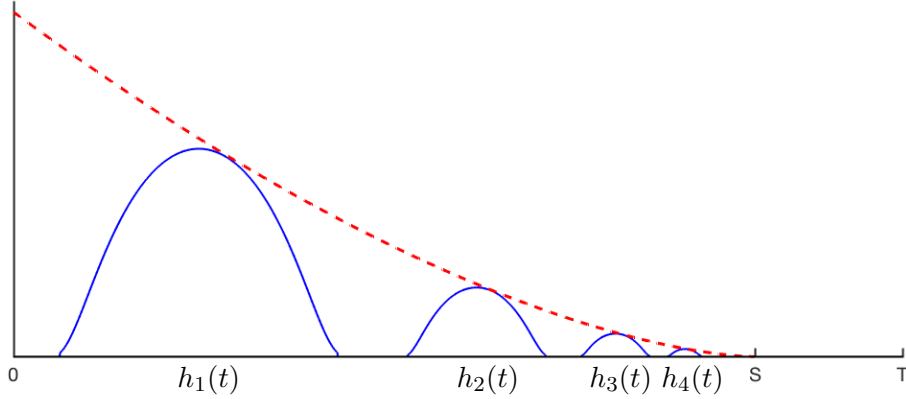


Figure 1: Examples of functions  $h_k$ ,  $k \geq 1$ , are plotted in blue with the choice of  $n$  being a Gaussian bump. The exponential decay of the amplitude of  $h_k$  as  $k$  increases is illustrated by the red dashed line. The values of  $h_k$  are not presented in scale.

#### 4.1 The local source-to-solution operator $L_V$ can be determined with one measurement

We construct a source  $h \in C_c^\infty((0, T); L^2(V))$  such that the local source-to-solution operator  $L_V$  is completely determined by the single function  $L_{V,T} h$ .

**Definition 14.** Fix a constant  $T > 0$  and let  $V \subset M$  be a nonempty open set with smooth boundary. Choose a number  $0 < S < T$ , a nonzero non-negative function  $n \in C_c^\infty(-1, 1)$ , a bounded sequence  $(\psi_k)_{k=1}^\infty \subset L^2(V)$  of functions that spans a dense subspace of  $L^2(V)$ , and a sequence  $(r(k))_{k=1}^\infty \subset \mathbb{Z}^+$  which contains every positive integer infinitely many times. Then define the source  $h$  by

$$h(t) := \sum_{k=1}^{\infty} \underbrace{\frac{S^k}{2^{k(k+2)} m_k} \cdot n \left( 2^{k+1} \left( \frac{t}{S} - 1 \right) + 3 \right)}_{=: h_k(t)} \psi_{r(k)} \quad (t \in \mathbb{R}), \quad (34)$$

where  $m_k := \max_{0 \leq l \leq k} \|n^{(l)}\|_\infty$ .

We provide illustration of the decay of functions  $h_k$  as  $k$  increases in Figure 4.1.

*Remark 3.* Boundedness and denseness in  $L^2(V)$  are qualities that are independent of the metric  $g|_V$ , because all Riemannian metrics on  $M$  induce the same topology on  $L^2(V)$ .

*Remark 4.* An example of a sequence that contains every positive integer infinitely many times is the sequence that begins

$$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, \dots$$

In order to prove Proposition 2, consider a manifold  $(\widetilde{M}, \widetilde{g})$  and an open set  $\widetilde{V} \subset \widetilde{M}$  with smooth boundary, and suppose they satisfy the conditions of Theorem 1. Let  $\theta : \text{cl}(\widetilde{V}) \rightarrow \text{cl}(V)$  be a diffeomorphism. Because of the compactness of  $\text{cl}(\widetilde{V})$  and  $\text{cl}(V)$ , the pullback  $\theta^*$  is a continuous operator from  $L^2(\text{cl}(V))$  onto  $L^2(\text{cl}(\widetilde{V}))$ . By the diffeomorphism invariance of the boundary,  $\theta^*$  is also a continuous operator from  $L^2(V)$  onto  $L^2(\widetilde{V})$ . Therefore, if  $h \in C_c^\infty((0, T); L^2(V))$ , then  $\theta^*h \in C_c^\infty((0, \infty); L^2(\widetilde{V}))$ .

It is convenient to introduce the conjugated operators  $\widetilde{L}_{\widetilde{V}}$  and  $\widetilde{L}_{\widetilde{V}, T}$  defined for  $f \in C_c^2((0, \infty); L^2(V))$  by

$$\widetilde{L}_{\widetilde{V}}f := (\theta^*)^{-1}L_{\widetilde{V}}(\theta^*f) \text{ and } \widetilde{L}_{\widetilde{V}, T}f := (\theta^*)^{-1}L_{\widetilde{V}, T}(\theta^*f).$$

Then

$$\widetilde{L}_{\widetilde{V}} : C_c^2((0, \infty); L^2(V)) \rightarrow C^1([0, \infty); L^2(V)) \cap L^\infty([0, \infty); L^2(V))$$

and

$$\widetilde{L}_{\widetilde{V}, T} : C_c^2((0, \infty); L^2(V)) \rightarrow C^1([0, T]; L^2(V)) \cap L^\infty([0, T]; L^2(V)),$$

and (5) and (6) are equivalent to  $L_{V, T}h = \widetilde{L}_{\widetilde{V}, T}h$  and  $L_V = \widetilde{L}_{\widetilde{V}}$ , respectively.

Proposition 2 will be proved in several steps. Let us first prove the compactness and smoothness of  $h$ .

**Proposition 15.** *The terms  $h_k \in C_c^\infty((0, \infty))$  in the series (34) satisfy*

$$\text{supp } h_k \subset ((1 - 2^{1-k})S, (1 - 2^{-k})S) \quad (k \in \mathbb{Z}^+). \quad (35)$$

*In particular, their supports are pairwise disjoint. Furthermore, the series converges uniformly in  $t \in \mathbb{R}$ , and defines a function  $h \in C_c^\infty((0, T); L^2(V))$ .*

*Proof.* Inclusion (35), which is seen to hold by a straightforward calculation, implies that  $h$  is defined pointwise and supported in  $(0, S] \subset (0, T)$ . For  $l = 0, 1, 2, \dots$ , the  $l$ -th derivative  $h_k^{(l)}$  with  $k$  large enough so that  $k \geq l$  and  $2^{k+1} \geq S$  can be estimated as

$$|h_k^{(l)}(t)| = \frac{2^{(k+1)l} S^k |n^{(l)}(2^{k+1}t/S - 2^{k+1} + 3)|}{S^l 2^{k(k+2)} \max_{0 \leq l \leq k} \|n^{(l)}\|_\infty} \leq \frac{1}{2^k} \quad (t \in \mathbb{R}). \quad (36)$$

Therefore  $\sum_{k=1}^\infty \|h_k^{(l)}\|_\infty < \infty$ , for every  $l = 0, 1, 2, \dots$ . Estimate (36) together with Proposition 24 stated in the Appendix implies the remaining claims.  $\square$

**Lemma 16.** *If  $f \in C_c^2((0, \infty); L^2(V))$  and  $t_0 \in \mathbb{R}$  is such that  $f_{t_0} := f(\cdot - t_0) \in C_c^2((0, \infty); L^2(V))$ , then*

$$L_V f_{t_0}(t) = \begin{cases} 0, & 0 \leq t < t_0, \\ L_V f(t - t_0), & t \geq \max\{0, t_0\}. \end{cases}$$

*Proof.* This follows from a straightforward change of variables in (32).  $\square$

Following proposition states the essential fact that if the support of the source  $f$  is included in the time interval  $(0, T')$ , then the future evolution of  $u^f|_V$  is determined completely by its evolution up to time  $T'$ .

**Proposition 17.** *Let  $T' > 0$  and consider  $f \in C_c^2((0, T'); L^2(V))$ . If  $L_{V, T'} f = \tilde{L}_{\tilde{V}, T'} f$ , then  $L_V f = \tilde{L}_{\tilde{V}} f$ .*

*Proof.* The proposition will be proved by extending  $L_V f$  holomorphically onto a region of the complex plane and applying the uniqueness of holomorphic continuation. For properties of vector-valued holomorphic functions, we refer to [52].

Fix  $\epsilon > 0$  small enough so that  $\text{supp } f \subset (0, T' - 2\epsilon)$ . We show that the  $L^2(V)$ -valued mapping

$$\mathbb{R} \supset (T' - \epsilon, \infty) \ni t \mapsto L_V f(t) \in L^2(V)$$

extends holomorphically onto the complex region  $\{z \in \mathbb{C} : \text{Re}(z) > T' - \epsilon\}$ . This is enough to prove the claim. Namely, in this case also  $\tilde{L}_{\tilde{V}} f$  extends holomorphically onto the region, and by assumption the extensions agree on  $(T' - \epsilon, T')$ . By uniqueness of holomorphic extension, they agree everywhere on the region, so in particular also on  $(T' - \epsilon, \infty)$ .

Consider complex-valued functions  $g_k$  on  $\mathbb{C}_+ \times [0, \infty)$ , where  $k \in \mathbb{Z}^+$ , defined by

$$g_k(z, \tau) := \langle f(\tau), \varphi_k \rangle_{L^2(M)} (z + T' - \epsilon - \tau)^{\alpha-1} E_{\alpha, \alpha}(-\lambda_k^\beta (z + T' - \epsilon - \tau)^\alpha).$$

By assumption  $f(\tau) = 0$  if  $\tau \geq T' - 2\epsilon$ , therefore the functions are well-defined. In addition, an inspection shows that

$$L_V f(z + T' - \epsilon) = \sum_{k=1}^{\infty} \int_0^{T'} g_k(z, \tau) d\tau \varphi_k|_V \quad (z \in (0, \infty)). \quad (37)$$

We show that the right-hand side of (37) is an  $L^2(V)$ -valued holomorphic function on  $\mathbb{C}_+$ .

By (i) of Proposition 4,  $E_{\alpha, \alpha}$  is bounded on  $\mathbb{C} \setminus \mathbb{C}_+$ . It follows that with a constant  $C = C(\alpha, \epsilon) > 0$  we have

$$|g_k(z, \tau)| \leq C |\langle f(\tau), \varphi_k \rangle_{L^2(M)}| \quad (z \in \mathbb{C}_+, \tau \geq 0, k \geq 1). \quad (38)$$

Consequently

$$\left| \int_0^{T'} g_k(z, \tau) d\tau \right|^2 \leq C^2 T' \int_0^{T'} |\langle f(\tau), \varphi_k \rangle_{L^2(M)}|^2 d\tau \quad (z \in \mathbb{C}_+, k \geq 1). \quad (39)$$

Let  $D_k \geq 0$  denote the right-hand side of (39). Then for all  $N > M > 0$  and  $z \in \mathbb{C}_+$  it holds that

$$\left\| \sum_{k=1}^N \int_0^{T'} g_k(z, \tau) d\tau \varphi_k - \sum_{k=1}^M \int_0^{T'} g_k(z, \tau) d\tau \varphi_k \right\|_{L^2(M)}^2 \leq \sum_{k=M+1}^N D_k,$$

and

$$\sum_{k=1}^{\infty} D_k = C^2 T' \int_0^{T'} \|f(\tau)\|_{L^2(M)}^2 d\tau < \infty.$$

It follows from the Cauchy criterion for uniform convergence that for every  $z \in \mathbb{C}_+$  the series

$$\sum_{k=1}^{\infty} \int_0^{T'} g_k(z, \tau) d\tau \varphi_k \quad (40)$$

converges in the topology of  $L^2(M)$ , and that the convergence is uniform in  $z \in \mathbb{C}_+$ .

Using the fact that for every  $\tau \geq 0$  the function  $g_k(\cdot, \tau)$  is holomorphic on  $\mathbb{C}_+$ , it is straightforward to verify with Morera's theorem and Fubini's theorem that the function

$$\mathbb{C}_+ \ni z \mapsto \int_0^{T'} g_k(z, \tau) d\tau \in \mathbb{C}$$

is also holomorphic. As a uniform limit of holomorphic functions, the  $L^2(M)$ -valued function defined on  $\mathbb{C}_+$  by the series (40) is holomorphic. Consequently also the function

$$\mathbb{C}_+ \ni z \mapsto \sum_{k=1}^{\infty} \int_0^{T'} g_k(z, \tau) d\tau \varphi_k|_V \in L^2(V) \quad (41)$$

is holomorphic. Because (41) extends (37) from  $(0, \infty)$  onto  $\mathbb{C}_+$ , the proof is finished.  $\square$

The following two results prepare for the proof of Proposition 2.

**Lemma 18.** *If*

$$L_{V,T} h = \tilde{L}_{\tilde{V},T} h, \quad (42)$$

*then  $L_V(h_k \psi_{r(k)}) = \tilde{L}_{\tilde{V}}(h_k \psi_{r(k)})$  for every  $k \in \mathbb{Z}^+$ .*

*Proof.* Assume that (42) holds. Consider an integer  $j \geq 0$  and for induction purposes assume that  $L_V(h_k \psi_{r(k)}) = \tilde{L}_{\tilde{V}}(h_k \psi_{r(k)})$  for  $1 \leq k \leq j$ . For  $j = 0$  this is vacuously true.

Suppose  $j > 0$  and let  $T' := (1 - 2^{-(j+1)})S < T$ . Inclusion (35) implies that for every  $l > j + 1$  the function  $h_l \psi_{r(l)}$  vanishes on  $(0, T')$ . This implies that

$$\sum_{k=1}^{j+1} L_{V, T'}(h_k \psi_{r(k)}) = L_{V, T'} h, \quad (43)$$

and an analogous equality holds for  $\tilde{L}_{\tilde{V}, T'}$ . Now equalities (42) and (43) and the fact that  $T' < T$  imply

$$\sum_{k=1}^{j+1} L_{V, T'}(h_k \psi_{r(k)}) = \sum_{k=1}^{j+1} \tilde{L}_{\tilde{V}, T'}(h_k \psi_{r(k)}). \quad (44)$$

By induction hypothesis the first  $j$  terms in the sums of (44) agree, therefore the last terms have to agree, also. This, and Proposition 17 imply  $L_V(h_{j+1} \psi_{r(j+1)}) = \tilde{L}_{\tilde{V}}(h_{j+1} \psi_{r(j+1)})$ , and the induction is finished.  $\square$

**Lemma 19.** *If  $L_{V, T} h = \tilde{L}_{\tilde{V}, T} h$ , then  $L_V(a\xi) = \tilde{L}_{\tilde{V}}(a\xi)$  for every  $a \in C_c^2((0, \infty))$  and  $\xi \in L^2(V)$ .*

*Proof.* The first step is to show that

$$L_V(a\psi_l) = \tilde{L}_{\tilde{V}}(a\psi_l) \quad (l \in \mathbb{Z}^+, a \in C_c^2((0, \infty))). \quad (45)$$

For that purpose, let us fix an integer  $l \in \mathbb{Z}^+$  and a function  $a \in C_c^2((0, \infty))$ , and choose a constant  $\delta = \delta(a) > 0$  such that  $a(t) = 0$  for  $t \in (-\infty, \delta]$ .

For every  $k \in \mathbb{Z}^+$ , define a scaled translate  $d_k$  of  $h_k$  by setting

$$d_k(t) := \frac{h_k(t + S)}{\int_{\mathbb{R}} h_k(\tau) d\tau} \quad (t \in \mathbb{R}).$$

Then  $d_k$  is a non-negative function and following hold:

$$\int_{\mathbb{R}} d_k(\tau) d\tau = 1 \text{ and } \text{supp } d_k \subset \left(-\frac{S}{2^{k-1}}, -\frac{S}{2^k}\right) \quad (k \in \mathbb{Z}^+). \quad (46)$$

It follows that the sequence  $(d_k * a)_{k=1}^{\infty}$  of convolutions and the sequence  $((d_k * a)')_{k=1}^{\infty}$  of their derivatives satisfy

$$d_k * a(t) \rightarrow a(t) \text{ and } (d_k * a)'(t) \rightarrow a'(t) \text{ as } k \rightarrow \infty \quad (t \in \mathbb{R}), \quad (47)$$

where both convergences are uniform in  $t \in \mathbb{R}$ .

From the inclusion of the support of  $d_k$  in (46), it follows that

$$a(t_0)d_k(\cdot - t_0) \in C_c^\infty((0, \infty)) \quad (2^{k-1} \geq S/\delta, t_0 \in \mathbb{R}). \quad (48)$$

From (48), Lemma 16, and Lemma 18, it follows that

$$L_V(a(t_0)d_k(\cdot - t_0)\psi_{r(k)}) = \tilde{L}_{\tilde{V}}(a(t_0)d_k(\cdot - t_0)\psi_{r(k)}) \quad (2^{k-1} \geq S/\delta, t_0 \in \mathbb{R}). \quad (49)$$

By the assumed property of the sequence  $r$ , there exists some  $k \in \mathbb{Z}^+$  such that  $2^{k-1} \geq S/\delta$  and  $r(k) = l$ . Choose any such  $k$ , and define a sequence  $(b_m)_{m=1}^\infty \in C_c^\infty((0, \infty))$  of functions by setting

$$b_m(t) := \frac{1}{m} \sum_{j \in \mathbb{Z}} a(j/m) d_k(t - j/m) \quad (t \in \mathbb{R}, m \in \mathbb{Z}^+). \quad (50)$$

Note that the sum is always in fact finite, and by (49) it holds that

$$L_V(b_m \psi_l) = \tilde{L}_{\tilde{V}}(b_m \psi_l) \quad (m \in \mathbb{Z}^+). \quad (51)$$

The Riemann sums (50) satisfy

$$b_m(t) \rightarrow d_k * a(t) \text{ and } b'_m(t) \rightarrow (d_k * a)'(t) \text{ as } m \rightarrow \infty \quad (t \in \mathbb{R}), \quad (52)$$

uniformly in  $t \in \mathbb{R}$  (see, e.g., Lemma 4.1.3 in [26]). Because all supports of the functions  $b_m$  as well as the support of  $d_k * a$  are included in some bounded interval, the uniform convergence (52) together with Proposition 13 and equality (51) imply that in  $L^2(V)$  it holds that

$$L_V(d_k * a \psi_l) = \lim_{m \rightarrow \infty} L_V(b_m \psi_l) = \lim_{m \rightarrow \infty} \tilde{L}_{\tilde{V}}(b_m \psi_l) = \tilde{L}_{\tilde{V}}(d_k * a \psi_l). \quad (53)$$

Once more we use the assumed property of the sequence  $r$  to choose an increasing sequence  $(k_j)_{j=1}^\infty$  of indices such that  $2^{k_j-1} \geq S/\delta$  and  $r(k_j) = l$  for every  $j \in \mathbb{Z}^+$ . Then (53) holds for every index  $k_j$ . If we let  $j \rightarrow \infty$ , using (47) and the same reasoning as above, we obtain

$$L_V(a \psi_l) = \lim_{j \rightarrow \infty} L_V(d_{k_j} * a \psi_l) = \lim_{j \rightarrow \infty} \tilde{L}_{\tilde{V}}(d_{k_j} * a \psi_l) = \tilde{L}_{\tilde{V}}(a \psi_l).$$

Finally we can use the denseness of  $\text{span}\{\psi_l : l \in \mathbb{Z}^+\} \subset L^2(V)$  and (45) to conclude by Proposition 13 that (45) holds also if  $\psi_l$  is replaced by an arbitrary function  $\xi \in L^2(V)$ .  $\square$

We are now ready to prove Proposition 2:

*Proof of Proposition 2.* We need to prove that  $L_V = \tilde{L}_{\tilde{V}}$ .

Pick  $f \in C_c^2((0, \infty); L^2(V))$  and let  $(\xi_k)_{k=1}^\infty \subset L^2(V)$  be an orthonormal basis. Define

$$f_N(t) := \sum_{k=1}^N \langle f(t), \xi_k \rangle_{L^2(V)} \xi_k \quad (t \in \mathbb{R}, N \in \mathbb{Z}^+).$$

Then for  $T' > 0$  large enough it holds that

$$(f_N)_{N=1}^\infty \subset C_c^2((0, T'); L^2(V)),$$

$$\sup_{\substack{N \in \mathbb{Z}^+, \\ t \in (0, T')}} \|f'_N(t)\|_{L^2(V)} \leq \sup_{t \in \mathbb{R}} \|f'(t)\|_{L^2(V)} < \infty, \text{ and}$$

$$f'_N(t) \rightarrow f'(t) \text{ as } N \rightarrow \infty, \text{ for every } t \in \mathbb{R}.$$

Consequently the conditions of Proposition 13 hold, and therefore for every  $t \geq 0$  we have

$$L_V f(t) = \lim_{N \rightarrow \infty} L_V f_N(t) = \lim_{N \rightarrow \infty} \tilde{L}_{\tilde{V}} f_N(t) = \tilde{L}_{\tilde{V}} f(t).$$

Here the convergence is in the topology of  $L^2(V)$ , the middle equality is due to Lemma 19 and the linearity of  $L_V$  and  $\tilde{L}_{\tilde{V}}$ , and other equalities are due to Proposition 13. Because  $f \in C_c^2((0, \infty); L^2(V))$  is arbitrary, the proof is finished.  $\square$

## 4.2 The local source-to-solution operator $L_V$ determines the manifold

Let  $(\lambda_k)_{k=1}^\infty$  and  $(\varphi_k)_{k=1}^\infty$  be as in Section 3.1. Let  $(q_k)_{k=1}^\infty \subset \mathbb{Z}^+$  be a sequence such that  $(\lambda_{q_k})_{k=1}^\infty$  is the strictly increasing sequence that contains all distinct eigenvalues of  $-\Delta_g$ , and let  $E_k \subset L^2(M)$  be the eigenspace corresponding to the eigenvalue  $\lambda_{q_k}$ . Furthermore, let  $P_k : L^2(M) \rightarrow L^2(M)$  be the orthogonal projection onto  $E_k$ , and define  $P_{V,k} : L^2(V) \rightarrow L^2(V)$  by

$$P_{V,k} u := (P_k u)|_V \quad (u \in L^2(V) \subset L^2(M)).$$

Then  $P_k \in B(L^2(M))$  and  $P_{V,k} \in B(L^2(V))$ , where the sets are the spaces of bounded linear operators on  $L^2(M)$  and  $L^2(V)$ , respectively. We consider these spaces as normed spaces with the operator norm.

Suppose that  $\varphi_{K+1}, \varphi_{K+2}, \dots, \varphi_{K+\dim E_k}$  is the subsequence of the orthonormal basis  $(\varphi_k)_{k=1}^\infty$  that spans the eigenspace  $E_k$ . Then

$$P_k u = \sum_{k=K+1}^{K+\dim E_k} \langle u, \varphi_k \rangle_{L^2(M)} \varphi_k \quad (u \in L^2(M)).$$

It follows from (32) that the local source-to-solution operator can be written as

$$L_V f(t) = \sum_{k=1}^{\infty} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{q_k}^\beta (t-\tau)^\alpha) P_{V,k} f(\tau) d\tau \quad (t \geq 0),$$

where the sum converges in  $L^2(V)$ , for every  $t \geq 0$ .

**Proposition 20.** *Consider the region  $\Omega := \mathbb{C} \setminus (-\infty, 0]$  and the  $B(L^2(V))$ -valued mapping  $H_V$  on  $\Omega$  defined by*

$$H_V(z) := \sum_{k=1}^{\infty} \frac{1}{z + \lambda_{q_k}^\beta} P_{V,k} \quad (z \in \Omega). \quad (54)$$

*Then the following hold:*

1. For every  $z \in \Omega$  the series (54) converges in  $B(L^2(V))$  in the operator norm topology, and the  $B(L^2(V))$ -valued function  $H_V$  is holomorphic on  $\Omega$ .
2. For every  $z_0 \in \mathbb{C}$  the following limit holds:

$$\lim_{\substack{z \rightarrow -z_0, \\ z \in \Omega}} (z + z_0)H_V(z) = \begin{cases} 0, & z_0 \notin \{\lambda_{q_k}^\beta : k \in \mathbb{Z}^+\}, \\ P_{V,k}, & z_0 = \lambda_{q_k}^\beta. \end{cases} \quad (55)$$

Note that as every complex number is a limit point of  $\Omega$ , the limit (55) can be considered also for  $z_0 \in [0, \infty)$ .

*Proof.* In what follows, it is convenient to explicitly write out the zero extension and restriction operators. Thus, let  $Z : L^2(V) \rightarrow L^2(M)$  and  $Z^* : L^2(M) \rightarrow L^2(V)$  be the operators that extend a function with zero from  $V$  to  $M$ , and restrict a function on  $M$  to  $V$ , respectively.

1. It follows from  $\lim_{k \rightarrow \infty} \lambda_{q_k}^\beta = \infty$  and the fact that the operators  $P_k$  project onto mutually orthogonal subspaces, that the function

$$\Omega \ni z \mapsto H(z) := \sum_{k=1}^{\infty} \frac{1}{z + \lambda_{q_k}^\beta} P_k \in B(L^2(M)) \quad (56)$$

is holomorphic on  $\Omega$ . As

$$H_V(z) = Z^* H(z) Z, \quad (57)$$

the function  $H_V$  is holomorphic on  $\Omega$ , as well.

2. Suppose that  $z_0 \notin \{\lambda_{q_k}^\beta : k \in \mathbb{Z}^+\}$ . Then there exists a constant  $\delta > 0$  such that if  $|z + z_0| < \delta$ , then

$$|z + \lambda_{q_k}^\beta| > \delta \quad (k \in \mathbb{Z}^+). \quad (58)$$

It is easy to see that for every  $z \in \Omega$  for which (58) holds, also

$$\|(z + z_0)H(z)\|_{B(L^2(M))} < \frac{|z + z_0|}{\delta}$$

holds. Letting  $z \rightarrow -z_0$  in  $\Omega$  and using the boundedness of  $Z$  and  $Z^*$  in (57) prove the first case of (55).

If  $z_0 = \lambda_{q_k}^\beta$ , we can write

$$(z + \lambda_{q_k}^\beta)H_V(z) = P_{V,k} + (z + \lambda_{q_k}^\beta) \sum_{\substack{l=1, \\ l \neq k}}^{\infty} \frac{1}{z + \lambda_{q_l}^\beta} P_{V,l}. \quad (59)$$

As  $z \rightarrow -\lambda_{q_k}^\beta$  in  $\Omega$ , the right-hand side of (59) tends to  $P_{V,k}$  by the same reasoning as above.  $\square$



The following proposition relates the function  $H_V$  to the Laplace transforms of  $L_V f$  and  $f$ . It plays an essential part in the proof of Theorem 3:

**Proposition 21.** *Let  $H_V$  be as in Proposition 20. For every source  $f \in C_c^2((0, \infty); L^2(V))$  and complex number  $s \in \mathbb{C}_+$ , the Laplace transform of  $f$  and  $L_V f$  exist at the point  $s$ , and they are related by the equality*

$$\mathcal{L}L_V f(s) = H_V(s^\alpha)\mathcal{L}f(s) \quad (s \in \mathbb{C}_+). \quad (60)$$

*Proof.* In this proof it is also convenient to use the extension and restriction operators  $Z$  and  $Z^*$  from the proof of Proposition 20.

Let  $f \in C_c^2((0, \infty); L^2(V))$  and let

$$u^{Zf} \in C^1([0, \infty); L^2(M)) \cap L^\infty([0, \infty); L^2(M))$$

be the strong solution of (1) with the source  $Zf \in C_c^2((0, \infty); L^2(M))$ . The boundedness of the function  $u^{Zf}$  implies that for every  $s \in \mathbb{C}_+$ , the  $L^2(M)$ -valued function  $[0, \infty) \ni \tau \mapsto e^{-s\tau}u^{Zf}(\tau)$  is integrable, so that the Laplace transform

$$\mathcal{L}u^{Zf}(s) := \int_0^\infty e^{-s\tau}u^{Zf}(\tau) d\tau$$

is defined for all  $s \in \mathbb{C}_+$ . Due to the compact support of  $f$ , an analogous reasoning shows that the Laplace transform  $\mathcal{L}Zf(s)$  is defined for all  $s \in \mathbb{C}$ .

We can write

$$u^{Zf}(t) = \sum_{k=1}^{\infty} (F_k * \langle Zf, \varphi_k \rangle_{L^2(M)})(t)\varphi_k \quad (t \geq 0),$$

where the functions  $F_k$  are defined in (22). By Proposition 23, we can take the Laplace transform of  $u^{Zf}$  componentwise. With (ii) of Proposition 4 and  $H$  as defined in (56), this results in

$$\begin{aligned} \mathcal{L}u^{Zf}(s) &= \sum_{k=1}^{\infty} \mathcal{L}(F_k * \langle Zf, \varphi_k \rangle_{L^2(M)})(s)\varphi_k \\ &= \sum_{k=1}^{\infty} \frac{1}{s^\alpha + \lambda_k^\beta} \langle \mathcal{L}Zf(s), \varphi_k \rangle_{L^2(M)}\varphi_k \\ &= \sum_{k=1}^{\infty} \frac{1}{s^\alpha + \lambda_{q_k}^\beta} P_k \mathcal{L}Zf(s) \\ &= H(s^\alpha)\mathcal{L}Zf(s) \quad (s \in \mathbb{C}_+). \end{aligned}$$

Using the fact that  $\mathcal{L}$  commutes with  $Z$  and  $Z^*$ , we obtain

$$\begin{aligned} \mathcal{L}L_V f(s) &= \mathcal{L}Z^*u^{Zf}(s) \\ &= Z^*H(s^\alpha)\mathcal{L}Zf(s) \\ &= H_V(s^\alpha)\mathcal{L}f(s) \quad (s \in \mathbb{C}_+). \end{aligned} \quad \square$$

To consider Theorem 3, let  $(\widetilde{M}, \widetilde{g})$ ,  $\widetilde{V} \subset \widetilde{M}$ ,  $\theta : \text{cl}(\widetilde{V}) \rightarrow \text{cl}(V)$ ,  $L_{\widetilde{V}}$ , and  $\widetilde{L}_{\widetilde{V}}$  be as in Section 4.1. Let  $(\widetilde{\lambda}_k)_{k=1}^{\infty} \subset [0, \infty)$  be the sequence of eigenvalues of  $-\Delta_{\widetilde{g}}$  (counted with multiplicities), and let  $(\widetilde{q}_k)_{k=1}^{\infty} \subset \mathbb{Z}^+$  be a sequence such that  $(\widetilde{\lambda}_{\widetilde{q}_k})_{k=1}^{\infty}$  is the strictly increasing sequence of all distinct eigenvalues of  $-\Delta_{\widetilde{g}}$ . Define operators  $(P_{\widetilde{V},k})_{k=1}^{\infty} \subset B(L^2(\widetilde{V}))$  and function  $H_{\widetilde{V}} : \Omega \rightarrow B(L^2(\widetilde{V}))$  analogously to  $(P_{V,k})_{k=1}^{\infty} \subset B(L^2(V))$  and  $H_V : \Omega \rightarrow B(L^2(V))$ , respectively.

Let

$$\widetilde{P}_{\widetilde{V},k} := (\theta^*)^{-1} P_{\widetilde{V},k} \theta^* \in B(L^2(V)) \quad (k \in \mathbb{Z}^+)$$

be the conjugated operators, and let  $\widetilde{H}_{\widetilde{V}}$  be the pointwise conjugated  $B(L^2(V))$ -valued function defined by

$$\widetilde{H}_{\widetilde{V}}(z) := (\theta^*)^{-1} H_{\widetilde{V}}(z) \theta^* = \sum_{k=1}^{\infty} \frac{1}{z + \widetilde{\lambda}_{\widetilde{q}_k}^{\beta}} \widetilde{P}_{\widetilde{V},k} \quad (z \in \Omega).$$

**Proposition 22.** *The local source-to-solution operator  $L_V$  uniquely determines the pairs  $(\lambda_{q_k}, P_{V,k})$ , i.e., if  $L_V = \widetilde{L}_{\widetilde{V}}$ , then*

$$\{(\lambda_{q_k}, P_{V,k}) : k \in \mathbb{Z}^+\} = \{(\widetilde{\lambda}_{\widetilde{q}_k}, \widetilde{P}_{\widetilde{V},k}) : k \in \mathbb{Z}^+\}. \quad (61)$$

*Proof.* The pointwise conjugated function  $\widetilde{H}_{\widetilde{V}}$  is holomorphic on  $\Omega$ , and the limit (55) holds if  $H_V$ ,  $P_{V,k}$  and  $\lambda_{q_k}$  are replaced by  $\widetilde{H}_{\widetilde{V}}$ ,  $\widetilde{P}_{\widetilde{V},k}$  and  $\widetilde{\lambda}_{\widetilde{q}_k}$ , respectively. Also, equality (60) holds if  $L_V$  and  $H_V$  are replaced by  $\widetilde{L}_{\widetilde{V}}$  and  $\widetilde{H}_{\widetilde{V}}$ , respectively.

Fix a nonzero non-negative function  $a(t) \in C_c^2((0, \infty))$  and pick an arbitrary function  $\xi \in L^2(V)$ . If  $L_V(a(t)\xi) = \widetilde{L}_{\widetilde{V}}(a(t)\xi)$ , then

$$\begin{aligned} (\mathcal{L}a(s))H_V(s^\alpha)\xi &= \mathcal{L}(L_V(a(t)\xi))(s) \\ &= \mathcal{L}(\widetilde{L}_{\widetilde{V}}(a(t)\xi))(s) \\ &= (\mathcal{L}a(s))\widetilde{H}_{\widetilde{V}}(s^\alpha)\xi \quad (s \in \mathbb{C}_+), \end{aligned} \quad (62)$$

where the first and last equality are due to (60). Because  $\xi \in L^2(V)$  is arbitrary and  $\mathcal{L}a(s) > 0$  for every  $s \in \mathbb{R}_+$ , equality (62) implies

$$H_V(s) = \widetilde{H}_{\widetilde{V}}(s) \quad (s \in \mathbb{R}_+). \quad (63)$$

Proposition 20 states that both sides of (63) are holomorphic functions on  $\Omega$ . Because  $\Omega$  is a region and the functions agree on  $(0, \infty)$ , they must agree everywhere on  $\Omega$ .

Due to the unique continuation principle, note that the functions  $(\varphi_k|_V)_{k=1}^{\infty}$  are linearly independent, and therefore  $P_{V,k} \neq 0$ . From (55) and the fact

that  $H_V = \tilde{H}_{\tilde{V}}$  on  $\Omega$  it follows that

$$\begin{aligned} P_{V,k} &= \lim_{\substack{s \rightarrow -\lambda_{q_k}^\beta \\ s \in \Omega}} (s + \lambda_{q_k}^\beta) H_V(s) \\ &= \lim_{\substack{s \rightarrow -\lambda_{q_k}^\beta \\ s \in \Omega}} (s + \lambda_{q_k}^\beta) \tilde{H}_{\tilde{V}}(s) \\ &= \begin{cases} 0, & \lambda_{q_k}^\beta \notin \{\tilde{\lambda}_{\tilde{q}_l}^\beta : l \in \mathbb{Z}^+\}, \\ \tilde{P}_{\tilde{V},l}, & \lambda_{q_k}^\beta = \tilde{\lambda}_{\tilde{q}_l}^\beta. \end{cases} \end{aligned}$$

Thus there must exist an index  $l \in \mathbb{Z}^+$  such that  $\lambda_{q_k} = \tilde{\lambda}_{\tilde{q}_l}$  and  $P_{V,k} = \tilde{P}_{\tilde{V},l}$ .

We have shown that the left-hand side of (61) is a subset of the right-hand side. The other direction follows from symmetry.  $\square$

In order to prove Theorem 3, we reduce the situation from the fractional diffusion equation (1) to that of the wave equation on the same manifold  $(M, g)$ :

$$\begin{aligned} (\partial_t^2 - \Delta_g)w(x, t) &= p(x, t), & (x, t) \in M \times (0, \infty), \\ w(x, 0) &= 0, & x \in M, \\ \partial_t w(x, 0) &= 0, & x \in M. \end{aligned} \tag{64}$$

For a source  $p \in C_c^\infty(V \times (0, \infty))$ , let  $w^p \in C^\infty(M \times (0, \infty))$  denote the unique solution of (64), and define the hyperbolic local source-to-solution operator  $L_V^{\text{hyp}} : C_c^\infty(V \times (0, \infty)) \rightarrow C^\infty(V \times (0, \infty))$  by

$$L_V^{\text{hyp}} p := w^p|_{V \times (0, \infty)}.$$

*Proof of Theorem 3.* Consider  $p \in C_c^\infty(V \times (0, \infty))$ . The solution  $w^p$  of the wave equation (64) can be written as

$$w^p(x, t) = \sum_{k=1}^{\infty} \int_0^t s_k(t - \tau) \langle p(\cdot, \tau), \varphi_k \rangle_{L^2(M)} \varphi_k(x) d\tau \quad (t \geq 0), \tag{65}$$

where

$$s_1(t) := t \text{ and } s_k(t) := \frac{\sin(\sqrt{\lambda_k} t)}{\sqrt{\lambda_k}}, \text{ for } k \geq 2,$$

and the series (65) converges in  $L^2(M \times [0, T])$ , for every  $T > 0$  (see Corollary 2 of [24]). Thus

$$L_V^{\text{hyp}} p(x, t) = \sum_{k=1}^{\infty} \int_0^t s_{q_k}(t - \tau) P_{V,k} p(x, \tau) d\tau \quad (t \geq 0), \tag{66}$$

where the series converges in  $L^2(V \times [0, T])$ , for every  $T > 0$ .

By Proposition 22, the local source-to-solution operator  $L_V$  for the fractional diffusion equation (1) uniquely determines the eigenvalues  $(\lambda_{q_k})_{k=1}^\infty \subset [0, \infty)$  of  $-\Delta_g$  and the corresponding operators  $(P_{V,k})_{k=1}^\infty \subset B(L^2(V))$ . From (66) it follows that these in turn uniquely determine the hyperbolic local source-to-solution operator  $L_V^{\text{hyp}}$  for the wave equation (64). By Theorem 2 of [24],  $L_V^{\text{hyp}}$  determines the manifold  $(M, g)$  up to a Riemannian isometry. The proof is finished.  $\square$

*Proof of Theorem 1.* Let  $h \in C_c^\infty((0, T); L^2(V))$  be a source as defined in Definition 14, and suppose (3) of Theorem 1 holds with this source. Then (5) of Proposition 2 holds, and consequently (6) holds for every source  $f \in C_c^2((0, \infty); L^2(V))$ . Therefore, by Theorem 3, the manifolds  $(M, g)$  and  $(\tilde{M}, \tilde{g})$  are Riemannian isometric.  $\square$

## A Calculus of Hilbert space valued functions of a real variable

Consider a measurable subset  $I \subset \mathbb{R}$  and a function  $y : I \rightarrow L^2(M)$ . We say that  $y$  is *integrable* if it is strongly measurable and  $\int_I \|y(\tau)\|_{L^2(M)} d\tau < \infty$ . The measure on  $I$  is the Lebesgue measure, and measurability of the norm is a consequence of the strong measurability of  $y$ . For  $1 \leq p \leq \infty$ , the space  $L^p(I; L^2(M))$  consists of those strongly measurable functions  $y : I \rightarrow L^2(M)$  for which  $\|y\|_{L^2(M)} \in L^p(I)$ . We recall that for  $y \in L^1(I; L^2(M))$  the (Bochner) integral  $\int_I y(\tau) d\tau \in L^2(M)$  is defined. For general theory of integration of functions with values in a Banach space, we refer the reader to [36].

Suppose then that  $I \subset \mathbb{R}$  is an interval with at least two points and fix a point  $t \in I$ . We recall that  $y$  is said to be *differentiable at the point  $t$* , if there exists a function  $\xi \in L^2(M)$  such that

$$\lim_{h \rightarrow 0} \left\| \frac{y(t+h) - y(t)}{h} - \xi \right\|_{L^2(M)} = 0.$$

If  $t$  is an endpoint of  $I$ , the limit is the appropriate one-sided limit.

The *derivative of  $y$  at  $t$* , denoted by  $y'(t)$ , is defined to be the function  $\xi \in L^2(M)$ . If  $y$  is differentiable at every point of  $I$  and the so obtained function  $y' : I \rightarrow L^2(M)$  is continuous,  $y$  is *continuously differentiable*. The space of all continuously differentiable functions is denoted by  $C^1(I; L^2(M))$ . Higher order derivatives and spaces  $C^k(I; L^2(M))$  are defined recursively exactly as in the case of scalar functions.

**Proposition 23.** *Consider a function  $y : I \rightarrow L^2(M)$ ,  $y(t) = \sum_{k=1}^\infty y_k(t)\psi_k$ , where  $I \subset \mathbb{R}$  is measurable,  $y_k : I \rightarrow \mathbb{C}$  are complex-valued functions,  $(\psi_k)_{k=1}^\infty \subset L^2(M)$  is an orthonormal basis, and the series converges in  $L^2(M)$  for every  $t \in I$ . Then the following hold:*

(i) The  $L^2(M)$ -valued function  $y$  is strongly measurable, if and only if all the complex-valued functions  $y_k : I \rightarrow \mathbb{C}$  are measurable.

(ii) If  $y \in L^1(I; L^2(M))$ , then

$$\int_I y(\tau) d\tau = \sum_{k=1}^{\infty} \left[ \int_I y_k(\tau) d\tau \right] \psi_k. \quad (67)$$

Note that the integrals on the right-hand side of (67) are ordinary Lebesgue integrals of complex-valued functions.

*Proof.* If  $y$  is strongly measurable and  $k \in \mathbb{Z}^+$ , the component function  $y_k = \langle y, \psi_k \rangle_{L^2(M)}$  is strongly measurable as the composition of a continuous function with a strongly measurable function. A scalar strongly measurable function on  $I$  is measurable.

On the other hand, if  $y_k : I \rightarrow \mathbb{C}$  is measurable, it is an almost everywhere limit of complex-valued step functions, and therefore the  $L^2(M)$ -valued map  $y_k \psi_k : I \ni t \mapsto y_k(t) \psi_k \in L^2(M)$  is an almost everywhere limit of  $L^2(M)$ -valued step maps. In other words,  $y_k \psi_k$  is strongly measurable. It follows that the partial sums  $\sum_{k=1}^N y_k \psi_k$  are strongly measurable, and therefore  $y$  as their pointwise limit in  $L^2(M)$  is strongly measurable.

If  $y \in L^1(I; L^2(M))$ , it is a property of the integral that

$$\left\langle \int_I y(\tau) d\tau, \xi \right\rangle_{L^2(M)} = \int_I \langle y(\tau), \xi \rangle_{L^2(M)} d\tau \quad (\xi \in L^2(M)).$$

Applying this with  $\xi = \psi_k$  proves (67).  $\square$

**Proposition 24.** Let  $X$  be a Banach space,  $(\xi_k)_{k=1}^{\infty} \subset X$  be a bounded sequence, and  $(h_k)_{k=1}^{\infty} \subset C^\infty(I)$  be a sequence of complex-valued functions, where  $I \subset \mathbb{R}$  is an open set. Suppose that the derivatives  $h_k^{(l)}$  satisfy

$$\sum_{k=1}^{\infty} \|h_k^{(l)}\|_{\infty} < \infty \quad (l = 0, 1, 2, \dots). \quad (68)$$

Then the series  $f := \sum_{k=1}^{\infty} h_k \xi_k$  converges uniformly on  $I$ , and  $f \in C^\infty(I; X)$ . Furthermore, the derivatives of  $f$  are obtained by term-wise differentiation, and also those series converge uniformly on  $I$ .

*Proof.* The terms  $h_k \xi_k : I \rightarrow X$  are continuous, and by the Weierstrass M-test (using the boundedness of  $\xi_k$  and (68)), the series  $\sum_{k=1}^{\infty} h_k \xi_k$  converges uniformly to  $f$ . It follows that  $f$  is continuous.

For  $N \in \mathbb{Z}^+$  we have

$$\left( \sum_{k=1}^N h_k \xi_k \right)' = \sum_{k=1}^N h_k' \xi_k,$$

and the same reasoning as above implies that

$$\left( \sum_{k=1}^N h_k \xi_k \right)' \rightarrow \sum_{k=1}^{\infty} h'_k \xi_k$$

uniformly as  $N \rightarrow \infty$ . It follows from standard results of differentiation (see, e.g., [36], Theorem 9.1) that  $f$  is differentiable and  $f' = \sum_{k=1}^{\infty} h'_k \xi_k$ .

An easy induction finishes the proof.  $\square$

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