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Zero-free neighborhoods around the unit circle for Kac polynomials

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Abstract

In this paper, we study how the roots of the Kac polynomials $W_n(z) = \sum_{k=0}^{n-1} \xi_k z^k$ concentrate around the unit circle when the coefficients of W_n are independent and identically distributed nondegenerate real random variables. It is well known that the roots of a Kac polynomial concentrate around the unit circle as $n \to \infty$ if and only if $\mathbb{E}[\log(1 + |\xi_0|)] < \infty$. Under the condition $\mathbb{E}[\xi_0^2] < \infty$, we show that there exists an annulus of width $\mathrm{O}(n^{-2}(\log n)^{-3})$ around the unit circle which is *free* of roots with probability $1 - \mathrm{O}((\log n)^{-1/2})$. The proof relies on small ball probability inequalities and the least common denominator used in [17].

Keywords Locally sub-Gaussian random variables · Salem–Zygmund type inequalities · Small ball probability · Zeros of random polynomials

Mathematics Subject Classification Primary 60G99 · 12D10; Secondary · 11CXX · 30C15

1 Introduction

The z-transform (a particular case is the discrete Fourier transform) is an important tool in signal analysis and speech recognition. In this context, the study of the zeros of z-transforms provides useful information on a signal. The existence of a region free of zeros around the unit circle is important to the proper working of the z-transform. For further details, see Chapter 3 in [4]. Roughly speaking, in this paper we find a region free of zeros around the unit circle for Kac polynomials, which are closely related to the discrete Fourier transform.

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Let $n \in \mathbb{N}$ and let $\xi_0, \ldots, \xi_{n-1}$ be independent and identically distributed (iid for short) nondegenerate real random variables (rvs for short) defined in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $\mathbb{E}$ the expectation with respect to the measure $\mathbb{P}$. The Kac polynomial W_n is defined as the random polynomial of degree $n-1$ given by

$$W_n(z) = \sum_{j=0}^{n-1} \xi_j z^j, \quad z \in \mathbb{C}.$$

In the sequel, we introduce the basic notation and terminology that will be used throughout this paper. For any $z \in \mathbb{C}$, let $|z|$ denote the modulus of z and $\arg(z)$ the argument of z. Choose $a, b \in \mathbb{R}$ such that $a \leq b$. Let $R_n(a, b)$ denote the number of roots of W_n in the annulus $\{z \in \mathbb{C} : a \leq |z| \leq b\}$ and, for any $\alpha, \beta \in [-\pi, \pi]$ such that $\alpha \leq \beta$, let $S_n(\alpha, \beta)$ denote the number of roots in $\{z \in \mathbb{C} : \alpha \leq \arg(z) \leq \beta\}$.

Shparo and Shur proved in [20] that under general conditions on the random coefficients (rcs for short), the roots of W_n concentrate around the unit circle with asymptotically uniform distribution in the argument as n increases. Moreover, Ibragimov and Zaporozhets showed in [7] that the rcs of W_n are nondegenerate satisfying $\mathbb{E}[\log(1 + |\xi_0|)] < \infty$ if and only if its roots are asymptotically concentrated near the unit circle. Later, Kabluchko and Zaporozhets provided in [9] a wide description of the localization of the roots for different conditions on the rcs. We point out that the localization of the roots of Kac polynomials determine the poor efficiency of some algorithms for speech recognition and signal processing applications; see [5] for further details.

Ibragimov and Zaporozhets proved in [7] that

$$\mathbb{P}\Big(\lim_{n\to\infty} \frac{1}{n} R_n\,(1-\delta, 1+\delta) = 1 \Big) = 1 \quad \text{holds for any } \delta \in (0, 1)$$

if and only if $\mathbb{E}[\log(1 + |\xi_0|)] < \infty$. They also proved that for any distribution ξ_0 and $\alpha, \beta \in (-\pi, \pi)$ the following holds:

$$\mathbb{P}\Big(\lim_{n\to\infty} \frac{1}{n} S_n\,(\alpha, \beta) = \frac{\beta - \alpha}{2\pi} \Big) = 1.$$

Shepp and Vanderbei studied in [19] the case of iid standard Gaussian coefficients and showed that

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}[R_n(e^{-\delta/n}, e^{\delta/n})] = \frac{1 + e^{-2\delta}}{1 - e^{-2\delta}} - \frac{1}{\delta} \quad \text{for any } \delta > 0. \tag{1}$$

Later, Ibragimov and Zeitouni, in [8], extended (1) to the case of iid coefficients, whose common distribution belongs to the domain of attraction of an α-stable law:

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}[R_n(e^{-\delta/n}, e^{\delta/n})] = \frac{1 + e^{-\alpha\delta}}{1 - e^{-\alpha\delta}} - \frac{2}{\alpha\delta} \quad \text{for any } \delta > 0. \tag{2}$$

Note that for any $\delta > 0$, as $\alpha \to 0^+$ we have $\frac{1+e^{-\alpha\delta}}{1-e^{-\alpha\delta}} - \frac{2}{\alpha\delta} \to 0$. Then (2) may tend to zero as $n \to \infty$ when ξ_0 has a slowly varying tail distribution. In fact, Götze and Zaporozhets showed in [6] that if $|\xi_0|$ has a slowly varying tail distribution, then

$$\lim_{n\to\infty} \mathbb{P}\big(R_n(e^{-\delta/n}, e^{\delta/n}) = 0\big) = 1 \quad \text{for any } \delta > 0,$$

i.e., the roots of a Kac polynomial with iid rcs with a slowly varying tail distribution hit the unit circle with almost zero probability.

In the case that W_n has iid rcs whose common distribution belongs to the domain of attraction of an α-stable law, limit (2) yields that, for $\delta > 0$, W_n has at least one root in the annulus $R_{\delta,n} := \{z \in \mathbb{C} : e^{-\delta/n} \le |z| \le e^{\delta/n}\}$ with positive probability for all large n and

$$\mathbb{P}\big(R_n(e^{-\delta/n}, e^{\delta/n}) = n\big) \le \frac{1 + e^{-\alpha\delta}}{1 - e^{-\alpha\delta}} - \frac{2}{\alpha\delta} + \mathrm{o}(1).$$

Therefore, a remarkable question is to determine if there exists an annulus inside of $R_{\delta,n}$ such that W_n has at least one root on it or not. The existence of roots *pretty close* to the unit circle is an important aspect in the analysis of signals. This helps us to understand the contribution of the phase information of a signal. We refer to [5] for further details.

Shepp and Vanderbei conjectured in [19] that, with high probability, the nearest root of W_n to the unit circle is at a distance of order $\mathrm{O}(n^{-2})$. Later, Konyagin and Schlag showed in [13] that the last conjecture holds true when the rcs have standard Gaussian or Rademacher (uniform distribution on $\{-1, 1\}$) distribution. To be more precise, in [13] it is shown that there exists a positive constant C such that for any $t > 0$

$$\limsup_{n\to\infty} \mathbb{P}\Big(\min_{||z|-1|\le tn^{-2}} |W_n(z)| \le tn^{-1/2}\Big) \le Ct. \tag{3}$$

Moreover, in (2.3) in [13] the limit

$$\mathbb{P}\left(\min_{x\in[0,1]} |W_n(x)| \le n^{-1/2}(\log n)^{-\gamma}\right) = \mathrm{o}(1), \quad \text{as } n \to \infty, \tag{4}$$

is established for $\gamma > 1/2$ and iid Gaussian rcs.

Karapetyan in [12] mentioned that it is possible to extend the above result under the assumption of nondegenerate real *sub-Gaussian* rcs, but only a sketch of the proof was given. Moreover, he claimed that the previous result can be extended under the finiteness of the third moment on the rcs. However, Karapetyan in [11] showed that for iid rcs with zero mean and finite third moment, it follows, for any $\epsilon \in (0, 1)$ and $n > 16C^{\frac{9936}{\epsilon^3}}$,

$$\mathbb{P}\Big(\min_{x\in[0,1]} \Big|\sum_{j=0}^{n-1} \xi_j e^{ijx}\Big| \ge n^{-1/2+\epsilon}\Big) \le \frac{1}{n^{\epsilon^2/180}}, \tag{5}$$

where the constant C depends only on the moments of ξ_0. The proof of (5) is long, technical and complicated.

Later, Barrera and Manrique [2] proved that if the moment generating function of iid coefficients exists in an open neighborhood around 0, then for any $t \ge 1$

$$\mathbb{P}\Big(\min_{||z|-1|\le tn^{-2}(\log n)^{-1/2-\gamma}} |W_n(z)| \le tn^{-1/2}(\log n)^{-\gamma}\Big) = \mathrm{O}((\log n)^{-\gamma+1/2}), \tag{6}$$

where $\gamma > 1/2$. The proof of (6) recovers the essential ideas of Konyagin and Schlag [13], who only considered the problem when the rcs have Rademacher or standard Gaussian distribution. Their proof is quite technical and involved. It is based on the Salem–Zygmund inequality for sub-Gaussian rvs.

To extend for more distributions, Barrera and Manrique [2] took advantage of the concept of *least common denominator* (lcd for short), which was developed in the study of the singularity of random matrices [17]. Roughly speaking, the lcd is a combinatorial measurement to understand the concentration of a sum of independent rvs in a small ball. Furthermore, under the assumptions of the finiteness of the second moment, using similar ideas from Barrera and Manrique [2], it is possible to find an annulus in which W_n does not have roots with high probability. We remark that the lcd has been converted into a useful tool that allows the analysis of various interesting problems. For instance, it is used in the study of isomorphism between graphs [15] and in the analysis of the condition number for random matrices [18]. In this paper, the lcd is used to understand how small the modulus of a random polynomial near the unit circle can be.

In this work, the lcd allows us to develop clear arguments to estimate how close the roots of a Kac polynomial are to the unit circle. To be more precise, when the rcs of a Kac polynomial are iid rvs with zero mean and finite second moment, the majority of the roots are at a distance of order $\mathrm{O}(n^{-2}(\log n)^{-3})$ with probability $1-\mathrm{O}((\log n)^{-1/2})$. The main obstacle for extending this result comes from the Salem–Zygmund inequality as we will see in Section 2.

The main result of this paper is the following.

Theorem 1.1 *Let $\{\xi_j : j \geq 0\}$ be a sequence of real iid nondegenerate real rvs satisfying*

$$\sup_{u\in\mathbb{R}} \mathbb{P}(|\xi_0 - u| \leq \gamma) \leq 1-q \quad \textit{and} \quad \mathbb{P}(|\xi_0| > M) \leq \frac{q}{2} \qquad \textbf{(H)}$$

for some $M > 0$, $\gamma > 0$ and $q \in (0,1)$. Suppose $\mathbb{E}[\xi_0] = 0$ and $\mathbb{E}[\xi_0^2] < \infty$. Then for all fixed $t \geq 1$ we have

$$\mathbb{P}\Big(\min_{||z|-1|\leq tn^{-2}(\log n)^{-3}} |W_n(z)| \leq tn^{-1/2}(\log n)^{-2}\Big) = \mathrm{O}((\log n)^{-1/2}), \qquad (7)$$

where the implicit constant depends on t and the distribution of ξ_0.

Remark 1.2 (1) In Theorem 1.1 we only assume the finiteness of the second moment, zero mean and condition (**H**) which include Rademacher and standard Gaussian rvs. As a direct consequence of Theorem 1.1, we have

$$\mathbb{P}\big(W_n \text{ has no roots on } \{z\in\mathbb{C} : ||z|-1| \leq tn^{-2}(\log n)^{-3}\}\big) = 1-\mathrm{O}((\log n)^{-1/2}).$$

(2) We point out that in (7) we consider the minimum of the modulus of the Kac polynomial over the set $\{||z|-1| \leq tn^{-2}(\log n)^{-3}\}$ which is properly contained in the region considered in (3), but it contains the region considered in (4). Nevertheless, we obtain the upper bound $\mathrm{O}((\log n)^{-1/2})$ which improves the bound given in (3).

This paper is organized as follows. In Sect. 2 we give an outline of the proof. In Sect. 3 we provide the proof of Theorem 1.1. Finally, in Appendix A we prove auxiliary results that we used throughout the paper.

2 Outline of the proof

In this section, we present the strategy used to prove Theorem 1.1. Our goal is to estimate $\mathbb{P}(A_n)$, where

$$A_n := \Big\{\min_{z\in\mathbb{C}:||z|-1|\leq tn^{-2}(\log n)^{-3}} |W_n(z)| \leq tn^{-1/2}(\log n)^{-2}\Big\}$$

and $t \geq 1$ is a fixed constant. First, motivated by the estimates given in [13, Section 2, p. 4964], we analyze the probability of the events

$$A_{n,\alpha} := \{|W_n(\exp(i2\pi x_\alpha))| \leq g_n\} \quad \text{for } x_\alpha = \frac{\alpha}{N_n},\ \alpha = 0,\ldots,N_n-1,$$

where N_n and g_n are appropriate functions of n (we provide a precise description of them later on). We anticipate that $N_n \approx n^2(\log(n))^3$, which is similar to the number of balls used in [13]. We point out that N_n needs to trade off g_n in order that the probability of $A_{n,\alpha}$ tends to zero, as $n \to \infty$.

Second, for each $\alpha = 0, \ldots, N_n - 1$ we analyze the arithmetic structure of the sequence $\{\exp(i2\pi jx_\alpha) : j = 0, \ldots, n-1\}$ and using the so-called small ball inequalities we prove that $\mathbb{P}(A_{n,\alpha}) \to 0$, as $n \to \infty$. The idea is to apply the Taylor Theorem to approximate W_n in small balls with centers at $\exp(i2\pi x_\alpha)$. This allows us to write the event A_n as the union of events of the form $A_{n,\alpha}$. However, we need to handle the maximum value for the derivative of W_n on the unit circle. The latter can be done by a Salem–Zygmund type inequality, which estimates the maximum possible value of a Kac polynomial on the unit circle.

Let $\|W_n'\|_\infty$ denote the supremum norm of W_n' over the unit circle. In the case of $\xi_0, \ldots, \xi_{n-1}$ being iid sub-Gaussian rvs, a Salem–Zygmund type inequality (in probability) gives

$$\mathbb{P}\left(\|W_n\|_\infty > C_p n^{1/2} (\log n)^{1/2}\right) = \mathrm{O}\left(n^{-2}\right) \tag{8}$$

for some suitable positive constant C_p, see for instance [10, Chapter 6, Theorem 2]. In [2], the authors showed that (8) holds for iid zero mean rvs with a finite moment generating function. In this paper, *we do not assume the existence of the moment generating function*. Instead, we assume the finiteness of the second moment. By applying the majorizing measure method, Weber [21] showed (8) in expectation. To be more precise, let $\xi_0, \xi_1, \ldots, \xi_{n-1}$ be iid zero mean rvs with finite second moment. Corollary 2 in [21] implies that there exists a positive constant $\tilde{C}$ (that only depends on $\mathbb{E}[\xi_0^2]$) such that

$$\mathbb{E}\left[\|W_n\|_\infty\right] \le \tilde{C} n^{1/2} (\log n)^{1/2} \quad \text{for any } n \in \mathbb{N}.$$

To improve Theorem 1.1 (using the lcd technique) to more general rcs, we require a refined version of the Salem–Zygmund inequality for rvs without a finite second moment. At the moment, the authors are not able to obtain a Salem–Zygmund type inequality for rvs without the finiteness of the second moment. Later, we apply small ball inequalities to show that

$$\mathbb{P}\left(|W_n\left(\exp\left(i2\pi x_\alpha\right)\right)| \le g_n\right) \to 0, \quad \text{as } n \to \infty.$$

Such inequalities allow us to consider more general rcs and provide a new proof of the main theorem in [13]. To apply small ball inequalities, we analyze the lcd for some specific matrix. In the sequel, we give the definition of the lcd for a matrix. Let $\log_+ x := \max\{\log x, 0\}$ for any $x > 0$.

Definition 2.1 (Least common denominator (lcd)) Let $L > 0$ be a positive number. Let $\|\cdot\|_2$ be the standard Euclidean norm and let $\operatorname{dist}\left(v, \mathbb{Z}^M\right)$ denote the distance between the vector $v \in \mathbb{R}^M$ and the set $\mathbb{Z}^M$. For a given matrix $V \in \mathbb{R}^{m\times M}$ the lcd is defined as

$$D(V) := \inf\left\{\|\Theta\|_2 : \Theta \in \mathbb{R}^m, \operatorname{dist}\left(V^T\Theta, \mathbb{Z}^M\right) < L\sqrt{\log_+ \frac{\|V^T\Theta\|_2}{L}}\right\}.$$

For a review of the concept of lcd, we recommend [17, Section 7]. For our purposes, in Definition 2.1 we take $m = 2$, $M = n$ and the matrix V is given by

$$V := \begin{bmatrix} 1 & \cos(2\pi x_\alpha) & \ldots & \cos((n-1)2\pi x_\alpha) \\ 0 & \sin(2\pi x_\alpha) & \ldots & \sin((n-1)2\pi x_\alpha) \end{bmatrix}.$$

Set $X = \left[\xi_0, \ldots, \xi_{n-1}\right]^T$. Observe that

$$\mathbb{P}\left(\|VX\|_2 \leq g_n\right) = \mathbb{P}\left(|W_n\left(\exp\left(i2\pi x_\alpha\right)\right)| \leq g_n\right).$$

Note that if $\det\left(VV^T\right) > 0$, Theorem 7.5 in [17] implies that for $a > 0$ and $t \geq 0$

$$\mathbb{P}\left(\|aVX\|_2 \leq t\right) \leq \frac{C^2L^2}{2a^2(\det(VV^T))^{1/2}}\left(t + \frac{1}{D(aV)}\right)^2, \tag{9}$$

where $L \geq \sqrt{2/q}$ and the constant C only depends on constants M, γ, q specified in Theorem 1.1. By Definition 2.1 it is not hard to deduce that, for any $a > 0$, $D(aV) \geq (1/a)D(V)$. Recall the inequality $(x+y)^2 \leq 2x^2 + 2y^2$ for any $x, y \in \mathbb{R}$. By (9) we deduce

$$\begin{aligned}\mathbb{P}\left(\|aVX\|_2 \leq t\right) &\leq \frac{C^2L^2t^2}{a^2\left(\det\left(VV^T\right)\right)^{1/2}} + \frac{C^2L^2}{a^2\left(\det\left(VV^T\right)\right)^{1/2}(D(aV))^2} \\ &\leq \frac{C^2L^2t^2}{a^2\left(\det\left(VV^T\right)\right)^{1/2}} + \frac{C^2L^2}{\left(\det\left(VV^T\right)\right)^{1/2}(D(V))^2}. \qquad (\mathbf{K})\end{aligned}$$

Since $x_\alpha = \frac{\alpha}{N_n}$, the arithmetic properties of x_α given by α and N_n should play an important role in the estimates. Depending on the greatest common divisor between α and N_n, $\gcd(\alpha, N_n)$, we deduce suitable positive lower bounds for $\det\left(V^TV\right)$ and $\operatorname{dist}\left(V^T\Theta, \mathbb{Z}^n\right)$ which together with (**K**) allow us to show that $\mathbb{P}\left(\|VX\|_2 \leq g_n\right)$ is sufficiently small.

Taylor's approximation

In the sequel, define the trigonometric random polynomial $T_n(x) := \sum_{j=0}^{n-1} \xi_j e^{ijx}$, $x \in \mathbb{R}$, and let T_n' denote its derivative with respect to x. To make the notation shorter, Δ_n denotes the following event:

$$\Delta_n := \left\{\max_{z\in\mathbb{C}:||z|-1|\leq 2tn^{-11/10}} |W_n(z)| \leq n^{3/2},\ \|T_n'\|_\infty \leq C_0 n^{3/2}\log n\right\},$$

where C_0 is a positive constant of which we will find the precise value later on. We also let $\mathbb{P}(A, B)$ denote the probability $\mathbb{P}(A \cap B)$ for any two events A and B. By the total probability law, we deduce

$$\begin{aligned}\mathbb{P}(A_n) \ &\leq\ \mathbb{P}(A_n, \Delta_n) + \mathbb{P}\left(\|T_n'\|_\infty > C_0 n^{\frac{3}{2}}\log n\right) \\ &\quad + \mathbb{P}\left(\max_{z\in\mathbb{C}:||z|-1|\leq 2tn^{-11/10}} |W_n(z)| > n^{3/2}\right).\end{aligned} \tag{10}$$

The Markov inequality yields

$$\begin{aligned}\mathbb{P}\left(\max_{z\in\mathbb{C}:||z|-1|\leq 2tn^{-11/10}} |W_n(z)| > n^{3/2}\right) &\leq \mathbb{P}\left(\sum_{j=0}^{n-1}|\xi_j|\left(1 + \frac{2t}{n^{1+1/10}}\right)^j > n^{3/2}\right) \\ \leq \frac{\mathbb{E}\left[\sum_{j=0}^{n-1}|\xi_j|\left(1+\frac{2t}{n^{1+1/10}}\right)^j\right]}{n^{3/2}} &\leq \frac{e^{2t}n\mathbb{E}\left[|\xi_0|\right]}{n^{3/2}} = \frac{e^{2t}\mathbb{E}\left[|\xi_0|\right]}{n^{1/2}}\end{aligned}$$

in other words,

$$\mathbb{P}\left(\max_{z\in\mathbb{C}:||z|-1|\leq 2tn^{-11/10}} |W_n(z)| > n^{3/2}\right) = \mathrm{O}\left(n^{-1/2}\right), \tag{11}$$

where the implicit constant depends on t and $\mathbb{E}[|\xi_0|]$. On the other hand, the Bernstein inequality (see [16, Theorem 14.1.1]) allows us to deduce for the second term on the right-side hand of (10) the inequality

$$\mathbb{P}\left(\|T_n'\|_\infty > C_0 n^{3/2}\log n\right) \leq \mathbb{P}\left(\|T_n\|_\infty > C_0 n^{1/2}\log n\right).$$

Since $\mathbb{E}[\xi_0] = 0$ and $\mathbb{E}\left[\xi_0^2\right] < \infty$, one can apply [21, Corollary 2] which together with the Markov inequality imply

$$\mathbb{P}\left(\|T_n\|_\infty > C_0 n^{1/2}\log n\right) \leq \frac{C(\mathbb{E}\left[\xi_0^2\right])^{1/2} n^{1/2}(\log n)^{1/2}}{C_0 n^{1/2}\log n} = \frac{C(\mathbb{E}\left[\xi_0^2\right])^{1/2}}{C_0(\log n)^{1/2}},$$

where C is a universal positive constant. Consequently, the Bernstein inequality yields

$$\mathbb{P}\left(\|T_n'\|_\infty > C_0 n^{3/2}\log n\right) = \mathrm{O}\left((\log n)^{-1/2}\right). \tag{12}$$

By (10), (11) and (12), we observe that to estimate $\mathbb{P}(A_n)$ we only need to analyze $\mathbb{P}(A_n, \Delta_n)$.

Remark 2.2 In the preceding reasoning we only used zero mean and the finiteness of the second moment of ξ_0. In particular, it holds for sub-Gaussian rvs, which includes Rademacher, standard Gaussian, and bounded rvs.

Arithmetic properties of x

In the sequel, we decompose the event $A_n \cap \Delta_n$ into regions for which the arithmetic properties of x_α are useful in allowing the use of the anti-concentration assumption(**H**) and allowing us to show that $\mathbb{P}(A_n, \Delta_n)$ tends to zero, as $n \to \infty$. We point out that in the following reasoning we use only assumption(**H**).

To achieve our goal, we consider a set of balls with center at points on the unit circle with an adequate radius. We distinguish two kinds of balls: the special balls with center at $1 + 0i$ and $-1 + 0i$, where the radius r is *large* ($r = 2tn^{-11/10}$), and the balls with center at points z with argument satisfying $n^{-11/10} < |\arg(z)| < \pi - n^{-11/10}$ and *small* radius ($r = 2tn^{-2}(\log n)^{-3}$).

Recall that for any $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Let $N = \lfloor n^2 (\log n)^3 \rfloor$ and $x_\alpha = \frac{\alpha}{N}$ for $\alpha = 0, 1, \ldots, N-1$. For $a \in \mathbb{C}$ and $s > 0$, let $\mathrm{B}(a, s)$ denote the closed ball with center a and radius s, i.e., $\mathrm{B}(a, s) = \{z \in \mathbb{C} : |z - a| \leq s\}$. Write $\mathbb{S}^1$ for the unit circle and let

$$\mathcal{A}\left(\mathbb{S}^1, tn^{-2}(\log n)^{-3}\right) := \left\{z \in \mathbb{C} : ||z| - 1| \leq tn^{-2}(\log n)^{-3}\right\}.$$

Notice that

$$\begin{aligned}\mathcal{A}\left(\mathbb{S}^1, tn^{-2}(\log n)^{-3}\right) = &\left\{z \in \mathcal{A} : n^{-11/10} < |\arg(z)| < \pi - n^{-11/10}\right\} \\ &\cup \left\{z \in \mathcal{A} : |\arg(z)| \leq n^{-11/10} \quad \text{or} \quad |\arg(z) - \pi| \leq n^{-11/10}\right\}.\end{aligned}$$

If $t \geq 1$, observe that

$$\begin{aligned}&\left\{z \in \mathcal{A} : |\arg(z)| \leq n^{-11/10} \quad \text{or} \quad |\arg(z) - \pi| \leq n^{-11/10}\right\} \\ &\quad \subset \mathrm{B}\left(-1 + 0i, 2tn^{-11/10}\right) \cup \mathrm{B}\left(1 + 0i, 2tn^{-11/10}\right).\end{aligned}$$

The preceding inclusion yields that any $z \in \mathcal{A}$ with *small argument* belongs in the union of the balls with center at $1 + 0i$ and $-1 + 0i$ with radius $2tn^{-11/10}$. On the other hand, for $z \in \mathcal{A}$ with *large argument*,

$$\left\{z \in \mathcal{A} : n^{-11/10} < |\arg(z)| < \pi - n^{-11/10}\right\} \subset \bigcup_{\substack{\alpha=1 \\ \alpha\,:\, n^{-11/10} < |2\pi x_\alpha| < \pi - n^{-11/10}}}^{N-1} \mathrm{B}\left(e^{i2\pi x_\alpha}, 2tn^{-2}(\log n)^{-3}\right).$$

Define

$$J_1(n, N) := \left\{\alpha \in [1, N-1] \cap \mathbb{N} : \gcd(\alpha, N) \geq n^{11/10}(\log n)^{-1/2}\right\},$$
$$J_2(n, N) := \left\{\alpha \in [1, N-1] \cap \mathbb{N} : n^{11/10}(\log n)^{-1/2} \geq \gcd(\alpha, N) \geq n(\log n)^{3}\right\},$$
$$J_3(n, N) := \left\{\alpha \in [1, N-1] \cap \mathbb{N} : n(\log n)^{3} \geq \gcd(\alpha, N) \geq n^{9/10}(\log n)^{3}\right\},$$

where $\gcd(\alpha, N)$ denotes the greatest common divisor of α and N.

For any $\alpha \in J_3(n, N)$, we have

$$n - \frac{1}{n(\log n)^3} \leq \frac{N}{\gcd(\alpha, N)} \leq n^{11/10}.$$

The preceding inequalities mean that the irreducible fraction of x_α is as small as a multiple of $n^{-11/10}$. Therefore,

$$\begin{aligned} &\bigcup_{\substack{\alpha=1 \\ \alpha\,:\, n^{-11/10} < |2\pi x_\alpha| < \pi - n^{-11/10}}}^{N-1} \mathrm{B}\left(e^{i2\pi x_\alpha}, 2tn^{-2}(\log n)^{-3}\right) \\ &= \bigcup_{\alpha \in J_1(n,N)} \mathrm{B}\left(e^{i2\pi x_\alpha}, 2tn^{-2}(\log n)^{-3}\right) \cup \bigcup_{\alpha \in J_2(n,N)} \mathrm{B}\left(e^{i2\pi x_\alpha}, 2tn^{-2}(\log n)^{-3}\right) \\ &\quad \cup \bigcup_{\alpha \in J_3(n,N)} \mathrm{B}\left(e^{i2\pi x_\alpha}, 2tn^{-2}(\log n)^{-3}\right). \end{aligned}$$

We emphasize that if $\alpha \in J_1(n, N) \cup J_2(n, N) \cup J_3(n, N)$, then

$$n^{-11/10} < |2\pi x_\alpha| < \pi - n^{-11/10}.$$

Consequently,

$$\begin{aligned} \mathbb{P}(A_n, \Delta_n) \leq\; &\mathbb{P}\left(\Delta_n, \min_{z \in \mathrm{B}\left(1+0i, 2tn^{-11/10}\right)} |W_n(z)| < tn^{-1/2}(\log n)^{-2}\right) \\ &+ \mathbb{P}\left(\Delta_n, \min_{z \in \mathrm{B}\left(-1+0i, 2tn^{-11/10}\right)} |W_n(z)| < tn^{-1/2}(\log n)^{-2}\right) \\ &+ \sum_{\alpha \in J_1(n,N)} \mathbb{P}(\Delta_n, \mathrm{B}_\alpha) + \sum_{\alpha \in J_2(n,N)} \mathbb{P}(\Delta_n, \mathrm{B}_\alpha) + \sum_{\alpha \in J_3(n,N)} \mathbb{P}(\Delta_n, \mathrm{B}_\alpha), \end{aligned} \tag{13}$$

where

$$\mathrm{B}_\alpha := \left\{\min_{z \in \mathrm{B}\left(e^{i2\pi x_\alpha}, 2tn^{-2}(\log n)^{-3}\right)} |W_n(z)| < tn^{-1/2}(\log n)^{-2}\right\}.$$

The right-hand side of (13) will be estimated as follows.

Lemma 2.3 *The following hold.*

$$\mathbb{P}\left(\Delta_n, \min_{z\in \mathrm{B}\left(1+0i, 2tn^{-11/10}\right)} |W_n(z)| < tn^{-1/2}(\log n)^{-2}\right) = \mathrm{O}\left(\frac{\log n}{n^{1/10}}\right)$$

and

$$\mathbb{P}\left(\Delta_n, \min_{z\in \mathrm{B}\left(-1+0i, 2tn^{-11/10}\right)} |W_n(z)| < tn^{-1/2}(\log n)^{-2}\right) = \mathrm{O}\left(\frac{\log n}{n^{1/10}}\right),$$

where the implicit constants in the big O *notation depend on L and t.*

Lemma 2.4 *Suppose* $\gcd(\alpha, N) \geq n^{11/10}(\log n)^{-1/2}$, *where* $N = \lfloor n^2(\log n)^3 \rfloor$. *Then for a suitable constant* $\tilde{C}$ *it follows that*

$$\sum_{\alpha\in J_1(n,N)} \mathbb{P}\left(|W_n\left(e^{i2\pi x_\alpha}\right)| \leq \tilde{C}tn^{-1/2}(\log n)^{-2}\right) = \mathrm{O}\left(\frac{(\log n)^4}{n^{1/20}}\right),$$

where the implicit constant in the big O *notation depends on L and t.*

Lemma 2.5 *Suppose* $\frac{n^{11/10}}{(\log n)^{1/2}} \geq \gcd(\alpha, N) \geq n(\log n)^3$, *where* $N = \lfloor n^2(\log n)^3 \rfloor$. *Then for a suitable constant* $\tilde{C}$ *it follows that*

$$\sum_{\alpha\in J_2(n,N)} \mathbb{P}\left(|W_n\left(e^{i2\pi x_\alpha}\right)| \leq \tilde{C}tn^{-1/2}(\log n)^{-2}\right) = \mathrm{O}\left(\frac{1}{\log n}\right),$$

where the implicit constant in the big O *notation depends on L and t.*

Lemma 2.6 *Suppose* $n(\log n)^3 \geq \gcd(\alpha, N) \geq n^{9/10}(\log n)^3$, *where* $N = \lfloor n^2(\log n)^3 \rfloor$. *Then for a suitable constant* $\tilde{C}$ *it follows that*

$$\sum_{\alpha\in J_3(n,N)} \mathbb{P}\left(|W_n\left(e^{i2\pi x_\alpha}\right)| \leq \tilde{C}tn^{-1/2}(\log n)^{-2}\right) = \mathrm{O}\left(\frac{1}{n^{1/10}}\right),$$

where the implicit constant in the big O *notation depends on L and t.*

In the sequel, we stress that Theorem 1.1 is just a consequence of what we have already stated up to here. Indeed, combining Lemma 2.3, Lemma 2.4, Lemma 2.5, Lemma 2.6, estimate (11) and estimate (12) in inequality (10) yields Theorem 1.1.

3 Proof of Theorem 1.1

In this section, we show that the left-hand side of inequality (13) is of order $\mathrm{O}((\log(n))^{-1/2})$.

3.1 Estimates on the balls centered at −1 and 1

Proof of Lemma 2.3 Let $z \in \mathrm{B}\left(1+0i, 2tn^{-11/10}\right)$. The Taylor Theorem implies

$$|W_n(z) - W_n(1)| \leq |z-1||W_n'(1)| + |R_2(z)|,$$

where $R_2(z)$ is the error of the Taylor approximation of order 2. On Δ_n,

$$\begin{aligned}|R_2(z)| &\le \frac{\left(2tn^{-1-1/10}\right)^2}{1-\mathrm{o}(1)} \max_{z\in \mathrm{B}\left(1+0i,2tn^{-11/10}\right)} |W_n(z)| \\ &\le \frac{4t^2n^{-2-1/5}n^{3/2}}{1-\mathrm{o}(1)} = \frac{4t^2n^{-1/2-1/5}}{1-\mathrm{o}(1)},\end{aligned}$$

where $\mathrm{o}(1) = 2tn^{-1-1/10}$. By the preceding inequality on Δ_n, we infer

$$\begin{aligned}|W_n(z) - W_n(1)| &\le 2tn^{-1-1/10}|W_n'(1)| + \frac{4t^2n^{-1/2-1/5}}{1-\mathrm{o}(1)} \\ &\le 2tn^{-1-1/10}\|T_n'\|_\infty + \frac{4t^2n^{-1/2-1/5}}{1-\mathrm{o}(1)} \\ &\le 2C_0tn^{1/2-1/10}\log n + \frac{4t^2n^{-1/2-1/5}}{1-\mathrm{o}(1)}.\end{aligned}$$

Hence,

$$\mathbb{P}\left(\Delta_n, \min_{z\in \mathrm{B}\left(1+0i,2tn^{-11/10}\right)} |W_n(z)| < tn^{-1/2}(\log n)^{-2}\right) \le \mathbb{P}\left(|W_n(1)| \le 2C_2tn^{1/2-1/10}\log n\right),$$

where $2C_2 = 2C_0 + 4t + 1$. Since $W_n(1) = \sum_{j=0}^{n-1}\xi_j$, [17, Corollary 7.6] implies

$$\mathbb{P}\left(|W_n(1)| \le 2C_2tn^{1/2-1/10}\log n\right) \le \frac{C_3L}{\|\mathbf{a}\|}\left(2C_2t + \frac{1}{D(\mathbf{a})}\right)$$

for $L \ge \sqrt{1/q}$, where C_3 is a positive constant and $D(\mathbf{a})$ is the lcd of

$$\mathbf{a} = \left(n^{1/2-1/10}\log n\right)^{-1}(1,\ldots,1) \in \mathbb{R}^n.$$

By [17, Proposition 7.4], $D(\mathbf{a}) \ge \frac{1}{2|\mathbf{a}|_\infty}$, where $|\mathbf{a}|_\infty$ denotes the maximum Euclidean norm of the columns of $\mathbf{a}$. Then $D(\mathbf{a}) \ge 1/2n^{1/2-1/10}\log n$ and

$$\begin{aligned}\mathbb{P}\left(|W_n(1)| \le 2C_2tn^{1/2-1/10}\log n\right) &\le \frac{C_3L\log n}{n^{1/10}}\left(2C_2t + \frac{2}{n^{1/2-1/10}\log n}\right) \\ &\le \frac{(2C_2t+2)\,C_3L\log n}{n^{1/10}}.\end{aligned}$$

Therefore,

$$\mathbb{P}\left(\Delta_n, \min_{z\in \mathrm{B}\left(1+0i,2tn^{-11/10}\right)} |W_n(z)| < tn^{-1/2}(\log n)^{-2}\right) = \mathrm{O}\left(\frac{\log n}{n^{1/10}}\right).$$

On the other hand, for $z \in B\left(-1+0i, 2tn^{-11/10}\right)$ a similar reasoning yields

$$\mathbb{P}\left(\Delta_n, \min_{z\in \mathrm{B}\left(-1+0i,2tn^{-11/10}\right)} |W_n(z)| < tn^{-1/2}(\log n)^{-2}\right) \le \mathbb{P}\left(|W_n(-1)| \le 2C_2tn^{1/2-1/10}\log n\right).$$

In this case, we need to analyze $W_n(-1) = \sum_{j=0}^{n-1}(-1)^j\xi_j$. Again taking $L \ge \sqrt{1/q}$ and applying [17, Corollary 7.6], we obtain

$$\mathbb{P}\left(|W_n(-1)| \le 2C_2tn^{1/2-1/10}\log n\right) \le \frac{C_3L}{\|\mathbf{b}\|}\left(2C_2t + \frac{1}{D(\mathbf{b})}\right),$$

where C_3 is a positive constant and $D(\mathbf{b})$ is the lcd of

$$\mathbf{b} = \left(n^{1/2-1/10}\log n\right)^{-1}\left(1,-1,1,\ldots,(-1)^{n-1}\right) \in \mathbb{R}^n.$$

By [17, Proposition 7.4], $D(\mathbf{b}) \geq 1/2n^{1/2-1/10}\log n$. So

$$\begin{aligned}\mathbb{P}\left(|W_n(-1)| \leq 2C_2tn^{1/2-1/10}\log n\right) &\leq \frac{C_3L\log n}{n^{1/10}}\left(2C_2t+\frac{2}{n^{1/2-1/10}\log n}\right)\\ &\leq \frac{(2C_2t+2)\,C_3L\log n}{n^{1/10}}.\end{aligned}$$

Therefore,

$$\mathbb{P}\left(\Delta_n,\min_{z\in\mathrm{B}(-1+0i,2tn^{-11/10})}|W_n(z)| < tn^{-1/2}(\log n)^{-2}\right) = \mathrm{O}\left(\frac{\log n}{n^{1/10}}\right).$$

□

3.2 Estimates of $\mathbb{P}\left(\Delta_n, \mathrm{B}_\alpha\right)$

In the sequel, we apply the Taylor Theorem repeatedly in order to reduce $\mathbb{P}(\Delta_n, \mathrm{B}_\alpha)$ to an estimate of the probability of how small a sum of iid rvs can be. The latter can be computed (estimated) using small ball inequalities.

Let $z \in \mathrm{B}\left(e^{i2\pi x_\alpha}, 2tn^{-2}(\log n)^{-3}\right)$ and suppose that Δ_n holds. The Taylor Theorem yields

$$\begin{aligned}|W_n(z)-W_n\left(e^{i2\pi x_\alpha}\right)| &\leq |z-e^{i2\pi x_\alpha}||W_n'\left(e^{i2\pi x_\alpha}\right)|+|R_2(z)|\\ &\leq 2tn^{-2}(\log n)^{-3}|W_n'\left(e^{i2\pi x_\alpha}\right)|+\frac{4t^2n^{-5/2}(\log n)^{-6}}{1-\mathrm{o}(1)}\\ &\leq \left(2tC_0+4t^2\right)n^{-1/2}(\log n)^{-2},\end{aligned}$$

where $\mathrm{o}(1) = 2tn^{-2}(\log n)^{-3}$. Hence

$$\mathbb{P}(\Delta_n, \mathrm{B}_\alpha) \leq \mathbb{P}\left(|W_n\left(e^{i2\pi x_\alpha}\right)| \leq 2tC_2n^{-1/2}(\log n)^{-2}\right). \tag{14}$$

To show that $\mathbb{P}(\Delta_n, \mathrm{B}_\alpha)$ tends to zero as n increases, we rewrite the sum $W_n\left(e^{i2\pi x_\alpha}\right)$ as the product of a matrix and a vector, and then we analyze the lcd of the corresponding matrix.

Define the $2\times n$ matrix V_α as follows:

$$V_\alpha := \begin{bmatrix} 1 & \cos(2\pi x_\alpha) & \ldots & \cos((n-1)2\pi x_\alpha)\\ 0 & \sin(2\pi x_\alpha) & \ldots & \sin((n-1)2\pi x_\alpha)\end{bmatrix}$$

and take $X = \left[\xi_0,\ldots,\xi_{n-1}\right]^T \in \mathbb{R}^n$. Notice that

$$\|V_\alpha X\|_2 = \left|\sum_{j=0}^{n-1}\xi_j e^{ij2\pi x_\alpha}\right| = |W_n\left(e^{i2\pi x_\alpha}\right)|.$$

Let $\Theta = r\left[\cos(\theta), \sin(\theta)\right]^T \in \mathbb{R}^2$, where $r > 0$ and $\theta \in [0, 2\pi)$. For fixed r and θ we have

$$V_\alpha^T\Theta = r\left[\cos(-\theta), \cos(2\pi x_\alpha-\theta), \ldots, \cos(2(n-1)\pi x_\alpha-\theta)\right]^T \in \mathbb{R}^n.$$

We also point out that $\|V_\alpha^T\Theta\|_2 \le r\sqrt{n}$. On the other hand, we observe that

$$\det\left(V_\alpha V_\alpha^T\right) = \det\begin{bmatrix} \sum_{j=0}^{n-1}\cos^2(j2\pi x_\alpha) & \frac{1}{2}\sum_{j=0}^{n-1}\sin(2\cdot j2\pi x_\alpha) \\ \frac{1}{2}\sum_{j=0}^{n-1}\sin(2\cdot j2\pi x_\alpha) & \sum_{j=0}^{n-1}\sin^2(j2\pi x_\alpha) \end{bmatrix}.$$

Bearing all this in mind, we can use the notion of lcd for high dimensions to obtain an accurate upper bound of the left-hand side of (14).

We recall that the events $\Delta_n \cap \mathrm{B}_\alpha$ are defined for

$$n^{-11/10} < |2\pi x_\alpha| < \pi - n^{-11/10}.$$

Therefore, to estimate the left-hand side of (14), we distinguish the following three cases.

3.2.1 Estimation on $J_1(n, N)$

Proof of Lemma 2.4 Notice that

$$\frac{N}{\gcd(\alpha, N)} \le \frac{n^2(\log n)^3}{n^{11/10}(\log n)^{-1/2}} = n^{9/10}(\log n)^{7/2}$$

and

$$|2\pi x_\alpha| = 2\pi\frac{\alpha}{N} = 2\pi\frac{\alpha/\gcd(\alpha, N)}{N/\gcd(\alpha, N)} \ge 2\pi\frac{1}{n^{9/10}(\log n)^{7/2}}.$$

Then $2\pi x_\alpha$ also satisfies $n^{-1} < |2\pi x_\alpha| < \pi - n^{-1}$ for all large n. By [13, Lemma 3.2, Part 1], there exist positive constants c_4, C_4 such that

$$c_4 n^2 \le \det\left(V_\alpha V_\alpha^T\right) \le C_4 n^2. \tag{15}$$

By Lemma A.1 in Appendix A we obtain that the number of indices $\alpha \in [1, N] \cap \mathbb{N}$ that satisfies the condition $\gcd(\alpha, N) \ge n^{11/10}(\log n)^{-1/2}$ is at most $\frac{N^{1+o(1)}}{n^{11/10}(\log n)^{-1/2}}$. By the definition of N we obtain

$$\frac{N^{1+o(1)}}{n^{11/10}(\log n)^{-1/2}} \le \frac{n^{2+o(1)}(\log n)^{7/2+o(1)}}{n^{11/10}} = n^{9/10+o(1)}(\log n)^{7/2+o(1)}. \tag{16}$$

By [17, Proposition 7.4], the lcd of V_α satisfies $D(V_\alpha) \ge \frac{1}{2|V_\alpha|_\infty}$, where $|V_\alpha|_\infty$ denotes the maximum Euclidean norm of the columns of V_α. Observe that $|V_\alpha|_\infty = 1$ and hence $D(V_\alpha) \ge 1/2$. Therefore, inequality (**K**), inequality (15) and inequality (16) yield

$$\begin{aligned}
&\sum_{\alpha\in J_1(n,N)} \mathbb{P}\left(|W_n\left(e^{i2\pi x_\alpha}\right)| \le 2tC_2 n^{-1/2}(\log n)^{-2}\right) \\
&\quad \le n^{9/10+o(1)}(\log n)^{7/2+o(1)}\left(\frac{2C^2L^2(2tC_2)^2}{\left(c_4n^2\right)^{1/2}\left(n^{1/2}(\log n)^2\right)^2} + \frac{2C^2L^2}{\frac{1}{4}\left(c_4n^2\right)^{1/2}}\right) \\
&\quad \le \frac{8C^2C_2^2L^2t^2}{c_4^{1/2}n^{11/10-o(1)}(\log n)^{1/2-o(1)}} + \frac{8C^2L^2(\log n)^{7/2+o(1)}}{c_4^{1/2}n^{1/10-o(1)}}
\end{aligned}$$

for all large n. Consequently,

$$\sum_{\alpha\in J_1(n,N)} \mathbb{P}\left(|W_n\left(e^{i2\pi x_\alpha}\right)| \le 2tC_2 n^{-1/2}(\log n)^{-2}\right) = \mathrm{O}\left(\frac{(\log n)^4}{n^{1/20}}\right),$$

where the implicit constant depends on L and t. □

3.2.2 Estimation on $J_2(n, N)$

Proof of Lemma 2.5 Notice that

$$n \geq \frac{N}{\gcd(\alpha, N)} \geq n^{9/10} (\log n)^{7/2} - \mathrm{o}(1), \tag{17}$$

where $\mathrm{o}(1) = \frac{(\log n)^{1/2}}{n^{11/10}}$. The latter implies

$$|2\pi x_\alpha| = 2\pi \frac{\alpha}{N} = 2\pi \frac{\alpha/\gcd(\alpha, N)}{N/\gcd(\alpha, N)} \geq 2\pi \frac{1}{n}.$$

Then $2\pi x_\alpha$ also satisfies $n^{-1} \leq |2\pi x_\alpha| \leq \pi - n^{-1}$ for all large n. By [13, Lemma 3.2, Part 1] there exist positive constants c_4, C_4 such that

$$c_4 n^2 \leq \det\left(V_\alpha V_\alpha^T\right) \leq C_4 n^2. \tag{18}$$

Note $x_\alpha = \frac{\alpha}{N} = \frac{\alpha'}{N'}$, where $\alpha = \alpha' \gcd(\alpha, N)$ and $N = N' \gcd(\alpha, N)$. Observe that $\gcd\left(\alpha', N'\right) = 1$. Since $N' \leq n$, for any θ we have

$$\left\{\exp\left(i\left(j2\pi\frac{\alpha'}{N'} - \theta\right)\right) : j = 0, \ldots, N'-1\right\} = \left\{\exp\left(i\left(j2\pi\frac{1}{N'} - \theta\right)\right) : j = 0, \ldots, N'-1\right\}.$$

Hence, without loss of generality, we assume that $x_\alpha = \frac{1}{N'}$. A straightforward computation yields

$$V_\alpha^T \Theta = r\left[\cos(-\theta), \cos(2\pi x_\alpha - \theta), \ldots, \cos(2(n-1)\pi x_\alpha - \theta)\right]^T \in \mathbb{R}^n.$$

Notice that the proof of Lemma A.2 in Appendix A holds true for any real positive number r. If $r \leq \frac{1}{32\pi x_\alpha}$, by Lemma A.2 in Appendix A, inequality (17) and remembering that $\|V_\alpha^T \Theta\|_2 \leq r\sqrt{n}$ we deduce

$$\begin{aligned} \frac{1}{128\pi}\left(n^{9/10}(\log n)^{7/2} - \mathrm{o}(1)\right) &\leq \frac{1}{128\pi x_\alpha} \leq \operatorname{dist}\left(V_\alpha^T \Theta, \mathbb{Z}^n\right) \\ &\leq L\sqrt{\log_+ \frac{\|V_\alpha^T \Theta\|_2}{L}} \leq L\sqrt{\log_+ \frac{r n^{1/2}}{L}} \leq L\sqrt{\log_+ \frac{n^{3/2}}{L}}, \end{aligned}$$

which yields a contradiction as $L \geq \sqrt{2/q}$ is fixed. Then for $r > \frac{1}{32\pi x_\alpha}$ we have

$$D(V_\alpha) \geq r > \frac{1}{32\pi}\left(n^{9/10}(\log n)^{7/2} - \mathrm{o}(1)\right).$$

Therefore, the preceding inequality together with inequality (**K**), inequality (18) and the fact that the cardinality of $J_2(n,N)$ is at most N allow us to deduce

$$\sum_{\alpha\in J_2(n,N)} \mathbb{P}\left(|W_n\left(e^{i2\pi x_\alpha}\right)| \leq 2tC_2 n^{-1/2}(\log n)^{-2}\right)$$
$$\leq n^2(\log n)^3\left(\frac{2C^2L^2(2tC_2)^2}{\left(c_4n^2\right)^{1/2}\left(n^{1/2}(\log n)^2\right)^2}\right)$$
$$+n^2(\log n)^3\left(\frac{2C^2L^2}{\left(c_4n^2\right)^{1/2}\left(\frac{1}{32\pi}\left(n^{9/10}(\log n)^{7/2}-\mathrm{o}(1)\right)\right)^2}\right)$$
$$\leq \frac{8C^2C_2^2L^2t^2}{c_4^{1/2}\log n}+\frac{2048c\pi^2C^2L^2}{c_4^{1/2}n^{2/5}(\log n)^4}$$

for all large n, where c_4 is a positive constant. As a consequence we obtain

$$\sum_{\alpha\in J_2(n,N)} \mathbb{P}\left(|W_n\left(e^{i2\pi x_\alpha}\right)| \leq 2tC_2 n^{-1/2}(\log n)^{-2}\right)=\mathrm{O}\left(\frac{1}{\log n}\right),$$

where the implicit constant depends on L and t. □

3.2.3 Estimation on $J_3(n, N)$

Proof of Lemma 2.6 This case requires a more refined analysis. Observe that

$$n^{11/10}\geq \frac{N}{\gcd(\alpha,N)}\geq n-\mathrm{o}(1),$$

where $\mathrm{o}(1)=\frac{1}{n^2(\log n)^3}$. Then $2\pi x_\alpha$ satisfies

$$n^{-11/10}\leq|2\pi x_\alpha|\leq(n-\mathrm{o}(1))^{-1}\quad\text{or}\quad \pi-(n-\mathrm{o}(1))^{-1}\leq|2\pi x_\alpha|\leq\pi-n^{-11/10}.$$

By [13, Lemma 3.2, Part 2], there exist positive constants c_4, C_4 such that

$$c_4n^{2-1/5}\leq\det\left(V_\alpha V_\alpha^T\right)\leq C_4n^2. \tag{19}$$

By Lemma A.1, the number of indexes $\alpha\in[1,N]\cap\mathbb{N}$ that satisfy the condition $n(\log n)^3\geq\gcd(\alpha,N)\geq n^{9/10}(\log n)^3$ is at most $n^{11/10+\mathrm{o}(1)}(\log(n))^{\mathrm{o}(1)}$, where $\mathrm{o}(1)\to 0$, as $n\to\infty$.

In the sequel, we analyze the lcd of V_α. In particular, we find an appropriate lower bound for the distance between $V_\alpha^T\Theta$ and the set $\mathbb{Z}^n$. Since $x_\alpha=\frac{\alpha}{N}=\frac{\alpha'}{N'}$ with $\gcd\left(\alpha',N'\right)=1$ and $N'\geq n-1$ for all large n, then we have that all the points in $\{\exp\left(i\left(j2\pi x_\alpha-\theta\right)\right):j=0,\ldots,n-1\}$ are. Let $r\in\mathbb{N}$ and consider the set of intervals of the form $\left[\frac{m}{r},\frac{m+1}{r}\right]$ for all $m\in[-r,r-1]\cap\mathbb{Z}$. Write I_m^r and J_m^r for the corresponding arcs on the unit circle such that their projections on the horizontal axis belong to the interval $\left[\frac{m}{r},\frac{m+1}{r}\right]$. If $4r\leq n$, then the Pigeonhole Principle implies that there exists at least one $M\in[-r,r-1]\cap\mathbb{Z}$ such that I_M or J_M contains at least $\frac{n}{4r}\geq 1$ elements of the set $\{\exp\left(i\left(j2\pi x_\alpha-\theta\right)\right):j=0,\ldots,n-1\}$.

In the sequel, we write

$$I_M^r:=\left\{j\in\{0,\ldots,n-1\}:\cos\left(j2\pi x-\theta\right)\in\left[\frac{M}{r},\frac{M+1}{r}\right]\in\left[\frac{M}{r},\frac{M+1}{r}\right]\right\}\neq\emptyset,$$

and, for each $j \in I_M^r$, we define

$$d_j = \min\left\{\left|\cos(j2\pi x_\alpha - \theta) - \frac{M}{r}\right|, \left|\cos(j2\pi x_\alpha - \theta) - \frac{M+1}{r}\right|\right\}.$$

Note that

$$\min_{0 \le l < k \le n-1} |l2\pi x_\alpha - k2\pi x_\alpha| \ge \frac{2\pi\alpha'}{N'} \ge \frac{2\pi}{N'}.$$

Let $L = \min\left\{\left\lfloor \frac{n}{8r} - \frac{3}{2}\right\rfloor, \left\lfloor \frac{N'}{8r} - \frac{1}{2}\right\rfloor\right\}$ and observe that for each $0 \le \lambda \le L$ there exists at least $j \in I_M^r$ such that $d_j \ge (2\lambda + 1)\frac{2\pi}{N'}$. Then

$$s_M^r := \sum_{j \in I_M^r} d_j \ge \sum_{\lambda=0}^{L} (2\lambda+1)\frac{2\pi}{N'} = \frac{2\pi(L+1)^2}{N'} \ge \frac{2\pi L^2}{N'}.$$

By the choice of L, if $r \le \lfloor n^{1/4} \rfloor$, then $\frac{2\pi L^2}{N'} \ge \frac{2\pi}{n^{11/10}} n^{3/2}$ for all large n.

Here, let v be a vector in $\mathbb{R}^n$ with entries $v_j = \cos(j2\pi x_\alpha - \theta)$ for each $j = 0, \ldots, n-1$. If r is a positive integer with $r \le \lfloor n^{1/4} \rfloor$, then by the previous discussion we deduce $\operatorname{dist}(rv, \mathbb{Z}^n) \ge 2\pi n^{2/5}$ for all large n. If r is any positive real number, observe that $\left[\frac{s}{r}, \frac{s+1}{r}\right] \subset \left[\frac{s}{\lceil r \rceil}, \frac{s+1}{\lceil r \rceil}\right]$, where $s \in \mathbb{N}$, and therefore our previous analysis holds true for any $r > 0$.

Suppose $r \le \lfloor n^{1/4} \rfloor$ and recall that $\|V_\alpha^T \Theta\|_2 \le r\sqrt{n}$ and that $L \ge \sqrt{2/q}$ is fixed. By the definition of lcd, for all large n, we obtain

$$2\pi n^{2/5} \le \operatorname{dist}\left(V_\alpha^T \Theta, \mathbb{Z}^n\right) \le L\sqrt{\log_+ \frac{\|V_\alpha^T \Theta_2\|_2}{L}} \le L\sqrt{\log_+ \frac{n^{3/4}}{L}},$$

which yields a contradiction for n large. Thus, $D(V_\alpha) \ge \lfloor n^{1/4} \rfloor$. Therefore, the preceding inequality together with inequality (**K**), inequality (19) and the fact that the cardinality of $J_3(n, N)$ is at most $n^{11/10+o(1)}(\log(n))^{o(1)}$ allow us to deduce

$$\begin{aligned}
&\sum_{\alpha \in J_3(n,N)} \mathbb{P}\left(|W_n\left(e^{i2\pi x_\alpha}\right)| \le 2tC_2 n^{-1/2}(\log n)^{-2}\right) \\
&\le n^{11/10+o(1)}(\log(n))^{o(1)}\left(\frac{2C^2L^2(2tC_2)^2}{\left(c_4 n^{2-1/5}\right)^{1/2}\left(n^{1/2}(\log n)^2\right)^2}\right) \\
&\quad + n^{11/10+o(1)}(\log(n))^{o(1)}\left(\frac{2C^2L^2}{\left(c_4 n^{2-1/5}\right)^{1/2}\left(n^{1/4}\right)^2}\right) \\
&\le \frac{8C^2C_2^2L^2t^2}{c_4^{1/2}n^{4/10}} + \frac{2C^2L^2}{c_4^{1/2}n^{1/10}}
\end{aligned}$$

for all large n. As a consequence we obtain

$$\sum_{\alpha \in J_3(n,N)} \mathbb{P}\left(|W_n\left(e^{i2\pi x_\alpha}\right)| \le 2tC_2 n^{-1/2}(\log n)^{-2}\right) = O\left(\frac{1}{n^{1/10}}\right),$$

where the implicit constant depends on L and t. ◻

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Appendix A: Arithmetic properties

This section contains the proofs of the results that we skipped in the paper in order to be more fluid.

Lemma A.1 *If* $m \geq 1$ *and* $M \in \mathbb{N}$, *then the cardinality of the set*

$$\Gamma_m^M := \{k \in [1, M] \cap \mathbb{N} : \ \gcd(k, M) \geq m\}$$

is at most $\frac{1}{\lfloor m \rfloor} M^{1+C(\log\log M)^{-1}}$, *where* C *is a positive constant.*

Proof Let T denote the Euler totient function. Observe that

$$\sum_{k \in \Gamma_m^M} 1 \leq \sum_{\substack{d=\lfloor m \rfloor \\ d|M}}^{M} T\left(\frac{M}{d}\right).$$

It is well known that $T(s) \leq s - \sqrt{s}$ for all $s \in \mathbb{N}$. Moreover, if $d(s)$ denotes the number of positive divisors of s, then [1, Theorem 13.12] implies that there exists a positive constant C such that $d(s) \leq s^{C(\log\log(s))^{-1}}$. Hence,

$$\sum_{k \in \Gamma_m^M} 1 \leq \left(\frac{M}{\lfloor m \rfloor} - \sqrt{\frac{M}{\lfloor m \rfloor}}\right) M^{C(\log\log(M))^{-1}} \leq \frac{1}{\lfloor m \rfloor} M^{1+C(\log\log M)^{-1}}$$

which yields the statement. □

Lemma A.2 *Let* $\theta \in [0, 2\pi)$ *and* $n \in \mathbb{N}$. *Let* $\mathcal{V} = (\mathcal{V}_j)_{j \in \{1,\ldots,n\}} \in \mathbb{R}^n$ *such that* $\mathcal{V}_j = r\cos(j2\pi x - \theta)$ *for* $j = 0, \ldots, n-1$, *where* $r \in \mathbb{N}$ *and* $x = 1/n$. *If* $\frac{1}{4\pi r x} \geq 8$, *then*

$$\text{dist}\left(\mathcal{V}, \mathbb{Z}^n\right) \geq \frac{1}{128\pi x}.$$

Proof Let $\theta \in [0, 2\pi)$ and $n \in \mathbb{N}$. Let $x = 1/n$ and we define the following sequence: $P_n = \{\exp(i(j2\pi x - \theta)) : j = 0, \ldots, n-1\}$, where i is the imaginary unit. Note that P_n is a set of points on the unit circle which can be viewed as vertices of a regular polygon with n sides inscribed in the unit circle.

Since the arguments of "two consecutive points" on P_n, $\exp(i(j2\pi x - \theta))$ and $\exp(i((j+1)2\pi x - \theta))$, are separated by a distance $2\pi x$, the number of points in P_n which are in any arc of length ℓ on the unit circle is at least $\frac{\ell}{2\pi x} - 2$.

Let $[y, y + 8\pi x]$ be a subinterval of $[-1, 1]$. We consider an arc $\widehat{I}$ on the unit circle such that its projection on the horizontal axis is $[y, y + 8\pi x]$. If the length of the arc $\widehat{I}$ is ℓ, then the number of values $\cos(j2\pi x - \theta)$, $j = 0, \ldots, n-1$, that belong to $(y, y + 8\pi x)$ is at least $\frac{1}{2}\left(\frac{\ell}{2\pi x} - 2\right)$. Observe that $\frac{1}{2}\left(\frac{\ell}{2\pi x} - 2\right) \geq 1$ when $\ell \geq 8\pi x$.

Let $r \in \mathbb{N}$ and $m \in [-(r-1), (r-1)] \cap \mathbb{Z}$. By the preceding explanation, for all positive integers $k \leq \frac{1}{8\pi r x}$, there exists $j \in \{0, \ldots, n-1\}$ such that

$$\cos(j2\pi x - \theta) \in \left(\frac{m}{r} + 8\pi(k-1)x, \frac{m}{r} + 8\pi k x\right) \subset \left[\frac{m}{r}, \frac{m+1}{r}\right].$$

In the sequel, set

$$I_m^r := \left\{j \in \{0, \ldots, n-1\} : \cos(j2\pi x - \theta) \in \left[\frac{m}{r}, \frac{m+1}{r}\right]\right\} \neq \emptyset$$

and, for each $j \in I_m^r$, define

$$d_j := \min\left\{\left|\cos(j2\pi x - \theta) - \frac{m}{r}\right|, \left|\cos(j2\pi x - \theta) - \frac{m+1}{r}\right|\right\}.$$

Let L be the biggest integer such that $8\pi L x \leq \frac{1}{2r}$, or, equivalently, $L = \left\lfloor \frac{1}{16\pi r x} \right\rfloor$. Observe that

$$L \geq \frac{1}{16\pi r x} - 1 \geq \frac{1}{32\pi r x} \quad \text{when} \quad \frac{1}{4\pi r x} \geq 8.$$

Then

$$s_m^r := \sum_{j \in I_m^r} d_j \geq \sum_{\lambda=1}^{L} 2\lambda(8\pi x) \geq 8\pi x L^2 \geq \frac{1}{128\pi r^2 x}.$$

Moreover,

$$\sum_{m=-(r-1)}^{m=r-1} s_m^r \geq \frac{2r-1}{128\pi r^2 x} \geq \frac{1}{128\pi r x},$$

where the last inequality follows since $\frac{2r-1}{r} \geq 1$ for $r \in \mathbb{N}$. Consequently, the distance between the vector $\mathcal{V} \in \mathbb{R}^n$ with entries $\mathcal{V}_j = r\cos(j2\pi x - \theta)$ for $j = 0, \ldots, n-1$ with $x = 1/n$ and the set $\mathbb{Z}^n$ is at least

$$r\left(\frac{1}{128\pi r x}\right) = \frac{1}{128\pi x} \quad \text{verifying that} \quad \frac{1}{4\pi r x} \geq 8 \quad \text{is fulfilled.}$$

□

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