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2022-07

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ORIGINAL ARTICLE

Salem–Zygmund inequality for locally sub-Gaussian random variables, random trigonometric polynomials, and random circulant matrices

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Abstract

In this manuscript we give an extension of the classic Salem–Zygmund inequality for locally sub-Gaussian random variables. As an application, the concentration of the roots of a Kac polynomial is studied, which is the main contribution of this manuscript. More precisely, we assume the existence of the moment generating function for the iid random coefficients for the Kac polynomial and prove that there exists an annulus of width

$$O(n^{-2}(\log n)^{-1/2-\gamma}), \quad \gamma > 1/2$$

around the unit circle that does not contain roots with high probability. As an another application, we show that the smallest singular value of a random circulant matrix is at least $n^{-\rho}$, $\rho \in (0, 1/4)$ with probability $1 - O(n^{-2\rho})$.

Keywords Zeros of random polynomials · Kac polynomials · Locally sub-Gaussian condition · Circulant matrices

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1 Introduction

A classical problem in Harmonic Analysis is the *quantification* of the magnitude of the modulus for a trigonometric polynomial on the unit circle. Erdös [10] studied the trigonometric polynomial $T_n(x) = \sum_{j=0}^{n-1} \alpha_j e^{ijx}$, $x \in [0, 2\pi]$, for choices of signs ± 1 for all α_j, and estimated how large $|T_n(x)|$ for $x \in [0, 2\pi)$ can be. Salem and Zygmund [30] proved that almost all choices of signs satisfy

$$c_1 (n \log n)^{\frac{1}{2}} \le \max_{x \in [0, 2\pi]} |T_n(x)| \le c_2 (n \log n)^{\frac{1}{2}} \quad \text{for some positive constants } c_1 \text{ and } c_2.$$

$$(1)$$

Inequalities of type (1) are known as Salem–Zygmund inequality. There are different versions of Salem–Zygmund inequality that appear in many areas of modern analysis, see [8]. In a probabilistic context, the common version of Salem–Zygmund inequality is usually established when the coefficients $\alpha_1, \ldots, \alpha_{n-1}$ of T_n are iid sub-Gaussian random variables, see Chapter 6 in [14]. In the present manuscript, we give an extension of Salem–Zygmund inequality for *locally sub-Gaussian* random coefficients. This extension allows us to study the localization of the roots of a random *Kac polynomial* and the probability for the singularity of a *random circulant matrix*.

1.1 Roots of random trigonometric polynomials

The study of the roots of a polynomial is an old topic in Mathematics. There are formulas to compute the roots for polynomials of degree 2, degree 3 (Tartaglia–Cardano's formula), degree 4 (Ferrari's formula), but due to Galois' work, for a generic polynomial of degree 5 or more it is not possible to find explicit formulas for computing its roots in terms of radicals.

For a random polynomial, Bloch and Polya [3] considered a random polynomial with iid Rademacher random variables (uniform distribution on $\{-1, 1\}$) and proved that the expected number of real zeros are $O\ (n^{1/2})$. In a series of papers between 1938 and 1939, Littlewood and Offord gave a better bound for the number of real roots of a random polynomial with iid random coefficients for the cases of Rademacher, Uniform$[-1, 1]$, and standard Gaussian [21]. Kac [13] established his famous integral formula for the density of the number of real roots of a random polynomial with iid coefficients with standard Gaussian distribution. Those were the first steps in the study of roots of random functions, which nowadays is a relevant part of modern Probability and Analysis. For further details, see [9] and the references therein.

The localization of the roots of a polynomial is in general a hard problem. However, there are relevant results in the theory of random polynomials [2]. For instance, for iid non-degenerate random coefficients with finite logarithm moment, the roots cluster asymptotically near the unit circle and the arguments of the roots are asymptotically uniform distributed. More precisely, Ibragimov and Zaporozhets [12] showed that for a *Kac polynomial*

$$G_n(z) = \sum_{j=0}^{n-1} \xi_j z^j \quad \text{for } z \in \mathbb{C}, \tag{2}$$

with (real or complex) iid non-degenerate coefficients satisfying $\mathbb{E}(\log(1 + |\xi_0|)) < \infty$, its roots are concentrated around the unit circle as $n \to \infty$, almost surely. Moreover, they proved that the condition $\mathbb{E}(\log(1 + |\xi_0|)) < \infty$ is necessary and sufficient for the roots of G_n to be asymptotically near the unit circle.

For iid standard Gaussian random coefficients of G_n, most of the roots are concentrated in an annulus of width $1/n$ centered in the unit circle. However, the nearest root to the unit circle is at least a distance $\mathrm{O}\,(n^{-2})$, for further details see [23]. Larry and Vanderbei [20] conjectured that the last statement holds not only for standard Gaussian coefficients but also for Rademacher coefficients. This conjecture was proved by Konyagin and Schlag [19]. Our Theorem 2.3 establishes that most of the roots of G_n are near to the unit circle in a distance at least $\mathrm{O}\,(n^{-2}(\log n)^{-1/2-\gamma})$ for $\gamma > 1/2$ with probability $1 - \mathrm{O}((\log n)^{-\gamma+1/2})$. Konyagin and Schlag [19] showed that if G_n has iid Rademacher or standard Gaussian random coefficients, then for all $\varepsilon > 0$ and large n the following expression

$$\min_{z \in \mathbb{C}: ||z|-1| < \varepsilon n^{-2}} |G_n(z)| \geq \varepsilon n^{-1/2}$$

holds with probability at least $1 - C\varepsilon$, for some positive constant C. Karapetyan [15, 16] studied the sub-Gaussian case, but up to our knowledge, his proof is not complete. Even so, using our extension of Salem–Zygmund inequality and the notion of *least common denominator*, which was developed to study the singularity of the random matrices [28], we show that for fixed $t \geq 1$,

$$\min_{z \in \mathbb{C}\,:\,||z|-1| \leq tn^{-2}(\log n)^{-1/2-\gamma}} |G_n(z)| \geq tn^{-1/2}(\log n)^{-\gamma},$$

with probability at least $1 - \mathrm{O}((\log n)^{-\gamma+1/2})$. The techniques using in the present paper are not the same using in Konyagin and Schlag [19]. The main result of Konyagin and Schlag only holds for Rademacher and Gaussian iid random coefficients. They did a refined analysis of the characteristic function and applying the so-called circle method. This approach is not straightforward to apply for more general random coefficients, even sub-Gaussian or with finite moment generating function (mgf for short).

The novelty of this manuscript is the use of the notion of least common denominator for cover more general random coefficients. This approach works for quite general random coefficients. However, the authors still working for relaxing the assumption of the existence of a mgf. The main obstacle for relaxing this assumption arises in the control of the maximum modulus over the unit circle of the random polynomial under the assumption of the existence of some $p-$moment. We emphasize that the proof is not a direct consequence of [29] since good estimates of the least common denominator typically are difficult to obtain. We remark that this

result and the main result in [19], up to our knowledge, are not direct consequences of the so-called concentration inequalities.

1.2 Random circulant matrices

Recall, an $n \times n$ complex circulant matrix, denoted by $\mathrm{circ}(c_0, \ldots, c_{n-1})$, has the form

$$
\mathrm{circ}(c_0, \ldots, c_{n-1}) := \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & \cdots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & \cdots & c_0 & c_1 \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{bmatrix},
$$

where $c_0, \ldots, c_{n-1} \in \mathbb{C}$. For $\xi_0, \ldots, \xi_{n-1}$ being random variables, we say that

$$
\mathcal{C}_n := \mathrm{circ}(\xi_0, \ldots \xi_{n-1})
$$

is an $n \times n$ random circulant matrix. The circulant matrices are a very common object in different areas of mathematics [11, 17, 26]. In particular, circulant matrices play a crucial role in the study of large-dimensional Toeplitz matrices [5, 31]. In the theory of the random matrices, the singularity is one aspect that has been intensively studied during recent years [4, 27, 28]. In the case of the random circulant matrices have Rademacher entries, Meckes [22] proved that the probability of a random circulant matrix is singular tends to zero when its dimension is growing.

As a consequence of our concentration result of the roots for Kac polynomials, for a random circulant matrix with iid zero-mean entries and finite mfg, it follows that for all fixed $t \geq 1$ and $\gamma > 1/2$, the smallest singular value $s_n(\mathcal{C}_n)$ of $\mathcal{C}_n$ satisfies

$$
s_n(\mathcal{C}_n) \geq t n^{-1/2} (\log n)^{-\gamma}
$$

with probability $1 - \mathrm{O}\left((\log n)^{-\gamma+1/2}\right)$. However, under weaker assumptions (see below the condition (H)), for $\rho \in (0, 1/4)$ we also show

$$
s_n(\mathcal{C}_n) \geq n^{-\rho}
$$

with probability $1 - \mathrm{O}(n^{-2\rho})$.

The manuscript is organized as follows. In Sect. 2 we state the main results and their consequences. In Section 3 we give the proof of a Salem–Zygmund inequality for random variables with mgf. In Sect. 4 with the help of Salem–Zygmund inequality and the notion of least common denominator we prove Theorem 2.3 about the location of the roots of a Kac polynomial. Finally, in Sect. 5 we prove Theorem 2.6 about that the smallest singular value of a random circulant is relatively large with high probability.

2 Main results

2.1 Salem–Zygmund inequality

Recall that a real-valued random variable ξ is said to be sub-Gaussian if its mgf is bounded by the mgf of a Gaussian random variable, i.e., there is $b > 0$ such that

$$\mathbb{E}(e^{t\xi}) \leq e^{b^2 t^2/2} \quad \text{for any } t \in \mathbb{R}.$$

When this condition is satisfied for a particular value of $b > 0$, we say that ξ is b-sub-Gaussian or sub-Gaussian with parameter b. In particular, it is straigforward to show that the mean of a sub-Gaussian random variable is necessarily equal to zero. For more details see [6] and the references therein.

According to [6], a random variable ξ is called *locally sub-Gaussian* when its mgf M_ξ exists in an open interval around zero. Due to this, it is possible to find constants $\alpha \geq 0$, $\delta \in (0, \infty]$ and $v \in \mathbb{R}$ such that

$$M_\xi(t) \leq e^{vt + \frac{1}{2}\alpha^2 t^2} \quad \text{for any } t \in (-\delta, \delta).$$

If the mean of ξ is zero and its variance σ^2 is finite and positive then we can take $v = 0$ and $\alpha^2 > \sigma^2$ for some $\delta > 0$ as the next lemma states.

Lemma 2.1 (*Locally sub-Gaussian r.v.*). *Let ξ be a random variable such that its mgf M_ξ exists in an interval around zero. Assume that $\mathbb{E}(\xi) = 0$ and $\mathbb{E}(\xi^2) = \sigma^2 > 0$. Then there is a positive constant δ such that*

$$M_\xi(t) \leq e^{\alpha^2 t^2/2} \quad \text{for any } t \in (-\delta, \delta) \text{ and } \alpha^2 > \sigma^2.$$

The preceding lemma is not suprising, see for instance Remark 2.7.9 in [34]. Since its proof is simple, we give it here for completeness of the presentation.

Proof Assume that $M_\xi(t)$ is well-defined for any $t \in (-\delta_1, \delta_1)$, for some $\delta_1 > 0$. Then $M_\xi(t)$ has derivatives of all orders at $t = 0$. Define $g(t) := e^{\alpha^2 t^2/2}$, for $t \in \mathbb{R}$. Then $g(0) = 1$, $g'(0) = 0$ and $g''(0) = \alpha^2$. Let $h(t) := g(t) - M_\xi(t)$, for all $t \in (-\delta_1, \delta_1)$. Since h is continuous and $h''(0) = \alpha^2 - \sigma^2 > 0$, then there exists $0 < \delta < \delta_1$ such that $h''(t) > 0$, for every $t \in (-\delta, \delta)$. Therefore, the function h is convex in the interval $(-\delta, \delta)$. As $h'(0) = 0$ then 0 is a local minimum of h. Therefore, it follows that $h(t) \geq h(0) = 0$, for every $t \in (-\delta, \delta)$. Thus, the result follows.

The classic Salem–Zygmund inequality is usually established for iid sub-Gaussian random variables. But thanks to Lemma 2.1 we are able to extend it to iid locally sub-Gaussian random variables as it is stated in Theorem 2.2. Even though, Theorem 2.2 is interesting on its own, we stress that it is also crucial for our approach using in the proof of the main result Theorem 2.3.

Before presenting Theorem 2.2, we introduce some useful notations. For simplicity, we keep the same notation between the Euclidean norm and the modulus

for the complex numbers. Denote by $\mathbb{T}$ the unit circle $\mathbb{R}/(2\pi\mathbb{Z})$. For any bounded function $f : \mathbb{T} \to \mathbb{C}$, the infinite norm of f is defined as $\|f\|_\infty = \sup_{x \in \mathbb{T}} |f(x)|$, and $\overset{\mathcal{D}}{=}$ means "equal in distribution".

Theorem 2.2 (*SalemZygmund inequality for locally sub-Gaussian random variables*). *Let ξ be a random variable with zero mean and finite positive variance. Assume that the mgf M_ξ of ξ exists in an open interval around zero. Let $\{\xi_k : k \geq 0\}$ be a sequence of iid random variables with $\xi_k \overset{\mathcal{D}}{=} \xi$ for every $k \geq 0$. Let $\phi : [0,1] \to \mathbb{R}$ is a non-zero continuous function. Consider $W_n(x) = \sum_{j=0}^{n-1} \xi_j \phi(j/n) e^{ijx}$ for any $x \in \mathbb{T}$. Then, for all large n*

$$\mathbb{P}\left(\|W_n\|_\infty \geq C_0 \left((\log n) \sum_{j=0}^{n-1} |\phi(j/n)|^2 \right)^{1/2} \right) \leq \frac{C_1}{n^2},$$

where C_0 and C_1 are positive constants that only depend on the mgf of ξ and the function ϕ.

Actually, under the assumption of finite second moment, a version of a Salem–Zygmund type inequality can be obtained in terms of the expected value of the infinite norm of a random trigonometric polynomial, for more details see [33]. Theorem 2.2 provides an upper bound of how large the infinite norm of a random trigonometric polynomial is in probability. Moreover, Theorem 2.2 gives a better bound than Corollary 2 in [33] as we see below.

Let $\{\xi_k : k \geq 0\}$ be a sequence of iid random variables such that $\mathbb{E}(\xi_0) = 0$ and $\mathbb{E}(\xi_0^2) = \sigma^2 > 0$. By Corollary 2 in [33] we have

$$\mathbb{E}\left(\max_{x \in \mathbb{T}} \left| \sum_{j=0}^{n-1} \xi_j e^{ijx} \right| \right) \leq C \min\left\{ (n \log(n+1)) \mathbb{E}(|\xi_0|^2))^{1/2}, n \mathbb{E}|\xi_0| \right\}$$

$$\leq C(n \log(n+1)) \mathbb{E}(|\xi_0|^2))^{1/2},$$

where C is a universal positive constant. By the Markov inequality we obtain

$$\mathbb{P}\left(\max_{x \in \mathbb{T}} \left| \sum_{j=0}^{n-1} \xi_j e^{ijx} \right| \geq C_0 (n \log n)^{1/2} \right) \leq \frac{C(n \log(n+1)) \mathbb{E}(|\xi_0|^2))^{1/2}}{C_0 (n \log n)^{1/2}}.$$

Note that the upper bound asymptotically equals a positive constant. On the other hand, under the assumptions of Theorem 2.2 we deduce

$$\mathbb{P}\left(\max_{x \in \mathbb{T}} \left| \sum_{j=0}^{n-1} \xi_j e^{ijx} \right| \leq C_0 (n \log n)^{1/2} \right) \geq 1 - \frac{C_1}{n^2}$$

for all large n, where C_0 and C_1 are positive constants that only depend on the mgf of ξ_0.

2.2 Kac polynomials

For using the concept of *least common denominator* we introduce the following condition. We say that a random variable ξ_0 satisfies the condition (H) if

$$\sup_{u \in \mathbb{R}} \mathbb{P}\{|\xi_0 - u| \leq 1\} \leq 1 - q \quad \text{and} \quad \mathbb{P}\{|\xi_0| > M\} \leq q/2 \quad \text{for some } M > 0 \text{ and } q \in (0, 1).$$

$$\text{(H)}$$

The notion of concentration function was introduced by P. Lévy in the context of the study of distributions of sums of random variables. For ξ_0 being not degenerate, zero mean with mgf, one can deduce that condition (H) is valid for some $M > 0$ and $q \in (0, 1)$. We refer to [32].

The main result of this manuscript is the following theorem.

Theorem 2.3 *Let ξ be a random variable with zero mean and finite positive variance. Assume that the mgf M_ξ of ξ exists in an open interval around zero. Let $\{\xi_k : k \geq 0\}$ be a sequence of iid random variables with $\xi_k \overset{\mathcal{D}}{=} \xi$ for every $k \geq 0$. Let*

$$\mathcal{M}_n := \left\{ \min_{z \in \mathbb{C} \,:\, ||z|-1| \leq tn^{-2}(\log n)^{-1/2-\gamma}} |G_n(z)| \leq tn^{-1/2}(\log n)^{-\gamma} \right\}.$$

Then for any fixed $t \geq 1$,

$$\mathbb{P}(\mathcal{M}_n) = \mathrm{O}\left((\log n)^{-\gamma+1/2} \right),$$

where $\gamma > 1/2$ and the implicit constant in the O-notation depends on t and the mgf of ξ.

Remark 2.4 Observe that all bounded random variables satisfy (H) in Theorem 2.3 (with a suitable scaling). In particular, the Rademacher distribution which corresponds to the uniform distribution on $\{-1, 1\}$ and the uniform distribution on the interval $[-1, 1]$ satisfy (H).

2.3 Random circulant matrices

It is well-known that any circulant matrix can be diagonalized in $\mathbb{C}$ using a Fourier basis. Indeed, let $\omega_n := \exp\left(i\frac{2\pi}{n}\right)$, $i^2 = -1$, and $F_n = \frac{1}{\sqrt{n}}(\omega_n^{jk})_{0 \leq j, k \leq n-1}$. The matrix F_n is called the *Fourier matrix* of order n. Note that F_n is a unitary matrix. By a straightforward computation it follows

$$\mathrm{circ}(c_0, \ldots, c_{n-1}) = F_n^* \mathrm{diag}\left(G_n(1), G_n(\omega_n), \ldots, G_n(\omega_n^{n-1})\right) F_n,$$

where G_n is the polynomial given by $G_n(z) := \sum_{k=0}^{n-1} c_k z^k$. Hence, the eigenvalues of $\mathrm{circ}(c_0, \ldots, c_{n-1})$ are $G_n(1), G_n(\omega_n), \ldots, G_n(\omega_n^{n-1})$, or equivalently

$$G_n(\omega_n^k) = \sum_{j=0}^{n-1} c_j \exp\left(i\frac{2\pi k j}{n}\right) \quad \text{for any} \quad k = 0,\ldots,n-1. \tag{3}$$

Expressions like (3) appear naturally in the study of Fourier transform of periodic functions. For a complete understanding of circulant matrices, we recommend the monograph [7].

In the sequel, we consider an $n \times n$ random circulant matrix $\mathcal{C}_n$, i.e., $\mathcal{C}_n := \operatorname{circ}(\xi_0,\ldots,\xi_{n-1})$, where $\xi_0,\ldots,\xi_{n-1}$ are independent random variables. The smallest singular value of the random circulant matrix $\mathcal{C}_n$ is given by

$$s_n(\mathcal{C}_n) = \min_{0 \le k \le n-1} |G_n(\omega_n^k)|. \tag{4}$$

We remark that in general the smallest singular value is not equal to the smallest eigenvalue modulus. Since $\mathcal{C}_n$ is a normal matrix, its singular values are the modulus of its eigenvalues. Thus, the following corollary is a direct consequence of Theorem 2.3.

Corollary 2.5 *Let ξ be a random variable with zero mean and finite positive variance. Assume that the mgf M_ξ of ξ exists in an open interval around zero. Let $\{\xi_k : k \ge 0\}$ be a sequence of iid random variables with $\xi_k \overset{\mathcal{D}}{=} \xi$ for every $k \ge 0$. Let $\mathcal{C}_n := \operatorname{circ}(\xi_0,\ldots,\xi_{n-1})$ be an $n \times n$ random circulant matrix and let $s_n(\mathcal{C}_n)$ be the smallest singular value of $\mathcal{C}_n$. Then, for all fixed $t \ge 1$ and $\gamma > 1/2$ we have*

$$\mathbb{P}\left(s_n(\mathcal{C}_n) \le t n^{-1/2}(\log n)^{-\gamma}\right) = \mathrm{O}\left((\log n)^{-\gamma+1/2}\right). \tag{5}$$

It is possible to weaken the assumptions of Corollary 2.5. Using similar reasoning as in the proof of Theorem 2.3 we obtain the following theorem.

Theorem 2.6 *Let ξ be a non-degenerate random variable which satisfies (H). Let $\{\xi_k : k \ge 0\}$ be a sequence of iid random variables with $\xi_k \overset{\mathcal{D}}{=} \xi$ for every $k \ge 0$. Let $\mathcal{C}_n := \operatorname{circ}(\xi_0,\ldots,\xi_{n-1})$ be an $n \times n$ random circulant matrix. Then, for each $\rho \in (0,1/4)$ we have*

$$\mathbb{P}(s_n(\mathcal{C}_n) \le n^{-\rho}) = \mathrm{O}(n^{-2\rho}).$$

3 Proof of Theorem 2.2. Salem–Zygmund inequality for locally sub-Gaussian random variables

Firstly, we provide the proof of the following claim which is an important fact that we use in the proof of Theorem 2.2.

Claim 1: There exists a random interval $I \subset \mathbb{T}$ of length $1/\rho_n$ with $\rho_n = 3n/8$ such that

$$|W_n(x)| \geq \frac{1}{2} \|W_n\|_\infty \quad \text{for any } x \in I.$$

Proof Let $p_n(x) := \sum_{j=0}^{n-1} b_j e^{ijx}$, $x \in \mathbb{T}$ be a trigonometric polynomial on $\mathbb{T}$, where $b_0, \ldots, b_{n-1}$ are real numbers. For $x \in \mathbb{T}$ write

$$g_n(x) := |p_n(x)|^2 = \left(\sum_{j=0}^{n-1} b_j \cos(jx) \right)^2 + \left(\sum_{j=0}^{n-1} b_j \sin(jx) \right)^2 \quad (6)$$

and

$$h_n(x) := \left(\sum_{j=0}^{n-1} jb_j \cos(jx) \right)^2 + \left(\sum_{j=0}^{n-1} jb_j \sin(jx) \right)^2.$$

Then

$$\|p_n\|_\infty^2 = \sup_{x \in \mathbb{T}} g_n(x) = \|g_n\|_\infty \quad \text{and} \quad \|p_n'\|_\infty^2 = \sup_{x \in \mathbb{T}} h_n(x). \quad (7)$$

Recall the Bernstein inequality $\|p_n'\|_\infty \leq n\|p_n\|_\infty$ (see for instance Theorem 14.1.1, Chapter 14, page 508 in [25]). For any $x \in \mathbb{T}$ we have

$$|g_n'(x)| \leq 4\|p_n\|_\infty \|p_n'\|_\infty \leq 4n\|p_n\|_\infty^2 = 4n\|g_n\|_\infty. \quad (8)$$

Since g is continuous, there exists $x_0 \in \mathbb{T}$ such that $g(x_0) = \|g_n\|_\infty$. Moreover, by the Mean Value Theorem and relation (8) we obtain

$$|g(x) - g(x_0)| \leq \|g_n'\|_\infty |x - x_0| \leq 4n\|g_n\|_\infty |x - x_0|$$

for any $x \in \mathbb{T}$. Take $I := [x_0 - \frac{3}{16n}, x_0 + \frac{3}{16n}] \subset \mathbb{T}$. Notice that the length of I is $\frac{3}{8n}$. The preceding inequality yields

$$|g(x) - g(x_0)| \leq \frac{3}{4} \|g_n\|_\infty \quad \text{for any } x \in I.$$

Since $g(x_0) = \|g_n\|_\infty$, the triangle inequality yields $(1/4)\|g_n\|_\infty \leq |g_n(x)|$ for any $x \in I$. The preceding inequality with the help of relation (6) and relation (7) implies

$$\frac{1}{2} \|p_n\|_\infty \leq |p_n(x)| \quad \text{for any } x \in I.$$

Now, we are ready to provide the proof of Theorem 2.2.

Proof of Theorem 2.2 By Lemma 2.1, there exists a $\delta > 0$ such that

$$M_\xi(t) \leq e^{\alpha^2 t^2/2} \quad \text{for any } t \in (-\delta, \delta), \text{ where } \alpha^2 > \sigma^2 > 0.$$

For each $j \in \{0, \ldots, n-1\}$, define $f_j(x) = \phi(j/n)e^{ijx}$, $x \in \mathbb{T}$. Let

$r_n := \sum_{j=0}^{n-1} |\phi(j/n)|^2$. At first, we suppose that the f_j are real (we consider only the real part or the imaginary part) and we write $S_n := \|W_n\|_\infty$. Since $\|f_j\|_\infty \le \|\phi\|_\infty =: K$ for every $j = 0, \ldots, n-1$, we obtain

$$e^{\alpha^2 t^2 r_n/2} = \prod_{j=0}^{n-1} e^{\alpha^2 t^2 \|f_j\|_\infty^2/2} \ge \prod_{j=0}^{n-1} e^{\alpha^2 t^2 |f_j(x)|^2/2} \ge \prod_{j=0}^{n-1} \mathbb{E}\left(e^{t\xi_j f_j(x)}\right)$$

$$= \mathbb{E}\left(\prod_{j=0}^{n-1} e^{t\xi_j f_j(x)}\right) = \mathbb{E}\left(e^{tW_n(x)}\right) \quad \text{for any } t \in (-\delta/K, \delta/K).$$

By **Claim 1**, there exists a random interval $I \subset \mathbb{T}$ of length $1/\rho_n$ with $\rho_n = 8n/3$ such that $W_n(x) \ge S_n/2$ or $-W_n(x) \ge S_n/2$ on I. Denote by μ the normalized Lebesgue measure on $\mathbb{T}$. Observe that

$$e^{tS_n/2} = \frac{1}{\mu(I)} \int_I e^{tS_n/2} dx \le \frac{1}{\mu(I)} \int_I \left(e^{tW_n(x)} + e^{-tW_n(x)}\right) dx.$$

Then, for every $t \in (-\delta/K, \delta/K)$ we have

$$\mathbb{E}\left(e^{tS_n/2}\right) \le \rho_n \mathbb{E}\left(\int_I \left(e^{tW_n(x)} + e^{-tW_n(x)}\right)\mu(dx)\right)$$

$$\le \rho_n \mathbb{E}\left(\int_{\mathbb{T}} \left(e^{tW_n(x)} + e^{-tW_n(x)}\right)\mu(dx)\right) \le 2\rho_n e^{\alpha^2 t^2 r_n/2}.$$

The preceding inequality yields

$$\mathbb{E}\left(\exp\left\{\frac{t}{2}\left(S_n - \alpha^2 t r_n - \frac{2}{t}\log(2\rho_n l)\right)\right\}\right) \le \frac{1}{l} \quad \text{for any } l > 0 \text{ and } t \in (-\delta/K, \delta/K),$$

which implies

$$\mathbb{P}\left(S_n \ge \alpha^2 t r_n + \frac{2}{t}\log(2\rho_n l)\right) \le \frac{1}{l} \quad \text{for any } l > 0 \text{ and } t \in (-\delta/K, \delta/K).$$

Note that $\lim_{n\to\infty} \frac{r_n}{n} = \int_0^1 |\phi(x)|^2 dx > 0$. By taking $l_n = cn^2$ where c is a positive constant, we have $\left|\frac{\log(2\rho_n l_n)}{\alpha^2 r_n}\right| < \delta^2/K^2$ for all large n. By choosing $t_n = \left(\frac{\log(2\rho_n l_n)}{\alpha^2 r_n}\right)^{1/2}$ we obtain

$$\mathbb{P}\left(S_n \ge 3\left(\alpha^2 r_n \log(2\rho_n l_n)\right)^{1/2}\right) \le \frac{1}{l_n} \quad \text{for all large } n.$$

Since $f_j = \text{Re}\,(f_j) + i\,\text{Im}\,(f_j)$, we get for all large n

$$\mathbb{P}\left(\| \operatorname{Re}(W_n)\|_\infty \ge 3\left(\alpha^2 \sum_{j=0}^{n-1} \| \operatorname{Re}(f_j)\|_\infty^2 \log(2\rho_n l_n)\right)^{1/2}\right) \le \frac{1}{l_n}$$

and

$$\mathbb{P}\left(\| \operatorname{Im}(W_n)\|_\infty \ge 3\left(\alpha^2 \sum_{j=0}^{n-1} \| \operatorname{Im}(f_j)\|_\infty^2 \log(2\rho_n l_n)\right)^{1/2}\right) \le \frac{1}{l_n}.$$

Finally, since $\rho_n = \frac{8n}{3}$, the choose of $l_n = \frac{3n^2}{16}$ yields

$$\mathbb{P}\left(\|W_n\|_\infty \ge 6\alpha\sqrt{3}(r_n \log n)^{1/2}\right) \le \frac{32}{3n^2} \quad \text{for all large } n.$$

4 Proof of Theorem 2.3. Localization of the roots for Kac polynomials

The proof is based on the small ball probability of linear combinations of iid random variables introduced by Rudelson and Vershynin in [29]. Throughout the proof, $\| \cdot \|_2$ denotes the Euclidean norm, $|\cdot|$ denotes the complex norm and $\det(\cdot)$ the determinant function that acts on the squared matrices. We consider the module π of a real number y, $y \bmod \pi$, which is defined as the set of numbers x such that $x - y = k\pi$ for some $k \in \mathbb{Z}$.

Definition 4.1 (Least common denominator (lcd for short)). Let L be any positive number and let V be any *deterministic* matrix of dimension $2 \times n$. The *least common denominator* (lcd) of V is defined as

$$D(V) := \inf\left\{ \|\theta\|_2 > 0 : \theta \in \mathbb{R}^2,\ \operatorname{dist}\left(V^T\theta, \mathbb{Z}^n\right) < L\sqrt{\log_+\left(\frac{\|V^T\theta\|_2}{L}\right)}\right\},$$

where $\operatorname{dist}(v, \mathbb{Z}^n)$ denotes the distance between the vector $v \in \mathbb{R}^n$ and the set $\mathbb{Z}^n$, and $\log_+ := \max\{\log, 0\}$.

For more details of the concept of lcd see Section 7 of [29]. Observe that $D(aV) = (1/|a|)D(V)$ for any $a \ne 0$. Indeed, from the definition of $D(aV)$ we have that $D(aV) \le \|\theta\|_2$ for any $\theta \in \mathbb{R}^2$ such that

$$\operatorname{dist}\left((aV)^T\theta, \mathbb{Z}^n\right) < L\sqrt{\log_+\left(\frac{\|(aV)^T\theta\|_2}{L}\right)} = L\sqrt{\log_+\left(\frac{\|V^T(a\theta)\|_2}{L}\right)}.$$

Therefore, from the definition of $D(V)$ we deduce $D(V) \le \|a\theta\|_2 = |a|\|\theta\|_2$. Since $a \ne 0$, then $(1/|a|)D(V) \le \|\theta\|_2$. Again, from the definition of $D(aV)$ we deduce that $(1/|a|)D(V) \le D(aV)$. On the other hand, from the definition of $D(V)$ we have that $D(V) \le \|\theta\|_2$ for any $\theta \in \mathbb{R}^2$ such that

$$\text{dist}\left(V^T\theta, \mathbb{Z}^n\right) < L\sqrt{\log_+\left(\frac{\|V^T\theta\|_2}{L}\right)} = L\sqrt{\log_+\left(\frac{\|(aV)^T(\theta/a)\|_2}{L}\right)}.$$

Therefore, from the definition of $D(aV)$ we deduce $D(aV) \leq \|\theta/a\|_2 = \|\theta\|_2/|a|$. Consequently, $|a|D(aV) \leq \|\theta\|_2$. Again, from the definition of $D(V)$ we deduce that $|a|D(aV) \leq D(V)$. Putting all these pieces together we obtain the next useful lemma.

Lemma 4.2 *For all $a \neq 0$, the lcd of any matrix $V \in \mathbb{R}^{2 \times n}$ satisfies $D(V) = |a|D(aV)$.*

Let X be a *random* vector of dimension $n \times 1$ whose entries are iid satisfying (H). Assume $\det(VV^T) > 0$. For any $a > 0$ and $t \geq 1$, by Theorem 7.5 (Section 7 in [29]) we have

$$\mathbb{P}\left\{\|VX\|_2 \leq \frac{t\sqrt{2}}{a}\right\} = \mathbb{P}\left\{\|aVX\|_2 \leq \sqrt{2}t\right\} \leq \frac{C^2L^2}{2a^2(\det(VV^T))^{1/2}}\left(t + \frac{\sqrt{2}}{D(aV)}\right)^2,$$

where $L \geq \sqrt{2/q}$ with q given in (H), $D(aV)$ is the least common denominator of aV, and the constant C only depends on M, q. Recall the well-known inequality $(x + y)^2 \leq 2x^2 + 2y^2$ for any $x, y \in \mathbb{R}$. By Lemma 4.2, it follows that $D(aV) = (1/a)D(V)$ for all $a > 0$. Therefore,

$$\begin{aligned}
\mathbb{P}\left\{\|aVX\|_2 \leq \sqrt{2}t\right\} &\leq \frac{C^2L^2}{a^2(\det(VV^T))^{1/2}}t^2 + \frac{2C^2L^2}{a^2(\det(VV^T))^{1/2}(D(aV))^2} \\
&\leq \frac{C^2L^2}{a^2(\det(VV^T))^{1/2}}t^2 + \frac{2C^2L^2}{(\det(VV^T))^{1/2}(D(V))^2}.
\end{aligned} \tag{9}$$

In order to obtain a meaningful upper bound for the left-hand side of the preceding inequality, it is needed to do a refined analysis of the following quantities: *a lower bound for* $\det(VV^T)$ and *a lower bound for $D(V)$*. Implicitly, in the definition of the $D(V)$ we also need to estimate $\|V^T\theta\|_2$ for some adequate $\theta \in \mathbb{R}^2$.

4.1 Small ball probability analysis

The following analysis explains the reason of introducing the concept of the least common denominator, which is a crucial part along the proof of Theorem 2.3. Recall

$$G_n(z) = \sum_{j=0}^{n-1} \xi_j z^j \quad \text{for } z \in \mathbb{C}.$$

For G_n, we associate a random trigonometric polynomial

$$W_n(x) = \sum_{j=0}^{n-1} \xi_j e^{ijx} \quad \text{for } x \in \mathbb{T},$$

where $\mathbb{T}$ denotes the unit circle $\mathbb{R}/(2\pi\mathbb{Z})$. Assume $n \geq 2$ and $\gamma > 1/2$. Let $N = \lfloor n^2 (\log n)^{1/2+\gamma} \rfloor$ and $x_\alpha = \alpha/N$ for $\alpha \in \{0, 1, 2, \ldots, N-1\}$. Let $t \geq 1$ be fixed and let $C_0 > 0$ be the suitable positive constant being given in Theorem 2.2. Define the following event

$$\mathcal{G}_n := \left\{ \|W_n'\|_\infty \leq C_0 n^{3/2} (\log n)^{1/2}, \max_{z \in \mathbb{C} \, : \, ||z|-1| \leq 2tn^{-2}} |G_n(z)| \leq n^{3/2} \right\},$$

where W_n' denotes the derivative of W_n on $\mathbb{T}$. For short, we also denote by $\mathbb{P}(A, B)$ the probability $\mathbb{P}(A \cap B)$ for any two events A and B. Recall

$$\mathcal{M}_n = \left\{ \min_{z \in \mathbb{C} \, : \, ||z|-1| \leq tn^{-2}(\log n)^{-1/2-\gamma}} |G_n(z)| \leq tn^{-1/2}(\log n)^{-\gamma} \right\}.$$

By the Boole–Bonferroni inequality we obtain

$$\mathbb{P}(\mathcal{M}_n) \leq \mathbb{P}(\mathcal{M}_n, \mathcal{G}_n) + \mathbb{P}\left(\|W_n'\|_\infty \geq C_0 n^{3/2} (\log n)^{1/2} \right)$$
$$+ \mathbb{P}\left(\max_{z \in \mathbb{C} \, : \, ||z|-1| \leq 2tn^{-2}} |G_n(z)| \geq n^{3/2} \right) \tag{10}$$
$$=: \mathbb{P}(\mathcal{M}_n, \mathcal{G}_n) + I_1 + I_2.$$

Our goal is to show that every probability on the right side of the above expression is decreasing to zero when n tends to infinity.

Using the Berstein inequality (Theorem 14.1.1 in [25]) and Theorem 2.2 for $\phi \equiv 1$, for all large n we have

$$\mathbb{P}\left(\|W_n'\|_\infty \geq C_0 n^{3/2} (\log n)^{1/2} \right) \leq \mathbb{P}\left(\|W_n\|_\infty \geq C_0 (n \log n)^{1/2} \right) \leq \frac{C_1}{n^2}.$$

On the other hand, using the Markov inequality we obtain

$$\mathbb{P}\left(\max_{z \in \mathbb{C} \, : \, ||z|-1| \leq 2tn^{-2}} |G_n(z)| \geq n^{3/2} \right) \leq \mathbb{P}\left(\sum_{j=0}^{n-1} |\xi_j| \left(1 + \frac{2t}{n^2} \right)^j \geq n^{3/2} \right)$$
$$\leq \frac{1}{n^{3/2}} \mathbb{E}\left(\sum_{j=0}^{n-1} |\xi_j| \left(1 + \frac{2t}{n^2} \right)^j \right) \leq \frac{e^{2t} \mathbb{E}(|\xi_0|) n}{n^{3/2}} = \frac{e^{2t} \mathbb{E}(|\xi_0|)}{n^{1/2}},$$

where the last inequality follows from the following fact: for any $j \in \{0, \ldots, n^2\}$ we have

$$\left(1 + \frac{2t}{n^2}\right)^j \le \left(1 + \frac{2t}{n^2}\right)^{n^2} \le e^{2t}.$$

Therefore,

$$I_1 + I_2 = \mathrm{O}\left(n^{-1/2}\right), \tag{11}$$

where the implicit constant depends on the distribution of ξ_0 and t. We stress that the rate of the convergence in (11) can be improved, however, the contributed term in the right-hand side of (10) is $\mathbb{P}(\mathcal{M}_n, \mathcal{G}_n)$.

In the sequel, we analyze the strategy to prove that $\mathbb{P}(\mathcal{M}_n, \mathcal{G}_n)$ is small. First, we construct a set of closed balls that covers $\{z \in \mathbb{C} : ||z| - 1| \le tn^{-2}\}$. For each closed ball, we reduce the event $\{\mathcal{M}_n, \mathcal{G}_n\}$ to a "simple event" using Taylor's Theorem. Finally, we use the concept of lcd to show that the probability of each "simple event" is sufficiently small.

The strategy is to consider a set of balls centered at a point on the unit circle with a suitable radius. We distinguish two kind of balls. The special balls centered in $1 + 0i$ and $-1 + 0i$, where the radius r is *large*, $r = 2tn^{-11/10}$, and the balls centered in points z with argument satisfying $n^{-11/10} < |\arg(z) \bmod \pi| < \pi - n^{-11/10}$ with *small* radius, $r = 2tn^{-2}$.

Recall that for any $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Let $N := \lfloor n^2 (\log n)^{1/2+\gamma} \rfloor$ and $x_\alpha := \frac{\alpha}{N}$ for $\alpha = 0, 1, \ldots, N - 1$. For $a \in \mathbb{C}$ and $s > 0$, denote by $\mathrm{B}(a, s)$ the closed ball with center a and radius s, i.e., $\mathrm{B}(a, s) = \{z \in \mathbb{C} : |z - a| \le s\}$. Denote by $\mathbb{S}^1$ the unit circle. Let

$$\mathcal{A}\left(\mathbb{S}^1, tn^{-2}(\log n)^{-1/2-\gamma}\right) := \left\{z \in \mathbb{C} : ||z| - 1| \le tn^{-2}(\log n)^{-1/2-\gamma}\right\}.$$

Note that

$$\mathcal{A}\left(\mathbb{S}^1, tn^{-2}(\log n)^{-1/2-\gamma}\right) = \left\{z \in \mathcal{A} : n^{-11/10} < |\arg(z)| < \pi - n^{-11/10}\right\}$$
$$\cup \left\{z \in \mathcal{A} : |\arg(z)| \le n^{-11/10} \quad \text{or} \quad |\arg(z) - \pi| \le n^{-11/10}\right\}.$$

Let $t \ge 1$ and observe that

$$\left\{z \in \mathcal{A} : |\arg(z)| \le n^{-11/10} \quad \text{or} \quad |\arg(z) - \pi| \le n^{-11/10}\right\}$$
$$\subset \mathrm{B}\left(-1 + 0i, 2tn^{-11/10}\right) \cup \mathrm{B}\left(1 + 0i, 2tn^{-11/10}\right).$$

The preceding inclusion yields that any $z \in \mathcal{A}$ with *small argument* belongs in the balls centered at $1 + 0i$ and $-1 + 0i$ with radius $2tn^{-11/10}$. On the other hand, for $z \in \mathcal{A}$ with *large argument* we have

$$\left\{ z \in \mathcal{A} : n^{-11/10} < |\arg(z)| < \pi - n^{-11/10} \right\}$$

$$\subset \bigcup_{\substack{\alpha = 1 \\ \alpha \ : \ n^{-11/10} < |2\pi x_\alpha \bmod \pi| < \pi - n^{-11/10}}}^{N-1} \mathrm{B}\left(e^{i2\pi x_\alpha}, 2tn^{-2}(\log n)^{-1/2-\gamma} \right).$$

We define $[N-1] := [1, N-1] \cap \mathbb{N}$ and

$$J_1(n, N) := \left\{ \alpha \in [N-1] : \gcd(\alpha, N) \geq n^{1+1/10}(\log n)^{-\gamma} \right\},$$

$$J_2(n, N) := \left\{ \alpha \in [N-1] : n^{1+1/10}(\log n)^{-\gamma} \geq \gcd(\alpha, N) \geq n(\log n)^{1/2+\gamma} \right\},$$

$$J_3(n, N) := \left\{ \alpha \in [N-1] : n(\log n)^{1/2+\gamma} \geq \gcd(\alpha, N) \geq n^{9/10}(\log n)^{1/2+\gamma} \right\},$$

where $\gcd(\alpha, N)$ denote the greatest common divisor between α and N. Observe that for any $\alpha \in J_3(n, N)$ we have

$$n - \frac{1}{n(\log n)^{1/2+\gamma}} \leq \frac{N}{\gcd(\alpha, N)} \leq n^{11/10}.$$

The preceding inequalities yield that the irreducible fraction of x_α is as small as a multiple of $n^{-11/10}$. Therefore,

$$\bigcup_{\substack{\alpha = 1 \\ \alpha \ : \ n^{-11/10} < |2\pi x_\alpha \bmod \pi| < \pi - n^{-11/10}}}^{N-1} \mathrm{B}\left(e^{i2\pi x_\alpha}, 2tn^{-2}(\log n)^{-1/2-\gamma} \right)$$

$$= \bigcup_{\alpha \in J_1(n,N)} \mathrm{B}\left(e^{i2\pi x_\alpha}, 2tn^{-2}(\log n)^{-1/2-\gamma} \right)$$

$$\cup \bigcup_{\alpha \in J_2(n,N)} \mathrm{B}\left(e^{i2\pi x_\alpha}, 2tn^{-2}(\log n)^{-1/2-\gamma} \right)$$

$$\cup \bigcup_{\alpha \in J_3(n,N)} \mathrm{B}\left(e^{i2\pi x_\alpha}, 2tn^{-2}(\log n)^{-1/2-\gamma} \right).$$

We emphasize that if $\alpha \in J_1(n, N) \cup J_2(n, N) \cup J_3(n, N)$, then we have

$$n^{-11/10} < |2\pi x_\alpha \bmod \pi| < \pi - n^{-11/10}.$$

Consequently,

$$\mathbb{P}\{\mathcal{M}_n, \mathcal{G}_n\} \leq \mathbb{P}\left\{\mathcal{G}_n, \min_{z \in \mathrm{B}(1+0i, 2tn^{-11/10})} |G_n(z)| < tn^{-1/2}(\log n)^{-\gamma}\right\}$$

$$+ \mathbb{P}\left\{\mathcal{G}_n, \min_{z \in \mathrm{B}(-1+0i, 2tn^{-11/10})} |G_n(z)| < tn^{-1/2}(\log n)^{-\gamma}\right\}$$

$$+ \sum_{\alpha \in J_1(n,N)} \mathbb{P}\{\mathcal{G}_n, \mathrm{B}_\alpha\} + \sum_{\alpha \in J_2(n,N)} \mathbb{P}\{\mathcal{G}_n, \mathrm{B}_\alpha\} + \sum_{\alpha \in J_3(n,N)} \mathbb{P}\{\mathcal{G}_n, \mathrm{B}_\alpha\},$$

$$(12)$$

where

$$\mathrm{B}_\alpha := \left\{ \min_{z \in \mathrm{B}\left(e^{i2\pi x_\alpha}, 2tn^{-2}(\log n)^{-1/2-\gamma}\right)} |G_n(z)| < tn^{-1/2}(\log n)^{-\gamma}\right\}.$$

4.1.1 Small ball analysis at the points $1 + 0i$ and $-1 + 0i$

On the two points $1 \pm 0i$ we have the largest two closed balls, which are considered in our set of balls. This is remarkable since the number of real roots of a Kac polynomial for some common random variables is at least $\mathrm{O}(\frac{\log n}{\log\log\log n})$ with high probability [24]. This means that the real roots of a Kac Polynomial are moving slowly to the unit circle.

On the one hand, let $z \in \mathrm{B}(1 + 0i, 2tn^{-11/10})$. By Taylor's Theorem we obtain

$$|G_n(z) - G_n(1)| \leq |z - 1||G'_n(1)| + |R_2(z)|,$$

where $R_2(z)$ is the error of the Taylor approximation of order 2. On the event $\mathcal{G}_n$ we have

$$|R_2(z)| \leq \frac{\left(2tn^{-1-1/10}\right)^2}{1 - \mathrm{o}(1)}\left[\max_{z \in \mathbb{C}\,:\,||z|-1| \leq 2tn^{-2}} |G_n(z)|\right]$$

$$\leq \frac{4t^2 n^{-2-1/5}n^{3/2}}{1 - \mathrm{o}(1)} = \frac{4t^2 n^{-1/2-1/5}}{1 - \mathrm{o}(1)},$$

where $\mathrm{o}(1) = 2tn^{-1-1/10}$. Assuming that $\mathcal{G}_n$ holds, the preceding inequality yields

$$|G_n(z) - G_n(1)| \leq 2tn^{-1-1/10}|G'_n(1)| + \frac{4t^2 n^{-1/2-1/5}}{1 - \mathrm{o}(1)} \leq 2tn^{-1-1/10}\|W'_n\|_\infty + \frac{4t^2 n^{-1/2-1/5}}{1 - \mathrm{o}(1)}$$

$$\leq 2C_0 tn^{1/2-1/10}(\log n)^{1/2} + \frac{4t^2 n^{-1/2-1/5}}{1 - \mathrm{o}(1)}.$$

Hence,

$$\mathbb{P}\left(\mathcal{G}_n, \min_{z \in \mathrm{B}(1+0i, 2tn^{-11/10})} |G_n(z)| \le tn^{-1/2}(\log n)^{-\gamma}\right)$$
$$\le \mathbb{P}\left(|G_n(1)| \le 2C_2 tn^{1/2-1/10}(\log n)^{1/2}\right),$$

where $2C_2 = 2C_0 t + 4t^2 + 1$. Since $G_n(1) = \sum_{j=0}^{n-1} \xi_j$, Corollary 7.6 in [29] implies for $L \ge \sqrt{1/q}$ (with q given in (H)) that

$$\mathbb{P}\left\{|G_n(1)| \le 2C_2 tn^{1/2-1/10}(\log n)^{1/2}\right\} \le \frac{C_3 L}{\|\mathbf{a}\|}\left(2C_2 t + \frac{1}{D(\mathbf{a})}\right),$$

where C_3 is a positive constant and $D(\mathbf{a})$ is the lcd of the vector

$$\mathbf{a} = (n^{1/2-1/10}(\log n)^{1/2})^{-1}(1, \ldots, 1) \in \mathbb{R}^n.$$

By Proposition 7.4 in [29] we have $D(\mathbf{a}) \ge \frac{1}{2}n^{1/2-1/10}(\log n)^{1/2}$. Therefore,

$$\mathbb{P}\left(|G_n(1)| \le 2C_2 tn^{1/2-1/10}(\log n)^{1/2}\right)$$
$$\le \frac{C_3 L(\log n)^{1/2}}{n^{1/10}}\left(2C_2 t + \frac{2}{n^{1/2-1/10}(\log n)^{1/2}}\right) \le \frac{(2C_2 t + 2)L(\log n)^{1/2}}{n^{1/10}}. \quad (13)$$

On the other hand, let $z \in \mathrm{B}(-1 + 0i, 2tn^{-11/10})$. Assuming that G_n holds, Taylor's Theorem implies

$$|G_n(z) - G_n(-1)| \le |z + 1|\,|G'_n(-1)| + |R_2(z)| \le 2tn^{-1-1/10}\|W'_n\|_\infty + \frac{4t^2 n^{-1/2-1/5}}{1 - o(1)}$$
$$\le (2C_0 t + 4t^2)n^{1/2-1/10}(\log n)^{1/2}.$$

Thus,

$$\mathbb{P}\left(\mathcal{G}_n, \min_{z \in \mathrm{B}(-1+0i, 2tn^{-11/10})} |G_n(z)| \le tn^{-1/2}(\log n)^{-\gamma}\right)$$
$$\le \mathbb{P}\left(|G_n(-1)| \le 2C_2 tn^{1/2-1/10}(\log n)^{1/2}\right).$$

Since $G_n(-1) = \sum_{j=0}^{n-1}(-1)^j \xi_j$, by Corollary 7.6 in [29] for $L \ge \sqrt{1/q}$ (with q given in (H)) we obtain

$$\mathbb{P}\left(|G_n(-1)| \le 2C_2 tn^{1/2-1/10}(\log n)^{1/2}\right) \le \frac{C_3 L}{\|\mathbf{b}\|}\left(2C_2 t + \frac{1}{D(\mathbf{b})}\right),$$

where C_3 is a positive constant and $D(\mathbf{b})$ is the lcd of the vector

$$\mathbf{b} = (n^{1/2-1/10}(\log n)^{1/2})^{-1}\left(1, -1, \ldots, (-1)^{n-1}\right) \in \mathbb{R}^n.$$

By Proposition 7.4 in [29], we have $D(\mathbf{b}) \ge \frac{1}{2}n^{1/2-1/10}(\log n)^{1/2}$. Therefore,

$$\mathbb{P}\Big(|G_n(-1)| \le 2C_2 tn^{1/2-1/10}(\log n)^{1/2}\Big)$$

$$\le \frac{C_3 L(\log n)^{1/2}}{n^{1/10}}\left(2C_2 t + \frac{2}{n^{1/2-1/10}(\log n)^{1/2}}\right) \le \frac{(2C_2 t + 2)L(\log n)^{1/2}}{n^{1/10}}. \quad (14)$$

Combining (13) and (14) we obtain

$$\mathbb{P}\left(\mathcal{G}_n, \min_{z\in B(1+0i,2tn^{-11/10})} |G_n(z)| \le tn^{-1/2}(\log n)^{-\gamma}\right)$$

$$+ \mathbb{P}\left(\mathcal{G}_n, \min_{z\in B(-1+0i,2tn^{-11/10})} |G_n(z)| \le tn^{-1/2}(\log n)^{-\gamma}\right) = O\Big(n^{-1/10}\Big).$$

4.1.2 Small ball analysis at $e^{i2\pi x_\alpha}$

In this part, we are focusing mainly on the complex roots of a Kac polynomial. We remark that the complex roots are more dispersed than the real roots, but they are approaching faster than the real roots to the unit circle. However, the complex roots do not approach extremely fast.

Let $z \in B\big(e^{i2\pi x_\alpha}, 2tn^{-2}(\log n)^{-1/2-\gamma}\big)$ and assume that $\mathcal{G}_n$ holds. By Taylor's Theorem we obtain

$$\big|G_n(z) - G_n\big(e^{i2\pi x_\alpha}\big)\big| \le \big|z - e^{i2\pi x_\alpha}\big|\big|G_n'\big(e^{i2\pi x_\alpha}\big)\big| + |R_2(z)|,$$

where $R_2(z)$ is the error of the Taylor approximation of order 2, and it satisfies

$$|R_2(z)| \le \frac{(2tn^{-2})^2}{1 - 2tn^{-2}}\left[\max_{z\in\mathbb{C}\,:\,||z|-1|<tn^{-2}} |G_n(z)|\right] \le \frac{4t^2 n^{-5/2}}{1 - 2tn^{-2}}.$$

Then

$$\big|G_n(z) - G_n\big(e^{i2\pi x_\alpha}\big)\big| \le 2tn^{-2}(\log n)^{-1/2-\gamma}\|W_n'\| + \frac{4t^2 n^{-5/2}}{1 - 2tn^{-2}}$$

$$\le 2C_0 tn^{-1/2}(\log n)^{-\gamma} + \frac{4t^2 n^{-5/2}}{1 - 2tn^{-2}}.$$

Hence,

$$\mathbb{P}(\mathcal{G}_n, B_\alpha) \le \mathbb{P}\big(\big|G_n\big(e^{i2\pi x_\alpha}\big)\big| \le 2C_4 tn^{-2}(\log n)^{-\gamma}\big),$$

where $2C_4 = 2C_0 + 4t + 1$. For proving that $\mathbb{P}(\mathcal{G}_n, B_\alpha)$ tends to zero as $n \to \infty$, we rewrite the sum $G_n(e^{i2\pi x_\alpha})$ as the product of a matrix by a vector. This simple

rewriting allows us to apply lcd techniques for matrices. To be precise, we define the $2 \times n$ matrix V_α as follows

$$V_\alpha := \begin{bmatrix} 1 & \cos(2\pi x_\alpha) & \ldots & \cos((n-1)2\pi x_\alpha) \\ 0 & \sin(2\pi x_\alpha) & \ldots & \sin((n-1)2\pi x_\alpha) \end{bmatrix}$$

and $X := [\xi_0, \ldots, \xi_{n-1}]^T \in \mathbb{R}^n$. Notice that

$$V_\alpha X = \left[\sum_{j=0}^{n-1} \xi_j \cos(j2\pi x_\alpha), \sum_{j=0}^{n-1} \xi_j \sin(j2\pi x_\alpha) \right]^T \in \mathbb{R}^2,$$

which implies

$$\|V_\alpha X\|_2 = \left| \sum_{j=0}^{n-1} \xi_j e^{ij2\pi x_\alpha} \right| = \left| G_n\left(e^{i2\pi x_\alpha}\right) \right|.$$

Let $\Theta = r[\cos(\theta), \sin(\theta)]^T \in \mathbb{R}^2$, where $r > 0$ and $\theta \in [0, 2\pi]$. For fixed r, θ, we have

$$V_\alpha^T \Theta = r[\cos(-\theta), \cos(2\pi x_\alpha - \theta), \ldots, \cos(2(n-1)\pi x_\alpha - \theta)]^T.$$

Note that $\|V_\alpha^T \Theta\|_2 \leq r\sqrt{n}$. On the other hand, we have

$$\det\left(V_\alpha V_\alpha^T\right) = \det \begin{bmatrix} \sum_{j=0}^{n-1} \cos^2(j2\pi x_\alpha) & \dfrac{1}{2}\sum_{j=0}^{n-1} \sin(2 \cdot j2\pi x_\alpha) \\ \dfrac{1}{2}\sum_{j=0}^{n-1} \sin(2 \cdot j2\pi x_\alpha) & \sum_{j=0}^{n-1} \sin^2(j2\pi x_\alpha) \end{bmatrix}.$$

Now, we are in the setting of inequality (9). Recall that x_α satisfies

$$n^{-11/10} < |2\pi x_\alpha \bmod \pi| < \pi - n^{-11/10}.$$

In the following we distinguish three cases for x_α.

4.1.3 Case 1. $\alpha \in J_1(n,N)$

Assume that $\gcd(\alpha, N) \geq n^{1+1/10}(\log n)^{-\gamma}$. Recall that $N = \lfloor n^2(\log n)^{1/2+\gamma} \rfloor$. Then we have

$$\frac{N}{\gcd(\alpha, N)} \leq \frac{n^2(\log n)^{1/2+\gamma}}{n^{1+1/10}(\log n)^{-\gamma}} = n^{1-1/10}(\log n)^{1/2+2\gamma}.$$

Note that $2\pi x_\alpha$ satisfies $n^{-1} < |2\pi x_\alpha \bmod \pi| < \pi - n^{-1}$ for all large n. By Lemma 3.2 part 1 in [19], there exist positive constants c_5, C_5 such that

$$c_5 n^2 \leq \det\left(V_\alpha V_\alpha^T\right) \leq C_5 n^2. \tag{15}$$

Before continue with our arguments, we estimate the number of indexes α where the

condition $\gcd(\alpha, N) \geq n^{1+1/10} (\log n)^{-\gamma}$ holds. The following lemma provides such estimate.

Lemma 4.3 *The number of indices α such that*

$$\gcd(\alpha, N) \geq \frac{n^{1+1/10}}{(\log n)^{\gamma}}$$

is at most

$$n^{1-1/10+o(1)} (\log n)^{1/2+2\gamma+o(1)}.$$

By Proposition 7.4 in [29], the lcd of V_α satisfies $D(V_\alpha) \geq 1/2$. Thus, by inequalities (9) and (15), and Lemma 4.3 we obtain

$$\sum_{\alpha \in J_1(n,N)} \mathbb{P}\left(\left| G_n\left(e^{i2\pi x_\alpha}\right) \right| \leq 2t C_4 n^{-1/2} (\log n)^{-\gamma} \right)$$

$$\leq 2n^{1-1/10+o(1)} (\log n)^{1/2+2\gamma+o(1)} \left(\frac{2C^2 L^2 (2tC_4)^2}{(c_5 n^2)^{1/2} (n^{1/2} (\log n)^\gamma)^2} + \frac{2C^2 L^2}{\frac{1}{4} (c_5 n^2)^{1/2}} \right)$$

$$= \frac{4C^2 L^2 (2tC_4)^2 (\log n)^{1/2+o(1)}}{c_5^{1/2} n^{1+1/10-o(1)}} + \frac{4C^2 L^2 (\log n)^{1/2+2\gamma+o(1)}}{\frac{1}{4} c_5^{1/2} n^{1/10-o(1)}}$$

$$\leq C_6 \frac{(\log n)^{1/2+2\gamma+o(1)}}{n^{1/10-o(1)}},$$

where $C_6 = 4 c_5^{-1/2} C^2 L^2 \left((2tC_4)^2 + 4 \right)$.

4.1.4 Case 2. $\alpha \in J_2(n,N)$

Assume that

$$n^{1+1/10} (\log n)^{-\gamma} \geq \gcd(\alpha, N) \geq n(\log n)^{1/2+\gamma}.$$

Since $N = \lfloor n^2 (\log n)^{1/2+\gamma} \rfloor$, we have

$$n \geq \frac{N}{\gcd(\alpha, N)} \geq n^{1-1/10} (\log n)^{1/2+2\gamma} - o(1), \tag{16}$$

where $o(1) = n^{-1-1/10} (\log n)^\gamma$. We observe that $2\pi x_\alpha$ is such that

$$n^{-1} \leq |2\pi x_\alpha \bmod \pi| \leq \pi - n^{-1}.$$

By Lemma 3.2 part 1 in [19] there exist positive constants c_5, C_5 such that

$$c_5 n^2 \leq \det\left(V_\alpha V_\alpha^T\right) \leq C_5 n^2.$$

Also, we observe that $x_\alpha = \frac{\alpha}{N} = \frac{\alpha'}{N'}$ where $\alpha = \alpha' \gcd(\alpha, N)$ and $N = N' \gcd(\alpha, N)$. Note that $\gcd(\alpha', N') = 1$. Since $N' \leq n$, for any θ we have

$$\left\{\exp\left(i\left(j2\pi\frac{\alpha'}{N'} - \theta\right)\right) : j = 0, \ldots, N'-1\right\} = \left\{\exp\left(i\left(j2\pi\frac{1}{N'} - \theta\right)\right) : j = 0, \ldots, N'-1\right\}.$$

The above observation allows us to assume that $x_\alpha = 1/N'$. To apply inequality (9) we need to estimate the lcd. The following lemma shows an arithmetic property of the values $\cos(j2\pi x_\alpha - \theta)$ for $j = 0, \ldots, N'$ which becomes crucial for estimating the lcd.

Lemma 4.4 *Fixed $\theta \in [0, 2\pi)$ and positive $m \in \mathbb{Z}$. Let $\mathcal{V}$ be a vector in $\mathbb{R}^m$ which entries are $\mathcal{V}_j = r\cos(j2\pi x - \theta)$ for $j = 0, \ldots, m-1$ with positive integer $r \geq 2$ and $x = 1/m$. Then*

$$\text{dist}(\mathcal{V}, \mathbb{Z}^m) \geq \frac{1}{48} \cdot \frac{1}{2\pi x} \quad \text{whenever} \quad \frac{1}{2r(2\pi x)} \geq 6.$$

Since it is needed to analyze

$$V_\alpha^T \Theta = r[\cos(-\theta), \cos(2\pi x_\alpha - \theta), \ldots, \cos(2(n-1)\pi x_\alpha - \theta)]^T$$

in the definition of the least common denominator, we can assume without loss of generality that r is a positive integer. In fact, by Proposition 7.4 in [29], we can take $r \geq 1/2$. For the case $2 > r \geq 1/2$, we can replicate the ideas in the proof of Lemma 4.4 to obtain that $\text{dist}\left(V_\alpha^T \Theta, \mathbb{Z}^n\right) \geq Cn^{1-1/10}$ for some positive constant C. If $r \geq 2$, we can use $\lfloor r \rfloor$ instead of r in Lemma 4.4.

If $r \leq \frac{1}{2 \cdot 6 \cdot 2\pi x_\alpha}$, by Lemma 4.4 and expression (16), we would obtain

$$\frac{1}{48} \cdot \frac{1}{2\pi} n^{1-1/10} (\log n)^{1/2+2\gamma} - o(1) \leq \frac{1}{48} \cdot \frac{1}{2\pi x_\alpha} \leq \text{dist}\left(V_\alpha^T \Theta, \mathbb{Z}^n\right)$$

$$\leq L\sqrt{\log_+ \frac{\|V_\alpha^T \Theta\|_2}{L}} \leq L\sqrt{\log_+ \frac{rn^{1/2}}{L}} \leq L\sqrt{\log_+ \frac{n^{3/2}}{L}},$$

which is a contradiction since $L \geq \sqrt{2/q}$ is fixed. Thus, we should have $r > \frac{1}{2 \cdot 6 \cdot 2\pi x_\alpha}$ which implies that lcd of V_α satisfies

$$D(V_\alpha) > \frac{1}{12} \cdot \frac{1}{2\pi} n^{1-1/10} (\log n)^{1/2+2\gamma} - o(1).$$

By inequality (9) we obtain

$$\sum_{\alpha \in J_2(n,N)} \mathbb{P}\left(\left|G_n\left(e^{i2\pi x_\alpha}\right)\right| \le 2tC_4 n^{-1/2}(\log n)^{-\gamma}\right)$$

$$\le n^2(\log n)^{1/2+\gamma}\left(\frac{2C^2L^2(2tC_4)^2}{(c_5n^2)^{1/2}(n^{1/2}(\log n)^\gamma)^2}\right)$$

$$+ n^2(\log n)^{1/2+\gamma}\left(\frac{2C^2L^2}{(c_5n^2)^{1/2}\left(\frac{1}{12}\cdot\frac{1}{2\pi}\cdot n^{1-1/10}(\log n)^{1/2+2\gamma}-\mathrm{o}(1)\right)^2}\right)$$

$$\le \frac{2C^2L^2(2tC_2)^2}{(\log n)^{\gamma-1/2}} + \frac{2C^2L^2}{c_5^{1/2}\left(\frac{1}{12}\cdot\frac{1}{2\pi}\right)^2 n^{1-1/5}(\log n)^{1/2+3\gamma}(1-\mathrm{o}(1))^2}.$$

$$\le \frac{C_7}{(\log n)^{\gamma-1/2}},$$

where $C_7 = 2C^2L^2\left((2tC_2)^2 + c_5^{-1/2}\right)$.

4.1.5 Case 3. $\alpha \in J_3(n,N)$

Assume that $n(\log n)^{1/2+\gamma} \ge \gcd(\alpha, N) \ge n^{9/10}(\log n)^{1/2+\gamma}$. Since that $N = \lfloor n^2(\log n)^{1/2+\gamma}\rfloor$, then

$$n^{11/10} \ge \frac{N}{\gcd(\alpha, N)} \ge n - \mathrm{o}(1),$$

where $\mathrm{o}(1) = \frac{1}{n(\log n)^{1/2+\gamma}}$. Note that $2\pi x_\alpha$ satisfies

$$n^{-11/10} \le |2\pi x_\alpha \bmod \pi| \le (n - \mathrm{o}(1))^{-1}$$

or

$$\pi - (n - \mathrm{o}(1))^{-1} \le |2\pi x_\alpha \bmod \pi| \le \pi - n^{-11/10}.$$

By Lemma 3.2 part 2 in [19], there exist positive constants c_5, C_5 such that

$$c_5 n^{2-1/5} \le \det\left(V_\alpha V_\alpha^T\right) \le C_5 n^2.$$

On the other hand, the number of indexes α which satisfy the condition over $\gcd(\alpha, N)$ is at most

$$4N\left(\frac{1}{n - \mathrm{o}(1)} - \frac{1}{n^{1+1/10}}\right) \le 4n(\log n)^{1/2+\gamma}\left(\frac{1}{1 - \mathrm{o}(1)} - \frac{1}{n^{1/10}}\right).$$

In order to use the inequality (9), we need to analyze the least common denominator of V_α for this case. In particular, we need to obtain a suitable lower bound for the distance between $V_\alpha^T\Theta$ and $\mathbb{Z}^n$. We use similar ideas using in the proof of Lemma 4.4.

As $x_\alpha = \frac{\alpha}{N} = \frac{\alpha'}{N'}$ with $\gcd(\alpha', N') = 1$ and $N' \geq n - 1$, then all the points in

$$\{\exp(i(j2\pi x_\alpha - \theta)) : j = 0, \ldots, n - 1\} \quad \text{are different.}$$

Let r be a positive integer and we consider the set of intervals of the form $\left[\frac{m}{r}, \frac{m+1}{r}\right]$ for all $m \in [-r, r] \cap \mathbb{Z}$. Let I_m and J_m be the corresponding arcs on the unit circle whose projection on the horizontal axis is the interval $\left[\frac{m}{r}, \frac{m+1}{r}\right]$. If $r < n$, by the pigeon-hole principle we have that there exists at least one I_M (or J_M) for some $M \in [-r, r] \cap \mathbb{Z}$, which contain at least $n/(2r)$ points $\exp(i(j2\pi x_\alpha - \theta))$ in it. For each $\cos(j2\pi x_\alpha - \theta) \in \left[\frac{M}{r}, \frac{M+1}{r}\right]$, it is defined

$$d_j = \min\left\{\left|\cos(j2\pi x_\alpha - \theta) - \frac{M}{r}\right|, \left|\cos(j2\pi x_\alpha - \theta) - \frac{M+1}{r}\right|\right\}.$$

Note among the values d_j at most two can be equal and

$$\min_{0 \leq l, k \leq n-1} \{|l2\pi x_\alpha - k2\pi x_\alpha|\} \geq 2\pi \frac{1}{N'}.$$

Observe that for each $0 \leq \lambda \leq L$, with $L = \min\left\{\left\lfloor\frac{n}{4 \cdot 2r} - \frac{3}{2}\right\rfloor, \left\lfloor\frac{N'}{2 \cdot 2r \cdot 2\pi} - \frac{1}{2}\right\rfloor\right\}$, there exists at least one d_j such that $d_j \geq (2\lambda + 1)2\pi \frac{1}{N'}$. So, the sum of all d_j is at least

$$\sum_{\lambda=0}^{L} (2\lambda + 1)2\pi \frac{1}{N'} \geq 2\pi \frac{L^2}{N'},$$

and taking $r \leq \lfloor n^{1/4} \rfloor$ it follows that

$$2\pi \frac{L^2}{N'} \geq 2\pi \cdot \frac{1}{n^{1+1/10}} \left(\frac{n^{3/4} - o(1)}{16\pi}\right)^2 \geq \frac{1}{128\pi} \left(n^{1/4 - 1/20} - o(1)\right)^2.$$

Now, let v be a vector in $\mathbb{R}^n$ whose entries are $v_j = \cos(j2\pi x_\alpha - \theta)$ for each $j = 0, \ldots, n - 1$. If a positive integer $r \leq \lfloor n^{1/4} \rfloor$, by the previous discussion it follows that the vector $rv = (rv_j)_{1 \leq j \leq n}$ satisfies

$$\text{dist}(rv, \mathbb{Z}^n) \geq \frac{1}{128\pi} \left(n^{1/4 - 1/20} - o(1)\right)^2.$$

Thus, if $r \leq \lfloor n^{1/4} \rfloor$ and taking a fixed $L \geq \sqrt{2/q}$, by the definition of lcd we would deduce that

$$\frac{1}{128\pi}\left(n^{1/4-1/20} - \mathrm{o}(1)\right)^2 \leq \mathrm{dist}\left(V_\alpha^T \Theta, \mathbb{Z}^n\right) \leq L\sqrt{\log_+ \frac{\|V_\alpha^T \Theta\|_2}{L}}$$

$$\leq L\sqrt{\log_+ \frac{rn^{1/2}}{L}} \leq L\sqrt{\log_+ \frac{n^{3/4}}{L}},$$

which implies that the lcd of V_α should satisfy $D(V_\alpha) \geq n^{1/4}$. By (9), we obtain

$$\sum_{\alpha \in J_3(n,N)} \mathbb{P}\left(\left|G_n\left(e^{i2\pi x_\alpha}\right)\right| \leq 2tC_4 n^{-1/2}(\log n)^{-\gamma}\right)$$

$$\leq 4n(\log n)^{1/2+\gamma}\left(\frac{1}{1-\mathrm{o}(1)} - \frac{1}{n^{1/10}}\right)\left(\frac{2C^2L^2(2tC_4)^2}{(c_5 n^{2-1/5})^{1/2}(n^{1/2}(\log n)^\gamma)^2}\right)$$

$$+ 4n(\log n)^{1/2+\gamma}\left(\frac{1}{1-\mathrm{o}(1)} - \frac{1}{n^{1/10}}\right)\left(\frac{2C^2L^2}{(c_5 n^{2-1/5})^{1/2}(n^{1/4})^2}\right)$$

$$= 4\left(\frac{1}{1-\mathrm{o}(1)} - \frac{1}{n^{1/10}}\right)\left(\frac{2C^2L^2(2tC_4)^2}{c_5^{1/2}n^{1-1/10}(\log n)^{\gamma-1/2}} + \frac{2C^2L^2(\log n)^{1/2+\gamma}}{c_5^{1/2}n^{1/2-1/10}}\right)$$

$$\leq C_8\left(\frac{1}{1-\mathrm{o}(1)}\right)\frac{(\log n)^{1/2+\gamma}}{n^{1/2-1/10}},$$

where $C_8 = 8c_5^{-1/2}C^2L^2\left((2tC_4)^2 + 1\right)$.

Combining Case 1, Case 2 and Case 3 we obtain

$$\sum_{\alpha \in \Lambda} \mathbb{P}(\mathcal{G}_n, B_\alpha) = \mathrm{O}\left((\log n)^{-\gamma+1/2}\right), \quad \text{where } \gamma > 1/2. \tag{17}$$

Hence, inequality (12) with the help of (13), (14) and (17) yields

$$\mathbb{P}(\mathcal{M}_n) = \mathrm{O}\left((\log n)^{-\gamma+1/2}\right), \quad \text{where } \gamma > 1/2.$$

The preceding estimate, inequality (10) and relation (11) imply Theorem 2.3.

5 Proof of Theorem 2.6. On the lower bound for the smallest singular value for random circulant matrices

Let $\rho \in (0, 1/4)$ be fixed. We define $x_k = k/n$, $k = 0, \ldots, n-1$. Note that

$$\mathbb{P}(s_n(\mathcal{C}_n) \leq n^{-\rho}) \leq \sum_{k=0}^{n-1} \mathbb{P}\left(\left|G_n\left(e^{i2\pi x_k}\right)\right| \leq n^{-\rho}\right)$$

$$\leq \mathbb{P}(|G_n(1)| \leq n^{-\rho}) + \mathbb{P}(|G_n(-1)| \leq n^{-\rho})$$

$$+ \sum_{\substack{k=0 \\ k \ : \ \gcd(k,n) \ > \ n^{1/2}}}^{n-1} \mathbb{P}\left(\left|G_n\left(e^{i2\pi x_k}\right)\right| \leq n^{-\rho}\right)$$

$$+ \sum_{\substack{k=0 \\ k \ : \ \gcd(k,n) \ \leq \ n^{1/2}}}^{n-1} \mathbb{P}\left(\left|G_n\left(e^{i2\pi x_k}\right)\right| \leq n^{-\rho}\right).$$

In the sequel, we prove that the right-hand side of the preceding inequality is $\mathrm{O}(n^{-2\rho})$. We consider the following three cases.

Case 1. The same reasoning using in Section 4.1.1 yields

$$\mathbb{P}(|G_n(1)| \leq n^{-\rho}) + \mathbb{P}(|G_n(-1)| \leq n^{-\rho}) = \ \mathrm{O}\left(n^{-1/2}\right).$$

Case 2. $\gcd(k,n) > n^{1/2}$. By similar reasoning using in the first case of the proof of Theorem 2.3, Section 4.1.3, we deduce

$$\sum_{\substack{k=0 \\ k \ : \ \gcd(k,n) \ > \ n^{1/2}}}^{n-1} \mathbb{P}\left(\left|G_n\left(e^{i2\pi x_k}\right)\right| \leq n^{-\rho}\right) \leq n^{1/2+o(1)}\left(\frac{2C^2L^2}{c_5^{1/2}n^{1+2\rho}} + \frac{2C^2L^2}{\frac{1}{2}c_5^{1/2}n}\right)$$

$$\leq \frac{2C^2L^2}{c_5^{1/2}n^{1/2+2\rho-o(1)}} + \frac{4C^2L^2}{c_5^{1/2}n^{1/2-o(1)}} \leq \frac{C_9}{n^{1/2-o(1)}},$$

where $C_9 = 4c_5^{-1/2}C^2L^2$.

Case 3. $\gcd(k,n) \leq n^{1/2}$.

By similar reasoning using in the second case of the proof of Theorem 2.3, Section 4.1.4, we obtain

$$\sum_{\substack{k=0 \\ k \ : \ \gcd(k,n) \ \leq \ n^{1/2}}}^{n-1} \mathbb{P}\left(\left|G_n\left(e^{i2\pi x_k}\right)\right| \leq n^{-\rho}\right) \leq n\left(\frac{2C^2L^2}{c_5^{1/2}n^{1+2\rho}} + \frac{2C^2L^2}{c_5^{1/2}n\left(\frac{1}{2\cdot6\cdot2\pi}n^{1/2}\right)^2}\right)$$

$$\leq \frac{2C^2L^2}{c_5^{1/2}n^{2\rho}} + \frac{1152\pi^2C^2L^2}{c_5^{1/2}n} \leq \frac{C_{10}}{n^{2\rho}},$$

where $C_{10} = c_5^{-1/2}C^2L^2(2 + 1152\pi^2)$.

The combination of all the preceding cases yields $\mathbb{P}(s_n(\mathcal{C}_n) \leq n^{-\rho}) = \mathrm{O}(n^{-2\rho})$ for any $\rho \in (0, 1/4)$.

Appendix A. Proofs of Lemma 4.3 and Lemma 4.4 Proof of

Lemma 4.3 Write $m := n^{1+1/10}(\log n)^{-\gamma}$. Let T be the Euler totient function. Then we have

$$\sum_{\substack{\alpha \,:\, \gcd(\alpha, N) \,\geq\, m \\ 0 \,\leq\, \alpha \,\leq\, N}} 1 \leq \sum_{\substack{d = \lfloor m \rfloor \\ d \mid N}}^{N} T\left(\frac{N}{d}\right).$$

Notice that $T(s) \leq s - \sqrt{s}$ for all $s \in \mathbb{N}$. Moreover, if $d(s)$ is the number of divisors of s, it is well-known (see Theorem 13.12 in [1]) that there exists an absolute constant $C > 0$ such that

$$d(s) \leq s^{C(\log\log(s))^{-1}}.$$

Hence,

$$\sum_{\substack{\alpha \,:\, \gcd(\alpha, N) \,\geq\, m \\ 0 \,\leq\, \alpha \,\leq\, N}} 1 \leq \left(\frac{N}{\lfloor m \rfloor} - \sqrt{\frac{N}{\lfloor m \rfloor}}\right) N^{C(\log\log(N))^{-1}} \leq \frac{1}{\lfloor m \rfloor} N^{1 + C(\log\log N)^{-1}}$$

$$\leq 2n^{1-1/10+o(1)}(\log n)^{1/2+2\gamma+o(1)},$$

where $o(1) = C(\log\log(N))^{-1}$.

Proof of Lemma 4.4 We define the following sequence

$$P = \{\exp(i(j2\pi x - \theta)) : j = 0, \ldots, m - 1\},$$

where i is the imaginary unit. Note that P is a set of points on the unit circle which can be seen as vertices of a regular polygon with m sides inscribed in the unit circle. Since the arguments of points $\exp(i(j2\pi x - \theta))$ are separated exactly by a distance $2\pi x$, the number of points $\exp(i(j2\pi x - \theta))$ which are in any arc on the unit circle is at least $\frac{l}{2\pi x} - 2$, where l is the length of the arc.

Let $[y, y + 3(2\pi x)]$ be a subinterval of $[-1, 1]$ and consider the arc A on the unit circle whose projection on the horizontal axis is $[y, y + 3(2\pi x)]$. If the length of the arc A is l, then the number of values $\cos(j2\pi x - \theta)$ which are still in $(y, y + 3(2\pi x))$ is at least $\frac{l}{2\pi x} - 2 \geq \frac{3(2\pi x)}{2\pi x} - 2 = 1$ since $l \geq 3(2\pi x)$.

Let $s \in [-(r-1), (r-1)] \cap \mathbb{Z}$. Note that there exists at least one value

$$\cos(j2\pi x - \theta) \in \left(\frac{s}{r} + 3(k-1)(2\pi x), \frac{s}{r} + 3k(2\pi x)\right) \subset \left[\frac{s}{r}, \frac{s+1}{r}\right]$$

for all positive integers $k \leq \frac{1}{3r(2\pi x)}$. In the sequel, we consider all the values $\cos(j2\pi x - \theta) \in \left[\frac{s}{r}, \frac{s+1}{r}\right]$ and define

$$d_j := \min\left\{\left|\cos(j2\pi x - \theta) - \frac{s}{r}\right|, \left|\cos(j2\pi x - \theta) - \frac{s+1}{r}\right|\right\}.$$

Let L be the biggest integer which satisfies $(3 \cdot 2\pi x)L \leq \frac{1}{2r}$, or equivalently, $L = \lfloor\frac{1}{2r(3\cdot 2\pi x)}\rfloor$. Therefore, the sum of d_j for all $\cos(j2\pi x - \theta) \in \left[\frac{s}{r}, \frac{s+1}{r}\right]$ is at least

$$\sum_{\lambda=1}^{L} 2\lambda(3 \cdot 2\pi x) = 6(2\pi x)\sum_{\lambda=1}^{L}\lambda \geq 6(2\pi x)\frac{L^2}{2}$$

$$\geq 3(2\pi x)\left(\frac{1}{2} \cdot \frac{1}{(2r)(3 \cdot 2\pi x)}\right)^2 = \frac{1}{12} \cdot \frac{1}{(2r)^2(2\pi x)},$$

where we used the following inequality

$$L \geq \frac{1}{2r(3 \cdot 2\pi x)} - 1 \geq \frac{1}{2} \cdot \frac{1}{2r(3 \cdot 2\pi x)},$$

which holds if $\frac{1}{2r(2\pi x)} \geq 6$. Let σ_s be the sum of d_j for each interval $\left[\frac{s}{r}, \frac{s+1}{r}\right]$, $s = -(r-1), \ldots, (r-1)$. As $r \geq 2$, then

$$\sum_{s=-(r-1)}^{r-1} \sigma_s \geq (2r-2)\left(\frac{1}{12} \cdot \frac{1}{(2r)^2(2\pi x)}\right) \geq \frac{1}{24} \cdot \frac{1}{(2r)(2\pi x)}.$$

By the previous analysis, we have that the distance between the vector $\mathcal{V} \in \mathbb{R}^m$ whose entries are $\mathcal{V}_j = r\cos(j2\pi x - \theta)$ for $j = 0, \ldots, m-1$ with $x = 1/m$ to $\mathbb{Z}^m$ is at least

$$r\left(\frac{1}{12} \cdot \frac{1}{(2r)(2\pi x)}\right) = \frac{1}{48} \cdot \frac{1}{2\pi x},$$

verifying that expression $\frac{1}{2r(2\pi x)} \geq 6$ is fulfilled.

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