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Calculus of Variations

Regularity of the free boundary for a parabolic cooperative system

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Abstract

In this paper we study the following parabolic system

$$\Delta \mathbf{u} - \partial_t \mathbf{u} = |\mathbf{u}|^{q-1}\mathbf{u}\,\chi_{\{|\mathbf{u}|>0\}}, \qquad \mathbf{u} = (u^1, \cdots, u^m)\ ,$$

with free boundary $\partial\{|\mathbf{u}| > 0\}$. For $0 \le q < 1$, we prove optimal growth rate for solutions $\mathbf{u}$ to the above system near free boundary points, and show that in a uniform neighbourhood of any a priori well-behaved free boundary point the free boundary is $C^{1,\alpha}$ in space directions and half-Lipschitz in the time direction.

Mathematics Subject Classification 35B65 · 35R35

1 Introduction

1.1 Background

In this paper we shall study for $0 \le q < 1$ the parabolic (free boundary) system

$$\Delta \mathbf{u} - \partial_t \mathbf{u} = f(\mathbf{u}) := |\mathbf{u}|^{q-1}\mathbf{u}\,\chi_{\{|\mathbf{u}|>0\}}, \qquad \mathbf{u} = (u^1, \cdots, u^m)\ , \tag{1.1}$$

where $\mathbf{u} : Q_1 \to \mathbb{R}^m$, $Q_1 = B_1(0) \times (-1, 1)$, with B_1 being the unit ball in $\mathbb{R}^n$, $n \ge 2$, $m \ge 2$, and $|\cdot|$ is the Euclidean norm on the respective spaces. System (1.1) relates to

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concentrations of species/reactants, where an increase in each species/reactant accelerates the extinction/reaction of all species/reactants. The special choice of our reaction kinetics would assure a constant decay/reaction rate in the case that u^i, for $i = 1, \cdots, m$ are of comparable size.

A diverse scalar parabolic free boundary problem has been subject of intense studies in more than half-century. On the other hand there are very few results for problems that involve systems (see [1, 5, 9]), and probably no results for the system related to equation (1.1).

The elliptic case of the above system is studied in [1, 9] or in the scalar case, when $m = 1$ in [8], where they prove optimal growth rate for the solutions as well as $C^{1,\alpha}$-regularity of the free boundary at points that are a priori well-behaved.

In this paper we shall study the parabolic system (1.1) from a regularity point of view. The analysis of the above parabolic system introduces several serious obstruction, and hence a straightforward generalization of the ideas and techniques of its elliptic counterpart is far from being obvious. Due to its technical nature, and the need for notations and definitions, we shall explain these difficulties below, during the course of developing the tools and ideas.

1.2 Main results and plan of the paper

Our results concern two main questions: Optimal growth of the solution $\mathbf{u}$ at free boundary points (Theorem 3.3), and the regularity of the free boundary (Theorem 6.5) at *well-behaved* points.[1]

To prove our results we use the regularity theory for the elliptic case, see [1, 9] and follow the ideas that have been used to treat parabolic free boundary problems, as in [4] that was used for the no-sign one phase scalar case. In doing so we encounter several technical problems, that we need to circumvent by enhancing the previous techniques. The first problem we encounter is the use of the balanced-energy monotonicity formula for proving quadratic growth estimates from the free boundary points. In parabolic setting, and specially in system case, the combination of balanced energy and Almgren's frequency is more delicate than the elliptic case done in [1].

The second problem we encounter concerns the regularity of the free boundary, where we are forced to use the epiperimetric inequality in elliptic setting. In order to do this we need to prove that $\partial_t \mathbf{u}$, the time derivative of $\mathbf{u}$, is Hölder regular for $q = 0$. When $q > 0$ we need some modification (see Sect. 5). This, however, can be proved at the so-called regular points. Indeed, since the set of regular points is open (in relative topology) we can use indirect argument to show that $\partial_t \mathbf{u}$ tends to zero at free boundary points close to a regular point. From here one can bootstrap a Hölder regularity theory for $|\partial_t \mathbf{u}|$. Once this is done we can invoke the epiperimetric inequality for equations with Hölder right hand side and deduce (in a standard way) the regularity of the free boundary in space. The Hölder regularity in time then follows by blow-up techniques, and indirect argument.

1.3 Notation

For clarity of exposition we shall introduce some notation and definitions here that are used frequently in the paper.

$\lfloor s \rfloor$ is the greatest integer below s, i.e. $s - 1 \leq \lfloor s \rfloor < s$.

Points in $\mathbb{R}^{n+1}$ are denoted by (x, t), where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

[1] Later we shall call them regular points.

Let $X = (x, t)$ and define $|X| := (|x|^2 + |t|)^{1/2}$.
$B_r(x)$ is the open ball in $\mathbb{R}^n$ with center x and radius r, $B_r := B_r(0)$.
$Q_r(x, t)$ denotes the open cylinder $B_r(x) \times (t - r^2, t + r^2)$ in $\mathbb{R}^{n+1}$.
$Q_r^+(x, t) = B_r(x) \times (t, t + r^2)$ (upper half cylinder).
$Q_r^-(x, t) = B_r(x) \times (t - r^2, t)$ (lower half cylinder).
$T_{r,a} := B_a \times \left(-4r^2, -r^2\right]$, $T_r := \mathbb{R}^n \times \left(-4r^2, -r^2\right]$.
$\partial Q_r(x, t)$ is the topological boundary.
$\partial_p Q_r(x, t)$ is the parabolic boundary, i.e., the topological boundary minus the top of the cylinder.
∇ denotes the spatial gradient, $\nabla = (D_{x_1}, \cdots, D_{x_n})$.
$\nabla \mathbf{u} = [\partial_i u^j]_{1\le i\le n, 1\le j\le m}$ is the derivative matrix of $\mathbf{u}$ with other notations

$$|\nabla \mathbf{u}|^2 = \sum_{i=1}^m |\nabla u^i|^2, \qquad \nabla \mathbf{u} : \nabla \mathbf{v} = \sum_{i=1}^m (\nabla u^i \cdot \nabla v^i),$$
$$\nabla \mathbf{u} \cdot \xi = \xi^t \nabla \mathbf{u} = (\nabla u^1 \cdot \xi, \cdots, \nabla u^m \cdot \xi), \qquad \text{for all } \xi \in \mathbb{R}^n.$$

We will denote the derivative of the function f by $f_{\mathbf{u}}$.
We fix the following constants throughout the paper

$$\kappa := \frac{2}{1-q}, \qquad \alpha = (\kappa(\kappa - 1))^{-\kappa/2}. \tag{1.2}$$

$\Gamma = \Gamma(\mathbf{u}) = \partial\{|\mathbf{u}| > 0\}$.
$\Gamma^\kappa(\mathbf{u}) = \{(x_0, t_0) \in \Gamma(\mathbf{u}) : \partial_t^i \partial_x^\mu \mathbf{u}(x_0, t_0) = 0 \text{ for all } 2i + |\mu| < \kappa\}$.
Ω_t, Γ_t, $\partial\Omega_t$ are t-sections of the corresponding sets in $\mathbb{R}^{n+1}$, at the level t.
$H = \Delta - \partial_t$ (the heat operator).
χ_Ω is the characteristic function of Ω.
We denote by $G(x, t)$ the backward heat kernel

$$G(x, t) = \begin{cases} (-4\pi t)^{-\frac{n}{2}} e^{\frac{|x|^2}{4t}}, & t < 0 \\ 0, & t \geq 0. \end{cases}$$

The following parabolic scalings at the point $X_0 = (x_0, t_0) \in \Gamma$ are used,

$$\mathbf{u}_{r,X_0}(x, t) := \frac{\mathbf{u}(rx + x_0, r^2 t + t_0)}{r^\kappa}, \qquad \mathbf{u}_r := \mathbf{u}_{r,(0,0)}(x, t).$$

We say that $\mathbf{u}$ is κ-backward self-similar if $\mathbf{u}_r = \mathbf{u}$ for all $r > 0$, or equivalently $Lu^i \equiv 0$, for $i = 1, ..., m$, where

$$Lv := \nabla v \cdot x + 2t\partial_t v - \kappa v.$$

For $\mathbf{u}$ a solution to the system (1.1) in $\mathbb{R}^n \times (-4, 0]$, with a polynomial growth, we denote by $\mathbb{W}$ the parabolic balanced energy

$$\mathbb{W}(\mathbf{u}, r) := \frac{1}{r^{2\kappa}} \int_{-4r^2}^{-r^2} \int_{\mathbb{R}^n} \left(|\nabla \mathbf{u}|^2 + \frac{\kappa |\mathbf{u}|^2}{2t} + \frac{2}{1+q} |\mathbf{u}|^{1+q} \right) G(x, t)\, dx\, dt, \tag{1.3}$$

for $0 < r < 1$. A change of variables implies that

$$\mathbb{W}(\mathbf{u}, r) = \mathbb{W}(\mathbf{u}_r, 1).$$

For a fixed point $X_0 = (x_0, t_0) \in \Gamma$, denote by

$$\mathbb{W}(\mathbf{u}, r; X_0) := \mathbb{W}(\mathbf{u}_{r,X_0}, 1).$$

For notational simplicity we set

$$\mathbb{M}(\mathbf{u}) := \mathbb{W}(\mathbf{u}, 1).$$

The class of half-space solutions $\mathbb{H}$ is defined as

$$\mathbb{H} := \left\{x \mapsto \alpha \max(x \cdot \nu, 0)^\kappa \mathbf{e} : \text{ where, } \nu \in \mathbb{R}^n, |\nu| = 1, \mathbf{e} \in \mathbb{R}^m, |\mathbf{e}| = 1\right\},$$

where α is defined in (1.2). A simple computation yields that $\mathbb{W}(\mathbf{h}, 1) =: A_q$ is constant for every $\mathbf{h} \in \mathbb{H}$.

We denote by $\mathcal{N}(r)$ the monotonicity function of Almgren

$$\mathcal{N}(r) = \mathcal{N}(r, h) := \frac{\int_{-4r^2}^{-r^2} \int_{\mathbb{R}^n} |\nabla h(x,t)|^2 G(x,t) dx dt}{\int_{-4r^2}^{-r^2} \int_{\mathbb{R}^n} \frac{1}{-t} |h(x,t)|^2 G(x,t) dx dt},$$

where h is of polynomial growth in x-variables.

2 Preliminary results and standard facts

2.1 Monotonicity formulas

In this section we shall present a few monotonicity formulas, that are the corner stone of our approach. The first of these is the standard balanced energy functional, that has strict monotonicity property for (global) solutions of our equation, unless the solution is backward self-similar of order κ. See [11] for the similar result for the scaler case.

Theorem 2.1 *(Monotonicity formula) Let* $\mathbf{u}$ *be a solution of* (1.1) *in* $\mathbb{R}^n \times (-4, 0)$*, with a polynomial growth at infinity. Then* $\mathbb{W}(\mathbf{u}, r)$ *is monotone nondecreasing in* r.

Proof Using the identity

$$\nabla v G = \nabla(vG) - \frac{x}{2t} vG,$$

we compute the derivative of $\mathbb{W}$ with respect to r

$$\begin{aligned}
\frac{d\mathbb{W}(\mathbf{u}, r)}{dr} =& \frac{d\mathbb{W}(\mathbf{u}_r, 1)}{dr} = \int_{-4}^{-1} \int_{\mathbb{R}^n} \frac{d}{dr} \left(|\nabla \mathbf{u}_r|^2 + \frac{\kappa |\mathbf{u}_r|^2}{2t} + \frac{2}{1+q} |\mathbf{u}_r|^{1+q} \right) G dx dt \\
=& 2 \int_{-4}^{-1} \int_{\mathbb{R}^n} \left(\nabla \mathbf{u}_r : \nabla \frac{d\mathbf{u}_r}{dr} + \frac{\kappa \mathbf{u}_r}{2t} \frac{d\mathbf{u}_r}{dr} + \mathbf{u}_r |\mathbf{u}_r|^{q-1} \frac{d\mathbf{u}_r}{dr} \right) G dx dt \\
=& 2 \int_{-4}^{-1} \int_{\mathbb{R}^n} \frac{d\mathbf{u}_r}{dr} \left(-\Delta \mathbf{u}_r + \mathbf{u}_r |\mathbf{u}_r|^{q-1} + \frac{\kappa \mathbf{u}_r}{2t} - \frac{x \cdot \nabla \mathbf{u}_r}{2t} \right) G dx dt
\end{aligned}$$

$$=\int_{-4}^{-1}\int_{\mathbb{R}^n}\frac{d\mathbf{u}_r}{dr}\left(-2\frac{\partial\mathbf{u}_r}{\partial t}+\kappa\frac{\mathbf{u}_r}{t}-\frac{x\cdot\nabla\mathbf{u}_r}{t}\right)Gdxdt$$

$$=r\int_{-4}^{-1}\int_{\mathbb{R}^n}\left(\frac{d\mathbf{u}_r}{dr}\right)^2\frac{G(x,t)}{-t}dxdt\geq 0.$$

□

The above monotonicity functional being limited to global solutions, needs to be enhanced in order for us to apply to a local setting. This is done by inserting a cutoff function into the functional, that in turn makes the functional almost monotone and calls for adding an extra term, as stated in the next theorem. See also [4] for the similar result in obstacle problem.

Theorem 2.2 *Given a solution* $\mathbf{u}$ *to* (1.1) *in* Q_1^-, *we consider the function* $\mathbf{v}:=\eta\mathbf{u}$, *where* $\eta\in C_0^\infty(B_{3/4})$ *is nonnegative,* $\eta\leq 1$, *and* $\eta=1$ *in* $B_{1/2}$. *Then there exists a non-negative function* F *depending on the given data, satisfying* $F(0+)=0$, *and such that* $\mathbb{W}(\mathbf{v},r)+F(r)$ *is monotone nondecreasing in* r *for* $0<r<1/2$.

Proof As in the previous theorem and applying the relation $\frac{d\mathbf{v}_r}{dr}=\frac{1}{r}L\mathbf{v}_r$, we get

$$\frac{d\mathbb{W}(\mathbf{v},r)}{dr}=\frac{d\mathbb{W}(\mathbf{v}_r,1)}{dr}=\frac{2}{r^{2\kappa+1}}\int_{-4r^2}^{-r^2}\int_{\mathbb{R}^n}L\mathbf{v}\left(-\Delta\mathbf{v}+\mathbf{v}|\mathbf{v}|^{q-1}+\frac{\kappa\mathbf{v}}{2t}-\frac{x\cdot\nabla\mathbf{v}}{2t}\right)Gdxdt$$

$$=\frac{2}{r^{2\kappa+1}}\int_{-4r^2}^{-r^2}\int_{\mathbb{R}^n}L\mathbf{v}\left(-\Delta\mathbf{v}+\mathbf{v}|\mathbf{v}|^{q-1}+\partial_t\mathbf{v}+\frac{L\mathbf{v}}{-2t}\right)Gdxdt$$

$$\geq\frac{2}{r^{2\kappa+1}}\int_{-4r^2}^{-r^2}\int_{\mathbb{R}^n}L\mathbf{v}\left(-\Delta\mathbf{v}+\mathbf{v}|\mathbf{v}|^{q-1}+\partial_t\mathbf{v}\right)Gdxdt,$$

Observe that $H\mathbf{v}=H\mathbf{u}=\mathbf{u}|\mathbf{u}|^{q-1}=\mathbf{v}|\mathbf{v}|^{q-1}$ in $B_{1/2}$, and $H\mathbf{v}(x,t)=0$ if $|x|>3/4$, hence

$$\frac{d\mathbb{W}(\mathbf{v},r)}{dr}\geq\frac{2}{r^{2\kappa+1}}\int_{-4r^2}^{-r^2}\int_{\mathbb{R}^n}L\mathbf{v}\left(-H\mathbf{v}+\mathbf{v}|\mathbf{v}|^{q-1}\right)Gdxdt$$

$$=\frac{2}{r^{2\kappa+1}}\int_{-4r^2}^{-r^2}\int_{B_{3/4}\setminus B_{1/2}}L\mathbf{v}\left(-H\mathbf{v}+\mathbf{v}|\mathbf{v}|^{q-1}\right)Gdxdt\geq-\frac{Ce^{\frac{-1}{64r^2}}}{r^{n+2\kappa-1}},$$

where we have used the following relations

$$L\mathbf{v}=(x\cdot\nabla\eta)\mathbf{u}+\eta L\mathbf{u},\quad\text{and}\quad H\mathbf{v}=\Delta\eta\mathbf{u}+\eta H\mathbf{u}+2\nabla\mathbf{u}\cdot\nabla\eta.$$

Now the statement of the lemma follows with

$$F(r)=C\int_0^r\tau^{-n-2\kappa+1}e^{\frac{-1}{64\tau^2}}d\tau.$$

□

We state the following standard result concerning regularity theory, leaving out the standard proof. See for example [12] for similar result.

Corollary 2.3 *Let* $\mathbf{u}$ *be a solution to our problem and suppose it has polynomial growth from a free boundary point* $X_0 = (x_0, t_0) \in \Gamma^\kappa(\mathbf{u})$. *Then the following hold:*

(1) The function $\mathbb{W}(\mathbf{u}, r; X_0)$ *has a right limit as* $r \to 0+$.
(2) Any blow up of $\mathbf{u}$ *at* (x_0, t_0) *is a* κ*-backward self-similar function.*
(3) The function $X_0 \mapsto \mathbb{W}(\mathbf{u}, 0+; X_0)$ *is upper semicontinous.*

Next we state, and for reader's convenience, prove Almgren's monotonicity formula. There are different versions of this formula in literature, see for example [6].

Lemma 2.4 *(Almgren's frequency formula) Let* h *be a non-zero caloric function in* $\mathbb{R}^n \times (-4, 0)$*, with polynomial growth, and recall the definition of Almgren's monotonicity function* $\mathcal{N}(r, h)$. *Then*

i) $\mathcal{N}'(r, h) \geq 0$, *for* $0 < r < 1$.
ii) If $\mathcal{N}(r, h) \equiv const := \mathcal{N}$, *then* h *is a backward self-similar caloric function of degree* $2\mathcal{N}$.
iii) For an integer number $\ell \geq 2$, *if* $\partial_t^j \partial_x^\mu h(0) = 0$ *for all* $2j + |\mu| \leq \ell - 1$, *we obtain* $2\mathcal{N}(0+, h) \geq \ell$. *Furthermore, equality* $2\mathcal{N}(r, h) = \ell$ *for some* $r > 0$ *implies that* h *is backward self-similar of degree* ℓ.

Proof We have

$$\mathcal{N}(r) := \frac{\int_{-4}^{-1}\int_{\mathbb{R}^n} |\nabla h_r|^2 G dx dt}{\int_{-4}^{-1}\int_{\mathbb{R}^n} \frac{1}{-t}|h_r|^2 G dx dt},$$

and

$$\mathcal{N}'(r) = \frac{2I_1 \int_{-4}^{-1}\int_{\mathbb{R}^n} \frac{1}{-t}|h_r|^2 G dx dt - 2I_2 \int_{-4}^{-1}\int_{\mathbb{R}^n} \frac{1}{-t} h_r \frac{dh_r}{dr} G dx dt}{\left(\int_{-4}^{-1}\int_{\mathbb{R}^n} \frac{1}{-t}|h_r|^2 G dx dt\right)^2},$$

where

$$I_1 := \int_{-4}^{-1}\int_{\mathbb{R}^n} \nabla h_r \cdot \nabla \frac{dh_r}{dr} G dx dt \quad \text{and} \quad I_2 := \int_{-4}^{-1}\int_{\mathbb{R}^n} |\nabla h_r|^2 G dx dt.$$

Let us recall that $\frac{dh_r}{dr} = \frac{1}{r} L h_r$. Using integration by parts, and taking into account that h is caloric, we obtain

$$\begin{aligned} I_1 &:= \int_{-4}^{-1}\int_{\mathbb{R}^n} \left(-\Delta h_r - \frac{x \cdot \nabla h_r}{2t}\right) \frac{dh_r}{dr} G dx dt \\ &= \int_{-4}^{-1}\int_{\mathbb{R}^n} \frac{-1}{2rt} \left((L h_r)^2 + \kappa h_r L h_r\right) G dx dt. \end{aligned}$$

By similar computations,

$$I_2 := \int_{-4}^{-1}\int_{\mathbb{R}^n} \left(-\Delta h_r - \frac{x \cdot \nabla h_r}{2t}\right) h_r G dx dt = \int_{-4}^{-1}\int_{\mathbb{R}^n} \frac{-1}{2t} (L h_r + \kappa h_r) h_r G dx dt. \tag{2.1}$$

Now consider

$$
\begin{aligned}
& r\mathcal{N}'(r)\left(\int_{-4}^{-1}\int_{\mathbb{R}^n}\frac{-1}{t}|h_r|^2 G\,dx\,dt\right)^2 \\
&= \left(\int_{-4}^{-1}\int_{\mathbb{R}^n}\frac{-1}{t}(Lh_r)^2 G\,dx\,dt\right)\left(\int_{-4}^{-1}\int_{\mathbb{R}^n}\frac{-1}{t}|h_r|^2 G\,dx\,dt\right) \\
&\quad - \left(\int_{-4}^{-1}\int_{\mathbb{R}^n}\frac{-1}{t}h_r Lh_r G\,dx\,dt\right)^2 \geq 0.
\end{aligned}
$$

Hence $\mathcal{N}$ is nondecreasing, and if $\mathcal{N}'=0$, then $Lh_r = ch_r$. Recalling that $Lh_r = x\cdot\nabla h_r + 2t\partial_t h_r - \kappa h_r$, we obtain $x\cdot\nabla h - 2t\partial_t h - (\kappa+c)h = 0$, which is equivalent to h being backward self-similar of degree $c+\kappa$. On the other hand, we have from (2.1)

$$
\mathcal{N}(r) = \frac{\int_{-4}^{-1}\int_{\mathbb{R}^n}|\nabla h_r|^2 G\,dx\,dt}{\int_{-4}^{-1}\int_{\mathbb{R}^n}\frac{1}{-t}|h_r|^2 G\,dx\,dt} = \frac{c+\kappa}{2},
$$

hence $c+\kappa = 2\mathcal{N}$.

The last statement of the lemma follows now by the contradiction argument. Suppose that $2\mathcal{N}(s) < \ell$ for some $s\in(0,1]$, it follows that $2\mathcal{N}(0+) < \ell$. By scaling

$$
w_r := \frac{h_r}{\left(\int_{-4}^{-1}\int_{\mathbb{R}^n}\frac{1}{-t}|h_r|^2 G\,dx\,dt\right)^{1/2}},
$$

we infer from the boundedness of $\mathcal{N}(r)$ that $\{w_r\}$ is bounded[2] in $L^2(-4,-1;W^{1,2}(B_R))$ for every $R>0$. Now we apply Lemma 7.2 for w_r and ∇w_r to get that $\{w_r\}$ is bounded in $L^2(-4,0;W^{1,2}(B_R))$. Indeed, for $-4<s<-2$ and $-1<t<0$ we can write

$$
\begin{aligned}
\int_{-1}^{0}\int_{B_R}|w_r(x,t)|^2 dx\,dt &\leq \int_{-1}^{0}\int_{B_R}\int_{-4}^{-2}\int_{\mathbb{R}^n}\frac{1}{2}e^{-\frac{|x|^2}{t+s}}\left(\frac{\sqrt{3}s}{s-t}\right)^n \\
&\quad\times |w_r(y,s)|^2 G(y,s)\,dy\,ds\,dt\,dx \\
&\leq e^{R^2/2}2^{2n+1}3^{n/2}|B_R|\int_{-4}^{-2}\int_{\mathbb{R}^n}\frac{1}{-s}|w_r(y,s)|^2 G(y,s)\,dy\,ds \\
&\leq e^{R^2/2}2^{2n+1}3^{n/2}|B_R|.
\end{aligned}
$$

Furthermore, the estimates on derivatives for caloric functions imply that $\{w_r\}$ is bounded in $L^2(-3,0;W^{2,2}(B_R))$. Consequently, by diagonalization technique there is a weakly convergence sequence $w_{r_m}\rightharpoonup w_0$ in $L^2(-3,0;W^{2,2}_{\text{loc}}(\mathbb{R}^n))$ as well as $w_{r_m}\to w_0$ strongly in

[2] Note that $G \geq \frac{e^{-R^2/4}}{(16\pi)^{n/2}}$ for $|x|\leq R$, and h is of polynomial growth.

$L^2(-3,0;W^{1,2}_{\text{loc}}(\mathbb{R}^n))$. Therefore, the limit w_0 is a caloric function satisfying $w_0(0) = \partial_t^j\partial_x^\mu w_0(0) = 0$ for all $2j+|\mu| \le \ell-1$. The later equality is a consequence of w_{r_m} being smooth and their derivatives being uniformly bounded by $\|w_{r_m}\|_{L^1((-4,0)\times B_R)}$. We claim now that for every fixed $0<r\le 1/3$,

$$\int_{-4r^2}^{-r^2}\int_{\mathbb{R}^n}\frac{1}{-t}|w_0|^2 G dxdt = \lim_{r_m\to 0}\int_{-4r^2}^{-r^2}\int_{\mathbb{R}^n}\frac{1}{-t}|w_{r_m}|^2 G dxdt = 1, \tag{2.2}$$

and

$$\int_{-4r^2}^{-r^2}\int_{\mathbb{R}^n}|\nabla w_0|^2 G dxdt = \lim_{r_m\to 0}\int_{-4r^2}^{-r^2}\int_{\mathbb{R}^n}|\nabla w_{r_m}|^2 G dxdt. \tag{2.3}$$

Suppose this is true, then we for $r<1/3$ we have

$$\mathcal{N}(r,w_0) = \lim_{r_m\to 0}\mathcal{N}(r,w_{r_m}) = \lim_{r_m\to 0}\mathcal{N}(rr_m,h) = \mathcal{N}(0+,h).$$

So, w_0 must be a backward self-similar function of degree $2\mathcal{N}(0+,h)<\ell$ for $0<t<1$. Since w_0 is caloric function, so $2\mathcal{N}(0+,h)\in\mathbb{N}$, comparing with $w_0(0)=\partial_t^j\partial_x^\mu w_0(0)=0$ for all $2j+|\mu|\le\ell-1$, this yields a contradiction with (2.2).

Therefore, $2\mathcal{N}(s,h)\ge\ell$ for $s\in(0,1]$. If $2\mathcal{N}(1,h)=\ell$, then $\mathcal{N}$ is constant on $(0,1)$ and thereby h is a backward self-similar function of degree ℓ.

To close the argument, we need to prove (2.2) and (2.3). This is a matter of computation and can be settled easily by Lemma 7.2. Indeed, we just need to show the following uniform convergence when $0<r\le 1/3$ is fixed and $r_m\to 0$,

$$\begin{aligned}\int_{-4r^2}^{-r^2}\int_{\mathbb{R}^n\setminus B_R}\frac{1}{-t}|w_{r_m}|^2 G dxdt &\le \left(\int_{-4r^2}^{-r^2}\int_{\mathbb{R}^n\setminus B_R}(2\sqrt{3})^n e^{\frac{|x|^2}{3-t}}\frac{1}{-t}G(x,t)dxdt\right)\\ &\quad\times\left(\int_{-4}^{-3}\int_{\mathbb{R}^n}\frac{1}{-s}|w_{r_m}(y,s)|^2 G(y,s)dyds\right)\\ &\le \int_{-4r^2}^{-r^2}\int_{\mathbb{R}^n\setminus B_R}\frac{\pi}{3}\left(\frac{3}{-\pi t}\right)^{n/2+1}\exp\left(\frac{|x|^2}{3-t}+\frac{|x|^2}{4t}\right)dxdt\\ &\le \int_{-4r^2}^{-r^2}\int_{\mathbb{R}^n\setminus B_R}\frac{\pi}{3}\left(\frac{3}{-\pi t}\right)^{n/2+1}\\ &\quad\times e^{\frac{|x|^2}{8t}}dxdt\to 0 \text{ as } R\to\infty.\end{aligned}$$

The proof of (2.3) is the same if we apply again Lemma 7.2 for the caloric function ∇w_{r_m}. □

2.2 Nondegeneracy

Proposition 2.5 *(Nondegeneracy) Let* $\mathbf{u}$ *be a solution of* (1.1) *with* $0 \le q < 1$. *Then there is a positive constant* $c = c(q, n)$ *such that if* $(x_0, t_0) \in \overline{\{|\mathbf{u}| > 0\}}$, *and* $Q_r^-(x_0, t_0) \subset Q_1$, *then*

$$\sup_{Q_r^-(x_0,t_0)} |\mathbf{u}| \ge cr^\kappa.$$

Proof Let $U(x, t) := |\mathbf{u}(x, t)|^{1-q}$. The proof follows in a standard way using

$$\Delta U - \partial_t U = (1-q) + (1-q)\frac{|\nabla \mathbf{u}|^2}{U^{\kappa-1}} - \frac{1+q}{1-q}\frac{|\nabla U|^2}{U}, \qquad \text{in } \{U > 0\}.$$

For any $(y, s) \in \{|\mathbf{u}| > 0\}$, (close to (x_0, t_0)), set $w(x, t) = c(|x - y|^2 + (s - t))$ for small constant $c > 0$ to be specified later. Then $h = U - w$ satisfies in $Q_r^-(y, s)$

$$\begin{aligned}\mathcal{L}h - \partial_t h &:= \Delta h - \partial_t h + \frac{1+q}{1-q}\left(\frac{\nabla(U+w)}{U}\cdot\nabla h - \frac{4c}{U}h\right)\\ &= (1-q) - 4c(\frac{2n+1}{4} + \frac{1+q}{1-q}) + (1-q)\frac{|\nabla \mathbf{u}|^2}{U^{\kappa-1}} + 4c^2\frac{1+q}{1-q}\frac{s-t}{U} \ge 0,\end{aligned}$$

provided that c is small enough. In particular h cannot attain a local maximum in $Q_r^-(y, s) \cap \{|\mathbf{u}| > 0\}$ according to the maximum principle for $\mathcal{L} - \partial_t$. On the other hand $h < 0$ on $\partial\{|\mathbf{u}| > 0\}$ and hence the positive maximum of h is attained on $\partial_p Q_r^-(y, s)$, and we conclude that

$$\sup_{\partial_p Q_r^-(y,s)} (U - w) \ge U(y, s) > 0,$$

which amounts to

$$\sup_{\partial_p Q_r^-(y,s)} U \ge cr^2.$$

Letting $(y, s) \to (x_0, t_0)$, we arrive at the statement of the proposition. □

3 Regularity of solutions

In this section we study the regularity of solutions (to equation (1.1)), which according to the parabolic regularity theory, are known to be $C_x^{1,\beta} \cap C_t^{0,(1+\beta)/2}$ for $q = 0$ and $C_x^{2,\beta} \cap C_t^{1,\beta/2}$ for $q > 0$. Here we will show the optimal growth for solutions from points where $\mathbf{u}$ vanish to the highest order for our problem. In order to study the optimal growth (regularity) of solution, we start with the following definition; see also [10].

Definition 3.1 The vanishing order of $\boldsymbol{u}$ at point X_0 is defined to be the largest value $\mathcal{V}(X_0)$ which satisfies

$$\limsup_{r\to 0^+} \frac{\|\boldsymbol{u}\|_{L^\infty(Q_r^-(X_0))}}{r^{\mathcal{V}(X_0)}} < +\infty.$$

One of main tools in studying a sublinear equation is Lemma 7.5, which is the dual of [3,Lemma 1.1] for the elliptic case. For the convenience of reader we put the proof in the Appendix. One of the useful result of Lemma 7.5 is that if $\mathbf{u}$ is a solution of (1.1) and $X_0 \in$

$\Gamma(\mathbf{u})$, then $\mathcal{V}(X_0) \in \{1, 2, 3, \cdots, \lfloor\kappa\rfloor, \kappa\}$. (recall also nondegeneracy property, Proposition 2.5). Moreover, we can find out easily that if $\mathcal{V}(X_0) = s \le \kappa$, then $\partial_t^i \partial_x^\mu \mathbf{u}(X_0)$ exists and vanishes for $2i + |\mu| < s$. Indeed, there is a self-similar vectorial polynomial P of degree s such that $|\mathbf{u}(X) - P(X)| \le C|X - X_0|^s$. Our main result for case $q > 0$ is that if $X_0 \in \Gamma^\kappa(\mathbf{u}) = \{X_0 \in \Gamma(\mathbf{u}) : \partial_t^i \partial_x^\mu \mathbf{u}(X_0) = 0 \text{ for all } 2i + |\mu| < \kappa\}$, then $\mathcal{V}(X_0) = \kappa$.

We start with the following lemma which is essential to obtain our result.

Lemma 3.2 *For any* $\mathbf{u}$ *solving* (1.1) *in* Q_2*, and satisfying the doubling*

$$\|\mathbf{u}\|_{L^\infty(Q_2^-)} \le 2^\kappa \|\mathbf{u}\|_{L^\infty(Q_1^-)}, \tag{3.1}$$

we have

$$\|\mathbf{u}\|_{L^\infty(Q_1^-)} \le \max\left\{1, C\|\mathbf{u}G^{1/(1+q)}\|_{L^{1+q}(Q_1^-)}\right\},$$

where C *is independent of* $\mathbf{u}$.

Proof Suppose the statement of the lemma fails. Then there is a sequence $\mathbf{u}_j$ satisfying the hypothesis of the lemma with

$$\|\mathbf{u}_j\|_{L^\infty(Q_1^-)} \ge 1, \qquad \text{and} \qquad \|\mathbf{u}_j\|_{L^\infty(Q_1^-)} \ge j\|\mathbf{u}_j G^{1/(1+q)}\|_{L^{1+q}(Q_1^-)}. \tag{3.2}$$

Define $\tilde{\mathbf{u}}_j = \mathbf{u}_j / \|\mathbf{u}_j\|_{L^\infty(Q_1^-)}$, and insert in (3.2), to arrive at

$$\frac{1}{j} \ge \|\tilde{\mathbf{u}}_j G^{1/(1+q)}\|_{L^{1+q}(Q_1^-)}. \tag{3.3}$$

Since $\tilde{\mathbf{u}}_j$ satisfies the doubling (3.1), then it yields

$$\|H(\tilde{\mathbf{u}}_j)\|_{L^\infty(Q_2^-)} \le 2^{\kappa q}\|\mathbf{u}_j\|_{L^\infty(Q_1^-)}^{q-1} \le 2^{\kappa q}.$$

Therefore we have a subsequence of $\tilde{\mathbf{u}}_j$ which converges to a limit function $\mathbf{u}_0$ satisfying

$$\|\tilde{\mathbf{u}}_0 G^{1/(1+q)}\|_{L^{1+q}(Q_1^-)} = 0, \qquad \|\tilde{\mathbf{u}}_0\|_{L^\infty(Q_1^-)} = 1,$$

which is obviously a contradiction. □

Theorem 3.3 *For* $\mathbf{u}$ *a solution to* (1.1)*, with* $(0, 0) \in \Gamma^\kappa(\mathbf{u})$*, there exists a constant* C *such that*

$$\sup_{Q_r^-} |\mathbf{u}| \le Cr^\kappa, \qquad \forall\, 0 < r < 1/2.$$

Proof **Case** $\kappa \notin \mathbb{N}$: The proof in this case follows by standard blow-up and the use of Liouville's theorem, and the only subtle point would be to prove the blow-up solution will vanish at the origin, of order κ; the latter is taken care of in Appendix. Here is how it works out. If the statement of the theorem fails, then there exists a sequence $r_j \to 0$ such that

$$\sup_{Q_r^-} |\mathbf{u}| \le jr^\kappa, \qquad \forall r \ge r_j, \qquad \sup_{Q_{r_j}^-} |\mathbf{u}| = jr_j^\kappa.$$

In particular the function $\tilde{\mathbf{u}}_j(x, t) = \frac{\mathbf{u}(r_j x, r_j^2 t)}{j r_j^\kappa}$ satisfies

$$\sup_{Q_R^-} |\tilde{\mathbf{u}}_j| \le R^\kappa, \qquad \text{for } 1 \le R \le \frac{1}{r_j},$$

with equality for $R = 1$, along with

$$H\tilde{\mathbf{u}}_j = \frac{f(\tilde{\mathbf{u}}_j)}{j^{1-q}} \longrightarrow 0 \text{ uniformly in } Q_R^-.$$

From this we conclude that there is a convergent subsequence, tending to a caloric function $\mathbf{u}_0$ with growth κ, i.e.

$$\sup_{Q_R^-} |\mathbf{u}_0| \le R^\kappa, \quad \forall R \ge 1, \quad \sup_{Q_1^-} |\mathbf{u}_0| = 1, \quad H\mathbf{u}_0 = 0, \tag{3.4}$$

and furthermore, $|\tilde{\mathbf{u}}_j(X)| \le C_0|X|$ in Q_1^- uniformly for some constant $C_0 > 0$ and all j. Thus $|H\tilde{\mathbf{u}}_j| \le |\tilde{\mathbf{u}}_j|^q \le C_0^q|X|^q$ in Q_1^-. Now if we apply Lemma 7.5 for each component of $\tilde{\mathbf{u}}_j = (\tilde{u}_j^1, \cdots, \tilde{u}_j^m)$, we obtain a caloric polynomial P_j^i of degree at most $\lfloor 2+q \rfloor = 2$ so that $|\tilde{u}_j^i(X) - P_j^i(X)| \le C_1C_0|X|^{2+q}$ in Q_1^- and the constant C_1 depends only on n, q and an upper bound on $\|\tilde{\mathbf{u}}_j\|_{L^\infty(Q_1^-)}$. Since $(0, 0) \in \Gamma^\kappa(\mathbf{u})$, so $P_j^i \equiv 0$ and then $|\tilde{u}_j^i(X)| \le C_1C_0|X|^{2+q}$. By a bootstrap argument we find out the uniform estimate $|\tilde{\mathbf{u}}_j(X)| \le C_\epsilon|X|^{\kappa-\epsilon}$ for every $\epsilon > 0$. Therefore, we get

$$\mathbf{u}_0(0, 0) = \partial_t^i \partial_x^\mu \mathbf{u}_0(0, 0) = 0, \quad \text{for all } 2i + |\mu| < \kappa. \tag{3.5}$$

Obviously (3.4) and (3.5), along with the fact that $\kappa \notin \mathbb{N}$, violates Liouville's theorem and we have a contradiction.

Case $\kappa \in \mathbb{N}$: Consider the function $\mathbf{v} = \eta\mathbf{u}$ where $\eta \in C_0^\infty(B_{3/4})$ satisfies $0 \le \eta \le 1$, and $\eta = 1$ in $B_{1/2}$. Fix $0 < r < \frac{1}{2}$, let $\rho_i := 2^{-i}r$, $i = 0, 1, 2, ...$, and define $\mathbf{v}_{\rho_i}(x, t) = \mathbf{v}(\rho_i x, \rho_i^2 t)/\rho_i^\kappa$, then

$$\begin{aligned}
&\int_{-r^2}^{0}\int_{B_r} |\mathbf{u}|^{1+q} G dx dt \\
&= \sum_{i=1}^{\infty} \int_{-\rho_{i-1}^2}^{-\rho_i^2} \int_{B_r} |\mathbf{v}|^{1+q} G dx dt \\
&= \sum_{i=1}^{\infty} \rho_i^{2\kappa} \int_{-4}^{-1}\int_{B_{2^i}} |\mathbf{v}_{\rho_i}|^{1+q} G dx dt \le \sum_{i=1}^{\infty} \rho_i^{2\kappa} \int_{-4}^{-1}\int_{\mathbb{R}^n} |\mathbf{v}_{\rho_i}|^{1+q} G dx dt \\
&= \frac{1+q}{2} \sum_{i=1}^{\infty} \rho_i^{2\kappa} \left(\mathbb{W}(\mathbf{v}_{\rho_i}, 1) - \int_{-4}^{-1}\int_{\mathbb{R}^n} \left(|\nabla \mathbf{v}_{\rho_i}|^2 + \frac{\kappa |\mathbf{v}_{\rho_i}|^2}{2t} \right) G dx dt \right) \\
&= \frac{1+q}{2} \sum_{i=1}^{\infty} \rho_i^{2\kappa} \left(\mathbb{W}(\mathbf{v}, \rho_i) - \int_{-4}^{-1}\int_{\mathbb{R}^n} \left(|\nabla (\mathbf{v}_{\rho_i} - \mathbf{p})|^2 + \frac{\kappa |\mathbf{v}_{\rho_i} - \mathbf{p}|^2}{2t} \right) G dx dt \right) \\
&\le \frac{1+q}{2} \sum_{i=1}^{\infty} \rho_i^{2\kappa} \left(\mathbb{W}(\mathbf{v}, 1) + F(1) + \int_{-4}^{-1}\int_{\mathbb{R}^n} \frac{\kappa |\mathbf{v}_{\rho_i} - \mathbf{p}|^2}{-2t} G dx dt \right)
\end{aligned}$$

$$\leq Cr^{2\kappa}\left(1+\int_{-1}^{0}\int_{\mathbb{R}^n}\frac{\kappa|\mathbf{v}_r-\mathbf{p}|^2}{-2t}Gdxdt\right), \tag{3.6}$$

where we have used Lemma 7.4, F is the function defined in Theorem 2.2 and $\mathbf{p}\in\mathcal{H}$, the space of all κ-backward self-similar caloric vector-functions. We now let $\mathbf{p}=\pi_r$, where

$$\pi_r=\operatorname{argmin}_{\mathbf{q}\in\mathcal{H}}\int_{-1}^{0}\int_{\mathbb{R}^n}\frac{|\mathbf{v}_r-\mathbf{q}|^2}{-t}Gdxdt,$$

and observe that

$$\int_{-1}^{0}\int_{\mathbb{R}^n}\frac{(\mathbf{v}_r-\pi_r)\cdot\mathbf{p}}{-t}Gdxdt=0,\qquad\text{for every }\mathbf{p}\in\mathcal{H}. \tag{3.7}$$

Now suppose, towards a contradiction, that there is a sequence $r_k\to 0$, such that

$$\sup_{Q_r^-}|\mathbf{u}|\leq kr^{\kappa},\qquad\forall r\geq r_k,\qquad\sup_{Q_{r_k}^-}|\mathbf{u}|=kr_k^{\kappa}.$$

Consider the scaling $\mathbf{u}_r(x,t)=\mathbf{u}(rx,r^2t)/r^{\kappa}$, where the sequence $\mathbf{u}_{r_k}$ satisfies the doubling condition (3.1) because

$$\|\mathbf{u}_{r_k}\|_{L^\infty(Q_2^-)}=2^{\kappa}\|\mathbf{u}_{2r_k}\|_{L^\infty(Q_1^-)}\leq 2^{\kappa}k=2^{\kappa}\|\mathbf{u}_{r_k}\|_{L^\infty(Q_1^-)}.$$

Therefore Lemma 3.2 and (3.6) implies that

$$M_k=\left(\int_{-1}^{0}\int_{\mathbb{R}^n}\frac{|\mathbf{v}_{r_k}-\pi_{r_k}|^2}{-t}Gdxdt\right)^{1/2}\longrightarrow\infty.$$

For $\mathbf{w}_k=\frac{\mathbf{v}_{r_k}-\pi_{r_k}}{M_k}$, we have

$$\int_{-1}^{0}\int_{\mathbb{R}^n}\frac{|\mathbf{w}_k|^2}{-t}Gdxdt=1. \tag{3.8}$$

Furthermore, we can show that $\{\nabla\mathbf{w}_kG^{1/2}\}$ is bounded in $L^2(-1,0;L^2(\mathbb{R}^n))$. In order to show this, we can write

$$\begin{aligned}
&\int_{-1}^{0}\int_{\mathbb{R}^n}\left(|\nabla\mathbf{w}_k|^2+\frac{\kappa|\mathbf{w}_k|^2}{2t}\right)Gdxdt\\
&=\frac{1}{M_k^2}\int_{-1}^{0}\int_{\mathbb{R}^n}\left(|\nabla\mathbf{v}_{r_k}|^2+\frac{\kappa|\mathbf{v}_{r_k}|^2}{2t}\right)Gdxdt\\
&=\frac{1}{M_k^2}\sum_{i=1}^{\infty}2^{-2i\kappa}\int_{-4}^{-1}\int_{\mathbb{R}^n}\left(|\nabla\mathbf{v}_{2^{-i}r_k}|^2+\frac{\kappa|\mathbf{v}_{2^{-i}r_k}|^2}{2t}\right)Gdxdt\\
&\leq\frac{1}{M_k^2}\sum_{i=1}^{\infty}2^{-2i\kappa}\mathbb{W}(\mathbf{v},2^{-i}r_k)\longrightarrow 0,
\end{aligned} \tag{3.9}$$

which together with (3.8) implies

$$\int_{-1}^{0}\int_{\mathbb{R}^n} |\nabla \mathbf{w}_k|^2 G dx dt = O(1). \tag{3.10}$$

On the other hand, we have

$$H\mathbf{w}_k = \frac{1}{M_k} H\mathbf{v}_{r_k} = \frac{1}{M_k}\left(r_k^2 \mathbf{u}_{r_k}\Delta\eta(r_k x) + \eta(r_k x)^{1-q} f(\mathbf{v}_{r_k}) + 2r_k \nabla\mathbf{u}_{r_k}\cdot\nabla\eta(r_k x)\right), \tag{3.11}$$

and also,

$$\begin{aligned}
&\frac{2}{1+q}\int_{-1}^{0}\int_{\mathbb{R}^n} |\mathbf{v}_{r_k}|^{1+q} G dx dt \\
&= \frac{2}{1+q}\sum_{i=1}^{\infty} 2^{-2i\kappa}\int_{-4}^{-1}\int_{\mathbb{R}^n} |\mathbf{v}_{2^{-i}r_k}|^{1+q} G dx dt \\
&= \sum_{i=1}^{\infty} 2^{-2i\kappa}\left(\mathbb{W}(\mathbf{v}_{2^{-i}r_k},1) - \int_{-4}^{-1}\int_{\mathbb{R}^n}\left(|\nabla \mathbf{v}_{2^{-i}r_k}|^2 + \frac{\kappa|\mathbf{v}_{2^{-i}r_k}|^2}{2t}\right) G dx dt\right) \\
&\leq \sum_{i=1}^{\infty} 2^{-2i\kappa}\left(\mathbb{W}(\mathbf{v},1) + F(1) + \int_{-4}^{-1}\int_{\mathbb{R}^n} \frac{\kappa|\mathbf{v}_{2^{-i}r_k} - \pi_{r_k}|^2}{-2t} G dx dt\right) \\
&= O(1) + \int_{-1}^{0}\int_{\mathbb{R}^n} \frac{\kappa|\mathbf{v}_{r_k} - \pi_{r_k}|^2}{-2t} G dx dt \\
&= O(1) + \frac{M_k^2}{1-q}.
\end{aligned}$$

Therefore by (3.11), we get for $q > 0$

$$\begin{aligned}
\int_{-1}^{0}\int_{\mathbb{R}^n} |H\mathbf{w}_k|^{(1+q)/q} G dx dt &\leq \frac{1}{M_k^{(1+q)/q}}\left(C(\eta, \|\mathbf{u}\|_{H^1(B_1,\mathbb{R}^m)}) + \int_{-1}^{0}\int_{\mathbb{R}^n} |\mathbf{v}_{r_k}|^{1+q} G dx dt\right) \\
&\leq \frac{1}{M_k^{(1+q)/q}}\left(C(\eta, \|\mathbf{u}\|_{H^1(B_1,\mathbb{R}^m)}, n, q) + \frac{M_k^2}{1-q}\right) \longrightarrow 0,
\end{aligned}$$

and $\|H\mathbf{w}_k\|_\infty \to 0$ for $q = 0$. Hence $\{\mathbf{w}_k\}$ is bounded in $W^{1,p}(-1, \frac{1}{R}; W^{2,p}(B_R))$ for all fixed $R > 0$ and by diagonalization technique there is a weakly convergent subsequence with limit $\mathbf{w}_0$ in $L^2(-1, 0; W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n))$, satisfying $H\mathbf{w}_0 = 0$. We claim next that the strong convergence

$$\frac{\mathbf{w}_k}{(-t)^{1/2}} G^{1/2} \to \frac{\mathbf{w}_0}{(-t)^{1/2}} G^{1/2}, \qquad \text{in } L^2(-1, 0; L^2(\mathbb{R}^n)), \tag{3.12}$$

holds, which follows if we prove the uniform convergence in k

$$\int_{-1}^{0}\int_{\mathbb{R}^n\setminus B_R} \frac{|\mathbf{w}_k|^2}{-t} G dxdt \to 0, \qquad \text{as } R\to\infty. \tag{3.13}$$

This can obviously be obtained by applying Lemma 7.3 and relations (3.8) and (3.10)

$$R^2\int_{-1}^{-0}\int_{\mathbb{R}^n\setminus B_R} \frac{|\mathbf{w}_k|^2}{-t} G dxdt \le \int_{-1}^{0}\int_{\mathbb{R}^n} |\mathbf{w}_k|^2 \frac{|x|^2}{-t} G dxdt = O(1).$$

Therefore we get

$$\int_{-1}^{0}\int_{\mathbb{R}^n} \frac{|\mathbf{w}_0|^2}{-t} G dxdt = 1,$$

and also by (3.7),

$$\int_{-1}^{0}\int_{\mathbb{R}^n} \frac{\mathbf{w}_0\cdot\mathbf{p}}{-t} G dxdt = 0, \qquad \text{for every } \mathbf{p}\in\mathcal{H}. \tag{3.14}$$

Moreover, from (3.9), (3.12) and weakly lower semicontinuity of norm we obtain for every $R>0$

$$\int_{-1}^{0}\int_{B_R}\left(|\nabla\mathbf{w}_0|^2+\frac{\kappa|\mathbf{w}_0|^2}{2t}\right) G dxdt \le \lim_{k\to\infty}\int_{-1}^{0}\int_{\mathbb{R}^n}\left(|\nabla\mathbf{w}_k|^2+\frac{\kappa|\mathbf{w}_k|^2}{2t}\right) G dxdt \le 0.$$

It implies that

$$\begin{aligned} 0 &\ge \int_{-1}^{0}\int_{\mathbb{R}^n}\left(|\nabla\mathbf{w}_0|^2+\frac{\kappa|\mathbf{w}_0|^2}{2t}\right) G dxdt \\ &= \sum_{i=1}^{\infty} 2^{-2i\kappa}\int_{-4}^{-1}\int_{\mathbb{R}^n}\left(|\nabla\mathbf{w}_{0,2^{-i}}|^2+\frac{\kappa|\mathbf{w}_{0,2^{-i}}|^2}{2t}\right) G dxdt. \end{aligned} \tag{3.15}$$

If we further have

$$|\partial_t^\ell\partial_x^\mu w_0^j(0,0)| = 0, \qquad \text{for } 2\ell+|\mu|\le\kappa-1, \tag{3.16}$$

then by Lemma 2.4, each component w_0^j of $\mathbf{w}_{0,2^{-i}}$ must satisfy

$$\int_{-4}^{-1}\int_{\mathbb{R}^n}\left(|\nabla w_0^j|^2+\frac{\kappa|w_0^j|^2}{2t}\right) G dxdt \ge 0.$$

Summing over j and comparing with (3.15), implies that w_0^j is a κ-backward self-similar caloric function. But (3.14) implies that $\mathbf{w}_0 = 0$ which contradicts (3.13).

To close the argument, we need to prove (3.16). This can be shown by invoking Lemma 7.5 to obtain uniform estimate $\|\mathbf{w}_k\|_{L^\infty(Q_r^-)} = o(r^{\kappa-1})$. To apply Lemma 7.5 it is necessary to show the uniform estimate $\|H\mathbf{w}_k\|_{L^\infty(Q_r^-)} = o(r^{\kappa-2-1/2})$. Since we have assumed

$\|\mathbf{u}_{r_k}\|_{L^\infty(Q_1^-)} \to \infty$ by contradiction, the scaled sequence $\tilde{\mathbf{u}}_k := \mathbf{u}_{r_k}/\|\mathbf{u}_{r_k}\|_{L^\infty(Q_1^-)}$ satisfies $H\tilde{\mathbf{u}}_k = f(\tilde{\mathbf{u}}_k)/\|\mathbf{u}_{r_k}\|^{1-q}_{L^\infty(Q_1^-)} \to 0$ and converges to a caloric function $\tilde{\mathbf{u}}_0$ as a subsequence. Moreover, $|\tilde{\mathbf{u}}_k(X)| \le C_0|X|$ for a constant $C_0 > 0$ and all k. Now apply Lemma 7.5 repeatedly to obtain the uniform estimate $|\tilde{\mathbf{u}}_k(X)| \le C_\epsilon|X|^{\kappa-\epsilon}$ for a small value ϵ. So, $|\mathbf{u}_{r_k}(X)| \le C_\epsilon\|\mathbf{u}_{r_k}\|_{L^\infty(Q_1^-)}|X|^{\kappa-\epsilon}$ and by Lemma 3.2 as well as (3.6)

$$|H\mathbf{w}_k(X)| \le \frac{1}{M_k}|f(\mathbf{u}_{r_k}(X))| \le \frac{C_\epsilon^q}{M_k}(1+M_k^2)^{q/(1+q)}|X|^{\kappa-2-\epsilon q} \le C|X|^{\kappa-2-\epsilon q}. \qquad \square$$

Remark 3.4 Although Theorem 3.3 shows the backward regularity, we can see obviously the regularity in forward problem. A line of proof can be considered toward a contradiction and assuming the sequence $r_j \to 0$ such that

$$\sup_{Q_r^+}|\boldsymbol{u}| \le jr^\kappa, \qquad \forall r \ge r_j, \qquad \sup_{Q_{r_j}^+}|\boldsymbol{u}| = jr_j^\kappa.$$

Then $\boldsymbol{u}_j(X) = \boldsymbol{u}(r_jX)/(jr_j^\kappa)$ converges to a caloric function $\boldsymbol{u}_0$ in $\mathbb{R}^n \times \mathbb{R}$ with polynomial growth, $\|\boldsymbol{u}_0\|_{L^\infty(Q_1^+)} = 1$ and $\boldsymbol{u}_0 \equiv 0$ for $t \le 0$ (we also apply here Theorem 3.3). This contradicts the uniqueness of heat equation solution with polynomial growth in forward problem.

4 Homogeneous global solutions

In this section we perform energy classification of regular free boundary points, that will be needed in order to establish the Hölder regularity of the time derivative $\partial_t\mathbf{u}$ in the next section. Indeed the main goal is to show that half-space solutions are isolated within certain topology. The proofs for the case $q = 0$ and $q > 0$ differs to some extent and hence we are forced to consider them separately. For the case $q = 0$ we need to consider two lemmas (Lemmas 4.1, and 4.2) that will give us the result. The proof for the case $q > 0$ takes a different turn, and is shown in the proof of Proposition 4.3.

Lemma 4.1 *Let $q = 0$ and $\mathbf{u}$ be a backward self-similar solution to (1.1). If $\{|\mathbf{u}| > 0\} \cap Q_1^- \subset \{x_n > -\delta\}$, where $\delta > 0$ is small, then $\mathbf{u} \in \mathbb{H}$.*

Proof Let $\mathbf{u}$ be a backward self-similar solution, and recall that the condition of homogeneity (for each component) is

$$Lu^k := 2t\partial_t u^k + x\cdot\nabla u^k - 2u^k = 0. \tag{4.1}$$

Hence we obtain the following equation for each component

$$\frac{2u^k - x\cdot\nabla u^k}{-2t} + \Delta u^k - \frac{u^k}{|\mathbf{u}|}\chi_{\{|\mathbf{u}|>0\}} = 0, \tag{4.2}$$

and

$$2t\Delta u^k + x\cdot\nabla u^k - 2t\frac{u^k}{|\mathbf{u}|}\chi_{\{|\mathbf{u}|>0\}} = 2u^k. \tag{4.3}$$

Denote by $\mathcal{L}_0 := -\Delta + x\cdot\nabla$ and $\mathcal{L} := \mathcal{L}_0 + \frac{1}{|\mathbf{u}|}$. Then for $t = -\frac{1}{2}$, any u^k is an eigenfunction of $\mathcal{L}$ in $\{U > 0\}$ corresponding to the eigenvalue $\lambda = 2$.

We want to show that $\lambda = 2$ is the first eigenvalue for $\mathcal{L}$, since then $u^k = c_k|\mathbf{u}|$ in each connected component of $\{|\mathbf{u}| > 0\}$, and we have a scalar problem for $|\mathbf{u}|$ when $t = -1/2$. It is sufficient to show that 2 is not larger than the second eigenvalue for $\mathcal{L}_0$.

We prove that for some $\delta > 0$, $\lambda_2(\mathcal{L}, \{x_n > -\delta\}) > 2$, which implies $\lambda_1(\mathcal{L}, \{x_n > -\delta\}) = 2$. Since[3] $\lambda_2(\mathcal{L}_0, \mathbb{R}^n_+) = 3$, we have

$$\lambda_2(\mathcal{L}, \{x_n > -\delta\}) > \lambda_2(\mathcal{L}_0, \{x_n > -\delta\}) \geq \lambda_2(\mathcal{L}_0, \mathbb{R}^n_+) - \omega(\delta) = 3 - \omega(\delta), \tag{4.4}$$

where $\omega(\delta)$ is the modulus of continuity of $\lambda_2(\mathcal{L}_0, \{x_n > -\delta\})$. By choosing $\delta > 0$ small, we will obtain $\lambda_2(\mathcal{L}, \{x_n > -\delta\}) > 2$, implying that $\lambda_1(\mathcal{L}, \{x_n > -\delta\}) = 2$. Hence $\mathbf{u} = \mathbf{c}|\mathbf{u}|$ in each connected component of $\{|\mathbf{u}| > 0\}$, where $\mathbf{c} \in \mathbb{R}^m$ depends on the component and $|\mathbf{c}| = 1$. It remains to observe that for $t = -1/2$, the function $U = |\mathbf{u}|$ is a homogeneous stationary solution to the equation $HU = \chi_{\{U>0\}}$, and therefore U is a half-space solution and $\mathbf{u} \in \mathbb{H}$. □

Lemma 4.2 *(Closeness to half-space) Let $q = 0$ and $\mathbf{u}$ be a backward self-similar solution to the system* (1.1) *with the property*

$$\iint_{Q_1^-} |\mathbf{u} - \mathbf{h}| G dx dt < \varepsilon, \tag{4.5}$$

where $\mathbf{h} = \frac{(x_n^+)^2}{2}\mathbf{e}_1$. Then

$$\{|\mathbf{u}| > 0\} \cap Q_{1/2}^- \subset \{(x,t) : x_n > -C\varepsilon^\beta\}. \tag{4.6}$$

for $C = C(n, m)$, and $\beta = \beta(n)$.

Proof The proof is standard and follows from the nondegeneracy. Let $(x_0, t_0) \in \{|\mathbf{u}| > 0\} \cap Q_{1/2}^-$, and $x_n^0 = -\varrho < 0$, then

$$\iint_{Q_\varrho^-} |\mathbf{u}| G dx dt \leq \iint_{Q_1^-} |\mathbf{u} - \mathbf{h}| G dx dt \leq \varepsilon. \tag{4.7}$$

By the nondegeneracy, there exists $X \in Q_\varrho^-(x_0, t_0)$, such that

$$|\mathbf{u}(X)| = \sup_{Q_\varrho^-(x_0,t_0)} |\mathbf{u}| \geq c_n \varrho^2.$$

Then for a small $r > 0$,

$$\inf_{Q_r^-(X)} |\mathbf{u}| \geq c_n\varrho^2 - C_n r^2 \geq C\varrho^2,$$

and

$$\varepsilon \geq \iint_{Q_r^-(X)} |\mathbf{u}| G dx dt \geq C\varrho^{n+4}. \tag{4.8}$$

Now (4.6) follows with $\beta = \frac{1}{n+4}$. □

The next proposition shows that the half-space solutions in $\mathbb{H}$ are isolated in the class of κ-backward self-similar solutions.

[3] This follows from a simple computation for one dimensional case, and the fact that eigenvalues decrease by symmetrisation, and translation invariance of the set $\mathbb{R}^n_+$ in directions orthogonal to $\mathbf{e}_n$.

Proposition 4.3 *The half-space solutions are isolated (in the topology of $L^2(-1,0;H^1(B_1;\mathbb{R}^m))$) within the class of backward self-similar solutions of degree κ.*

Proof The proof follows from Lemmas 4.1 and 4.2 for $q=0$. When $q>0$, we assume toward a contradiction that there exists a sequence of backward self-similar solutions of degree κ, say $\mathbf{u}_i$, such that

$$0<\inf_{\mathbf{h}\in\mathbb{H}}\|\mathbf{u}_i-\mathbf{h}\|_{L^2(-1,0;H^1(B_1;\mathbb{R}^m))}=\|\mathbf{u}_i-\hat{\mathbf{h}}\|_{L^2(-1,0;H^1(B_1;\mathbb{R}^m))}=:\delta_i\to 0,\quad \text{as } i\to\infty,$$

where $\hat{\mathbf{h}}=\alpha(x_n^+)^\kappa\mathbf{e}_1$. When passing to a subsequence, $(\mathbf{u}_i-\hat{\mathbf{h}})/\delta_i=:\mathbf{w}_i\rightharpoonup\mathbf{w}$ weakly in $L^2(-1,0;H^1(B_1;\mathbb{R}^m))$, the limit $\mathbf{w}$ is still a backward self-similar function of degree κ.

Furthermore, for $\phi\in C_0^\infty(Q_1^-;\mathbb{R}^m)$ we have

$$\begin{aligned}\int_{-1}^{0}\int_{B_1}-\nabla\mathbf{w}_i:\nabla\phi+\mathbf{w}_i\cdot\partial_t\phi\,dxdt&=\frac{1}{\delta_i}\int_{-1}^{0}\int_{B_1}\Big(f(\mathbf{u}_i)-f(\hat{\mathbf{h}})\Big)\cdot\phi\,dxdt\\&=\frac{1}{\delta_i}\int_{-1}^{0}\int_{B_1}\int_0^1\frac{d}{d\tau}f(\hat{\mathbf{h}}+\tau(\mathbf{u}_i-\hat{\mathbf{h}}))\cdot\phi\,d\tau dxdt\\&=\int_{-1}^{0}\int_{B_1}\int_0^1 f_{\mathbf{u}}(\hat{\mathbf{h}}+\tau\delta_i\mathbf{w}_i)(\mathbf{w}_i)\cdot\phi\,d\tau dxdt\end{aligned}$$

If $\operatorname{supp}\phi\subset B_1^-\times(-1,0)$, we conclude that

$$\begin{aligned}\int_{-1}^{0}\int_{B_1^-}-\nabla\mathbf{w}_i:\nabla\phi+\mathbf{w}_i\cdot\partial_t\phi\,dxdt&=\int_{-1}^{0}\int_{B_1^-}\int_0^1 f_{\mathbf{u}}(\hat{\mathbf{h}}+\tau\delta_i\mathbf{w}_i)(\mathbf{w}_i)\cdot\phi\,d\tau dxdt\\&=\int_{-1}^{0}\int_{B_1^-}\int_0^1 f_{\mathbf{u}}(\tau\delta_i\mathbf{w}_i)(\mathbf{w}_i)\cdot\phi\,d\tau dxdt\\&=\frac{1}{q}\delta_i^{q-1}\int_{-1}^{0}\int_{B_1^-}f_{\mathbf{u}}(\mathbf{w}_i)(\mathbf{w}_i)\cdot\phi\,dxdt,\end{aligned}$$

let $i\to\infty$ to obtain

$$\int_{-1}^{0}\int_{B_1^-}f_{\mathbf{u}}(\mathbf{w})(\mathbf{w})\cdot\phi\,dxdt=0.$$

Then $\mathbf{w}\equiv 0$ in $B_1^-\times(-1,0)$. Now for every $\operatorname{supp}\phi\subset B_1^+\times(-1,0)$,

$$\int_{-1}^{0}\int_{B_1^+}-\nabla\mathbf{w}:\nabla\phi+\mathbf{w}\cdot\partial_t\phi\,dxdt=\int_{-1}^{0}\int_{B_1^+}f_{\mathbf{u}}(\hat{\mathbf{h}})(\mathbf{w})\cdot\phi\,dxdt.$$

Thus $H\mathbf{w} = f_{\mathbf{u}}(\hat{\mathbf{h}})(\mathbf{w})$ in $B_1^+ \times (-1, 0)$. Now let $w^j := \mathbf{w} \cdot \mathbf{e}_j$ for $1 \le j \le m$, then

$$Hw^j = q\kappa(\kappa-1)(x_n^+)^{-2}w^j, \quad \text{for } j = 1,$$

and

$$Hw^j = \kappa(\kappa-1)(x_n^+)^{-2}w^j, \quad \text{for } j > 1.$$

Next extend w^j to a backward self-similar function of degree κ in $\{x_n < 0\}$ and define

$$\tilde{w}^j(x', x_n, t) := \begin{cases} w^j(x', x_n, t), & x_n > 0, \\ -w^j(x', x_n, t), & x_n < 0, \end{cases}$$

which is a backward self-similar weak solution of degree κ and satisfies

$$H\tilde{w}^j = \begin{cases} q\kappa(\kappa-1)|x_n|^{-2}\tilde{w}^j, & \text{for } j = 1, \\ \kappa(\kappa-1)|x_n|^{-2}\tilde{w}_j, & \text{for } j > 1. \end{cases} \tag{4.9}$$

If we consider any multiindex $\mu \in \mathbb{Z}_+^{n-1} \times \{0\}$ and any nonnegative integer $\gamma \in \mathbb{Z}_+$ as well as the higher order partial derivatives $\partial_t^\gamma \partial_x^\mu \tilde{w}^j =: \zeta$ then ζ is a backward self-similar function of order $\kappa - |\mu|_1 - 2\gamma$ and satisfies again in the same equation in $\mathbb{R}^n \times (-\infty, 0)$. From the integrability and homogeneity we infer that $\partial_t^\gamma \partial_x^\mu \tilde{w}^j \equiv 0$ for $\kappa - |\mu|_1 - 2\gamma + 1 \le -n/2$. Thus $(x', t) \mapsto \tilde{w}^j(x', x_n, t)$ is a polynomial and the homogeneity imply the existence of a polynomial p such that $w^j(x', x_n, t) = x_n^\kappa w^j(\frac{x'}{x_n}, 1, \frac{t}{x_n^2}) = x_n^\kappa p(\frac{x'}{x_n}, \frac{t}{x_n^2})$ for $x_n > 0$. Next choose γ such that $\partial_t^\gamma p = r(x') \neq 0$, then according to the H^1-integrability of $\partial_t^\gamma w^j = x_n^{\kappa-2\gamma} r(\frac{x'}{x_n})$ we know that $\kappa - 2\gamma - \deg r > \frac{1}{2}$. Take the multiindex $\mu \in \mathbb{Z}_+^{n-1} \times \{0\}$ such that $|\mu|_1 = \deg r$ and $\partial_x^\mu r \neq 0$, and insert $\partial_t^\gamma \partial_x^\mu w^j = \partial_x^\mu r x_n^{\kappa-2\gamma-|\mu|_1}$ in equation (4.9), which implies that

$$(\kappa - 2\gamma - |\mu|_1)(\kappa - 2\gamma - |\mu|_1 - 1) = q\kappa(\kappa-1), \text{ for } j = 1,$$

$$(\kappa - 2\gamma - |\mu|_1)(\kappa - 2\gamma - |\mu|_1 - 1) = \kappa(\kappa-1), \quad \text{for } j > 1,$$

and hence

$$2\gamma + |\mu|_1 = 1, \text{ or } 2\kappa - 2, \text{ for } j = 1,$$

$$2\gamma + |\mu|_1 = 0, \text{ or } 2\kappa - 1, \text{ for } j > 1.$$

The condition $\kappa - 2\gamma - \deg r > \frac{1}{2}$ and $\kappa > 2$ yields that the only possible case is $\gamma = 0$ and $|\mu|_1 = 1$ for $j = 1$ and $|\mu|_1 = 0$ for $j > 1$. We obtain that $w^1(x, t) = x_n^\kappa(d + \ell \cdot x'/x_n)$ and $w^j(x, t) = \ell_j x_n^\kappa$ for $j > 1$. Comparing with the equation (4.9) implies that we must have $d = 0$. To sum up, we find that $\mathbf{w}(x, t) = (x_n^{\kappa-1}\ell_1 \cdot x', \ell_2 x_n^\kappa, \cdots, \ell_m x_n^\kappa)$ for some $\ell_1 \in \mathbb{R}^{n-1}$ and $\ell_2, \cdots, \ell_m \in \mathbb{R}$.

Recall that we have chosen $\hat{\mathbf{h}}$ as the best approximation of $\mathbf{u}_i$ in $\mathbb{H}$. So, it follows that for $\mathbf{h}_\nu(x) := \alpha \max(x \cdot \nu, 0)^\kappa \mathbf{e}_1$,

$$(\mathbf{w}_i, \mathbf{h}_\nu - \hat{\mathbf{h}})_{L^2(-1,0;H^1(B_1;\mathbb{R}^m))} \le \frac{1}{2\delta_i} \|\mathbf{h}_\nu - \hat{\mathbf{h}}\|^2_{L^2(-1,0;H^1(B_1;\mathbb{R}^m))}. \tag{4.10}$$

Now let $\nu \to \mathbf{e}_n$, so that $\frac{\nu-\mathbf{e}_n}{|\nu-e_n|}$ converges to the vector ξ (where $\xi\cdot\mathbf{e}_n=0$), then

$$o(1) \geq \int_{-1}^{0}\int_{B_1} (\mathbf{w}_i\cdot\mathbf{e}_1)\kappa(x_n^+)^{\kappa-1}(x\cdot\xi)+ \nabla(\mathbf{w}_i\cdot\mathbf{e}_1)\cdot\left[\kappa(x_n^+)^{\kappa-1}\xi+\kappa(\kappa-1)(x_n^+)^{\kappa-2}(x\cdot\xi)\mathbf{e}_n\right]dxdt.$$

Choosing $\xi=(\ell_1,0)$ and passing to the limit in i, we obtain that

$$0 \geq \int_{-1}^{0}\int_{B_1} \kappa(x_n^+)^{2\kappa-2}(x'\cdot\ell_1)^2+\kappa(x_n^+)^{2\kappa-2}|\ell_1|^2 +\kappa(\kappa-1)^2(x_n^+)^{2\kappa-4}(x'\cdot\ell_1)^2\,dxdt.$$

Hence, $\ell_1=0$, and then $\mathbf{w}\cdot\mathbf{e}_1=0$.

If we apply once more the relation (4.10) for $\mathbf{h}_\theta=\alpha(x_n^+)^\kappa\mathbf{e}_t$ istead of $\mathbf{h}_\nu$, where $\mathbf{e}_\theta=(\cos\theta)\mathbf{e}_1\pm(\sin\theta)\mathbf{e}_j$, and let $\theta\to0$. We obtain

$$(\mathbf{w}_i,\pm\alpha(x_n^+)\kappa\mathbf{e}_j)_{W^{1,2}(Q_1^-;\mathbb{R}^m)}\leq 0.$$

Therefore,

$$\ell_j\|(x_n^+)^\kappa\|^2_{W^{1,2}(Q_1^-;\mathbb{R}^m)}=0,$$

and then $\ell_j=0$.

So far, we have proved that $\mathbf{w}\equiv0$. In order to obtain a contradiction to the assumption $\|\mathbf{w}_i\|_{L^2(-1,0;H^1(B_1;\mathbb{R}^m))}=1$, it is therefore sufficient to show the strong convergence of $\nabla\mathbf{w}_i$ to $\nabla\mathbf{w}$ in $L^2(-1,0;L^2(B_1;\mathbb{R}^m))$ as a subsequence $i\to\infty$. But by compact imbedding on the boundary

$$\begin{aligned}
\int_{-1}^{0}\int_{B_1}|\nabla\mathbf{w}_i|^2\,dxdt &= \int_{-1}^{0}\int_{\partial B_1}\mathbf{w}_i\cdot(\nabla\mathbf{w}_i\cdot x)\,d\mathcal{H}^{n-1}dt-\int_{-1}^{0}\int_{B_1}\mathbf{w}_i\cdot\Delta\mathbf{w}_i\,dxdt\\
&= \int_{-1}^{0}\int_{\partial B_1}\kappa|\mathbf{w}_i|^2-2t\partial_t\mathbf{w}_i\cdot\mathbf{w}_i\,d\mathcal{H}^{n-1}dt-\int_{-1}^{0}\int_{B_1}\mathbf{w}_i\cdot\partial_t\mathbf{w}_i\,dxdt\\
&\quad -\frac{1}{\delta_i^2}\int_{-1}^{0}\int_{B_1}(\mathbf{u}_i-\hat{\mathbf{h}})\cdot(f(\mathbf{u}_i)-f(\hat{\mathbf{h}}))\,dxdt\\
&\leq \int_{-1}^{0}\int_{\partial B_1}\kappa|\mathbf{w}_i|^2-t\partial_t|\mathbf{w}_i|^2\,d\mathcal{H}^{n-1}dt-\frac{1}{2}\int_{-1}^{0}\int_{B_1}\partial_t|\mathbf{w}_i|^2\,dxdt\\
&= \int_{-1}^{0}\int_{\partial B_1}(\kappa+1)|\mathbf{w}_i|^2\,d\mathcal{H}^{n-1}dt-\int_{\partial B_1}|\mathbf{w}_i(x,-1)|^2\,d\mathcal{H}^{n-1}\\
&\quad +\frac{1}{2}\int_{B_1}|\mathbf{w}_i(x,-1)|^2-|\mathbf{w}_i(x,0)|^2\,dx
\end{aligned}$$

$$\leq \int_{-1}^{0}\int_{\partial B_1} (\kappa+1)|\mathbf{w}_i|^2\, d\mathcal{H}^{n-1}dt + \frac{1}{2}\int_{B_1} |\mathbf{w}_i(x,-1)|^2\, dx$$
$$= \int_{-1}^{0}\int_{\partial B_1} (\kappa+1)|\mathbf{w}_i|^2\, d\mathcal{H}^{n-1}dt$$
$$+ \frac{1}{n+2\kappa+2}\int_{-1}^{0}\int_{B_1} |\mathbf{w}_i|^2 dxdt \to 0,$$

as a subsequence $i \to \infty$. (Note that we have used the homogeneity property of $\mathbf{w}_i$ in the last line.) □

Definition 4.4 (Regular points) We say that a point $z = (x, t) \in \Gamma^\kappa(\boldsymbol{u})$ is a regular[4] free boundary point for $\boldsymbol{u}$ if at least one blowup limit of $\boldsymbol{u}$ at z belongs to $\mathbb{H}$. We denote by $\mathcal{R}$ the set of all regular free boundary points in $\Gamma(\boldsymbol{u})$.

Proposition 4.5 *If $z_0 = (x_0, t_0) \in \mathcal{R}$, then all blowup limits of $\mathbf{u}$ at z_0 belong to $\mathbb{H}$.*

Proof Suppose there are two sequences $r_i, \rho_i \to 0$ such that the scaling $\mathbf{u}(x_0+r_i\cdot, t_0+r_i^2\cdot)/r_i^\kappa$ and $\mathbf{u}(x_0 + \rho_i\cdot, t_0 + \rho_i^2\cdot)/\rho_i^\kappa$ converges respectively to $\mathbf{u}_0 \in \mathbb{H}$ and a $\tilde{\mathbf{u}}_0 \notin \mathbb{H}$. Furthermore, we can assume that $r_{i+1} < \rho_i < r_i$. By a continuity argument we can find $\rho_i < \tau_i < r_i$ when i is large enough such that $dist\left(\mathbf{u}(x_0 + \tau_i\cdot, t_0 + \tau_i^2\cdot)/\tau_i^\kappa, \mathbb{H}\right) = \theta\, dist(\tilde{\mathbf{u}}_0, \mathbb{H})$ for an arbitrary $\theta \in (0, 1/2)$. The boundedness of $\mathbf{u}(x_0 + \tau_i\cdot, t_0 + \tau_i^2\cdot)/\tau_i^\kappa$ implies that every limit $\mathbf{u}^*$ of that is a κ-backward self-similar solution such that $dist(\mathbf{u}^*, \mathbb{H}) = \theta\, dist(\tilde{\mathbf{u}}_0, \mathbb{H})$, which for small θ contradicts the isolation property Proposition 4.3. □

Remark 4.6 We will conclude later the uniqueness of blowups at regular points of free boundary by Theorem 6.4 in Sect. 6.

The following lemma and theorem show that the half-space solutions have the lowest energy among the global self-similar solutions for the case $q = 0$.

Lemma 4.7 *Let $q = 0$ and $\mathbf{u}$ be a backward self-similar solution to the system (1.1), satisfying $|\mathbf{u}| > 0$ a.e.. Then*

$$\mathbb{M}(\mathbf{u}) = \int_{-4}^{-1}\int_{\mathbb{R}^n} |\mathbf{u}| G dxdt \geq \frac{15}{2}.$$

Proof Observe that by homogeneity of $\mathbf{u}$ we have $\mathbb{M}(\mathbf{u}) = \mathbb{W}(\mathbf{u}, r)$, for any r. Integration by parts, and (again) homogeneity of $\mathbf{u}$ implies

$$\mathbb{M}(\mathbf{u}) = \int_{-4}^{-1}\int_{\mathbb{R}^n} \left(|\nabla \mathbf{u}|^2 + \frac{|\mathbf{u}|^2}{t} + 2|\mathbf{u}|\right) G dxdt = \int_{-4}^{-1}\int_{\mathbb{R}^n} |\mathbf{u}| G dxdt.$$

[4] They are also called low-energy points. When $q = 0$, these points have the lowest energy (see Theorem 4.8). It is not generally true for $q > 0$ and they have the lowest energy when the coincidence set has nonempty interior (see Theorem 4.9). In this case, half-space solutions have the energy $\mathbb{M}(\mathbf{h}) = \frac{\alpha^{1+q}}{\kappa(\kappa-1)}\frac{4^\kappa - 1}{\sqrt{\pi}}2^{2\kappa-3}\Gamma(\kappa - \frac{1}{2})$, where Γ is the Gamma function here. The time-dependent global solution $\theta(x, t) = (\frac{-2t}{\kappa})^{\kappa/2}\mathbf{e}$ has the energy $\mathbb{M}(\theta) = \frac{2^{\kappa-1}}{\kappa^\kappa(\kappa-1)}(4^\kappa - 1)$ which is less than the energy of half-space solutions for $\kappa \geq 5/2$.

Let $U = |\mathbf{u}| > 0$ a.e., as we observed before, $\Delta U - \partial_t U \geq 1$. Hence

$$\begin{aligned} 3 =& \int_{-4}^{-1}\int_{\mathbb{R}^n} G dx dt \leq \int_{-4}^{-1}\int_{\mathbb{R}^n} (\Delta U - \partial_t U) G dx dt \\ =& \int_{-4}^{-1}\int_{\mathbb{R}^n} -U\Delta G + U\partial_t G - \partial_t(UG) dx dt = -\int_{\mathbb{R}^n} U(x,t)G(x,t)dx|_{t=-4}^{t=-1} \\ =& -\pi^{-\frac{n}{2}}\int_{\mathbb{R}^n} U(\sqrt{-4t}z,t)e^{-|z|^2}dz|_{t=-4}^{t=-1} = +4\pi^{-\frac{n}{2}}\int_{\mathbb{R}^n} tU(z,-1/4)e^{-|z|^2}dz|_{t=-4}^{t=-1} \\ =& 12\pi^{-\frac{n}{2}}\int_{\mathbb{R}^n} U(z,-1/4)e^{-|z|^2}dz. \end{aligned} \tag{4.11}$$

Employing again the homogeneity of U, we obtain

$$\begin{aligned} \mathbb{M}(\mathbf{u}) = \int_{-4}^{-1}\int_{\mathbb{R}^n} U(x,t)G(x,t)dx dt =& -4\pi^{-\frac{n}{2}}\int_{-4}^{-1}\int_{\mathbb{R}^n} tU(z,-1/4)e^{-|z|^2}dz dt \\ =& 30\pi^{-\frac{n}{2}}\int_{\mathbb{R}^n} U(z,-1/4)e^{-|z|^2}dz \geq \frac{15}{2}, \end{aligned}$$

where we used (4.11) in the last step. □

Theorem 4.8 *Let $q = 0$ and $\mathbf{u}$ be a backward self-similar solution to the system (1.1). Then $\mathbb{M}(\mathbf{u}) \geq \frac{15}{4}$ and the equality holds if and only if $\mathbf{u} \in \mathbb{H}$.*

Proof ***Step 1:*** We show that $\mathbb{M}(\mathbf{u}) = \frac{15}{4}$ if $\mathbf{u}$ is a half-space solution. If $\mathbf{u} \in \mathbb{H}$, then

$$\mathbb{M}(\mathbf{u}) = \int_{-4}^{-1}\int_{\mathbb{R}^n} |\mathbf{u}| G dx dt = \frac{1}{2}\int_{-4}^{-1} -t dt = \frac{15}{4}.$$

Step 2: If $U = |\mathbf{u}| > 0$ a.e., then $\mathbb{M}(\mathbf{u}) \geq \frac{15}{2}$ by Lemma 4.7. Suppose that $|\{U = 0\}| > 0$ and $\mathbb{M}(\mathbf{u}) < \frac{15}{4}$. By the nondegeneracy and quadratic decay estimates, we imply that the interior of $\{\mathbf{u} \equiv 0\}$ is nonempty. Then we may choose $Q_r^-(Y) \subset \{\mathbf{u} \equiv 0\}$ in such a way that there exists a point $Z \in \partial_p Q_r^-(Y) \cap \Gamma(\mathbf{u})$. Moreover, since $\mathbf{u}$ is backward self-similar we can assume that the point Z is very close to the origin and satisfies $\mathbb{W}(\mathbf{u}, 0+; Z) < 15/4$, by the upper semicontinuity of the balanced energy. Hence any blow-up at Z is a half-space solution by Lemma 4.1 which contradicts the result in Step 1.

Step 3: It remains to show that if $\mathbb{M}(\mathbf{u}) = \frac{15}{4}$ for a backward self-similar solution $\mathbf{u}$, then $\mathbf{u}$ is a halfspace solution. Let $X_0 = (x_0, t_0) \in \Gamma$ be as in *Step 2*, i.e. such that $Q_{r_0}^-(Y) \subset \{\mathbf{u} \equiv 0\}$, for a small $r > 0$. If $X_0 = 0$, then $\mathbf{u} \in \mathbb{H}$ by Lemma 4.1. Assume that $|X_0| > 0$. Let $\mathbf{u}_0$ be a blow-up of $\mathbf{u}$ at X_0, then $\mathbf{u}_0 \in \mathbb{H}$. Hence

$$\frac{15}{4} = \mathbb{M}(\mathbf{u}_0) = \mathbb{W}(\mathbf{u}, 0+; X_0) \leq \mathbb{W}(\mathbf{u}, +\infty; X_0) = \mathbb{W}(\mathbf{u}, +\infty; 0) = \mathbb{M}(\mathbf{u}) = \frac{15}{4}, \tag{4.12}$$

since $\mathbb{W}(\mathbf{u}, +\infty, X)$ does not depend on $X \in \Gamma$. Indeed, by homogeneity of $\mathbf{u}$ at the origin,

$$\frac{\mathbf{u}(rx + x_0, r^2t + t_0)}{r^2} = \mathbf{u}(x + \frac{x_0}{r}, t + \frac{t_0}{r^2}) \to \mathbf{u}(x, t), \text{ as } r \to +\infty.$$

Therefore (4.12) implies that $\mathbb{W}(\mathbf{u}, r; X_0)$ does not depend on r and $\mathbf{u}$ is backward self-similar with centre at X_0, hence $\mathbf{u}(x_0 + rx, t_0 + r^2t) = r^2\mathbf{u}(x_0 + x, t_0 + t)$. On the other hand, $\mathbf{u}(x_0+rx, t_0+r^2t) = r^2\mathbf{u}(x_0/r+x, t_0/r^2+t)$, and therefore $\mathbf{u}(x_0/r+x, t_0/r^2+t) = \mathbf{u}(x_0 + x, t_0 + t)$. Letting $r \to +\infty$, we obtain $\mathbf{u}(x, t) = \mathbf{u}(x_0 + x, t_0 + t)$, for any (x, t), hence $\mathbf{u}$ satisfies the assumptions in Lemma 4.1, and $\mathbf{u} \in \mathbb{H}$. □

The next theorem is a version of Theorem 4.8 for case $q > 0$ to show that half-space solutions have the lowest energy among the backward self-similar solutions whose coincidence set has nonempty interior.

Theorem 4.9 *Let* $\mathbf{u}$ *be a backward self-similar solution to the system* (1.1) *satisfying* $\{|\mathbf{u}| = 0\}^\circ \neq \emptyset$. *Then* $\mathbb{M}(\mathbf{u}) \geq A_q$, *and equality implies that* $\mathbf{u}$ *is a half-space solution; here* $A_q = \mathbb{M}(\mathbf{h})$ *for every* $\mathbf{h} \in \mathbb{H}$.

Proof The proof is indirect. Consider the self-similar solution $\mathbf{u}$ with $\mathbb{M}(\mathbf{u}) < A_q$. Assume that $\{|\mathbf{u}| = 0\}$ contains the cube Q and $X_0 = (x_0, t_0) \in \partial Q \cap \partial\{|\mathbf{u}| > 0\}$ and $t_0 < 0$. From here we deduce that all derivatives of $\mathbf{u}$ at X_0 vanish if they exist. If we start with the initial regularity and the estimate $|\mathbf{u}(X)| \leq C|X - X_0|$, we are able to apply Lemma 7.5 iteratively and obtain $|\mathbf{u}(X)| \leq C_\epsilon|X - X_0|^{\kappa-\epsilon}$. This implies that $X_0 \in \Gamma^\kappa$. Also from self-similarity of $\mathbf{u}$, we infer that

$$\mathbb{W}(\mathbf{u}, 0+; X_0) = \lim_{r\to 0^+} \mathbb{W}(\mathbf{u}, r; X_0) = \lim_{r\to 0^+} \mathbb{W}(\mathbf{u}, \frac{r}{m}; X_0^m) = \mathbb{W}(\mathbf{u}, 0+; X_0^m),$$

where $X_0^m := (\frac{x_0}{m}, \frac{t_0}{m^2})$. By the upper semicontinuity of the function $X \mapsto \mathbb{W}(\mathbf{u}, 0+; X)$, we get

$$\mathbb{W}(\mathbf{u}, 0+; X_0) = \limsup_{m\to\infty} \mathbb{W}(\mathbf{u}, 0+; X_0^m) \leq \mathbb{W}(\mathbf{u}, 0+; 0) \leq \mathbb{W}(\mathbf{u}, 1; 0) = \mathbb{M}(\mathbf{u}) < A_q.$$

Thus every blow-up limit $\mathbf{u}_0$ of $\mathbf{u}$ at the point X_0 satisfies the inequality $\mathbb{M}(\mathbf{u}_0) < A_q$. Note that by the nondegeneracy property $\mathbf{u}_0 \not\equiv 0$. Now the self-similarity of $\mathbf{u}$ tells us that $\mathbf{u}_0$ must be time-independent. To see that let $\mathbf{u}_r(x, t) := \mathbf{u}(x_0 + rx, t_0 + r^2t)/r^\kappa$ which converges to $\mathbf{u}_0$ in some sequence. According to the self-similarity of $\mathbf{u}$, we have

$$\begin{aligned}&\nabla\mathbf{u}(x_0 + rx, t_0 + r^2t) \cdot (x_0 + rx) + 2(t_0 + r^2t)\partial_t\mathbf{u}(x_0 + rx, t_0 + r^2t)\\&\quad = \kappa\mathbf{u}(x_0 + rx, t_0 + r^2t),\end{aligned}$$

so,

$$r\nabla\mathbf{u}_r(x, t) \cdot (x_0 + rx) + 2(t_0 + r^2t)\partial_t\mathbf{u}_r(x, t) = r^2\kappa\mathbf{u}_r(x, t),$$

and passing to the limit, we obtain $t_0\partial_t\mathbf{u}_0(x, t) = 0$. Therefore, $\mathbf{u}_0$ is a κ-homogeneous global solution of $\Delta\mathbf{u} = f(\mathbf{u})$ and violates Proposition 4.6 in [9], the elliptic version of this theorem. To find the elliptic energy of $\mathbf{u}_0$, we can write for every κ-homogeneous time-independent solution $\mathbf{v}$,

$$
\begin{aligned}
\mathbb{M}(\mathbf{v}) =& \frac{1-q}{1+q} \int_{-4}^{-1} \int_{\mathbb{R}^n} |\mathbf{v}(x)|^{1+q} G(x,t) dx dt \\
=& \frac{1-q}{1+q} \pi^{-\frac{n}{2}} \int_{-4}^{-1} \int_{\mathbb{R}^n} |\mathbf{v}(\sqrt{-4t} z)|^{1+q} e^{-|z|^2} dz dt \\
=& \frac{1-q}{1+q} \frac{4^{2\kappa} - 4^{\kappa}}{4\kappa} \pi^{-\frac{n}{2}} \int_{\mathbb{R}^n} |\mathbf{v}(z)|^{1+q} e^{-|z|^2} dz \\
=& \frac{1-q}{1+q} \frac{4^{2\kappa} - 4^{\kappa}}{4\kappa} \pi^{-\frac{n}{2}} \int_0^{\infty} \int_{\partial B_1} |\mathbf{v}(\hat{z})|^{1+q} r^{\kappa(1+q)+n-1} e^{-r^2} d\hat{z} dr \\
=& \frac{1-q}{1+q} c_{q,n} \int_{\partial B_1} |\mathbf{v}(\hat{z})|^{1+q} d\hat{z} \\
=& \frac{1-q}{1+q} \tilde{c}_{q,n} \int_{B_1} |\mathbf{v}(z)|^{1+q} dz = \tilde{c}_{q,n} M(\mathbf{v}),
\end{aligned}
$$

where $M(\mathbf{v})$ is the adjusted energy for the elliptic case which is used in Proposition 4.6 in [9]. In particular, for $\mathbf{h} \in \mathbb{H}$, we find out $M(\mathbf{u}_0) < M(\mathbf{h})$.

Finally, to prove the second part of the statement we consider the backward self-similar solution $\mathbf{u}$ satisfying $\mathbb{M}(\mathbf{u}) = A_q$, $Q \subset \{|u| = 0\}$ and $X_0 = (x_0, t_0) \in \partial Q \cap \partial\{|\mathbf{u}| > 0\}$ for some $t_0 < 0$. As in the first part of the proof we obtain that every blow-up limit $\mathbf{u}_0$ of $\mathbf{u}$ at the point X_0 satisfies the inequality $\mathbb{M}(\mathbf{u}_0) \le A_q$, that $\mathbf{u}_0$ is a κ-homogeneous time-independent solution and $\{|\mathbf{u}_0| = 0\}^\circ \neq \emptyset$. Thus according to Proposition 4.6 in [9], $\mathbf{u}_0$ must be a half-space solution. Therefore, every blow-up limit of $\mathbf{u}$ at the point $X_0^m = (\frac{x_0}{m}, \frac{t_0}{m^2})$ must be a half-space solution. Assuming $\mathbf{u} \notin \mathbb{H}$, we find by a continuity argument for an arbitrary $\theta \in (0, 1)$ a sequence $\rho_m \to 0$ such that

$$
dist(\rho_m^{-\kappa} \mathbf{u}(x_0/m + \rho_m \cdot, t_0/m^2 + \rho_m^2 \cdot), \mathbb{H}) = \theta\, dist(\mathbf{u}, \mathbb{H}) > 0.
$$

It follows that $\mathbf{u}(x_0/m + \rho_m \cdot, t_0/m^2 + \rho_m^2 \cdot)/\rho_m^\kappa$ converges to a backward self-similar solution $\mathbf{u}^*$ along a subsequence as $m \to \infty$, because

$$
\mathbb{W}(\mathbf{u}^*, r; 0) = \lim_{m\to\infty} \mathbb{W}(\mathbf{u}, r\rho_m; X_0^m) \ge \mathbb{W}(\mathbf{u}, 0+, X_0^m) = A_q,
$$

and for every $0 < \rho$

$$
\mathbb{W}(\mathbf{u}^*, r, 0) = \lim_{m\to\infty} \mathbb{W}(\mathbf{u}, r\rho_m; X_0^m) \le \lim_{m\to\infty} \mathbb{W}(\mathbf{u}, \rho\,; X_0^m) = \mathbb{W}(\mathbf{u}, \rho\,; 0).
$$

Then $\mathbb{W}(\mathbf{u}^*, r; 0) = A_q$ for all $r > 0$ and $\mathbf{u}^*$ must be a self-similar solution. The conclusion is that $dist(\mathbf{u}^*, \mathbb{H}) = \theta\, dist(\mathbf{u}, \mathbb{H})$ which for small θ contradicts the isolation property Proposition 4.3. □

Here, we show that the regular points are an open set in $\Gamma(\mathbf{u}) = \partial\{|\mathbf{u}| > 0\}$.

Proposition 4.10 *The regular set $\mathcal{R}$ is open relative to $\Gamma(\mathbf{u})$.*

Proof Assume that there is a sequence $X_i = (x_i, t_i) \in \Gamma(\mathbf{u}) \setminus \mathcal{R}$ converging to $X_0 = (x_0, t_0) \in \mathcal{R}$. We can find a sequence $\tau_i \to 0$ and a subsequence of X_i such that[5]

$$dist\left(\mathbf{u}(x_i + \tau_i \cdot, t_i + \tau_i^2 \cdot)/\tau_i^\kappa, \mathbb{H}\right) = \frac{c}{2^{2\kappa+1}}, \tag{4.13}$$

where c is the constant defined in Proposition 2.5 and the distance is measured in $L^\infty(Q_1^-)$. The uniform boundedness of set $\mathbb{H}$ implies the convergence $\mathbf{u}(x_i + \tau_i \cdot, t_i + \tau_i^2 \cdot)/\tau_i^\kappa$ in a subsequence to a global solution $\mathbf{u}^*$. For convenience assume that $\|\mathbf{u}_{\tau_i, X_i} - \mathbf{h}\|_{L^\infty(Q_1^-)} \leq c/4^\kappa$ for $\mathbf{h}(x, t) = \alpha(x_+^1)^\kappa$. Then

$$|\mathbf{u}_{\tau_i, X_i}(x, t)| \leq \frac{c}{4^\kappa} \tag{4.14}$$

for all $(x, t) \in Q_1^-$ where $x_1 \leq 0$. According to the nondegeneracy property, Proposition 2.5, we know that $\sup_{Q_r^-(Z)} |\mathbf{u}_{\tau_i, X_i}| \geq cr^\kappa$ for all $Z \in \overline{\{|\mathbf{u}_{\tau_i, X_i}| > 0\}}$ such that $Q_r^-(Z) \subseteq Q_1^-$. Comparing with (4.14) for $r > 1/4$, we deduce that $\mathbf{u}_{\tau_i, X_i} \equiv 0$ in $\{(x, t) \in Q_{1/2}^- : x_1 \leq -1/4\}$. Therefore, the coincidence set $\{|\mathbf{u}^*| = 0\}$ has a nonempty interior and there exists cube $Q \subseteq \{|\mathbf{u}^*| = 0\}$ and $Y_0 \in \partial Q \cap \partial\{|\mathbf{u}^*| > 0\}$. According to Theorem 4.8 and Theorem 4.9, Y_0 is a regular point for $\mathbf{u}^*$ provided its energy is not larger than A_q. To see this, we fix $r \leq 1$ and consider the energy value

$$\begin{aligned} \mathbb{W}(\mathbf{u}^*, 0+; Y_0) \leq \mathbb{W}(\mathbf{u}^*, r; Y_0) &= \lim_{i\to\infty} \mathbb{W}(\mathbf{u}\eta, r\tau_i; X_i + \tau_i Y_0) \\ &\leq \lim_{i\to\infty} \mathbb{W}(\mathbf{u}\eta, \rho; X_i + \tau_i Y_0) + F(\rho) \\ &= \mathbb{W}(\mathbf{u}\eta, \rho; X_0) + F(\rho), \end{aligned}$$

where $\rho > 0$ is an arbitrary constant. Then $\mathbb{W}(\mathbf{u}^*, 0+; Y_0) \leq \mathbb{W}(\mathbf{u}^*, r; Y_0) \leq \mathbb{W}(\mathbf{u}\eta, 0+; X_0) = A_q$ and Y_0 is a regular point. So, $\mathbb{W}(\mathbf{u}^*, 0+; Y_0) = A_q$, and $\mathbb{W}(\mathbf{u}^*, 0+; Y_0) = \mathbb{W}(\mathbf{u}^*, r; Y_0) = A_q$ for every $r \leq 1$. Therefore, $\mathbf{u}^*$ is self-similar in Q_1^- with respect to the point $Y_0 = (y_0, s_0)$. Now apply again Theorem 4.8 and Theorem 4.9 to find out $\mathbf{u}^*$ is a half-space solution with respect to Y_0, say $\mathbf{u}^*(x, t) = \alpha((x^1 - y_0^1)_+)^\kappa$ for $t \leq s_0$. The uniqueness of solution of (1.1) (Lemma 7.1) yields that the equality holds for $t \leq 0$. Notice that $|\mathbf{u}^*(0, 0)| = 0$ since $X_i \in \Gamma(\mathbf{u})$. Therefore, $y_0^1 = 0$ and $\mathbf{u}^*(x, t) = \alpha(x_+^1)^\kappa$, which contradicts (4.13). □

5 Hölder regularity of $\partial_t \mathbf{u}$

Our way of approach, as mentioned in the introduction, is to use elliptic regularity theory for the free boundary problems. This approach is based on using the epiperimetric inequality for the elliptic systems as done in [1, 9]. The reduction of parabolic problem to the elliptic case was successfully used in [12]. The idea is that near regular points of the free boundary, where the blow-up regime is half-space, the time derivative of the solutions vanishes faster than the order of scaling which is $\kappa = 2/(1 - q)$. This enables us to apply the epiperimetric inequality.

Our strategy is to prove that $\partial_t \mathbf{u}$ is subcaloric and vanishes continuously on the free boundary (when $q = 0$). So we can deduce the Hölder regularity for it. This method needs a modification for $q > 0$. We start by following lemma which is essential in the case $q > 0$.

[5] The distance ranges between almost zero to infinity, depending on τ_i.

Lemma 5.1 *Let $(x_0, t_0) \in \Gamma^\kappa$ be a regular free boundary point of* $\mathbf{u}$. *Then for every $\epsilon > 0$, there exists $r_0 > 0$ such that*

$$|\mathbf{u}|^2|\nabla\mathbf{u}|^2 \le (1+\epsilon)\left|\sum_{k=1}^{m} u^k\nabla u^k\right|^2, \quad \text{in } Q_{r_0}(x_0,t_0). \tag{5.1}$$

Proof By contradiction consider the sequence $(x_j, t_j) \to (x_0, t_0)$ at which inequality (5.1) does not hold. Let $d_j := \sup\{r : Q_r^-(x_j,t_j) \subset \{|\mathbf{u}| > 0\}\}$ and $(y_j, s_j) \in \partial_p Q_{d_j}^-(x_j,t_j) \cap \Gamma(\mathbf{u})$. According to the openness of regular points, see Proposition 4.10, we imply that (y_j, s_j) are regular points of free boundary. Now, employing the growth estimates of solutions near Γ^κ, Theorem 3.3 (as well as Remark 3.4), and possibly passing to a subsequence, we may assume that

$$\frac{\mathbf{u}(d_jx + x_j, d_j^2t + t_j)}{d_j^\kappa} := \mathbf{u}_j(x,t) \to \mathbf{u}_0(x,t),$$

and

$$((y_j - x_j)/d_j, (s_j - t_j)/d_j^2) := (\tilde{y}_j, \tilde{s}_j) \to (\tilde{y}_0, \tilde{s}_0) \in \partial_p Q_1^-.$$

Therefore, inequality (5.1) can not be true for $\mathbf{u}_j$ at point $(0, 0)$. We will show that $\mathbf{u}_0$ is a half-space solution with respect to $(\tilde{y}_0, \tilde{s}_0)$, i.e.

$$\mathbf{u}_0(x,t) = \alpha((x-\tilde{y}_0)\cdot\nu)_+^\kappa\mathbf{e}, \text{ in } \mathbb{R}^n \times (-\infty, \tilde{s}_0], \tag{5.2}$$

for some unite vectors $\nu \in \mathbb{R}^n$, $\mathbf{e} \in \mathbb{R}^m$. By the uniqueness of forward problem, Lemma 7.1, the representation (5.2) is valid for $t \in (-\infty, 0]$ and $\mathbf{u}_0$ must satisfy the equality $|\mathbf{u}_0|^2|\nabla\mathbf{u}_0|^2 = \left|\sum_{k=1}^m u_0^k\nabla u_0^k\right|^2$. The contradiction proves the lemma.

In order to show that $\mathbf{u}_0$ is a half-space solution, let $\varrho > 0$ be a small number, such that $Q_\varrho(x_0,t_0) \cap \Gamma$ consists only of regular points; see Proposition 4.10. For every $r > 0$ denote by

$$w_r(X) := \mathbb{W}(\mathbf{u}\eta, r; X).$$

Then w_r is continuous, and has a pointwise limit, as $r \to 0$. Since $Q_\varrho(x_0,t_0) \cap \Gamma$ consists only of regular points, then

$$\lim_{r\to 0+} w_r(x,t) = A_q, \quad \text{for } (x,t) \in Q_\varrho(x_0,t_0) \cap \Gamma. \tag{5.3}$$

Furthermore, by monotonicity formula $w_r(x,t) + F(r)$ is a nondecreasing function in r, hence by Dini's monotone convergence theorem, the convergence in (5.3) is uniform. Thus

$$\mathbb{W}(\mathbf{u}_0, r; \tilde{y}_0, \tilde{s}_0) = \lim_{j\to\infty}\mathbb{W}(\mathbf{u}_j\eta, r; \tilde{y}_j, \tilde{s}_j) = \lim_{j\to\infty}\mathbb{W}(\mathbf{u}\eta, d_jr; y_j, s_j) = A_q,$$

for any $r > 0$. Hence $\mathbf{u}_0$ is backward self-similar with respect to $(\tilde{y}_0, \tilde{s}_0)$. To finish the argument, note that $(\tilde{y}_j, \tilde{s}_j)$ is a regular point of $\mathbf{u}_j$ and consider the convergence $\mathbf{u}_j \to \mathbf{u}_0$ in Q_2^-. Then the interior of $\{\mathbf{u}_0 = 0\}$ is not empty and by Theorem 4.9 we infer that $\mathbf{u}_0$ must be a half-space solution with respect to the point $(\tilde{y}_0, \tilde{s}_0)$. □

Lemma 5.2 *Let*

$$g(x,t) := |\partial_t\mathbf{u}(x,t)|^2|\mathbf{u}(x,t)|^{-2q},$$

(i) If $q = 0$, then $g(x,t) = |\partial_t\mathbf{u}(x,t)|^2$ is a subcaloric function in the set $\{|\mathbf{u}| > 0\}$.
(ii) If $0 < q < 1$ and $(x_0,t_0) \in \Gamma^\kappa$ is a regular free boundary point. Then there exists $0 < r_0$ and $\theta \ge 2$ such that g^θ is a subcaloric function in the set $\{|\mathbf{u}| > 0\} \cap Q_{r_0}(x_0,t_0)$.

Proof (i) By direct calculations;

$$\Delta|\partial_t \mathbf{u}|^2 = \sum_{k=1}^m \Delta(\partial_t u^k)^2 = 2\sum_{k=1}^m \partial_t u^k \Delta\partial_t u^k + 2\sum_{k=1}^m |\nabla\partial_t u^k|^2$$

and

$$\partial_t|\partial_t \mathbf{u}|^2 = 2\sum_{k=1}^m \partial_t u^k \partial_{tt}^2 u^k.$$

Hence calculating and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} H(|\partial_t \mathbf{u}|^2) &= 2\sum_{k=1}^m \partial_t u^k H(\partial_t u^k) + 2\sum_{k=1}^m |\nabla\partial_t u^k|^2 \\ &= 2\sum_{k=1}^m \partial_t u^k \frac{\partial}{\partial t}\left(\frac{u^k}{|\mathbf{u}|^{1-q}}\right) + 2\sum_{k=1}^m |\nabla\partial_t u^k|^2 \\ &= 2\sum_{k=1}^m \left(\frac{(\partial_t u^k)^2}{|\mathbf{u}|^{1-q}} - (1-q)\frac{u^k\partial_t u^k \sum_{j=1}^m u^j\partial_t u^j}{|\mathbf{u}|^{3-q}}\right) + 2\sum_{k=1}^m |\nabla\partial_t u^k|^2 \\ &= \frac{2}{|\mathbf{u}|^{3-q}}\left(|\mathbf{u}|^2|\partial_t\mathbf{u}|^2 - (1-q)(\mathbf{u}\cdot\partial_t\mathbf{u})^2\right) + 2\sum_{k=1}^m |\nabla\partial_t u^k|^2 \geq 0. \end{aligned}$$

(ii) Since $H(g^\theta) = \theta g^{\theta-2}(gHg + (\theta-1)|\nabla g|^2)$, it is enough to show that

$$gHg + (\theta-1)|\nabla g|^2 \geq 0. \tag{5.4}$$

Note that this relation is valid for $\theta \geq 2$ in $\{|\mathbf{u}| > 0\}$ regardless of whether $\partial_t\mathbf{u}$ vanishes or not. We can write

$$Hg = |\mathbf{u}|^{-2q}H(|\partial_t\mathbf{u}|^2) + |\partial_t\mathbf{u}|^2 H(|\mathbf{u}|^{-2q}) + 2\nabla(|\partial_t\mathbf{u}|^2)\cdot\nabla(|\mathbf{u}|^{-2q}).$$

From part (i), we know that

$$H(|\partial_t\mathbf{u}|^2) = \frac{2}{|\mathbf{u}|^{3-q}}\left(|\mathbf{u}|^2|\partial_t\mathbf{u}|^2 - (1-q)(\mathbf{u}\cdot\partial_t\mathbf{u})^2\right) + 2\sum_{k=1}^m |\nabla\partial_t u^k|^2,$$

and by a direct calculation we obtain,

$$H(|\mathbf{u}|^{-2q}) = -2q|\mathbf{u}|^{-q-1} - 2q|\mathbf{u}|^{-2q-2}|\nabla\mathbf{u}|^2 + 4q(1+q)|\mathbf{u}|^{-2q-4}\left|\sum_{k=1}^m u^k\nabla u^k\right|^2.$$

Then

$$\begin{aligned} \frac{1}{2}gHg =& (1-q)|\partial_t\mathbf{u}|^2|\mathbf{u}|^{-3q-3}\left(|\mathbf{u}|^2|\partial_t\mathbf{u}|^2 - (\mathbf{u}\cdot\partial_t\mathbf{u})^2\right) - q|\partial_t\mathbf{u}|^4|\mathbf{u}|^{-4q-2}|\nabla\mathbf{u}|^2 \\ &+ |\partial_t\mathbf{u}|^2|\mathbf{u}|^{-4q}|\nabla\partial_t\mathbf{u}|^2 + 2q(1+q)|\partial_t\mathbf{u}|^4|\mathbf{u}|^{-4q-4}\left|\sum_{k=1}^m u^k\nabla u^k\right|^2 \\ &- 4q|\partial_t\mathbf{u}|^2|\mathbf{u}|^{-4q-2}\left(\sum_{k=1}^m \partial_t u^k\nabla\partial_t u^k\right)\cdot\left(\sum_{k=1}^m u^k\nabla u^k\right). \end{aligned}$$

According to Lemma 5.1, we can assume that

$$|\mathbf{u}|^2|\nabla \mathbf{u}|^2 \le (1+\epsilon)\left|\sum_{k=1}^m u^k \nabla u^k\right|^2,$$

in a neighborhood of (x_0, t_0) for some $\epsilon > 0$ which is determined later. Therefore, in order to prove (5.4) in this neighborhood we have

$$\begin{aligned}\frac{1}{2} g H g \ge & |\mathbf{u}|^{-4q}\left|\sum_{k=1}^m \partial_t u^k \nabla \partial_t u^k\right|^2 + q(2q+1-\epsilon)|\partial_t \mathbf{u}|^4 |\mathbf{u}|^{-4q-4}\left|\sum_{k=1}^m u^k \nabla u^k\right|^2 \\ & - 4q|\partial_t \mathbf{u}|^2|\mathbf{u}|^{-4q-2}\left(\sum_{k=1}^m \partial_t u^k \nabla \partial_t u^k\right)\cdot\left(\sum_{k=1}^m u^k\nabla u^k\right) \\ \ge & -2(\theta-1)\left||\mathbf{u}|^{-2q}\left(\sum_{k=1}^m \partial_t u^k \nabla \partial_t u^k\right) - q|\partial_t \mathbf{u}|^2 |\mathbf{u}|^{-2q-2}\left(\sum_{k=1}^m u^k \nabla u^k\right)\right|^2 \\ = & -\frac{1}{2}(\theta-1)|\nabla g|^2\end{aligned}$$

where the last inequality holds when $2\theta q \le (2\theta-1)(1-\epsilon)$. We can choose suitable $\epsilon > 0$ provided $2\theta > \frac{1}{1-q}$. □

Now we prove that the time derivative vanishes continuously on the regular part of the free boundary.

Lemma 5.3 *Let g be the function defined in Lemma 5.2 and suppose $(x_0, t_0) \in \Gamma^\kappa$ is a regular free boundary point, then*

$$\lim_{(x,t)\to(x_0,t_0)} g(x,t) = 0.$$

Proof Let $(x_j, t_j) \to (x_0, t_0)$ be a maximizing sequence in the sense that

$$\lim_{j\to+\infty} g(x_j, t_j) = \limsup_{(x,t)\to(x_0,t_0)} g(x,t) := m^2 > 0.$$

Let $d_j := \sup\{r : Q_r^-(x_j, t_j) \subset \{|\mathbf{u}| > 0\}\}$ and $(y_j, s_j) \in \partial_p Q_{d_j}^-(x_j, t_j) \cap \Gamma$. Following the same lines of proof as that of Lemma 5.1, we may assume

$$\frac{\mathbf{u}(d_j x + x_j, d_j^2 t + t_j)}{d_j^\kappa} := \mathbf{u}_j(x,t) \to \mathbf{u}_0(x,t),$$
$$((y_j - x_j)/d_j, (s_j - t_j)/d_j^2) := (\tilde{y}_j, \tilde{s}_j) \to (\tilde{y}_0, \tilde{s}_0) \in \partial_p Q_1^-,$$

and

$$\mathbf{u}_0(x,t) = \alpha((x-\tilde{y}_0)\cdot\nu)_+^\kappa \mathbf{e}, \text{ in } \mathbb{R}^n \times (-\infty, 0]. \tag{5.5}$$

Since $Q_1^- \subset \{|\mathbf{u}_j| > 0\}$, then $Q_1^- \subset \{|\mathbf{u}_0| > 0\}$, and the convergence is uniform in Q_1^-. Hence

$$\begin{aligned}|\partial_t \mathbf{u}_0(0,0)|\,|\mathbf{u}_0(0,0)|^{-q} &= \lim_{j\to\infty} |\partial_t \mathbf{u}_j(0,0)|\,|\mathbf{u}_j(0,0)|^{-q} \\ &= \lim_{j\to\infty} |\partial_t \mathbf{u}(x_j,t_j)|\,|\mathbf{u}(x_j,t_j)|^{-q} = m,\end{aligned}$$

and for all $(x,t) \in Q_1^-$,

$$|\partial_t \mathbf{u}_0(x,t)|\,|\mathbf{u}_0(x,t)|^{-q} = \lim_{j\to\infty} |\partial_t \mathbf{u}_j(x,t)|\,|\mathbf{u}_j(x,t)|^{-q}$$
$$= \lim_{j\to\infty} |\partial_t \mathbf{u}(d_j x + x_j, d_j^2 t + t_j)|\,|\mathbf{u}(d_j x + x_j, d_j^2 t + t_j)|^{-q} \le m.$$

Since $|\partial_t \mathbf{u}|^2$ is subcaloric for $q = 0$ or g^θ for $q > 0$ (Lemma 5.2), we can apply the maximum principle to arrive at $|\partial_t \mathbf{u}_0(x,t)| = m|\mathbf{u}_0(x,t)|^q$ in the connected component of Q_1^-, containing the origin, which contradicts (5.5). □

Now using a standard iterative argument one can prove the Hölder regularity of the time derivative.

Lemma 5.4 *Let g be the function defined in Lemma 5.2 and suppose $(x_0,t_0) \in \Gamma$ is a regular free boundary point. Then g is a Hölder continuous function in a neighbourhood of (x_0,t_0).*

Proof Lemma 5.2 and Lemma 5.3 together imply that g (or g^θ for $0 < q < 1$) is a continuous subcaloric function in a neighbourhood of regular points (we extend g to zero in $\{\mathbf{u} = 0\}$). Since the coincidence set $\{\mathbf{u} = 0\}$ close to regular points are uniformly large, we may invoke Lemma A4 in [2], which states that if $h \le M$ in $Q_1 := B_1 \times (0,1)$ is a continuous subcaloric function and

$$\frac{|Q_1 \cap \{h < M/2\}|}{|Q_1|} > \lambda > 0,$$

then there exists $0 < \gamma = \gamma(\lambda) < 1$ such that

$$h(0,1) < \gamma M.$$

Since g is continuous subcaloric, we obtain that

$$\sup_{Q_{r/2}} g(x,t) \le \gamma \sup_{Q_r} g(x,t).$$

Fix $(x,t) \in Q_{r_0/2}(x_0,t_0)$, then there exists $k \ge 1$ such that $2^{-k-1}r_0 < |x| \le 2^{-k}r_0$, and

$$g(x,t) \le \gamma \sup_{Q_{2^{-k+1}r_0}} g(x,t) \le \gamma^k \sup_{Q_{r_0}} g(x,t) \le \frac{1}{\gamma}\left(\frac{|x|}{r_0}\right)^{-\frac{\ln\gamma}{\ln 2}} \sup_{Q_{r_0}} g(x,t).$$

Hence the function g is Hölder continuous with the exponent $\beta = -\frac{\ln\gamma}{2\ln 2}$. □

Corollary 5.5 *Let $\mathbf{u}$ be a solution to (1.1) and suppose that $(x_0,t_0) \in \Gamma^\kappa(\mathbf{u})$ is a regular point, then there exists constants C, $0 < r_0 < 1$ and $0 < \beta < 1$ such that*

$$\sup_{Q_r^-(x_0,t_0)} |\partial_t \mathbf{u}| \le C r^{\kappa-2+\beta}, \qquad \forall\, 0 < r < r_0.$$

6 Regularity of the free boundary

We consider the following local (fixed time) version of balanced energy;

$$W_{t_0}(\mathbf{u}, r, x_0) := \frac{1}{r^{n+2\kappa-2}} \int_{B_r(x_0)} |\nabla \mathbf{u}(x, t_0)|^2 + \frac{2}{1+q} |\mathbf{u}(x, t_0)|^{1+q} dx - \frac{\kappa}{r^{n+2\kappa-1}} \int_{\partial B_r(x_0)} |\mathbf{u}(x, t_0)|^2 d\mathcal{H}^{n-1}.$$

Proposition 6.1 *Let $(x_0, t_0) \in \Gamma^\kappa$ be a regular free boundary point, then there exist constants $C > 0$ and $0 < \beta < 1$, such that*

$$\left| W_{t_0}(\mathbf{u}, r_2, x_0) - W_{t_0}(\mathbf{u}, r_1, x_0) - 2 \int_{r_1}^{r_2} r \int_{\partial B_1(0)} \left| \frac{d}{dr} \mathbf{u}_{r,t_0} \right|^2 d\mathcal{H}^{n-1} dr \right| \le C |r_2^\beta - r_1^\beta|.$$

Proof Let us denote by $\mathbf{u}_{r,t_0} := \frac{\mathbf{u}(rx+x_0,t_0)}{r^\kappa}$, then

$$W_{t_0}(\mathbf{u}, r, x_0) = \int_{B_1(0)} |\nabla \mathbf{u}_{r,t_0}|^2 + \frac{2}{1+q} |\mathbf{u}_{r,t_0}|^{1+q} dx - \kappa \int_{\partial B_1(0)} |\mathbf{u}_{r,t_0}|^2 d\mathcal{H}^{n-1}.$$

Hence

$$\begin{aligned} \frac{d}{dr} W_{t_0}(\mathbf{u}, r, x_0) &= 2 \int_{B_1(0)} \nabla \mathbf{u}_{r,t_0} \nabla \frac{d}{dr} \mathbf{u}_{r,t_0} + \frac{\mathbf{u}_{r,t_0}}{|\mathbf{u}_{r,t_0}|^{1-q}} \frac{d}{dr} \mathbf{u}_{r,t_0} dx \\ &\quad - 2\kappa \int_{\partial B_1(0)} \mathbf{u}_{r,t_0} \frac{d}{dr} \mathbf{u}_{r,t_0} d\mathcal{H}^{n-1} \\ &= 2 \int_{B_1(0)} -\partial_t \mathbf{u}(rx + x_0, t_0) \frac{d}{dr} \mathbf{u}_{r,t_0} dx + 2r \int_{\partial B_1(0)} \left| \frac{d}{dr} \mathbf{u}_{r,t_0} \right|^2 d\mathcal{H}^{n-1}. \end{aligned} \tag{6.1}$$

Letting $r_2 > r_1 > 0$ and integrating (6.1) in the interval (r_1, r_2), we obtain

$$\begin{aligned} &\left| W_{t_0}(\mathbf{u}, r_2, x_0) - W_{t_0}(\mathbf{u}, r_1, x_0) - 2 \int_{r_1}^{r_2} r \int_{\partial B_1(0)} \left| \frac{d}{dr} \mathbf{u}_{r,t_0} \right|^2 d\mathcal{H}^{n-1} dr \right| \\ &\le 2 \int_{r_1}^{r_2} \int_{B_1(0)} |\partial_t \mathbf{u}(rx + x_0, t_0)| \left| \frac{d}{dr} \mathbf{u}_{r,t_0} \right| dx dr \\ &\le C \int_{r_1}^{r_2} \int_{B_1(0)} \frac{|\partial_t \mathbf{u}(rx + x_0, t_0)|}{r} dx dr \\ &\le C_1 \int_{r_1}^{r_2} r^{\beta-1} dr = \frac{C_1}{\beta} (r_2^\beta - r_1^\beta). \end{aligned}$$

□

The following epiperimetric inequality from [1] and [9] will be used to treat the parabolic system.

Theorem 6.2 *(Epiperimetric inequality) There exists* $\varepsilon \in (0,1)$ *and* $\delta > 0$ *such that if* $\mathbf{c} = \mathbf{c}(x)$ *is a backward self-similar function of degree* κ *satisfying*

$$||\mathbf{c} - \mathbf{h}||_{W^{1,2}(B_1,\mathbb{R}^m)} + ||\mathbf{c} - \mathbf{h}||_{L^\infty(B_1,\mathbb{R}^m)} \le \delta, \text{ for some } \boldsymbol{h} \in \mathbb{H},$$

then there exists $\mathbf{v} \in W^{1,2}(B_1; \mathbb{R}^m)$ *such that* $\mathbf{v} = \mathbf{c}$ *on* ∂B_1 *and*

$$M(\mathbf{v}) - M(\mathbf{h}) \le (1-\varepsilon)\left(M(\mathbf{c}) - M(\mathbf{h})\right),$$

where

$$M(\mathbf{v}) := \int_{B_1} (|\nabla \mathbf{v}|^2 + \frac{2}{1+q}|\mathbf{v}|^{1+q})dx - \kappa \int_{\partial B_1} |\mathbf{v}|^2 d\mathcal{H}^{n-1}.$$

Proposition 6.3 *(Energy decay, uniqueness of blow-up limits) Let* $(x_0, t_0) \in \Gamma$ *be a regular point, and* $\mathbf{u}_0$ *be any blow-up of* $\mathbf{u}$ *at* (x_0, t_0)*. Suppose that the epiperimetric inequality holds with* $0 < \varepsilon < 1$ *for each*

$$\mathbf{c}_r(x, t_0) := |x|^\kappa \mathbf{u}_{r,t_0}(x/|x|, t_0) = \frac{|x|^\kappa}{r^\kappa} \mathbf{u}(x_0 + rx/|x|, t_0)$$

and for all $r \le r_0$*. Then there exists* $C > 0$ *and* $0 < \gamma < 1$ *such that*

$$\left|W_{t_0}(\mathbf{u}, r, x_0) - W_{t_0}(\mathbf{u}, 0+, x_0)\right| \le Cr^\gamma, \text{ for small } r > 0, \tag{6.2}$$

and

$$\int_{\partial B_1(0)} \left|\frac{\mathbf{u}(rx + x_0, t_0)}{r^\kappa} - \mathbf{u}_0(x)\right| d\mathcal{H}^{n-1} \le Cr^{\gamma/2}, \tag{6.3}$$

therefore $\mathbf{u}_0$ *is the unique blow-up limit of* $\mathbf{u}$ *at the point* (x_0, t_0)*.*

Proof Let $e(r) := W_{t_0}(\mathbf{u}, r, x_0) - W_{t_0}(\mathbf{u}, 0+, x_0)$, then

$$
\begin{aligned}
e'(r) = &-\frac{n+2\kappa-2}{r^{n+2\kappa-1}} \int\limits_{B_r(x_0)} |\nabla \mathbf{u}(x,t_0)|^2 + \frac{2}{1+q}|\mathbf{u}(x,t_0)|^{1+q} dx \\
&+ \frac{\kappa(n+2\kappa-1)}{r^{n+2\kappa}} \int\limits_{\partial B_r(x_0)} |\mathbf{u}(x,t_0)|^2 d\mathcal{H}^{n-1} \\
&+ \frac{1}{r^{n+2\kappa-2}} \int\limits_{\partial B_r(x_0)} |\nabla \mathbf{u}(x,t_0)|^2 + 2|\mathbf{u}(x,t_0)| d\mathcal{H}^{n-1} \\
&- \frac{\kappa}{r^{n+2\kappa-1}} \left(2 \int\limits_{\partial B_r(x_0)} (\nabla \mathbf{u}(x,t_0)\cdot \nu)\cdot \mathbf{u}(x,t_0) d\mathcal{H}^{n-1} \right. \\
&\left. + \frac{n-1}{r} \int\limits_{\partial B_r(x_0)} |\mathbf{u}(x,t_0)|^2 d\mathcal{H}^{n-1} \right) \\
= &-\frac{n+2\kappa-2}{r}\big(e(r) + W_{t_0}(\mathbf{u},0+,x_0)\big) \\
&+ \frac{1}{r} \int\limits_{\partial B_1(x_0)} |\nabla \mathbf{u}_r(x,t_0)|^2 + \frac{2}{1+q}|\mathbf{u}_r(x,t_0)|^{1+q} d\mathcal{H}^{n-1} \\
&- \frac{2\kappa}{r} \int\limits_{\partial B_1(x_0)} (\nabla \mathbf{u}_r(x,t_0)\cdot \nu)\cdot \mathbf{u}_r(x,t_0) d\mathcal{H}^{n-1} \\
&- \frac{\kappa(n-2)}{r} \int\limits_{\partial B_1(x_0)} |\mathbf{u}_r(x,t_0)|^2 d\mathcal{H}^{n-1} \\
\geq &-\frac{n+2\kappa-2}{r}\big(e(r) + W_{t_0}(\mathbf{u},0+)\big) \\
&+ \frac{1}{r} \int\limits_{\partial B_1(0)} \left(|\nabla_\theta \mathbf{u}_r|^2 + \frac{2}{1+q}|\mathbf{u}_r|^{1+q} - (\kappa(n-2)+\kappa^2)|\mathbf{u}_r|^2 \right) d\mathcal{H}^{n-1} \\
= &-\frac{n+2\kappa-2}{r}\big(e(r) + W_{t_0}(\mathbf{u},0+)\big) \\
&+ \frac{1}{r} \int\limits_{\partial B_1(0)} \left(|\nabla_\theta \mathbf{c}_r|^2 + \frac{2}{1+q}|\mathbf{c}_r|^{1+q} - (\kappa(n-2)+\kappa^2)|\mathbf{c}_r|^2 \right) d\mathcal{H}^{n-1} \\
= &\frac{n+2\kappa-2}{r}\big(M(\mathbf{c}_r) - e(r) - W_{t_0}(\mathbf{u},0+,x_0)\big) \\
\geq &\frac{n+2\kappa-2}{r}\left(\frac{M(\mathbf{v}) - W_{t_0}(\mathbf{u},0+,x_0)}{1-\varepsilon} - e(r) \right),
\end{aligned}
$$

where we employed the epiperimetric inequality in the last step. Now let us observe that $\mathbf{u}_{r,t_0}$ minimises the following energy

$$
J(\mathbf{v}) := \int\limits_{B_1(0)} |\nabla \mathbf{v}|^2 + \frac{2}{1+q}|\mathbf{v}|^{1+q} dx + 2r^{2-\kappa} \int\limits_{B_1(0)} \mathbf{v}\cdot \partial_t \mathbf{u}(x_0+rx,t_0) dx,
$$

where $\mathbf{v} = \mathbf{u}_{r,t_0}$ on $\partial B_1(0)$. Hence

$$\begin{aligned} M(\mathbf{v}) &= \int_{B_1} |\nabla \mathbf{v}|^2 + \frac{2}{1+q}|\mathbf{v}|^{1+q} dx - \kappa \int_{\partial B_1} |\mathbf{v}|^2 d\mathcal{H}^{n-1} \\ &= J(\mathbf{v}) - 2r^{2-\kappa} \int_{B_1} \mathbf{v} \cdot \partial_t \mathbf{u}(x_0 + rx, t_0) dx - \kappa \int_{\partial B_1} |\mathbf{u}_{r,t_0}|^2 d\mathcal{H}^{n-1} \\ &\geq M(\mathbf{u}_{r,t_0}) + 2r^{2-\kappa} \int_{B_1} (\mathbf{u}_{r,t_0} - \mathbf{v}) \cdot \partial_t \mathbf{u}(x_0 + rx, t_0) dx. \end{aligned}$$

Now we may conclude that

$$\begin{aligned} e'(r) \geq & \frac{n+2\kappa-2}{r(1-\varepsilon)} \Big(M(\mathbf{u}_{r,t_0}) - W_{t_0}(\mathbf{u}, 0+, x_0) + 2r^{2-\kappa} \\ & \times \int_{B_1} (\mathbf{u}_{r,t_0} - \mathbf{v}) \cdot \partial_t \mathbf{u}(x_0 + rx, t_0) dx \Big) \\ & - \frac{(n+2\kappa-2)e(r)}{r} = \frac{(n+2\kappa-2)e(r)}{r(1-\varepsilon)} - \frac{(n+2\kappa-2)e(r)}{r} \\ & + 2\frac{(n+2\kappa-2)}{(1-\varepsilon)} r^{1-\kappa} \int_{B_1} (\mathbf{u}_{r,t_0} - \mathbf{v}) \cdot \partial_t \mathbf{u}(x_0 + rx, t_0) dx \\ \geq & \frac{\varepsilon(n+2\kappa-2)e(r)}{r(1-\varepsilon)} - Cr^{\beta-1}, \end{aligned}$$

by Corollary 5.5. It follows from (6) that

$$\frac{d}{dr}\left(e(r) r^{-\frac{\varepsilon(n+2\kappa-2)}{1-\varepsilon}}\right) \geq -Cr^{\beta-1-\frac{\varepsilon(n+2\kappa-2)}{1-\varepsilon}}.$$

Integrating the last inequality from r to 1, we obtain

$$e(1) - e(r) r^{-\frac{\varepsilon(n+2\kappa-2)}{1-\varepsilon}} \geq -\frac{C}{\beta - \frac{\varepsilon(n+2\kappa-2)}{1-\varepsilon}} \left(1 - r^{\beta - \frac{\varepsilon(n+2\kappa-2)}{1-\varepsilon}}\right),$$

and therefore

$$e(r) \leq e(1) r^{\frac{\varepsilon(n+2\kappa-2)}{1-\varepsilon}} - \frac{C}{\beta - \frac{\varepsilon(n+2\kappa-2)}{1-\varepsilon}} \left(r^{\frac{\varepsilon(n+2\kappa-2)}{1-\varepsilon}} - r^{\beta}\right) \leq C_0 r^{\gamma},$$

where $\gamma := \min\left(\beta, \frac{\varepsilon(n+2\kappa-2)}{1-\varepsilon}\right)$, and $C_0 > 0$ depends only on the given parameters. The proof of (6.2) is now complete, and we proceed to the proof of (6.3).

Let $2^{-l} < r_1 \le 2^{-l+1} \le 2^{-k} < r_2 < 2^{-k+1}$, where $k, l \in \mathbb{N}$. It is easy to see that

$$\int_{\partial B_1(0)} \left| \frac{\mathbf{u}(x_0 + r_1 x, t_0)}{r_1^\kappa} - \frac{\mathbf{u}(x_0 + r_2 x, t_0)}{r_2^\kappa} \right| d\mathcal{H}^{n-1} \le \int_{\partial B_1(0)} \int_{r_1}^{r_2} \left| \frac{d}{dr} \mathbf{u}_{r,t_0} \right| dr d\mathcal{H}^{n-1}$$

$$\le \sum_{j=l}^{k} \int_{\partial B_1(0)} \int_{2^{-j}}^{2^{-j+1}} \left| \frac{d}{dr} \mathbf{u}_{r,t_0} \right| dr d\mathcal{H}^{n-1}$$

$$\le C_n \sum_{j=l}^{k} \left(\int_{\partial B_1(0)} \int_{2^{-j}}^{2^{-j+1}} r \left| \frac{d}{dr} \mathbf{u}_{r,t_0} \right|^2 dr d\mathcal{H}^{n-1} \right)^{\frac{1}{2}}$$

By Proposition 6.1 and relation (6.2), we can estimate

$$\int_{\partial B_1(0)} \int_{2^{-j}}^{2^{-j+1}} r \left| \frac{d}{dr} \mathbf{u}_{r,t_0} \right|^2 dr d\mathcal{H}^{n-1} \le C(2^{(-j+1)\gamma} - 2^{-j\gamma}) \le C 2^{-\gamma j}.$$

Hence

$$\int_{\partial B_1(0)} \left| \frac{\mathbf{u}(x_0 + r_1 x, t_0)}{r_1^\kappa} - \frac{\mathbf{u}(x_0 + r_2 x, t_0)}{r_2^\kappa} \right| d\mathcal{H}^{n-1} \le C \sum_{j=l}^{k} 2^{-\gamma j/2}$$

$$= C \frac{2^{-\gamma l/2} - 2^{-\gamma(k+1)/2}}{1 - 2^{-\gamma/2}} \le \frac{C}{2^{\gamma/2} - 1} (r_2^{\gamma/2} - r_1^{\gamma/2}),$$

and (6.3) follows. □

The following theorem has been proved as Theorem 4.7 in [9] (for $q > 0$) and Theorem 4 in [1] (for $q = 0$).

Theorem 6.4 *Let C_h be a compact set of points $x_0 \in \Gamma_{t_0}^\kappa$ with the following property: at least one blow-up limit $\mathbf{u}_0$ of $\mathbf{u}(rx + x_0, t_0)/r^\kappa$ is a half-space solution, say $\mathbf{u}_0(x) = \alpha \max(x \cdot \nu(x_0, t_0), 0)^\kappa \mathbf{e}(x_0, t_0)$ for some $\nu(x_0, t_0) \in \partial B_1 \subset \mathbb{R}^n$ and $\mathbf{e}(x_0, t_0) \in \partial B_1 \subset \mathbb{R}^m$. Then there exist r_0 and $C < \infty$ such that*

$$\int_{\partial B_1} \left| \frac{\mathbf{u}(rx + x_0, t_0)}{r^\kappa} - \alpha \max(x \cdot \nu(x_0, t_0), 0)^\kappa \mathbf{e}(x_0, t_0) \right| d\mathcal{H}^{n-1} \le C r^{\gamma/2},$$

for every $x_0 \in C_h$ and every $r \le r_0$.

Theorem 6.5 *In a neighbourhood of regular points the free boundary is $C^{1,\alpha}$ in space and $C^{0,1/2}$ in time.*

Proof First, consider the normal vectors $\nu(x_0, t_0)$ and $\mathbf{e}(x_0, t_0)$ defined in Theorem 6.4, we show that $(x_0, t_0) \mapsto \nu(x_0, t_0)$ and $(x_0, t_0) \mapsto \mathbf{e}(x_0, t_0)$ are Hölder continuous with exponent $\beta = \frac{\gamma}{\gamma + 2\kappa}$.

Therefore, it follows that for each time section the free boundary is $C^{1,\beta}$, provided the free boundary point is a regular point. This in turn implies that the free boundary is a graph in the time direction, close to such points. To see that the free boundary is half-Lipschitz in time, we may perform a blow-up at free boundaries, along with a contradiction argument. This is standard and left to the reader. □

7 Appendix

Lemma 7.1 *(Uniqueness of forward problem) Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be global solutions of* (1.1) *in* $\mathbb{R}^n \times (-\infty, t_0]$ *which have polynomial growth. If* $\mathbf{u}(\cdot, s) = \mathbf{v}(\cdot, s)$ *for some* $s < t_0$*, then* $\mathbf{u}(\cdot, t) = \mathbf{v}(\cdot, t)$ *for all* $s \le t \le t_0$.

Proof Multiply $H(\mathbf{u} - \mathbf{v}) = f(\mathbf{u}) - f(\mathbf{v})$ by $(\mathbf{u} - \mathbf{v})G$ and integrate

$$0 \le \int_s^\tau \int_{\mathbb{R}^n} (f(\mathbf{u}) - f(\mathbf{v})) \cdot (\mathbf{u} - \mathbf{v}) G dx dt = -\int_s^\tau \int_{\mathbb{R}^n} \Big[\frac{1}{2}\partial_t |\mathbf{u} - \mathbf{v}|^2 + |\nabla(\mathbf{u} - \mathbf{v})|^2 + \left(\nabla(\mathbf{u} - \mathbf{v}) \cdot \frac{x}{2t}\right) \cdot (\mathbf{u} - \mathbf{v})\Big] G dx dt.$$

Let $\mathbf{w} := \mathbf{u} - \mathbf{v}$, then

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^n} |\mathbf{w}(x,\tau)|^2 G(x,\tau) dx &\le \int_s^\tau \int_{\mathbb{R}^n} \frac{1}{2}|\mathbf{w}|^2 \partial_t G - \left[|\nabla \mathbf{w}|^2 + \left(\nabla \mathbf{w} \cdot \frac{x}{2t}\right) \cdot \mathbf{w}\right] G dx dt \\ &\le \int_s^\tau \int_{\mathbb{R}^n} \left[-\frac{|x|^2 + 2nt}{8t^2}|\mathbf{w}|^2 - |\nabla \mathbf{w}|^2 + |\nabla \mathbf{w}|\, |\frac{x}{2t}|\, |\mathbf{w}|\right] G dx dt \\ &\le \int_s^\tau \int_{\mathbb{R}^n} \frac{n}{-4t}|\mathbf{w}|^2 G dx dt =: \phi(\tau).\end{aligned}$$

Therefore, $-\frac{2\tau}{n}\phi'(\tau) \le \phi(\tau)$ and so $\frac{d}{d\tau}\left[(-\tau)^{n/2}\phi(\tau)\right] \le 0$ for $s < \tau < 0$. From $\phi(s) = 0$, we conclude that $\phi(\tau) \equiv 0$. □

Lemma 7.2 *Let* h *be a caloric function in* $\mathbb{R}^n \times (-4, 0]$. *Then for* $s < t \le 0$ *we have the following estimate*

$$e^{\frac{|x|^2}{t+s}} |h(x,t)|^2 \le \left(\frac{\sqrt{3}s}{s-t}\right)^n \int_{\mathbb{R}^n} |h(y,s)|^2 G(y,s) dy.$$

Proof By the representation of the caloric function, we have

$$h(x,t) = \int_{\mathbb{R}^n} h(y,s) G(x-y, s-t) dy.$$

Then

$$|h(x,t)|^2 \le \left(\int_{\mathbb{R}^n} |h(y,s)|^2 G(y,s) dy\right)\left(\int_{\mathbb{R}^n} \frac{(G(x-y,s-t))^2}{G(y,s)} dy\right) \tag{7.1}$$

On the other hand, we can write

$$\begin{aligned}\frac{(G(x-y,s-t))^2}{G(y,s)} &= \left(\frac{-s}{4\pi(t-s)^2}\right)^{n/2} \exp\left(-\frac{|x-y|^2}{2(t-s)} - \frac{|y|^2}{4s}\right) \\ &\le \left(\frac{-s}{4\pi(t-s)^2}\right)^{n/2} \exp\left(-\frac{|x|^2}{2(t-s)} + \frac{x \cdot y}{t-s} - \frac{(t+s)|y|^2}{4s(t-s)}\right)\end{aligned}$$

$$\leq \left(\frac{-s}{4\pi(t-s)^2}\right)^{n/2} \exp\left(\left(-\frac{1}{2}+\epsilon\right)\frac{|x|^2}{t-s} + \left(\frac{1}{4\epsilon} - \frac{t+s}{4s}\right)\frac{|y|^2}{t-s}\right).$$

For every $\epsilon > \frac{s}{t+s}$, we obtain that

$$\exp\left(\left(\frac{1}{2}-\epsilon\right)\frac{|x|^2}{t-s}\right)\int_{\mathbb{R}^n} \frac{(G(x-y,s-t))^2}{G(y,s)}dy$$
$$\leq \left(\frac{-s}{4\pi(t-s)^2}\right)^{n/2}\int_{\mathbb{R}^n} \exp\left(\left(\frac{1}{4\epsilon} - \frac{t+s}{4s}\right)\frac{|y|^2}{t-s}\right)dy$$
$$= \left(\frac{-s}{(t-s)\left(\frac{t+s}{s} - \frac{1}{\epsilon}\right)}\right)^{n/2}.$$

Now let $\epsilon = \frac{3s-t}{2(t+s)}$, so by (7.1) the proof will be done. □

Lemma 7.3 *Assume that* $\mathbf{w} \in L^2(Q_4)$ *has polynomial growth and* $t < 0$ *fixed, then*

$$\int_{\mathbb{R}^n} |\mathbf{w}(x,t)|^2 \frac{|x|^2}{-t} G(x,t)dx \leq 4\int_{\mathbb{R}^n} \left(n|\mathbf{w}(x,t)|^2 - 4t|\nabla \mathbf{w}(x,t)|^2\right) G(x,t)dx.$$

Proof Using the relation $\nabla G(x,t) = \frac{x}{2t}G(x,t)$ to obtain

$$\int_{\mathbb{R}^n} |\mathbf{w}(x,t)|^2 \frac{|x|^2}{-t} G(x,t)dx = -2\int_{\mathbb{R}^n} |\mathbf{w}|^2 (x\cdot\nabla G)dx = 2\int_{\mathbb{R}^n} \mathrm{div}(|\mathbf{w}|^2 x)Gdx$$
$$= 2n\int_{\mathbb{R}^n} |\mathbf{w}|^2 Gdx + 4\int_{\mathbb{R}^n} \mathbf{w}\cdot(\nabla\mathbf{w}\cdot x)Gdx$$
$$\leq 2n\int_{\mathbb{R}^n} |\mathbf{w}|^2 Gdx + \int_{\mathbb{R}^n} |\mathbf{w}|^2 \frac{|x|^2}{-2t}Gdx$$
$$+ \int_{\mathbb{R}^n} (-8t)|\nabla\mathbf{w}|^2 Gdx.$$

Now we can easily prove the lemma. □

Lemma 7.4 *Let* $\mathbf{u}$ *be a function defined in* $\mathbb{R}^n \times [-R, 0)$ *(for some* $a, R > 0$*) with polynomial growth, and* $\mathbf{p}$ *be a* κ*-backward self-similar caloric vector-function. Then for* $-R < t_1 < t_2 < 0$

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \left(|\nabla(\mathbf{p}-\mathbf{u})|^2 + \frac{\kappa|\mathbf{p}-\mathbf{u}|^2}{2t}\right)Gdxdt = \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \left(|\nabla\mathbf{u}|^2 + \frac{\kappa|\mathbf{u}|^2}{2t}\right)Gdxdt.$$

Proof Since $\nabla G(x,t) = \frac{x}{2t}G(x,t)$, we have

$$(\nabla\mathbf{u} : \nabla\mathbf{v})G = \nabla\mathbf{u} : \nabla(\mathbf{v}G) - (\nabla\mathbf{u}\cdot x)\cdot\frac{\mathbf{v}G}{2t}.$$

Obviously, $|\nabla(\mathbf{p}-\mathbf{u})|^2 = |\nabla\mathbf{u}|^2 - 2\nabla\mathbf{p} : \nabla\mathbf{u} + |\nabla\mathbf{p}|^2$, hence

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \left(|\nabla(\mathbf{p}-\mathbf{u})|^2 + \frac{\kappa|\mathbf{p}-\mathbf{u}|^2}{2t}\right) G dx dt \\
&= \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \left(|\nabla\mathbf{u}|^2 + \frac{\kappa|\mathbf{u}|^2}{2t}\right) G dx dt \\
&\quad + \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \nabla\mathbf{p} : (\nabla\mathbf{p} - 2\nabla\mathbf{u}) G dx dt \\
&\quad + \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \kappa \frac{\mathbf{p}\cdot(\mathbf{p}-2\mathbf{u})}{2t} G dx dt \\
&= \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \left(|\nabla\mathbf{u}|^2 + \frac{\kappa|\mathbf{u}|^2}{2t}\right) G dx dt \\
&\quad - \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \left(\Delta\mathbf{p} + \frac{1}{2t} x\cdot\nabla\mathbf{p} - \frac{\kappa\mathbf{p}}{2t}\right)\cdot(\mathbf{p}-2\mathbf{u}) G dx dt \\
&= \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \left(|\nabla\mathbf{u}|^2 + \frac{\kappa|\mathbf{u}|^2}{2t}\right) G dx dt
\end{aligned}$$

where we used integration by parts and that

$$\Delta\mathbf{p} + \frac{1}{2t} x\cdot\nabla\mathbf{p} - \frac{\kappa\mathbf{p}}{2t} = \partial_t\mathbf{p} + \frac{1}{2t} x\cdot\nabla\mathbf{p} - \frac{\kappa}{2}\frac{\mathbf{p}}{t} = \frac{1}{2t} L\mathbf{p} = 0. \qquad \square$$

The following lemma is an extension of Lemma 1.1 in [3] to the parabolic case.

Lemma 7.5 *Consider $\beta > 0$ to be noninteger, and let $u(x,t)$ be a function satisfying*

$$|H(u)(X)| \leq C_*|X|^\beta.$$

Then there is a caloric polynomial P of degree at most $\lfloor\beta\rfloor + 2$ such that

$$\|u - P\|_{L^\infty(Q_r^-)} \leq C C_* r^{\beta+2}, \qquad \text{for } r \in (0,1),$$

where constant C depends only on n, β and $\|u\|_{L^\infty(Q_1^-)}$.

Proof We can assume that $\|u\|_{L^\infty(Q_1^-)} \leq 1$ and $C_* \leq \delta$, where δ is small enough and will be determined later. (Replace u by $u(R^{-1}x, R^{-2}t)$ for a large fixed constant R to find $|H(u)| \leq \delta|X|^\beta$) The proof of lemma is based on the following claim.

Claim: There exists $0 < \rho < 1$ and a sequence of caloric polynomials P_k such that

$$\|u - P_k\|_{L^\infty(Q^-_{\rho^k})} \leq \rho^{k(\beta+2)},$$

and

$$|\partial_x^\mu \partial_t^\ell (P_k - P_{k-1})(0,0)| \leq C_0 \rho^{(k-1)(\beta+2-|\mu|-2\ell)}, \quad \text{if } |\mu| + 2\ell < \beta + 2.$$

A straight forward implication of this claim is that the sequence $\{P_k\}$ converges uniformly in Q_1 to a polynomial P of degree at most $\lfloor\beta\rfloor + 2$ which clearly satisfies

$$\begin{aligned}\|u-P\|_{L^\infty(Q^-_{\rho^k})} \le& \|u-P_k\|_{L^\infty(Q^-_{\rho^k})} + \sum_{i=k+1}^{\infty} \|P_i - P_{i-1}\|_{L^\infty(Q^-_{\rho^k})}\\ \le& \rho^{k(\beta+2)} + \sum_{i=k+1}^{\infty} \sum_{|\mu|+2\ell<\beta+2} C_0 \rho^{(i-1)(\beta+2-|\mu|-2\ell)} \rho^{k(|\mu|+2\ell)}\\ \le& \rho^{k(\beta+2)} + \sum_{|\mu|+2\ell<\beta+2} C_0\rho^{k(\beta+2)} \le C_{n,\beta} C_0 \rho^{k(\beta+2)}.\end{aligned}$$

Therefore, the lemma will be proved for $C := \frac{1}{\delta} C_{n,\beta} C_0 \rho^{-(\beta+2)}$.

Now we prove the claim. It is obviously true for $k = 0$ (just take $P_0 \equiv P_{-1} \equiv 0$). We now assume that it holds for k and we prove it for $k+1$. Define

$$v(X) := \frac{u(\rho^k x, \rho^{2k} t) - P_k(\rho^k x, \rho^{2k} t)}{\rho^{k(\beta+2)}}.$$

Then by inductive hypothesis $|v| \le 1$ in Q_1^-. In addition,

$$|H(v)| = \left|\frac{H(u)(\rho^k x, \rho^{2k} t)}{\rho^{k\beta}}\right| \le C_* \le \delta.$$

If we apply Lemma 6.1 in [7], there exist $\delta = \delta(\epsilon)$ and function w satisfying

$$|v - w| \le \epsilon, \quad \text{in } Q^-_{1/2},$$

and

$$\begin{cases} H(w) = 0 \text{ in } Q^-_{1/2},\\ w = v \qquad \text{on } \partial_p Q^-_{1/2}.\end{cases}$$

Now consider a polynomial $\hat{P}$ of degree at most $\lfloor\beta\rfloor + 2$ such that $\partial_x^\mu \partial_t^\ell \hat{P}(0,0) = \partial_x^\mu \partial_t^\ell w(0,0)$ for $|\mu| + \ell < \beta + 2$. Since $\|w\|_{L^\infty(Q^-_{1/2})} \le \|v\|_{L^\infty(Q^-_1)} \le 1$, by estimates on derivatives for caloric functions $|\partial_x^\mu \partial_t^\ell \hat{P}(0,0)| \le C_0$ for a universal constant C_0. Obviously, $\hat{P}$ is caloric and

$$\|w - \hat{P}\|_{L^\infty(Q^-_\rho)} \le C_0 \rho^{\lfloor\beta\rfloor+3}.$$

In particular, if we choose ρ sufficiently small so that $C_0\rho^{\lfloor\beta\rfloor+3} \le \frac{1}{2}\rho^{\beta+2}$ and then choose ϵ such that $\epsilon \le \frac{1}{2}\rho^{\beta+2}$, we arrive at

$$\|v - \hat{P}\|_{L^\infty(Q^-_\rho)} \le \rho^{\beta+2},$$

or equivalently

$$\|u - P_{k+1}\|_{L^\infty(Q^-_{\rho^{k+1}})} \le \rho^{(k+1)(\beta+2)}, \quad P_{k+1}(X) := P_k(X) + \rho^{k(\beta+2)} \hat{P}(\rho^{-k} x, \rho^{-2k} t).$$

We also have

$$|\partial_x^\mu \partial_t^\ell (P_{k+1} - P_k)(0,0)| \le \rho^{k(\beta+2-|\mu|-\ell)} |\partial_x^\mu \partial_t^\ell \hat{P}(0,0)| \le C_0 \rho^{k(\beta+2-|\mu|-\ell)}. \qquad \square$$

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