A Hierarchy of Spacetime Symmetries: Holes to Heraclitus*

JB Manchak and Thomas Barrett

Abstract

We present a hierarchy of symmetry conditions within the context of general relativity. The weakest condition captures a sense in which spacetime is free of symmetry "holes" of a certain type. All standard models of general relativity satisfy the condition but we show that violations can occur if the Hausdorff assumption is dropped. On the other extreme, the strongest condition of the hierarchy is satisfied whenever a model is completely devoid of symmetries. In these "Heraclitus spacetimes," no pair of distinct points can be mapped (even locally) into one another. We prove that such spacetimes exist. We also show a sense in which Heraclitus spacetimes are completely determined by their local properties. We close with a brief comment on the prospect of using the symmetries of a spacetime as a guide to how much "structure" it possesses.

1 Introduction

In the instructive and influential second chapter of his book *World Enough and Spacetime*, John Earman constructs an elegant hierarchy of classical spacetime theories (e.g. Leibnizian, Newtonian). The hierarchy tracks both the geometric structures involved (e.g. temporal metric, inertial structure) as well as the associated spacetime symmetries (e.g. translations, rotations). Stepping back, one finds that "as the space-time structure becomes richer, the symmetries become narrower, the list of absolute quantities increases, and more and more questions about motion become meaningful" (Earman 1989, p. 36).

Following Earman, here we also construct a hierarchy of spacetime symmetries. But instead of comparing the symmetry properties of different spacetime theories, we restrict attention to one particular spacetime theory – general relativity – and compare the symmetry properties of different spacetime models within that theory. In this way, the symmetry hierarchy we present is akin to the hierarchy of causal conditions that has long been used in the foundations of general relativity (Hawking and Ellis 1973).

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In what follows, we begin with a few mathematical preliminaries concerning spacetime isometries. We then present the hierarchy of symmetry conditions. The weakest condition captures a sense in which a spacetime is free of symmetry "holes" of a certain type (Halvorson and Manchak 2022). It turns out that all standard models of general relativity satisfy the condition but we show that violations can occur if the Hausdorff assumption is dropped. On the other extreme, the strongest condition of the hierarchy is satisfied whenever a model is completely devoid of symmetries. In these "Heraclitus spacetimes," no pair of distinct points can be mapped (even locally) into one another. We prove that such spacetimes exist. We also show a sense in which Heraclitus spacetimes are completely determined by their local properties. We close with a brief comment on the prospect of using the symmetries of a spacetime as a guide to how much "structure" it possesses (cf. North 2021).

2 Spacetime Isometries

Unless otherwise flagged, a *spacetime* is a pair (M, g_{ab}) where M is a smooth, n-dimensional (for $n \geq 2$), connected, Hausdorff manifold without boundary and g_{ab} is a smooth, Lorentzian metric on M of signature $(-, +, ..., +)$. Given a pair of spacetimes (M, g_{ab}) and (M', g'_{ab}), we say a diffeomorphism $\psi : M \to M'$ is an *isometry* if $\psi^*(g'_{ab}) = g_{ab}$ where ψ^* is the pull-back associated with ψ.

One can identify the collection of isometries from a spacetime (M, g_{ab}) to itself by letting $(M', g'_{ab}) = (M, g_{ab})$ in the definition. This collection of isometries are the "global symmetries" of a given spacetime. (The notion of the "local" symmetries of a spacetime is more subtle and will be considered in due course.) Of course, any spacetime (M, g_{ab}) has a trivial global symmetry: the identity map $\psi : M \to M$ defined by $\psi(p) = p$ for all $p \in M$. For some spacetimes, the identity map is its only global symmetry (see the discussion of "giraffe" spacetimes below). But virtually all example spacetimes found in textbooks have additional global symmetries, e.g. the translations, rotations, and boosts in Minkowski spacetime.

Within this context, it might be useful to consider an influential construction used in discussions of the "hole argument" (Earman and Norton 1989). Let (M, g_{ab}) be any spacetime, let $O \subset M$ be an open set whose compact closure is a proper subset of M. It is well-known that there exists a diffeomorphism $\psi : M \to M$ which is non-trivial – it is not the identity map – but which is the identity map on the restricted domain $M - O$. It is immediate that this "hole" diffeomorphism ψ counts as an isometry between the spacetimes (M, g_{ab}) and $(M, \psi_*(g_{ab}))$. But note the following facts: (i) the identity map is not an isometry between (M, g_{ab}) and $(M, \psi_*(g_{ab}))$ (cf. Weatherall 2018) and (ii) the map ψ is not an isometry from (M, g_{ab}) to itself (Halvorson and Manchak 2022). This latter fact means that the "hole" diffeomorphism ψ fails to be a global symmetry of the spacetime (M, g_{ab}).

3 Symmetry Hierarchy

In what follows, six symmetry conditions will be considered. As we explore them, the diagram below will be a useful guide. Arrows correspond to implication relations. It is an open question whether a locally giraffe spacetime must be Heraclitus. But otherwise, if two conditions in the diagram are not connected by an arrow (or series of arrows), then the corresponding implication relation does not hold (examples will be given to show this). Not shown in the diagram is the fact that the giraffe condition is equivalent to the conjunction of the point rigid and fixed point conditions.

$$\begin{tikzcd}
& \text{Heraclitus} \arrow[d, Rightarrow] & \\
& \text{Locally Giraffe} \arrow[d, Rightarrow] & \\
& \text{Giraffe} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{Point Rigid} \arrow[dr, Rightarrow] & & \text{Fixed Point} \arrow[dl, Rightarrow] \\
& \text{Rigid} &
\end{tikzcd}$$

4 Global Symmetries

We begin with the weakest condition whose formulation draws on the "hole" construction considered in the previous section. The condition essentially requires that when global spacetime symmetries are fixed in an open region – however small – they are fixed everywhere. Following Geroch (1969), we will usually refer to such spacetimes as "rigid" to avoid confusion with issues related the hole argument and also with other types of spacetime "holes" related to causal determinism and prediction (Geroch 1977; Manchak 2014).

Definition 1. A spacetime (M, g_{ab}) is *rigid* (or symmetry hole-free) if, for any open set $O \subseteq M$ and any isometry $\psi : M \to M$, if ψ is the identity map on O, then ψ is the identity map on M.

Proposition 1. Any spacetime is rigid.

A proof of the proposition is given in Halvorson and Manchak (2022) which draws on a general rigidity theorem due to Geroch (1969). Given that every spacetime is rigid, the condition would seem to be quite weak. But it is worth appreciating that violations of rigidity can easily occur if the Hausdorff condition is relaxed. In other words, there can be "symmetry holes" in such non-standard spacetimes: fixing symmetries in an open set does not fix the symmetries everywhere.

Proposition 2. Some non-Hausdorff spacetimes are not rigid.

Proof. First, we construct the "plane with two origins". Let $U_1 = \mathbb{R}^2$ and let $p = (0,0) \in \mathbb{R}^2 = U_1$. Let $M = U_1 \cup \{p'\}$ for any abstract object p' and let $U_2 = M - \{p\}$. Let $\varphi_1 : U_1 \to \mathbb{R}^2$ be the identity map and let $\varphi_2 : U_2 \to \mathbb{R}^2$ be such that $\varphi_2(q) = q$ for all $q \neq p'$ and $\varphi_2(p') = p$. Since $\varphi_1[U_1] = \varphi_2[U_2] = \mathbb{R}^2$, we find that (U_1, φ_1) and (U_2, φ_2) are charts on M. Moreover, these charts cover M. Consider the maps $\varphi_1 \circ \varphi_2^{-1} : \varphi_2[U_1 \cap U_2] \to \mathbb{R}^2$ and $\varphi_2 \circ \varphi_1^{-1} : \varphi_1[U_1 \cap U_2] \to \mathbb{R}^2$. We find that $\varphi_1[U_1 \cap U_2] = \varphi_2[U_1 \cap U_2] = \mathbb{R}^2 - \{p\}$ and $\varphi_1 \circ \varphi_2^{-1} = \varphi_2 \circ \varphi_1^{-1}$ is just the inclusion map $\iota : \mathbb{R}^2 - \{p\} \to \mathbb{R}^2$ and hence smooth. So the two charts are compatible. Let $\mathcal{C}$ be the collection of all charts on M compatible with (U_1, φ_1) and (U_2, φ_2). So $(M, \mathcal{C})$ is a manifold. (See Hicks (1965) for treatment of non-Hausdorff manifolds.)

We now verify that $p, p' \in M$ are non-Hausdorff witness points: for any open set $U_p \subseteq M$ containing p and any open set $U_{p'} \subseteq M$ containing p', we have $U_p \cap U_{p'} \neq \varnothing$. Suppose not. Since $U_p \cap U_{p'} = \varnothing$, we know $p' \notin U_p$ and $p \notin U_{p'}$. So $U_p \subseteq U_1$ and $U_{p'} \subseteq U_2$. So $\varphi_1[U_p]$ and $\varphi_2[U_{p'}]$ are open sets in $\mathbb{R}^2$ containing p. So the region $\varphi_1[U_p] \cap \varphi_2[U_{p'}]$ is a non-empty open set in $\mathbb{R}^2$. Let $r \neq p$ be a point in $\varphi_1[U_p] \cap \varphi_2[U_{p'}]$. So $\varphi_1^{-1}(r) \in U_p$ and $\varphi_2^{-1}(r) \in U_{p'}$. But we have defined φ_1 and φ_2 so that $\varphi_1^{-1}(q) = \varphi_2^{-1}(q)$ for all $q \in \mathbb{R}^2 - \{p\}$. So the point $\varphi_1^{-1}(r) = \varphi_2^{-1}(r)$ is contained in $U_p \cap U_{p'}$ which is absurd since $U_p \cap U_{p'} = \varnothing$. So $p, p' \in M$ are non-Hausdorff witness points.

Two facts will be useful in what follows: (i) for any chart $(U, \varphi) \in \mathcal{C}$, it is not the case that $p, p' \in U$ and (ii) for any smooth function $\alpha : M \to \mathbb{R}$, we have $\alpha(p) = \alpha(p')$. Suppose there were a chart $(U, \varphi) \in \mathcal{C}$ such that $p, p' \in U$. Then, the open set $\varphi[U] \in \mathbb{R}^2$ contains the points $\varphi(p)$ and $\varphi(p')$. Because $\mathbb{R}^2$ is Hausdorff, there are open neighborhoods $O_p \subset \varphi[U]$ of $\varphi(p)$ and $O_{p'} \subset \varphi[U]$ of $\varphi(p')$ such that $O_p \cap O_{p'} = \varnothing$. So $U_p \cap U_{p'} = \varnothing$ where $U_p = \varphi^{-1}[O_p]$ and $U_{p'} = \varphi^{-1}[O_{p'}]$ are open subsets of M. But this contradicts the fact that $p, p' \in M$ are non-Hausdorff witness points. So we have established (i). Now let $\alpha : M \to \mathbb{R}$ be any smooth function. So $\alpha \circ \varphi_1^{-1} : \mathbb{R}^2 \to \mathbb{R}$ and $\alpha \circ \varphi_2^{-1} : \mathbb{R}^2 \to \mathbb{R}$ are both smooth. But we have defined φ_1 and φ_2 so that $\varphi_1^{-1}(q) = \varphi_2^{-1}(q)$ for all $q \in \mathbb{R}^2 - \{p\}$. So $\alpha \circ \varphi_1^{-1}(q) = \alpha \circ \varphi_2^{-1}(q)$ for all $q \in \mathbb{R}^2 - \{p\}$. Since $\alpha \circ \varphi_1^{-1}$ and $\alpha \circ \varphi_2^{-1}$ are both smooth functions on $\mathbb{R}^2$ and agree everywhere but the point p, they must also agree at p. We have $\alpha(p) = \alpha \circ \varphi_1^{-1}(p) = \alpha \circ \varphi_2(p)^{-1} = \alpha(p')$ which establishes (ii).

Let $\psi : M \to M$ be the bijection defined by $\psi(q) = q$ for all $q \in M - \{p, p'\}$, $\psi(p) = p'$, $\psi(p') = p$. Let $\alpha : M \to \mathbb{R}$ be any smooth function and let (U, φ) be any chart in $\mathcal{C}$. So $\alpha \circ \varphi^{-1} : \varphi[U] \to \mathbb{R}$ is smooth. We show $\alpha \circ \psi : M \to \mathbb{R}$ is smooth. This we can do by showing that $\alpha \circ \psi \circ \varphi^{-1} : \varphi[U] \to \mathbb{R}$ is just the smooth map $\alpha \circ \varphi^{-1}$. If U contains neither p nor p', then it is immediate that $\psi \circ \varphi^{-1} = \varphi^{-1}$ and therefore $\alpha \circ \psi \circ \varphi^{-1} = \alpha \circ \varphi^{-1}$. Suppose U contains p but not p'. (An analogous argument can be given if U contains p' but not p. From (i) above, we know U cannot contain both p and p'.) We have defined ψ such that $\alpha \circ \psi \circ \varphi^{-1}(q) = \alpha \circ \varphi^{-1}(q)$ for all $q \neq \varphi(p)$ in $\varphi[U]$. What about $\varphi(p)$? From (ii) above, we know that $\alpha(p) = \alpha(p')$. So $\alpha \circ \psi \circ \varphi^{-1}(\varphi(p)) = \alpha(p') = \alpha(p) = \alpha \circ \varphi^{-1}(\varphi(p))$. So $\alpha \circ \psi \circ \varphi^{-1} = \alpha \circ \varphi^{-1}$ on all of $\varphi[U]$. So $\alpha \circ \psi$ is smooth. Since α was chosen arbitrarily, the map ψ is smooth. Since $\psi^{-1} = \psi$, we know that ψ^{-1} is also smooth. So ψ is a diffeomorphism.

Now let (U_1, η_{ab}) be two-dimensional Minkowski spacetime. Let (M, g_{ab}) be a non-Hausdorff version of this spacetime defined by letting $g_{ab|q} = \eta_{ab|q}$ for all $q \neq p'$ in M and $g_{ab|p'} = \eta_{ab|p}$. So the diffeomorphism $\psi : M \to M$ defined above – which is not the identity map – is an isometry. But $\psi(p) = p$ for all p in the open set $M - \{p, p'\}$. So (M, g_{ab}) is not rigid. □

The rigidity condition requires that when global spacetime symmetries are fixed in an open region – however small – they are fixed everywhere. One can naturally strengthen the condition by requiring that when global spacetime symmetries are fixed at a single point, they are fixed everywhere. Consider the following.

Definition 2. A spacetime (M, g_{ab}) is *point rigid* if, for any point $p \in M$ and any isometry $\psi : M \to M$, if $\psi(p) = p$, then ψ is the identity map.

Proposition 3. Any point rigid spacetime is rigid. The implication does not go in the other direction.

The proof of the first statement is immediate from the definitions. The following example shows that some rigid spacetimes are not point rigid.

Example 1. Let $(\mathbb{R}^2, \eta_{ab})$ be two-dimensional Minkowski spacetime in standard (t, x) coordinates. By proposition 1, it is rigid. Let $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ be the isometry defined by $\psi(t, x) = (t, -x)$. So $\psi(p) = p$ for $p = (0, 0)$ but ψ is not the identity map. So the spacetime fails to be point rigid.

Another natural way to strengthen the rigidity condition is to require that, at least at some points, the global spacetime symmetries are completely fixed. Consider the following.

Definition 3. A spacetime (M, g_{ab}) has a *fixed point* if there is a point $p \in M$ such that $\psi(p) = p$ for any isometry $\varphi : M \to M$.

Proposition 4. Any spacetime with a fixed point is rigid. The implication does not go in the other direction.

The proof of the first statement is immediate since any spacetime is rigid. Example 1 shows that some rigid spacetimes fail to have a fixed point. Let

$(\mathbb{R}^2, \eta_{ab})$ be two-dimensional Minkowski spacetime in standard (t, x) coordinates. Consider the isometry $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\psi(t, x) = (t + 1, x)$. Since $\psi(p) \neq p$ for all $p \in \mathbb{R}^2$, the spacetime fails to have a fixed point. Now, what is the relationship between the fixed point condition and the point rigid condition? It turns out they are independent. Consider the following.

Proposition 5. Some spacetimes with a fixed point fail to be point rigid. Some point rigid spacetimes fail to have a fixed point.

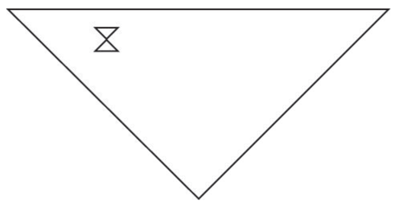

Figure 1: Example 2

Example 2. Let $(\mathbb{R}^2, \eta_{ab})$ be two-dimensional Minkowski spacetime in standard (t, x) coordinates. Consider (M, η_{ab}) where $M = \{(t, x) : 0 < t < 1, x^2 < t^2\}$ (see Figure 1). Aside from the identity map, there is only one other isometry $\psi : M \to M$ defined by the reflection $\psi(t, x) = (t, -x)$. So for any isometry $\psi : M \to M$, we have $\psi(p) = p$ for the point $p = (1/2, 0)$ showing that the spacetime has a fixed point. But since the identity map is not the only isometry such that $\psi(p) = p$, the spacetime is not point rigid.

Example 3. Let $(\mathbb{R}^2, \eta_{ab})$ be two-dimensional Minkowski spacetime in standard (t, x) coordinates. For each integer n, excise the compact region enclosed by the points $(0, n)$, $(1/2, n)$, and $(0, n + 1/2)$. Let the resulting spacetime be (M, η_{ab}) (see Figure 2). For each integer n, there is an isometry $\psi_n : M \to M$ defined by $\psi_n(t, x) = (t, x + n)$. But these are the only isometries by construction. It follows that the spacetime fails to have a fixed point but is point rigid.

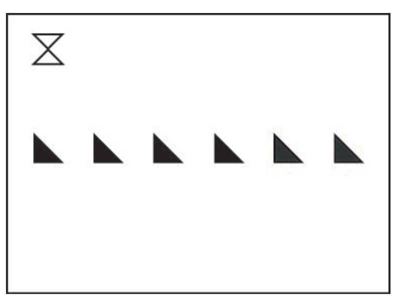

Figure 2: Example 3

Now we come to the strongest condition concerning global symmetries: the requirement that they are completely fixed. Consider the following.

Definition 4. A spacetime (M, g_{ab}) is *giraffe* if the only isometry $\varphi : M \to M$ is the identity map.

One way to construct a giraffe spacetime is to take Minkowski spacetime and excise a compact region "shaped like a giraffe" (Malament, private communication). The shape of a sufficiently asymmetric giraffe (as opposed to a sphere or cube) ensures that there are no global symmetries. A less interesting but more tractable giraffe spacetime will be constructed later on (Example 4). How strong is the giraffe condition? It has been claimed that "everyone knows" giraffe spacetimes are generic in some sense (D'Ambra and Gromov 1991, p. 21). But the meaning of "generic" is not made precise and a general proof remains elusive (Mounound 2015). How is the giraffe condition related to the other symmetry conditions considered so far? We have the following.

Proposition 6. Any giraffe spacetime is point rigid and has a fixed point. The implications do not go in the other direction.

The proof of the first statement is immediate from the definitions. Example 2 is a spacetime with a fixed point which fails to be giraffe. Example 3 is a point rigid spacetime which fails to be giraffe. When considered separately, both the point rigid and the fixed point conditions are strictly weaker than the giraffe condition. However, the conjunction of these conditions turns out to be strong enough to imply the giraffe condition. Consider the following.

Proposition 7. A spacetime is giraffe if and only if it is both point rigid and has a fixed point.

Proof. One direction is trivial. Suppose a spacetime (M, g_{ab}) is both point rigid and has a fixed point. Let $\psi : M \to M$ be any isometry. Since the spacetime has a fixed point, there is a point $p \in M$ such that $\psi(p) = p$. Because the spacetime is point rigid, we know that for all $q \in M$, if $\psi(q) = q$, then ψ is the identity map. Since $\psi(p) = p$, it follows that ψ is the identity map. □

5 Local Symmetries

We now turn to the notion of the "local" symmetries of spacetime. There are a number of conditions one might consider. For example, one might explore those involving the non-existence of the "infinitesimal isometries" associated with Killing vector fields (Malament 2012, p. 86). Given a spacetime (M, g_{ab}), we say a smooth vector field λ^a on M is a *Killing field* if $\pounds_\lambda g_{ab} = \mathbf{0}$. Here, the Lie derivative term $\pounds_\lambda g_{ab}$ represents the "rate of change" of the metric along the flow maps determined by λ^a. Now consider a spacetime (M, g_{ab}) which contains no "local Killing fields" in the sense that for every open connected set $O \subseteq M$, the spacetime (O, g_{ab}) has no Killing fields aside from the zero vector field on M. One might be tempted to declare such a spacetime free of local symmetries. But one must keep in mind that the full collection of spacetime symmetries "may include some discrete isometries (such as reflections in a plane) which are not generated by Killing vector fields" (Hawking and Ellis 1973, p. 44). This will be important later on.

Another, more general, approach to the "local symmetries" of spacetime makes use of the machinery built up so far concerning global symmetries. A natural condition along these lines is the requirement that any open connected region has only trivial global symmetries when considered as a spacetime in its own right. Consider the following.

Definition 5. A spacetime (M, g_{ab}) is *locally giraffe* if, for any connected open set $O \subseteq M$ the spacetime (O, g_{ab}) is giraffe.

Proposition 8. Any locally giraffe spacetime is giraffe. The implication does not go in the other direction.

The proof of the first statement is immediate from the definitions. The following example is giraffe but not locally giraffe.

Example 4. Let $(\mathbb{R}^2, \eta_{ab})$ be two-dimensional Minkowski spacetime in standard (t, x) coordinates. Consider (M, η_{ab}) where $M = \{(t, x) : 0 < t < 1, 0 < x, x^2 < t^2\}$ (see Figure 3). This spacetime is just the $x > 0$ portion of Example 2. The identity map is the only isometry showing the spacetime is giraffe. But consider the connected open set $O = \{(t, x) \in M : t + x < 1\}$. The spacetime (O, η_{ab}) is not giraffe since there is an isometry $\psi : O \to O$ defined by $\psi(t, x) = (-t + 1, x)$ which reflects O about the $t = 1/2$ line.

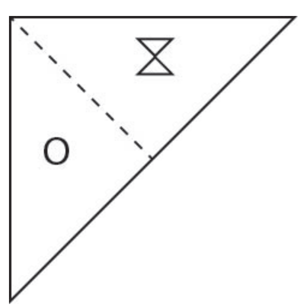

Figure 3: Example 4

We now come to the strongest condition in the symmetry hierarchy which requires that no pair of distinct points can be isometrically mapped – even locally – into one another. Consider the following.

Definition 6. A spacetime (M, g_{ab}) is *Heraclitus* if, for any distinct points $p, q \in M$, and any open neighborhoods $O_p, O_q \subseteq M$ of p and q respectively, there is no isometry $\psi : O_p \to O_q$ such that $\psi(p) = q$.

A Heraclitus spacetime is utterly devoid of symmetries – global and local. Since any neighborhoods of any distinct points fail to be isometric, each event is unlike any other. One might say that in such a spacetime "it is impossible to step in the same river twice." One can show that any Heraclitus spacetime is locally giraffe but it is an open question whether the implication goes in the other direction. We have the following.

Proposition 9. Any Heraclitus spacetime is locally giraffe.

Proof. Let (M, g_{ab}) be a spacetime which fails to be locally giraffe. Then there is some connected open set $O \subseteq M$ such that (O, g_{ab}) is not giraffe. So there is an isometry $\psi : O \to O$ which is not the identity map. So for some point $q \in O$, we have $\psi(q) = r$ where $r \neq q$. So there are distinct points $q, r \in M$ and open neighborhoods $O_q = O$ and $O_r = O$ of q and r respectively such that there is an isometry $\psi : O_q \to O_r$ with $\psi(q) = r$. So (M, g_{ab}) fails to be Heraclitus. □

We close this section by giving an equivalent definition of a Heraclitus spacetime which does not make reference to points and their neighborhoods. This will be useful in what follows.

Definition 7. A spacetime (M, g_{ab}) is *Heraclitus** if, for any open sets $U, V \subseteq M$ and any isometry $\psi : U \to V$, it follows that (i) $U = V$ and (ii) ψ is the identity map.

Proposition 10. A spacetime is Heraclitus if and only if it is Heraclitus*.

Proof. Suppose a spacetime (M, g_{ab}) fails to be Heraclitus. So there are distinct points $p, q \in M$, and open neighborhoods $O_p, O_q \subseteq M$ of p and q respectively, such that there is an isometry $\psi : O_p \to O_q$ with $\psi(p) = q$. Let U and V be the open sets O_p and O_q respectively. If $U \neq V$ then (M, g_{ab}) fails to satisfy (i) in the definition of a Heraclitus* spacetime. Suppose then that $U = V$. Since $p, q \in U$ are distinct and $\psi(p) = q$, then (M, g_{ab}) fails to satisfy (ii) in the definition of a Heraclitus* spacetime. So (M, g_{ab}) fails to be Heraclitus*.

Now suppose (M, g_{ab}) fails to be Heraclitus*. So for some open sets $U, V \subseteq M$ there is an isometry $\psi : U \to V$ such that either (i) $U \neq V$ or (ii) ψ fails to be the identity map. Suppose (i) $U \neq V$. So either there is a point $p \in U$ which fails to be in V or there is a point $r \in V$ which fails to be in U. Suppose the first possibility obtains (an analogous argument can be made for the other case). So $\psi(p) = q$ for some point $q \neq p$. So there are distinct points p and q and open neighborhoods $O_p = U$ and $O_q = V$ of p and q respectively, such that there is an isometry $\psi : O_p \to O_q$ with $\psi(p) = q$. So (M, g_{ab}) fails to be Heraclitus. Now suppose that $U = V$ but (ii) ψ fails to be the identity map. Then there will be distinct points $p, q \in U$ such that $\psi(p) = q$. So there are open neighborhoods $O_p = U$ and $O_q = U$ of p and q respectively, such that there is an isometry $\psi : O_p \to O_q$ with $\psi(p) = q$. So (M, g_{ab}) fails to be Heraclitus. $\square$

6 Existence of Heraclitus Spacetimes

There is a vast literature on "inhomogeneous cosmology" which investigates a variety of asymmetric models of the universe (Ellis 2011). Even so, it seems that all of the examples considered make use of various "symmetries which are sufficiently strong to render the field equations tractable" (Collins and Szafron 1979 p. 2347). One might therefore wonder about the possibility of finding a spacetime without symmetries at all. Do Heraclitus spacetimes even exist?

In a paper entitled "A Metric with No Symmetries or Invariants," Koutras and McIntosh (1996) present a peculiar spacetime. Consider the manifold $M = \mathbb{R}^4$ in (u, w, x, y) coordinates and let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary smooth function. The metric g_{ab} on M is given by the following.

$$\begin{aligned} g_{ab} \quad &= \quad 2x\nabla_{(a}w\nabla_{b)}u - 2w\nabla_{(a}u\nabla_{b)}x \\ &\quad +[2f(u)x(x^2 + y^2) - w^2]\nabla_a u\nabla_b u - \nabla_a x\nabla_b x - \nabla_a y\nabla_b y \end{aligned}$$

One can show that the spacetime (M, g_{ab}) admits no local Killing fields. This is a remarkable property. But here it is instructive to recall that there are discrete isometries that are not generated by Killing fields. In the present case, one can easily verify that (M, g_{ab}) has a global isometry $\psi : M \to M$ defined by the reflection $\psi(u, w, x, y) = (u, w, x, -y)$. So not only does the spacetime fail to be Heraclitus, it isn't even point rigid. Stepping back, it may be that restricting attention to the $y > 0$ portion of M will result in a Heraclitus spacetime for an appropriate choice of the function $f(u)$. In any case, here we present a simple Heraclitus example in order to get a better grip on the condition. This example will be needed later on to prove an even stronger existence result.

We first we collect together a number of facts and prove a lemma. Consider the manifold $\mathbb{R}^2$ in (t, x) coordinates and let ∇ be the associated coordinate derivative operator. Let $(\mathbb{R}^2, \eta_{ac})$ be two-dimensional Minkowski spacetime where $\eta_{ac} = -\nabla_a t\nabla_c t + \nabla_a x\nabla_c x$. So $\nabla_a \eta_{nm} = \mathbf{0}$. For convenience let $t^a = (\frac{\partial}{\partial t})^a$ and $x^a = (\frac{\partial}{\partial x})^a$. Let χ^a be the "position field" on $\mathbb{R}^2$ relative to ∇ and the origin $(0, 0)$; this is the unique, smooth vector field on $\mathbb{R}^2$ that vanishes at the origin and satisfies the condition $\nabla_a \chi^n = \delta^n_a$ (Malament 2012, p. 66). At any point $(t, x) \in \mathbb{R}^2$, one can verify that $\chi^a = tt^a + xx^a$. Let $h_{ac} = \nabla_a t\nabla_c t + \nabla_a x\nabla_c x$. We find that $\nabla_a h_{nm} = \mathbf{0}$ and $h_{nm}\xi^n\xi^m \geq 0$ for all vectors ξ^n. Let $M = \mathbb{R}^2 - \{(0, 0)\}$ and let $\Omega : M \to \mathbb{R}$ be the smooth, strictly positive function f^{-1} where $f = h_{nm}\chi^n\chi^m$. One can verify that $h_{nm}\chi^n\chi^m = t^2 + x^2$ and so $\Omega = (t^2 + x^2)^{-1}$. We have the following facts which will be useful later on.

$$
\begin{aligned}
\nabla_a f &= \nabla_a[h_{nm}\chi^n\chi^m] = h_{nm}[\chi^n\delta_a^m + \chi^m\delta_a^n] = 2h_{an}\chi^n \\
\nabla_a\nabla_c f &= \nabla_a[2h_{cn}\chi^n] = 2h_{cn}\delta_a^n = 2h_{ac} \\
\nabla_a\Omega &= \nabla_a f^{-1} = -f^{-2}\nabla_a f = -2f^{-2}h_{an}\chi^n = -2\Omega^2 h_{an}\chi^n \\
\nabla_a\nabla_c\Omega &= \nabla_a[-f^{-2}\nabla_c f] = 2f^{-3}(\nabla_a f)\nabla_c f - f^{-2}\nabla_a\nabla_c f. \\
&= 8\Omega^3 h_{an}h_{cm}\chi^n\chi^m - 2\Omega^2 h_{ac} \\
\eta^{de}h_{dn}h_{em} &= [-t^d t^e + x^d x^e][\nabla_d t\nabla_n t + \nabla_d x\nabla_n x][\nabla_e t\nabla_m t + \nabla_e x\nabla_m x] \\
&= [-t^d t^e + x^d x^e][\nabla_d t\nabla_n t\nabla_e t\nabla_m t + \nabla_d t\nabla_n t\nabla_e x\nabla_m x \\
&\quad +\nabla_d x\nabla_n x\nabla_e t\nabla_m t + \nabla_d x\nabla_n x\nabla_e x\nabla_m x] \\
&= -\nabla_n t\nabla_m t + \nabla_n x\nabla_m x = \eta_{nm} \\
\eta^{de}h_{de} &= [-t^d t^e + x^d x^e][\nabla_d t\nabla_e t + \nabla_d x\nabla_e x] = -1 + 1 = 0 \\
\eta^{de}(\nabla_d\Omega)\nabla_e\Omega &= \eta^{de}[-2\Omega^2 h_{dn}\chi^n][-2\Omega^2 h_{em}\chi^m] \\
&= 4\Omega^4\eta_{nm}\chi^n\chi^m \\
\eta^{de}\nabla_d\nabla_e\Omega &= \eta^{de}[8\Omega^3 h_{dn}h_{em}\chi^n\chi^m - 2\Omega^2 h_{de}] \\
&= 8\Omega^3\eta_{nm}\chi^n\chi^m
\end{aligned}
$$

Lemma 1. Let (M, g_{ac}) be the spacetime $(M, \Omega^2\eta_{ac})$. Let $R : M \to \mathbb{R}$ be the Ricci scalar associated with g_{ac} and let $Q : M \to \mathbb{R}$ be the scalar defined by $Q = g^{ac}(\tilde{\nabla}_a R)\tilde{\nabla}_c R$ where $\tilde{\nabla}$ is the derivative operator associated with g_{ac}. It is the case that $R = -8\eta_{nm}\chi^n\chi^m = 8(t^2 - x^2)$ and $Q = -32R\Omega^{-2}$.

Proof. Because (M, g_{ac}) is conformally flat and two-dimensional, we find that $R = -2\Omega^{-2}\eta^{ac}\nabla_a\nabla_c\ln\Omega$ where ∇ is the coordinate derivative operator compatible with η_{ac} (see Wald 1984, p. 446). Let $\tilde{\nabla}$ be the derivative operator compatible with g_{ac} and note that $\tilde{\nabla}_a R = \nabla_a R$ since the action of any two derivative operators agree on a scalar field. Using the facts from above, we have the following as claimed.

$$
\begin{aligned}
R &= -2\Omega^{-2}\eta^{ac}\nabla_a\nabla_c\ln\Omega \\
&= -2\Omega^{-2}\eta^{ac}[-\Omega^{-2}(\nabla_a\Omega)\nabla_c\Omega + \Omega^{-1}\nabla_a\nabla_c\Omega] \\
&= 2\Omega^{-4}\eta^{ac}(\nabla_a\Omega)\nabla_c\Omega - 2\Omega^{-3}\eta^{ac}\nabla_a\nabla_c\Omega \\
&= 8\Omega^{-4}\Omega^4\eta_{nm}\chi^n\chi^m - 16\Omega^{-3}\Omega^3\eta_{nm}\chi^n\chi^m \\
&= 8\eta_{nm}\chi^n\chi^m - 16\eta_{nm}\chi^n\chi^m = -8\eta_{nm}\chi^n\chi^m = 8[t^2 - x^2] \\
\tilde{\nabla}_a R &= \nabla_a R = -8\nabla_a[\eta_{nm}\chi^n\chi^m] = -8\eta_{nm}[\chi^n\delta_a^m + \chi^m\delta_a^n] = -16\eta_{an}\chi^n \\
Q &= g^{ac}(\tilde{\nabla}_a R)\tilde{\nabla}_c R = \eta^{ac}\Omega^{-2}[-16\eta_{an}\chi^n][-16\eta_{cm}\chi^m] \\
&= 256\Omega^{-2}[\delta_n^c\chi^n][\eta_{cm}\chi^m] = 256\Omega^{-2}\eta_{cm}\chi^c\chi^m = -32\Omega^{-2}R \qquad \square
\end{aligned}
$$

Proposition 11. There exists a Heraclitus spacetime.

Proof. Let (M, g_{ac}) be defined as in Lemma 1: $M = \mathbb{R}^2 - \{(0,0)\}$ and $g_{ac} = \Omega^2\eta_{ac}$ where $\eta_{ac} = -\nabla_a t\nabla_c t + \nabla_a x\nabla_c x$ and $\Omega = f^{-1}$ for $f = (h_{nm}\chi^n\chi^m)$. We will show that the spacetime (N, g_{ac}) is Heraclitus where $N = \{(t, x) \in M : t > 0, x > 0, t^2 > x^2\}$ (see Figure 4). Let $p = (t_p, x_p)$ and $q = (t_q, x_q)$ be any distinct points in N and let $O_p, O_q \subseteq N$ be open neighborhoods of p and q respectively. Suppose there were an isometry $\psi : O_p \to O_q$ such that $\psi(p) = q$. We show a contradiction.

Consider the Ricci scalar $R : N \to \mathbb{R}$ associated with g_{ac} and the scalar $Q : N \to \mathbb{R}$ defined by $Q = g^{ac}(\tilde{\nabla}_a R)\tilde{\nabla}_c R$ where $\tilde{\nabla}$ is the derivative operator associated with g_{ac}. From Lemma 1, we see that $R = 8(t^2 - x^2) > 0$ on N which we will need later on. Since R, $\tilde{\nabla}$, and g^{ac} are all invariant under the isometry ψ, we know that Q is also invariant under ψ. Since $\psi(p) = q$, we have $R(p) = R(q)$ and $Q(p) = Q(q)$. In what follows, we will show that $Q(p) = Q(q)$ implies $f(p) = f(q)$ which, together with $R(p) = R(q)$, will require that $p = q$ (see Figure 4).

From Lemma 1, we know that $R = 8(t^2 - x^2)$ and $Q = -32R\Omega^{-2}$. Since $R(p) = R(q)$ we know (i) $t_p^2 - x_p^2 = t_q^2 - x_q^2$. Since $Q(p) = Q(q)$ we know $R(p)\Omega(p)^{-2} = R(q)\Omega(q)^{-2}$. Since $R(p) = R(q)$ and $R > 0$ on N, we know $\Omega(p)^{-2} = \Omega(q)^{-2}$. Since $\Omega > 0$ on N, we know $\Omega^{-1}(p) = \Omega(q)^{-1}$ and therefore $f(p) = f(q)$. So (ii) $t_p^2 + x_p^2 = t_q^2 + x_q^2$. Using equations (i) and (ii), a bit of algebra shows $t_p^2 = t_q^2$ and $x_p^2 = x_q^2$. Since both $t > 0$ and $x > 0$ on N, we have $t_p = t_q$ and $x_p = x_q$. So $p = (t_p, x_p) = (t_q, x_q) = q$ which is impossible since p and q are distinct. So there is no isometry $\psi : O_p \to O_q$ such that $\psi(p) = q$. So (N, g_{ac}) is Heraclitus. □

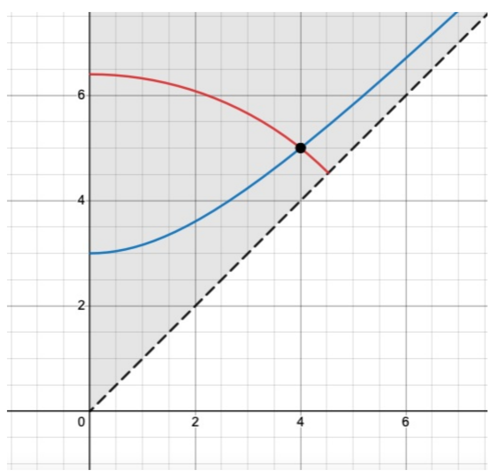

Figure 4: The shaded region is the manifold N. The isometry ψ must map any point p to a point q with the same R value (blue line) and the same f value (red line). So $p = q$.

The use of scalar curvature invariants in the proof just given suggests a special type of symmetry condition even stronger than the Heraclitus condition

(David Malament, private communication). Consider the following definition.

Definition 8. Let (M, g_{ab}) be a spacetime and let $\mathcal{S}$ be the collection of scalar curvature invariant functions on M. We say (M, g_{ab}) *individuates points* if there do not exist distinct points $p, q \in M$ such that $f(p) = f(q)$ for all $f \in \mathcal{S}$.

If a spacetime fails to be Heraclitus, then it will contain distinct points which, by virtue of the local isometry between them, will have the same values for all scalar curvature invariants. Thus, the spacetime must also fail to individuate points. We have the following.

Proposition 12. If a spacetime individuates points, it is Heraclitus.

What about the other direction? Does there exist a Heraclitus spacetime which fails to individuate points? This is an open question. Curiously, the spacetime presented by Koutras and McIntosh (1996) is such that all of its scalar curvature invariants vanish everywhere. So the spacetime does not individuate points. Thus, if an appropriately truncated version of this spacetime does count as Heraclitus, then the two conditions are not equivalent. In any case, the example spacetime given in the proof of Proposition 12 also ensures the following stronger existence result.

Proposition 13. A spacetime which individuates points exists.

7 Heraclitus Properties

Here we show a sense in which Heraclitus spacetimes are completely determined by their local properties. Once we have defined "local property" in this context, we will present a recovery result: given any collection of local spacetime properties, there is at most one Heraclitus spacetime (up to isometry) with exactly those local properties. Consider the following.

Definition 9. Spacetimes (M, g_{ab}) and (M', g'_{ab}) are *locally isometric* if each point $p \in M$ has an open neighborhood O which is isometric to some open set $O' \subseteq M'$ and, correspondingly, with the roles of (M, g_{ab}) and (M', g'_{ab}) interchanged.

One can use this definition to make precise the notion of local spacetime properties (Manchak 2009). Consider the collection $\mathscr{U}$ of all spacetimes. In the natural way, a spacetime property can be regarded as a sub-collection of $\mathscr{U}$. We now have the following.

Definition 10. A spacetime property $\mathscr{P} \subseteq \mathscr{U}$ is *local* if, for any locally isometric spacetimes $(M, g_{ab}), (M', g'_{ab}) \in \mathscr{U}$, we have $(M, g_{ab}) \in \mathscr{P}$ if and only if

$(M', g'_{ab}) \in \mathscr{P}$.

We now show that if a pair of Heraclitus spacetimes are locally isometric, they must be isometric. Consider the following lemma (O'Neill 1983, p. 5).

Lemma 2. Let M and N be manifolds. For each index $\alpha \in A$, let O_α be an open set on M and let $\psi_\alpha : O_\alpha \to N$ be a smooth map. If, for all $\alpha, \beta \in A$, $\psi_\alpha = \psi_\beta$ on $O_\alpha \cap O_\beta$, then the unique map $\psi : \bigcup O_\alpha \to N$ defined such that $\psi_{|O_\alpha} = \psi_\alpha$ for all $\alpha \in A$ must be smooth.

Given a pair of locally isometric Heraclitus spacetimes, we can use the lemma to "patch together" the local isometries to construct a unique global isometry. The process is analogous to putting together a puzzle where the picture is so asymmetric that one knows exactly where each piece must go.

Proposition 14. If Heraclitus spacetimes are locally isometric, then they are isometric.

Proof. Let (M, g_{ab}) and (M', g'_{ab}) be locally isometric Heraclitus spacetimes. Because the spacetimes are locally isometric, for each point $p \in M$, we can fix once and for all an associated open neighborhood $O_p \subseteq M$ and an isometry $\psi_p : O_p \to O'_p$ where O'_p is an open set in M'. Let p, q be any points in M and suppose there is a point $r \in O_p \cap O_q$. Let $U' = \psi_p[O_p \cap O_q]$ and $V' = \psi_q[O_p \cap O_q]$. Since ψ_p and ψ_q are isometries, we know that $\psi_q \circ \psi_p^{-1} : U' \to V'$ is an isometry which maps $\psi_p(r)$ to $\psi_q(r)$. Since (M', g'_{ab}) is Heraclitus, it follows that $\psi_p(r) = \psi_q(r)$. So $\psi_p = \psi_q$ on the region $O_p \cap O_q$ for any $p, q \in M$. Since $\bigcup O_p = M$, it follows from Lemma 2 that the unique map $\psi : M \to M'$ defined such that $\psi_{|O_p} = \psi_p$ for all $p \in M$ must be smooth.

Next we show that ψ is a bijection. Let p, q be any points in M and suppose that $\psi(p) = \psi(q)$. So $\psi_p(p) = \psi_q(q)$ where $\psi_p : O_p \to O'_p$ and $\psi_q : O_q \to O'_q$ are the isometries associated with p and q. Let $U = \psi_p^{-1}[O'_p \cap O'_q]$ and $V = \psi_q^{-1}[O'_p \cap O'_q]$. Since ψ_p and ψ_q are isometries, we know $\psi_q^{-1} \circ \psi_p : U \to V$ is an isometry which maps p to q. Since (M, g_{ab}) is Heraclitus, it follows that $p = q$ and thus ψ is injective. Now let p' be any point in M'. Because the spacetimes are locally isometric, there is an isometry $\varphi : N' \to N$ where N' is an open neighborhood of p' and N is an open set in M. Let $p \in M$ be the point $\varphi(p')$ and consider its associated isometry $\psi_p : O_p \to O'_p$. Let $U' = \varphi^{-1}[O_p \cap N]$ and $V' = \psi_p[O_p \cap N]$. Since φ and ψ_p and are isometries, we know that $\psi_p \circ \varphi : U' \to V'$ is an isometry which maps p' to $\psi_p(p)$. Since (M', g'_{ab}) is Heraclitus, it follows that $p' = \psi_p(p)$. Because $\psi_p(p) \in \psi[M]$, we know $p' \in \psi[M]$ and thus ψ is surjective. So ψ is a bijection.

Next we show that ψ^{-1} is smooth. For each $p \in M$, we can consider the inverse of its associated isometry: $\psi_p^{-1} : O'_p \to O_p$. Let p, q be any points in M. Suppose there is a point $r' \in O'_p \cap O'_q$. The map ψ is defined such that $\psi_{|O_p} = \psi_p$ for all $p \in M$. So ψ must send the point $\psi_p^{-1}(r') \in O_p$ to the point $r' \in O'_p$. Similarly, ψ must send the point $\psi_q^{-1}(r') \in O_q$ to the point $r' \in O'_q$.

Since ψ is injective, we know $\psi_p^{-1}(r') = \psi_q^{-1}(r')$. So $\psi_p^{-1} = \psi_q^{-1}$ on the region $O'_p \cap O'_q$ for any $p, q \in M$. Since ψ is surjective, it follows that $\bigcup O'_p = M'$. So ψ^{-1} is the unique map from $\bigcup O'_p = M'$ to M defined such that $\psi^{-1}_{|O'_p} = \psi_p^{-1}$ for all $p \in M$. By Lemma 2, ψ^{-1} must be smooth.

Since ψ is a smooth bijection with a smooth inverse, it is a diffeomorphism. The final step is to verify that it is an isometry. Consider any point $p \in M$ and its associated isometry $\psi_p : O_p \to O'_p$. We know $\psi_p^*(g'_{ab}) = g_{ab}$ on the region O_p where ψ_p^* is the pull-back associated with ψ_p. Since $\psi_{|O_p} = \psi_p$, we know $\psi^*(g'_{ab}) = g_{ab}$ on O_p where ψ^* is the pull-back associated with ψ. Since p was chosen arbitrarily, $\psi^*(g'_{ab}) = g_{ab}$ on all of M and thus ψ is an isometry. □

From Proposition 14, we have the following result which captures a sense in which Heraclitus spacetimes are completely determined by their local properties.

Corollary 1. Given any collection of local spacetime properties, there is at most one Heraclitus spacetime (up to isometry) with exactly those local properties.

Proof. Given a collection of local properties, suppose there were non-isometric Heraclitus spacetimes (M, g_{ab}) and (M', g'_{ab}) with exactly those local properties. Proposition 14 requires that the spacetimes are not locally isometric. Let $\mathscr{P} \subset \mathscr{U}$ be the collection of spacetimes locally isometric to (M, g_{ab}). We find that $\mathscr{P}$ is a local property possessed by (M, g_{ab}) but not (M', g'_{ab}). So the spacetimes cannot have the same collection of local properties: a contradiction. □

8 Symmetry and Structure

We have isolated a number of precise senses in which general relativity has models with "few symmetries". It is worth making a brief remark about how these results come to bear on a recent debate about symmetry and structure in spacetime theories.

There is a dogma in foundations of spacetime theories that says that the symmetries of a spacetime are a guide to its structure. Recall the passage from Earman (1989, p. 36) in the introduction: "As the space-time structure becomes richer, the symmetries become narrower". In addition, Jill North (2009, p. 87) writes that "stronger structure...admits a smaller group of symmetries." And more recently North (2021, p. 50) says that one of the litmus tests for the presence of more structure on an object is that the "associated group of structure-preserving transformations becomes narrower". This idea is behind the scenes in almost all contemporary discussions of symmetry and structure (Bradley and Weatherall 2020; Wilhelm 2021). But our results here put pressure on this position. We will consider the following simple condition, which is entailed by the dogma:

($\star$) If two spacetimes have the same symmetries, then they have the same amount of structure.

If the symmetries of a spacetime are a good guide to its amount of structure, as the dogma claims, then condition ($\star$) will be true. Two spacetimes with the same symmetries must have the same amount of structure. Of course, much turns here on what one means by "symmetry". But our results demonstrate that ($\star$) is false on some of the most natural ways of understanding what a symmetry is.

Suppose first that we understand a symmetry of a spacetime to be an automorphism of that spacetime, i.e. an isometry from that spacetime to itself. The existence of a giraffe spacetime (see Example 4) captures a sense in which ($\star$) is false (Barrett et al. 2022). One can simply take the giraffe spacetime and add to it any random tensor field that is not definable in terms of the metric. The resulting spacetime has more structure than the giraffe spacetime that we began with, but it has the same automorphisms, since the only automorphism of the giraffe spacetime was the identity map. In light of this result, one might attempt to salvage the dogma by moving to a more general notion of 'symmetry'. A local automorphism of a spacetime (M, g_{ab}) is a smooth map $f : O \to O$ that preserves the metric on M, where O is some open set of M. The definition generalizes in the natural way to spacetimes with structures in addition to g_{ab}. Local automorphisms are simply the automorphisms of local regions of the spacetime. If we consider local automorphisms to be symmetries of the spacetime, then more maps count as symmetries, and so we provide ourselves with more information with which to compare amounts of structure between spacetimes. But even local automorphisms do not provide a good guide to the amount of structure that a spacetime has, since two spacetimes with different amounts of structure might nonetheless have the same local automorphisms. The existence of spacetimes that are locally giraffe (implied by Proposition 11) demonstrate that ($\star$) is false on this understanding. One simply adds to a locally giraffe spacetime any random tensor field that is not definable in terms of the metric. The resulting spacetime has more structure than the one that we began with, but it has the same local automorphisms, since it follows from the local giraffe condition that the only local automorphisms were identity maps to begin with.

There is yet another way one might try to salvage the dogma. We will say that a local homomorphism of a spacetime (M, g_{ab}) is a map $f : O_1 \to O_2$ that preserves the metric on M, where O_1 and O_2 are open sets of M. As above, this definition naturally generalizes to spacetimes with additional structures. All local automorphisms are local homomorphisms, but not vice versa. If we consider local homomorphisms to be symmetries of a spacetime, then we allow ourselves to appeal to *even more* maps in order to compare amounts of structure. Once again, however, ($\star$) comes out false. The existence of a Heraclitus spacetime (Proposition 11) shows precisely this. One simply adds to a Heraclitus spacetime any random tensor field that is not definable in terms of the metric. The resulting spacetime has more structure than the one that we began with, but it has the same local homomorphisms, since the only local homomorphisms of a Heraclitus spacetime were identity maps to begin with. As we mentioned above, Heraclitus spacetimes are utterly devoid of symmetries, and so their mere existence puts pressure on the above dogma about symmetry and amount of structure.

We will conclude by mentioning one final attempt to salvage the dogma that one might make that seems to avoid the difficulties posed by Heraclitus spacetimes. It has recently been suggested that when considering amounts of structure we should change the question we are asking (Barrett 2021). Instead of asking whether one object has more structure than another, we should ask whether one *kind* of object has more structure than another *kind* of object. For example, instead of asking whether a Heraclitus spacetime has more or less structure than a Heraclitus spacetime with additional tensor field on it, we ask whether a manifold with metric has more or less structure than a manifold with metric and an additional tensor field. In order to answer this new question, one looks to the entire classes of objects of those two kinds, along with all of the structure-preserving maps between them. In essence, we are here again liberalizing what we mean by "symmetry"; one now considers all structure-preserving maps between objects of the same kind as "symmetries". One conjectures that this move will save the dogma – indeed, it has been emphasized elsewhere that symmetries in this most general sense do suffice to capture facts about definability (Barrett 2018, 2021) – but it requires a stark conceptual revision. In order to judge amounts of structure, one is now not just looking at maps from the spacetime to itself. Rather, one has to take a much more holistic approach. Only by looking at maps from our spacetime to and from other spacetimes of the same kind (and maps between these other spacetimes too) can one hope to use symmetries as a guide to amounts of structure.

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