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Ehrenfest Theorems, Deformation Quantization, and the Geometry of Inter-Model Reduction

Joshua Rosaler

Abstract This study attempts to spell out more explicitly than has been done previously the connection between two types of formal correspondence that arise in the study of quantum-classical relations: one the one hand, deformation quantization and the associated continuity between quantum and classical algebras of observables in the limit $\hbar \to 0$, and, on the other, a certain generalization of Ehrenfest's Theorem and the result that expectation values of position and momentum evolve approximately classically for narrow wave packet states. While deformation quantization establishes a direct continuity between the abstract algebras of quantum and classical observables, the latter result makes in-eliminable reference to the quantum and classical state spaces on which these structures act - specifically, via restriction to narrow wave packet states. Here, we describe a certain geometrical re-formulation and extension of the result that expectation values evolve approximately classically for narrow wave packet states, which relies essentially on the postulates of deformation quantization, but describes a relationship between the *actions* of quantum and classical algebras and groups over their respective state spaces that is non-trivially distinct from deformation quantization. The goals of the discussion are partly pedagogical in that it aims to provide a clear, explicit synthesis of known results; however, the particular synthesis offered aspires to some novelty in its emphasis on a certain general type of mathematical and physical relationship between the state spaces of different models that represent the same physical system, and in the explicitness with which it details the above-mentioned connection between quantum and classical models.

Keywords Quantum · Classical · Deformation Quantization · Ehrenfest's Theorem · Reduction

1 Introduction

The relationship between quantum and classical mechanics is characterized by many distinct correspondences: deformation quantization, Ehrenfest's Theorem, the WKB approximation, environmental decoherence, consistent histories, various speculative accounts of quantum measurement, and many others. It remains unclear precisely how these correspondences are related to one another, or how they might be placed within a more unified picture of the relationship between quantum and classical theories. The importance of clarifying the relationships among these disparate results, and their distinct roles within a more unified understanding of quantum-classical relations, has been emphasized repeatedly by Landsman, whose own work has furthered the project of providing such a synthesis in many important ways; see, for example, [8], [9], [10].

Institute for Theoretical Particle Physics and Cosmology, RWTH Aachen University

The goal of the present discussion is to clarify more explicitly than we believe has been done previously the connection between two types of correspondence in this web of disparate results: between, on the one hand, a certain generalization of the result, which can be derived from Ehrenfest's Theorem, that for narrowly peaked wave functions, expectation values of position and momentum follow approximately Hamiltonian trajectories, and, on the other hand, the specific continuity between the abstract algebras of quantum and classical observables, and between the groups of canonical and unitary transformations that these algebras respectively generate, associated with deformation quantization and the limit $\hbar \to 0$. The goals of the discussion are partly pedagogical insofar as it aims at clarification and synthesis of existing results. However, the investigation aspires to some novelty in the particular synthesis that it provides - in particular, in its application of a certain simple, general type of mathematical and physical relationship between the state spaces of different models that describe the same physical system, whereby one model "reduces to" another in the sense that the latter furnishes a strictly more accurate and detailed description of the system's behavior. This relationship rests on the existence of a certain "bridging" function B from the state space of the more accurate, more encompassing, "low-level" model to the state space of the less accurate, less encompassing "high-level" model, which serves to identify the low-level model's representation of the state of those physical degrees of freedom of the system that are approximately represented in the state space of the high-level model. It also rests on the identification of a privileged subset d of the low-level model's state space such that, on timescales over which low-level state remains within this subset, the trajectory that the low-level state evolution induces over the high-level state space (via the function B) matches the corresponding solution of the high-level dynamics within a small empirically determined margin of error.

Adopting this general framing, we show that over the subset d, the function B establishes an approximate Lie algebra homomorphism from the set of vector field generators of unitary flows in Hilbert space to the set of vector field generators of canonical flows in phase space, and an approximate Lie group homomorphism from the set of unitary transformations that preserve d to the associated set of canonical transformations. Both approximate homomorphisms rest on the postulates of deformation quantization and become exact in certain limits where $\hbar \to 0$ and where the position and momentum widths of the quantum state vanish. This continuity between the state space actions of unitary and canonical transformations, and of their associated algebras, is clearly distinct from the continuity that arises in the context of deformation quantization, since it depends crucially on restrictions on the quantum state, while deformation quantization, which directly relates abstract groups and algebras without reference to their actions over particular spaces, imposes no such restrictions.

Our discussion is outlined as follows. Section 2 briefly describes the basic framework, concerning the relationship between state space formulations of two models describing the same physical system, that will be used to explore the connection between quantum and classical models. Section 3 provides a alternative geometrical derivation and interpretation of Ehrenfest's Theorem, and its corollary that expectation values evolve approximately classically when wave packets remain narrow, based on this general framework. Section 4 generalizes this formulation, which relates the actions of dynamical unitary and canonical transformations, to a relationship between the actions of general continuous unitary and canonical transformations. Section 5 shows that the push-forward mapping B^* from vector fields over Hilbert space to vector fields over phase space establishes an approximate, restricted Lie algebra homomorphism from vector field generators of unitary transformations over Hilbert space to vector field generators of canonical transformations over phase space, which holds only over the subset d of Hilbert space consisting (roughly speaking) of narrow wave packet states. On the basis of this result, it further shows that the the function B establishes an approximate, restricted Lie group homomorphism from the set of unitary transformations that act within the subset d of narrow wave packets in Hilbert space (which constitute a neighborhood of the identity transformation) to the set of corresponding canonical transformations. It is further shown that these approximate homomorphisms become exact in certain limits where $\hbar \to 0$ and where the position and momentum widths of the quantum state are concurrently approach zero. We should note that our discussion bears some similarity to the analysis of group contractions given by Wigner and Inonu, but is clearly distinct in its essential reliance on the mapping B between state spaces and the restriction to the subset d of the low-level state space [6].

2 A Template for Reduction between Dynamical Systems Models

In many cases, the behavior of a single system can be accurately represented by distinct models, often from different theories, where one model offers a strictly more accurate and more widely applicable description of the system's behavior than the other. For example, the motion of a single charged particle can be accurately described by mathematical models from classical mechanics, [1] quantum mechanics, relativistic quantum mechanics, and quantum field theory. Each successive model in this sequence offers a purportedly more accurate and more detailed description of the electron's behavior than the previous one. It is natural to ask: for each pair of models in this succession, what sort of mathematical relationship does the low-level (i.e., purportedly more fundamental) model bear to high-level (i.e., purportedly less fundamental) model such that it is capable of representing all features of the electron that are represented by the high-level model at least as accurately as the high-level model does? That is, by virtue of what mathematical relationship between models does the high-level model "reduce to" [2] the low-level model?

There is a simple template that applies in many cases where both high- and low-level models can be formulated as continuous, deterministic dynamical systems, and that can be extended with slight modification to other kinds of model as well. Let the high-level model $M_h = (S_h, D_h)$ of the system K be specified by some state space S_h possessing the structure of a differentiable manifold endowed with a metric (whose primary purpose is to provide a quantitative notion of proximity between different states) and some deterministic rule of dynamical evolution $D_h : S_h \to S_h$ given by the one-parameter group of differentiable functions $D_h(x_h, t)$. Likewise, let the low-level model $M_l = (S_l, D_l)$ of the same system K be specified by a different state space S_l also possessing the structure of a differentiable manifold, and some rule of dynamical evolution $D_l : S_l \to S_l$ given by the one-parameter group of differentiable functions $D_l(x_l, t)$. Moreover, assume that the dynamical evolution in each model satisfies the relations $D(x, t_1 + t_2) = D(D(x, t_1), t_2)$ for all $t_1, t_2 \in \mathbb{R}$, with $D(x, 0) = x$.

In general, the state spaces S_h and S_l may describe different degrees of freedom of the system K, with the degrees of freedom represented by S_h depending on the degrees of freedom represented by S_l in such a way that there can be no difference in the state of the S_h degrees of freedom without some corresponding difference in the state of the S_l degrees of freedom (just as, for example, there can be no difference in the center of mass of an object without a difference in the positions of its constituent particles). In the context of a direct mathematical relationship between the two models, this dependence relation can often be represented by a fixed, time-independent function B that maps from the low-level state space S_l to the high-level state space S_h. The quantity $B(x_l)$, where $x_l \in S_l$, furnishes

[1] Note that the classical Lorentz Force Law, rather than a quantum mechanical equation, is often used to describe the trajectories of charged particles in an accelerator.

[2] There are two conflicting conventions concerning the use of the term "reduce." According to the convention used here, it is the less encompassing low-level model that "reduces to" the more encompassing low-level model. Precedent for this usage can be found, for example, in Anderson's famous article, "More is Different" [1]. According to the second convention, it is the more encompassing low-level model that reduces to the less-encompassing high-level model. Precedent for this usage can be found in the often-made claim that quantum mechanics "reduces to" classical mechanics, or that special relativity "reduces to" Newtonian mechanics. Here, we employ the first of these conventions, so that it is the classical model of a system that purportedly reduces to the quantum model of that same system.

the low-level model's representation of those features of the system K that are represented by the high-level state $x_h \in S_h$; the time evolution $B(x_l(t))$ of this quantity is determined by the low-level dynamics, since the evolution $x_l(t)$ is determined by the low-level dynamics and the function B is fixed. If the low-level model is to furnish at least as accurate a representation of these features of K as the high-level model does, it must be the case that for each physically realistic [3] solution $x_h(t)$ of the high-level model, there exists some solution $x_l(t)$ of the low-level model such that $B(x_l(t))$ tracks the corresponding features of K at least as accurately as $x_h(t)$. In turn, this will require that the induced trajectory $B(x_l(t))$ and the high-level trajectory $x_h(t)$ be approximately equal, at least over those timescales τ for which $x_h(t)$ approximates a physically realistic trajectory for K. More formally, for any physically realistic trajectory $D_h(x_h, t)$, there must exist an $x_l \in S_l$ such that $x_h = B(x_l)$ and such that

$$B(D_l(x_l, t)) \approx D_h(B(x_l), t) \tag{1}$$

for $0 \leq t \leq \tau$, where τ is the timescale over which $D_h(B(x_l), t)$ approximates a physically realistic trajectory of K (see Figure 1). [4]

In many cases, the trajectories $D(x, t)$ in both high- and low-level models are integral curves of some dynamical vector field V over S. That is, they are solutions of a first-order differential equation of the form $\frac{dx}{dt} = V|_x$. In such cases, one can show that (1) holds by showing that, by virtue of the low-level dynamics, the quantity $B(x_l)$ approximately satisfies the high-level equations of motion for all x_l in some subset d of S_l - i.e., that

$$\frac{dB(x_l(t))}{dt} \approx V_h|_{B(x_l)} \tag{2}$$

for $x_l \in d \subset S_l$. As we show in Appendix B, by integrating both sides of this relation, one can then recover (1) for timescales T over which the low-level state x_l remains in d. The quantity $B(x_l)$ will succeed at tracking the relevant features of K over timescales where x_h does if $T \geq \tau$ - that is, if the timescale over which the low-level state remains in the subset d of the low-level state space is greater than or equal to the timescale over which the high-level trajectory tracks the relevant features of the system. [5]

[3] To further clarify what is meant by "physically realistic" here, note that a solution of a classical model according to which the system achieves speeds greater than that of light would *not* be physically realistic. By contrast, virtually any low-momentum solution with velocity substantially less than that of light would be counted as physically realistic, in that it provides a reasonably accurate approximation to the system's behavior. Of course, since every model represents the system that it describes only up to some margin of error, the precise boundaries of what counts as physically realistic will depend on somewhat abitrary error thresholds.

[4] More precisely, the requirement $B(D_l(x_l, t)) \approx D_h(B(x_l), t)$ should be understood as the requirement that $|B(D_l(x_l, t)) - D_h(B(x_l), t)| < \delta$, where δ is the allowable margin of error associated with the requirement of approximate equality. The choice of δ is constrained by how accurately the high-level model is known to represent the behavior of the relevant degrees of freedom of the system K in question, since we would like the low-level model's representation of these features to be at least as accurate as the high-level model's representation.

[5] To give an example of (2) not discussed further in this article, consider the case of a slow-moving electron, which can be accurately described both by non-relativistic Pauli and relativistic Dirac models of a spin-1/2 particle. In such a case, we expect the Dirac model to furnish a strictly more accurate description of the system than the Pauli model, although the Pauli model works quite well. The function B connecting the state spaces of the two models here will be the function from the low-level space of Dirac 4-spinors to the high-level space of Pauli 2-spinors that projects a 4-spinor onto its upper two components. The relevant subset d of the low-level space will be the set of 4-spinors with non-relativistic momentum. The vector field components V_h and V_l of the high- and low-level models in this case can be read directly off of the equations of motion of the two models (the Pauli and Dirac equations, respectively), which are both first-order in time.

To give another example of (2), consider the relationship between the quantum mechanical model of N non-relativistic free particles and the quantum field theoretic model of a free Klein-Gordon field. The function B in this case maps the QFT state $|\Psi\rangle$ into the quantum mechanical state with wave function $\psi(x_1, ..., x_n) \equiv \langle 0|\hat{\phi}(x_1)...\hat{\phi}(x_n)|\Psi\rangle$.The relevant subset d of the low-level state space in this case is given by the set of N-particle states of the field which include only non-relativistic momenta. The vector field components V_h and V_l of the high- and low-level models in this case can be read directly off of the equations

The relation (2), in turn, will be satisfied if the high-level vector field V_h evaluated at $B(x_l)$ is approximately equal to the push-forward under B of the low-level vector field V_l evaluated at x_l - that is, if

$$V_h|_{B(x_l)} \approx B^*(V_l|_{x_l}) \tag{3}$$

or, in component form,

$$V_h^{\mu}\big|_{B(x_l)} \approx \frac{\partial B^{\mu}}{\partial x_l^{\nu}} V_l^{\nu}\big|_{x_l} \tag{4}$$

- for all x_l in d (See Figure 2). This can be seen by applying the Chain Rule $\frac{dB^{\mu}}{dt} = \frac{\partial B^{\mu}}{\partial x_l^{\nu}} \frac{dx_l^{\nu}}{dt}$ to the left-hand side of (2), and employing the substitution $V_l^{\nu} = \frac{dx_l^{\nu}}{dt}$, which is based on the low-level equation of motion.

Thus, if we take d to be *defined* as the subset of S_l for which the push-forward condition (4) holds (for an appropriate margin of approximate equality), then condition (1) will hold for all $x_l \in d$ and for all t over the (x_l-dependent) timescale T over which $D_l(x_l, t)$ remains in d. [6] Since $D_h(x_h, t) = \exp(V_h t)x_h'|_{x_h}$ and $D_l(x_l, t) = \exp(V_l t)x_l'|_{x_l}$, relation (1) can we written in terms of the vector fields V_h and V_l as

$$B\left((\exp(V_l t)x_l')\,|_{x_l}\right) \approx (\exp(V_h t)x_h')\,|_{B(x_l)}. \tag{5}$$

Reduction requires that the timescale T on which the induced trajectory (LHS) approximates the high-level trajectory (RHS) be greater than or equal to the timescale τ over which the high-level trajectory approximates a physically realistic trajectory for the system K - i.e., that $T \geq \tau$.

To summarize, in many cases where two dynamical systems models describe the same physical system K, the quantity $B(x_l)$, whose time evolution is determined by the low-level dynamics, represents the same features of K that the high-level state x_h does, and so functions in a sense as the low-level model's surrogate for x_h. If for states in some subset d of S_l the high-level dynamical vector field V_h evaluated at $B(x_l)$ approximates the push-forward under B of the low-level dynamical vector field evaluated at x_l, then the high-level trajectory $B(D_l(x_l, t))$ induced on S_h through B by the low-level dynamics will approximate the corresponding high-level trajectory $D_h(B(x_l), t)$ over the timescale where $D_l(x_l, t)$ remains in d. For M_l to furnish at least as accurate representation of the system K as M_h, this timescale should be at least as long as the timescale over which the high-level trajectory $D_h(B(x_l), t)$ approximates a physically realistic trajectory for K itself.

3 Ehrenfest's Theorem: A Geometrical View

Let us now consider a system K that can be described both classically and quantum mechanically - say, a heavy charged particle with a localized wave packet propagating in a fixed electrostatic field. Let the high-level model in this case be a classical model whose state space is N-particle phase space Γ and whose dynamics are given by Hamilton's equations with Hamiltonian $H = \frac{p^2}{2m} + V(x)$. Let the low-level model be a quantum mechanical model whose state space is the Hilbert space $\mathcal{H}$ of N spinless particles and whose dynamics are

of motion of the two models (the Schrodinger equations for free N-particle quantum mechanics and free Klein-Gordon quantum field theory, respectively), which are both first-order in time. In both of the examples discussed here, the pair of models in question satisfies the relation (2), or equivalently (4). We will discuss yet another example of these general relations, connecting classical and quantum models, in the next section on Ehrenfest's Theorem.

[6] If the relation (4) holds within margin of error ϵ for all $x_l \in d$, then the approximate equality (1) will hold within margin of error ϵT for all $0 \leq t < T$, where T is the timescale on which the low-level state remains in d.

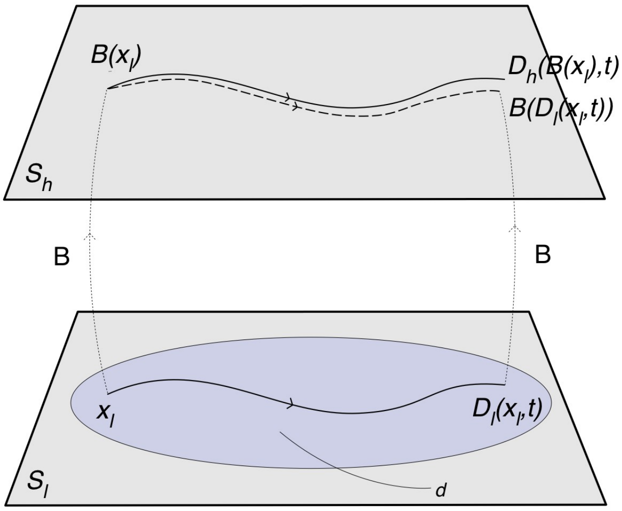

Fig. 1 Reduction between the high-level model $M_h = (S_h, D_h)$ and the low-level model $M_l = (S_l, D_l)$ requires that for every physically realistic high-level trajectory $D_h(x_h, t)$, there exist a low-level trajectory $D_l(x_l, t)$ such that x_h is the image under B of some $x_l \in S_l$ and such that the induced high-level trajectory $B(D_l(x_l, t))$ approximates $D_h(x_h, t)$ over those timescales where $D_h(x_h, t)$ continues to approximate a realistic trajectory for the system. In many cases, the approximate equality of the two trajectories will hold exclusively for x_l in some restricted subset d of S_l.

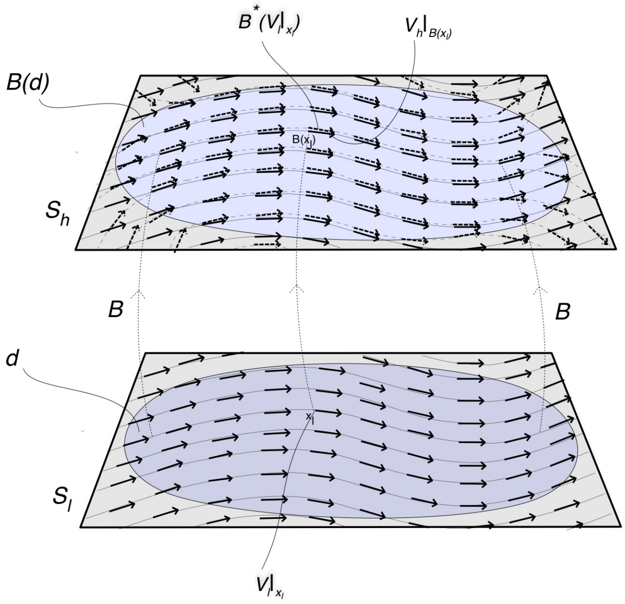

Fig. 2 If the high-level dynamical vector field V_h evaluated at $B(x_l)$ is approximately equal to the push-forward of the low-level vector field V_l (dotted arrows in upper state space) evaluated at x_l, for all $x_l \in d$, then the trajectory induced on S_h through B by the integral curves of V_l in d will approximate the integral curves of V_h in the image domain $B(d) \subset S_h$.

given by Schrodinger's equation with Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$. In the classical model, the vector field that generates the dynamical evolution on Γ is

$$V_h = \frac{\partial H}{\partial p^\mu}\frac{\partial}{\partial q^\mu} - \frac{\partial H}{\partial q^\mu}\frac{\partial}{\partial p^\mu} = \frac{p^\mu}{m}\frac{\partial}{\partial q^\mu} - \frac{\partial V}{\partial q^\mu}\frac{\partial}{\partial p^\mu}, \tag{6}$$

with summation over repeated indices. The high-level dynamical equations $\frac{dx_h^\mu}{dt} = V_h|_{x_h}$ are given by Hamilton's equations $\frac{dq^\mu}{dt} = \frac{p^\mu}{m}$, $\frac{dp^\mu}{dt} = -\frac{\partial V}{\partial q^\mu}$. The vector field over $\mathcal{H}$ that generates the unitary quantum evolution is

$$\begin{aligned} V_l &= \frac{1}{i\hbar}\hat{H}|\psi\rangle\frac{\partial}{\partial|\psi\rangle} - \frac{1}{i\hbar}\langle\psi|\hat{H}\frac{\partial}{\partial\langle\psi|} \\ &\equiv \frac{1}{i\hbar}H^{ab}w^b\frac{\partial}{\partial w^a} - \frac{1}{i\hbar}\bar{w}^a H^{ab}\frac{\partial}{\partial\bar{w}^b}, \end{aligned} \tag{7}$$

where $w^a \equiv \langle e^a|\psi\rangle$ for a given orthonormal basis $\{|e^a\rangle\}$ of $\mathcal{H}$, and $|\phi\rangle\frac{\partial}{\partial|\psi\rangle} \equiv \langle e^a|\phi\rangle\frac{\partial}{\partial\langle e^a|\psi\rangle}$, $\langle\phi|\frac{\partial}{\partial\langle\psi|} \equiv \langle\phi|e^a\rangle\frac{\partial}{\partial\langle\psi|e^a\rangle}$ for any $|\phi\rangle \in \mathcal{H}$ (one can think of $\frac{\partial}{\partial|\psi\rangle}$ as the gradient operator whose components are $\frac{\partial}{\partial w^a}$, and likewise for $\frac{\partial}{\partial\langle\psi|}$ with respect to the $\frac{\partial}{\partial\bar{w}^a}$).

One might initially expect the low-level dynamical vector field over $\mathcal{H}$ to be given by

$$\begin{aligned} V_l &= \frac{1}{i\hbar}\hat{H}|\psi\rangle\frac{\partial}{\partial|\psi\rangle} \\ &\equiv \frac{1}{i\hbar}H^{ab}w^b\frac{\partial}{\partial w^a}, \end{aligned} \tag{8}$$

since the corresponding low-level dynamical equations $\frac{dx_l^\mu}{dt} = V_l|_{x_l}$ then take the form of Schrodinger's equation, $\frac{d|\psi\rangle}{dt} = \frac{1}{i\hbar}\hat{H}|\psi\rangle$. However, because $\mathcal{H}$ is a complex manifold, its full tangent space is spanned by the vectors $\{\frac{\partial}{\partial w^a}, \frac{\partial}{\partial\bar{w}^a}\}$. The set $\{\frac{\partial}{\partial w^a}\}$ does not span the full set of derivations operating on differentiable functions over $\mathcal{H}$, but only the set of derivations operating on functions that are *complex* differentiable and so do not depend on the $\bar{w}^a$. However, many functions of physical interest in quantum theory, including expectation values of Hermitian observables, are differentiable but not complex differentiable and therefore depend on both the w^a and the $\bar{w}^a$: note that $\langle\psi|\hat{A}|\psi\rangle = \bar{w}^a A^{ab} w^b$. For this reason, if we wish to differentiate such a function along the unitary flow associated with a Hermitian observable such as $\hat{H}$, it is necessary to include both holomorphic and anti-holomorphic basis elements of the tangent space, $\frac{\partial}{\partial w^a} and \frac{\partial}{\partial\bar{w}^a}$, in the vector field that generates this flow.

We can see this more concretely by differentiating the expectation value $\langle\psi|\hat{A}|\psi\rangle$ of a Hermitian operator $\hat{A}$ along the unitary flow $|\psi(s)\rangle = e^{-\frac{i}{\hbar}\hat{C}s}|\psi\rangle$ associated with the Hermitian operator $\hat{C}$. A simple application of the Chain Rule gives,

$$\begin{aligned} \frac{d}{ds}\langle\psi(s)|\hat{A}|\psi(s)\rangle &= \left(\frac{d|\psi\rangle}{ds}\frac{\partial}{\partial|\psi\rangle} + \frac{d\langle\psi|}{ds}\frac{\partial}{\partial\langle\psi|}\right)\langle\psi(s)|\hat{A}|\psi(s)\rangle \\ &= \left(-\frac{i}{\hbar}\hat{C}|\psi\rangle\frac{\partial}{\partial|\psi\rangle} + \frac{i}{\hbar}\langle\psi|\hat{C}\frac{\partial}{\partial\langle\psi|}\right)\langle\psi(s)|\hat{A}|\psi(s)\rangle \\ &= \langle\psi(s)|\frac{1}{i\hbar}[\hat{A},\hat{C}]|\psi(s)\rangle. \end{aligned} \tag{9}$$

The second line shows explicitly that both the holomorphic and anti-holomorphic basis elements $\frac{\partial}{\partial|\psi\rangle}$ and $\frac{\partial}{\partial\langle\psi|}$ of $T_{|\psi\rangle}\mathcal{H}$ are needed to differentiate expectation values, and non-holomorphic functions generally, along a unitary flow in $\mathcal{H}$. In the case where $\hat{C} = \hat{H}$ and $s = t$, the vector field that diffierentiates arbitrary differentiable functions along the flow generated by $\hat{C}$ is given by the expression (7).

Employing the framework for inter-model reduction described in the previous section, the "bridge function" B that we consider here carries a quantum state into the classical phase space point associated with expectation values of position and momentum:

$$\begin{aligned} &B : \mathcal{H} \longrightarrow \Gamma \\ &B : |\psi\rangle \longmapsto \left(\langle\psi|\hat{q}|\psi\rangle, \langle\psi|\hat{p}|\psi\rangle\right). \end{aligned} \tag{10}$$

As we will see, the relevant subset d of the low-level state space, over which we can expect the quantity B to approximately satisfy the high-level dynamical equations (i.e., Hamilton's equations), and over which the push-forward condition (3) holds between the vector fields (7) and (6), is

$$d = \{|\psi\rangle \in \mathcal{H}| \ \sigma_q < l_V\}, \tag{11}$$

where l_V depends on the characteristic scale of spatial variation of the potential V and σ_q is the spatial width of the quantum state. [7]

Let us now confirm that the condition (3) holds between the vector fields (7) and (6), with B given by (10) and d given by (11). We begin by calculating the push-forward of V_l under B:

$$\begin{aligned} B^*(V_l) &= \frac{\partial B^\mu}{\partial x_l^\nu} V_l^\nu \frac{\partial}{\partial x_h^\mu} \\ &= \left[\frac{\partial B^{q^\mu}}{\partial w^a}(-\frac{i}{\hbar}H^{ab}w^b) + \frac{\partial B^{q^\mu}}{\partial \bar{w}^a}(\frac{i}{\hbar}\bar{w}^b H^{ba})\right]\frac{\partial}{\partial q^\mu} \\ &\quad + \left[\frac{\partial B^{p^\mu}}{\partial w^a}(-\frac{i}{\hbar}H^{ab}w^b) + \frac{\partial B^{p^\mu}}{\partial \bar{w}^a}(\frac{i}{\hbar}\bar{w}^b H^{ba})\right]\frac{\partial}{\partial p^\mu} \\ &= \frac{1}{i\hbar}\left[\bar{w}^c (q^\mu)^{ca} H^{ab} w^b - \bar{w}^b H^{ba} (q^\mu)^{ac} w^c\right]\frac{\partial}{\partial q^\mu} \\ &\quad + \frac{1}{i\hbar}\left[\bar{w}^c (p^\mu)^{ca} H^{ab} w^b - \bar{w}^b H^{ba} (p^\mu)^{ac} w^c\right]\frac{\partial}{\partial p^\mu} \\ &= \frac{1}{i\hbar}\langle\psi|[\hat{q}^\mu, \hat{H}]|\psi\rangle\frac{\partial}{\partial q^\mu} + \frac{1}{i\hbar}\langle\psi|[\hat{p}^\mu, \hat{H}]|\psi\rangle\frac{\partial}{\partial p^\mu} \\ &= \langle\psi|\frac{\partial\hat{H}}{\partial p^\mu}|\psi\rangle\frac{\partial}{\partial q^\mu} - \langle\psi|\frac{\partial\hat{H}}{\partial q^\mu}|\psi\rangle\frac{\partial}{\partial p^\mu} \\ &= \langle\psi|\frac{\hat{p}^\mu}{m}|\psi\rangle\frac{\partial}{\partial q^\mu} - \langle\psi|\frac{\partial\hat{V}}{\partial q^\mu}|\psi\rangle\frac{\partial}{\partial p^\mu}. \end{aligned} \tag{12}$$

Since $\frac{dB^\mu(x_l(t))}{dt} = \frac{\partial B^\mu}{\partial x_l^\nu}\frac{dx_l^\nu}{dt} = \frac{\partial B^\mu}{\partial x_l^\nu}V_l^\nu$, it follows from this last line that

$$\frac{d\langle\hat{q}^\mu\rangle}{dt} = \frac{\langle\hat{p}^\mu\rangle}{m} \tag{13}$$

$$\frac{d\langle\hat{p}^\mu\rangle}{dt} = -\langle\frac{\partial\hat{V}}{\partial q}\rangle, \tag{14}$$

[7] The length scale l_V can be quantified precisely in terms of the potential V and its derivatives, by requiring that $\langle\frac{\partial\hat{V}}{\partial q}\rangle \approx \frac{\partial V}{\partial q}\big|_{\langle\hat{q}\rangle}$. Assuming that the state $|\psi\rangle$, whose$|\psi(x)|^2$ distribution has standard deviation σ_q, can be approximated by a Gaussian of width σ_q, one can then expand the expectation value $\langle\frac{\partial\hat{V}}{\partial q}\rangle$ in powers of σ_q. The zeroth-order term in this expansion is $\frac{\partial V}{\partial q}\big|_{\langle\hat{q}\rangle}$. Requiring higher-order terms to be vanishingly small at each order imposes a set of constraints requiring σ_q to be smaller than some set of length scales (one for each order) defined in terms of V and its spatial derivatives. The value of l_V may be defined as the smallest of these length scales.

where the second of these relations is none other than Ehrenfest's Theorem. In fact , these relations follow more directly as a special case of (9), with $s = t$, $\hat{A} = (\hat{x}, \hat{p})$, and $\hat{C} = \hat{H}$. However, it is important for the purposes of our discussion below to highlight the role of the push-forward relation B^* between high- and low-level dynamical vector fields, since this same push-forward mapping will turn out to provide an approximate homomorphism from the vector field Lie algebra associated with the unitary group action over Hilbert space to the vector field Lie algebra associated with the canonical group action over phase space. [8]

Evaluated over the domain d of wave packets narrowly peaked in position, we have that $\langle\psi|\frac{\partial \hat{V}}{\partial q^\mu}|\psi\rangle = \frac{\partial V}{\partial q^\mu}\big|_{\langle\hat{q}\rangle} + \mathcal{O}(\sigma_q) \approx \frac{\partial V}{\partial q^\mu}\big|_{\langle\hat{q}\rangle} = \frac{\partial H}{\partial q^\mu}\big|_{\langle\hat{q}\rangle,\langle\hat{p}\rangle}$. Because the operator $\hat{p}^\mu$ is linear in $\hat{p}$, we also have $\frac{\langle p^\mu\rangle}{m} = \langle\frac{\partial H}{\partial p^\mu}\rangle = \frac{\partial H}{\partial p^\mu}\Big|_{\langle\hat{q}\rangle,\langle\hat{p}\rangle}$. The final two expressions in (12) then yield

$$\begin{aligned}
B^*(V_l) &= \frac{\partial H}{\partial p^\mu}\bigg|_{\langle\hat{q}\rangle,\langle\hat{p}\rangle}\frac{\partial}{\partial q^\mu} - \frac{\partial H}{\partial q^\mu}\bigg|_{\langle\hat{q}\rangle,\langle\hat{p}\rangle}\frac{\partial}{\partial p^\mu} + \mathcal{O}(\sigma_q) \\
&= \frac{\langle p^\mu\rangle}{m}\frac{\partial}{\partial q^\mu} - \frac{\partial V}{\partial q^\mu}(\langle\hat{q}\rangle)\frac{\partial}{\partial p^\mu} + \mathcal{O}(\sigma_q) \\
&\approx \frac{\langle p^\mu\rangle}{m}\frac{\partial}{\partial q^\mu} - \frac{\partial V}{\partial q^\mu}(\langle\hat{q}\rangle)\frac{\partial}{\partial p^\mu} \\
&= V_h\big|_{B(x_l)}.
\end{aligned} \tag{15}$$

From this we see that relation (3) is satisfied with respect to the domain d and bridge function B in this case. As we saw in the previous section, the relation $B^*(V_l) \approx V_h\big|_{B(x_l)}$ entails that the low-level model's surrogate for x_h, $B(x_l)$, approximately satisfies the high-level model's equations of motion, so that $\frac{dB(x_l)}{dt} \approx V_h\big|_{B(x_l)}$. In the present context, this relationship entails that expectation values of position and momentum approximately satisfy the equations for the integral curves of V_h - i.e., Hamilton's equations:

$$\frac{d\langle\hat{q}^\mu\rangle}{dt} = \frac{\langle\hat{p}^\mu\rangle}{m} \tag{16}$$

$$\frac{d\langle\hat{p}^\mu\rangle}{dt} \approx -\frac{\partial V}{\partial q}\bigg|_{\langle\hat{q}\rangle}. \tag{17}$$

If V is of quadratic or lower order, the second, approximate equality will hold exactly and for all states. However, if V is higher than quadratic order, this approximate equality will only hold only on timescales over which the spatial width σ_q of the time-evolved quantum state $e^{-i\hat{H}t}|\psi\rangle$ remains less than l_V.

The above relations entail that on the timescale T where the quantum state remains in the subset d, the evolution induced on classical phase space via the function B by the Schrodinger evolution approximates the phase space evolution prescribed by Hamilton's equations - that is, that the relation (1) will be satisfied over timescale T. Formally, this reads

$$e^{\{H,\cdot\}t}(q,p)\big|_{\langle\hat{q}\rangle,\langle\hat{p}\rangle} \approx \Big(\langle\psi|e^{\frac{i}{\hbar}\hat{H}t}\ \hat{q}\ e^{-\frac{i}{\hbar}\hat{H}t}|\psi\rangle, \langle\psi|e^{\frac{i}{\hbar}\hat{H}t}\ \hat{p}\ e^{-\frac{i}{\hbar}\hat{H}t}|\psi\rangle\Big) \tag{18}$$

[8] More conventional derivations of Ehrenfest's Theorem can be found in most graduate and undergraduate texts on quantum mechanics. A discussion of its application to the case of narrow wave packets can be found in [3].

for $0 \leq t \leq T$, where T is the timescale over which $e^{-\frac{i}{\hbar}\hat{H}t}|\psi\rangle \in d$. [9] Reduction in this case requires that $T \geq \tau$, where τ is timescale over which the classical model provides a good approximation to the behavior of the system K. [10]

In this section, we have re-cast results associated with Ehrenfest's Theorem within the more general geometrical framework for reduction described in the previous section. In the next section, we will see how this type of relationship can be extended from the case of quantum and classical time evolution to include relations between continuous unitary and canonical transformations more generally.

4 Beyond Ehrenfest's Theorem: Approximately Canonical Phase Space Flow Induced by Continuous Unitary Transformations

The relations (15) and (18) illustrate a particular type of relationship between a one-parameter group of unitary transformations and the corresponding one-parameter group of canonical transformations, associated respectively with quantum and classical time evolutions. The relation (15) connects the vector fields that generate these groups, while the relation (18) connects the group transformations themselves. In both cases, the relationship is mediated by the function B from the quantum to the classical state space, and depends on a restriction to the subset d of Hilbert space. In this section, we will see that with slight modifications, this type of correspondence can be extended to relations between arbitrary one-parameter unitary group actions over Hilbert space and the corresponding one-parameter canonical group actions over phase space. Like the previous section's discussion of Ehrenfest's Theorem, this more general correspondence is mediated by the function B from Hilbert space to phase space specified in (10), and rests on a restriction to a certain subset d of Hilbert space. However, by contrast with the subset defined in (11), the relevant subset of Hilbert space is now defined by the requirement that wave packets be narrowly peaked in both position *and* momentum. Because the uncertainty principle $\sigma_q \sigma_p \geq \frac{\hbar}{2}$ restricts the degree to which a quantum state can be simultaneously localized in both of these variables, the formal limit $\hbar \to 0$ - in which such simultaneous localization becomes possible to an arbitrary degree of precision (if only in a mathematical, rather than a physical, sense) - will figure centrally into our discussion.

Just as one can associate to the function H the dynamical Hamiltonian vector field (6), more generally one can associate to each classical observable $f \in C^\infty(\Gamma)$ the Hamiltonian vector field

$$U_h = \frac{\partial f}{\partial p^\mu}\frac{\partial}{\partial q^\mu} - \frac{\partial f}{\partial q^\mu}\frac{\partial}{\partial p^\mu} \tag{21}$$

over Γ. The vector field U_h generates a one-parameter group of canonical transformations $G_h(x_h, s) \equiv \exp(U_h s)x'_h\big|_{x_h} = e^{\{f,\cdot\}s}(q,p)\big|_{q_0,p_0}$. Likewise, to each quantum observable, asso-

[9] The exponentiated Poisson bracket is defined by

$$\left(e^{\{f,\cdot\}s}u\right)\big|_{q_0,p_0} \equiv u\big|_{q_0,p_0} + s\{f,u\}\big|_{q_0,p_0} + \frac{1}{2!}s^2\{f,\{f,u\}\}\big|_{q_0,p_0} \tag{19}$$

$$+ \frac{1}{3!}s^3\{f,\{f,\{f,u\}\}\}\big|_{q_0,p_0}\cdots \tag{20}$$

where $f, u \in C^\infty(\Gamma)$, $s \in \mathbb{R}$. The notation $e^{\{H,\cdot\}t}(q,p)\big|_{\langle\hat{q}\rangle,\langle\hat{p}\rangle}$ indicates that the exponential is being applied component-wise to the functions q and p in $C^\infty(\Gamma)$, and that the resulting expression is then evaluated at $(\langle\hat{q}\rangle, \langle\hat{p}\rangle)$.

[10] A more precise account of the relationship between quantum and classical models of a system like a baseball or a charged particle would take into account the interaction between the system and its environment in the context of the quantum model, where effects of entanglement with external degrees of freedom can make a dramatic difference to the system's evolution. The extensive field of decoherence theory has shown how such effects can be modeled; see, for example, [17], [7], [14], and references therein. For a general conceptual discussion of the role of decoherence in quantum mechanics, see [2].

ciated with some Hermitian operator $\hat{f}$ on $\mathcal{H}$, is associated a one-parameter group of unitary transformations generated by the vector field

$$U_l \equiv -\frac{i}{\hbar}\hat{f}|\psi\rangle\frac{\partial}{\partial|\psi\rangle} + \frac{i}{\hbar}\langle\psi|\hat{f}\frac{\partial}{\partial\langle\psi|} \tag{22}$$

over $\mathcal{H}$. The vector field U_l generates a one-parameter group of unitary transformations $G_l(x_l, s) \equiv \exp(U_l s)x_l'\big|_{x_l} = e^{-\frac{i}{\hbar}\hat{f}s}|\psi_0\rangle$.

The process of quantization maps a given classical observable f into a unique Hermitian operator $\hat{f}$ on Hilbert space. Different types of quantization, such as deformation quantization and geometric quantizaiton, impose different constraints on this mapping. Here we will focus on deformation quantization. Studies of deformation quantization tend to rely on one of two equivalent mathematical formalisms. The first treats quantum observables as elements of a C^* algebra, often associated with operators acting on some complex Hilbert space. The second formalism, which is employed more frequently in the literature on deformation quantization, relies on so-called phase space formulations of quantum mechanics, which treat quantum observables as functions over classical phase space (and, therefore, in a manner that more closely resembles the conventional treatment of classical observables). In the C^* algebraic formalism, the algebra of quantum observables possesses two sorts of product: for two elements $\hat{f}$ and $\hat{g}$ of the C^* algebra, the first product corresponds to simple multiplication of operators, $\hat{f}\hat{g} = \widehat{fg} + \mathcal{O}(\hbar)$, while the second corresponds to the usual commutator, $\frac{1}{i\hbar}[\hat{f}, \hat{g}] = \widehat{\{f, g\}} + \mathcal{O}(\hbar)$. In the formalism of phase space quantum mechanics, the simple product of operators in the C^* algebraic formalism corresponds to a so-called star product $f \star_\hbar g = fg + \mathcal{O}(\hbar)$ between functions over classical phase space, which converges to classical pointwise multiplication in the limit $\hbar \to 0$. Likewise, the operator commutator in the C^* algebraic approach corresponds in the formalism of phase space quantum mechanics to a so-called star bracket $\{f, g\}_\hbar \equiv f \star g - g \star f = \{f, g\} + \mathcal{O}(\hbar)$, which converges to the classical Poisson bracket $\{f, g\}$ in the limit $\hbar \to 0$. Our discussion here will focus on operator-based formulations of deformation quantization. Within this setting, we assume a quantization map that carries a classical observable f, given by a smooth function on phase space, to the quantum observable $\hat{f}$ associated with the formal power series expansion that is obtained given by replacing monomial terms $q^n p^m$ in the Taylor expansion of f with "Hermitized" operators $\mathrm{Herm}(\hat{q}^n\hat{p}^m)$:

$$\hat{f} \equiv \sum_{k=1}^{\infty}\sum_{n+m=k}\frac{1}{n!m!}\frac{\partial^{n+m} f}{\partial q^n \partial p^m}\bigg|_{0,0}\mathrm{Herm}(\hat{q}^n\hat{p}^m), \tag{23}$$

where $\mathrm{Herm}(\hat{q}^n\hat{p}^m)$ is a re-ordering of the non-Hermitian operator $\hat{q}^n\hat{p}^m$ that renders it Hermitian. [11] For example, in the Weyl quantization scheme, we have $\mathrm{Herm}(\hat{q}^n\hat{p}^m) = \frac{1}{(n+m)!}\sum_{\sigma\in S_{n+m}}\sigma(\hat{q}, \overset{\times n}{\ldots}, \hat{q}, \hat{p}, \overset{\times m}{\ldots}, \hat{p})$ where for any operators $\hat{F}_1, ..., \hat{F}_n$ and any $\sigma \in S_n$, $\sigma(\hat{F}_1, ..., \hat{F}_n) = \hat{F}_{\sigma_1}...\hat{F}_{\sigma_n}$. [12] Other prescriptions, such as $\mathrm{Herm}(\hat{q}^n\hat{p}^m) = \frac{1}{2}\left(\hat{q}^n\hat{p}^m + \hat{p}^m\hat{q}^n\right)$, are also possible. However, inequivalent prescriptions differ only at $\mathcal{O}(\hbar)$ - that is, $\mathrm{Herm}_1(\hat{q}^n\hat{p}^m) = \mathrm{Herm}_2(\hat{q}^n\hat{p}^m) + \mathcal{O}(\hbar) = \hat{q}^n\hat{p}^m + \mathcal{O}(\hbar)$. The quantization map of the form (23) satisfies

$$\hat{f}\hat{g} = \widehat{fg} + \mathcal{O}(\hbar) \tag{24}$$

$$[\hat{f}, \hat{g}] = i\hbar\widehat{\{f, g\}} + \mathcal{O}(\hbar^2), \tag{25}$$

[11] See, for example, [11] for more detailed discussion of this quantization strategy.

[12] See [15] for extended discussion of Weyl quantization of general smooth functions f using formal power series.

as required by the axioms of deformation quantization. [13] [14]

This section highlights two central claims, the first pertaining to relations between vector field generators of continuous unitary and canonical transformations on $\mathcal{H}$ and Γ, respectively, and the second pertaining to relations between the corresponding one-parameter group actions on these spaces. Both claims make heavy use of the function B invoked in our discussion of Ehrenfest's Theorem, given by (10), and of the set $d \subset \mathcal{H}$ of quantum states that are narrowly peaked in *both* position and momentum (instead of just in position, as before):

$$d = \{|\psi\rangle \in \mathcal{H} | \sigma_q < l_q, \sigma_p < l_p\} \tag{26}$$

where σ_q and σ_p are the position- and momentum-space widths [15] of the quantum state $|\psi\rangle$ and l_q and l_p are the upper limits on the position and momentum widths of states in d, with $l_q l_p > \frac{\hbar}{2}$.

The first claim, concerning relations between vector field generators of continuous unitary and canonical transformations, is that

$$B^*(U_l\big|_{x_l}) = U_h\big|_{B(x_l)} + \mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p) + \mathcal{O}(\hbar). \tag{27}$$

or, in component form,

$$\frac{\partial B^\mu}{\partial x_l^\nu} U_l^\nu\big|_{x_l} = U_h^\mu\big|_{B(x_l)} + \mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p) + \mathcal{O}(\hbar), \tag{28}$$

for $x_l \in d$ as defined in (26). Henceforth, the reader should take d to refer to the subset of Hilbert space defined by (26) rather than (11). In terms of the observables $\hat{f}$ and f, (27) reads,

$$\begin{aligned} &\langle\psi|\frac{\partial\hat{f}}{\partial p^\mu}|\psi\rangle\frac{\partial}{\partial q^\mu} - \langle\psi|\frac{\partial\hat{f}}{\partial q^\mu}|\psi\rangle\frac{\partial}{\partial p^\mu} \\ &= \frac{\partial f}{\partial p^\mu}\bigg|_{\langle\hat{q}\rangle,\langle\hat{p}\rangle}\frac{\partial}{\partial q^\mu} - \frac{\partial f}{\partial q^\mu}\bigg|_{\langle\hat{q}\rangle,\langle\hat{p}\rangle}\frac{\partial}{\partial p^\mu} + \mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p) + \mathcal{O}(\hbar). \end{aligned} \tag{29}$$

For states in d, the values of σ_q and σ_q will be "small" in the sense that they are bounded from above, respectively, by l_q and l_p. If the terms $\mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p) + \mathcal{O}(\hbar)$ are collectively small enough in magnitude to be neglected, we recover the approximate equality,

$$\begin{aligned} U_h\big|_{B(x_l)} &\approx B^*(U_l\big|_{x_l}) \\ U_h^\mu\big|_{B(x_l)} &\approx \frac{\partial B^\mu}{\partial x_l^\nu} U_l^\nu\big|_{x_l} \end{aligned} \tag{30}$$

[13] For a detailed formulation of the axioms of deformation quantization in the setting of operator algebras, see, for example, [8] or [12].

[14] It is worth noting here that early efforts at quantization sought a mapping from quantum to classical observables that satisfied the more restrictive requirement $[\hat{f}, \hat{g}] = i\hbar\widehat{\{f, g\}}$. However, the Groenewold-van Hove no-go Theorem entails that this is not generally possible for classical observables outside of the subalgebra of polynomials of order two. More precisely, it states there is no map $f \mapsto \hat{f}$ from functions on $\mathbb{R}^2$ (the phase space of a single spatial degree of freedom) to self-adjoint operators on $L^2(\mathbb{R})$ satisfying both the requirement that 1) $[\hat{f}, \hat{g}] = i\hbar\widehat{\{f, g\}}$ and 2) that $\hat{q}$ and $\hat{p}$, defined as the images of q and p under the mapping, coincide with the usual Schrodinger position and momentum operators, for any Lie subalgebra of the functions on $\mathbb{R}^2$ larger than the subalgebra of polynomials of degree less than or equal to two. See, for example, [4], Ch. 5 and [16], Ch. 17 for further discussion of this result.

[15] Here, we define the position width of a quantum state to be the standard deviation of position with respect to the distribution $|\psi(q)|^2$ and likewise for the momentum width with respect to the distribution $|\tilde{\psi}(p)|^2$. However, it should be noted here that other definitions of the state width are also possible, and give rise to alternate statements of the uncertainty principle, as Hilgevoord and Uffink show in [5].

for $x \in d$. If $\hbar$ is treated as a formal parameter that can be varied, rather than as a fixed constant, the terms $\mathcal{O}(\sigma_q)+\mathcal{O}(\sigma_p)+\mathcal{O}(\hbar)$ can be made to vanish in the collective limit where $\hbar \to 0$, $\sigma_q \to 0$, and $\sigma_p \to 0$, where the constraint $\sigma_q\sigma_p \geq \frac{\hbar}{2}$ is respected as the limit is taken. [16] However, for real physical systems, the value of $\hbar$ is *fixed* at its measured value, and there is therefore a lower bound on the collective magnitude of the terms $\mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p) + \mathcal{O}(\hbar)$. The interpretation of the limit $(\hbar, \sigma_q, \sigma_p) \to 0$ taken subject to the constraint $\sigma_q\sigma_p \geq \frac{\hbar}{2}$, and the conditions under which these error terms can be neglected for $\hbar$ fixed at its physical value, are discussed further in Section **??**.

Relation (29) can be shown to hold in largely the same manner as relation (15) of the previous section. Begin by calculating the pushforward under B of the vector field U_l defined in (22):

$$\begin{aligned}
B^*(U_l) &= -\frac{i}{\hbar}\langle\psi|[\hat{q}^\mu, \hat{f}]|\psi\rangle\frac{\partial}{\partial q^\mu} - \frac{i}{\hbar}\langle\psi|[\hat{p}^\mu, \hat{f}]|\psi\rangle\frac{\partial}{\partial p^\mu} \\
&= \langle\psi|\left(\frac{\hat{\partial f}}{\partial p^\mu} + \mathcal{O}(\hbar)\right)|\psi\rangle\frac{\partial}{\partial q^\mu} - \langle\psi|\left(\frac{\hat{\partial f}}{\partial q^\mu} + \mathcal{O}(\hbar)\right)|\psi\rangle\frac{\partial}{\partial p^\mu} \\
&= \langle\psi|\frac{\hat{\partial f}}{\partial p^\mu}|\psi\rangle\frac{\partial}{\partial q^\mu} - \langle\psi|\frac{\hat{\partial f}}{\partial q^\mu}|\psi\rangle\frac{\partial}{\partial p^\mu} + \mathcal{O}(\hbar),
\end{aligned} \tag{31}$$

where in going from the first to the second line we have made use of the fact that $[\hat{q}, \hat{f}(\hat{q},\hat{p})] = i\hbar\frac{\hat{\partial A}}{\partial p} + \mathcal{O}(\hbar^2)$ and $[\hat{p}, \hat{f}(\hat{q},\hat{p})] = -i\hbar\frac{\hat{\partial f}}{\partial q} + \mathcal{O}(\hbar^2)$. For states in the domain d defined by (26), a given Hermitian observable $\hat{h}$ is related to its classical counterpart h not only by the quantization map (23), but also in terms of expectation values by

$$\langle\psi|\hat{h}|\psi\rangle = h\left(\langle\hat{q}\rangle, \langle\hat{p}\rangle\right) + \mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p) + \mathcal{O}(\hbar), \tag{32}$$

as shown in Appendix A. Since σ_q and σ_p will be small for states in d, the terms $\mathcal{O}(\sigma_q)$ and $\mathcal{O}(\sigma_p)$ will also tend to be small, inasmuch as they vanish with σ_q and σ_p, respectively. [17] From (32) and (31), it follows that

$$\begin{aligned}
B^*(U_l) &= \left.\frac{\partial f}{\partial p^\mu}\right|_{\langle\hat{q}\rangle,\langle\hat{p}\rangle}\frac{\partial}{\partial q^\mu} - \left.\frac{\partial f}{\partial q^\mu}\right|_{\langle\hat{q}\rangle,\langle\hat{p}\rangle}\frac{\partial}{\partial p^\mu} \\
&\qquad + \mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p) + \mathcal{O}(\hbar) \\
&= U_h\big|_{B(x_l)} + \mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p) + \mathcal{O}(\hbar).
\end{aligned} \tag{33}$$

For $\hbar$, σ_q, and σ_q sufficiently small, this yields,

$$B^*(U_l) \approx U_h\big|_{B(x_l)} \tag{34}$$

or in component form,

[16] Strictly speaking, the vanishing of the terms $\mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p) + \mathcal{O}(\hbar)$ in this limit also requires that the series expansions defining the operators $\frac{\hat{\partial f}}{\partial p^\mu}$ and $\frac{\hat{\partial f}}{\partial q^\mu}$ converge. Proofs given here are formal rather than rigorous; where necessary, the reader may assume that the classical and quantum observables in question are restricted to those for which (23) converges for a given choice of Hermitization prescription. However, in keeping with common practice, we do not address the detailed requirements for convergence of formal power series such as (23).

[17] A similar result is proven by Landsman for the specific case of minimum uncertainty wave packets [8], [9]. Here, we do not assume that the quantum states in d are necessarily minimum uncertainty wave packets; it is possible that $\sigma_q\sigma_p > \frac{\hbar}{2}$ for states in d. We only require that states in d satisfy the constraint that $\sigma_q < l_q$ and $\sigma_p < l_p$. Since σ_q and σ_p can be varied independently of each other and of $\hbar$ (apart from the constraint $\sigma_q\sigma_p \geq \frac{\hbar}{2}$) these three variables are treated here as independent parameters.

$$\langle\psi|\frac{\partial\hat{f}}{\partial p^{\mu}}|\psi\rangle\frac{\partial}{\partial q^{\mu}}-\langle\psi|\frac{\partial\hat{f}}{\partial q^{\mu}}|\psi\rangle\frac{\partial}{\partial p^{\mu}}$$
$$\approx\frac{\partial f}{\partial p^{\mu}}\bigg|_{\langle\hat{q}\rangle,\langle\hat{p}\rangle}\frac{\partial}{\partial q^{\mu}}-\frac{\partial f}{\partial q^{\mu}}\bigg|_{\langle\hat{q}\rangle,\langle\hat{p}\rangle}\frac{\partial}{\partial p^{\mu}}. \tag{35}$$

In short, when evaluated over the domain $d\subset\mathcal{H}$, the push-forward under B of the quantum-mechanical vector field over $\mathcal{H}$ associated with the Hermitian operator $\hat{f}$ approximates the classical Hamiltonian vector field over Γ associated with the corresponding phase space function f. Thus, the push-forward transformation B^* approximately maps an element of the Lie algebra of vector fields that generate unitary group actions over Hilbert space into the unique corresponding element of the Lie algebra of vector fields that generate canonical transformations over phase space.

In the limit $(\hbar,\sigma_q,\sigma_p)\to 0$ taken subject to the constraint $\sigma_q\sigma_p\geq\frac{\hbar}{2}$, the error terms $\mathcal{O}(\sigma_q)+\mathcal{O}(\sigma_p)+\mathcal{O}(\hbar)$ vanish and the approximate equality (35) becomes exact. It should be noted that for fixed $(\hbar,\sigma_q,\sigma_p)$, the error terms $\mathcal{O}(\sigma_q)+\mathcal{O}(\sigma_p)+\mathcal{O}(\hbar)$ may fall beneath one's fixed margin of approximate equality for some choices of f but not others. In particular, these higher-order terms may be non-negligible in size if the function f varies rapidly on the scale of σ_q, σ_p, or $\hbar$. However, as $(\hbar,\sigma_q,\sigma_p)\to 0$ subject to the constraint $\sigma_q\sigma_p\geq\frac{\hbar}{2}$, σ_q, σ_p, and $\hbar$ will at some point become small by comparison with the scale of variation of the function f. Heuristically, the terms $\mathcal{O}(\sigma_q)+\mathcal{O}(\sigma_p)+\mathcal{O}(\hbar)$ become negligible for an ever-expanding subset of the corresponding low-level algebra of vector fields over $\mathcal{H}$, expanding to encompass the full algebra in this limit.

As we will see in the next section, the push-forward mapping is also an approximate Lie algebra homomorphism over the domain d in that it approximately preserves the composition structure of vector fields in these algebras. This homomorphism, which is clearly distinct from (although closely related to) the approximate Lie algebra homomorphism furnished by a given deformation quantization, likewise becomes exact in the limit $(\hbar,\sigma_q,\sigma_p)\to 0$, taken subject to the constraint $\sigma_q\sigma_p\geq\frac{\hbar}{2}$.

We now review several consequences of the relations (27) and (34), which are derived in Appendix (B). If $s\in\mathbb{R}$ parameterizes the continuous transformations generated by both U_l and U_h, it follows from (28) that $\frac{dB(x_l)}{ds}\approx U_h|_{B(x_l)}$ for $x_l\in d$ up to terms $\mathcal{O}(\sigma_q)+\mathcal{O}(\sigma_p)+\mathcal{O}(\hbar)$ - i.e., that

$$\frac{d\langle\hat{q}\rangle}{ds}=\frac{\partial f}{\partial p}\bigg|_{\langle\hat{q}\rangle,\langle\hat{p}\rangle}+\mathcal{O}(\sigma_q)+\mathcal{O}(\sigma_p)+\mathcal{O}(\hbar) \tag{36}$$
$$\frac{d\langle\hat{p}\rangle}{ds}=-\frac{\partial f}{\partial q}\bigg|_{\langle\hat{q}\rangle,\langle\hat{p}\rangle}+\mathcal{O}(\sigma_q)+\mathcal{O}(\sigma_p)+\mathcal{O}(\hbar). \tag{37}$$

for $|\psi\rangle\in d$. For $\hbar$, σ_q, and σ_q sufficiently small, this gives

$$\frac{d\langle\hat{q}\rangle}{ds}\approx\frac{\partial f}{\partial p}\bigg|_{\langle\hat{q}\rangle,\langle\hat{p}\rangle}$$
$$\frac{d\langle\hat{p}\rangle}{ds}\approx-\frac{\partial f}{\partial q}\bigg|_{\langle\hat{q}\rangle,\langle\hat{p}\rangle}. \tag{38}$$

That is, under the low-level unitary transformation generated by $\hat{f}$, the quantity $(\langle\psi(s)|\hat{q}|\psi(s)\rangle,\langle\psi(s)|\hat{p}|\psi(s)\rangle)$ approximately satisfies the first-order equations for integral curves of the high-level vector field U_h.

At the level of finite group transformations, the relation (35), or alternatively (38), implies a relation analogous to (1). When acting within $d\subset\mathcal{H}$, the unitary transformation generated by U_l induces via B a transformation on Γ that approximates the canonical transformation generated by U_h. More formally, defining

$$G_h(x_h, s) \equiv e^{\{f,\cdot\}s}(q,p)\big|_{q_0,p_0} \tag{39}$$

$$G_l(x_l, s) \equiv e^{-\frac{i}{\hbar}\hat{f}s}|\psi_0\rangle, \tag{40}$$

with f and $\hat{f}$ related by (23), it follows from (30) that

$$B(G_l(x_l,s)) = G_h(B(x_l), s) + \epsilon \tag{41}$$

for s such that $G_l(x_l, s') \in d$ for all $0 \le s' \le s$, where the error $\epsilon \to 0$ as $\hbar \to 0$, and $l_q \to 0$, $l_p \to 0$ concurrently, subject to the constraint $l_q l_p > \frac{\hbar}{2}$. In Section 5.3, we will see that for fixed s, ϵ vanishes in a certain carefully defined limit in which $\hbar$, l_q, l_p, and the initial state widths σ_q and σ_p associated with x_l all approach zero. For the moment, let us assume that the values of $\hbar$, l_q, and l_p are set at some small but fixed values (where $\hbar$, again, is regarded as a formal parameter). In this case, we may write,

$$B(G_l(x_l,s)) \approx G_h(B(x_l), s), \tag{42}$$

if $G_l(x_l, s') \in d$ for all $0 \le s' \le s$. This result extends (1) to general continuous unitary and canonical transformations. In terms of the quantum and classical observables $\hat{f}$ and f, this reads

$$e^{\{f,\cdot\}s}(q,p)\big|_{\langle\hat{q}\rangle,\langle\hat{p}\rangle} \approx \left(\langle\psi|e^{\frac{i}{\hbar}\hat{f}s}\hat{q}e^{-\frac{i}{\hbar}\hat{f}s}|\psi\rangle, \langle\psi|e^{\frac{i}{\hbar}\hat{f}s}\hat{p}e^{-\frac{i}{\hbar}\hat{f}s}|\psi\rangle\right). \tag{43}$$

Note that, for a given $x_l = |\psi_0\rangle \in d$, the approximate equality between the phase space trajectory $B(G_l(x_l,s))$ induced by a unitary transformation and the phase space trajectory associated with the action of the corresponding canonical transformation $G_h(B(x_l), s)$ does not hold for arbitrarily large values of the parameter s, but only for unitary transformations in some neighborhood of the identity (namely, those that keep the initial state x_l in d). However, we will see in 5.3 that in a certain limit, this approximate equality of induced and high-level trajectories becomes exact and holds for all transformations that are continuously connected to the identity.

It is also worth highlighting here that rotations, spatial translations, and Galiliean boosts are included in both the canonical and unitary groups of transformations. One can easily check that the relations (30) and (43) hold with exact rather than approximate equality for all states in $\mathcal{H}$ and for all values of the associated continuous transformation parameters. However, for more general unitary transformations and their associated canoncial transformations, these relations hold only approximately, only over the restricted domain d, and only for transformations in a restricted neighborhood of the identity.

5 Approximate, Restricted Lie Algebra and Lie Group Homomorphisms

In this section, we will see that in the collective limit where $\hbar \to 0$, $\sigma_q \to 0$, $\sigma_p \to 0$ subject to the constraint $\sigma_q\sigma_p \ge \frac{\hbar}{2}$, the push forward mapping B^* furnishes a homomorphism from the Lie algebra of vector fields of the form (22) over $d \subset \mathcal{H}$ to the algebra of Hamiltonian vector fields over Γ. [18] Likewise, we will see that in a similar limit, the function B furnishes a homomorphism from the group of finite unitary transformations of the form (40) acting within d to the group of finite canonical transformations of the form (39) acting over Γ.

[18] Note that $B(d) = \Gamma$ - that is, the image of the domain d under B is the whole phase space Γ.

5.1 An Approximate Lie Algebra Homomorphism

Let us first discuss the claim concerning vector field Lie algebras over $\mathcal{H}$ and Γ, from which the corresponding claim regarding associated finite Lie group transformations will follow. Define

$$U_h \equiv \frac{\partial f}{\partial p^\mu}\frac{\partial}{\partial q^\mu} - \frac{\partial f}{\partial q^\mu}\frac{\partial}{\partial p^\mu} \tag{44}$$

$$V_h \equiv \frac{\partial g}{\partial p^\mu}\frac{\partial}{\partial q^\mu} - \frac{\partial g}{\partial q^\mu}\frac{\partial}{\partial p^\mu} \tag{45}$$

$$U_l \equiv -\frac{i}{\hbar}\hat{f}|\psi\rangle\frac{\partial}{\partial|\psi\rangle} + \frac{i}{\hbar}\langle\psi|\hat{f}\frac{\partial}{\partial\langle\psi|} \tag{46}$$

$$V_l \equiv -\frac{i}{\hbar}\hat{g}|\psi\rangle\frac{\partial}{\partial|\psi\rangle} + \frac{i}{\hbar}\langle\psi|\hat{g}\frac{\partial}{\partial\langle\psi|}, \tag{47}$$

with $f, g \in C^\infty(\Gamma)$, and where both the pairs f and $\hat{f}$, and g and $\hat{g}$, are related by a deformation quantization mapping of the form (23).

The first core claim of this section, then, is that the push-forward under B of the Lie product of the low-level vector fields U_l and V_l (here, given by the low-level vector field commutator) evaluated at $x_l \in d$ is approximately equal to the Lie product of the high-level vector fields U_h and V_h (given by the high-level vector field commutator) evaluated at $B(x_l)$, up to terms $\mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p) + \mathcal{O}(\hbar)$. Formally, this reads,

$$[U_h, V_h]|_{B(x_l)} = B^*([U_l, V_l]|_{x_l}) + \mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p) + \mathcal{O}(\hbar), \tag{48}$$

or in component form,

$$\left(V_h^\mu\partial_\mu U_h^\nu - U_h^\mu\partial_\mu V_h^\nu\right)\Big|_{B(x_l)} \tag{49}$$

$$= \frac{\partial B^\nu}{\partial x_l^\eta}\Big|_{x_l}\left(V_l^\kappa\partial_\kappa U_l^\eta - U_l^\kappa\partial_\kappa V_l^\eta\right)\Big|_{x_l} \tag{50}$$

$$+ \mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p) + \mathcal{O}(\hbar) \tag{51}$$

for $x_l \in d$ for all pairs of vector fields of the form (U_l, V_l). For $\hbar$, σ_q, and σ_q sufficiently small,, the push-forward map B^* is an approximate Lie algebra homomorphism over the domain d:

$$\boxed{[U_h, V_h]|_{B(x_l)} \approx B^*([U_l, V_l]|_{x_l})} \tag{52}$$

or, in component form,

$$\left(V_h^\mu\partial_\mu U_h^\nu - U_h^\mu\partial_\mu V_h^\nu\right)\Big|_{B(x_l)} \tag{53}$$

$$\approx \frac{\partial B^\nu}{\partial x_l^\eta}\Big|_{x_l}\left(V_l^\kappa\partial_\kappa U_l^\eta - U_l^\kappa\partial_\kappa V_l^\eta\right)\Big|_{x_l} \tag{54}$$

for $x_l \in d$ (see Figure 3). As above, it should be noted that for fixed $(\hbar, \sigma_q, \sigma_p)$, the error terms $\mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p) + \mathcal{O}(\hbar)$ may fall beneath one's fixed margin of approximate equality for some choices of U_l and V_l but not others. In particular, these higher-order terms may be non-negligible in size if the functions f, g vary rapidly on the scale of σ_q, σ_p, or $\hbar$. However, as $(\hbar, \sigma_q, \sigma_p) \to 0$ subject to the constraint $\sigma_q\sigma_p \geq \frac{\hbar}{2}$, the terms $\mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p) + \mathcal{O}(\hbar)$ will

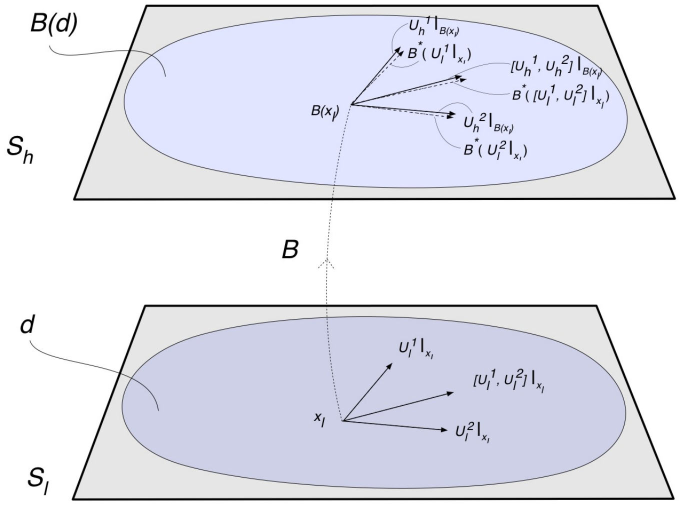

Fig. 3 Over the domain d specified in (26), the push-forward of the Lie product of two low-level quantum-mechanical vector fields U_l and V_l evaluated at x_l approximates the Lie product of the corresponding high-level classical vector fields U_h and V_h evaluated at $B(x_l)$.

become negligible for an ever-expanding subset of the full low-level algebra of vector fields over $\mathcal{H}$, expanding to encompass the full algebra in this limit.

We can see that relation (48) holds as follows. The high- and low-level vector field commutators $[U_h, V_h]|_{B(x_l)}$ and $[U_l, V_l]|_{x_l}$, respectively, are

$$[U_h, V_h]|_{B(x_l)} = \frac{\partial\{f,g\}}{\partial q^\mu}\bigg|_{\langle\hat{x}\rangle,\langle\hat{p}\rangle}\frac{\partial}{\partial p^\mu} - \frac{\partial\{f,g\}}{\partial p^\mu}\bigg|_{\langle\hat{x}\rangle,\langle\hat{p}\rangle}\frac{\partial}{\partial q^\mu} \tag{55}$$

$$[U_l, V_l]|_{x_l} = \left(-\frac{i}{\hbar}\right)^2 [\hat{f},\hat{g}]|\psi\rangle\frac{\partial}{\partial|\psi\rangle} + \left(\frac{i}{\hbar}\right)^2 \langle\psi|[\hat{g},\hat{f}]\frac{\partial}{\partial\langle\psi|}. \tag{56}$$

The push forward of $[U_l, V_l]|_{x_l}$ under B is

$$\begin{aligned} &B^*([U_l, V_l]|_{x_l}) \\ &= \left(\frac{1}{i\hbar}\right)^2 \langle\psi|[\hat{q}^\mu,[\hat{f},\hat{g}]]|\psi\rangle\frac{\partial}{\partial q^\mu} + \left(\frac{1}{i\hbar}\right)^2 \langle\psi|[\hat{p}^\mu,[\hat{f},\hat{g}]]|\psi\rangle\frac{\partial}{\partial p^\mu}. \end{aligned} \tag{57}$$

The commutator $\frac{1}{i\hbar}[\hat{f},\hat{g}]$ satisfies

$$\frac{1}{i\hbar}[\hat{f},\hat{g}] = \widehat{\{f,g\}} + \mathcal{O}(\hbar) \tag{58}$$

in accordance with the axioms of deformation quantization, as can readily be confirmed using the definition (23). Substitution into (57) yields

$$\begin{aligned} &B^*([U_l, V_l]|_{x_l}) \\ &= \frac{1}{i\hbar}\langle\psi|[\hat{q}^\mu,\widehat{\{f,g\}}]|\psi\rangle\frac{\partial}{\partial q^\mu} + \frac{1}{i\hbar}\langle\psi|[\hat{p}^\mu,\widehat{\{f,g\}}]|\psi\rangle\frac{\partial}{\partial p^\mu} + \mathcal{O}(\hbar) \\ &= \langle\psi|\frac{\partial\widehat{\{f,g\}}}{\partial p^\mu}|\psi\rangle\frac{\partial}{\partial q^\mu} - \langle\psi|\frac{\partial\widehat{\{f,g\}}}{\partial q^\mu}|\psi\rangle\frac{\partial}{\partial p^\mu} + \mathcal{O}(\hbar). \end{aligned} \tag{59}$$

For $x_l = |\psi\rangle$ in the domain d, we have

$$\begin{aligned}
&B^*([U_l, V_l]|_{x_l}) \\
&= \frac{\partial\{f,g\}}{\partial p^\mu}\Big|_{\langle\hat{q}\rangle,\langle\hat{p}\rangle}\frac{\partial}{\partial q^\mu} - \frac{\partial\{f,g\}}{\partial q^\mu}\Big|_{\langle\hat{q}\rangle,\langle\hat{p}\rangle}\frac{\partial}{\partial p^\mu} + \mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p) + \mathcal{O}(\hbar) \\
&= [U_h, V_h]|_{B(x_l)} + \mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p) + \mathcal{O}(\hbar),
\end{aligned} \tag{60}$$

yielding the desired result (48). Again, for $(\hbar, \sigma_q, \sigma_p)$ sufficiently small, this yields the approximate equality (52), and exact equality in the limit where $\hbar \to 0$, $\sigma_q \to 0$, $\sigma_p \to 0$ in keeping with the constraint $\sigma_x\sigma_p \geq \frac{\hbar}{2}$.

5.2 An Approximate Lie Group Homomorphism

As we have seen, to each of the classical observables f and g corresponds a one-parameter group of canonical transformations on Γ:

$$G_h^1(x_h, u) = \exp(U_h u)x_h'\big|_{x_h} = e^{\{f,\cdot\}u}(q,p)\big|_{q_0,p_0} \tag{61}$$

$$G_h^2(x_h, v) = \exp(V_h v)x_h'\big|_{x_h} = e^{\{g,\cdot\}v}(q,p)\big|_{q_0,p_0} \tag{62}$$

where $x_h = (q_0, p_0)$ and $u, v \in \mathbb{R}$. Likewise, to each of the quantum observables $\hat{f}$ and $\hat{g}$ corresponds a one-parameter group of unitary transformations on $\mathcal{H}$:

$$G_l^1(x_l, u) = \exp(U_l u)x_l'\big|_{x_l} = e^{-\frac{i}{\hbar}\hat{f}u}|\psi_0\rangle \tag{63}$$

$$G_l^2(x_l, v) = \exp(V_l v)x_l'\big|_{x_l} = e^{-\frac{i}{\hbar}\hat{g}v}|\psi_0\rangle, \tag{64}$$

where $x_l = |\psi_0\rangle$, and u, v are the same real parameters as in (61) and (62).

The second core claim of this section, then, is that if $G_l^2(G_l^1(x_l, u'), v') \in d$ for all $0 \leq u' \leq u$ and $0 \leq v' \leq v$,

$$G_h^2(G_h^1(B(x_l), u), v) = B(G_l^2(G_l^1(x_l, u), v)) + \epsilon \tag{65}$$

where the error ϵ will be small if $\hbar$, l_q, and l_p are. As already mentioned, we will see in Section 5.3 that for fixed (u, v), ϵ vanishes in a certain limit in which $\hbar$, l_q, l_p, and the initial state widths σ_q and σ_p associated with x_l all approach zero. Assuming that the values of $\hbar$, l_q, l_p are set at some small but fixed values, we may write,

$$\boxed{G_h^2(G_h^1(B(x_l), u), v) \approx B(G_l^2(G_l^1(x_l, u), v))} \tag{66}$$

if $G_l^2(G_l^1(x_l, u'), v') \in d$ for all $0 \leq u' \leq u$ and $0 \leq v' \leq v$. Note that for a given x_l and for fixed $(\hbar, l_q, l_p)$, (66) holds only for parameter values (u, v) in some restricted neighborhood of $(0, 0)$ (which corresponds to the identity transformation). However, we will see that in the special limit just described, this neighborhood grows to enompass all real values of (u, v).

Insofar as B maps the action over Hilbert space associated with the product of two unitary transformations into a transformation on phase space that approximates the product of the corresponding two canonical transformations, it functions as a partial, approximate homomorphism from unitary group actions over the subset d of Hilbert space to canonical group actions over phase space. One sense in which the homomorphism is only partial is that it holds only for transformations in some restricted neighborhood of the identity, since continuous unitary transformations do not generally preserve the set d for arbitrarily large parameter values. However, we will see that in a certain limit, this neighborhood expands to include the full set of unitary transformations continuously connected to the identity. Only in this limit does the approximate homomorphism furnished by B become exact and extend to cover the full group of continuous unitary transformations.

The relation (65), which connects unitary group actions over Hilbert space and canonical group actions over phase space, can be shown to hold by virtue of the corresponding relation (48) relating the actions of these groups' Lie algebras. Re-rewritten in terms of vector field generators and the exponential map, the relation (65) that we wish to derive is,

$$B(\exp(V_l v)\exp(U_l u)x_l'\big|_{x_l}) = \exp(V_h v)\exp(U_h u)x_h\big|_{B(x_l)} + \epsilon \tag{67}$$

for all (u, v) such that $\exp(V_l v')\exp(U_l u')x_l'\big|_{x_l} \in d$ for all $0 \leq u' \leq u$ and $0 \leq v' \leq v$. Defining

$$W_h = \frac{1}{w}\log(\exp(V_h v)\exp(U_h u)) \tag{68}$$

$$W_l = \frac{1}{w}\log(\exp(V_l v)\exp(U_l u)) \tag{69}$$

so that $\exp(W_h w) = \exp(V_h v)\exp(u_h u)$ and $\exp(W_l w) = \exp(V_l v)\exp(U_l u)$, with $w \in \mathbb{R}$, we wish to show that

$$B(\exp(W_l w)x_l'\big|_{x_l}) = \exp(W_h w)x_h'\big|_{B(x_l)} + \epsilon \tag{70}$$

if $\exp(W_l w')x_l'\big|_{x_l} \in d$ for all $0 \leq w' \leq w$. As we have seen, to show this, it suffices to show that

$$W_h\big|_{B(x_l)} = B^*(W_l\big|_{x_l}) + \mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p) + \mathcal{O}(\hbar) \tag{71}$$

for all $x_l \in d$. To show this, in turn, we may invoke the Baker-Campbell-Hausdorff Lemma, which provides an expansion the vector fields W in terms of the vector fields U and V:

$$\begin{aligned} wW_h &\equiv \log(\exp(V_h v)\exp(U_h u)) \\ &= \sum_{n>0}\frac{(-1)^{n-1}}{n}\sum_{\substack{r_i+s_i>0\\1\leq i\leq n}}\frac{\left(\sum_{i=1}^n (r_i+s_i)\right)^{-1}}{r_1!s_1!...r_n!s_n!} \\ &\qquad v^{r_1}u^{s_1}...v^{r_n}u^{s_n}\left[V_h^{r_1}U_h^{s_1}...V_h^{r_n}U_h^{s_n}\right], \end{aligned} \tag{72}$$

$$\begin{aligned} wW_l &= \log(\exp(V_l v)\exp(U_l u)) \\ &= \sum_{n>0}\frac{(-1)^{n-1}}{n}\sum_{\substack{r_i+s_i>0\\1\leq i\leq n}}\frac{\left(\sum_{i=1}^n (r_i+s_i)\right)^{-1}}{r_1!s_1!...r_n!s_n!} \\ &\qquad v^{r_1}u^{s_1}...v^{r_n}u^{s_n}\left[V_l^{r_1}U_l^{s_1}....V_l^{r_n}U_l^{s_n}\right]. \end{aligned} \tag{73}$$

The multi-bracket $[V^{r_1}U^{s_1}...V^{r_n}U^{s_n}]$ in these expressions is defined by

$$
\begin{aligned}
&[V^{r_1}U^{s_1}...V^{r_n}U^{s_n}] \equiv && (74)\\
&\quad \underbrace{[V,[V,...[V}_{r_1},\underbrace{[U,[U,...[U}_{s_1},...\underbrace{[V,[V,...[V}_{r_n},\underbrace{[U,[U,...[U,U}_{s_n}]]...]], && (75)
\end{aligned}
$$

where, in the present context, the commutator on the right-hand side is the vector field commutator, and where this term is zero if $s_n > 1$, or if $s_n = 0$ and $r_n > 1$ [13]. By linearity of the push-forward map, the condition (71) will follow if we can show that

$$
\begin{aligned}
[V_h^{r_1}U_h^{s_1}...V_h^{r_n}U_h^{s_n}]\,\big|_{B(x_l)} = B^*\Big([V_l^{r_1}U_l^{s_1}...V_l^{r_n}U_l^{s_n}]\,\big|_{x_l}\Big)& \\
+\,\mathcal{O}(\sigma_q)+\mathcal{O}(\sigma_p)+\mathcal{O}(\hbar)& \qquad (76)
\end{aligned}
$$

for all $x_l \in d$. Note that

$$
\begin{aligned}
&\left[V_h^{r_1}U_h^{s_1}...V_h^{r_n}U_h^{s_n}\right] = && (77)\\
&\quad \frac{\partial}{\partial p^\mu}\left(\{g^{r_1}f^{s_1}...g^{r_n}f^{s_n}\}\right)\frac{\partial}{\partial q^\mu} - \frac{\partial}{\partial q^\mu}\left(\{g^{r_1}f^{s_1}...g^{r_n}f^{s_n}\}\right)\frac{\partial}{\partial p^\mu} && (78)
\end{aligned}
$$

where the multibracket $\{g^{r_1}f^{s_1}...g^{r_n}f^{s_n}\}$ on the right-hand side is defined analogously to (74) using the Poisson bracket rather than the vector field commutator. Similarly,

$$
\begin{aligned}
\left[V_l^{r_1}U_l^{s_1}...V_l^{r_n}U_l^{s_n}\right] = \left(-\frac{i}{\hbar}\right)^{\sum_i^n(r_i+s_i)}\left[\hat{g}^{r_1}\hat{f}^{s_1}...\hat{g}^{r_n}\hat{f}^{s_n}\right]|\psi\rangle\frac{\partial}{\partial|\psi\rangle}& \\
+\left(\frac{i}{\hbar}\right)^{\sum_i^n(r_i+s_i)}\langle\psi|\left[\hat{f}^{s_n}\hat{g}^{r_n}...\hat{f}^{s_1}\hat{g}^{r_1}\right]\frac{\partial}{\partial\langle\psi|}& \qquad (79)
\end{aligned}
$$

where the multibracket $\left[\hat{g}^{r_1}\hat{f}^{s_1}...\hat{g}^{r_n}\hat{f}^{s_n}\right]$ on the right-hand side is defined analogously to (74) using the commutator of Hilbert space operators rather than the vector field commutator. To show (76), we begin by taking the pushforward of (79):

$$
\begin{aligned}
&B^*(\left[V_l^{r_1}U_l^{s_1}...V_l^{r_n}U_l^{s_n}\right])\\
&= \langle\psi|\left[\hat{q}^\mu,\left[\hat{g}^{r_1}\hat{f}^{s_1}...\hat{g}^{r_n}\hat{f}^{s_n}\right]\right]|\psi\rangle\frac{\partial}{\partial q^\mu} + \langle\psi|\left[\hat{p}^\mu,\left[\hat{g}^{r_1}\hat{f}^{s_1}...\hat{g}^{r_n}\hat{f}^{s_n}\right]\right]|\psi\rangle\frac{\partial}{\partial p^\mu}\\
&\qquad + \mathcal{O}(\hbar)\\
&= \langle\psi|\frac{\partial\{g^{r_1}f^{\widehat{s_1...g}^{r_n}}f^{s_n}\}}{\partial p^\mu}|\psi\rangle\frac{\partial}{\partial q^\mu} - \langle\psi|\frac{\partial\{g^{r_1}f^{\widehat{s_1...g}^{r_n}}f^{s_n}\}}{\partial q^\mu}|\psi\rangle\frac{\partial}{\partial p^\mu}\\
&\qquad + \mathcal{O}(\hbar). \qquad (80)
\end{aligned}
$$

In going from the second to the third line, we have made use of the fact that

$$
\left[\hat{g}^{r_1}\hat{f}^{s_1}...\hat{g}^{r_n}\hat{f}^{s_n}\right] = i\hbar\{g^{r_1}f^{\widehat{s_1...g}^{r_n}}f^{s_n}\} + \mathcal{O}(\hbar^2), \qquad (81)
$$

which follows from iterated application of (58). Evaluating $B^*([V_l^{r_1}U_l^{s_1}...V_l^{r_n}U_l^{s_n}])$ over the domain d, we then have

$$
\begin{aligned}
&B^*(\left[V_l^{r_1}U_l^{s_1}...V_l^{r_n}U_l^{s_n}\right])\\
&= \frac{\partial}{\partial q^\mu}\left(\left\{g^{r_1}f^{s_1}...g^{r_n}f^{s_n}\right\}\right)\Big|_{\langle\hat{q}\rangle,\langle\hat{p}\rangle}\frac{\partial}{\partial p^\mu} - \frac{\partial}{\partial p^\mu}\left(\left\{g^{r_1}f^{s_1}...g^{r_n}f^{s_n}\right\}\right)\Big|_{\langle\hat{q}\rangle,\langle\hat{p}\rangle}\frac{\partial}{\partial q^\mu}\\
&\qquad\qquad + \mathcal{O}(\hbar)+\mathcal{O}(\sigma_q)+\mathcal{O}(\sigma_p)\\
&= \left[V_h^{r_1}U_h^{s_1}...V_h^{r_n}U_h^{s_n}\right]\Big|_{B(x_l)} + \mathcal{O}(\hbar)+\mathcal{O}(\sigma_q)+\mathcal{O}(\sigma_p), \qquad (82)
\end{aligned}
$$

thus ensuring that (76) holds. Recall that by linearity of the pushforward map and the expansions (72) and (73), this relation entails (71), which in turn entails (67), the result that we initially sought to derive. Note that the error ϵ in (70) and (67) will be small if the values of $\hbar$, l_q, and l_p are. As we now discuss, for given parameter values (u, v), ϵ vanishes in a certain limit where $\hbar$, l_q, l_p, and the initial state widths σ_q and σ_p associated with x_l all approach zero.

5.3 Wave Packet Spreading Under General Unitary Transformations

We have seen that for fixed non-zero values of $(\hbar, l_q, l_p)$, the group of continuous unitary transformations on $\mathcal{H}$ will preserve the domain d only for transformations in some neighborhood of the identity map. That is, for a given initial state $|\psi_0\rangle$ in d and a given generator $\hat{f}$, the set of states $e^{-\frac{i}{\hbar}\hat{f}u}|\psi_0\rangle$ will remain continuously in d for values of u between zero and some finite upper bound, since unitary transformations generally may alter the position and momentum widths of quantum states, carrying them beyond one or both of the limits l_q and l_p that define d. However, we will see in this section that in a certain limit, this upper bound can be increased arbitrarily, so that the approximate homomorphism B between quantum and classical group actions expands to encompass the full set of continuous unitary transformations.

To see this, let us first calculate the rates of wave packet spreading in position and momentum under the evolution associated with an arbitrary one-parameter group of unitary transformations generated by the quantization $\hat{f}$ of some classical observable f. First, let us calculate the rate of change with respect to u of the variance in position under the one-parameter group of transformations $e^{-\frac{i}{\hbar}\hat{f}u}$, where $u \in \mathbb{R}$ and we assume without loss of generality that the quantum state $|\psi_0\rangle$ at $u = 0$ is initially centered about the spatial origin, so that $\langle\hat{q}\rangle = 0$ [19] :

$$\begin{aligned}
\frac{d}{du}Var(\hat{q}) &= \frac{d}{du}\left[\langle\psi_0|e^{\frac{i}{\hbar}\hat{f}u}\ \hat{q}^2\ e^{-\frac{i}{\hbar}\hat{f}u}|\psi_0\rangle - \left(\langle\psi_0|e^{\frac{i}{\hbar}\hat{f}u}\ \hat{q}\ e^{-\frac{i}{\hbar}\hat{f}u}|\psi_0\rangle\right)^2\right] \\
&= \frac{d}{du}\langle\psi_0|e^{\frac{i}{\hbar}\hat{f}u}\ \hat{q}^2\ e^{-\frac{i}{\hbar}\hat{f}u}|\psi_0\rangle \\
&= \frac{1}{i\hbar}\langle\psi_0|[\hat{q}^2, \hat{f}]|\psi_0\rangle \\
&= \langle\psi_0|\left(\hat{q}\widehat{\frac{\partial f}{\partial p}} + \widehat{\frac{\partial f}{\partial p}}\hat{q}\right)|\psi_0\rangle \\
&= 2\langle\hat{q}\widehat{\frac{\partial f}{\partial p}}\rangle + \mathcal{O}(\hbar) \\
&= 2\langle\hat{q}\rangle\langle\widehat{\frac{\partial f}{\partial p}}\rangle + \mathcal{O}(\hbar) + \mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p) \\
&= \mathcal{O}(\hbar) + \mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p)
\end{aligned} \tag{83}$$

In going from the first to the second line, we have made use of the assumption that $\langle\hat{q}\rangle = 0$; in going from the second to the third line, the relation (9); in going from the third to the fourth, the relation $[\hat{q}, \hat{f}] = i\hbar\widehat{\frac{\partial f}{\partial p}}$, from which it follows that $[\hat{q}^2, \hat{f}] = i\hbar\left(\hat{q}\widehat{\frac{\partial f}{\partial p}} + \widehat{\frac{\partial f}{\partial p}}\hat{q}\right)$; in going from the fourth to the fifth, the relation $\hat{q}\widehat{\frac{\partial f}{\partial p}} = \widehat{\frac{\partial f}{\partial p}}\hat{q} + \mathcal{O}(\hbar)$; in going from the fifth to the sixth, the relation $\langle\hat{q}\widehat{\frac{\partial f}{\partial p}}\rangle = \langle\hat{q}\rangle\langle\widehat{\frac{\partial f}{\partial p}}\rangle + \mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p)$. [20] By taking $(\hbar, \sigma_q, \sigma_p)$ sufficiently small, it is possible to ensure that the initial rate of spreading in position under the unitary transformation generated by $\hat{f}$ is arbitrarily close to zero.

[19] It is always possible to translate one's coordinate system so that the origin lies at the wave packet's centroid.

[20] This last relation follows from the fact that both $\langle\hat{q}\widehat{\frac{\partial f}{\partial p}}\rangle$ and $\langle\hat{q}\rangle\langle\widehat{\frac{\partial f}{\partial p}}\rangle$ are equal to $q\frac{\partial f}{\partial p}\big|_{\langle\hat{q}\rangle,\langle\hat{p}\rangle}$ up to $\mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p)$, as shown in Appendix A, and therefore can only differ by terms of $\mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p)$.

Higher derivatives of the position variance with respect to u also take the form $\mathcal{O}(\hbar) + \mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p)$, and therefore also vanish with $(\hbar, \sigma_q, \sigma_p)$. To see this, note that

$$\frac{d^n}{du^n} Var(\hat{q}) = \frac{1}{(i\hbar)^n} \langle \psi_0 | [...[[\hat{q}^2, \underbrace{\hat{f}], \hat{f}], ..., \hat{f}]}_{n\ times} | \psi_0 \rangle, \tag{84}$$

which follows from iterated application of (9). Each term in the expansion of this commutator consists of a product of two powers of $\hat{q}$ and n powers of $\hat{f}$. The expectation value of each such term is equal to $\langle \hat{q} \rangle^2 \langle \hat{f} \rangle^n + \mathcal{O}(\hbar) + \mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p)$, with different terms differing at $\mathcal{O}(\hbar) + \mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p)$. Since $\langle \hat{q} \rangle = 0$ by assumption, it follows that

$$\frac{d^n}{du^n} Var(\hat{q}) = \mathcal{O}(\hbar) + \mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p). \tag{85}$$

for all postive integers n. By making $(\hbar, \sigma_q, \sigma_p)$ sufficiently small, all derivatives at $t = 0$ of the position-space variance can be taken arbitrarily close to zero. Since

$$\sigma_q^2(u) = e^{u \frac{d}{du'}} \sigma_q^2(u')\big|_{u'=0} \tag{86}$$

$$= \sigma_q^2(0) + u \frac{d\sigma_q^2}{du'}\big|_0 + \frac{1}{2!} u^2 \frac{d^2 \sigma_q^2}{du'^2}\big|_0 + ..., \tag{87}$$

the size of $\sigma_q^2(u)$ for fixed u can be made arbitrarily small by making all of the derivatives $\frac{d^n \sigma_q}{du^n}$ arbitrarily small. As the derivatives of $\sigma_q^2(u)$ with respect to u at $u = 0$ become smaller, the value of u for which $\sigma_q^2(u)$ first reaches or exceeds l_q^2 (assuming $\sigma_q(0) < l_q$) grows without bound. By a similar sequence of steps, one can derive analogous results for the momentum variance $Var(\hat{p}) = \sigma_p^2(u)$ with respect to the bound l_p that defines d.

Thus, for a given choice of (l_q, l_p), in the limit $(\hbar, \sigma_q(0), \sigma_p(0)) \rightarrow 0$ constrained by $\sigma_q(0)\sigma_p(0) \geq \frac{\hbar}{2}$, the set of parameter values u for which $e^{-\frac{i}{\hbar}\hat{f}u}|\psi_0\rangle \in d$ (assuming $|\psi_0\rangle \in d$) extends to include the whole real line. Since this applies to arbitrary generators $\hat{f}$ in the algebra of quantum observables, the neighborhood of unitary transformations around the identity that map a given initial state $|\psi_0\rangle \in d$ continuously to other states in d expands to include the full group of unitary transformations on $\mathcal{H}$ continuously connected to the identity. It is worth emphasizing once again that this applies for fixed non-zero (l_q, l_p) in the constrained limit $(\hbar, \sigma_q(0), \sigma_p(0)) \rightarrow 0$.

However, even in this constrained limit, the margins of error within which the relations (30) and (52) hold do not vanish since l_q and l_p are assumed to be finite and non-zero. However, by taking the limit $(l_q, l_p) \rightarrow 0$ *after* the constrained limit $(\hbar, \sigma_q(0), \sigma_p(0)) \rightarrow 0$, it is possible to ensure that the relations (30), (42), (52), and (66) all hold simultaneously and with exact equality, and that (42) and (66) hold exactly for the full group of continuous unitary transformations on $\mathcal{H}$, since the low-level state will be guaranteed to remain in d for all such transformations in this limit.

6 Conclusion

This investigation has attempted to show how a certain general template for "reduction" between two models of the same physical system serves to clarify the relationship between deformation quantization and a certain geometrical extension of the well-known result that expectation values of position and momentum traverse approximately classical trajectories when the quantum state is narrowly peaked in position. Whereas the specific sort of continuity between quantum and classical formalisms associated with deformation quantization concerns only the abstract algebras of observables that generate unitary and canonical transformations, the correspondence described here relies essentially on the structure of quantum and classical state spaces over which specific *actions* of these abstract algebras and their associated groups are defined. In particular, while deformation quantization rests only on the formal limit $\hbar \to 0$, the continuity associated with the correspondence described here rests both on this limit and on certain limits where the position and momentum widths of the quantum state vanish along with $\hbar$. This latter feature provides one respect in which the correspondence described here is non-trivially distinct from that associated with deformation quantization. Moreover, this correspondence provides a unified frame within which to understand the distinct-but-related roles played by deformation quantization on the one hand and results associated with Ehrenfest's Theorem and the restriction to narrow wave packet states on the other.

A Proof of (32)

In this section, we prove that the expectation value of a Hermitian operator $\hat{h}$ of the form (23) evaluated on a state $|\psi\rangle \in d$ with position space width σ_q and momentum space width σ_p takes the form

$$\langle\psi|\hat{h}|\psi\rangle = h\left(\langle\hat{q}\rangle, \langle\hat{p}\rangle\right) + \mathcal{O}(\hbar) + \mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p), \tag{88}$$

where h is the unique real-valued function over classical phase space corresponding to the operator $\hat{h}$. From the expression (23), we can see that to show this it suffices to show that

$$\langle\psi|\mathrm{Herm}(\hat{q}^n\hat{p}^m)|\psi\rangle = \langle\hat{q}\rangle^n\langle\hat{p}\rangle^m + \mathcal{O}(\hbar) + \mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p). \tag{89}$$

First, note that

$$\mathrm{Herm}(\hat{q}^n\hat{p}^m) = \hat{q}^n\hat{p}^m + \mathcal{O}(\hbar). \tag{90}$$

Second, for our purposes it is reasonable to approximate the momentum space expansion of the state $|\psi\rangle$ localized to width σ_p in momentum by a Gaussian of width σ_p (for example, through a truncated expansion of the momentum-space wave function in terms of Hermite polynomials). In the momentum basis, we approximate $|\psi\rangle$ as,

$$|\psi\rangle = \frac{1}{(2\pi)^{1/4}\sigma_p^{1/2}} \int dp\; e^{-\frac{(p-p_0)^2}{4\sigma_p^2}}\, e^{ipq_0}|p\rangle. \tag{91}$$

Acting on $|\psi\rangle$ with $\hat{p}$, we have

$$\begin{aligned}
\hat{p}|\psi\rangle &= \frac{1}{(2\pi)^{1/4}\sigma_p^{1/2}} \int dp\; p\; e^{-\frac{(p-p_0)^2}{4\sigma_p^2}}\, e^{ipq_0}|p\rangle && (92)\\
&= p_0|\psi\rangle + \sigma_p^2\left(\frac{e^{ip_0q_0}}{(2\pi)^{1/4}\sigma_p^{1/2}} \int dp'\; p'\; e^{-\frac{p'^2}{4}}\, e^{i\sigma_p p' q_0}|\sigma_p p' + p_0\rangle\right) && (93)\\
&= p_0|\psi\rangle + \mathcal{O}(\sigma_p), && (94)
\end{aligned}$$

where in the second line, we have employed the change of variables $p' = \frac{p-p_0}{\sigma_p}$, and the integral in the second line is a vector of magnitude proportional to $\frac{1}{\sigma_p^{1/2}}$. As a whole, the error term therefore vanishes as σ_p. Iterating this result m times, we have to first order in σ_p,

$$\hat{p}^m|\psi\rangle = p_0^m|\psi\rangle + \mathcal{O}(\sigma_p). \tag{95}$$

Through a similar calculation, we find that $\hat{q}|\psi\rangle = q_0|\psi\rangle + \mathcal{O}(\sigma_q)$ and to first order in σ_q, $\hat{q}^n|\psi\rangle = q_0^n|\psi\rangle + \mathcal{O}(\sigma_q)$. It then follows that

$$\hat{q}^n\hat{p}^m|\psi\rangle = q_0^n p_0^m|\psi\rangle + \mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p). \tag{96}$$

From (90), we then have that

$$\mathrm{Herm}(\hat{q}^n\hat{p}^m) = q_0^n p_0^m|\psi\rangle + \mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p) + \mathcal{O}(\hbar) \tag{97}$$

Noting that $q_0 = \langle \hat{q} \rangle$ and $p_0 = \langle \hat{p} \rangle$, we then have

$$\langle \psi | \mathrm{Herm}(\hat{q}^n \hat{p}^m) | \psi \rangle = \langle \hat{q} \rangle^n \langle \hat{p} \rangle^m + \mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p) + \mathcal{O}(\hbar) \tag{98}$$

Inserting into relation (23), it follows that

$$\langle \hat{h} \rangle = h(\langle \hat{q} \rangle, \langle \hat{p} \rangle) + \mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p) + \mathcal{O}(\hbar), \tag{99}$$

as we aimed to show.

B Proof of (41)

Assume that

$$U_h^\mu\big|_{B(x_l)} = \frac{\partial B^\mu}{\partial x_l^\nu} U_l^\nu\big|_{x_l} + \delta(x_l) \tag{100}$$

for all $x_l \in d$, where $\delta(x_l)$ is some x_l-dependent error term. In the example studied here, $\delta(x_l)$ will consist of terms $\mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p) + \mathcal{O}(\hbar)$, where σ_q and σ_p are the widths of the state $x_l = |\psi\rangle$. We wish to show that

$$G_h(B(x_l), s) = B(G_l(x_l, s)) + \epsilon \tag{101}$$

if $G_l(x_l, s') \in d$ for all $0 \leq s' \leq s$, where the error ϵ vanishes as $\delta(x_l)$ vanishes uniformly within d. Since $\delta(x_l)$ consists of terms $\mathcal{O}(\sigma_q) + \mathcal{O}(\sigma_p) + \mathcal{O}(\hbar)$, ϵ will be negligible for sufficiently small values of $\hbar$, l_q, and l_p (where $l_q l_p > \frac{\hbar}{2}$).

If $x_l(s) \equiv G_l(x_l, s)$ is an integral curve of the low-level vector field U_l, then from (100) it follows that

$$\frac{dB(x_l(s))}{ds} = U_h|_{B(x_l(s))} + \delta(x_l), \tag{102}$$

for all $x_l \in d$. This can be seen through the sequence of steps: $\frac{dB(x_l)}{ds} = \frac{\partial B}{\partial x_l}\frac{dx_l}{ds} = \frac{\partial B}{\partial x_l} U_l|_{x_l} = U_h^\mu\big|_{B(x_l)} + \delta(x_l)$, where the first equality follows from the Chain Rule, the second from the fact that $\frac{dx_l}{ds} = U_l|_{x_l}$, and the third from (100). We can approximate $U_h|_{B(x_l(s))} \approx \frac{d}{ds'}\big|_{s'=s} D_h(B(x_l), s')$ for s such that U_h does not vary substantially on the scale $\big|D_h(B(x_l), s) - B(D_l(x_l, s))\big|_h$.[21] The relation can then be rewritten (102) in terms of the transformations $G_l(x_l, s)$ and $G_h(B(x_l), s)$ as follows:

$$\frac{d}{ds'} B(G_l(x_l, s')) \approx \frac{d}{ds'} G_h(B(x_l), s') + \delta(x_l(s')), \tag{103}$$

Integrating both sides from 0 to s, we then have

$$B(G_l(x_l, s)) - B(G_l(x_l, 0)) \approx G_h(B(x_l), s) - G_h(B(x_l), 0) \tag{104}$$

$$+ \int_0^s ds'\ \delta(x_l(s')). \tag{105}$$

Since $B(G_l(x_l, 0)) = G_h(B(x_l), 0) = B(x_l)$, this gives

$$B(G_l(x_l, s)) \approx G_h(B(x_l), s) + \epsilon, \tag{106}$$

where $\epsilon \equiv \int_0^s ds'\ \delta(x_l(s'))$, and vanishes as $\delta(x_l) \to 0$ uniformly in d.

[21] Here we are ignoring a certain second-order effect. Differences between $D_h(B(x_l), s)$ and $B(D_l(x_l, s))$ for increasing s accumulate not only as a result of the slight difference $\delta(x_l)$ between the generating vector fields $U_h^\mu\big|_{B(x_l)}$ and $\frac{\partial B^\mu}{\partial x_l^\nu} U_l^\nu\big|_{x_l}$, but also as a result of the fact that the points in S_h at which the high-level vector field and the push-forward vector field are integrated are slightly different for $s > 0$: U_h is integrated along $D_h(B(x_l), s)$, while the push-forward vector field $\frac{\partial B^\mu}{\partial x_l^\nu} U_l^\nu$ is integrated $B(D_l(x_l, s))$; in general, these points only coincide at $s = 0$. This additional error is proportional to the derivative $\frac{\partial U_h}{\partial x_l}$. As long as $D_h(B(x_l), s)$ and $B(D_l(x_l, s))$ do not differ by more than the scale over which U_h changes substantially, we may ignore this error, which also vanishes as $\delta(x_l) \to 0$ uniformly in d.

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