Variational Principle for Optimal Quantum Controls in Quantum Metrology

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We develop a variational principle to determine the quantum controls and initial state that optimizes the quantum Fisher information, the quantity characterizing the precision in quantum metrology. When the set of available controls is limited, the exact optimal initial state and the optimal controls are, in general, dependent on the probe time, a feature missing in the unrestricted case. Yet, for time-independent Hamiltonians with restricted controls, the problem can be approximately reduced to the unconstrained case via Floquet engineering. In particular, we find for magnetometry with a time-independent spin chain containing three-body interactions, even when the controls are restricted to one- and two-body interaction, that the Heisenberg scaling can still be approximately achieved. Our results open the door to investigate quantum metrology under a limited set of available controls, of relevance to many-body quantum metrology in realistic scenarios.

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Introduction.—At the heart of quantum technology, quantum metrology aims at improving the precision of parameter estimation [1–5]. Recent theoretical advances in quantum metrology [6–11] have led to the introduction of optimal control protocols that maximize the quantum Fisher information (QFI), the key quantity in characterizing the precision in quantum metrology. The use of control protocols for enhanced parameter estimation has been extensively explored in many practical physical setups [8,9,11–13].

The identification of the optimal control protocol for parameter estimation [8,9] can be done by exploiting the notion of adiabatic continuation of quantum states. It resembles the engineering of shortcuts to adiabaticity (STA) by counterdiabatic driving [14–18], whereby the system Hamiltonian is supplemented by some controls that enforce adiabatic continuation along a prescribed trajectory. In the context of STA, the auxiliary controls enforce parallel transport along the eigenstates of the uncontrolled system Hamiltonian. By contrast, optimal controls for quantum metrology enforce adiabatic continuation of the eigenstates of an operator defined by the parametric derivative of the estimation Hamiltonian [19]. This is intuitive, as such operator guarantees maximal distinguishability of states with a slight change of the estimated parameter. The connection of optimal control for quantum metrology and STA is not only fruitful in providing a geometric justification of the required controls, but, as we show in this Letter, it also suggests solutions to common challenges.

In STA, the exact identification of the counterdiabatic control protocol is not feasible in many-body systems in which spectral properties of the system Hamiltonian cannot be found, e.g., in problems such as quantum optimization. In addition, the set of available controls in a given experimental setup is often restricted. This generally precludes the implementation of the exact STA protocol. This is particularly important in the many-particle systems when exact counterdiabatic controls involve nonlocal multiple-body interactions beyond one-body and pairwise potentials, which are hard to realize in practice [20]. In addition, the use of variational methods provides then an alternative, by designing optimal protocols that are realizable with a restricted set of controls [21–29]. Likewise, the exact construction of optimal protocols in many-body quantum metrology [11] (i) requires access to the spectrum of the parametric derivative of the many-body Hamiltonian, which is hardly accessible, in general, and (ii) may require optimal controls that are nonlocal and hard to implement in the laboratory. Even in single-qubit metrology using a nitrogen vacancy center, superconducting qubit, or quantum dot, certain control operations may be hard to implement. It is thus required to develop a new formalism of optimal control for quantum metrology beyond the state of the art [8], by taking into account the ubiquitous limitations on the controls and circumventing the requirement to access the spectral properties of the Hamiltonian derivative. This is the problem solved in this Letter by means of a novel variational approach that relies on a metrological action. We note that our Letter is unrelated to other

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variational approaches introduced in quantum metrology that do not involve quantum control [30,31].

Specifically, we assume that the space of control Hamiltonians is spanned by a set of local basis operators and introduce a metrological action, which includes contributions of the quantum Fisher information and the Schrödinger equation, which is considered as a constraint to the optimization problem. We derive the optimal control conditions and show that it reduces to the unrestricted protocol in Ref. [8] if the basis operators generate the full space of the Hermitian operators. Furthermore, we show that when the control Hamiltonian is restricted, the *exact* optimal solution for the initial state and the control Hamiltonian depends on the probe time. However, we find that for multiplicative time-independent Hamiltonians, even with limited controls, one may identify *approximate* optimal controls, which reduce to the unrestricted protocol and therefore become independent of the probe time, using the high-frequency expansion in Floquet engineering [32–38]. In particular, we show in an example in magnetometry that one can avoid local three-body interaction terms making use only of local two-body control Hamiltonians.

Optimal initial states and controls through variational principle.—Consider the quantum estimation of a parameter λ in a general Hamiltonian $H_\lambda(t)$. In this setting, one may find that the QFI may decrease as the probe time increases [39]. This motivates the introduction of quantum controls $H_c(t)$ to enhance the QFI [6,8]. When the quantum controls are introduced, the unitary evolution $U(t)$ is generated by $H_{\mathrm{tot}}(t) = H_\lambda(t) + H_c(t)$. Let the initial time and fixed final probe time be 0 and t_f, respectively. The generator for parameter estimation is given by $G_{t_f}[U] = \int_0^{t_f} U^\dagger(\tau)\partial_\lambda H_\lambda(\tau)U(\tau)d\tau$ [8] and the quantum Fisher information I is given by the variance of the generator [4], i.e., $I[U] = \mathrm{Var}(G_{t_f}[U])$.

The optimization of the QFI $I[U]$ over the initial state yields the optimal initial state $[|\varphi_+(t_f)\rangle + |\varphi_-(t_f)\rangle]/\sqrt{2}$, where $|\varphi_\pm(t_f)\rangle$ are referred to as the maximum and minimum eigenstates, as they are associated with the maximum and minimum eigenvalues $\mu_\pm(t_f)$ of $G_{t_f}[U]$ (see the Supplemental Material [40]). The maximum value of the QFI over the initial states $|\psi_0\rangle$ is $\max_{|\psi_0\rangle} I = \|G_{t_f}[U]\|^2/4$, where the norm of an operator is defined by the difference between its maximum and minimum eigenvalues [5,41,42]. Our next goal is to maximize the quantum Fisher information $I[U]$ over all possible unitary dynamics under the condition that U and H_c satisfy the Schrödinger equation $i\partial_t U(t) = H_{\mathrm{tot}}(t)U(t)$ as a constraint. We further require the control Hamiltonian H_c to be spanned by a limited set of available linearly independent terms as $\{\mathcal{X}_i\}_{i=1}^{d_c}$. We denote $\mathcal{V}_c \equiv \mathrm{span}\{\mathcal{X}_i\}_{i=1}^{d_c}$ and expand $H_c \in \mathcal{V}_c$ as $H_c(\tau) = \sum_{i=1}^{d_c} c_i(\tau)\mathcal{X}_i$. This procedure amounts to removing the constraint on H_c, reducing the optimization over H_c to the optimization over $c_i(\tau)$ [43].

We shall denote $I_0 = \max_{U,H_c,|\psi_0\rangle} I[U]$. In principle, I_0 can be computed through the variational principle by constructing an appropriate action. We note that $I[U]$ is quartic in U since it is quadratic in $G_{t_f}[U]$, which is itself quadratic in U. This makes the variational calculus of $I[U]$ with respect to U tedious. To facilitate the calculation, we observe that

$$I_0 = \max_{U,H_c}\max_{|\varphi_{a,b}\rangle}(\langle\varphi_a|G_{t_f}[U]|\varphi_a\rangle - \langle\varphi_b|G_{t_f}[U]|\varphi_b\rangle)^2 \tag{1}$$

under the constraint of the Schrödinger equation and the condition that $|\varphi_{a,b}\rangle$ are normalized. The introduction of two more optimization variables $|\varphi_{a,b}\rangle$ allows us to remove the square in Eq. (1) and transform the original optimization problem to the following equivalent one, $\max_{|\varphi_{a,b}\rangle,U,H_c} S_I[\Delta\rho, U]$, under aforementioned constraints, with the "information action" being defined as

$$S_I[\Delta\rho, U] \equiv \langle\varphi_a|G_{t_f}[U]|\varphi_a\rangle - \langle\varphi_b|G_{t_f}[U]|\varphi_b\rangle = \int_0^{t_f}\mathrm{Tr}\{\Delta\rho U^\dagger(\tau)\partial_\lambda H_\lambda(\tau)U(\tau)\}d\tau, \tag{2}$$

and $\Delta\rho \equiv |\varphi_a\rangle\langle\varphi_a| - |\varphi_b\rangle\langle\varphi_b|$. The introduction of two additional auxiliary variables $|\varphi_{a,b}\rangle$ effectively renders $S_I[U]$ quadratic in U, unlike $I[U]$, facilitating the variational calculus with respect to U. Upon introducing the Lagrangian multipliers $\mu_{a,b}$ and $\Lambda(\tau)$, we obtain the following "metrological action":

$$S_M(|\varphi_a\rangle, |\varphi_b\rangle, U, H_c) \equiv S_I[\Delta\rho, U] + S_S[U, H_c] - \mu_a[\langle\varphi_a|\varphi_a\rangle - 1] + \mu_b[\langle\varphi_b|\varphi_b\rangle - 1], \tag{3}$$

where the "Schrödinger action" is defined as

$$S_S[U, H_c] \equiv \int_0^{t_f}\mathrm{Tr}\{\Lambda(\tau)[\mathrm{i}\dot{U}(\tau)U^\dagger(\tau) - H_\lambda(\tau) - H_c(\tau)]\}d\tau. \tag{4}$$

We emphasize that $|\varphi_{a,b}\rangle$, U and H_c are independent variables. The optimization over $|\varphi_{a,b}\rangle$ can be easily implemented by differentiation with respect to them, which yields

$$G_{t_f}[U]|\varphi_\alpha\rangle = \mu_\alpha|\varphi_\alpha\rangle, \qquad \alpha = a, b. \tag{5}$$

As shown in Sec. I in the Supplemental Material [40], in order for I_0 to take the global maximum values over $|\varphi_{a,b}\rangle$, $\mu_{a,b}$ and $|\varphi_{a,b}\rangle$ must be the maximum and minimum eigenvalues and eigenvectors of $G_{t_f}[U]$, respectively.

Variation with respect to H_c and U gives the trace condition $\mathrm{Tr}\{\Lambda(\tau)\mathcal{X}_i\} = 0$ for $i \in [1, d_c]$ and the differential equation $\dot{\Lambda}(\tau) - \mathrm{i}[\Lambda(\tau), H_{\mathrm{tot}}(\tau)] + \mathrm{i}[\Delta\rho(\tau), \partial_\lambda H_\lambda(\tau)] = 0$, with the final condition $\Lambda(t_f) = 0$, where $\Delta\rho(\tau) \equiv U(\tau)\Delta\rho U^\dagger(\tau)$. One can solve for $\Lambda(\tau)$ the differential equation and substitute the result into the trace condition to find

$$\mathrm{Tr}\{\mathcal{X}_i \partial_\lambda[\Delta\rho(\tau)]\} = 0, \qquad i \in [1, d_c]. \tag{6}$$

Equations (5) and (6) are our central results. The form of these equations in the parameter-independent rotating frame can be found in [40]. They give the optimal initial state and optimal dynamics that maximize the QFI when the quantum controls are restricted to the subspace spanned by $\{\mathcal{X}_i\}$. In what follows, we discuss their implications and applications.

The unrestricted control and the general feature of exact restricted controls.—As a first application of our results, let us assume there is no restriction on the control Hamiltonians, that is, $\{\mathcal{X}_j\}$ spans the whole space of traceless Hermitian operators. We shall see how the Pang-Jordan protocol in Ref. [8] is reproduced. In this case, Eq. (6) is equivalent to $\partial_\lambda[\Delta\rho(\tau)] = 0$. Taking the time derivative on both sides yields

$$\partial_\tau \partial_\lambda[\Delta\rho(\tau)] = -\mathrm{i}[\partial_\lambda H_\lambda(\tau), \Delta\rho(\tau)] = 0, \quad \forall\, \tau \in [0, t_f], \tag{7}$$

where we use the Liouville equation for $\Delta\rho(\tau)$. Since $|\varphi_{a,b}(\tau)\rangle \equiv U(\tau)|\varphi_{a,b}\rangle$ are associated with the nondegenerate eigenvalues ± 1 of $\Delta\rho(\tau)$, we conclude that $|\varphi_{a,b}(\tau)\rangle$ must also be eigenvectors of $\partial_\lambda H_\lambda(\tau)$ at all times. That is, $\partial_\lambda H_\lambda(\tau)|\varphi_{a,b}(\tau)\rangle = \nu_{a,b}(\tau)|\varphi_{a,b}(\tau)\rangle, \ \forall\, \tau \in [0, t_f]$, where $\nu_{a,b}(\tau)$ is the eigenvalue of $\partial_\lambda H_\lambda(\tau)$. It is then straightforward to check that $|\varphi_{a,b}\rangle$ are eigenvectors of $G_{t_f}[U]$ with eigenvalue $\int_0^{t_f} \nu_{a,b}(\tau) d\tau$. Thus, any unitary dynamics that preserves the adiabatic evolution of any pair of eigenstates of $\partial_\lambda H_\lambda(\tau)$ satisfies Eq. (7) and is an extremal solution satisfying $\delta S_M = 0$. To further maximize the QFI among the manifold of extremal solutions, one needs to further optimize the difference between $\int_0^{t_f} \nu_a(\tau)$ and $\int_0^{t_f} \nu_b(\tau)$. This requires $\nu_a(\tau)$ and $\nu_b(\tau)$ to be the maximal and minimum eigenvalues of $\partial_\lambda H_\lambda(\tau)$ at all times. When the unitary dynamics preserves the adiabatic evolution of all eigenstates of $\partial_\lambda H_\lambda(\tau)$, i.e., $U(\tau) = \sum_\alpha |\varphi_\alpha(\tau)\rangle\langle\varphi_\alpha(0)|$, where $|\varphi_\alpha\rangle$ denotes the eigenvectors of $\partial_\lambda H_\lambda(\tau)$, one recovers the Pang-Jordan control Hamiltonian [8] $H_c(\tau) = \mathrm{i}\sum_\alpha |\dot{\varphi}_\alpha(\tau)\rangle\langle\varphi_\alpha(\tau)| - H_\lambda(\tau)$. Note that, if there is any level crossing in $\nu_a(\tau)$ at some instant time in $[0, t_f]$, the way of labeling the eigenstates and eigenvalues is not unique. Choosing $\nu_+(\tau)$ and $\nu_-(s)$ always as the maximum and minimum eigenvalues of $\partial_\lambda H_\lambda(\tau)$ at all times, the resulting QFI is the greatest among all the different ways of labeling the eigenstates. With this labeling, the first-order time derivative of $|\varphi_a(\tau)\rangle$ is discontinuous, which results in a δ pulse in the control Hamiltonian H_c. This provides an alternative understanding of the σ_x-like pulses in Refs. [8,9,13].

In the general case, $\{\mathcal{X}_i\}_{i=1}^{d_c}$ do not span the whole space of Hermitian operators, and the generic optimal solutions $U(\tau)$ and $|\varphi_{a,b}\rangle$ implicitly depend on the probe time t_f. This is due to the dependence of the generator $G_{t_f}[U]$ on t_f, which may make the eigenvectors $|\varphi_{a,b}\rangle$ determining the optimal initial state also dependent on t_f. The dependence on t_f for the optimal unitary $U(\tau)$ can then be seen from Eq. (6). With these observations, we take the derivative with respective to t_f to obtain the following differential-integral equation:

$$\partial_{t_f} G_{t_f} |\varphi_{\alpha,t_f}\rangle + G_{t_f} |\partial_{t_f} \varphi_{\alpha,t_f}\rangle = \partial_{t_f} \mu_{\alpha,t_f} |\varphi_{\alpha,t_f}\rangle + \mu_{\alpha,t_f} |\partial_{t_f} \varphi_{\alpha,t_f}\rangle, \tag{8}$$

where we have suppressed the dependence on U in the generator G_{t_f} for simplicity, and $\alpha = a,\ b$, $\partial_{t_f} G_{t_f} = \mathrm{i}\underline{\partial}_{t_f} U^\dagger_{t_f}(t_f)\partial_\lambda U_{t_f}(t_f) + \mathrm{i}U^\dagger_{t_f}(t_f)\underline{\partial}_{t_f}\partial_\lambda U_{t_f}(t_f) + U^\dagger_{t_f}(t_f)\times \partial_\lambda H_\lambda(t_f) U_{t_f}(t_f)$, where $\underline{\partial}_{t_f}$ denotes the derivative with respect the subscript t_f instead of the one in the parentheses [40]. Generally, Eq. (8) is difficult to solve analytically, while numerical calculation is tractable. However, if $U_{t_f}(\tau)$ and $|\varphi_{a,t_f}\rangle$ are independent of the subscript variable t_f, Eq. (8) reduces to $U^\dagger(t_f)\partial_\lambda H_\lambda(t_f)U(t_f)|\varphi_\alpha\rangle = \partial_{t_f}\mu_{\alpha,t_f}|\varphi_\alpha\rangle$. It then follows that $|\varphi_a(t_f)\rangle = U(t_f)|\varphi_a\rangle$ is an eigenstate of $\partial_\lambda H_\lambda(t_f)$ for all t_f, with eigenvalue $\partial_{t_f}\mu_{\alpha,t_f}$. The solution reduces again to the Pang-Jordan protocol [8].

This suggests that, when the control Hamiltonian is restricted to some nontrivial subspace of the Hermitian operators, such that $\partial_\lambda \Delta\rho(\tau)$ does not always vanish on $[0, t_f]$, both the exact optimal controls and the exact optimal initial states depend on the probing time t_f, making it challenging to find them analytically. In particular, one can show that, when only $U(\tau)$ depends on t_f but $|\varphi_\alpha\rangle$ is independent of t_f, this is the case as there exists a time such that $\partial_\lambda \Delta\rho(\tau) \neq 0$ [40].

Approximate solution via Floquet engineering.—We next show that for a time-independent Hamiltonian H_λ the restricted optimal controls can be approximately engineered by the high-frequency control, known as Floquet engineering [32–38]. For time-independent Hamiltonians with unrestricted controls, the minimum optimal control Hamiltonian is the one that cancels the parts in H_λ that do not commute with $\partial_\lambda H_\lambda$ [8]. However, if any of the required control Hamiltonians are not available, it is not clear how to reach the Heisenberg scaling, $I_0^{\mathrm{HS}} = 4n^2 t_f^2$, where n is the number of probes. Here, we search for a static control $H_{c,0}$ and high-frequency driving controls $H_c^{(d)}(t)$, where

$H_c^{(d)}(t) \equiv \sum_{l\neq 0} H_{c,l} e^{il\omega t}$, so that $H_{\rm tot}(t) = H_\lambda + H_{c,0} + H_c^{(d)}(t)$, and show that the unavailable controls in the minimum optimal control Hamiltonian can be actually constructed approximately through a high-frequency expansion. Let us first move to the rotating frame associated with $\mathcal{U}(t) = e^{iK(t)}$, where $K(t)$ is the so-called kick operator [33]. In it, the total Hamiltonian is given by the Floquet effective Hamiltonian H_F. When the driving frequency is high enough, an expansion in orders of $1/\omega$ can be performed [33,36]. To the first order of $1/\omega$, one finds that $K(t) = [1/(i\omega)]\sum_{l\neq 0}(1/l)(H_{c,l}e^{il\omega t} - 1) + O(1/\omega^2)$ [44] and

$$H_F = H_\lambda + H_{c,0} + 1/\omega \sum_{l=1}^{\infty} [H_{c,l}, H_{c,-l}]/l + O(1/\omega^2). \qquad (9)$$

One can explicitly show that the kick operator $K(t)$ consists of the $H_{c,l}$ with $l \neq 0$ and it is independent of the estimation parameter. Thus, the QFI remains unchanged in the Floquet rotating frame. The generator becomes $G_{t_f} = \int_0^{t_f} e^{-iH_F\tau}\partial_\lambda H_\lambda e^{iH_F\tau} d\tau$ [40]. The key idea is that the unavailable controls in the original static frame can be constructed through the commutator in Eq. (9). This can be best illustrated using a simple qubit example with the Hamiltonian $H_\lambda = \lambda\sigma_z/2 + \Delta\sigma_x/2$ and $\mathcal{V}_c = \{\sigma_y, \sigma_z\}$. The term that does not commute with $\partial_\lambda H_\lambda$ is $\Delta\sigma_x/2$, which is not available in $\mathcal{V}_c$. Therefore, we consider $H_{c,0} = c_0^y\sigma_y + c_0^z\sigma_z$ and $H_c^{(d)}(t) = \sum_{l\neq 0}(c_l^y\sigma_y + c_l^z\sigma_z)e^{il\omega t}$, where for $l \neq 0$, $c_l^y = c_{-l}^{y*}$, and $c_l^z = c_{-l}^{z*}$ to guarantee the Hermiticity of $H_c^{(d)}(t)$. According to Eq. (9), to the first order of $1/\omega$, we find

$$H_F = (\lambda/2 + c_0^z)\sigma_z + \left(\Delta/2 - 4/\omega \sum_{l=1}^{\infty} \mathrm{Im}[c_l^y c_l^{z*}]/l\right) \times \sigma_x + c_0^y\sigma_y. \qquad (10)$$

The approximate optimal control fulfills $c_0^y = 0$ and

$$\omega = 8/\Delta \sum_{l=1}^{\infty} \mathrm{Im}[c_l^y c_l^{z*}]/l, \qquad (11)$$

which we call the "amplitude-frequency matching" (AFM) condition. The validity of the high-frequency expansion requires that ω should be the largest frequency in the original Hamiltonian, i.e., that $\omega \gg \lambda$, Δ, c_l^y, and c_l^z. Experimentally, for a laser frequency that satisfies this condition, one can always tune the amplitude so that Eq. (11) is satisfied. Conversely, one can also choose a proper laser frequency for given amplitudes so that Eq. (11) is fulfilled. We emphasize that when Eq. (11) is satisfied and the initial state in the Floquet rotating frame is $(|0\rangle + |1\rangle)/\sqrt{2}$, the optimal control conditions (5) and (6) are approximately satisfied

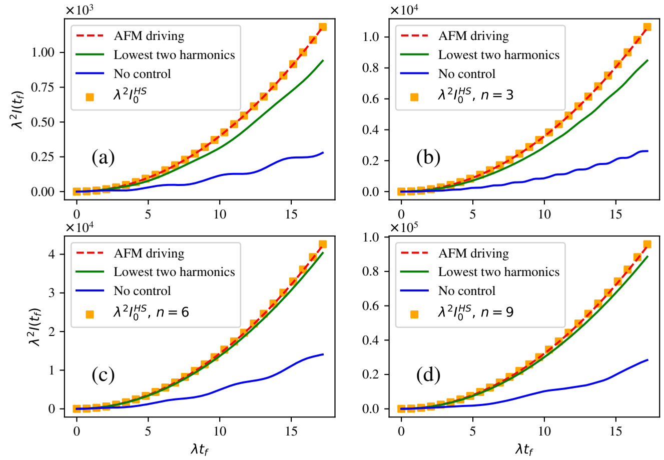

FIG. 1. QFI for various scenarios. (a) A single qubit with the sensing Hamiltonian $H_\lambda = \lambda\sigma_z/2 + \Delta\sigma_x/2$ (b)–(d) The spin chain with the sensing Hamiltonian (13). For (a)–(d) $\lambda = \Delta = 1$, the frequency of the drive $\omega = 1826.67$. The red lines satisfy the AFM condition (11) or (16) with $c_l^y = \tilde{c}_l^z = c_l^{xy} = \tilde{c}_l^{zx} = 10$ when $1 \leq l \leq 5$ and vanish for $l > 5$. The total simulation time is 5000 times the fundamental periodic $2\pi/\omega$. For the case of lowest two harmonics in green lines, $c_l^y, \tilde{c}_l^z, c_l^{xy}, \tilde{c}_l^{zx}$ are nonvanishing only when $l = 1$, 2 and take values 10.

up to the order of $1/\omega$ in the Floquet rotating frame. The initial state in the lab and Floquet rotating frames is the same as $K(0) = 0$. Going back to the lab frame, we generate a solution to Eqs. (5) and (6) when the controls are restricted. This illustrates the power of our approach beyond the Pang-Jordan protocol [8].

A simple choice for the amplitudes involves taking for $l \geq 1$ both c_l^y and $\tilde{c}_l^z = ic_l^z$ to be real. This yields

$$H_c^{(d)}(t) = 2\sum_{l=1}^{\infty} [c_l^y \cos(l\omega t)\sigma_y + \tilde{c}_l^z \sin(l\omega t)\sigma_z]. \qquad (12)$$

As one can see from Fig. 1(a), for parameters satisfying the AFM condition (11), the Heisenberg scaling is achieved. As a result, the first-order term in the high-frequency expansion makes it possible to construct the σ_x term by commuting the operators in $\mathcal{V}_c$. This idea can be generalized to many-body systems, as shown in the following.

Restricted control in a quantum spin chain.—Consider the sensing of magnetic field using a spin chain

$$H_\lambda = \frac{J}{2}\sum_{i=1}^{n} \sigma_i^x\sigma_{i+1}^x + \frac{\Delta}{2}\sum_{i=1}^{n} \sigma_i^x\sigma_{i+1}^x\sigma_{i+2}^x + \frac{\lambda}{2}\sum_{i=1}^{n}\sigma_i^z, \qquad (13)$$

which contains both two- and three-body interactions. We assume periodic boundary conditions for simplicity and consider the set of allowed controls $\mathcal{V}_c = \{\sigma_i^a, \sigma_i^a\sigma_{i+1}^b\}$, $(a, b = x, y, z)$, involving only one-body and nearest neighbor two-body operators. When the controls are unrestricted, the minimum optimal control Hamiltonian contains local two-body and local three-body terms. The first part in the control Hamiltonian would cancel the nearest neighbor two-body terms, i.e., $H_{c,0} = -J/2 \times \sum_{i=1}^{n} \sigma_i^x \sigma_{i+1}^x$. However, the term consisting of three-body operators cannot be canceled directly through the allowed control set $\mathcal{V}_c$.

Our goal is to construct the three-body operators using the first-order correction in Eq. (9), that is, we expect to induce $[H_l, H_{-l}] \propto \sum_{i=1}^{n} \sigma_i^x \sigma_{i+1}^x \sigma_{i+2}^x$. The commutator between one- and two-body operators cannot produce a three-body operator. To generate a three-body operator, we must commute the two-body operators in $\mathcal{V}_c$ [40]. Therefore, one can construct $H_c(t) = \sum_{l\neq 0} H_{c,l} e^{il\omega t}$, where $H_{c,l} = \sum_{i=1}^{n} (c_{li}^{xy} \sigma_i^x \sigma_{i+1}^y + c_{li,zx} \sigma_i^z \sigma_{i+1}^x)$, $c_{li}^{xy} = c_{-li}^{xy*}$, and $c_{li}^{zx} = c_{-li}^{zx*}$ to ensure the Hermiticity of $H_c(t)$ [40]. When c_{li}^{xy} and c_{li}^{zx} are, respectively, real and purely imaginary, the commutator $[H_{c,l}, H_{c,-l}] = -4\sum_{i=1}^{n} \mathrm{Im}(c_{li}^{xy} c_{li+1}^{zx*}) \sigma_i^x \sigma_{i+1}^x \sigma_{i+2}^x$. The most general choice of the coefficients c_{li}^{xy} and c_{li}^{zx}, which generates the σ^x-type three-body interaction, is provided in Sec. V in the Supplemental Material [40]. For simplicity, we further assume these coefficients are homogeneous across the chain and take $c_{li}^{xy} = c_l^{xy}$ and $\tilde{c}_l^{zx} = ic_{li}^{zx}$ to be real for $l \geq 1$. This yields the high-frequency driving control Hamiltonian

$$H_c^{(d)}(t) = 2\sum_{l=1}^{\infty}\sum_{i=1}^{n} [c_l^{xy} \cos(l\omega t) \sigma_i^x \sigma_{i+1}^y + \tilde{c}_l^{zx} \sin(l\omega t) \sigma_i^z \sigma_{i+1}^x], \tag{14}$$

while the effective Hamiltonian obtained as the leading term in the high-frequency expansion becomes

$$H_F = \frac{\lambda}{2}\sum_{i=1}^{n} \sigma_i^z + \left[\frac{\Delta}{2} - \frac{4}{\omega}\sum_{l=1}^{\infty} \frac{1}{l}(c_l^{xy} \tilde{c}_l^{zx})\right] \sigma_i^x \sigma_{i+1}^x \sigma_{i+2}^x. \tag{15}$$

The three-body term is then canceled by tuning c_l^{xy} and $\tilde{c}_l^{zx}$ such that

$$\omega = 8/\Delta \sum_{l=1}^{\infty} (c_l^{xy} \tilde{c}_l^{zx}/l). \tag{16}$$

As with the qubit case, the optimal initial state in both the lab and Floquet rotating is the gigahertz state. Equations (5) and (6) are thus approximately satisfied up to the order of $1/\omega$. As shown in Figs. 1(b)–1(d), for parameters satisfying the AFM condition (16), the Heisenberg scaling I_0^{HS} is achieved. Furthermore, even when one just takes the lowest two harmonics in the driving, QFI is not very far below I_0^{HS}. In Fig. 2, I_0^{HS} is achieved with Eq. (14) when the parameters satisfy Eq. (16). Again, the precision using only the lowest two harmonics can approach I_0^{HS} very closely.

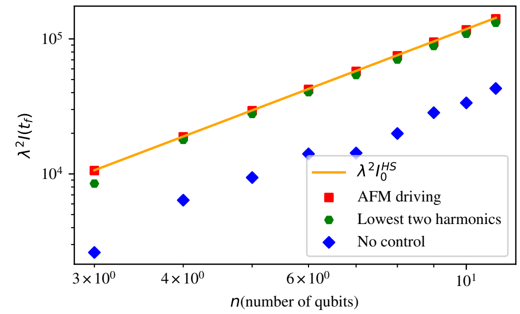

FIG. 2. Numerical calculation of the QFI with respect to the number of qubit n for the sensing of magnetic field using the spin-chain Hamiltonian (13). The initial state preparation and the values of the parameters λ, Δ, ω, c_l^{xy}, and $\tilde{c}_l^{zx}$ are the same as Fig. 1 with $J = 0$. (b)–(d) $t_f = 17.2$ is the total simulation time in Fig. 1.

Finally, we emphasize that a similar technique can be applied to design more general driving protocols to cancel the effect of other types of three-body interactions [40]. Experimental platforms where three-body interactions either appear naturally or can be potentially engineered—such as NMR systems [45,46], Kitaev spin liquid [47], superconducting circuits [48], and quantum gas systems [49–52]—can be potentially used to test the metrological protocol discussed here.

In summary, we have introduced a variational approach to quantum parameter estimation and derived the optimal control equations under which the precision is optimal when the available control Hamiltonians are limited. This approach readily yields the optimal initial state and Hamiltonian controls that are generally dependent on the probe time, in contrast with the unconstrained case. The implementation of the constrained optimal protocol in many-body systems can be eased by Floquet engineering, as we have demonstrated in applications to magnetometry. We hope that our results inspire new theoretical and technological advances in quantum metrology with quantum many-body systems. Many questions are open for further investigation, such as determining the ultimate scaling bounds of the QFI under restricted local controls and the application of our method to critically enhanced quantum metrology [53].

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[44] We choose the normalization $K(0) = 0$ to all orders of $1/\omega$, which is different from the normalization $1/T \int_0^T K(t)dt = 0$ used in Refs. [33,36]. Therefore, the resulting expression of $K(t)$ is different from the one in Refs. [33,36], up to some irrelevant constant, which does not affect the form of the Floquet effective Hamiltonian H_F.
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