Parallel Distributed Compensation for Piecewise Bilinear Models and Recurrent Fuzzy Systems Based on Piecewise Quadratic Lyapunov Functions

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Abstract: Piecewise Bilinear Models and Recurrent Fuzzy Systems are universal approximators for any smooth nonlinear dynamics. One of their advantage is the efficient representation of the modeled system dynamics by means of rule-bases or look-up-tables. In this paper, it is shown how to obtain provably stabilizing controllers by means of piecewise quadratic Lyapunov functions. Interpolating controllers with affine local controllers are considered for interpolation, akin to the concept of parallel distributed compensation widely used for control of Takagi-Sugeno systems.

1 Introduction

In order to analyze and control nonlinear systems, it is well known that besides exact methods, approximate approaches may be utilized. In the context of fuzzy logic-based modeling and control of dynamic systems, Takagi-Sugeno (T-S) systems [1] are probably the most prominent models allowing for an approximate system representation. In addition, Piecewise Bilinear Models (PBM) [2], [3] have been investigated over the past decade, which can be seen as special case of T-S systems. As universal approximators, they are also able to represent any smooth nonlinear dynamics, yet compared to T-S systems, their advantage is in the efficient representation of the system dynamics as look-up table. This makes PBM particularly interesting for industrial applications, where many times functional dependencies are given only approximately as look-up tables.

Independently of PBM, Recurrent Fuzzy Systems (RFS) [4] have evolved at the same time. Although the primary intention for this system class was to obtain a linguistically interpretable model based on fuzzy automata, due to recent developments they are utilized for control purposes as well. As found by the authors, both system classes are indeed similar and in special cases even identical, which makes it amendable to answer common questions related to both system classes in a common framework.

For both system classes, a number of literature exists dealing with the question of controller synthesis for the respected models. In the context of PBM, two main lines of research were followed up until now: The first is based on look-up table controllers which are derived for a given PBM by means of feedback-linearization [5] or a design by means of the vertex placement principle [6]. In a second approach, feedforward controllers are considered, which are obtained from feedback error learning [7].

In the case of RFS, fuzzy controllers [8], piecewise-polynomial controllers [9] and switching controllers [10] were discussed, whereas the latter approach focused especially on robust control.

In this paper, we now discuss the synthesis of provably stabilizing controllers for PBM and RFS, whereas the concept of parallel distributed compensation is used. The stability criterion is based on piecewise quadratic Lyapunov functions, which in the context of T-S systems were treated, e.g., in [11]. Based on the latter work, a similar discussion of stability analysis for PBM and RFS was carried out in detail in [12], which is now the starting point of the controller synthesis presented in this paper.

The remainder is organized as follows: Sec. 2 shortly reviews necessary definitions of PBM and RFS, whereas a review of the stability criterion under consideration is given in Sec. 3. The synthesis of state feedback is then given in Sec. 4, whereas practical hints considering the implementation are given in Sec. 5. The method is applied to the inverted pendulum example in Sec. 6, and concluding remarks are given in Sec. 7.

2 Preliminaries

In this section, the basic definitions of PBM and RFS are reviewed as detailed, e.g., in [2] and [13]. It is thereby shown that both system classes follow the same idea of universal approximators for smooth nonlinear dynamics and in some cases and under mild constraints are even of identical structure.

2.1 Piecewise Bilinear Systems

In order to model nonlinear system dynamics by means of PBM, it is assumed beforehand that these are input-affine, i.e.,

$$\dot{\mathbf{x}} = \mathbf{f}_{\mathrm{nl}}(\mathbf{x}) + \mathbf{G}_{\mathrm{nl}}(\mathbf{x})\mathbf{u}. \tag{1}$$

By discretizing (1) at finitely many discrete vertices $\mathbf{v_j}$, such that

$$\mathbf{x} = \sum_{\mathbf{j}} \mathbf{v_j} \prod_{i=1}^{n} w_{j_i}(x_i), \tag{2}$$

$\sum_{\mathbf{j}} w_{j_i} = 1$, $w_{j_i}(x_i) \in [0,1]$, the samples $\dot{\mathbf{v}}_\mathbf{j} = \mathbf{f}_{\mathrm{nl}}(\mathbf{v_j})$ and $\mathbf{G}_{\mathrm{nl}}(\mathbf{v_j})$ thus obtained may be stored in a look-up table. Thus, PBM become particularly suitable for practical applications, and can at the same time be given as rules of the form

$$\begin{aligned} &\text{If } \mathbf{x} = \mathbf{v_j}, \\ &\text{then } \dot{\mathbf{x}} = \dot{\mathbf{v}}_\mathbf{j} + \mathbf{G}_{\mathrm{nl}}(\mathbf{v_j})\mathbf{u}. \end{aligned} \tag{3}$$

Hence, by interpolating between rules (3) in every dimension of the state space using weights $w_{j_i}(x_i)$, the dynamics of the PBM

$$\dot{\mathbf{x}} = \sum_{\mathbf{j}} (\dot{\mathbf{v}}_{\mathbf{j}} + \mathbf{G}(\mathbf{v}_{\mathbf{j}})\mathbf{u}) \prod_{i=1}^{n} w_{j_i}(x_i) \tag{4}$$

are obtained, which are given here in *parameter form* [2]. Due to the sampling, a grid is introduced by the vertices $\mathbf{v}_{\mathbf{j}}$, whereas each rectangular region between $\mathbf{v}_{\mathbf{j}}$ and $\mathbf{v}_{\mathbf{j+1}}$ is abbreviated $R_{\mathbf{j}} = \text{conv}\,\{\mathbf{v}_{\mathbf{j}}, \mathbf{v}_{\mathbf{j}+1}\}$. Although these regions are hypersquares in general, the term *rectangle* is used for simplicity.

Similar to the dynamic function, the output function may be taken into consideration by means of the same sampling technique, which is omitted here due to space restrictions.

2.2 Recurrent Fuzzy Systems

In contrast to PBM, RFS allow for the modeling of any smooth nonlinear dynamics

$$\dot{\mathbf{x}} = \mathbf{f}_{\text{nl}}(\mathbf{x}, \mathbf{u}) \tag{5}$$

without assuming input-affinity. By sampling (5) on a grid at discrete vertex positions $(\mathbf{v}_{\mathbf{j}}, \mathbf{v}_{\mathbf{q}})$ akin to the PBM, gradients

$$\dot{\mathbf{v}}_{\mathbf{j},\mathbf{q}} = \mathbf{f}_{\text{nl}}(\mathbf{v}_{\mathbf{j}}, \mathbf{v}_{\mathbf{q}}) \tag{6}$$

are obtained. Furthermore, by introducing vectors of linguistic values $\mathbf{L}^{\mathbf{x}}_{\mathbf{j}}$, $\mathbf{L}^{\mathbf{u}}_{\mathbf{q}}$ and $\mathbf{L}^{\dot{\mathbf{x}}}_{\mathbf{j},\mathbf{q}}$ which are associated with crisp vertices $(\mathbf{v}_{\mathbf{j}}, \mathbf{v}_{\mathbf{q}})$ and gradients $\dot{\mathbf{v}}_{\mathbf{j},\mathbf{q}}$, the dynamics of the RFS

$$\begin{aligned} \text{If } \mathbf{x} = \mathbf{L}^{\mathbf{x}}_{\mathbf{j}} \text{ and } \mathbf{u} = \mathbf{L}^{\mathbf{u}}_{\mathbf{q}}, \\ \text{then } \dot{\mathbf{x}} = \mathbf{L}^{\dot{\mathbf{x}}}_{\mathbf{j},\mathbf{q}} \end{aligned} \tag{7}$$

are obtained for each vertex. In order to obtain again crisp gradient values from the RFS, membership functions $w_{j_i}(x_i), w_{q_p}(u_p) \in [0, 1]$ are introduced fuzzifying the linguistic variables. Following [13], the algebraic product is then chosen for aggregation and implication, such that the premise weights

$$w_{\mathbf{j},\mathbf{q}}(\mathbf{x},\mathbf{u}) = w_{\mathbf{j}}(\mathbf{x})w_{\mathbf{q}}(\mathbf{u}) = \prod_{i=1}^{n} w_{j_i}(x_i) \prod_{p=1}^{m} w_{q_p}(u_p) \tag{8}$$

are obtained. Evaluating the implication as

$$w_{\mathbf{j},\mathbf{q}}^{\text{imp}}(\mathbf{x},\mathbf{u}) = \dot{\mathbf{v}}_{\mathbf{j},\mathbf{q}} w_{\mathbf{j},\mathbf{q}}(\mathbf{x},\mathbf{u}) \tag{9}$$

and using the algebraic sum for accumulation,

$$w_{\mathbf{j},\mathbf{q}}^{\text{acc}} = \sum_{\mathbf{j},\mathbf{q}} w_{\mathbf{j},\mathbf{q}}^{\text{imp}}(\mathbf{x},\mathbf{u}) \tag{10}$$

is obained. Finally, by making use of the center of singleton defuzzification, crisp values

$$\dot{\mathbf{x}} = \frac{w_{\mathbf{j},\mathbf{q}}^{\text{acc}}(\mathbf{x},\mathbf{u})}{\sum_{\mathbf{j},\mathbf{q}} w_{\mathbf{j},\mathbf{q}}(\mathbf{x},\mathbf{u})} \tag{11}$$

are obtained. If furthermore $\sum_{\mathbf{j},\mathbf{q}} w_{\mathbf{j},\mathbf{q}}(\mathbf{x},\mathbf{u}) = 1$ holds, again a parametric form

$$\dot{\mathbf{x}} = \sum_{\mathbf{j},\mathbf{q}} \dot{\mathbf{v}}_{\mathbf{j},\mathbf{q}} \prod_{i=1}^{n} w_{j_i}(x_i) \prod_{p=1}^{m} w_{q_p}(u_p) \tag{12}$$

akin to (4) is obtained, showing the similarity between PBM and RFS.

2.3 Comparison

Comparing the system structure of PBM and RFS, it becomes obvious that their main difference is in the treatment of the input space. Whereas in the case of PBM, the class of system dynamics are restricted to input-affine systems, RFS allow for more general dynamics. Nevertheless, the restriction to input-affine systems is reasonable under a practical viewpoint, since this assumption holds for a variety of technical systems. In addition, the controller synthesis is considerably facilitated with this restriction.

From a structural point of view, it can be noted that for the special cases of unforced systems and single-input systems, PBM and RFS are fully equivalent. For $m > 1$, i.e., systems with multiple inputs, this equivalence does not hold in general. On the other hand, RFS can be rendered input-affine by means of a dynamic transformation, such that a conversion into PBM is again possible.

For both system classes, nothing was said until now about the type of membership function. In general, any kind of membership function is applicable fulfilling he properties $w_{j_i} \in [0,1]$ and $\sum_{\mathbf{j}} w_{j_i} = 1$. On the other hand, the restriction to ramp- and triangular-shaped membership functions is considered due to several reasons: First, the interpolation by means of piecewise affine functions is easy to implement. Second, these membership functions are nonzero only in a local region, supporting the idea of a piecewise locally defined system. In addition, the simplicity of triangular membership functions allows for the derivation of system dynamics in a piecewise polynomial form, which can furthermore be expressed in a matrix form, being particularly suitable for implementation.

3 Stability Analysis

This section outlines the stability analysis of PBM and RFS by means of piecewise quadratic Lyapunov functions as discussed in [12]. It will form the basis for the controller synthesis, which is detailed in Sec. 4. Piecewise quadratic Lyapunov have already been utilized in the context of hybrid systems [14] or affine T-S fuzzy systems [11]. We closely follow these discussions, but apply the approach to open-loop PBM and RFS

$$\dot{\mathbf{x}} = \sum_{\mathbf{j}} \dot{\mathbf{v}}_{\mathbf{j}} \prod_{i=1}^{n} w_{j_i}(x_i). \tag{13}$$

In the following an equilibrium $(\mathbf{x}^*, \mathbf{u}^*)$ is assumed at the origin and incident with a vertex, which is without loss of generality, because each PBM and RFS may be augmented with a vertex at $(\mathbf{x}^*, \mathbf{u}^*)$ without changing the overall dynamics.

Starting with the well-known Lyapunov equations

$$V(\mathbf{x}) > 0, \quad \forall \mathbf{x} \neq \mathbf{0}, \tag{14a}$$

$$\dot{V}(\mathbf{x}) < 0, \quad \forall \mathbf{x} \neq \mathbf{0}, \tag{14b}$$

$$V(\mathbf{0}) = 0, \tag{14c}$$

a local approximation

$$V(\mathbf{x}) \approx r_{\mathrm{j}} + 2\mathbf{q}_{\mathrm{j}}^T\mathbf{x} + \mathbf{x}^T\mathbf{P}_{\mathrm{j}}\mathbf{x}, \quad \mathbf{x} \in R_{\mathrm{j}} \tag{15}$$

is imposed on the Lyapunov function. A reformulation in terms of matrix multiplications then yields

$$V(\mathbf{x}) \approx \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}^T \begin{bmatrix} \mathbf{P}_{\mathrm{j}} & \mathbf{q}_{\mathrm{j}} \\ \mathbf{q}_{\mathrm{j}}^T & r_{\mathrm{j}} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} = \overline{\mathbf{x}}^T\overline{\mathbf{P}}_{\mathrm{j}}\overline{\mathbf{x}}, \quad \mathbf{x} \in R_{\mathrm{j}}. \tag{16}$$

Furthermore, it is assumed that

$$r_{\mathrm{j}} = 0, \quad \mathbf{q}_{\mathrm{j}} = \mathbf{0}, \quad \forall R_{\mathrm{j}} \in \mathcal{R}_0, \tag{17}$$

in order for (14c) to hold. Therein, $\mathcal{R}_0$ denotes the set of rectangles containing the origin. (16) can then be considered as piecewise-defined function

$$\Rightarrow V(\mathbf{x}) = \begin{cases} \mathbf{x}^T\mathbf{P}_{\mathrm{j}}\mathbf{x}, & R_{\mathrm{j}} \in \mathcal{R}_0, \\ \overline{\mathbf{x}}^T\overline{\mathbf{P}}_{\mathrm{j}}\overline{\mathbf{x}}, & R_{\mathrm{j}} \notin \mathcal{R}_0. \end{cases} \tag{18}$$

It is crucial for (18) to be continuous across rectangle borders in order to guarantee stability. As shown in [11], this can be ensured by decomposing the matrices of Lyapunov function candidates according to

$$\mathbf{P}_{\mathrm{j}} = \mathbf{F}_{\mathrm{j}}^T\mathbf{T}\mathbf{F}_{\mathrm{j}}, \quad R_{\mathrm{j}} \in \mathcal{R}_0, \tag{19a}$$

$$\overline{\mathbf{P}}_{\mathrm{j}} = \overline{\mathbf{F}}_{\mathrm{j}}^T\mathbf{T}\overline{\mathbf{F}}_{\mathrm{j}}, \quad R_{\mathrm{j}} \notin \mathcal{R}_0, \tag{19b}$$

where $\overline{\mathbf{F}}_{\mathrm{j}} = \begin{bmatrix} \mathbf{F}_{\mathrm{j}} & \mathbf{f}_{\mathrm{j}} \end{bmatrix}$ and $\mathbf{f}_{\mathrm{j}} = \mathbf{0}$ for $R_{\mathrm{j}} \in \mathcal{R}_0$ are introduced, fulfilling the facet conditions

$$\overline{\mathbf{F}}_{\mathrm{i}}\overline{\mathbf{x}} = \overline{\mathbf{F}}_{\mathrm{j}}\overline{\mathbf{x}}, \quad \mathbf{x} \in R_{\mathrm{i}} \cap R_{\mathrm{j}}. \tag{20}$$

Then,

$$\dot{V}(\mathbf{x}) = \begin{cases} 2\mathbf{x}^T \mathbf{P_j} \dot{\mathbf{x}}, & R_\mathbf{j} \in \mathcal{R}_0, \\ 2 \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}^T \overline{\mathbf{P}}_\mathbf{j} \begin{bmatrix} \dot{\mathbf{x}} \\ 1 \end{bmatrix}, & R_\mathbf{j} \notin \mathcal{R}_0 \end{cases} \tag{21}$$

follows from (14b). Herein, we now substitute the state and state derivative by (2) and (13), which are expressed as convex hull of vertices and gradients, i.e.,

$$\begin{aligned} \mathbf{x} &= \sum_{\mathbf{j}\backslash\mathbf{0}} \mathbf{v_j} \cdot w_\mathbf{j}(\mathbf{x}) + \mathbf{0} \cdot w_\mathbf{0}(\mathbf{x}) \\ &= \sum_{\mathbf{j}\backslash\mathbf{0}} \mathbf{v_j} \cdot w_\mathbf{j}(\mathbf{x}) = \operatorname*{conv}_{\mathbf{j}\backslash\mathbf{0}} \{\mathbf{v_j}\}, \quad \mathbf{x} \in R_\mathbf{j} \end{aligned} \tag{22}$$

and

$$\begin{aligned} \dot{\mathbf{x}} &= \sum_{\mathbf{j}\backslash\mathbf{0}} \dot{\mathbf{v}}_\mathbf{j} \cdot w_\mathbf{j}(\mathbf{x}) + \underbrace{\dot{\mathbf{v}}_\mathbf{0}}_{=\mathbf{0}} \cdot w_\mathbf{0}(\mathbf{x}) \\ &= \sum_{\mathbf{j}\backslash\mathbf{0}} \dot{\mathbf{v}}_\mathbf{j} \cdot w_\mathbf{j}(\mathbf{x}) = \operatorname*{conv}_{\mathbf{j}\backslash\mathbf{0}} \{\dot{\mathbf{v}}_\mathbf{j}\}, \quad \mathbf{x} \in R_\mathbf{j}. \end{aligned} \tag{23}$$

For simplicity, the shorthand notation $\mathbf{j}\backslash\mathbf{0}$ was introduced meaning all vertices $\mathbf{j}$ not corresponding to the equilibrium.

Substituting (22) and (23) into (21) then yields

$$\begin{aligned} \dot{V}(\mathbf{x}) &\in \begin{cases} 2 \operatorname*{conv}_{\mathbf{i}\backslash\mathbf{0}} \{\mathbf{v_i}^T\} \mathbf{P_k} \operatorname*{conv}_{\mathbf{j}\backslash\mathbf{0}} \{\dot{\mathbf{v}}_\mathbf{j}\}, & R_\mathbf{k} \in \mathcal{R}_0, \\ 2 \operatorname*{conv}_{\mathbf{i}\backslash\mathbf{0}} \left\{ \begin{bmatrix} \mathbf{v_i} \\ 1 \end{bmatrix}^T \right\} \overline{\mathbf{P}}_\mathbf{k} \operatorname*{conv}_{\mathbf{j}\backslash\mathbf{0}} \left\{ \begin{bmatrix} \dot{\mathbf{v}}_\mathbf{j} \\ 1 \end{bmatrix} \right\}, & R_\mathbf{k} \notin \mathcal{R}_0, \end{cases} \\ &= \begin{cases} 2 \operatorname*{conv}_{\mathbf{i},\mathbf{j}\backslash\mathbf{0}} \{\mathbf{v_i}^T \mathbf{P_k} \dot{\mathbf{v}}_\mathbf{j}\}, & \mathbf{v_i}, \mathbf{v_j} \in R_\mathbf{k} \in \mathcal{R}_0, \\ 2 \operatorname*{conv}_{\mathbf{i},\mathbf{j}\backslash\mathbf{0}} \left\{ \begin{bmatrix} \mathbf{v_i} \\ 1 \end{bmatrix}^T \overline{\mathbf{P}}_\mathbf{k} \begin{bmatrix} \dot{\mathbf{v}}_\mathbf{j} \\ 1 \end{bmatrix} \right\}, & \mathbf{v_i}, \mathbf{v_j} \in R_\mathbf{k} \notin \mathcal{R}_0. \end{cases} \end{aligned} \tag{24}$$

Hence, (13) is asymptotically stable, if

$\forall\ R_{\mathrm{k}} \in \mathcal{R}_0$:

$$\mathbf{P}_{\mathrm{k}} \succ 0, \tag{25a}$$

$$\mathbf{v}_{\mathrm{i}}^T \mathbf{P}_{\mathrm{k}} \dot{\mathbf{v}}_{\mathrm{j}} < 0, \quad \mathbf{v}_{\mathrm{i}}, \mathbf{v}_{\mathrm{j}} \in R_{\mathrm{k}} \backslash \mathbf{0}, \tag{25b}$$

$\forall\ R_{\mathrm{k}} \notin \mathcal{R}_0$:

$$\overline{\mathbf{P}}_{\mathrm{k}} \succ 0, \tag{25c}$$

$$\begin{bmatrix} \mathbf{v}_{\mathrm{i}} \\ 1 \end{bmatrix}^T \overline{\mathbf{P}}_{\mathrm{k}} \begin{bmatrix} \dot{\mathbf{v}}_{\mathrm{j}} \\ 1 \end{bmatrix} < 0, \quad \mathbf{v}_{\mathrm{i}}, \mathbf{v}_{\mathrm{j}} \in R_{\mathrm{k}}. \tag{25d}$$

In order to account for the locality of these equations, the S-procedure (see, e.g., [15]) is utilized, such that inequalities (25) have to hold on R_{k} only instead of the entire state space. This relaxation comes at the cost that it renders the stability conditions sufficient only, because the S-procedure is itself in general only a sufficient conditions.

To apply the S-procedure for each rectangular regions R_{j}, they are described by means of $2n$ bounding hyperplanes, such that

$$\overline{\mathbf{E}}_{\mathrm{j}} \overline{\mathbf{x}} \geq \mathbf{0}, \quad \mathbf{x} \in R_{\mathrm{j}}, \tag{26}$$

with $\overline{\mathbf{E}}_{\mathrm{j}} \in \mathbb{R}^{2n \times n+1}$ as detailed in [11]. Then, stability conditions for system (13) in relaxed form read

$\forall\ R_{\mathrm{k}} \in \mathcal{R}_0$:

$$\mathbf{P}_{\mathrm{k}} \succ 0, \tag{27a}$$

$$\mathbf{v}_{\mathrm{i}}^T \mathbf{P}_{\mathrm{k}} \dot{\mathbf{v}}_{\mathrm{j}} < 0, \quad \mathbf{v}_{\mathrm{i}}, \mathbf{v}_{\mathrm{j}} \in R_{\mathrm{k}} \backslash \mathbf{0}, \tag{27b}$$

$\forall\ R_{\mathrm{k}} \notin \mathcal{R}_0$:

$$\overline{\mathbf{P}}_{\mathrm{k}} - \overline{\mathbf{E}}_{\mathrm{k}}^T \mathbf{S}_{\mathrm{k}} \overline{\mathbf{E}}_{\mathrm{k}} \succ 0, \tag{27c}$$

$$\begin{bmatrix} \mathbf{v}_{\mathrm{i}} \\ 1 \end{bmatrix}^T \overline{\mathbf{P}}_{\mathrm{k}} \begin{bmatrix} \dot{\mathbf{v}}_{\mathrm{j}} \\ 1 \end{bmatrix} < 0, \quad \mathbf{v}_{\mathrm{i}}, \mathbf{v}_{\mathrm{j}} \in R_{\mathrm{k}}, \tag{27d}$$

with $\mathbf{S_k} \geq \mathbf{0}$ being component-wise non-negative. It has to be emphasized that only (27c) has to be relaxed by means of the S-procedure, whereas (27b) and (27d) are scalar inequalities and thus inherently local.

As additional constraints, the region of attraction $\mathcal{E} = \{\mathbf{x} \mid V(\mathbf{x}) \leq 1\}$ has to be contained in the set of rectangles $\mathcal{R}$. This was discussed, e.g., in [16] for T-S systems. For $\mathcal{R}$ being a symmetric polytope around the origin, this interpolation region can be rewritten as

$$\mathcal{R} = \{\mathbf{x} \,|\, |\mathbf{a}_i^T \mathbf{x}| \leq 1, \quad i = 1, \dots, 2n\}. \tag{28}$$

Then, following [15], the additional constraint

$$\mathbf{a}_i^T \mathbf{P}_{\mathbf{k}}^{-1} \mathbf{a}_i \leq 1, \quad \forall\, \mathbf{a}_i \in R_{\mathbf{k}} \in \mathcal{R}_0, \tag{29a}$$

$$\begin{bmatrix} \mathbf{a}_i \\ 1 \end{bmatrix}^T \overline{\mathbf{P}}_{\mathbf{k}}^{-1} \begin{bmatrix} \mathbf{a}_i \\ 1 \end{bmatrix} \leq 1, \quad \forall\, \mathbf{a}_i \in R_{\mathbf{k}} \notin \mathcal{R}_0, \tag{29b}$$

for all $R_{\mathbf{k}} \cap \partial\mathcal{R} \neq \emptyset$ guarantees that $\mathcal{E} \subseteq \mathcal{R}$. Applying Schur complement, (29) can be rewritten as

$$\begin{bmatrix} \mathbf{P_k} & \mathbf{a}_i \\ \mathbf{a}_i^T & 1 \end{bmatrix} \succ 0, \quad \forall\, \mathbf{a}_i \in R_{\mathbf{k}} \in \mathcal{R}_0, \tag{30a}$$

$$\begin{bmatrix} \overline{\mathbf{P}}_{\mathbf{k}} & \overline{\mathbf{a}}_i \\ \overline{\mathbf{a}}_i^T & 1 \end{bmatrix} \succ 0, \quad \forall\, \mathbf{a}_i \in R_{\mathbf{k}} \notin \mathcal{R}_0, \tag{30b}$$

with $\overline{\mathbf{a}}_i = \begin{bmatrix} \mathbf{a}_i^T & 1 \end{bmatrix}^T$. These conditions may lead to conservative results for sets $\mathcal{R}$ being non-symmetric around the origin. A way to circumvent this problem is proposed in [12].

4 Synthesis of Parallel Distributed Compensation

Based on the previously discussed stability analysis, this section is devoted to the synthesis of local affine controllers stabilizing a given PBM/RFS. Because the input space has to be considered for the controller synthesis, the discussion will be carried out for the more general RFS, which include PBM as special case.

We assume for each vertex $\mathbf{v_j} \in \mathcal{R} \subset \mathbb{R}^n$ an affine local controller $\mathbf{K_j x}+\mathbf{k_j}$, with $\mathbf{k_j} = \mathbf{0}$ for $\mathbf{v_j} = \mathbf{x}^*$. Then, by means of a linear interpolation, the control law

$$\mathbf{u} = \mathbf{k}(\mathbf{x}) = \sum_{\mathbf{j}} (\mathbf{K_j x} + \mathbf{k_j}) \prod_{i=1}^{n} w_{j_i}(x_i) \tag{31}$$

is obtained, where a notation similar to the parametric form for PBM/RFS was used. Note that local controllers were only chosen for each vertex in the state space, rather than the input-state space to avoid the controller from being an implicit function. For convenience, we will also denote (31) by

$$\mathbf{k}(\mathbf{x}) \in \underset{\mathbf{v_j}, \mathbf{x} \in R_\mathbf{k}}{\text{conv}} \{\mathbf{K_j x} + \mathbf{k_j}\}. \tag{32}$$

Starting from the dynamics (12), a partial matrix form

$$\dot{\mathbf{x}} = \sum_{\mathbf{j},\mathbf{q}} \dot{\mathbf{v}}_{\mathbf{j},\mathbf{q}} \cdot w_\mathbf{j}(\mathbf{x}) \cdot w_\mathbf{q}(\mathbf{u}) = \mathbf{f}_{\mathbf{j},\mathbf{q}}(\mathbf{x}, \mathbf{u}) \tag{33}$$

is derived, where for simplicity $\mathbf{f}_{\mathbf{j},\mathbf{q}}$ denotes the local dynamics being active in rectangle $R_{\mathbf{j},\mathbf{q}}$. Then, by means of a linearization

$$\mathbf{A}_{\mathbf{j},\mathbf{q}} = \frac{\partial}{\partial \mathbf{x}} \mathbf{f}_{\mathbf{j},\mathbf{q}}(\mathbf{x}, \mathbf{0}) \bigg|_{\mathbf{x}=\mathbf{v_j}}, \tag{34a}$$

$$\mathbf{a}_{\mathbf{j},\mathbf{q}} = \mathbf{f}_{\mathbf{j},\mathbf{q}}(\mathbf{v_j}, \mathbf{0}) - \mathbf{A}_{\mathbf{j},\mathbf{q}} \mathbf{v_j}, \tag{34b}$$

$$\mathbf{B}_{\mathbf{j},\mathbf{q}} = \frac{\partial}{\partial \mathbf{u}} \mathbf{f}_{\mathbf{j},\mathbf{q}}(\mathbf{v_j}, \mathbf{u}) \bigg|_{\mathbf{u}=\mathbf{0}} \tag{34c}$$

for each $\mathbf{v_j} \in R_\mathbf{k}$, (33) is restated as

$$\dot{\mathbf{x}} \in \underset{\mathbf{q}}{\text{conv}} \left\{ \underset{\mathbf{v_j}, \mathbf{x} \in R_\mathbf{k}}{\text{conv}} \{\mathbf{A}_{\mathbf{j},\mathbf{q}} \mathbf{x} + \mathbf{a}_{\mathbf{j},\mathbf{q}} + \mathbf{B}_{\mathbf{j},\mathbf{q}} \mathbf{u}\} \right\}, \tag{35}$$

where it is assumed that $\mathbf{a}_{\mathbf{j},\mathbf{q}} = \mathbf{0}$ for $\mathbf{v_j} \in \mathcal{R}_0$, i.e., the region of rectangles including the origin. Note that for $\mathbf{v}_{\mathbf{j},\mathbf{q}} = \mathbf{0}$ to be an equilibrium, this is also a necessary conditions.

By substituting (32) into (35), the dynamics of the closed-loop system

$$\begin{aligned}\dot{\mathbf{x}} &\in \underset{\mathbf{v_i},\mathbf{v_j},\mathbf{x}\in R_\mathbf{k};\mathbf{q}}{\operatorname{conv}} \left\{\mathbf{A_{j,q}x} + \mathbf{a_{j,q}} + \mathbf{B_{j,q}}\left(\mathbf{K_i x} + \mathbf{k_i}\right)\right\} \\ &= \begin{cases} \underset{\mathbf{v_i},\mathbf{v_j},\mathbf{x}\in R_\mathbf{k};\mathbf{q}}{\operatorname{conv}} \left\{\mathbf{A_{i,j,q}x}\right\}, & \mathbf{v_i},\mathbf{v_j} \in \mathcal{R}_0, \\ \underset{\mathbf{v_i},\mathbf{v_j},\mathbf{x}\in R_\mathbf{k};\mathbf{q}}{\operatorname{conv}} \left\{\overline{\mathbf{A}}_\mathbf{i,j,q}\overline{\mathbf{x}}\right\}, & \mathbf{v_i},\mathbf{v_j} \notin \mathcal{R}_0, \end{cases}\end{aligned} \tag{36a}$$

are obtained as differential inclusion, with

$$\mathbf{A_{i,j,q}} = \mathbf{A_{j,q}} + \mathbf{B_{j,q}K_i}, \qquad \mathbf{v_i},\mathbf{v_j} \in \mathcal{R}_0, \tag{36b}$$

$$\overline{\mathbf{A}}_\mathbf{i,j,q} = \begin{bmatrix}\mathbf{A_{j,q}} + \mathbf{B_{j,q}K_i} & \mathbf{a_{j,q}} + \mathbf{k_i} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}, \qquad \mathbf{v_i},\mathbf{v_j} \notin \mathcal{R}_0. \tag{36c}$$

Then by means of a piecewise quadratic Lyapunov function (18),

$$\begin{aligned}\dot{V}(\mathbf{x}) &= \begin{cases} \operatorname{He}\left\{\mathbf{x}^T\mathbf{P_k}\dot{\mathbf{x}}\right\}, & R_\mathbf{k} \in \mathcal{R}_0, \\ \operatorname{He}\left\{\overline{\mathbf{x}}^T\overline{\mathbf{P}}_\mathbf{k}\begin{bmatrix}\dot{\mathbf{x}} \\ 1\end{bmatrix}\right\}, & R_\mathbf{k} \notin \mathcal{R}_0, \end{cases} \\ &\in \begin{cases} \mathbf{x}^T \underset{\mathbf{v_i},\mathbf{v_j},\mathbf{x}\in R_\mathbf{k};\mathbf{q}}{\operatorname{conv}} \left\{\operatorname{He}\left\{\mathbf{P_k A_{i,j,q}}\right\}\right\}\mathbf{x}, & R_\mathbf{k} \in \mathcal{R}_0, \\ \overline{\mathbf{x}}^T \underset{\mathbf{v_i},\mathbf{v_j},\mathbf{x}\in R_\mathbf{k};\mathbf{q}}{\operatorname{conv}} \left\{\operatorname{He}\left\{\overline{\mathbf{P}}_\mathbf{k}\overline{\mathbf{A}}_\mathbf{i,j,q}\right\}\right\}\overline{\mathbf{x}}, & R_\mathbf{k} \notin \mathcal{R}_0 \end{cases}\end{aligned} \tag{37}$$

follows from substituting (36a) into (21). Thus, $\dot{V}(\mathbf{x}) < 0$ is ensured, if

$$\operatorname{He}\left\{\mathbf{P_k A_{i,j,q}}\right\} \prec 0, \quad R_\mathbf{k} \in \mathcal{R}_0, \tag{38a}$$

$$\operatorname{He}\left\{\overline{\mathbf{P}}_\mathbf{k}\overline{\mathbf{A}}_\mathbf{i,j,q}\right\} \prec 0, \quad R_\mathbf{k} \notin \mathcal{R}_0 \tag{38b}$$

holds. In contrast to the gradient conditions (27b) and (27d) for the stability analysis, (37) can no longer be reduced to finitely many scalar inequalities at vertices. As a consequence, the S-procedure has to be utilized for (38) in order to take locality of the matrix inequalities into consideration. Thus, (38) holds if there exists symmetric matrices $\mathbf{S}_{\mathrm{a},\mathbf{k}}$ with non-negative components such that

$$\operatorname{He}\left\{\mathbf{P_k A_{i,j,q}}\right\} + \mathbf{E}_\mathbf{k}^T\mathbf{S}_{\mathrm{a},\mathbf{k}}\mathbf{E_k} \prec 0, \quad R_\mathbf{k} \in \mathcal{R}_0, \tag{39a}$$

$$\operatorname{He}\left\{\overline{\mathbf{P}}_\mathbf{k}\overline{\mathbf{A}}_\mathbf{i,j,q}\right\} + \overline{\mathbf{E}}_\mathbf{k}^T\mathbf{S}_{\mathrm{a},\mathbf{k}}\overline{\mathbf{E}}_\mathbf{k} \prec 0, \quad R_\mathbf{k} \notin \mathcal{R}_0. \tag{39b}$$

Akin to the conditions for stability analysis, the restriction of the region of attraction to the domain, i.e., $\mathcal{E} \subseteq \mathcal{R}$, is taken into consideration by means of (30). In order to maximize the size of the region of attraction, auxiliary ellipsoids $\varepsilon_i = \mathbf{x}^T\mathbf{M}_i\mathbf{x}$ for each sector $\mathcal{S}_i$ around $\mathbf{x}^*$ are inscribed into the region of attraction. With $\mathbf{n}_i^T$ being a normal vector pointing into the sector $\mathcal{S}_i$, the radius of each ellipse is maximized by $\min \mathbf{n}_i^T\mathbf{M}_i\mathbf{n}_i$. In addition, the constraint of auxiliary ellipses being contained in the region of attraction, i.e., $\cup_i\varepsilon_i \subseteq \mathcal{E}$, may be formulated as

$$\begin{cases} \mathbf{x}^T\mathbf{P_k}\mathbf{x} \leq \mathbf{x}^T\mathbf{M}_i\mathbf{x}, & \mathbf{x} \in R_\mathbf{k} \in \mathcal{R}_0, \\ \overline{\mathbf{x}}^T\overline{\mathbf{P}}_\mathbf{k}\overline{\mathbf{x}} \leq \overline{\mathbf{x}}^T \begin{bmatrix} \mathbf{M}_i & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} \overline{\mathbf{x}}, & \mathbf{x} \in R_\mathbf{k} \notin \mathcal{R}_0 \end{cases} \tag{40}$$

for each $R_\mathbf{k} \cap \mathcal{S}_i$. Equivalently, by using the S-procedure, (40) can be written as LMI

$$\begin{cases} \mathbf{M}_i - \mathbf{P_k} - \mathbf{E}_\mathbf{k}^T\mathbf{S}_{\mathrm{b},\mathbf{k}}\mathbf{E_k} \succ 0, & R_\mathbf{k} \in \mathcal{R}_0, \\ \begin{bmatrix} \mathbf{M}_i & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} - \overline{\mathbf{P}}_\mathbf{k} - \overline{\mathbf{E}}_\mathbf{k}^T\mathbf{S}_{\mathrm{b},\mathbf{k}}\overline{\mathbf{E}}_\mathbf{k} \succ 0, & R_\mathbf{k} \notin \mathcal{R}_0. \end{cases} \tag{41}$$

Then, the controller synthesis problem for PBM and RFS is solved by means of the following theorem, summarizing the aforementioned conditions:

Theorem 1. *A PBM or RFS is piecewise quadratically stable, if there exist local controller matrices* $\mathbf{K_j}, \mathbf{k_j}$ *for each vertex* $\mathbf{v_j} \in \mathcal{R}$, *and symmetric matrices* $\mathbf{T}$, $\mathbf{S_k} > \mathbf{0}$, $\mathbf{S}_{\mathrm{a},\mathbf{k}} > \mathbf{0}$, $\mathbf{S}_{\mathrm{b},\mathbf{k}} > \mathbf{0}$, *such that*

$$\min \sum_i \mathbf{n}_i^T\mathbf{M}_i\mathbf{n}_i, \tag{42a}$$

$\forall\ R_\mathbf{k} \in \mathcal{R}_0$*:*

$$\mathbf{P_k} - \mathbf{E}_\mathbf{k}^T\mathbf{S_k}\mathbf{E_k} \succ 0, \tag{42b}$$

$$\mathrm{He}\left\{\mathbf{P_k}\mathbf{A_{i,j,q}}\right\} + \mathbf{E}_\mathbf{k}^T\mathbf{S}_{\mathrm{a},\mathbf{k}}\mathbf{E_k} \prec 0, \quad \mathbf{v_i}, \mathbf{v_j} \in R_\mathbf{k}, \tag{42c}$$

$$\begin{bmatrix} \mathbf{P_k} & \mathbf{a}_i \\ \mathbf{a}_i^T & 1 \end{bmatrix} \succ 0, \quad R_\mathbf{k} \cap \partial\mathcal{R} \neq \emptyset, \tag{42d}$$

$$\mathbf{M}_i - \mathbf{P_k} - \mathbf{E}_\mathbf{k}^T \mathbf{S}_{\mathrm{b},\mathbf{k}} \mathbf{E_k} \succ 0, \tag{42e}$$

$\forall\ R_\mathbf{k} \notin \mathcal{R}_0$:

$$\overline{\mathbf{P}}_\mathbf{k} - \overline{\mathbf{E}}_\mathbf{k}^T \mathbf{S_k} \overline{\mathbf{E}}_\mathbf{k} \succ 0, \tag{42f}$$

$$\mathrm{He}\left\{\overline{\mathbf{P}}_\mathbf{k}\overline{\mathbf{A}}_\mathbf{i,j,q}\right\} + \overline{\mathbf{E}}_\mathbf{k}^T \mathbf{S}_{\mathrm{a},\mathbf{k}} \overline{\mathbf{E}}_\mathbf{k} \prec 0, \quad \mathbf{v_i}, \mathbf{v_j} \in R_\mathbf{k} \tag{42g}$$

$$\begin{bmatrix} \overline{\mathbf{P}}_\mathbf{k} & \overline{\mathbf{a}}_i \\ \overline{\mathbf{a}}_i^T & 1 \end{bmatrix} \succ 0, \quad R_\mathbf{k} \cap \partial\mathcal{R} \neq \emptyset, \tag{42h}$$

$$\begin{bmatrix} \mathbf{M}_i & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} - \overline{\mathbf{P}}_\mathbf{k} - \overline{\mathbf{E}}_\mathbf{k}^T \mathbf{S}_{\mathrm{b},\mathbf{k}} \overline{\mathbf{E}}_\mathbf{k} \succ 0 \tag{42i}$$

holds for all $R_\mathbf{q} \in \mathcal{U}$. Then, the maximum region of attraction $\mathcal{E}$ is given by $V(\mathbf{x}) \leq 1$, and $\mathcal{E} \subseteq \mathcal{R}$.

If $\mathbf{K_j} = \mathbf{0}$ is chosen, then a controller is obtained, which is structurally equivalent to a Mamdani type fuzzy controller or look-up-table controller. Nevertheless, in general it is desirable to have more degrees of freedom by allowing non-zero $\mathbf{K_j}$, leading to an improved performance compared to pure static fuzzy controllers.

5 Implementational Aspects

Due to product terms in $\mathbf{P_k}$ and $\mathbf{K_j}$, $\mathbf{k_j}$, the optimization problem (42) contains bilinear matrix inequalities (BMI). Therefore, only suboptimal results can be obtained in contrast to the stability analysis. Because of the piecewise definition of the system, a linearizing transformation as in the case of linear systems with unique quadratic Lyapunov function appears to be impossible. Among the solution strategies proposed in the literature for solving BMI, we mention the path-following approach [17] linearizing the optimization problem and solving it in an iterative manner, as well as the rank minimization strategy [18], both being local solution algorithms. Global solution strategies have been proposed in [19] based on a branch-and-bound technique and in [20] based on the benders decomposition.

To limit the implementational burden, we make use of a V-K-iteration scheme as proposed in [21], which iteratively solves (42) by keeping either $\mathbf{P_j}$ or $\mathbf{K_j}, \mathbf{k_j}$ fixed while solving the LMI-problem for the remaining variables. Obviously, the solution thereby obtained relies heavily on the initial solution. Therefore, we propose to compute an initial controller heuristically akin to the initial solution proposed for the ILMI algorithm for T-S systems in [22]. The aim is to obtain for each vertex $\mathbf{v_j} \in \mathcal{R}$ a mean affine system

$$\dot{\mathbf{v}}_\mathbf{j} \approx \tilde{\mathbf{A}}_\mathbf{j}\mathbf{x} + \tilde{\mathbf{a}}_\mathbf{j} + \tilde{\mathbf{B}}_\mathbf{j}\mathbf{u}, \tag{43}$$

with $\tilde{\mathbf{A}}_\mathbf{j} = \frac{1}{N_q}\sum_\mathbf{q} \frac{1}{2^n}\sum_{\mathbf{k}:\mathbf{v_j}\in R_\mathbf{k}} \mathbf{A}_{\mathbf{j},\mathbf{q}}$, $\tilde{\mathbf{a}}_\mathbf{j} = \frac{1}{N_q}\sum_\mathbf{q} \frac{1}{2^n}\sum_{\mathbf{k}:\mathbf{v_j}\in R_\mathbf{k}} \mathbf{a}_{\mathbf{j},\mathbf{q}}$, $\tilde{\mathbf{B}}_\mathbf{j} = \frac{1}{N_q}\sum_\mathbf{q} \frac{1}{2^n}\sum_{\mathbf{k}:\mathbf{v_j}\in R_\mathbf{k}} \mathbf{B}_{\mathbf{j},\mathbf{q}}$ where $\mathbf{A}_{\mathbf{j},\mathbf{q}}, \mathbf{a}_{\mathbf{j},\mathbf{q}}, \mathbf{B}_{\mathbf{j},\mathbf{q}}$ are defined by (34a). Note that the necessity to average over linearizations around a vertex $\mathbf{v_j}$ for each of the 2^n adjacent rectangles arises, since an approximate gradient is to be obtained independently of a particular rectangle $R_\mathbf{k}$, and because the utilized triangular membership functions are not differentiable in $\mathbf{v_j}$.

For each local controller $\mathbf{K_j}\mathbf{x} + \mathbf{k_j}$, the offset part is set according to

$$\mathbf{k_j} = -\tilde{\mathbf{B}}_\mathbf{j}^+ \tilde{\mathbf{a}}_\mathbf{j} \tag{44}$$

with $\tilde{\mathbf{B}}_\mathbf{j}^+ = \left(\tilde{\mathbf{B}}_\mathbf{j}^T \tilde{\mathbf{B}}_\mathbf{j}\right)^{-1} \tilde{\mathbf{B}}_\mathbf{j}$ denoting the Moore–Penrose pseudoinverse of $\tilde{\mathbf{B}}_\mathbf{j}$. For the remaining linear subsystem, $\mathbf{K_j}$ is then determined by means of pole placement. With these initial solutions for $\mathbf{K_j}$ and $\mathbf{k_j}$, (42) is solved for $\mathbf{P_j}$, then by fixing $\mathbf{P_j}$, new local controllers are computed. The procedure is repeated until no further improvements can be made.

As for the stability analysis, relaxing the S-procedure may lead to a reduced number of LMI variables. For non-symmetric regions, appropriate modifications of (42d), (42h) akin to (30) will prevent conservative solutions with regard to the stability region.

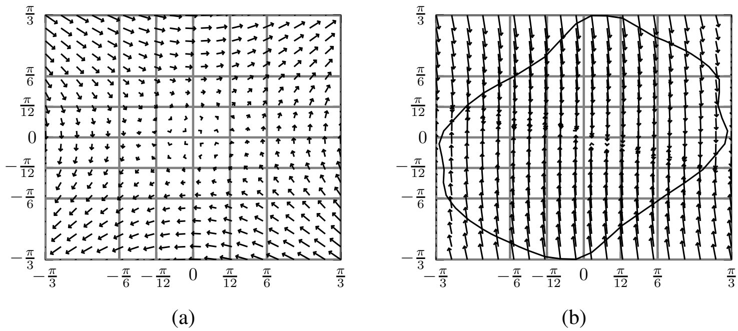


Figure 1: Phase portraits of (a) open-loop inverted pendulum modeled as PBM/RFS and (b) controlled inverted pendulum

6 Numerical Example

The controller synthesis algorithm outlined in Theorem 1 is applied to the well-known dynamics of a pendulum mounted on a cart

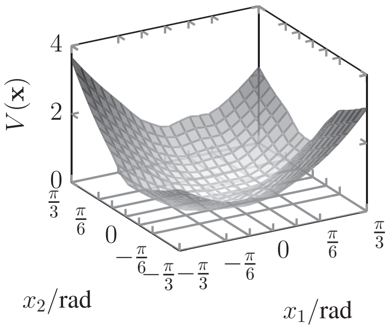
$$\dot{x}_1 = x_2, \tag{45a}$$

$$\dot{x}_2 = -\sin(x_1 + \pi) - \cos(x_1 + \pi)u, \tag{45b}$$

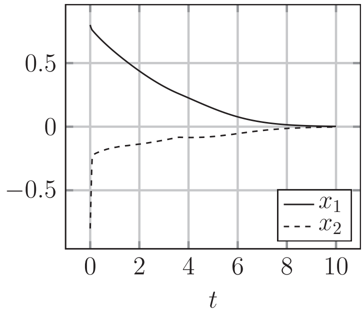
whereas the dynamics of the cart is neglected.

From these nonlinear dynamics, a PBM/RFS is derived by sampling (45) at vertices $(\mathbf{v_j}, \mathbf{v_q})$, whereas the vertices in each dimension are chosen to be $v_{j_1} = v_{j_2} = \{0, \pm\frac{\pi}{12}, \pm\frac{\pi}{6}, \pm\frac{\pi}{3}\}$ and $v_q = \{-10, 0, 10\}$. Since the system has only one input, it may be represented equivalently as PBM and RFS. Its open-loop dynamics are depicted in Fig. 1a, revealing the origin being an anti-stable equilibrium that is to be stabilized.

In order to solve the bilinear matrix inequality optimization problem (42), initial local controllers are first computed for the approximating affine local systems (43). The affine part of these initial controllers is given by (44), whereas for the remaining linear subsystems, the controller matrices $\mathbf{K_j}$ are chosen such that the poles of closed-loop local systems are at $s_i =$



(a)



(b)

Figure 2: (a) Lyapunov function for controlled inverted pendulum and (b) state development of controlled inverted pendulum for initial condition of $\mathbf{x}_0 = [0.8,\ -0.8]^T$.

$\{-4, -5\}$. With this initial solution, (42) is then solved iteratively, resulting in a stable controlled system, whose phase portrait is documented in Fig. 1b. Therein, the region of attraction is shown as well, being fully included in the system domain. The final Lyapunov function is also shown in Fig. 2a.

By applying the resulting controller to the ground truth dynamics (45) with initial conditions $\mathbf{x}_0 = [0.8,\ -0.8]^T$, the development of the states as shown in Fig. 2b was obtained during simulation. As expected, the closed-loop dynamics reveal an asymptotically stable behavior.

7 Conclusion

A control method was proposed, which is applicable for PBM and RFS, leading to a provable stabilization of equilibria of the system. Since the concept of parallel distributed compensation was used, the dynamics of the controlled system will again be smooth, i.e., no chattering will occur, even though the system dynamics are defined piecewise. The stability analysis was based on piecewise quadratic Lyapunov functions without introducing any other assumptions. Thus, with increasing precision of the PBM/RFS, the approximation of the Lyapunov function becomes increasingly precise as well. The piecewise definition on the other hand comes at the cost of the use of the S-Procedure, introducing a slight degree of conservatism, which renders the stability conditions only sufficient. Nevertheless, it was shown that the stability conditions can be extended towards controller synthesis in terms of bilinear matrix inequalities. To solve this non-convex problem in an iterative manner, a way for finding good initial solutions was proposed.

A thorough comparison with existing control concepts for general smooth nonlinear systems remains a topic for future research. Preliminary results indicate, that the framework of PBM/RFS as finite element approach is able to clearly outperform existing methods in nonlinear control, since no assumptions have to be made about the type of the Lyapunov function and dynamic function to be approximated. On the other hand, compared to other universal approximators, it is still possible to prove system properties such as stability.

Literatur

[1] Takagi, T.; Sugeno, M.: Fuzzy identification of systems and its applications to modeling and control. *IEEE Transactions on Systems, Man And Cybernetics* 15 (1985) 1, S. 116–132.

[2] Sugeno, M.: On stability of fuzzy systems expressed by fuzzy rules with singleton consequents. *IEEE Transactions on Fuzzy Systems* 7 (1999) 2, S. 201–224.

[3] Sugeno, M.; Taniguchi, T.: On improvement of stability conditions for continuous Mamdani-like fuzzy systems. *IEEE Transactions on Systems, Man, and Cybernetics. Part B, Cybernetics* 34 (2004) 1, S. 120–131.

[4] Adamy, J.; Kempf, R.: Regularity and chaos in recurrent fuzzy systems. *Fuzzy Sets and Systems* 140 (2003) 2, S. 259–284.

[5] Taniguchi, T.; Sugeno, M.: Robust stabilization of nonlinear systems modeled with piecewise bilinear systems based on feedback linearization. In: *Advances on Computational Intelligence*, S. 111–120. Springer. 2012.

[6] Eciolaza, L.; Sugeno, M.: On-line design of LUT controllers based on desired closed loop plant: Vertex Placement Principle. In: *IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, S. 1–8. IEEE. 2012.

[7] Eciolaza, L.; Taniguchi, T.; Sugeno, M.; Filev, D.; Wang, Y.: Piecewise Bilinear models for feedback error learning: On-line feedforward controller design. In: *IEEE International Conference on Fuzzy Systems*, S. 1–8. 2013.

[8] Gering, S.; Adamy, J.: Fuzzy control of continuous-time recurrent fuzzy systems. *Fuzzy Sets and Systems* (2014).

[9] Gering, S.; Schwung, A.; Gußner, T.; Adamy, J.: Sum of Squares Approaches for Control of Continuous-Time Recurrent Fuzzy Systems. In: *Proc. of the 21st IEEE Mediterranean Conference on Control and Automation*, S. 271–277. Chania, Greece. 2013.

[10] Gering, S.; Adamy, J.: Robust Stabilization of Recurrent Fuzzy Systems via Switching Control. In: *Proceedings of the IEEE Conference on Fuzzy Systems*. Beijing, China. 2014.

[11] Johansson, M.; Rantzer, A.; Arzen, K.-E.: Piecewise quadratic stability of fuzzy systems. *IEEE Transactions on Fuzzy Systems* 7 (1999) 6, S. 713–722.

[12] Gering, S.; Eciolaza, L.; Adamy, J.; Sugeno, M.: A Piecewise Approximation Approach to nonlinear Systems: Stability and Region of Attraction. *Transactions on Fuzzy Systems (Submited)* (2014).

[13] Adamy, J.; Flemming, A.: Equilibria of continuous-time recurrent fuzzy systems. *Fuzzy Sets and Systems* 157 (2006) 22, S. 2913–2933.

[14] Johansson, M.; Rantzer, A.: Computation of piecewise quadratic Lyapunov functions for hybrid systems. *IEEE Transactions on Automatic Control* 43 (1998) 4, S. 555–559.

[15] Boyd, S.; Ghaoui, L. E.; Feron, E.; Balakrishnan, V.: *Linear Matrix Inequalities in System and Control Theory*. Society for Industrial and Applied Mathematics Philadelphia. 1994.

[16] Diepold, K.; Pieczona, S.: Abschätzung des Einzugsbereiches für T-S Systeme: Beachtung des Gültigkeitssektors. In: *Proceedings 23. Workshop Computational Intelligence*. Dortmund. 2013.

[17] Hassibi, A.; How, J.; Boyd, S.: A path-following method for solving BMI problems in control. In: *Proceedings of the American Control Conference*, Bd. 2, S. 1385–1389. San Diego, California, USA. 1999.

[18] Ibaraki, S.; Tomizuka, M.: Rank minimization approach for solving BMI problems with random search. In: *Proceedings of the American Control Conference*, Bd. 3, S. 1870–1875. 2001.

[19] Goh, K.-C.; Safonov, M.; Papavassilopoulos, G.: A global optimization approach for the BMI problem. In: *Proceedings of the 33rd IEEE Conference on Decision and Control*, Bd. 3, S. 2009 –2014. 1994.

[20] Beran, E.; Vandenberghe, L.; Boyd, S.: A Global BMI Algorithm Based on the Generalized Benders Decomposition. 1997.

[21] El Ghaoui, L.; Balakrishnan, V.: Synthesis of fixed-structure controllers via numerical optimization. In: *Proceedings of the 33rd Conference on Decision and Control*, Bd. 3, S. 2678–2683. 1994.

[22] Kim, E.; Kim, S.: Stability analysis and synthesis for an affine fuzzy control system via LMI and ILMI: continuous case. *IEEE Transactions on Fuzzy Systems* 10 (2002) 3, S. 391–400.