Model for triple-point trajectory of shock wave reflection over cylindrical concave wedge

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Nomenclature

H_{m} = Mach stem height

$H_{\mathrm{m}}^{\max}$ = maximum Mach stem height

K_i = intersection point for triple-point trajectories of insert shock wave and Mach stem at the i-th wedge

M = Mach number of incident shock wave

n = quantity of small wedges

O_i = tip of the i-th wedge

R_0 = dimensionless wall curvature radius

T_i = Mach reflection triple-point of insert shock wave at the i-th wedge

T_{M} = Mach reflection triple-point of Mach stem

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γ = specific heat ratio

$\Delta\theta_{\mathrm{w}}$ = wedge angle difference

θ_{t} = wall tangential angle

$\theta_{\mathrm{t}}^{\max}$ = wall tangential angle for maximum Mach stem height

θ_{t}^{*} = wall tangential angle for Mach reflection-regular reflection transition

θ_{w} = planar wedge angle

θ_{w}^{i} = the *i*-th wedge angle

$\theta_{\text{w-2shock}}^{*}$ = critical wedge angle of Mach reflection calculated by the two-shock theory

$\theta_{\text{w-3shock}}^{*}$ = critical wedge angle of Mach reflection calculated by the three-shock theory

χ = triple-point trajectory angle

χ_i = triple-point trajectory angle for insert shock wave at the *i*-th wedge

χ_{M} = triple-point trajectory angle for Mach stem

χ_0 = corresponding trajectory angle when θ_{w} approaches to zero in $\chi-\theta_{\mathrm{w}}$ relation

χ_{2shock} = corresponding trajectory angle of $\theta_{\text{w-2shock}}^{*}$ in $\chi-\theta_{\mathrm{w}}$ relation

I. Introduction

The study of unsteady shock wave reflection over curved surfaces draws much attention due to its practical importance for various applications, such as propulsion system design and aviation safety [1-4]. As the foundation of this study, the reflection problem with the simplest cylindrical concave walls has been investigated for several decades. The relevant studies were also applied for explosion and detonation ignition in recent years, since this phenomenon involves drastic changes in pressure and temperature [5-6]. In previous research, the reflection process over a cylindrical concave wedge was described as a series of continuous variations of reflection types: direct-Mach reflection to stationary- Mach reflection to inverse-Mach reflection and finally to transitioned regular reflection [7]. To predict the trajectory of the Mach reflection triple-point and the transition angle of Mach-to-regular reflection, both the Chester–Chisnell–Whitham theory [8] and the "corner-signal" concept [9] were employed to construct models [10, 11]. Although the results indicated that the theoretical transition angle agrees with the experimental observation, the triple-point trajectory, however, cannot be predicted precisely, especially for strong shock waves with high Mach numbers.

In contrast to the shock wave reflection over a curved surface, the pseudo-steady shock reflection over a planar wedge is self-similar, so the classic two-shock and three-shock theories [12] can be adopted to predict the triple-point trajectory as well as the critical wedge angle for Mach reflection with reasonable accuracy [13]. Furthermore, the shock wave reflection over a double wedge with different conditions can also be analyzed by these theories [14]. Based on these theoretical studies, a model was constructed to predict the triple-point trajectory of strong shock wave reflection over a cylindrical concave wedge in this work. The detailed calculation process as well as a validation of the model was presented, and the characteristics of the triple-point trajectory were then analyzed by applying the model.

II. Model setup

The two-shock and three-shock theories were first proposed by von Neumann to describe the regular reflection and the flow field near the Mach reflection triple-point, respectively [12]. By giving two constraining conditions, the three-shock theory can be used to describe the entire Mach reflection process [13]: (1) the Mach stem is straight and normal to the wedge; (2) the triple-point trajectory originates from the tip of the wedge. Since the pseudo-steady Mach reflection of a shock wave can be regarded as a steady Mach reflection by transforming the coordinates, as Fig. 1 shows, the flow field parameters as well as the trajectory angle χ can be obtained by applying the three-shock theory with given shock wave Mach number M and wedge angle θ_w. Figure 2 plots the calculated $\chi-\theta_w$ relation for $M = 5.4$ and $\gamma = 1.44$, the detailed calculation process is introduced in [7]. It is worth noting that the part behind the critical wedge angle $\theta^*_{w\text{-}3shock}$ of Mach reflection is meaningless for pseudo-steady Mach reflection because the regular reflection is formed with the wedge angle, but the negative trajectory angle χ exists in the unsteady reflection process over a cylindrical concave wedge, which corresponds to the inverse-Mach reflection. To verify the accuracy of the theoretical $\chi-\theta_w$ relation, the shock wave reflection process with $\theta_w=15^\circ$ was simulated under the condition of $M = 5.4$ and $\gamma = 1.44$, and the density schlieren overlays with a red line showing the triple-point trajectory are displayed in Fig. 3. The theoretical trajectory angle χ for $\theta_w=15^\circ$ is 14.12°, approaching to the numerical result of 15.21°.

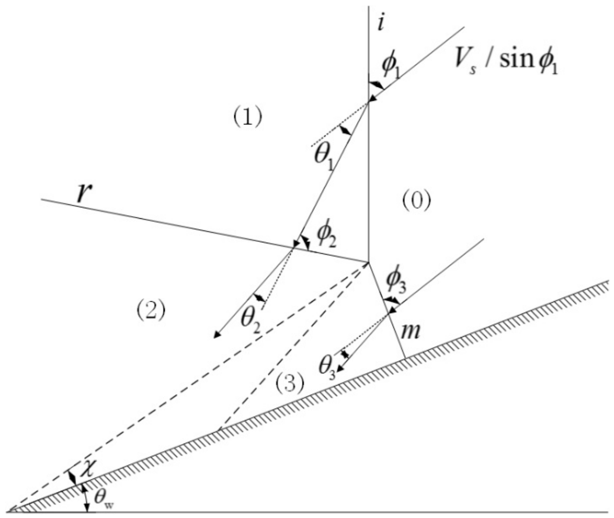


Figure 1. Schematic of pseudo-steady Mach reflection structure over a planar wedge.

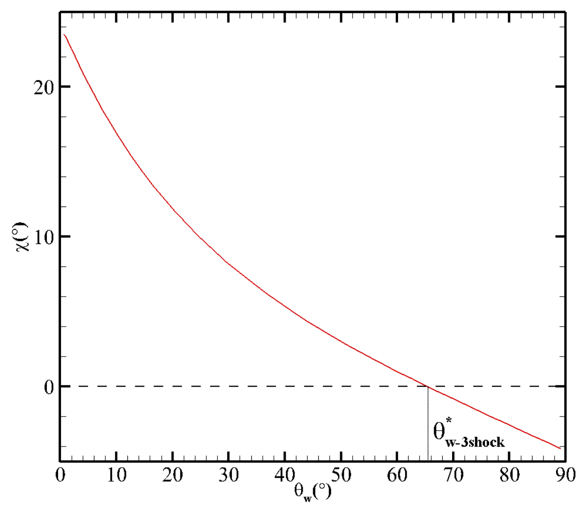


Figure 2. $\chi-\theta_w$ relation for $M = 5.4$ and $\gamma = 1.44$.

($\theta^*_{w\text{-}3shock}$: Critical wedge angle of Mach reflection by the three-shock theory)

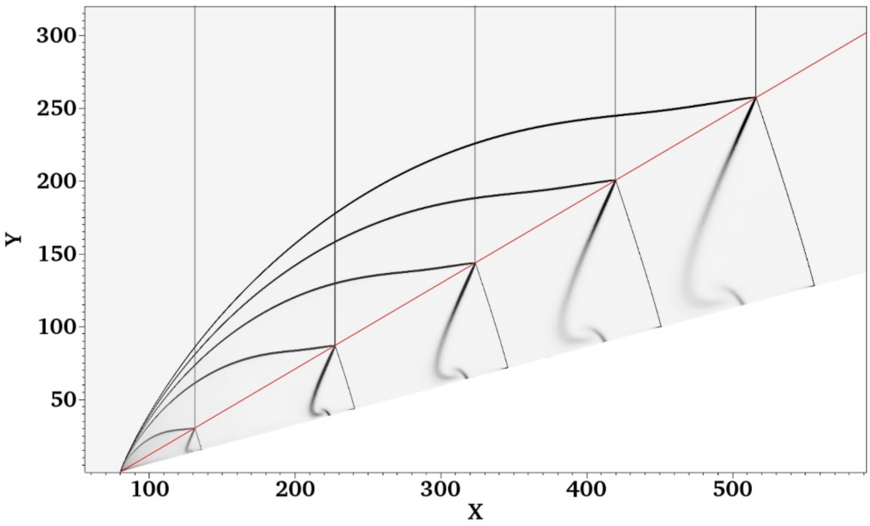
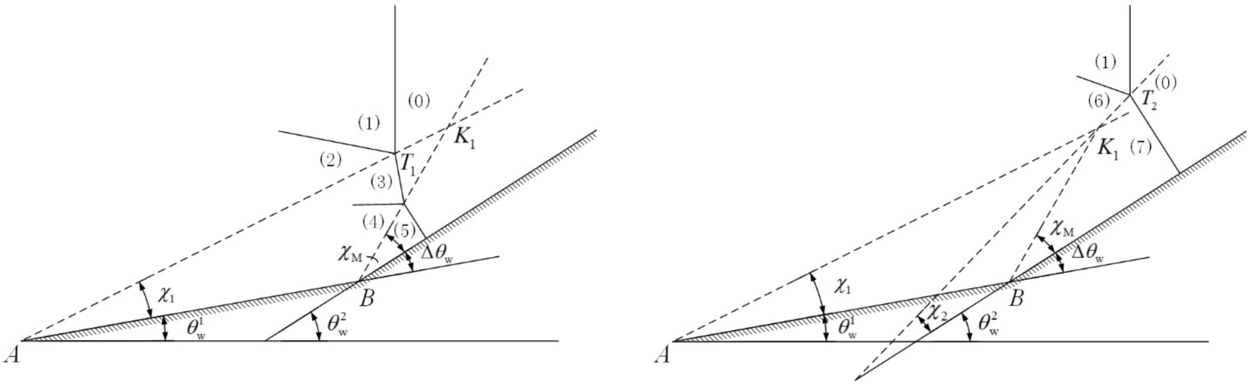


Figure 3. Density schlieren overlays of shock wave reflection over a planar wedge with θ_{w}=15°.

Based on the pseudo-steady reflection and $\chi-\theta_{\mathrm{w}}$ relation, the reflection structure and the triple-point trajectory over a double planar wedge can also be predicted [14]. Figure 4 displays the reflection process over a double wedge under the condition of $\theta_{\mathrm{w}}^{2}>\theta_{\mathrm{w}}^{1}$. As Fig. 3(a) shows, when the incident shock wave propagates to wedge *A* with wedge angle θ_{w}^{1}, the Mach reflection is formed and triple-point T_1 moves along the straight trajectory with angle χ_1. As the Mach stem reaches wedge *B* along with the incident shock, another Mach reflection is also formed on the Mach stem, generating the triple-point T_{M}. Since the Mach stem is perpendicular to wedge *A*, the newly formed triple-point trajectory angle χ_{M} actually corresponds to the wedge angle difference $\Delta\theta_{\mathrm{w}}$. Then the trajectories of triple-points T_1 and T_{M} finally intersect at point K_1, as shown in Fig. 4(b). A single Mach stem perpendicular to wedge B is formed, and the triple-point will then propagate along the trajectory with the angle of χ_2 generated by the wedge with the angle of θ_{w}^{2}. According to the reflection process, the triple-point trajectory over the double wedge can be predicted by obtaining the trajectory angles χ_1, χ_{M} and χ_2 and determining the location of the intersection point K_1.



(a) Mach reflection on incident shock wave and Mach stem simutaneously

(b) Mach reflections intersect

Figure 4. Schematic of reflection process over a double planar wedge with $\theta_\mathrm{w}^2 > \theta_\mathrm{w}^1$.

For a shock wave reflection over a cylindrical concave wedge, the wedge can be approximated by a larger number of small planar wedges with equivalent length, as displayed in Fig. 5, so the original reflection process is transformed to the reflection over a multiple segments wedge. From the reflection over a double wedge, it is inferred that the triple-point trajectory over a multiple segments wedge can be obtained by determining the location of *K* for each adjacent double wedge. For instance, by defining that the tip of the *i*-th small wedge has the coordinates of $O_i\left(x_i, y_i\right)$, the coordinates of the first intersection $K_1\left(x_{k1}, y_{k1}\right)$ can be obtained by the simultaneous equations of the incident shock and Mach stem triple-point trajectories:

$$\begin{cases} y - y_1 = \chi_1\left(x - x_1\right) \\ y - y_2 = \chi_\mathrm{M}\left(x - x_2\right) \end{cases} \tag{1}$$

Then the *i*-th intersection $K_i\left(x_{ki}, y_{ki}\right)$ can be confirmed by:

$$\begin{cases} y - y_{k(i-1)} = \chi_i\left(x - x_{k(i-1)}\right) \\ y - y_{i+1} = \chi_\mathrm{M}\left(x - x_{i+1}\right) \end{cases} \tag{2}$$

where χ_i represents to the triple-point trajectory angle for the insert shock wave at the *i*-th wedge, which corresponds to the *i*-th wedge angle θ_w^i in the $\chi - \theta_\mathrm{w}$ relation. Hence, if the quantity of the constituent small wedges is sufficient, the triple-point trajectory over a cylindrical concave wedge can be calculated approximately by confirming all the coordinates of K_i. It is worth noting that, since χ_M is the trajectory angle of the Mach stem, theoretically the calculation of χ_{M} should use the Mach number of the Mach stem instead of the incident shock. However, considering that the deviation of the trajectory angle formed by the Mach stem and the incident shock is minimal under such a high Mach number, the χ_{M} can be approximated by using M in the calculation. Furthermore, since the concave wedge is divided equally by the small incremental wedges, the $\Delta\theta_{\mathrm{w}}$ between each two adjacent wedges is equal, resulting a constant χ_{M} for all the composed double wedges, thus simplifying the calculation.

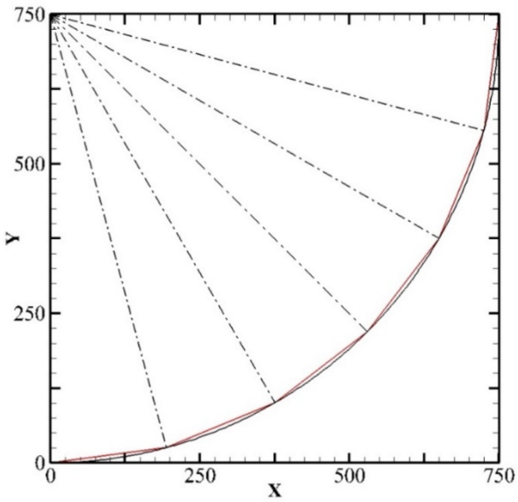


Figure 5. Schematic of cylindrical concave wedge approximated by multiple wedges with equivalent length.

III. Results and discussions

A. Theoretical triple-point trajectory and validation

A model to predict the trajectory of the reflection triple-point can be built by MATLAB based on the analysis above. To verify the accuracy of the model, a simulation of shock wave reflection over a cylindrical wedge was conducted with $M = 5.4$, $\gamma = 1.44$ and wall curvature radius $R_0 =$ 750. The density schlieren of the reflection process is shown in Fig. 6. The trajectory was recorded by tracking the coordinates of the triple-point with constant time intervals. Simultaneously, the theoretical trajectory was calculated by the model under the same condition with the quantity of small wedges $n = 2000$. The comparison of the trajectories as well as the converted Mach stem height H_{m} are presented in Fig. 7. It can be observed that the trajectories converge in the range of the wall tangential angle $\theta_t < 52^\circ$, whereas the H_m calculated by the model deviates from the simulation result with larger value for increasing θ_t. Additionally, the calculated trajectory does not realize Mach-to-Regular (MR-RR) transition, which is unreasonable. The reason is supposed that the $\chi - \theta_w$ relation inferred by the three-shock theory is imprecise for large θ_w. Indeed, the previous research has reported that the calculated χ becomes inaccurate compared with the experiment results as θ_w increases [15], and the study of Hornung claimed that the critical wedge angle of the Mach reflection, i.e., the corresponding wedge angle of $\chi = 0$, approaches to the result calculated by two-shock theory instead of three-shock theory, where the critical wedge angle $\theta^*_{\text{w-2shock}}$ and $\theta^*_{\text{w-3shock}}$ for these two theories can be obtained by plotting the shock-polar curves of the incident and reflected shock [16]. Simulations were also conducted to verify the report of Hornung. The shock wave reflections with $\theta_w = 50^\circ$ and 52° were simulated, and the results are shown in Fig. 8. It is found that a Mach reflection is still formed in the case of $\theta_w = 50^\circ$, while as θ_w increases to 52°, a regular reflection is formed, indicating that the critical wedge angle is between 50° to 52°. Since the $\theta^*_{\text{w-2shock}}$ and $\theta^*_{\text{w-3shock}}$ for $M = 5.4$ and $\gamma = 1.44$ are 51.14° and 65.51°, respectively, it is confirmed that the critical wedge angle is close to the theoretical $\theta^*_{\text{w-2shock}}$ instead of $\theta^*_{\text{w-3shock}}$, which agrees with Hornung's study.

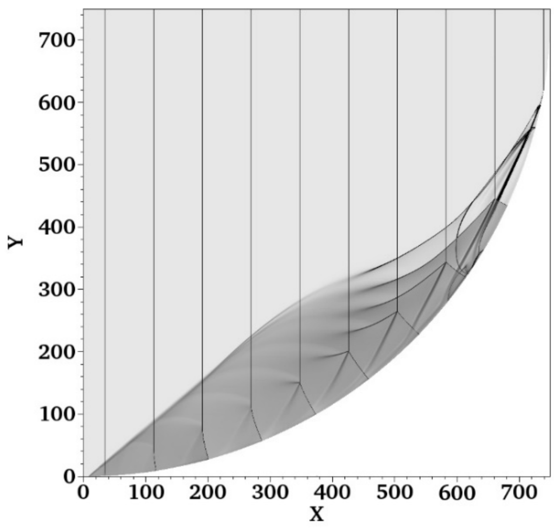
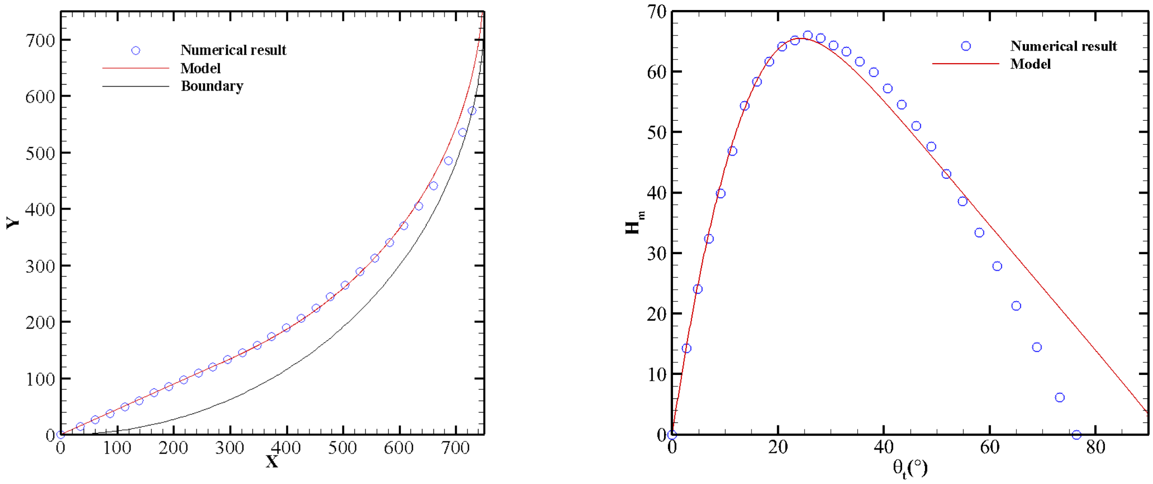
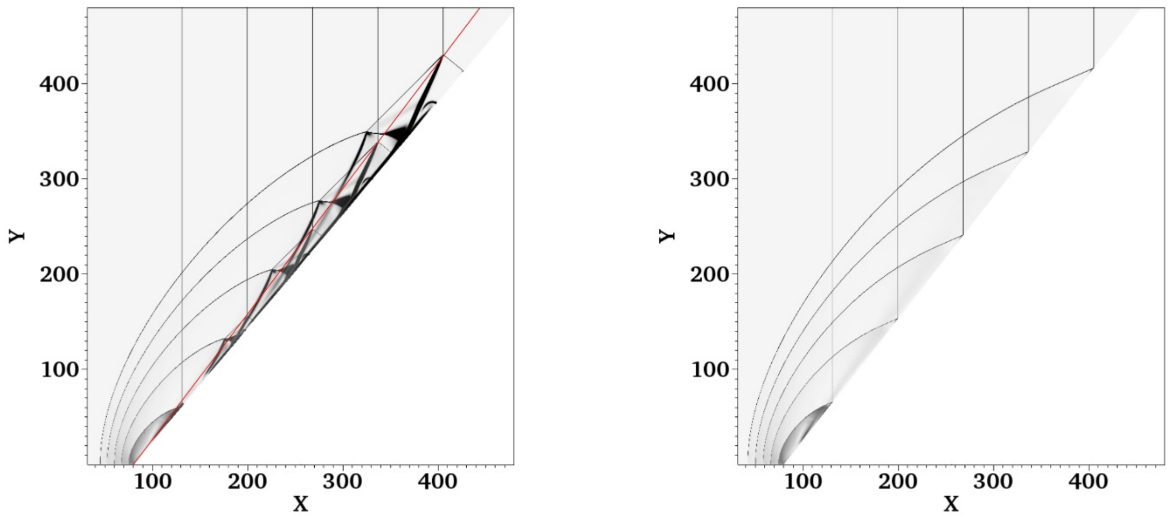


Figure 6. Density schlieren overlays of shock wave reflection over a cylindrical concave wedge.



(a) Triple-point trajectory (b) $H_m - \theta_t$ relation

Figure 7. Comparison of triple-point trajectory and H_m for simulation and model.



(a) $\theta_w = 50^\circ$ (b) $\theta_w = 52^\circ$

Figure 8. Shock wave reflection over planar wedges with θ_w close to the critical wedge angle.

According to the studies above, the $\chi - \theta_w$ relation needs to be modified to insure the χ is accurate for small wedge angle and equal to zero when $\theta_w = \theta^*_{w\text{-2shock}}$. A simple modification of adding a correction term of $-\frac{\chi_{2\text{shock}}}{\theta^*_{w\text{-2shock}}}\theta_w$ can meet the requirement, where $\chi_{2\text{shock}}$ represents the corresponding trajectory angle of $\theta^*_{w\text{-2shock}}$ in the original $\chi - \theta_w$ relation. For instance, the correction term for $M = 5.4$ and $\gamma = 1.44$ is calculated to be $-\frac{2.8}{51.2}\theta_w$. To verify the modification, the simulations of shock wave reflection over planar wedges was conducted with various θ_w under the conditions of $M = 5.4$ and $\gamma = 1.44$. The modified $\chi - \theta_w$

relation as well as the trajectory angle χ for each case are shown in Fig. 9. The numerical results are coincident with the relation curve, which indicates that the modification is valid. The trajectory and H_{m} calculated by the modified $\chi-\theta_{\mathrm{w}}$ relation are plotted in Fig. 10. It can be observed that the modified theoretical trajectory agrees well with the numerical result in the entire range. The calculated MR-RR transition angle is $\theta_{\mathrm{t}}^{*}=79.93^{\circ}$, which has a percentage error of 4.55% compared with the numerical result of $\theta_{\mathrm{t}}^{*}=76.45^{\circ}$, and the maximum percentage error of H_{m} is 6.35%, indicating the modification improves the accuracy of the model significantly, which can be applied to predict the triple-point trajectory of the shock wave reflection over the cylindrical concave wedge. It is worth noting that similar with the previous models [10, 11], the present model cannot predict the effect of the wall curvature radius R_0. In fact, the effect of R_0 on the reflection process is controversial, and the mechanism is still uncertain so far.

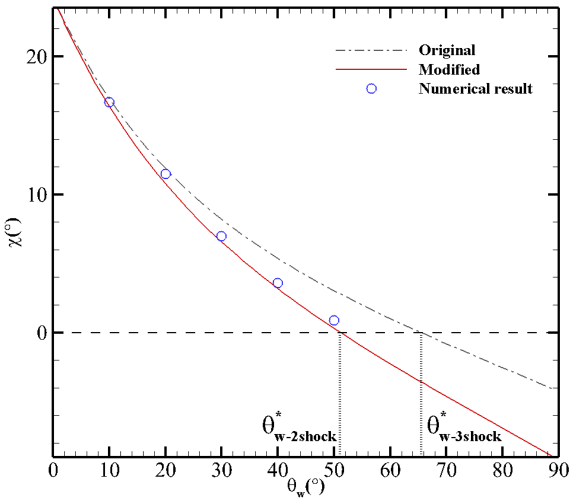
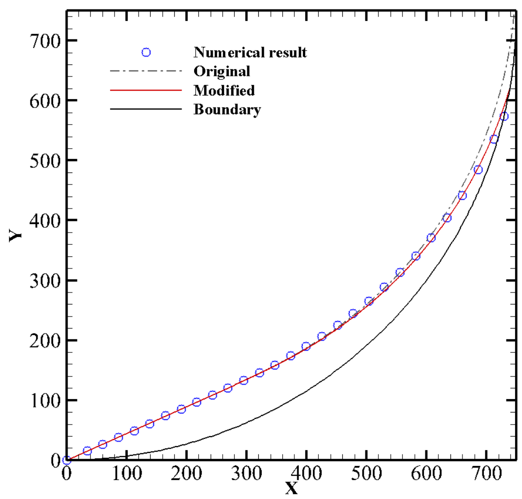
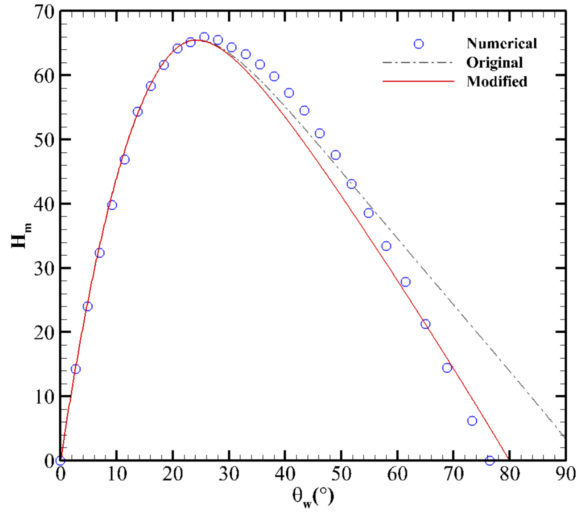


Figure 9. Comparison of the modified $\chi-\theta_{\mathrm{w}}$ relation and the numerical results with M=5.4 and γ= 1.44.



(a) Triple-point trajectory



(b) $H_m - \theta_t$ relation

Figure 10. Comparison of triple-point trajectory and H_m for simulation and modified model.

B. Characteristics of triple-point trajectory

Using the present model to predict accurately the shock reflection over a cylindrical concave wedge, the characteristics of the triple-point trajectory can be analyzed. Figure 11 shows the $H_m - \theta_t$ relations calculated by the model for different M. It can be observed that the H_m for the same θ_t and the transition angle θ_t^* decrease along with the increase of M, but the deviation is negligible, which coincides with the previous experimental study [11]. Thus, it appears that for the shock wave reflection with high Mach number, the triple-point trajectories for different M can actually be approximated by one trajectory with certain accuracy, which can simplify the analysis with varying M.

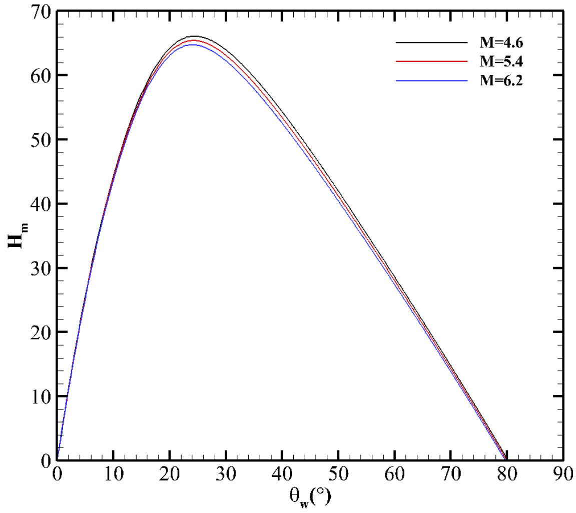


Figure 11. Theoretical $H_{\mathrm{m}}-\theta_{\mathrm{t}}$ relations for different M.

The model also indicates that the triple-point trajectory is not entirely a curve. In fact, the triple-point propagates along a straight line at the beginning of the reflection process. Figure 12 presents the trajectory in the range of $\theta_{\mathrm{t}}<30^{\circ}$ with $M=5.4$ and $R_0=750$. The data points calculated by the model represent the intersection points K of the triple-point trajectories generated by the Mach reflection of the incident shock wave and the Mach stem over each double wedge. It is found that the distance between the wedge tip O_1 (0,0) and the first data point K_1 is much longer than that between each adjacent two data points. Hence, according to the reflection over a double wedge, the triple-point trajectory before K_1 is a straight line with the slope of $\tan\chi_0$, where χ_0 denotes the trajectory angle when θ_{w} approaches to zero in the $\chi-\theta_{\mathrm{w}}$ relation. Moreover, it is also found that the corresponding wall tangential angle of K_1 is always larger than χ_0. Considering that the $H_{\mathrm{m}}-\theta_{\mathrm{t}}$ relation curve only has one maximum point, it can be confirmed that the location of the point with the maximum height of Mach stem $H_{\mathrm{m}}^{\max}$ is on the straight part of the trajectory, corresponding to the wall tangential angle of $\theta_{\mathrm{t}}^{\max}=\chi_0$. Thus the $H_{\mathrm{m}}^{\max}$ can also be obtained to be $H_{\mathrm{m}}^{\max}=R_0(1-\cos\chi_0)$. As a verification, the $\theta_{\mathrm{t}}^{\max}$ and $H_{\mathrm{m}}^{\max}$ were calculated for $M=5.4$ and $R_0=750$ with the results of 24.1° and 63.37. The theoretical results have percentage errors of 1.86% and 1.15% compared with the numerical results of $\theta_{\mathrm{t}}^{\max}=23.66$ and $H_{\mathrm{m}}^{\max}=64.63$, respectively, which presents high accuracy. Hence, the $\theta_{\mathrm{t}}^{\max}$ and $H_{\mathrm{m}}^{\max}$

can be obtained just by the $\chi - \theta_w$ relation instead of the model.

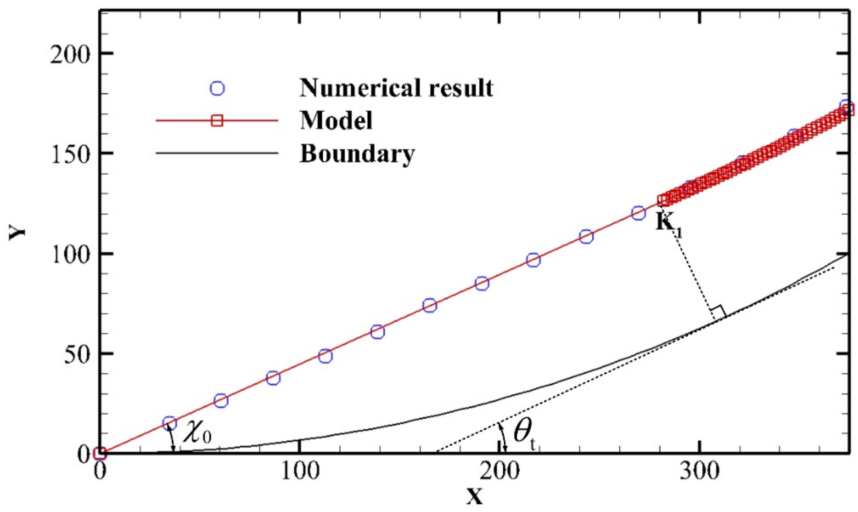


Figure 12. Triple-point trajectory in the range of $\theta_t < 30^\circ$.

IV. Conclusions

In the present study, a theoretical model to predict the triple-point trajectory of strong shock wave Mach reflection over a cylindrical concave wedge was constructed based on the reflection over a double wedge and the three-shock theory. Together with a modification by applying the two-shock theory, the triple-point trajectory calculated by the model for the entire shock reflection is validated with the simulation result. The model denotes that the triple-point trajectory as well as the transition angle of Mach reflection to regular reflection are almost invariable with high Mach number, and the trajectory is straight at the beginning of the reflection process with corresponding trajectory angle when the wedge angle approaches to zero. The point with the maximum height of Mach stem is confirmed on the straight part of the trajectory, and the location as well as the maximum Mach stem height is predicted accurately.

Acknowledgements

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