N-soft sets: semantics and aggregation

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Goals

- ▷ Understand the semantics of N-soft sets (Fatimah, Rosadi, Hakim, A., 2018).

  This is necessary for example, for a correct evaluation of the alternatives, or for interpreting aggregation.

- ▷ Two different approaches to the aggregation of N-soft sets.

  Aggregation is interesting for example, for multi-agent decision-making.

Why semantical analyses?

From Dubois and Prade “The three semantics of fuzzy sets”, Fuzzy Sets and Systems (1997).

> *there is no uniformity in the interpretation of what a membership grade means. (...) Most negative statements expressed in the literature turn around the question of interpreting and eliciting membership grades. Our claim in this position paper is that, far from being a weakness, the existence of several understandings of what a membership grade may mean proves the potential richness of the concept of fuzzy set (...)*
> *Three main semantics for membership functions seem to exist in the literature: similarity, preference and uncertainty.*

Recommended bibliography

▷ J.C.R.A.: "The semantics of N-soft sets, their applications, and a coda about three-way decision", Information Sciences 606 (2022), 837-852. *Open Access.*

Largely based on J. Yang, Y. Yao: "Semantics of soft sets and three-way decision with soft sets", Knowledge-Based Systems 194 (2020), 105538.

▷ J.C.R.A., G. Santos-García, M. Akram: "OWA aggregation operators and multi-agent decisions with N-soft sets", Expert Systems with Applications 203 (2022), 117430. *Open Access.*

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N-soft sets

Soft sets and N-soft sets

Conceptual definition (finite setting). Consider $T = \{t_1, \ldots, t_q\}$, a set of 'attributes'.

An N-soft set on a set $O = \{o_1, \ldots, o_p\}$ is defined by Table 1.

Table 1: Representation of an N-soft set (Fatimah, Rosadi, Hakim, A., 2018).

(F, T, N)	t_1	$\ldots\ldots$	t_q
o_1	r_{11}	$\ldots\ldots$	r_{1q}
$\vdots$	$\vdots$		$\vdots$
o_p	r_{p1}	$\ldots\ldots$	r_{pq}

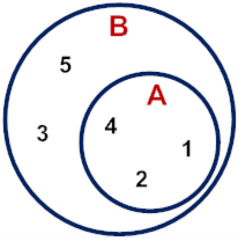
Each r_{ij} (a 'grade') is in $G = \{0, 1, \ldots, N-1\} \longleftarrow$ a convenient default.

When $N = 2$ we have a soft set (Molodtsov, 1999).

Crisp sets vs. soft sets: an example

Let $B = \{1, 2, 3, 4, 5\}$.

(a) $A = \{1, 2, 4\}$ is a crisp subset of B.



Identified by its characteristic function $\chi_A \colon B \longrightarrow \{0, 1\}$ with $\chi_A(1) = \chi_A(2) = \chi_A(4) = 1$, $\chi_A(3) = \chi_A(5) = 0$.

Or a vector with 5 components and binary values: $(1, 1, 0, 1, 0)$.

(b) A soft set over B is identified by several vectors with 5 components (one column vector for each relevant attribute) and binary values.

Crisp sets vs. N-soft sets: an example

(c) An N-soft set over B is identified by several vectors with 5 components (one column vector for each relevant attribute) and values from $G = \{0, 1, \ldots, N-1\}$.

Example with 3 characterizing attributes:

$(F, T, 4)$	t_1	t_2	t_3
1	1	1	2
2	3	2	0
3	0	1	2
4	2	3	2
5	1	0	3

What can we capture with this table? semantics of attributes and values.

Important: Real examples are given in various references.

Semantics of N-soft sets

Structure of the discussion

Two levels of discussion $\begin{cases} \text{The semantics of the attributes.} \\ \text{The semantics of the grades.} \end{cases}$

Two interpretations for each level.

First semantical interpretation of attributes: multi-context

The original interpretation of soft sets (replicated for *N*-soft sets).

An *N*-soft set offers a taxonomy: it classifies, describes or categorizes the alternatives based on their characteristic features.

N-soft sets are distinguished by their ability to rate the level of satisfaction of the attributes.

Second semantical interpretation of attributes: possible worlds

Due to Yang and Yao (2020) for soft sets, it can be replicated for *N*-soft sets too.

The set of attributes is formed by possible worlds for the interpretation of a partially-known concept.

Also here, *N*-soft sets allow us to rate the level of achievement under each possible world.

Example. In a gala dinner, the suitability of the dishes on a menu depends on the list of guests. If we do not know exactly who will show up, the situation is described by a soft set or an *N*-soft set.

In an MSc program, the adequacy of the elective courses depends on the list of students. Which type of students will enrol?

First semantical interpretation of grades: levels or ratings

This is the original interpretation of N-soft sets.

Grades are labels representing a “level of fulfilment”, like hotel stars, referee reports, language skills, or student’s marks.

Heterogeneity is allowed.

PAPER EVALUATION: COMMENTS RETURNED TO AUTHOR(S)

TECHNICAL MERIT:			
Importance	Valuable	○ ○ ◉ ○ ○	Useless
Content	Original	○ ○ ◉ ○ ○	Derivative
Depth	Deep	○ ○ ◉ ○ ○	Shallow
PRESENTATION:			
Style	Readable	○ ○ ◉ ○ ○	Incoherent
Organization	Precise	○ ◉ ○ ○ ○	Ambiguous
Presentation	Orderly	○ ◉ ○ ○ ○	Confusing
References	Complete	○ ◉ ○ ○ ○	Incomplete
OVERALL:			
Overall Evaluation	Excellent	○ ○ ◉ ○ ○	Dreadful

Second semantical interpretation of grades: many-valued logic

This interpretation owes to A. (2022).

Exclusive for N-soft sets ($N \geqslant 3$), meaningless for soft sets.

Basic assumption of the soft set model: every object can be unequivocally associated to each characteristic that it possesses.

Already in 3-valued systems of propositional logic, propositions must not be either true or false.

The rejection of the law of excluded middle means: objects exist that neither satisfy nor do not satisfy a property.

They are within the purview of N-soft set theory with $N \geqslant 3$. ‘Grades’ become values of truth – both under multi-context and possible worlds semantics.

The semantical interpretation in logical terms: an example

Four values of truth: 0 for "totally false", 3 for "totally true".

And 1 and 2 represent "more false than true" and "more true than false", respectively.

$(F, T, 4)$	t_1	t_2	t_3
o_1	1	1	2
o_2	3	2	0
o_3	0	1	2
o_4	2	3	2
o_5	1	0	3

For example: 1 means that the statement "o_1 satisfies property t_1" is more false than true (first semantic interpretation of attributes).

Alternatively, that the claim "o_1 is suitable under possible world t_1" is more false than true (second semantic interpretation of attributes).

Three-valued logic: some examples

Two-valued logic can be extended to three-valued logics in various reasonable ways. Pioneered by Łukasiewicz (1920).

The next table shows the primitives of some three-valued logics: $\frac{1}{2}$ denotes indeterminacy or possibility (1 holds for truth, 0 for falseness).

		Łukasiewicz				Bochvar				Kleene				Heyting				Reichenbach			
a	b	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$
0	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$
0	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0
$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$
1	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Three-valued logics in practice

Structured Query Language (SQL) has become the standard language for retrieving, updating, and removing information from relational databases.

SQL implements three logical results, and there is a state or marker identified by the reserved word NULL, indicating that a data value is not found in the database.

The truth tables that SQL applies for the combination of logical states (AND or $\wedge$, OR or $\vee$, and NOT or $\neg$) correspond to the Kleene and Łukasiewicz three-valued logics.

Four-valued logics in practice

IEEE established a four-valued logic with the standard IEEE 1364 (Verilog) in order to model signal values in digital circuits.

Truth and falseness are retrieved from various sources (like databases or multi-person inputs).

Incomplete information happens when no answer is found.

Simultaneous false and true answers produce contradictory information.

Truth values in Belnap's four-valued logic (1977): $\{T, F, N, B\}$ (true, false, none, both).

$\wedge$	T	B	N	F
T	T	B	N	F
B	B	B	F	F
N	N	F	N	F
F	F	F	F	F

$\vee$	T	B	N	F
T	T	T	T	T
B	T	B	T	B
N	T	T	N	N
F	T	B	N	F

Aggregation of N-soft sets

The problem

A list of N-soft sets $\{(F_1, T, N), \ldots, (F_k, T, N)\}$ on $O = \{o_1, \ldots, o_p\}$ with a common set of attributes $T = \{t_1, \ldots, t_q\}$.

(F_1, T, N)	t_1	$\ldots\ldots$	t_q
o_1	r^1_{11}	$\ldots\ldots$	r^1_{1q}
$\vdots$	$\vdots$	$\ddots$	$\vdots$
o_p	r^1_{p1}	$\ldots\ldots$	r^1_{pq}

$\ldots\ldots$

(F_k, T, N)	t_1	......	t_q
o_1	r^k_{11}	$\ldots\ldots$	r^k_{1q}
$\vdots$	$\vdots$	$\ddots$	$\vdots$
o_p	r^k_{p1}	$\ldots\ldots$	r^k_{pq}

Question. What is a sensible aggregate N-soft set of this information?

1st semantical interpretation of grades: levels I

Procedure: cell-by-cell application of an ordinal version of the OWA operator (Lizasoain and Moreno, 2013) on the grades.

The general expression needs the utilization of a t-norm and a t-conorm plus the definition of 'distributive weighting vector'.

A particular expression (standard t-norm and t-conorm) is:

for any distributive weighting vector $(\alpha_1, \ldots, \alpha_k) \in G^k$, cell-by-cell aggregation with

$$F_\alpha(r_{ij}^1, \ldots, r_{ij}^k) = \max\left(\min(r_{ij}^{\sigma(1)}, \alpha_1), \ldots, \min(r_{ij}^{\sigma(k)}, \alpha_k)\right) \text{ for every } i, j.$$

The permutation σ of $\{1, \ldots, k\}$ guarantees $r_{ij}^{\sigma(1)} \geqslant \ldots \geqslant r_{ij}^{\sigma(k)}$.

Examples of this operator. Max, min, median.

1st semantical interpretation of grades: levels II

Example. Tabular representation of three 4-soft sets.

Distributive weighting vector $(2, 3, 0)$.

$(F_1, T, 4)$	t_1	t_2	t_3
o_1	1	1	2
o_2	3	2	0
o_3	0	1	2
o_4	2	3	2
o_5	1	0	3

$(F_2, T, 4)$	t_1	t_2	t_3
o_1	1	0	3
o_2	2	3	0
o_3	0	0	3
o_4	2	1	2
o_5	2	0	2

$(F_3, T, 4)$	t_1	t_2	t_3
o_1	1	1	3
o_2	3	3	0
o_3	0	1	3
o_4	2	2	2
o_5	2	0	3

To aggregate emphasized values: We order values $(3, 1, 2)$ as $(3, 2, 1)$.

$\max(\min(3, 2), \min(2, 3), \min(1, 0)) = 2$.

2nd semantical interpretation of grades: many-valued logic I

Procedure: aggregation of values of truth with conjunctive / disjunctive connective in Łukasiewicz N-valued logic.

Truth values $\{0, 1, \ldots, N-1\}$.

Negation is computed by subtraction from $N-1$:

$\neg 0 = N-1$, $\neg 1 = N-2$, ..., $\neg(N-2) = 1$, $\neg(N-1) = 0$.

The truth value of $a \to b$ is $a \to b = \min(N-1, N-1+b-a)$.

The other logical connectives are derived from these by rules inclusive of the following instances:

$a \vee b = (a \to b) \to b = \max(a, b)$

$a \wedge b = \neg(\neg a \vee \neg b) = \min(a, b)$

$a \leftrightarrow b = (a \to b) \wedge (b \to a) = N-1-|a-b|$

Particular examples may call for the utilization of alternative logics.

2nd semantical interpretation of grades: many-valued logic II

Example. A special session of a conference receives two sets of reports on five articles.

$O = \{o_1, \ldots, o_5\}$ is the universe of articles.

$T = \{t_1, t_2, t_3\}$ is the set of attributes that a perfect candidate paper should meet: “enough scientific quality”, “suitable for the special session”, and “adequate quality of presentation”.

The reports use 4 values of truth to declare whether it is ‘true’ that an article satisfies each of the desirable properties.

2nd semantical interpretation of grades: many-valued logic III

Tabular representation of two 4-soft sets and their aggregate output (conjunction operator - conservative position).

$(F_1, T, 4)$	t_1	t_2	t_3
o_1	1	1	2
o_2	3	2	0
o_3	0	1	2
o_4	2	3	2
o_5	1	0	3

$(F_2, T, 4)$	t_1	t_2	t_3
o_1	1	0	3
o_2	2	3	0
o_3	0	0	3
o_4	2	1	2
o_5	2	0	2

$(F, T, 4)$	t_1	t_2	t_3
o_1	1	0	2
o_2	2	2	0
o_3	0	0	2
o_4	2	1	2
o_5	1	0	2

2nd semantical interpretation of grades: many-valued logic IV

Tabular representation of two 4-soft sets and their aggregate output (disjunction operator - optimistic position).

$(F_1, T, 4)$	t_1	t_2	t_3
o_1	1	1	2
o_2	3	2	0
o_3	0	1	2
o_4	2	3	2
o_5	1	0	3

$(F_2, T, 4)$	t_1	t_2	t_3
o_1	1	0	3
o_2	2	3	0
o_3	0	0	3
o_4	2	1	2
o_5	2	0	2

$(F', T, 4)$	t_1	t_2	t_3
o_1	1	1	3
o_2	3	3	0
o_3	0	1	3
o_4	2	3	2
o_5	2	0	3

Conclusions

Conclusions

▷ The semantical analysis of N-soft sets is quite rich (both in terms of the 'attributes' and 'grades') and interacts with the field of logics.

▷ The aggregation of N-soft sets allows for various interesting approaches.

Also with the help of other models like hesitant N-soft sets or fuzzy N-soft sets.

▷ Many other issues have been explored in the aforementioned papers, like the implications for decision-making, the construction of WAOWA scores, or the embedding of incomplete soft sets into 3-soft sets (under three-valued semantics of the grades).

▷ Other related topics like N-soft topology (Riaz, Çağman, Zareef, Aslam, 2019) might benefit from these insights in the future.

Thank you!

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