Dissertation

PDE-Constrained Equilibrium Problems under Uncertainty

Existence, Optimality Conditions and Regularization

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PDE-Constrained Equilibrium Problems under Uncertainty: Existence, Optimality Conditions and Regularization

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Zusammenfassung

In dieser Arbeit werden PDE-beschränkte Gleichgewichtsprobleme unter Unsicherheiten analysiert. Im Detail diskutieren wir eine Klasse von risikoneutralen verallgemeinerten Nash-Gleichgewichtsproblemen sowie eine Klasse von risikoaversen Nash Gleichgewichtsproblemen.

Sowohl für die risikoneutralen PDE-beschränkten Optimierungsprobleme mit punktweisen Zustandsschranken als auch für die risikoneutralen verallgemeinerten Nash Gleichgewichtsprobleme wird die Existenz von Lösungen beziehungsweise Nash Gleichgewichten bewiesen und Optimalitätsbedingungen hergeleitet. Die Betrachtung von Ungleichheitsbedingungen an den stochastischen Zustand führt in beiden Fällen zu Komplikationen bei der Herleitung der Lagrange-Multiplikatoren. Nur durch höhere Regularität des stochastischen Zustandes können wir auf die bestehende Optimalitätstheorie für konvexe Optimierungsprobleme zurückgreifen. Die niedrige Regularität des Lagrange-Multiplikators stellt auch für die numerische Lösbarkeit dieser Probleme ein große Herausforderung dar. Wir legen den Grundstein für eine erfolgreiche numerische Behandlung risikoneutraler Nash Gleichgewichtsproblem mittels Moreau-Yosida Regularisierung, indem wir zeigen, dass dieser Regularisierungsansatz konsistent ist. Die Moreau-Yosida Regularisierung liefert eine Folge von parameterabhängigen Nash Gleichgewichtsproblemen und der Grenzübergang im Glättungsparameter zeigt, dass die stationären Punkte des regularisierten Problems gegen ein verallgemeinertes Nash Gleichgewicht des ursprünglich Problems schwach konvergieren. Die Theorie legt also nahe, dass auf der Moreau-Yosida Regularisierung eine numerische Methode aufgebaut werden kann. Darauf aufbauend werden Algorithmen vorgeschlagen, die aufzeigen, wie risikoneutrale PDE-beschränkte Optimierungsprobleme mit punktweisen Zustandsschranken und risikoneutrale PDE-beschränkte verallgemeinerte Nash Gleichgewichtsprobleme gelöst werden können.

Für die Modellierung der Risikopräferenz in der Klasse von risikoaversen Nash Gleichgewichtsprobleme verwenden wir kohärente Risikomaße. Da kohärente Risikomaße im Allgemeinen nicht glatt sind, ist das resultierende PDE-beschränkte Nash Gleichgewichtsproblem ebenfalls nicht glatt. Daher glätten wir die kohärenten Risikomaße mit Hilfe einer Epi-Regularisierungstechnik. Sowohl für das ursprüngliche Nash Gleichgewichtsproblem als auch für die geglätteten parameterabhängigen Nash Gleichgewichtsprobleme wird die Existenz von Nash Gleichgewichten gezeigt, sowie Optimalitätsbedingungen hergeleitet. Wir liefern wertvolle Resultate dafür, dass dieser Glättungsansatz sich für die Entwicklung eines numerischen Verfahren eignet, indem wir beweisen können, dass sowohl eine Folge von stationären Punkten als auch eine Folge von Nash Gleichgewichten des epi-regularisierten Problems eine schwach

konvergente Teilfolge hat, deren Grenzwert ein Nash Gleichgewicht des ursprünglichen Problems ist.

Contents

Introduction

This thesis focuses on the theoretical study of PDE-constrained Nash equilibrium problems (NEP) and generalized Nash equilibrium problems (GNEP) under uncertainty. The Nash equilibrium as most prominent discipline of game theory is these days present in many fields of science, particularly in economics and social sciences. Also engineering sciences have discovered the Nash equilibrium concept as a means to design technical systems, since many practical problems require the simultaneous minimization of multiple objectives. The idea is to have these objectives competing against each other in a noncooperative game. Since many models in engineering and natural sciences are defined by partial differential equations (PDEs), it is naturally to assume that each player faces a PDE-constrained optimization problem. Rigorously mathematically modeling real-world phenomena leads to the fact that we have to incorporate the omnipresence of uncertainty in the governing PDE whether it be a consequence of noisy measurements, estimated parameter values, or model ambiguity. We include the uncertainty into our mathematical models by random variables. Each player is represented by an objective and controls their individual deterministic strategy to anticipate future uncertainties. We thus consider PDE-constrained (generalized) Nash equilibrium problems under uncertainty.

General Problem Setting

Summarizing the discussions above, in an abstract sense, we consider an N-player NEP or GNEP in which the ith player's problem takes the form

$$\min_{z_i \in Z^i_{ad}} \{\mathcal{R}_i(J_i(S(z_i, z_{-i}))) + \wp_i(z_i) \mid g(S(z_i, z_{-i}, \omega)) \in K \ \mathbb{P}\text{-a.s.}\} .$$

Here, J_i is an appropriate convex (dis)utility function for player i representing the state-dependent part of the objective function, e.g., a tracking-type term and $\wp_i$ is a deterministic part used to regularize the control, e.g., the Tikhonov regularization. The functional $\mathcal{R}_i$ denotes a risk measure. $S(z)$ is the z-dependent random field solution of a linear elliptic PDE with uncertain inputs, while the control $z_i \in Z^i_{ad}$ does not depend on the uncertain inputs. We use the typical convention $z = (z_i, z_{-i})$ to emphasize the ith component, i.e., z_{-i} denotes the components of z excluding z_i. Z^i_{ad} and K are closed convex sets. If the convex (with respect to $-K$) mapping $g(\cdot, z_{-i})$ is nontrivial , then we incorporate state constraints in the model and consider a GNEP instead of a NEP. This is an important distinction in the modeling of the game. When considering no state constraints in addition to the coupled PDE, the game has an equivalent formulation as a variational inequality. The enhancement by state constraints yields an equivalent formulation as a quasivariational inequality.

The term risk-neutral GNEP arises when $\mathcal{R}_i$ is the expected value $\mathbb{E}$. Then the expected (dis)utility is considered. Letting $\overline{z}$ be a generalized Nash equilibrium for the risk-neutral problem, player i would expect $\overline{z}_i$ to be the best response to $\overline{z}_{-i}$ on average.

The term risk-averse NEP arises when $\mathcal{R}_i$ is a coherent risk measure. Letting $\overline{z}$ be a Nash equilibrium for the risk-averse problem, player i would expect $\overline{z}_i$ to be the best response to $\overline{z}_{-i}$ taking into account their individual risk tolerance.

State of the Art

The inclusion of random inputs leads to parametric or random PDEs as part of our optimization problems, cf. [12, 22, 71, 87, 90]. In light of uncertainty, it is essential to manage the risk associated with an optimal design or control problem. There are a multitude of constructs to handle the uncertainty in the optimization setting, to name a few: A possibility would be the distributionally robust optimization approach, see, e.g., [59, 85]. Alternatively one could introduce stochastic order constraints as for example probability constraints [72, 84]. There are, e.g., the studies [26, 35] which make use of probability constraints. This may lead to an infinite number of highly non-linear and non-differentiable constraints. Instead, we employ risk measures [78] to treat the uncertainty in the objective functions. This brings us into a framework that is much closer to the classical PDE-constrained optimization. In addition using a coherent risk measure preserve the convexity of the original objective function. If the coherent risk measure is also proper, then the risk-averse optimization problems have equivalent dual, minimax formulations. PDE-constrained optimization under uncertainty and related topics in uncertainty quantification are challenging areas in applied mathematics with relevant applications in the engineering sciences. It is a growing field with many recent contributions in theory and algorithms, see, e.g., [3, 15, 18, 32, 34, 58, 60–63, 65, 81, 83, 91].

Until recently, much of the work has focused on numerical approximation and solution algorithms for the risk-neutral case ($\mathcal{R}_i = \mathbb{E}$), for example, the authors in [14, 17, 31, 58, 60, 61]. However, this is the first attempt at studying risk-neutral PDE-constrained GNEPs with uncertain inputs. The "G" in GNEP arises due to a shared state constraint of almost everywhere type. This may arise in the study of some risk measures (see [30]) but not in the form presented here. In the deterministic setting, we mention here the pioneering works [20,39,73–76,89]. Note that the models in these papers do not consider bound constraints, in particular there are no state constraints. For recent work on GNEPs in Banach spaces in the deterministic setting, we refer the reader to [50, 51, 56, 57] and the references therein. On the treatment of state constraints in PDE-constrained optimization under uncertainty, see, e.g., [26,35] and the recent preprint [33].

Outline of the Thesis

Parts of this thesis have been submitted in the preprint [29]. The rest of this thesis is structured as follows: In Chapter 1 we lay the necessary foundations. In Section 1.2 we introduce function spaces, in particular Sobolev and Bochner spaces. An introduction to NEPs and GNEPs is given in Section 1.3, where Section 1.3.1 focuses on the variational reformulation of jointly convex GNEPs.

In Chapter 2, we outline the class of PDE-constrained NEPs and GNEPs with uncertain inputs, we are dealing with in the thesis. This includes the formulation of the random field state equations in Section 2.1 as well as the introduction of risk measures in Section 2.2. In Section 2.1 the class of second order linear elliptic PDEs with uncertain data functions and coefficients is formulated in a precise setting. Two formulations are presented: Either a weak formulation of a deterministic PDE can be solved for $\mathbb{P}$-a.e realization $\omega \in \Omega$ or a fully weak formulation can be considered by integrating over the probability space Ω. The latter formulation is essential for our further proceeding. The solution of the PDE is then an element of a suitable Bochner space. Assumptions guaranteeing existence and uniqueness of solutions are given. We show the equivalence of both formulations. Finally, a proof of the higher regularity of the solution is presented, which later is crucial in order to use the existing optimality theory for convex optimization in Banach spaces applied to the risk-neutral GNEPs and risk-averse NEPs. In Section 2.2 we introduce the axiomatic definition of coherent risk measures. The uncertainty enters via the state variable in the individual objective functionals. Since there is no ordering on Lebesgue space, we cannot minimize the random field objective functionals directly. We choose risk-measures to remedy this problem. In addition, we can thereby incorporate risk preferences of the players. This brings us into a framework that is much closer to the classical PDE-constrained optimization. Finally, in Section 2.3 the general formulation of the classes of stochastic equilibrium problems is given. In Section 2.3.1, we analyze the properties of the control-to-state map. The requirements and proofs for well-definedness and differentiability of the random field objective functionals, which is in fact a superposition operator between Bochner and Lebesgue spaces, are presented in Section 2.3.2.

In Chapter 3, we focus on the class of risk-neutral generalized Nash equilibrium problems under uncertainty. The individual player's problem is a risk-neutral PDE-constrained optimization problem with a tracking-type objective function. The term risk-neutral results from the fact that we consider the expected value in the objective function. In addition to pointwise constraints on the strategies, we consider a pointwise constraint on the state variable. In Section 3.1 the existence of Nash equilibria for this problem is demonstrated, where we restrict ourselves to the concept of variational equilibria as introduced in Section 1.3.1. Further, we derive first-order optimality conditions under a in the optimization traditional constraint qualification, namely a Slater-type CQ. When uncertain inputs are involved, this problem of state constraints becomes much more challenging. The existing optimality theory for convex optimization problems applied to the risk-neutral generalized Nash equilibrium problems only guarantees that the multiplier associated with the state constraint is of low regularity. In particular, a standard adjoint equation is not available. In order to derive a function-space based numerical approach, we extend an well-known approximation scheme based on Moreau–Yosida-type regularization, in which the original problem is approximated by a parameter-dependent sequence of standard Nash equilibrium problems in Section 3.2. In Section 3.2.3 we require a constraint qualification tailored for the characteristic of a GNEP, namely the strict uniform feasible response constraint qualification (SUFR) introduced in [50]. Passing to the limit in the relaxation parameter is crucial for the justification of the numerical methods in the fully continuous setting. Due to SUFR, we can demonstrate the convergence of weak accumulation points to variational equilibria for the original problem.

In Chapter 4, we focus on risk-neutral PDE-constrained optimization subject to state constraints which are in accordance with the individual player's problems in-Chapter 3. This individual problems themselves have yet to appear (only a few preprints) in the literature and are themselves an interesting field of study. In Section 4.1, we prove existence of solutions and derive optimality conditions. We discuss an alternative way of formulating the state constraints, namely probability constraints and we link the Moreau-Yosida regularization technique and probability constraints using concentration inequalities in Section 4.2.

In Chapter 5 we provide a theoretical presentation of an algorithm to indicate how risk-neutral PDE-constrained optimization problems subject to pointwise state constraints and risk neutral PDE-constrained GNEPs under uncertainty might be solved. It is all from a analytical point of view. We focus on a suggestion for the numerical approach of the individual optimization problems in Section 5.1. We suggest to approximate the expected value/ PDEs with uncertain inputs with Monte Carlo approximation and then we explicitly show how the resulting deterministic regularized problem can be theoretically solved with semismooth Newton method. In Section 5.2, we illustrate how the approach from Section 5.1 can be applied to a two player GNEP. We suggest to use a Krasnoselskii-Mann-type alternating the dueling agents use the solver from Section 5.1. A numerical study is beyond the scope of this work as the focus of this work is the analytical approach.

In Chapter 6 we focus on the class of risk-averse Nash equilibrium problems under uncertainty. We model each player's risk preference with coherent risk measures as introduced in Section 2.2. In Section 6.2 we prove existence of Nash equilibria and derive optimality conditions. Considering risk-averse players forces the objective functions to be nonsmooth. To remedy this problem, we present an approximation scheme that provides us with a parameter-dependent sequence of Nash equilibrium problems in Section 6.3. This motivates the concept of paths of equilibria. We briefly recall the general smoothing approach of the epi-regularization as introduced in [65], before proving the existence of Nash equilibria and deriving optimality conditions for the epi-regularized Nash equilibrium problems in Section 6.3.2. Finally, in Section 6.3.3 we demonstrate the convergence of stationary points of the epi-regularized NEPs to a Nash equilibrium of the original NEP as well as that a sequence of Nash equlibria for the epi-regularized NEPs has a weakly converging subsequence whose limit is a Nash equilibrium of the original NEP. Therefore, we can show that epi-regularized Nash equilibrium problems provide a consistent approximation.

The thesis is concluded in Chapter 7, where we also point out future research aspects which where not covered in this work.

Chapter 1

Preliminaries

In this chapter, we present definitions and results that are needed in the course of this thesis. In the first section we fix some notation, in Section 1.2 we discuss the necessary function spaces and finally, in Section 1.3 we introduce Nash Equilibrium Problem and Generalized Nash Equilibrium Problem which we are going to treat.

1.1 Notational Conventions

At first, we fix several notational conventions. As usual, let $\mathbb{N}$ and $\mathbb{R}$ denote the set of all natural and real numbers. The triple $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space, where Ω is the sample space of possible outcomes, $\mathcal{F}$ the Borel σ-alegra of Ω for a fixed topology on Ω and $\mathbb{P}$ is a probability measure. For a (real) Banach space V we denote the expectation of a random element $X : \Omega \to V$ by

$$\mathbb{E}_{\mathbb{P}}[X] = \int_{\Omega} X(\omega)\, \mathrm{d}\mathbb{P}(\omega) \in V.$$

For some nonempty subset $C \subset V$, $\mathcal{I}_C : V \to \mathbb{R} \cup \{\infty\}$ represents the standard indicator function, which satisfies $\mathcal{I}_C(x) = 0$ if $x \in C$ and $+\infty$ otherwise. For an arbitrary convex set K, we define the standard convex normal cone by

$$\mathcal{N}_K(x) = \begin{cases} \{x^* \in V^* \mid \langle x^*, y - x\rangle \leq 0, \quad \forall y \in K\}, & \text{if } x \in K, \\ \emptyset, & \text{otherwise.} \end{cases}$$

The (set-theoretic) characteristic function associated with some subset A is denoted by χ or χ_A, where $\chi_A(x) = 1$ if $x \in A$ and 0 otherwise.

Strong convergence of a sequence is denoted by $\to$, weak-convergence by $\rightharpoonup$, and weak-*-convergence by $\overset{*}{\rightharpoonup}$.

The *closed* ε-ball with center x in some normed space is denoted $\mathbb{B}_\varepsilon(x)$. The superscript $*$ is used to denote the adjoint operator or dual space.

The frequently used relation $a \lesssim b$ is understood as $a \leq Cb$ with a constant $C > 0$ that does not depend on a and b.

For two Banach spaces V and W, the set of all bounded linear operators from V to W will be denoted by $\mathcal{L}(V, W)$.

1.2 Function Spaces

We start by defining the necessary function spaces. In particular we consider Lebesgue spaces, Sobolev spaces, Bochner spaces and some measure space. We always assume that the physical domain $D \subset \mathbb{R}^d$ with $d = 1, 2$, or 3 is a domain, i.e. a connected open set. There is a lot of extensive literature to deepen everybody's understanding of this issue, to name a few [44]. The Lebesgue space is defined as follows.

Definition 1.1. Let $p > 0$. Then, the usual Lebesgue space $L^p_\mu(D)$ with underlying measure μ is defined by

$$L^p_\mu(D) = \{f : D \to \mathbb{R} \,|\, f \text{ measurable and } \|f\|_{L^p_\mu(D)} < \infty\},$$

where

$$\|f\|_{L^p_\mu(D)} = \begin{cases} \left(\int\limits_D |f(x)|^p \, d\mu(x)\right)^{1/p}, & p < \infty, \\ \mu\text{-}\operatorname*{ess\,sup}_{x \in D} |f(x)|, & p = \infty. \end{cases}$$

Note that according to the definition of the Lebesgue space, $L^p_\mu(D)$ considers equivalence classes of functions that are equal μ-almost everywhere. For $p \geq 1$ the L^p_μ-spaces are Banach spaces. The space $L^2_\mu(D)$ is a Hilbert space with inner product

$$\langle f, g\rangle_{L^2_\mu(D)} = \int\limits_D f(x)\overline{g(x)} \, d\mu(x),$$

where $\overline{g}$ denotes the complex conjugate of g. We remark, that for $1 < p < \infty$ the space $L^p_\mu(D)$ is reflexive. There exists a natural isomorphismus, which associates $g \in L^q_\mu(D)$ with a functional $L_g \in L^p_\mu(D)^*$, where $\frac{1}{p} + \frac{1}{q} = 1$. This isomorphismus is defined by

$$f \mapsto L_g(f) = \int\limits_D f(x)g(x) \, d\mu(x).$$

In order to characterize the dual space in the case of $p = \infty$, we introduce the space of bounded additive measures **ba**. We recall here for ease of reference, cf. [42, 20.27 Definition] or [21].

Definition 1.2. Let $(D, \mathcal{B}, \mu)$ be a σ-finite measure space. The space $\mathbf{ba}(D, \mathcal{B}, \mu)$ denotes the set of all real-valued set-functions $\tau : \mathcal{B} \to \mathbb{R}$ such that

(i) $\sup\{|\tau(A)| | A \in \mathcal{B}\} < \infty$,

(ii) $\tau(A \cup B) = \tau(A) + \tau(B)$ for $A, B \in \mathcal{B}$ with $A \cap B = \emptyset$ and

(iii) $\tau(A) = 0$ if $A \in \mathcal{B}$ is μ-null, i.e. $\tau << \mu$.

The norm of $\tau \in \mathbf{ba}(D, \mathcal{B}, \mu)$ is given by $|\tau|(D)$, the total variation of τ on $\mathcal{B}$ defined by

$$|\tau|(A) = \sup \left\{ \sum_{i=1}^{n} |\tau(A_i)| |\{A_i\}_{i=1}^{n} \subset \mathcal{F}, A_i \cap A_j = \emptyset, i \neq j \text{ s.t. } \bigcup_{i=1}^{n} A_i = A \right\}.$$

The key result for our analysis related to this space is the existence of an isometric isomorphism between $L^{\infty}_{\mu}(D)^*$ and $\mathbf{ba}(D, \mathcal{B}, \mu)$, cf. [21, Thm. IV.8.16] or [42, 20.33 Theorem, 20.34 Theorem]. A more prominent example of measure space is the space of all bounded countably additive measures which we denoted by $\mathcal{M}(D, \mathcal{B}, \mu)$. This space is also called the space of regular Borel measures and is the dual space of the continuous functions $C(D)$. Clearly, The space $\mathbf{ba}(D, \mathcal{B}, \mu)$ is slightly larger than $\mathcal{M}(D, \mathcal{B}, \mu)$.

Finally on the subject of the Lebesgue space, when there is the Lebesgue measure λ considered, we omit the subscript λ and simply write $L^p(D)$.

To define Sobolev spaces, we introduce the concept of weak derivatives. We call $\alpha \in \mathbb{N}_0^d$ a multi index with absolute value $|\alpha| := \sum_{i=1}^{d} \alpha_i$. The space of test functions $C_0^{\infty}(D)$ is defined as the collection of all infinitely often differentiable functions with compact support. A function $f_\alpha \in L^{1,loc}(D)$ is called the αth weak derivative of $f \in L^{1,loc}(D)$ if it holds

$$\int_D f_\alpha(x)\varphi(x)\ d(x) = (-1)^{|\alpha|} \int_D f(x) D^\alpha \varphi(x)\ d(x) \qquad \text{for all } \varphi \in C_0^{\infty}(D),$$

where $L^{1,loc}(D)$ is defined as the space of all locally integrable functions given by

$$L^{1,loc}(D) := \{f | f \in L^1(K) \text{ for all } K \subset\subset D\}.$$

Note that the weak derivative is uniquely determined almost everywhere if it exists. For differentiable functions in the classical sense, the weak and classical derivative coincide.

Definition 1.3. Let $m \in \mathbb{N}_0$ and $p \geq 1$. The Sobolev spaces $W_p^m(D)$ includes all functions having weak derivatives up to order m in $L^p(D)$:

$$W_p^m(D) = \{f : D \to \mathbb{R} \,|\, D^\alpha f \in L^p(D) \text{ for all } 0 \leq |\alpha| \leq m\},$$

where

$$\|f\|_{W_p^m(D)} = \begin{cases} \left(\sum_{|\alpha| \leq m} \|D^\alpha f\|^p_{L^p(D)} \right)^{1/p}, & p < \infty, \\ \sup_{|\alpha| \leq m} ||D^\alpha f||_{L^p(D)}, & p = \infty. \end{cases}$$

Similar to the Lebesgue spaces, Sobolev spaces are Banach spaces. The dual of the Sobolev space is defined to be $W_p^m(D)^* = W_q^{-m}(D)$, where $\frac{1}{p} + \frac{1}{q} = 1$. The spaces $W_2^m(D)$ are Hilbert spaces, this motivates the notation $H^m(D) = W_2^m(D)$. In this case the inner product has the form

$$\langle f, g\rangle_{H^m(D)} = \sum_{|\alpha|\leq m} \langle D^\alpha f, D^\alpha g\rangle_{L^2(D)}.$$

The dual space of $H^m(D)$ is $H^{-m}(D)$. When it comes to elliptic PDEs, we have to consider Sobolev spaces with incorporated boundary conditions. The Sobolev space with zero boundary conditions $H_0^m(D)$ can be defined as the closure of $C_0^\infty(D)$ with respect to the $H^m(D)$-norm. We specify the definition of domains, since the regularity of the solution of a PDE depends on the smoothness of the boundary of the domain. For further details, we refer to [40].

Definition 1.4. Let either $k \in \mathbb{N} \setminus \{0\}$, $\alpha \in [0, 1]$ or $k = 0$, $\alpha = 1$. The domain $D \subset \mathbb{R}^d$ is called $C^{k,\alpha}$ domain, if for each $x \in \partial D = \Gamma$ there exists a neighborhood $U \subset \mathbb{R}^d$ and a bijective mapping $\varphi : U \to B_1(0)$ such that it holds

$$\begin{aligned}
&\varphi \in C^{k,\alpha}(\overline{U}), \qquad \varphi^{-1} \in C^{k,\alpha}(\overline{U}),\\
&\varphi(U \cap \Gamma) = \{\xi \in B_1(0) | \xi_d = 0\},\\
&\varphi(U \cap D) = \{\xi \in B_1(0) | \xi_d > 0\},\\
&\varphi(U \cap (\mathbb{R}^d \setminus D)) = \{\xi \in B_1(0) | \xi_d < 0\}.
\end{aligned}$$

$C^{0,1}$- domains are called Lipschitz domains.

In the course of this thesis, we will study second-order linear elliptic PDEs with random or uncertain inputs. Solutions of this special class of PDEs are vector-valued or Banach space valued functions. To define these solution spaces, we introduce Bochner spaces. These are mainly Lebesgue space for Banach space valued functions. The name is directly linked to the underlying Bochner integral. The Bochner space is defined analogously to the scalar-valued Lebesgue space and reads as follows.

Definition 1.5. Let $p > 0$ and V a Banach space. Then, the usual Bochner space $L^p_\mu(D; V)$ with underlying measure μ is defined by

$$L^p_\mu(D; V) = \{f : D \to V \mid f \text{ strongly measurable and } \|f\|_{L^p_\mu(D;V)} < \infty\},$$

where

$$\|f\|_{L^p_\mu(D;V)} = \begin{cases} \left(\int\limits_D \|f(x)\|_V^p \, d\mu(x)\right)^{1/p}, & p < \infty,\\ \mu\text{- ess}\sup\limits_{x\in D} \|f(x)\|_V, & p = \infty. \end{cases}$$

As in the scalar-valued case we view $f \in L^p_\mu(D;V)$ as an equivalence class of functions that are equal almost everywhere. Similar to the Lebesgue space, $L^p_\mu(D;V)$ is a Banach space. If V is a Hilbert space, then $L^2_\mu(D;V)$ is a Hilbert space and the inner product is given by

$$\langle f, g\rangle_{L^2_\mu(D;V)} = \int_D \langle f(x), \overline{g(x)}\rangle_V \; d\mu(x).$$

Many properties from the Lebesgue space carry over to the Bochner space.

We recall here that for $1 \le p, q \le \infty$ such that $1/p + 1/q = 1$, it is known that the topological dual fulfills $L^p_\mu(D;V)^* \simeq L^q_\mu(D;V^*)$. If V is reflexive, then so is $L^p_\mu(D;V)$ for $1 < p < \infty$.

1.3 Nash Equilibrium Problem and Generalized Nash Equilibrium Problem

The Nash equilibrium and the generalized Nash equilibrium are a concept for a particular class of games. Given N players, for $N \in \mathbb{N}$, the ith player faces the following optimization problem:

$$\begin{aligned} &\min && \mathcal{J}_i(z_i, z_{-i}) \text{ over } z_i \in Z_i \\ &\text{subject to} && z_i \in Z^i_{ad} \end{aligned} \tag{1.3.1}$$

for $i = 1, \ldots, N$. Here $Z_1, \ldots, Z_N$ are reflexive and separable Banach spaces and $Z^i_{ad} \subset Z_i$ denotes the ith player's admissible set of control. Each player controls a decision vector z_i. We use the typical convention $(z_i, z_{-i}) = z \in Z_1 \times \cdots \times Z_N$ to emphasize the ith component, i.e. $z_{-i} = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_N)$ denotes the components of z excluding z_i. We set $Z = Z_1 \times \cdots \times Z_N$. Thus the ith players objective function $\mathcal{J}_i$ does not only depend on their individual decision z_i, but also on the decisions of their competitors z_{-i}. The optimal choice of z_i depends on the choices of the other players. There are many solution concepts, the most commonly used solution concepts are equilibrium concepts, most famously Nash equilibrium introduced in 1950 by Nash [69]. In that sense, we refer to the problem in (1.3.1) as Nash equilibrium problem, for short NEP. We define solutions for this NEPs or Nash equilibria as follows.

Definition 1.6. A Nash equilibrium is a admissible decision vector $\bar{z} \in Z_{ad} = Z^1_{ad} \times \cdots \times Z^N_{ad}$ provided that for each $i = 1, \ldots, N$ the decision $\bar{z}_i$ fulfills

$$\bar{z}_i \in \underset{v_i \in Z_i}{\arg\min} \{\mathcal{J}_i(v_i, \bar{z}_{-i}) \text{ s.t. } v_i \in Z^i_{ad}\}.$$

In other words, at a Nash equilibrium no player can reduce the value of their objective function by unilaterally changing their decision. This concept describes a stable condition. In many situation modeling a problem as NEP is insufficiently since the choice of the competitors not only affects the value of the objective function but also the choice. We will therefore extend the model as follows: For $N \in \mathbb{N}$, the ith player faces the following optimization problem:

$$\begin{aligned} &\min && \mathcal{J}_i(z_i, z_{-i}) \text{ over } z_i \in Z_i \\ &\text{subject to} && z_i \in C_i(z_{-i}). \end{aligned} \tag{1.3.2}$$

Here, the set-valued multifunction $C_i(z_{-i}) : Z_{ad}^{-i} \rightrightarrows Z_{ad}^i$ denotes the constraint set of each player i for $i = 1, \dots, N$. Together, we have $C(z) = \prod_{i=1}^N C_i(z_{-i})$. This type of problem is referred to as generalized Nash equilibrium problem, for short GNEP. In analogy to the definition of the Nash equilibrium, we arrive at the following definition of a generalized Nash equilibrium (cf. [19]).

Definition 1.7. A generalized Nash equilibrium is a admissible decision vector $\bar{z} \in Z_{ad}$ provided that for each $i = 1, \dots, N$ the decision $\bar{z}_i$ fulfills

$$\bar{z}_i \in \arg\min_{v_i \in Z_i} \{\mathcal{J}_i(v_i, \bar{z}_{-i}) \ \text{ s.t. } \ v_i \in C_i(\bar{z}_{-i})\}.$$

The class of NEP and GNEP we will focus throughout this work, is called jointly convex NEP or jointly convex GNEP, respectively In this situation, the constraint set normally reads as

$$C_i(z_{-i}) = \{v_i \in Z_{ad}^i \mid G(v_i, z_{-i}) \in K\},$$

where K is a closed convex cone, $G(\cdot, z_{-i})$ is a convex mapping with respect to the convex set $-K$ (for further details see [13, Section 2.3.5]). Moreover, we assume that Z_{ad}^i and the individual objective function $\mathcal{J}_i$ are convex for $i = 1, \dots, N$.

1.3.1 Variational Reformulation and Variational Equilibria

We will use the concept of variation equilibria. This class is a specific class of Nash equilibria and in many cases they can be computed numerically. The concept is strongly related to the concept of normalized equilibria introduced by Rosen [80]. Note that Rosen's notion of normalized equilibria is based on a formulation using Lagrange multipliers. The variational equilibria is based on a variational reformulation which, in turn, is based on the so-called Nikaido-Isoda function $\Psi : Z \times Z \to \mathbb{R}$. For the introduced jointly convex GNEP the Nikaido-Isoda function is defined by

$$\Psi(z, v) = \sum_{i=1}^N \mathcal{J}_i(z_i, z_{-i}) - \mathcal{J}_i(v_i, z_{-i}).$$

This function represents the sum of unilateral objective improvements between the decision vectors z and v. In addition, we set $M = Z_{ad} \cap G^{-1}(K)$ and define an auxiliary function $V : M \to \mathbb{R}$ by

$$\begin{aligned} V(u) &= \max \{\Psi(z,v) | v \in Z \text{ such that } (v_i, z_{-i}) \in M \text{ for } i = 1, \dots, N\} \\ &= \max \{\Psi(z,v) | v \in C(z)\}. \end{aligned} \tag{1.3.3}$$

The auxiliary function V can be seen as gap function. Note that M is nonempty, bounded, closed and convex due to the assumptions on the jointly convex GNEP. This leads to the following convenient characterization of generalized Nash equilibria.

Lemma 1.8. *A point $\bar{z} \in U$ is a generalized Nash equilibrium of* (1.3.2) *if and only if $\bar{z} \in M$ and $V(\bar{z}) = 0$.*

Proof. See [70, Lemma 3.1.]. □

We define a variational equilibria as in [51, Section 3]. Therefore, we define the set-valued collective best-response function $\hat{\mathcal{R}} : M \rightrightarrows M$ by

$$\begin{aligned} \hat{\mathcal{R}}(z) &= \arg\max_{v \in Z} \{\Psi(z,v) | v \in M\} \\ &= \arg\min_{v \in Z} \left\{ \sum_{i=1}^{N} \mathcal{J}_i(v_i, z_{-i}) | v \in M \right\} \end{aligned}$$

and the auxiliary function $\widehat{V} : M \to \mathbb{R}$ by

$$\widehat{V}(z) = \Psi(z, \widehat{\mathcal{R}}(u)) = \max_{v} \{\Psi(z,v) | v \in M\}. \tag{1.3.4}$$

Note that in(1.3.3), the definition of V, the parameter z perturbs the objective functional and the feasible set. While in (1.3.4), the definition of $\widehat{V}$, the parameter z only perturbs the objective function.

Definition 1.9. A point $\bar{z} \in Z$ is a variational equilibrium of the jointly convex GNEP in (1.3.2) if $\bar{z} \in M$ and $\widehat{V}(\bar{z}) = 0$.

Note that $V(z) \geq 0$ and $\widehat{V}(z) \geq 0$ for all $z \in M$ and that variational equilibria are generalized Nash equilibria, since the contraposition $V(z) > 0$ implies $\widehat{V}(z) > 0$ holds. Finally, the best-response function $\widehat{\mathcal{R}}$ leads to the following convenient characterization of variational equilibria.

Lemma 1.10. *A point $\bar{z} \in M$ is a variational equilibrium of the jointly convex GNEP in* (1.3.2) *if and only if $\bar{z}$ is a fixed point of the best-response function $\widehat{\mathcal{R}}$, i.e. $\bar{z} \in \widehat{\mathcal{R}}(\bar{z})$.*

Proof. Let $\bar{z} \in Z$ be a variational equilibrium, then $\bar{z} \in M \subset Z$ and

$$0 = \widehat{V}(\bar{z}) = \Psi(\bar{z}, \widehat{\mathcal{R}}(\bar{z})).$$

Hence, $\bar{z} \in \widehat{\mathcal{R}}(\bar{z})$. By simply reversing this argument we obtain equivalence. □

This characterization reduces the proof of existence of a variational equilibrium to a fixed point problem.

Chapter 2

A Class of Generalized Nash Equilibrium Problems under Uncertainty

In this chapter, we introduce the stochastic equilibrium problems considered throughout the thesis. This is the first attempt at studying PDE-constrained NEPs and GNEPs with uncertain inputs. The first two sections are dedicated to the study of second-order linear elliptic PDEs with random or uncertain inputs. In addition, we introduce the necessary risk measures. With these essential ingredients, we are able to state the formal problem formulation of the risk-neutral or risk-averse (generalized) Nash equilibrium problems under uncertainty in Section 2.3.

2.1 A Class of Linear Elliptic PDE with Uncertain Coefficients

In this section, we focus on the state equation of our model problem and discuss a class of partial differential equations (PDEs) for which the input data are uncertain. We will focus on second order linear elliptic PDEs with uncertain data functions and coefficients. We consider only homogeneous Dirichlet boundary data. Inhomogeneous Dirichlet boundary conditions can also be treated, which would require a harmonic lifting step. This discussion closely follows [66].

Unless otherwise stated, let $D \subset \mathbb{R}^d$ with $d = 1, 2$, or 3 be some bounded domain. Throughout this thesis unless we consider the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is a triple of a sample space Ω containing all possible outcomes, the σ-algebra $\mathcal{F}$ as a set of subsets of Ω, called events and $\mathbb{P}$ a probability measure. We are interested in finding a function $u : D \times \Omega \to \mathbb{R}$, which solves for $\mathbb{P}$-almost every $\omega \in \Omega$ the following parameter-dependent elliptic partial differential equation

$$\begin{aligned} -\operatorname{div}\left(a(\omega,x)\nabla u(\omega,x)\right) &= B(\omega)z + f(\omega,x) && \text{in } D \\ u(\omega,x) &= 0 && \text{on } \partial D. \end{aligned} \tag{2.1.1}$$

Here f and a are functions from $\Omega \times D$ to $\mathbb{R}$. The solution of the elliptic boundary value problem (2.1.1), as well as the right hand side and the coefficient function a are assumed to be random fields. Also, z itself is not random, but the operator B potentially is.

We gather the required regularity properties of the data in the following assumption.

Assumption 2.1. (i) $D \subset \mathbb{R}^d$ with $d = 1, 2$, or 3 is an open bounded set with Lipschitz boundary.

(ii) Ω is an arbitrary set, $\mathcal{F}$ is an associated σ-algebra and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$ that makes $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space.

(iii) The diffusion coefficient $a \in L^\infty_{\mathbb{P}\times\lambda}(\Omega \times D)$ is uniformly bounded away from zero, i.e. there exit constants $\underline{\alpha}, \overline{\alpha} \in (0, \infty)$ such that

$$\underline{\alpha} \leq a(\omega, x) \leq \overline{\alpha} \tag{2.1.2}$$

for $\mathbb{P}$-almost every $\omega \in \Omega$ and $x \in D$.

(iv) For the forcing term, we assume that $f \in L^2_{\mathbb{P}}(\Omega; L^2(D))$.

(v) For all $z \in L^2(D)$ the control operator fulfills $B(\cdot)z \in L^2_{\mathbb{P}}(\Omega; L^2(D))$.

2.1.1 Weak Formulation

In order to determine the weak formulation of (2.1.1), we multiply (2.1.1) by an appropriate test function v and integrate over the spatial domain. Applying Green's identity yields

$$\int_D a(\omega, x)\nabla u_\omega(x) \cdot \nabla v(x) \,\mathrm{d}x = \int_D (B(\omega)z(x) + f(\omega, x))\, v(x) \,\mathrm{d}x \tag{2.1.3}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ and for all $v \in H^1_0(D)$. Here, the notation $u_\omega(x)$ should emphasise, the pointwise perspective. For $\mathbb{P}$-a.e. $\omega \in \Omega$, we introduce the parameter-dependent bilinear form $b(\omega; \cdot, \cdot) : H^1_0(D) \times H^1_0(D) \to \mathbb{R}$ and the parameter-dependent linear functional $g(\omega; \cdot) : H^1_0(D) \to \mathbb{R}$ given by

$$b(\omega; y, v) = \int_D a(\omega, x)\nabla u(x) \cdot \nabla v(x) \,\mathrm{d}x \text{ and } g(\omega; v) = \int_D (B(\omega)z(x) + f(\omega, x))\, v(x) \,\mathrm{d}x \tag{2.1.4}$$

respectively. Then the parameter-dependent weak formulation of (2.1.1) reads as: For $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\text{Find } u_\omega \in H^1_0(D) \quad \text{such that} \quad b(\omega; u_\omega, v) = g(\omega; v) \quad \text{in } H^{-1}(D), \tag{2.1.5}$$

for all $v \in H_0^1(D)$.

In addition to this $\mathbb{P}$-pointwise formulation, it is possible to introduce an equivalent version in Bochner space that provides us, amongst other things, with measurability and integrability of u. In order to derive this fully weak formulation, we take the expectation of the parameter-dependent weak formulation (2.1.3) and we arrive at:

$$\text{Find } u \in L^2_{\mathbb{P}}(\Omega; H_0^1(D)) \quad \text{such that} \quad b(u,v) = g(v) \tag{2.1.6}$$

for all $v \in L^2_{\mathbb{P}}(\Omega; H_0^1(D))$, where the bilinear form $b(\cdot,\cdot) : L^2_{\mathbb{P}}(\Omega; H_0^1(D)) \times L^2_{\mathbb{P}}(\Omega; H_0^1(D)) \to \mathbb{R}$ is defined by

$$b(u,v) = \mathbb{E}_{\mathbb{P}}\left[\int_D a(\cdot,x)\nabla u(\cdot,x) \cdot \nabla v(\cdot,x)\,\mathrm{d}x\right] \tag{2.1.7}$$

and the right hand side $g(\cdot) : L^2_{\mathbb{P}}(\Omega; H_0^1(D)) \to \mathbb{R}$ is given by

$$g(v) = \mathbb{E}_{\mathbb{P}}\left[\int_D (B(\cdot)z(x) + f(\cdot,x))\, v(\cdot,x)\,\mathrm{d}x\right]. \tag{2.1.8}$$

The weak formulation (2.1.6) is formulated equivalently as operator equation as a consequence of Riesz's representation theorem (see [40, Lemma 6.91.(a)]):

$$\text{Find } u \in L^2_{\mathbb{P}}(\Omega; H_0^1(D)) \quad \text{such that} \quad \mathbf{A}u = \mathbf{B}z + f \quad \text{in } L^2_{\mathbb{P}}(\Omega; H^{-1}(D)), \tag{2.1.9}$$

where we denote the control operator $\mathbf{B} : \Omega \to \mathcal{L}(L^2(D), H^{-1}(D))$ by

$$\langle \mathbf{B}(\omega)z, v\rangle = \int_D (B(\omega)z)(x)v(x)\ \mathrm{d}x$$

for $v \in H_0^1(D)$. The isomorphism $\mathbf{A} \in \mathcal{L}\left(L^2_{\mathbb{P}}(\Omega; H_0^1(D)), L^2_{\mathbb{P}}(\Omega; H^{-1}(D))\right)$ is uniquely determined by the bilinear form $b(\cdot,\cdot)$ in (2.1.7) through $b(v,w) = \langle Av, w\rangle$, for all $v, w \in L^2_{\mathbb{P}}(\Omega; H_0^1(D))$, provided that $b(\cdot,\cdot)$ is continuous and bilinear. The dual pairing reads as

$$\langle \mathbf{A}y, v\rangle_{L^2_{\mathbb{P}}(\Omega;H^{-1}(D)) \times L^2_{\mathbb{P}}(\Omega;H_0^1(D))} = \mathbb{E}_{\mathbb{P}}\left[\langle A(\omega)y(\omega), v(\omega)\rangle_{H^{-1}(D)\times H_0^1(D)}\right]. \tag{2.1.10}$$

Provided that the mapping $\omega \mapsto b(\omega; v, w)$ is measurable for all $v, w \in H_0^1(D)$ and that $b(\omega;\cdot,\cdot)$ in (2.1.4) is continuous and bilinear for $\mathbb{P}$-a.e. $\omega \in \Omega$, then $A(\omega)$ in (2.1.10) coincides with the bounded linear operator $A(\omega) : H_0^1(D) \to H^{-1}(D)$ which is uniquely determined by $b(\omega; v, w) = \langle A(\omega)v, w\rangle_{H^{-1}(D)\times H_0^1(D)}$ for all $y, v \in H_0^1(D)$. Then the parameter-dependent operator equation reads as

$$\text{Find } u_\omega \in H_0^1(D) \quad \text{such that} \quad A(\omega)u_\omega = B(\omega)z + f(\omega) \quad \text{in } H^{-1}(D) \tag{2.1.11}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$. For more on the deterministic setting leading to the operator formulation, we refer the reader to [52, Theorem 1.31, section 1.3.2.2]).

2.1.2 Existence and Uniqueness of Weak Solution

This section deals with the existence and uniqueness of weak solutions for the weak formulation (2.1.6).

In order to obtain a unique solution in $L^2_{\mathbb{P}}(\Omega; H^1_0(D))$, we will need continuity and ellipticity of the bilinear form in (2.1.7), which we recall here for ease of reference, cf. [66, Theorem 3.2.1] and the references therein.

Lemma 2.2. *Let (i) to (iii) in Assumption 2.1 hold. Then the bilinear form $b(\cdot,\cdot)$ defined in (2.1.7) is continuous and $L^2_{\mathbb{P}}(\Omega; H^1_0(D))$-elliptic.*

Proof. First, recall that

$$H^1_0(D) \overset{d}{\hookrightarrow} L^2(D) \text{ yields } L^2_{\mathbb{P}}(\Omega; H^1_0(D)) \overset{d}{\hookrightarrow} L^2_{\mathbb{P}}(\Omega; L^2(D)),$$

where $\hookrightarrow$ denotes the continuous and $\overset{d}{\hookrightarrow}$ the continuous and dense embedding. Moreover, $L^2_{\mathbb{P}}(\Omega; L^2(D)) \cong L^2_{\mathbb{P}\times\lambda}(\Omega \times D)$ where λ denotes the Lebesgue measure. We first prove the assumption of Tonelli's theorem [82, 13.8 Theorem]. Due to Hölder's inequality, we obtain

$$\begin{aligned}\int_{\Omega\times D} |a(\omega,x)\nabla u(\omega,x)\cdot\nabla v(\omega,x)|\,\mathrm{d}x\otimes\mathrm{d}\mathbb{P}(\omega) &\le \|a\|_{L^\infty_{\mathbb{P}\times\lambda}(\Omega\times D)}\,\langle\nabla y,\nabla v\rangle_{L^2_{\mathbb{P}\times\lambda}(\Omega\times D)}\\ &\le \|a\|_{L^\infty_{\mathbb{P}\times\lambda}(\Omega\times D)}\|\nabla y\|_{L^2_{\mathbb{P}\times\lambda}(\Omega\times D)}\|\nabla v\|_{L^2_{\mathbb{P}\times\lambda}(\Omega\times D)}.\end{aligned}$$

Thus, $a\nabla y\cdot\nabla v \in L^1_{\mathbb{P}\times\lambda}(\Omega\times D)$. By applying Fubini's theorem [82, 13.9 Corollary]

$$\begin{aligned}|b(u,v)| &= \left|\int_\Omega\int_D a(\omega,x)\nabla u(\omega,x)\cdot\nabla v(\omega,x)\,\mathrm{d}x\,\mathrm{d}\mathbb{P}(\omega)\right|\\ &\le \int_{\Omega\times D} |a(\omega,x)\nabla u(\omega,x)\cdot\nabla v(\omega,x)|\,\mathrm{d}x\otimes\mathrm{d}\mathbb{P}(\omega)\\ &\le \|a\|_{L^\infty_{\mathbb{P}\times\lambda}(\Omega\times D)}\int_{\Omega\times D}|\nabla u(\omega,x)\cdot\nabla v(\omega,x)|\,\mathrm{d}x\otimes\mathrm{d}\mathbb{P}(\omega)\\ &\le \|a\|_{L^\infty_{\mathbb{P}\times\lambda}(\Omega\times D)}\int_\Omega\int_D|\nabla u(\omega,x)||\nabla v(\omega,x)|\,\mathrm{d}x\,\mathrm{d}\mathbb{P}(\omega).\end{aligned}$$

By repeated use of Hölder's inequality we obtain

$$\begin{aligned}|b(u,v)| &\le \|a\|_{L^\infty_{\mathbb{P}\times\lambda}(\Omega\times D)}\int_\Omega \|\nabla u(\omega,\cdot)\|_{L^2(D)}\|\nabla v(\omega,\cdot)\|_{L^2(D)}\,\mathrm{d}\mathbb{P}(\omega)\\ &\le \|a\|_{L^\infty_{\mathbb{P}\times\lambda}(\Omega\times D)}\int_\Omega \|u(\omega,\cdot)\|_{H^1(D)}\|v(\omega,\cdot)\|_{H^1(D)}\,\mathrm{d}\mathbb{P}(\omega)\\ &\le \|a\|_{L^\infty_{\mathbb{P}\times\lambda}(\Omega\times D)}\|u\|_{L^2_{\mathbb{P}}(\Omega;H^1_0(D))}\|v\|_{L^2_{\mathbb{P}}(\Omega;H^1_0(D))}.\end{aligned}$$

Thus, $b(\cdot,\cdot)$ is a continuous bilinear form. In order to prove the ellipticity of $b(\cdot,\cdot)$, Fubini's theorem and the assumption on $a(\cdot,\cdot)$ in (2.1.2) implies for all $u \in L^2_{\mathbb{P}}(\Omega; H^1_0(D))$, we have

$$\begin{aligned}
b(u,u) &= \int_\Omega \int_D a(\omega,x)\nabla u(\omega,x)\cdot\nabla u(\omega,x)\,\mathrm{d}x\,\mathrm{d}\mathbb{P}(\omega)\\
&= \int_{\Omega\times D} a(\omega,x)\nabla u(\omega,x)\cdot\nabla u(\omega,x)\,\mathrm{d}x\otimes\mathrm{d}\mathbb{P}(\omega)\\
&\geq \underline{\alpha}\int_{\Omega\times D} \nabla u(\omega,x)\cdot\nabla u(\omega,x)\,\mathrm{d}x\otimes\mathrm{d}\mathbb{P}(\omega)\\
&= \underline{\alpha}\int_\Omega \|\nabla u(\omega,\cdot)\|^2_{L^2(D)}\,\mathrm{d}\mathbb{P}(\omega)\\
&= \underline{\alpha}\int_\Omega \frac{1}{2}\|\nabla u(\omega,\cdot)\|^2_{L^2(D)} + \frac{1}{2}\|\nabla u(\omega,\cdot)\|^2_{L^2(D)}\,\mathrm{d}\mathbb{P}(\omega).
\end{aligned}$$

Finally, by applying Poincaré's inequality,

$$\begin{aligned}
b(u,u) &\geq \underline{\alpha}\int_\Omega \frac{1}{2c_P}\|u(\omega,\cdot)\|^2_{L^2(D)} + \frac{1}{2}\|\nabla u(\omega,\cdot)\|^2_{L^2(D)}\,\mathrm{d}\mathbb{P}(\omega)\\
&\geq \frac{\underline{\alpha}}{2}\min\left\{1,\frac{1}{c_P}\right\}\int_\Omega \|u(\omega,\cdot)\|^2_{H^1(D)}\,\mathrm{d}\mathbb{P}(\omega)\\
&= \frac{\underline{\alpha}}{2}\min\left\{1,\frac{1}{c_P}\right\}\|u(\cdot,\cdot)\|^2_{L^2_{\mathbb{P}}(\Omega;H^1(D))}. \qquad (2.1.12)
\end{aligned}$$

□

The sufficient condition on $B(\cdot)z$ and f for g to be a linear form reads as:

Lemma 2.3. *If (i), (ii), (iv) and (v) in Assumption 2.1 hold, then the linear functional $g(\cdot)$ defined in* (2.1.8) *is continuous.*

Proof. Due to the Gelfand triple

$$H^1_0(D) \overset{d}{\hookrightarrow} L^2(D) \overset{d}{\hookrightarrow} H^{-1}(D),$$

we know that $f \in L^2_{\mathbb{P}}(\Omega; H^{-1}(D))$ and $B(\cdot)z \in L^2_{\mathbb{P}}(\Omega; H^{-1}(D))$. Hölder's inequality for Bochner spaces implies

$$\begin{aligned}
|g(v)| &= \Big|\int_\Omega \langle f(\omega,\cdot) + B(\omega)z(\cdot), v(\omega,\cdot)\rangle_{H^{-1}(D)\times H^1_0(D)}\,\mathrm{d}\mathbb{P}(\omega)\Big|\\
&\leq \|f(\cdot,\cdot) + B(\cdot)z(\cdot)\|_{L^2_{\mathbb{P}}(\Omega;H^{-1}(D))}\|v(\cdot,\cdot)\|_{L^2_{\mathbb{P}}(\Omega;H^1_0(D))}\\
&\leq \left(\|f(\cdot,\cdot)\|_{L^2_{\mathbb{P}}(\Omega;H^{-1}(D))} + \|B(\cdot)z(\cdot)\|_{L^2_{\mathbb{P}}(\Omega;H^{-1}(D))}\right)\|v(\cdot,\cdot)\|_{L^2_{\mathbb{P}}(\Omega;H^1_0(D))}
\end{aligned}$$

for all test function $v \in L^2_{\mathbb{P}}(\Omega; H^1_0(D))$. □

Thus, the linear functional $g(\cdot)$ is a duality pairing between $B(\cdot)z$ and f and a test function. A straightforward application of the Lax-Milgram lemma [5, 6.2 Lax-Milgram theorem] allows one to state the well posedness of problem (2.1.3) in the following proposition.

Proposition 2.4. *Let Assumption 2.1 hold. If the bilinear form $b(\cdot,\cdot)$ and the linear functional $g(\cdot)$ are defined by* (2.1.7) *and* (2.1.8)*, respectively, then the variational problem* (2.1.6) *has a unique weak solution $u = \mathbf{A}^{-1}(\mathbf{B}z + f) \in L^2_{\mathbb{P}}(\Omega; H^1_0(D))$ and*

$$\|u\|_{L^2_{\mathbb{P}}(\Omega;H^1_0(D))} \le \frac{1}{\underline{\alpha}}\left(\|\mathbf{B}z\|_{L^2_{\mathbb{P}}(\Omega;H^{-1}(D))} + \|f\|_{L^2_{\mathbb{P}}(\Omega;H^{-1}(D))}\right).$$

Proof. Apply Lax-Milgram theorem to the bilinear form $b(\cdot,\cdot)$ and the linear functional $g(\cdot)$. Due to Lemma 2.2 and Lemma 2.3 the assumptions of Lax-Milgram theorem [5, 4.2 Theorem or 6.2 Lax-Milgram theorem] are satisfied. The estimation follows from $u = \mathbf{A}^{-1}(\mathbf{B}z + f)$ and the fact that $\|\mathbf{A}^{-1}\|_{L^2_{\mathbb{P}}(\Omega;H^{-1}(D))\to L^2_{\mathbb{P}}(\Omega;H^1_0(D))} \le \frac{1}{\underline{\alpha}}$. □

Note that under the Assumption 2.1, the parameter-dependent weak problem (2.1.5) has a unique weak solution $u_\omega \in H^1_0(D)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.

In order to prove the equivalence between the parameter-dependent weak form (2.1.5) and the fully weak form (2.1.6), we will make use of Filippov's Theorem. For the reader's convenience, we state Filippov's Theorem here.

Theorem 2.5. *[11, Theorem 8.2.10] Let (T,Σ,μ) be a complete σ-finite measure space and let X, Y be complete separable metric spaces. Additionally, suppose $g : T \times X \to Y$ is a Carathéodory function, i.e., $g(t,\cdot)$ is continuous for a.e. $t \in T$ and $g(\cdot,x)$ is measurable for all $x \in X$. Finally, let $\Gamma : T \rightrightarrows X$ be a measurable multifunction with nonempty closed images. Then for any measurable function $h : T \to Y$ satisfying $h(t) \in g(t,\Gamma(t))$ for a.a. $t \in T$, there exists a measurable function $\gamma : T \to X$ satisfying $\gamma(t) \in \Gamma(t)$ for all $t \in T$ and $h(t) = g(t,\gamma(t))$ for a.a. $t \in T$.*

In what follows, we will use Filippov's Theorem to prove measurability of our parameter-dependent PDE solution.

Proposition 2.6. *Let Assumption 2.1 hold. The parameter-dependent weak form* (2.1.5) *and the fully weak form* (2.1.6) *are equivalent.*

Proof. Let u solve (2.1.6). Recall that

$$L^2_{\mathbb{P}}(\Omega; H^1_0(D)) \cong L^2_{\mathbb{P}}(\Omega) \otimes H^1_0(D),$$

here the space $L^2_{\mathbb{P}}(\Omega) \otimes H^1_0(D)$ is defined by the completion of the tensor product in the projective norm (see for example [41, Section 4]). Then we use $v(x,\omega) = (\chi_A \otimes \varphi)(\omega,x) = \chi_A(\omega)\varphi(x)$ such that $A \in \mathcal{F}$ and $\varphi \in H^1_0(D)$ (or $\varphi \in C^\infty_0(D)$) as test function and we have

$$\int_A\int_D a(\omega,x)\nabla u(\omega,x)\cdot\nabla\varphi(x)\,\mathrm{d}x\mathrm{d}\mathbb{P}(\omega) = \int_A\int_D ((B(\omega)z)(x) + f(\omega,x))\varphi(x)\,\mathrm{d}x\mathrm{d}\mathbb{P}(\omega),$$

for every $\varphi \in H_0^1(D)$ and consequently due to the fundamental lemma of calculus of variation

$$\int_D a(\omega,x)\nabla u(\omega,x)\cdot\nabla\varphi(x)\,\mathrm{d}x = \int_D ((B(\omega)z)(x)+f(\omega,x))\varphi(x)\,\mathrm{d}x, \quad \mathbb{P}-a.s.$$

for every $\varphi \in H_0^1(D)$.

The reverse direction from $\mathbb{P}$-pointwise weak solutions to a solution of (2.1.6) is adapted from the nonlinear setting in [65, Theorem 2.1]. For $\mathbb{P}$-a.e. $\omega \in \Omega$, the Lax-Milgram theorem ensures that $A(\omega)^{-1}(B(\omega)z+f(\omega)) \in H_0^1(D)$ exists and is unique for all $z \in L^2(D)$. Note that $\omega \mapsto A(\omega)u$ and $\omega \mapsto B(\omega)z+f(\omega)$ are measurable for all $u \in H_0^1(D)$ and $z \in L^2(D)$, and $u \mapsto A(\omega)u$ is continuous from $H_0^1(D)$ into $H^{-1}(D)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. In order to prove measurability of the PDE solution as mapping $A(\cdot)^{-1}(B(\cdot)z+f(\cdot)) : \Omega \to H_0^1(D)$, we want to apply Filippov's Theorem. We appeal to the formalism in Theorem 2.5. Here, we replace (T,Σ,μ) by $(\Omega,\mathcal{F},\mathbb{P})$ and set $g(\omega,u)=A(\omega)u$, $\Gamma : \Omega \rightrightarrows H_0^1(D)$ defined by $\Gamma(\omega)=H_0^1(D)$ for all $\omega \in \Omega$ and $h(\omega)=B(\omega)z+f(\omega)$. Clearly, it holds that $h(\omega) \in g(\omega,\Gamma(\omega))$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Then, an application of Filippov's Theorem ensures that there exists a measurable function $\gamma : \Omega \to H_0^1(D)$ such that $\gamma(\omega) \in \Gamma(\omega)$ and $h(\omega) = g(\omega,\gamma(\omega))$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Thus, we have $\gamma = \mathbf{A}^{-1}(\mathbf{B}z+f)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $A(\cdot)^{-1}(B(\cdot)z+f(\cdot))$ is measurable. We have to prove that the $\mathbb{P}$-pointwise weak solution are Bochner integrable. The assumption (iii) in Assumption 2.1 on the diffusion coefficient ensures

$$\begin{aligned}\|A(\cdot)^{-1}(B(\cdot)z+f)\|^2_{H_0^1(D)} &\le \frac{1}{\underline{\alpha}}\left\langle B(\cdot)z+f(\cdot), A(\cdot)^{-1}(B(\cdot)z+f(\cdot))\right\rangle_{H^{-1}(D),H_0^1(D)} \\ &\le \frac{1}{\underline{\alpha}}\|B(\cdot)z+f(\cdot)\|_{H^{-1}(D)}\|A(\cdot)^{-1}(B(\cdot)z+f(\cdot))\|_{H_0^1(D)}\end{aligned}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$. Therefore, we have that

$$\|A(\cdot)^{-1}(B(\cdot)z+f(\cdot))\|^2_{H_0^1(D)} \le \frac{1}{\underline{\alpha}^2}\|B(\cdot)z+f(\cdot)\|^2_{H^{-1}(D)}. \tag{2.1.13}$$

Taking the expectation of (2.1.13) yields

$$\int_\Omega \|A(\omega)^{-1}(B(\omega)z+f(\omega))\|^2_{H_0^1(D)}\mathrm{d}\mathbb{P}(\omega) \le \frac{1}{\underline{\alpha}^2}\int_\Omega \|B(\omega)z+f(\omega)\|^2_{H^{-1}(D)}\mathrm{d}\mathbb{P}(\omega).$$

Thus, $\|\mathbf{A}^{-1}(\mathbf{B}z+f)\|_{L^2_{\mathbb{P}}(\Omega;H_0^1(D))} < \infty$. Finally, applying [43, Theorem 3.7.4] ensures that $\mathbf{A}^{-1}(\mathbf{B}z+f) \in L^2_{\mathbb{P}}(\Omega; H_0^1(D))$. □

The preceding result justify that we will either work with (2.1.11) or (2.1.9) throughout the thesis.

2.1.3 Higher Regularity

In this section we discuss under which assumption the weak solution of (2.1.6) belongs to the spaces $\in L^\infty_{\mathbb{P}}(\Omega; H^1_0(D) \cap H^2(D))$ and $C(\Omega; H^2(D) \cap H^1_0(D))$.

We gather the required properties on (the coefficients of) the involved operators, the right hand side of the problem and the domain in the following assumption.

Assumption 2.7. (i) Let $D \subset \mathbb{R}^d$ be a bounded and open subset whose boundary ∂D is of class $C^{1,1}$.

or

(ii) Let $D \subset \mathbb{R}^d$ be a convex, bounded and open with Lipschitz boundary.

(iii) Let $a \in L^\infty_{\mathbb{P}}(\Omega; C^{0,1}(\bar{D}))$.

(iv) Let $f \in L^\infty_{\mathbb{P}}(\Omega; L^2(D))$.

(v) For all $z \in L^2(D)$ let $B(\cdot)z \in L^\infty_{\mathbb{P}}(\Omega; L^2(D))$.

Assumption 2.7 rads as follows, it holds either (i), (iii), (iv) and (v) or (ii), (iii), (iv) and (v).

First, we look at the spatial regularity. For $\mathbb{P}$-a.e. $\omega \in \Omega$, there are two results of the existence and uniqueness of a weak solution $u(\omega, \cdot) \in H^1_0(D)$ of (2.1.5) satisfying $u(\omega, \cdot) \in H^2(\Omega)$. This can be obtained by smoothness of the boundary or when ∂D is nonsmooth, by assuming that D is a convex polyhedron.

Proposition 2.8. *If in addition to Assumption 2.1, (i) in Assumption 2.7 holds, then there exists a unique solution $u(\omega, \cdot) \in H^1_0(D)$ of* (2.1.5) *satisfying $u(\omega, \cdot) \in H^2(D)$ and*

$$\|u(\omega, \cdot)\|_{H^2(D)} \leq C(\omega) \left(\|u(\omega, \cdot)\|_{H^1_0(D)} + \|B(\omega)z\|_{L^2(D)} + \|f(\omega, \cdot)\|_{L^2(D)} \right), \quad (2.1.14)$$

where $C(\omega) = C\left(\partial D, d, \underline{\alpha}, \|a(\omega, \cdot)\|_{C^{0,1}(\bar{D}^{d \times d})}\right)$.

The assertion of the regularity result in the interior of the domain $u(\omega, \cdot) \in H^1_0(D) \cap H^2_{loc}(D)$ follows from Friedrich's Theorem as for instance formulated in [5, A12.2 Theorem]. The smoothness assumption on the boundary yields the higher boundary smoothness of the weak solution in the neighborhood of the boundary. If ∂D is sufficiently smooth then the additional regularity of the weak solution holds up to the boundary.

For convex domains the result reads as

Proposition 2.9. *If in addition to Assumption 2.1 and (ii) in Assumption 2.7 holds, then there exists a unique weak solution $u(\omega, \cdot) \in H^2(D) \cap H^1_0(D)$ of* (2.1.5) *and*

$$\|u(\omega, \cdot)\|_{H^2(D)} \leq C(\omega) \left(\|u(\omega, \cdot)\|_{H^1_0(D)} + \|B(\omega)z\|_{L^2(D)} + \|f(\omega, \cdot)\|_{L^2(D)} \right), \quad (2.1.15)$$

where $C(\omega) = C\left(D, d, \underline{\alpha}, \|a(\omega, \cdot)\|_{C^{0,1}(\bar{D}^{d \times d})}\right)$.

The proof is based on the possibility of approximating a given convex domain D from the inside by a domain with smoother boundary which is a result from Eggleston [23].

Remark 2.10. For further details on these well-known regularity results, we refer to [38, Thm. 3.2.1.2], [5, A12.3 Theorem] and especially to [38, Thm. 3.1.3.3, Lem. 3.1.3.2, Thm. 3.1.3.1] for the estimation bounds. Note that Proposition 2.8 is an improvement of the original result of [36, Theorem 8.12] where the boundary ∂D has to be of class C^2.

Now, we state the Bochner space regularity result.

Proposition 2.11. *Let Assumption 2.1 and Assumption 2.7 hold. For any $z \in L^2(D)$, there exists a unique solution $u \in L^\infty_{\mathbb{P}}(\Omega; H^2(D) \cap H^1_0(D))$ of (2.1.6). Moreover, $u \in L^\infty_{\mathbb{P}}(\Omega; H^2(D) \cap H^1_0(D))$ and the following a priori bound holds*

$$\|u\|_{L^\infty_{\mathbb{P}}(\Omega;H^2(D)\cap H^1_0(D))} \leq C \left(\|f\|_{L^\infty_{\mathbb{P}}(\Omega;L^2(D))} + \|B(\cdot)z\|_{L^\infty_{\mathbb{P}}(\Omega;L^2(D))}\right) \tag{2.1.16}$$

Here, C is independent of ω.

Proof. For $\mathbb{P}$-a.e. $\omega \in \Omega$ we have (2.1.14) and (2.1.15), where the same estimate $C(\omega) = C\left(\partial D, d, \underline{\alpha}, \|A(\cdot,\omega)\|_{C^{0,1}(\bar{D})}\right)$ holds when ∂D is smooth or D is a convex polyhedron. The "constant" $C(\omega)$ is indeed a bounded and measurable function in ω. This follows from the fact that the term $\|a(\cdot,\omega)\|_{C^{0,1}(\bar{D})}$ is measurable, uniformly bounded away from zero, and appears in $C(\omega)$ either in the numerator or denominator of scaling constant in the upper bound. Continuing, for $\mathbb{P}$-a.e. $\omega \in \Omega$, we have

$$\begin{aligned}
\|u(\omega,\cdot)\|_{H^2(D)\cap H^1_0(D)} &= \max\left\{\|u(\omega,\cdot)\|_{H^2(D)}, \|u(\omega,\cdot)\|_{H^1_0(D)}\right\} \\
&< \|u(\omega,\cdot)\|_{H^2(D)} + \|u(\omega,\cdot)\|_{H^1_0(D)} \\
&\leq C(\omega)\left(\|u(\omega,\cdot)\|_{H^1_0(D)} + \|B(\omega)z\|_{L^2(D)} + \|f(\omega,\cdot)\|_{L^2(D)}\right) \\
&\quad + C_1\left(\|B(\omega)z\|_{H^{-1}(D)} + \|f(\omega,\cdot)\|_{H^{-1}(D)}\right).
\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}
\|u(\omega,\cdot)\|_{H^2(D)\cap H^1_0(D)} &\leq C(\omega)C_1\left(\|B(\omega)z\|_{H^{-1}(D)} + \|f(\omega,\cdot)\|_{H^{-1}(D)}\right) \\
&\quad + C(\omega)\left(\|B(\omega)z\|_{L^2(D)} + \|f(\omega,\cdot)\|_{L^2(D)}\right) \\
&\quad + C_1\left(\|B(\omega)z\|_{H^{-1}(D)} + \|f(\omega,\cdot)\|_{H^{-1}(D)}\right).
\end{aligned}$$

Finally, due to the Gelfand triple $H^1_0(D) \hookrightarrow L^2(D) \hookrightarrow H^{-1}(D)$, we have

$$\|u(\omega,\cdot)\|_{H^2(D)\cap H^1_0(D)} \leq \widehat{C}(\omega)\left(\|B(\omega)z\|_{L^2(D)} + \|f(\omega,\cdot)\|_{L^2(D)}\right),$$

where

$$\widehat{C}(\omega) := 3\max\{C(\omega), C_1, C_{\text{emb}}\}^3$$

and C_{emb} is the embedding constant for $L^2(D)$ into $H^{-1}(D)$. Passing to the $\mathbb{P}$-essential supremum yields

$$\operatorname*{ess\,sup}_{\omega\in\Omega}\|u(\omega,\cdot)\|_{H^2(D)\cap H_0^1(D)} \precsim \operatorname*{ess\,sup}_{\omega\in\Omega}\|B(\omega)z\|_{L^2(D)}+\operatorname*{ess\,sup}_{\omega\in\Omega}\|f(\omega,\cdot)\|_{L^2(D)} < \infty.$$

Thus, $u \in L^\infty_{\mathbb{P}}(\Omega; H_0^1(D) \cap H^2(D))$ and (2.1.16) follows. □

The current regularity assumptions on the random inputs only provide essential boundedness, we well need more parametric regularity of the solutions.

Assumption 2.12 (Higher Parametric Regularity)**.** The set Ω is a compact Polish space. For any $z \in L^2(D)$, there exists a unique solution $u \in C(\Omega; H^2(D) \cap H_0^1(D))$. Then, the solution mapping is a continuous affine mapping from $L^2(D)^N$ into $C(\Omega; H_0^1(D) \cap H^2(D))$.

The need for Ω to be a compact Polish space will be evident in the course of the thesis and we will highlight this in the relevant place. The continuity assumption in ω can be guaranteed under mild assumptions on $A(x,\omega)$, $B(\omega)$ and $f(x,\omega)$. For proof we refer to the results in [53, Section 6]. The main idea behind these results is to embed the random PDE inside a parametric fixed point equation and apply classic results on parametric dependence of solutions to fixed point equations.

2.2 Risk Measures

In this section, we discuss the so-called risk measures $\mathcal{R}$ with its help we will incorporate risk preferences of the players. For more details concerning risk measures, we refer to [7], [93], [28] and [86, chapter 6].

We first recall some basics of convex analysis, for this let $\mathcal{X} = L^p_{\mathbb{P}}(\Omega)$ with $p \in [1,\infty)$.

Definition 2.13. Let $F : \mathcal{X} \to \bar{\mathbb{R}}$ and let $X, Y \in \mathcal{X}$.

(i) F is proper if $F(X) > -\infty$ for all $X \in \mathcal{X}$ and the domain of F is given by

$$\mathit{dom}(F) = \{X \in \mathcal{X} \ : \ F(X) < \infty\} \neq \emptyset.$$

(ii) F is lower semicontinuous or closed if its epigraph

$$\mathit{epi}\ F = \{(X,\alpha) \in \mathcal{X} \times \mathbb{R} \ : \ F(X) \leq \alpha\}$$

is closed in the product topology on $\mathcal{X} \times \mathbb{R}$.

(iii) The conjugate function or Legendre-Fenchel conjugate of F is the mapping $F^* : \mathcal{X}^* \to \bar{\mathbb{R}}$ given by

$$F^*(\vartheta) = \sup_{X \in \mathcal{X}^*} \{\mathbb{E}[X\vartheta] - F(X)\}.$$

(iv) The conjugate of F^* (the biconjugate of F) is the mapping $F^{**} : \mathcal{X} \to \bar{\mathbb{R}}$ defined as

$$F^{**}(X) = \sup_{\vartheta \in \mathcal{X}^*} \{\mathbb{E}[X\vartheta] - F^*(\vartheta)\}.$$

(v) If F is convex, then its subdifferential at X is

$$\partial F(X) = \{(\vartheta \in \mathcal{X} \ : \ F(Y) - F(X) \geq \mathbb{E}[\vartheta(Y - X)] \quad \forall Y \in \mathcal{X}\}.$$

The axiomatic definition of coherent measure of risks reads as follows.

Definition 2.14. A functional $\mathcal{R} : \mathcal{X} \to \bar{\mathbb{R}}$ is said to be a coherent measure of risk, provided the following conditions are satisfied:

(R1) Convexity:

$$\mathcal{R}(tX + (1-t)Y) \leq t\mathcal{R}(X) + (1-t)\mathcal{R}(Y)$$

s for all $X, Y \in \mathcal{X}$ and all $T \in [0, 1]$.

(R2) Monotonicity: If $X, Y \in \mathcal{X}$ and $X \leq Y$ $\mathbb{P}$-a.e., then $\mathcal{R}(X) \leq \mathcal{R}(Y)$.

(R3) Translation Equivariance: If $a \in \mathbb{R}$ and $X \in \mathcal{X}$, then $\mathcal{R}(X + a) = \mathcal{R}(X) + a$.

(R4) Positive Homogeneity: If $t > 0$ and $X \in \mathcal{X}$, then $\mathcal{R}(tX) = t\mathcal{R}(X)$.

A particular popular coherent risk measure is the conditional value-at-risk (CVaR) (cf. [86, Example 6.15]). CVaR_{β_i} for a confidence level $\beta_i \in (0, 1)$ formulated as scalar optimization problem is given by

$$\mathrm{CVaR}_{\beta_i}[X] = \inf_{t \in \mathbb{R}} \left\{ t + \frac{1}{1 - \beta_i} \mathbb{E}_{\mathbb{P}}\left[(X - t)_+\right] \right\}, \tag{2.2.1}$$

where $(\cdot)_+$ denotes the pointwise maximum $\max\{0, \cdot\}$. CVaR_{β_i} for a confidence level $\beta_i \in (0, 1)$ builds on the concept of the the value-at-risk (VaR) which is defined as the β-quantile of a random variable. CVaR_{β_i} is the expected value of the β-tail distribution.

The above axioms for coherent risk measures ensure that numerous desirable properties hold.

Lemma 2.15. *Suppose* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a complete probability space. Let* $\mathcal{X} = L^p_{\mathbb{P}}(\Omega)$ *with* $p \in [1, \infty)$ *and let* $\mathcal{R} : \mathcal{X} \to \bar{\mathbb{R}}$ *be a coherent measure of risk. Moreover, assume that* $\mathcal{R}$ *is either finite on* $\mathcal{X}$ *or* $\operatorname{int} dom(\mathcal{R}) \neq \emptyset$*. Then* $\mathcal{R}$ *has the following properties:*

(i) $\mathcal{R}$ *is continuous (and lower semicontinuous).*

(ii) $\mathcal{R}$ *is subdifferentiable on* $dom((R))$*.*

(iii) $\mathcal{R}$ *is Hadamard directional differentiable with directional derivative at* $Z \in \mathcal{X}$ *in direction* $H \in \mathcal{X}$ *and the derivative is given by*

$$\mathcal{R}'[Z; H] = \sup_{\vartheta \in \partial \mathcal{R}(Z)} \mathbb{E}_{\mathbb{P}}[\vartheta H].$$

Proof. (i) If $\mathcal{R}$ satisfies (R1) and (R2), then it is continuous (cf. [86, Prop. 6.5.] for finite dimension).

(ii) Since $\mathcal{X} \subset L^1_{\mathbb{P}}(\Omega)$, we have that $\mathcal{R}$ is finite valued. Together with properties (R1) and (R2), we have that $\mathcal{R}$ is subdifferentiable.

(iii) Since $\mathcal{R}$ is continuous at Z, [13, Proposition 2.126(v)] implies the assertion. □

In general, it holds that $\mathcal{R}^{**} \leq \mathcal{R}$. But under certain conditions, we have a dual characterization of measures of risk.

Corollary 2.16. *If* $\mathcal{R} : \mathcal{X} \to \bar{\mathbb{R}}$ *is convex, proper and lower semicontinuous, then* $\mathcal{R}^{**} = \mathcal{R}$*, i.e.* $\mathcal{R}$ *has the representation*

$$\mathcal{R}(X) = \sup_{\vartheta \in \mathcal{X}^*} \left\{ \mathbb{E}[\vartheta X] - \mathcal{R}^*(\vartheta) \right\}.$$

Proof. This is a direct consequence of the Fenchel-Moreau Theorem ([86, Theorem 7.71]). □

Note if $\mathcal{R} : \mathcal{X} \to \bar{\mathbb{R}}$ is convex, proper and lower semicontinuous, then the conjugate function $\mathcal{R}^*$ is proper.

Theorem 2.17. *[86, Theorem 6.4.] Suppose that* $\mathcal{R} : \mathcal{X} \to \bar{\mathbb{R}}$ *is convex, proper an lower semicontinuous. Then the representation*

$$\mathcal{R}(X) = \sup_{\vartheta \in \mathfrak{U}} \{ \mathbb{E}[\vartheta X] - \mathcal{R}^*(\vartheta) \} \tag{2.2.2}$$

holds with $\mathfrak{U} = dom(\mathcal{R}^*) \subset \mathcal{X}^*$*. Moreover, we have that:*

(i) Condition (R2) holds if and only if every $\vartheta \in \mathfrak{U}$ *is nonnegative, i.e.* $\vartheta(\omega) \geq 0$ *for a.e.* $\omega \in \Omega$*.*

(ii) Condition (R3) holds if and only if $\mathbb{E}[\vartheta] = 1$ *for every* $\vartheta \in \mathfrak{U}$.

(iii) Condition (R4) holds if and only if $\mathcal{R}$ *is the support function of the set* $\mathfrak{U}$*, i.e. can be represented in the form*

$$\mathcal{R}(X) = \sup_{\vartheta \in \mathfrak{U}} \mathbb{E}[\vartheta X] \tag{2.2.3}$$

for all $X \in \mathcal{X}$.

There a several useful correspondences between convex sets and convex function. In order to see the dual characterization in the previous theorem, we briefly recall the basics of support functions. Let X be a Banach space. The simplest variant associates each convex set $C \subset X$ with the indicator function $\mathcal{I}_C(\cdot)$ of C, where

$$\mathcal{I}_C(x) = \begin{cases} 0, & x \in C \\ \infty, & x \notin C. \end{cases}$$

The support function $\mathcal{I}_C^*(\cdot)$ of a convex set $C \subset X$ is defined by

$$\mathcal{I}_C^*(x) = \sup \left\{ \langle x, y \rangle_X \ : \ y \in C \right\}.$$

The convexity is clear, since the support function is by definition the pointwise supremum of a certain collection of linear functions, namely the function $\langle \cdot, y \rangle_X$ as y ranges over C (cf. [77, Theorem 5.5]).

Lemma 2.18. *Let* X *be a Banach space. For a closed convex set* $C \subset X$*, it holds the indicator function* $\mathcal{I}_C(\cdot)$ *and the support function* $\mathcal{I}_C^*(\cdot)$ *are conjugate to each other.*

Proof. The conjugate of $\mathcal{I}_C(\cdot)$ is by definition given by

$$\mathcal{I}_C^*(x^*) = \sup_{x \in X} \left\{ \langle x, x^* \rangle - \mathcal{I}_C(x) \right\} = sup_{x \in C} \langle x, x^* \rangle.$$

Then by [77, Theorem 12.2], the conjugate of $\mathcal{I}_C^*(x^*)$ satisfies

$$\left(\mathcal{I}_C^*(\cdot)\right)^* = cl\ \mathcal{I}_C^*(\cdot) = \mathcal{I}_{cl\, C}(\cdot) = \mathcal{I}_C(\cdot),$$

where *cl* denotes the topological closure. □

In particular, the previous theorem says that $\mathcal{I}_C^*(x^*)$ is a lower semicontinuous function of x^*.

Corollary 2.19. *Let* X *be a Banach space.*

(i) Let $f : X \to \bar{\mathbb{R}}$ *be any positively homogeneous convex function which is not identically* $+\infty$*. Then* $cl\ f$ *is the support function of a certain closed convex set* C*, namely*

$$C = \left\{ x^* \in X^* \ : \ \langle x, x^* \rangle \le f(x)\ \forall x \in X \right\}.$$

(ii) *The support functions of the nonempty bounded convex set are the finite positively homogeneous convex functions.*

For proofs, we refer to [77, Corollary 13.2.1 and Corollary 13.2.1], the infinite dimensional result can be easily extended.

Now, we state the proof of Theorem 2.17.

Proof. (i) Now suppose that assumption (R2) holds. It follows that $\mathcal{R}^* = +\infty$ for every $\vartheta \in \mathcal{X}$, which is not nonnegative. Indeed, if $\vartheta \in \mathcal{X}$ is not nonnegative, then there exists a set $\Delta \in \mathcal{F}$ of positive measure such that $\vartheta(\omega) < 0$ for all $\omega \in \Delta$. Consequently, for $\bar{X} = \chi_\Delta \in \mathcal{X}$ we have that $\left\langle \vartheta, \bar{X} \right\rangle < 0$. Take any X in the domain of $\mathcal{R}$, such that $\mathcal{R}(X)$ is finite, and consider $X_t = X - t\bar{X}$. Then for $t \geq 0$, we have $X \geq X_t$ a.e. $\omega \in \Omega$ and assumption (R2) implies that $\mathcal{R}(X) \geq \mathcal{R}(X_t)$. Consequently,

$$\mathcal{R}^*(\vartheta) \geq \sup_{t \in \mathbb{R}_+} \{\mathbb{E}[\vartheta X_t] - \mathcal{R}(X_t)\} \geq \sup_{t \in \mathbb{R}_+} \{\mathbb{E}[\vartheta X] - t\mathbb{E}[\vartheta \bar{X}] - \mathcal{R}(X_t)\} = \infty$$

since $\mathbb{E}[\vartheta \bar{X}] \leq 0$ implies that $-t\mathbb{E}[\vartheta \bar{X}] \geq 0$. Conversely, suppose that $\vartheta \in \mathfrak{U}$ is nonnegative. Then for every $\vartheta \in \mathfrak{U}$ and $X \geq X'$ a.e., we have that $\mathbb{E}[\vartheta X] \geq \mathbb{E}[\vartheta X']$. By (2.2.2), this implies that if $X \geq X'$ a.e., then

$$\begin{aligned}
\mathcal{R}(X) &= \sup_{\vartheta \in \mathfrak{U}} \{\mathbb{E}[\vartheta X] - \mathcal{R}^*(\vartheta)\} \\
&\geq \sup_{\vartheta \in \mathfrak{U}} \{\mathbb{E}[\vartheta X'] - \mathcal{R}^*(\vartheta)\} \\
&= \mathcal{R}(X').
\end{aligned}$$

This completes the proof of assertion (i).

(ii) Now suppose that assumption (R3) holds. Then for every $X \in dom(\mathcal{R})$, due to the linearity of the expected value, we have

$$\begin{aligned}
\mathcal{R}^*(\vartheta) &\geq \sup_{a \in \mathbb{R}} \{\mathbb{E}[\vartheta(X + a)] - \mathcal{R}(x + a)\} \\
&= \sup_{a \in \mathbb{R}} \{\mathbb{E}[\vartheta X] + a\mathbb{E}[\vartheta] - \mathcal{R}(x) - a\} = \infty.
\end{aligned}$$

It follows that $\mathcal{R}^*(\vartheta) \to +\infty$ for $a \to -\infty$ for any $\vartheta \in \mathcal{X}^*$ such that $\mathbb{E}[\vartheta] \neq 1$. Conversely, if $\mathbb{E}[\vartheta] = 1$, then $\mathbb{E}[\vartheta(X + a)] = \mathbb{E}[\vartheta X] + a$, and hence condition (R3) follows by (2.2.2). This completes the proof of (ii).

(iii) Clearly, if (2.2.3) holds, then $\mathcal{R}$ is positively homogeneous, since $\mathfrak{U} = dom(\mathcal{R}^*)$ is a nonempty bounded convex set (see Corollary 2.19 (ii)). Conversely, if $\mathcal{R}$ is positively homogeneous, then its conjugate function is the indicator function of a closed convex subset of $\mathcal{X}^*$ (see Corollary 2.19 (i)) since a finite convex function is necessarily closed (cf. [77, Corollary 7.4.2]).

□

Lastly, we would like to state a result regarding a characterization of the subdifferential of $\mathcal{R}$.

Lemma 2.20. *Let $\mathcal{R} : \mathcal{X} \to \mathbb{R}$ be a coherent measure of risk, then $\mathcal{R}$ is subdifferentiable at any $C \in \mathbb{R}$ and*

$$\partial\mathcal{R}(C) = \mathrm{dom}(\mathcal{R}^*).$$

Proof. Let $\vartheta \in \mathrm{dom}(\mathcal{R}^*)$, i.e. $\mathcal{R}^*(\vartheta) = 0$. It holds that

$$0 = \mathcal{R}^*(\vartheta) = \sup_{X \in \mathcal{X}} \{\mathbb{E}[\vartheta X] - \mathcal{R}(X)\} \geq \mathbb{E}[\vartheta X] - \mathcal{R}(X)$$

For any $C \in \mathbb{R}$, since $\mathbb{E}[\vartheta] = 1$ and (R3) we obtain

$$\begin{aligned} 0 &\geq \mathbb{E}[\vartheta X] - C - \mathcal{R}(X) + C \\ &= \mathbb{E}[\vartheta X] - \mathbb{E}[\vartheta C] - \mathcal{R}(X) + C \\ &= \mathbb{E}[\vartheta X] - \mathbb{E}[\vartheta C] - \mathcal{R}(X - C) \end{aligned}$$

Thus $\mathbb{E}[\vartheta(X - C)] \leq \mathcal{R}(X) - \mathcal{R}(C)$ for all $X \in \mathcal{X}$ and therefore $\vartheta \in \partial\mathcal{R}(C)$. Conversely, suppose $\vartheta \in \partial\mathcal{R}(C)$ for $C \in \mathbb{R}$. In general, the Fenchel-Young inequality says

$$\mathcal{R}(C) + \mathcal{R}^*(\vartheta) \geq \mathbb{E}[\vartheta C].$$

It holds, that equality in the Fenchel-Young inequality characterizes subgradients. Indeed, we have

$$\begin{aligned} \vartheta \in \partial\mathcal{R}(C) &\Leftrightarrow \mathcal{R}(Z) \geq \mathcal{R}(C) + \mathbb{E}[\vartheta(Z - C)] \\ &\Leftrightarrow \mathbb{E}[\vartheta C] - \mathcal{R}(C) \geq \mathbb{E}[\vartheta Z] - \mathcal{R}(Z) \\ &\Leftrightarrow \mathbb{E}[\vartheta C] - \mathcal{R}(C) = \sup_{Z} \{\mathbb{E}[\vartheta Z] - \mathcal{R}(Z)\} \\ &\Leftrightarrow \mathbb{E}[\vartheta C] - \mathcal{R}(C) = \mathcal{R}^*(\vartheta) \\ &\Leftrightarrow \mathcal{R}(C) + \mathcal{R}^*(\vartheta) = \mathbb{E}[\vartheta C] \end{aligned}$$

for all $Z \in \mathcal{X}$ (cf. [24, Proposition 5.1], [25, Proposition III.2.2]). It follows that $\mathcal{R}^*(\vartheta) = \mathbb{E}[\vartheta C] - \mathcal{R}(C)$ is finite, since $\mathcal{R}$ is finite, $\vartheta \in \mathcal{X}^* = L^q_{\mathbb{P}}(\Omega) \subset L^1_{\mathbb{P}}(\Omega)$ and therefore $\mathbb{E}[\vartheta C] < \infty$. Thus, $\vartheta \in \mathrm{dom}(\mathcal{R}^*)$. ☐

This results in the following properties of the domain of $\mathcal{R}^*$.

Corollary 2.21. *Let $\mathcal{R} : \mathcal{X} \to \mathbb{R}$ be a coherent measure of risk, then $\mathrm{dom}(\mathcal{R}^*)$ fulfills the following properties*

(i) $\mathrm{dom}(\mathcal{R}^*)$ *is nonempty.*

(ii) $\operatorname{dom}(\mathcal{R}^*)$ *is convex.*

(iii) $\operatorname{dom}(\mathcal{R}^*)$ *is bounded.*

(iv) $\operatorname{dom}(\mathcal{R}^*)$ *is weakly* closed, more specifically weakly* compact.*

Proof. Since $\partial\mathcal{R}(0) = dom(\mathcal{R}^*)$, the properties are a consequence of the subdifferential. The subdifferential of a convex continuous finite function is nonempty, convex, bounded and weakly* close [24, Proposition 5.2], [25, Proposition III.2.6] and hence the subdifferential is weakly* compact due to Banach-Alaoglu [86, Theorem 7.70]. For more details, we refer to [86, Theorem 7.74]. □

In the literature, $\operatorname{dom}(\mathcal{R}^*)$ is often called risk envelope, which goes back to [79] where the term risk envelope was introduced.

2.3 Problem Formulation

We now introduce a noncooperative game with N players with $N \in \mathbb{N}$. In this framework N player choose feasible strategies z_i which influence a common state u in order to optimize their utility functions $\mathcal{J}_i$ over a common state constraint. Here, in particular, they rather optimize optimize their risk in a certain way. The risk preferences of the players is expressed by a (coherent) risk measure $\mathcal{R}_i : \mathcal{X} \to \bar{\mathbb{R}}$ denotes for $i = 1, \dots N$, where $\mathcal{X}$ is a space of random variables. In the present of uncertainty, the individual ith player solves the following optimization problem

$$\min \mathcal{J}_i(u, (z_i, z_{-i})) = \mathcal{R}_i\left[\frac{1}{2}\|T_i u - u_d^i\|^2_{L^2(D)}\right] + \frac{\nu_i}{2}\|z_i\|^2_{L^2(D)} \tag{2.3.1}$$

subject to

$$\begin{aligned} -\operatorname{div}\left(a(\omega, x)\nabla u(\omega, x)\right) &= B(\omega)z + f(\omega, x) \quad && \text{in } D \\ u(\omega, x) &= 0 && \text{on } \partial D \end{aligned} \tag{2.3.2}$$

plus the pointwise control constraints

$$z_i \in Z^i_{\text{ad}} = \left\{v \in L^2(D) \;:\; a_i(x) \le z_i(x) \le b_i(x) \text{ a.e. } x \in D\right\}. \tag{2.3.3}$$

In this case, we consider a risk-neutral or risk-averse stochastic PDE-constrained Nash equilibrium problem (NEP).

If the constraints include a pointwise state constraint given by

$$u(\omega, x) \ge \psi(\omega, x) \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega, \text{ a.e. } x \in D, \tag{2.3.4}$$

we consider a risk-neutral or risk-averse stochastic PDE-constrained generalized Nash equilibrium problem (GNEP).

Here, we again require Assumption 2.1 for each $i = 1, \dots, N$. In addition, we make the following additional assumptions on the problem data.

Assumption 2.22. (i) Let the cost of control be positive weighted, i.e. $\nu_i > 0$ for $i = 1, \dots, N$.

(ii) The upper and lower bounds on the control a_i and b_i, respectively, are in $L^{2+\sigma}(D)$ for $\sigma > 0$. Furthermore, we assume $a_i < b_i$ and $b_i > 0$ for $i = 1, \dots, N$.

(iii) The function u_d^i indicates the desired state for $i = 1, \dots, N$.

(iv) For the state bound $\psi \in C(\overline{\Omega \times D})$ there exists $\varepsilon > 0$ such that

$$\psi|_{\partial D}(\omega) \leq -\varepsilon \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

(v) The control mapping $B : \Omega \to \mathcal{L}(L^2(D)^N, L^2(D))$ is measurable and essentially bounded, i.e. $B \in L^\infty_{\mathbb{P}}(\Omega; \mathcal{L}(L^2(D)^N, L^2(D))$. Moreover, as a mapping from Ω to $\mathcal{L}(L^2(D), H^{-1}(D))$, B is completely continuous in the sense that for $\mathbb{P}$-a.e. $\omega \in \Omega$ we have

$$z_k \rightharpoonup z \text{ in } L^2(D)^N \text{ then } B(\omega)z \to B(\omega)z \text{ in } H^{-1}(D).$$

(vi) The operator B has the additive representation

$$B(\omega)(z_i, z_{-i}) = B_1(\omega)z_1 + \cdots + B_1(\omega)z_N \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Then, clearly every B_i satisfies (v) for $i = 1, \dots, N$.

Some remarks are in order. It is not necessary for our analysis in Chapter 3 and in Chapter 6 to restrict ourselves to the tracking-type objective. We could also proceed in a more general manner as suggested in [64] under appropriate convexity, continuity, and growth conditions. We choose the cost of control ν_i, the so-called Tikhonov regularization parameter, truly greater than zero in order to proof existence for the GNEP. Clearly, in case of the optimization setting it is unproblematic to set $\nu_i = 0$ since the set of admissible control is bounded. The space of the decision variables $Z_i = L^2(D)$ is a separable, reflexive and real Banach space. The set of admissible decision of each player is given by (2.3.3)

and we set $\mathcal{Z}_{\text{ad}} = Z^1_{\text{ad}} \times \cdots \times Z^N_{\text{ad}}$.

Lemma 2.23. *Let Assumption 2.22 hold. The set of admissible controls Z^i_{ad} as defined in* (2.3.3) *is nonempty, bounded, closed and convex for all $i = 1, \dots, N$.*

Proof. The set Z^i_{ad} is bounded by definition and nonempty, since (ii) in Assumption 2.22 holds. Thus a_i and b_i are contained in Z^i_{ad}. In order to see the convexity, consider $f_1, f_2 \in Z^i_{\text{ad}}$ and $\lambda \in (0, 1)$. By definition $a_i \leq f_1 \leq b_i$ and $a_i \leq f_2 \leq b_i$, hence

$$\lambda a_i \leq \lambda f_1 \leq \lambda b_i \quad \text{and} \quad (1-\lambda)a_i \leq (1-\lambda)f_2 \leq (1-\lambda)b_i.$$

Adding the inequalities yields $a_i \leq \lambda f_1 + (1-\lambda) f_2 \leq b_i$. Now let $\{f_n\}_{n\in\mathbb{N}} \subset Z_{\text{ad}}^i$ be a convergent sequence, i.e. $f_n \to f^*$ in $L^2(D)$. Then there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ that converges pointwise a.e. to f^*, i.e.

$$f_{n_k}(x) \to f^*(x) \quad \text{a.e. } x \in D.$$

Since $a_i \leq f_{n_k} \leq b_i$ for all k and a.e. $x \in D$, the same also applies to the function f^*. Hence, $f^* \in Z_{\text{ad}}^i$. □

Note that $\mathcal{Z}_{\text{ad}}$ defined as a finite Cartesian product of nonempty, bounded, closed and convex sets inherits the properties. Recall that in Section 2.1, we accurately discussed the state equation namely the class of second order linear elliptic PDEs with uncertain data functions and coefficients. With this knowledge we define the linear control-to-state map to be

$$S : L^2(D)^N \to L^2_{\mathbb{P}}(\Omega; H^1_0(D)) \tag{2.3.5}$$

in the weak sense given by

$$S(z_i, z_{-i}) + u_f = \mathbf{A}^{-1}\mathbf{B}(z_i, z_{-i}) + \mathbf{A}^{-1} f. \tag{2.3.6}$$

Here, for $z = 0$ we denote the weak solution of (2.3.2) by u_f. Due to this decomposition in (2.3.6), we move the additional part u_f in the objective functional. Then for $i = 1, \dots, N$ the reduced NEP reads as

$$\min_{z_i \in Z_{\text{ad}}^i} \left\{ \mathbb{E}_{\mathbb{P}} \left[J_i(S(z_i, z_{-i}) + u_f, (z_i, z_{-i})) \right] \right\} \tag{2.3.7}$$

and the reduced GNEP reads as

$$\min_{z_i \in Z_{\text{ad}}^i} \left\{ \mathbb{E}_{\mathbb{P}} \left[J_i(S(z_i, z_{-i}) + u_f, (z_i, z_{-i})) \right] \,\middle|\, S(z_i, z_{-i}) + u_f \geq \psi \text{ for } \quad \text{a.e. } (x, \omega) \in D \times \Omega \right\}. \tag{2.3.8}$$

2.3.1 Properties of the Control-to-State Map

In this section, we gather several essential properties of the control-to-state map.

Lemma 2.24. *Let Assumption 2.1 and Assumption 2.22 hold. Then the linear operators $\mathbf{B}_i$ as a map from $L^2(D)$ into $L^2_{\mathbb{P}}(\Omega; H^{-1}(D))$ and the adjoints $\mathbf{B}_i^*$ as a map from $H^1_0(D)$ into $L^2_{\mathbb{P}}(\Omega; L^2(D))$ are bounded and completely continuous for $i = 1, \dots, N$.*

Proof. For proofs look at [65, Lemma 2.1] and [5, 12.6 Schauder's Theorem]. □

Since $L^2(D)$ and $H_0^1(D)$ are Hilbert spaces as well as reflexive Banach spaces, the completely continuous operators $\mathbf{B}_i$ and $\mathbf{B}_i^*$ are compact for $i = 1, \dots, N$.

We discuss the integrability and regularity of the control-to-state map in the following corollary.

Corollary 2.25. *Under the Assumption 2.1, Assumption 2.7 and Assumption 2.22 we have:*

(i) As a mapping from $L^2(D)$ to $L^q_{\mathbb{P}}(\Omega; H_0^1(D))$ with $q \in [1, \infty)$, the control-to-state map S is completely continuous (or compact), bounded, and linear.

(ii) As a mapping from $L^2(D)$ to $L^\infty_{\mathbb{P}}(\Omega; H_0^1(D) \cap H^2(D)))$, the control-to-state map S is bounded and linear.

Proof. In case (i) complete continuity follows from Lemma 2.24 whereas the rest is a consequence of Proposition 2.4. In case (ii) linearity follows trivially from the definition of $S(z)$ whereas boundedness is a consequence of (2.1.16) and Assumption 2.7. □

Case (i) can also be seen as special case of [65, Prop 2.3]. In case (ii), we exploit existing results on elliptic regularity theory to prove achieve integrability and regularity of the random field solutions.

In what follows, we want to discuss the differentiability of the control-to-state map.

Lemma 2.26. *The control-to-state map $S : L^2(D) \to L^2_{\mathbb{P}}(\Omega; H_0^1(D))$ is continuously Frechet differentiable with respect to z_i and the derivative is given by*

$$S_i'(u_i, u_{-i})h = A^{-1}B_ih$$

for all $h \in L^2(D)$ and $i = 1, \dots, N$.

Proof. Every linear bounded operator is continuously Frechet differentiable and the derivative is the operator itself. □

2.3.2 Properties of the Objective Functions

Now we want to turn our attention on the objective functions. We introduce the following notation

$$\mathcal{R}_i \left[J_i(S(z_i, z_{-i}) + u_f)\right] + \wp(z_i) = \mathcal{R}_i \left[\frac{1}{2}\|T_i(u + u_f) - u_d^i\|^2_{L^2(D)}\right] + \frac{\nu_i}{2}\|z_i\|^2_{L^2(D)}$$

for $i = 1, \dots, N$.

We start with the cost functional $\wp_i(z_i) = \frac{\nu_i}{2}\|z_i\|^2_{L^2(D)}$.

Lemma 2.27. *The cost functional $\wp_i(z_i) = \frac{\nu_i}{2}\|z_i\|^2_{L^2(D)}$ is convex, proper, continuous and continuously Gateaux differentiable, where the the directional derivative is given by $\wp_i'(u_i; h) = \nu_i(z_i, h)_{L^2(D)}$.*

Proof. Since $\frac{\nu_i}{2}\|z_i\|^2_{L^2(D)} = \frac{\nu_i}{2}\left((\cdot)^2 \circ \|\cdot\|_{L^2(D)}\right)(z_i)$ is a composition of a convex, continuous function and a convex, continuous and non-decreasing function, it is also convex and continuous. Now, let $h \in L^2(D)$ then the limit

$$\begin{aligned}
\lim_{t\searrow 0} \frac{1}{t}\left(\wp_i(z_i+th) - \wp_i(z_i)\right) &= \lim_{t\searrow 0} \frac{1}{t}\left(\frac{\nu_i}{2}\|z_i+th\|^2_{L^2(D)} - \frac{\nu_i}{2}\|z_i\|^2_{L^2(D)}\right)\\
&= \lim_{t\searrow 0} \frac{1}{t}\left(\frac{\nu_i t^2}{2}\|h\|^2_{L^2(D)} + \nu_i t(z_i,h)_{L^2(D)}\right)\\
&= \lim_{t\searrow 0} \left(\frac{\nu_i t}{2}\|h\|^2_{L^2(D)} + \nu_i (z_i,h)_{L^2(D)}\right)\\
&= \nu_i(z_i,h)_{L^2(D)}
\end{aligned}$$

exists. Clearly, the directional derivative $\wp_i'(z_i;h) = \nu_i(z_i,h)_{L^2(D)}$ is linear and continuous in h. □

Next, we discuss the properties of the random quantity of interest, which is in fact a superposition operator between Bochner and Lebesgue spaces.

For $i = 1,\ldots,N$, given the mapping

$$\begin{aligned}
&J_i : \Omega \times L^2(D) \to \mathbb{R}\\
&\qquad (\omega, y) \mapsto \frac{1}{2}\|T_i(u + u_f(\omega)) - u_d^i\|^2_{L^2(D)},
\end{aligned} \tag{2.3.9}$$

we define the superposition operator or also called Nemytskij operator by the same name

$$\begin{aligned}
&J_i : L^2_{\mathbb{P}}(\Omega; L^2(D)) \to L^1_{\mathbb{P}}(\Omega)\\
&\qquad y \mapsto \frac{1}{2}\|y(\cdot) + A^{-1}(\cdot)f(\cdot) - y_d^i\|^2_{L^2(D)}.
\end{aligned} \tag{2.3.10}$$

Before we discuss the properties of the superposition operator, we start by making the following assumption.

Assumption 2.28. In the proofs of the following corollaries, we limit ourselves to the case, where the atomic part of the probability measure $\mathbb{P}$ is not considered. This includes e.g. domains equipped with the Lebesgue measure.

First, we show the well-definedness and continuity of the superposition operator.

Corollary 2.29. *The Nemytskij operator J_i in* (2.3.10) *is well-defined and continuous.*

Proof. In order to show the continuity of the Nemytskij operator, we have to show that the mapping J_i in (2.3.9) is a Carathéodory function and fulfills a growth condition (cf. [37, section 2, Theorem 4] and references in [64, Assumption 3.1]). Clearly, $J_i(\omega,\cdot) : L^2(D) \to \mathbb{R}$ is a continuous function for $\mathbb{P}$-a.e. $\omega \in \Omega$ as composition of continuous functions. In order to show that $J_i(\cdot, u) : \Omega \to \mathbb{R}$ is measurable for $u \in L^2(D)$, we set

$$\begin{aligned}\frac{1}{2}\|T_i(u+u_f(\omega)) - u_d^i\|^2_{L^2(D)} &= \left[\frac{1}{2}\|\cdot\|^2_{L^2(D)} \circ \left(\cdot + T_i u - u_d^i\right) \circ T_i u_f(\cdot)\right](\omega) \\ &= [\rho \circ \varrho \circ \varsigma](\omega).\end{aligned}$$

Note that $\rho : L^2(D) \to \mathbb{R}$ is continuous, $\varrho : L^2(D) \to L^2(D)$ is linear and $\varsigma : L^2_{\mathbb{P}}(\Omega; H^1_0(D)) \to H^1_0(D) \subset L^2(D)$ is measurable. Hence, the composition of measurable functions and linear transformation is a $\mathcal{F}$-measurable function. In order to see the growth condition, we observe that

$$\begin{aligned}J_i(\omega, u) &= \frac{1}{2}\|T_i(u+u_f(\omega)) - u_d^i\|^2_{L^2(D)} \\ &= \frac{1}{2}\|T_i u\|^2_{L^2(D)} + (u, T_i u_f(\omega) - u_d^i)_{L^2(D)} + \frac{1}{2}\|T_i u_f(\omega) - u_d^i\|^2_{L^2(D)} \\ &\leq \|T_i u\|^2_{L^2(D)} + \|T_i u_f(\omega) - u_d^i\|^2_{L^2(D)}.\end{aligned}$$

Then we set $\alpha(\omega) = \|T_i u_f(\omega) - u_d^i\|^2_{L^2(D)}$. Since $T_i u_f \in L^2_{\mathbb{P}}(\Omega; L^2(D))$ and $\|T_i u_f\|_{L^2(D)} \in L^2_{\mathbb{P}}(\Omega)$, we have $\alpha(\cdot) \in L^1_{\mathbb{P}}(\Omega)$. This guarantees that the Nemytskij operator $J_i : L^2_{\mathbb{P}}(\Omega; L^2(D)) \to L^1_{\mathbb{P}}(\Omega)$ is well-defined and continuous. This is a well-known result due to Krasnosel'skii. □

Next, we show the differentiability of the superposition operator.

Corollary 2.30. *The Nemytskij operator J_i in* (2.3.10) *is continuously Frechet differentiable and the Frechet derivative $J_i' : L^2_{\mathbb{P}}(\Omega; L^2(D)) \to \mathcal{L}\left(L^2_{\mathbb{P}}(\Omega; L^2(D)); L^1_{\mathbb{P}}(\Omega)\right)$ is given by*

$$J_i'(u; h)(\omega) = (h, T_i^*(T_i u(\omega) + T_i u_f(\omega) - u_d^i))_{L^2(D)}.$$

Proof. Our approach is based on [37, section 3, Theorem 7]. Thus, in order to the differentiability of the Nemytskij operator J_i, we have to show that the mapping J_i in (2.3.9) is Frechet differentiable with respect to $u \in L^2(D)$ and that the Frechet derivative satisfies the Carathéodory condition. Moreover, the Nemytskij operator defined by the Frechet derivative need to be continuous. We have

$$\begin{aligned}
&\lim_{t\searrow 0}\frac{1}{t}\left(J_i(\cdot,u+th)-J_i(\cdot,u)\right)\\
&\quad=\lim_{t\searrow 0}\frac{1}{t}\left(\frac{1}{2}\|T_i(u+th+u_f)-u_d^i\|^2_{L^2(D)}-\frac{1}{2}\|T_i(u+u_f)-u_d^i\|^2_{L^2(D)}\right)\\
&\quad=\lim_{t\searrow 0}\frac{1}{t}\left(\frac{1}{2}\|T_i(u+th)\|^2_{L^2(D)}+(tT_ih,T_iu_f-u_d^i)_{L^2(D)}-\frac{1}{2}\|T_iu\|^2_{L^2(D)}\right)\\
&\quad=\lim_{t\searrow 0}\frac{1}{t}\left((T_iu,tT_ih)_{L^2(D)}+\frac{1}{2}\|tT_ih\|^2_{L^2(D)}+(tT_ih,T_iu_f-u_d^i)_{L^2(D)}\right)\\
&\quad=\lim_{t\searrow 0}\left((T_iy,T_ih)_{L^2(D)}+\frac{t}{2}\|T_ih\|^2_{L^2(D)}+(T_ih,T_iu_f-u_d^i)_{L^2(D)}\right)\\
&\quad=(T_ih,T_iu+T_iu_f-u_d^i)_{L^2(D)}\\
&\quad=(h,T_i^*(T_iu+T_iu_f-u_d^i))_{L^2(D)}
\end{aligned}$$

and the directional derivative $J_i'(u;h)=(h,T_i^*(T_iu+T_iu_f(\cdot)-u_d^i))_{L^2(D)}:\Omega\to\mathbb{R}$ is linear and continuous in h. Moreover, the remainder fulfills $\frac{1}{2}\|T_ih\|^2_{L^2(D)}\in o(\|h\|_{L^2(D)})$, where o denotes the Landau notation. Recall $f\in o(g)$ means f is dominated by g asymptotically, i.e.

$$\lim_{x\to a}\left|\frac{f(x)}{g(x)}\right|.$$

Thus, the mapping J_i in (2.3.9) is continuously Frechet differentiable. The Frechet derivative $J_i'(\cdot,\cdot):\Omega\times L^2(D)\to\mathcal{L}\left(L^2(D);\mathbb{R}\right)$ satisfies the Carathéodory condition. For the continuity, let $\{u_k\}\subset L^2(D)$ with $u_k\to u^*$ in $L^2(D)$, then

$$\begin{aligned}
J_i'(\omega,u_k)-J_i'(\omega,u^*)&=u_k+u_f(\omega)-u_d^i-u^*-u_f(\omega)+u_d^i\\
&=u_k-u^*\to 0\ \text{ for } k\to\infty\ \text{ in } L^2(D).
\end{aligned}$$

Moreover, the Frechet derivative $J_i'(\cdot,u):\Omega\to\mathcal{L}\left(L^2(D);\mathbb{R}\right)$ with $J_i'(\cdot,u)=u+u_f(\cdot)-u_d^i$ is measurable. Let the Nemytskij operator B defined by $B(u)(\omega)=J_i'(\omega,u(\omega))$ from $L^2_{\mathbb{P}}(\Omega;L^2(D))$ into $L^2_{\mathbb{P}}(\Omega;\mathcal{L}(L^2(D);\mathbb{R}))$. The sufficient and necessary condition for B to be continuous is the growth condition for $J_i'(\cdot,\cdot):\Omega\times L^2(D)\to\mathcal{L}\left(L^2(D);\mathbb{R}\right)$ reads as

$$\begin{aligned}
\sup_{h\in L^2(D),\ \|h\|_{L^2(D)}=1}|J_i'(\omega,u)|&=\sup_{h\in L^2(D),\ \|h\|_{L^2(D)}=1}(T_i^*(T_iu+T_iu_f-u_d^i),h)_{L^2(D)}\\
&\le\sup_{h\in L^2(D),\ \|h\|_{L^2(D)}=1}\|T_i^*(T_iu+T_iu_f-u_d^i)\|_{L^2(D)}\|h\|_{L^2(D)}\\
&\le\|T_i^*T_iu\|_{L^2(D)}+\|T_i^*(T_iu_f-u_d^i)|_{L^2(D)}.
\end{aligned}$$

Now, we set $\gamma(\omega)=\|T_i^*(T_iu_f-u_d^i)\|_{L^2(D)}$ and clearly $\gamma\in L^2_{\mathbb{P}}(\Omega)$.
Thus $B:L^2_{\mathbb{P}}(\Omega;L^2(D))\to L^2_{\mathbb{P}}(\Omega;\mathcal{L}(L^2(D);\mathbb{R}))$ with $B(u)=(T_i^*(T_iu(\cdot)+T_iu_f(\cdot)-u_d^i),\cdot)_{L^2(D)}$ is continuous. □

Remark 2.31. Even so, we have restricted the proofs for the sake of comprehensibility to more specific spaces, this does not mean that it is not possible to consider probability measure with an atomic part. For a proof in a very general setting, we refer to [6] as a suitable reference.

Chapter 3

Risk-Neutral PDE-Constrained GNEPs

The results in this section can be obtained from the corresponding preprint [29]. We now introduce the class of noncooperative games with N risk-neutral players in terms of Eqs. (2.3.1) to (2.3.4) The individual ith player is assumed to solve the following optimization problem

$$\begin{aligned} \min \quad & \mathbb{E}_{\mathbb{P}}\left[\frac{1}{2}\|T_i u - u_d^i\|_{L^2(D)}^2\right] + \frac{\nu_i}{2}\|z_i\|_{L^2(D)}^2 \text{ over } (z_i, u) \in Z_{\text{ad}}^i \times L_{\mathbb{P}}^2(\Omega; H_0^1(D)) \\ \text{s.t.} \quad & \boldsymbol{A}(\omega)u = \boldsymbol{B}(\omega)(z_i, z_{-i}) + f(\omega) \qquad \mathbb{P}\text{-a.s.} \\ & \qquad u \geq \psi \qquad\qquad\qquad\qquad\quad \text{for } (\mathcal{L} \times \mathbb{P})\text{-a.e. } (x, \omega) \in D \times \Omega. \end{aligned}$$

We build on Section 2.3 and therefore Assumption 2.1, Assumption 2.7 and Assumption 2.22 are the required assumptions for the data. Considering the risk measure $\mathcal{R}_i$ to be the expected value $\mathbb{E}_{\mathbb{P}}$ is called risk neutral optimization in stochastic programming. A risk neutral player makes decisions that are neither risk-averse nor risk seeking. This means that a risk neutral player is indifferent between choices with equal expectations even if one choice is riskier.

In light of the assumptions, we may also formulate the jointly convex PDE-constrained GNEP in terms of the following reduced space problems. For $i = 1, \ldots, N$, we have

$$\min_{z_i \in Z_{\text{ad}}^i} \{\mathbb{E}_{\mathbb{P}}\left[J_i(S(z_i, z_{-i}) + u_f, (z_i, z_{-i}))\right] \mid S(z_i, z_{-i}) + u_f \geq \psi \text{ for } (\mathcal{L} \times \mathbb{P})\text{-a.e.}\} \quad (3.0.1)$$

where the solution operator S is defined as in (2.3.5) and (2.3.6).

3.1 Existence and Optimality Conditions

We first prove existence of generalized Nash equilibrium for (3.0.1) and then we derive optimality conditions for a certain type of equilibria. We restrict ourselves to the concept of variational equilibria as introduced in Section 1.3.1.

For our GNEP the Nikaido-Isoda function is given by

$$\Psi(z,v) = \sum_{i=1}^{N} E_{\mathbb{P}}\left[J_i(S(z_i,z_{-i})+u_f,(z_i,z_{-i}))\right] - E_{\mathbb{P}}\left[J_i(S(v_i,z_{-i})+u_f,(v_i,z_{-i}))\right].$$

The potentially set-valued collective best-response function to a strategy vector $\widehat{\mathcal{R}}: \mathcal{Z}_{\text{ad}} \rightrightarrows \mathcal{Z}_{\text{ad}}$ is given by

$$\widehat{\mathcal{R}}(z) = \arg\max\left\{\Psi(z,v) \,|\, v \in \mathcal{Z}_{\text{ad}} \text{ such that } S(v_i,v_{-i})+u_f \geq \psi\right\}.$$

Due to this characterization, the proof of the existence of Nash equilibria converts to a fixed point problem. The essential ingredient is the fixed point theorem of Kakutani-Fan-Glicksberg, see e.g. [4, Corollary 17.55].

Theorem 3.1. *Let Assumption 2.1, Assumption 2.7 and Assumption 2.22 hold. The set of variational equilibria of the jointly convex GNEP* (3.0.1) *is weakly compact and nonempty.*

Proof. We proceed as in [51, Theorem 3.2], in order to apply the fixed point theorem of Kakutani-Fan-Glicksberg on $\widehat{\mathcal{R}}$. We introduce the set

$$M = \left\{v \in \mathcal{Z}_{\text{ad}} \text{ such that } S(v_i,v_{-i})+u_f \geq \psi\right\}.$$

Since every objective functional $\mathcal{J}_i$ is weakly lower semicontinuous for $i = 1,\dots,N$ and M is bounded, the direct method of the calculus of variations (cf. [67, Theorem 1.1] and [67, Remark 1.3]) ensures that $\widehat{\mathcal{R}}$ has nonempty values. Finally, due to the convexity, the set of maximizers and by this $\widehat{\mathcal{R}}$ has convex images. To ensure compactness, we recast the problem in the space X_i, where X_i is $L^2(D)$ endowed with the weak topology. Note that X is a real locally convex topological space. The equivalence of weak and strong closure for convex sets in reflexive Banach spaces implies that Z_{ad}^i is closed in X_i. Moreover, the weak compactness of closed and bounded convex subsets in reflexive Banach spaces implies that each set Z_{ad}^i is convex and compact in X_i or equivalently sequentially compact (see [94, Satz VIII.6.1(Satz von Eberlein-Shmulyan)]). Consequently, if we take $\mathcal{Z}_{\text{ad}} = Z_{ad}^1 \times \cdots \times Z_{ad}^N$ and $X = X_1 \times \cdots \times X_N$, then $\mathcal{Z}_{\text{ad}} \subset X$, where $\mathcal{Z}_{\text{ad}}$ is also nonempty, convex and compact in X. Due to the latter property, the weak topology is metrizable on $\mathcal{Z}_{\text{ad}}$ (see [94, Lemma VIII.6.2]).

In order to see the closedness of the graph of $\widehat{\mathcal{R}}$, we consider a closed subspace $C \subset M$ and a sequence $\{z^n\}_{n\in\mathbb{N}} \subset \widehat{\mathcal{R}}^{-1}(C)$ with $z^n \to \bar{z}$ in X (i.e. $z^n \rightharpoonup \bar{z}$ in $L^2(D)^N$). For every z^n we choose $v^n \in C \cap \widehat{\mathcal{R}}(z^n)$. We can show that M is sequentially compact. Hence, there exists a convergent subsequence $v^{n_k} \to \bar{v}$ in X with $\bar{v} \in C$. For some arbitrary $w \in M$ it holds that

$$\sum_{i=1}^{N} E_{\mathbb{P}}\left[J_i(S(v_i^{n_k},z_{-i}^{n_k})+u_f,(v_i^{n_k},z_{-i}^{n_k}))\right] \leq \sum_{i=1}^{N} E_{\mathbb{P}}\left[J_i(S(w_i,z_{-i}^{n_k})+u_f,(w_i,z_{-i}^{n_k}))\right].$$

We can argue that

$$\begin{aligned}
&\sum_{i=1}^{N} E_{\mathbb{P}}[J_i(S(\bar{v}_i, \bar{z}_{-i}) + u_f, (\bar{v}_i, \bar{z}_{-i}))] \\
&\qquad\qquad \leq \liminf_{k\to\infty} \sum_{i=1}^{N} E_{\mathbb{P}}\left[J_i(S(v_i^{n_k}, z_{-i}^{n_k}) + u_f, (v_i^{n_k}, z_{-i}^{n_k}))\right] \\
&\qquad\qquad \leq \liminf_{k\to\infty} \sum_{i=1}^{N} E_{\mathbb{P}}\left[J_i(S(w_i, z_{-i}^{n_k}) + u_f, (w_i, z_{-i}^{n_k}))\right].
\end{aligned}$$

This is a consequence of the properties of the expectation, the objectives J_i and the solution operator S. In particular, it is essential that S is completely continuous into $L^1_{\mathbb{P}}(\Omega; H^1_0(D))$. Using again the complete continuity, we have

$$\liminf_{k\to\infty} \sum_{i=1}^{N} E_{\mathbb{P}}\left[J_i(S(w_i, z_{-i}^{n_k}) + u_f, (w_i, z_{-i}^{n_k}))\right] \leq \sum_{i=1}^{N} E_{\mathbb{P}}\left[J_i(S(w_i, \bar{z}_{-i}) + u_f, (w_i, \bar{z}_{-i}))\right].$$

It follows that $\bar{v} \in \widehat{\mathcal{R}}(\bar{z})$. Thus, $\bar{z} \in \widehat{\mathcal{R}}^{-1}(\bar{v}) \subset \widehat{\mathcal{R}}^{-1}(C)$, which proves the sequential closedness of the graph of $\widehat{\mathcal{R}}$ or equivalently the closedness in X ([94, Theorem B.1.2]). We now apply Kakutani-Fan-Glicksberg's fixed point theorem. The set of Nash equilibria of the GNEP is nonempty and compact in X and thus, weakly compact in $L^2(D)^N$. □

In order to derive first order optimality condition for variational equilibria for (3.0.1), we introduce some notation and rewrite

$$\min_{v_i \in Z^i_{\text{ad}}} \left\{ \mathbb{E}_{\mathbb{P}}\left[J_i(\boldsymbol{A}^{-1}\boldsymbol{B}(v_i, z_{-i}) + u_f, (v_i, z_{-i}))\right] \mid G(v_i, z_{-i}) \in K \right\}$$

and appeal to the general Lagrangian formalism in [13, Chap. 3], where we set

$$G(v_i, v_{-i}) = \iota \boldsymbol{A}^{-1}\boldsymbol{B}(v_i, v_{-i}) + \iota u_f - \psi \text{ and } K = L^\infty_\pi(\Xi)_+.$$

Here $(\Xi, \mathcal{B}, \pi)$ denotes the product (measure) space of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and D equipped with the standard Borel algebra as well as the Lebesgue measure. Further $\iota : L^\infty_{\mathbb{P}}(\Omega; H^1_0(D) \cap H^2(D))) \to L^\infty_\pi(\Xi)$ is the continuous embedding and K is the convex cone of all positive essentially bounded strongly measurable functions. In the definition of ι, we first use the continuous embedding of $L^\infty_{\mathbb{P}}(\Omega; H^1_0(D) \cap H^2(D))$ into $L^\infty_{\mathbb{P}}(\Omega; L^\infty(D))$ and then the continuous embedding of $L^\infty_{\mathbb{P}}(\Omega; L^\infty(D))$ into $L^\infty_\pi(\Xi)$. In the notation of [13], we set

$$F_z(v_i, v_{-i}) = \sum_{i=1}^{N} \mathbb{E}_{\mathbb{P}}\left[J_i(\boldsymbol{A}^{-1}\boldsymbol{B}(v_i, z_{-i}) + u_f, (v_i, z_{-i}))\right],$$

which yields the parametric Lagrangian

$$L_z(v_i, v_{-i}, \mu) = F_z(v_i, v_{-i}) + \langle G(v_i, v_{-i}), \mu \rangle.$$

Now we state the optimality condition for the risk-neutral generalized Nash equilibrium.

Theorem 3.2. *Let Assumption 2.1, Assumption 2.7 and Assumption 2.22 hold. If there exists a* $(z_i^0, z_{-i}^0) \in \mathcal{Z}_{\text{ad}}$ *and a constant* $\kappa > 0$ *such that*

$$S(z_i^0, z_{-i}^0) + u_f - \psi > \kappa \tag{3.1.1}$$

then there exists a measure $\bar{\mu} \in \mathbf{ba}(\Xi, \mathcal{B}, \pi)$ *such that*

*(i) (**Nonpositivity**)* $\bar{\mu}$ *is an element of the polar cone of* $L^\infty_\pi(\Xi)_+$.

*(ii) (**Complementarity**)* $\bar{\mu}$ *fulfills*

$$\int_\Xi G(\bar{z}_i, \bar{z}_{-i})(x,\omega)\, d\bar{\mu}(x,\omega) = 0.$$

*(iii) (**Subgradient Conditions**) For* $i = 1, \ldots, N$ *the general inclusion holds*

$$0 \in \mathbb{E}_\mathbb{P}[\boldsymbol{B}_i^* \boldsymbol{A}^{-*} T_i^* (T_i S(\bar{z}_i, \bar{z}_{-i}) + T_i u_f - u_d^i)] + \nu_i \bar{z}_i + \mathcal{N}_{Z^i_{ad}}(\bar{z}_i) + \boldsymbol{B}_i^* (\boldsymbol{A}^{-*} \iota^* \bar{\mu}).$$

Conversely, if there exists a pair $(\bar{z}, \bar{\mu})$ *such that (i)-(iii) hold, then* $\bar{z}$ *is generalized Nash equilibrium of* (3.0.1).

Proof. We work with the general Lagrangian formalism. We first note that $\bar{z} \in \widehat{\mathcal{R}}(\bar{z})$. This is equivalent to

$$\begin{aligned}
\bar{z} &\in \arg\max_{v \in \mathcal{Z}_{\text{ad}}} \{\Psi(\bar{z}, v) \,|\, G(v_i, \bar{z}_{-i}) \in K\} \\
&= \arg\min_{v \in \mathcal{Z}_{\text{ad}}} \left\{ \sum_{i=1}^N \mathbb{E}_\mathbb{P} \left[J_i(\boldsymbol{A}^{-1}\boldsymbol{B}(v_i, \bar{z}_{-i}) + u_f, (v_i, \bar{z}_{-i})) \right] \,|\, S(v_i, \bar{z}_{-i}) + u_f \geq \psi \right\}
\end{aligned}$$

Since (3.1.1) is equivalent to the constraint qualification $0 \in \operatorname{int}\{G(\mathcal{Z}_{\text{ad}}) - K\}$, it follows from [13, Thm. 3.6] that

$$0 \in \partial_z L_{\bar{z}}(\bar{z}_i, \bar{z}_{-i}, \bar{\mu}) + \mathcal{N}_{\mathcal{Z}_{\text{ad}}}(\bar{z}_i, \bar{z}_{-i}) \text{ and } \bar{\mu} \in \mathcal{N}_K(G(\bar{z}_i, \bar{z}_{-i})).$$

Assertions (i) and (ii) are implied by $\bar{\mu} \in \mathcal{N}_K(G(\bar{z}_i, \bar{z}_{-i}))$ since K is a closed, convex cone. To obtain the subgradient conditions in (iii), we first note that

$$\langle G(\cdot), \bar{\mu}\rangle'(\bar{z}; \delta z) = \langle \iota \boldsymbol{A}^{-1}\boldsymbol{B}(\delta z), \bar{\mu}\rangle.$$

For the objective function, it holds that

$$\begin{aligned}
\partial F_{\bar{z}}(\bar{z}_i, \bar{z}_{-i}) &= \partial \left(\sum_{i=1}^N \mathbb{E}_\mathbb{P} \left[J_i(\boldsymbol{A}^{-1}\boldsymbol{B}((\cdot)_i, \bar{z}_{-i}) + u_f, ((\cdot)_i, \bar{z}_{-i})) \right] \right)(\bar{z}) \\
&= \sum_{i=1}^N \partial \left(\mathbb{E}_\mathbb{P} \left[J_i(\boldsymbol{A}^{-1}\boldsymbol{B}((\cdot)_i, \bar{z}_{-i}) + u_f, ((\cdot)_i, \bar{z}_{-i})) \right] \right)(\bar{z}) \\
&= \prod_{i=1}^N \left(\partial_i \mathbb{E}_\mathbb{P} \left[J_i(\boldsymbol{A}^{-1}\boldsymbol{B}(\cdot, \bar{z}_{-i}) + u_f, (\cdot, \bar{z}_{-i})) \right] \right)(\bar{z}_i).
\end{aligned} \tag{3.1.2}$$

In order to see that the sum of the subdifferentials equals the product in (3.1.2), we refer to the proof of [51, Theorem 3.7]. Analogously to (4.1.2), we can write

$$F'_{\bar{z}}(\bar{z}_i, \bar{z}_{-i})\delta z_i = (\mathbb{E}_{\mathbb{P}}[(\boldsymbol{B}_i^* \boldsymbol{A}^{-*} T_i^* (T_i \boldsymbol{A}^{-1} \boldsymbol{B}(\bar{z}_i, \bar{z}_{-i}) + T_i u_f - u_d^i)] + \nu_i \bar{z}_i, \delta z_i).$$

Moreover, [10, section 4.6] enables us to write the normal cones as

$$\mathcal{N}_{\mathcal{Z}_{\mathrm{ad}}}(\bar{z}_i, \bar{z}_{-i}) = \mathcal{N}_{\prod_{i=1}^N Z_{\mathrm{ad}}^i}(\bar{z}_i, \bar{z}_{-i}) = \prod_{i=1}^N \mathcal{N}_{Z_{\mathrm{ad}}^i}(\bar{z}_i).$$

For $i = 1, \ldots, N$, we have the componentwise subgradient condition

$$0 \in \mathbb{E}_{\mathbb{P}}[\boldsymbol{B}_i^* \boldsymbol{A}^{-*} T_i^* (T_i S(\bar{z}_i, \bar{z}_{-i}) + T_i u_f - u_d^i)] + \nu_i \bar{z}_i + \mathcal{N}_{Z_{ad}^i}(\bar{z}_i) + \boldsymbol{B}_i^* (\boldsymbol{A}^{-*} \iota^* \bar{\mu}).$$

□

With the existing optimality theory for convex optimization problems, we only obtain a measure-valued multiplier $\bar{\mu}$, although the regularity of the random states is relatively high. This is due to the incorporated inequality constraints on the random state. In contrast to time-dependent problems, we do not gain additional compactness problems from energy estimates. This makes even the study of optimality conditions difficult. Structural assumptions can be made that give more regularity in the random parameters, but there is no real way to guarantee boundedness in these spaces when considering perturbations of the control.

3.2 Moreau-Yosida Regularization Technique

In order to solve the problems numerically, we resort to a regularization due to the low multiplier regularity. Following the approach in the studies by Hintermüller and Kunisch [47, 48], we approximate (3.0.1) by replacing the state constraint with a Moreau-Yosida regularization of the associated indicator function as a result the violation of the inequality constraints on the state are incorporated into the objective functional.

3.2.1 Approximation of Risk-Neutral PDE-Constrained GNEPs

For a given $\theta > 0$ and a function $f : V \to \mathbb{R}$ defined on a Hilbert space $(V, (\cdot, \cdot)_V)$, the Moreau envelope or Moreau [68] -Yosida [95] regularization $f_\theta : V \to \bar{\mathbb{R}}$ is given by

$$f_\theta(v) = \inf_{u \in V} \left\{ f(v) + \frac{1}{2\theta} \|u - v\|_V^2 \right\}.$$

The Moreau envelope f_θ is a smoothed or regularized form of f. It is continuously differentiable, even when f is not. Furthermore, the sets of minimizer of f and f_θ coincide.

In our case, we consider $f = \mathcal{I}_C$, where C is given by

$$C = \left\{ v \in L^2_{\mathbb{P}}(\Omega; H^1_0(D)) \,|\, v \geq \psi \text{ a.e } D,\ \mathbb{P}\text{-a.e. } \Omega \right\}.$$

Clearly, the subset C is nonempty, convex and closed. For the Moreau-Yosida regularization of the indicator function $(\mathcal{I}_C)_\theta$ we obtain

$$\begin{aligned}(\mathcal{I}_C)_\theta\,(u) &= \inf\left\{ \mathcal{I}_C(v) + \frac{1}{2\theta}\|u - v\|^2_{L^2_{\mathbb{P}}(\Omega;H^1_0(D))} \,|\, v \in L^2_{\mathbb{P}}(\Omega; H^1_0(D)) \right\} \\ &= \inf\left\{ \frac{1}{2\theta}\|u - v\|^2_{L^2_{\mathbb{P}}(\Omega;H^1_0(D))} \,|\, v \in L^2_{\mathbb{P}}(\Omega; H^1_0(D)),\, v \geq \psi \text{ a.e } D,\ \mathbb{P}\text{-a.e. } \Omega \right\}.\end{aligned}$$

Unfortunately, there exists no explicit form for the minimizer. We will remedy the problem by considering the set

$$\begin{aligned}\bar{C} &= \left\{ v \in L^2_{\mathbb{P}}(\Omega; L^2(D)) \,|\, v \geq \psi \text{ a.e } D,\ \mathbb{P}\text{-a.e. } \Omega \right\} \\ &\cong \left\{ v \in L^2_\pi(\Xi) \,|\, v \geq \psi \ \pi\text{-a.e. } \Xi \right\}.\end{aligned}$$

instead. Then we have

$$(\mathcal{I}_{\bar{C}})_\theta\,(u) = \inf\left\{ \frac{1}{2\theta}\|u - v\|^2_{L^2_\pi(\Xi)} \,|\, v \in L^2_\pi(\Xi),\, v \geq \psi \ \pi\text{-a.e. } \Xi \right\}.$$

The minimizer of $(\mathcal{I}_{\bar{C}})_\theta\,(u)$ is given by $v = u + (\psi - u)_+$, where $(\cdot)_+ = \max(0, \cdot)$ in the pointwise almost everywhere sense. Finally, we have

$$\begin{aligned}(\mathcal{I}_{\bar{C}})_\theta\,(u) &= \frac{1}{2\theta}\|u - u + (\psi - u)_+\|^2_{L^2_\pi(\Xi)} \\ &= \frac{1}{2\theta}\|(\psi - u)_+\|^2_{L^2_\pi(\Xi)}.\end{aligned}$$

For $\gamma = \frac{1}{\theta}$ and $\theta > 0$, the regularized problem of player i in the risk-neutral PDE-constrained GNEP (3.0.1) reads as

$$\min_{z \in Z^i_{\text{ad}}} \left\{ \mathbb{E}_{\mathbb{P}}\left[J_i(S(z_i, z_{-i}) + u_f, (z_i, z_{-i})) + \frac{\gamma}{2}\|(\psi - (S(z_i, z_{-i}) + u_f))_+\|^2_{L^2(D)} \right] \right\}. \quad (3.2.1)$$

We will refer to this γ-dependent game as NEP_γ.

3.2.2 Existence and Optimality Conditions

The proof of the existence of a Nash equilibrium for every γ-dependent NEP_γ follows with almost identical arguments to those in Theorem 3.1, namely the use of the Kakutani-Fan-Glicksberg fixed point theorem for example as shown in [50, Theorem 2.3]. We state the first-order conditions in the following theorem for ease of reference. Note that the form is similar, although it is numerically easier to handle.

Theorem 3.3. *Let Assumption 2.1, Assumption 2.7 and Assumption 2.22 hold. The set of variational equilibria of the jointly convex NEP*$_\gamma$ (3.2.1) *is weakly compact and nonempty. If* $\bar{z}^\gamma \in \mathcal{Z}_{\text{ad}}$ *is a Nash equilibrium, then we have the following necessary and sufficient optimality conditions: For each* $i = 1, \dots, N$ *we have*

$$0 \in \mathbb{E}_{\mathbb{P}}[\boldsymbol{B}_i^* \boldsymbol{A}^{-*} T_i^*(T_i S(\bar{z}_i^\gamma, \bar{z}_{-i}^\gamma) + T_i u_f - u_d^i)] + \nu_i \bar{z}_i^\gamma + \mathcal{N}_{Z_{ad}^i}(\bar{z}_i^\gamma) + \mathbb{E}_{\mathbb{P}}[\boldsymbol{B}_i^*(\boldsymbol{A}^{-*}\bar{\mu}^\gamma)], \quad (3.2.2)$$

where

$$\bar{\mu}^\gamma = -\gamma(\psi - (S(\bar{z}_i^\gamma, \bar{z}_{-i}^\gamma) + u_f))_+.$$

Theorem 3.3 allows us to use the adjoint state techniques.

Therefore we introduce the adjoint variables $\bar{\lambda}_i^\gamma \in L^2_{\mathbb{P}}(\Omega; H_0^1(D))$ for $i = 1, \dots, N$ and the associated parameter-dependent adjoint equations reads as

$$\int_D a(\omega, x) \nabla \bar{\lambda}_i^\gamma(\omega, x) \cdot \nabla \varphi(x) \, \mathrm{d}x = \int_D (T_i^*(T_i \bar{u}^\gamma + T_i u_f - u_d^i) + \bar{\mu}^\gamma) \varphi(x) \, \mathrm{d}x, \quad (3.2.3)$$

$\varphi \in H_0^1(D)$ and $\bar{u} = S(\bar{z}_i^\gamma, \bar{z}_{-i}^\gamma)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Then the fully weak adjoint equation formulated as operator equation reads as

$$\boldsymbol{A}^* \bar{\lambda}_i^\gamma(z_i) = T_i^*(T_i S(z_i, \bar{z}_{-i}^\gamma) - T_i u_d^i) - \bar{\mu}^\gamma,$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$. If we test the adjoint equation with $S(z_i, u_{-i}) - S(u_i, u_{-i})$, we obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[\left\langle \boldsymbol{A}^* \bar{\lambda}_i^\gamma(z_i), S(z_i, \bar{z}_{-i}^\gamma) - S(\bar{z}_i^\gamma, \bar{z}_{-i}^\gamma)\right\rangle\right] &= \mathbb{E}_{\mathbb{P}}\left[\left\langle \boldsymbol{A}^* \bar{\lambda}_i^\gamma(z_i), \boldsymbol{A}^{-1}\boldsymbol{B}_i(z_i - \bar{z}_i^\gamma)\right\rangle\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[\left\langle \bar{\lambda}_i^\gamma(z_i), \boldsymbol{B}_i(z_i - \bar{z}_i^\gamma)\right\rangle\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[\left(\boldsymbol{B}_i^* \bar{\lambda}_i^\gamma(z_i), z_i - \bar{z}_i^\gamma\right)\right].
\end{aligned}$$

This simplifies the formulation of the first order optimality condition (3.2.2) and thus the computation of the gradient to

$$0 \in \mathbb{E}_{\mathbb{P}}[\boldsymbol{B}_i^* \bar{\lambda}_i^\gamma] + \nu_i \bar{z}_i^\gamma + \mathcal{N}_{Z_{ad}^i}(\bar{z}_i^\gamma).$$

3.2.3 Asymptotic Consideration

In this section, we analyze the behavior of the NEP$_\gamma$ in (3.2.1) for $\gamma \to \infty$. This asymptotic consideration is of great interest for theoretical consideration as well as numerical treatment. We closely follow the approach in [50]. In order to guarantee consistency of the relaxed problems, we oblige the fulfillment of a constraint qualification as introduced in [50].

Definition 3.4. We say that (3.0.1) satisfies the strict uniform feasible response constraint qualification (SUFR), if there exists an $\varepsilon > 0$ for all $i = 1, \dots, N$ such that for all $z_{-i} \in \mathcal{Z}_{\text{ad}}^{-i}$ there exists a $v_i \in Z_{\text{ad}}^i$ such that

$$S(v_i, z_{-i}) + u_f \geq \psi + \varepsilon \quad \mathbb{P}\text{-a.s., a.e. } D.$$

So far, mainly in Theorem 3.2, it was sufficient to require the fulfillment of a traditional constraint qualifications, namely the existence of a Slater point. Note that traditional constraint qualifications such as the existence of a Slater point or in nonlinear programming in infinite dimensions Robinson's constraint qualification were developed for optimization problems. They ensure the existence of a Lagrange multiplier and indicate certain stability of the constraint set around the optimal solution. However since for the following argumentation the characteristic of a GNEP can not be left out of consideration, we require a much more robust CQ such as SUFR in order to exhibit the local stability which we need to bound the adjoint states and the constraint multipliers. SUFR in words means, we require that every player has a feasible response to any strategies of their opponents such that together they strictly uniformly fulfill the common state constraint. Finally, we assume that Assumption 2.12 holds, since we will need more regularity of the solutions on the random inputs, as the current regularity assumptions only provide essential boundedness.

We now state the main result of this section. We pass to the limit in the smoothing parameter γ to return to stationary points of the original problem.

Theorem 3.5. *Suppose the GNEP* (3.0.1) *satisfies the Slater condition* (4.1.1) *and SUFR. If in addition Assumption 2.12 holds, then there exist sequences $\gamma_n \to \infty$ and*

- $\{z^{\gamma_n}\}_{n\in\mathbb{N}} \subset L^2(D)^N$,
- $\{u^{\gamma_n}\}_{n\in\mathbb{N}} \subset L^2_{\mathbb{P}}(\Omega; H^1_0(D) \cap H^2(D))$,
- $\{\lambda^{\gamma_n}\}_{n\in\mathbb{N}} \subset L^2_{\mathbb{P}}(\Omega; H^1_0(D) \cap H^2(D))^N$,
- $\{\eta^{\gamma_n}\}_{n\in\mathbb{N}} \subset L^2(D)^N$,

such that $(z_i^{\gamma_n}, u^{\gamma_n}, \lambda_i^{\gamma_n}, \eta_i^{\gamma_n}, \mu^{\gamma_n})$ satisfies (3.2.2) *as stated in Theorem 3.3 for each $i = 1, \dots, N$. This sequence admits a limit point*

$$(z^*, u^*, \Lambda^*, \eta^*, \rho^*) \in L^2(D)^N \times L^2_{\mathbb{P}}(\Omega; H^1_0(D) \cap H^2(D)) \times L^2(D)^N \times L^2(D)^N \times \mathcal{M}(\overline{\Xi}),$$

where, for all $i = 1, \dots, N$, we have

$$z^{\gamma_n} \to z^* \quad \text{in } L^2(D)^N, \tag{3.2.4a}$$
$$u^{\gamma_n} \to u^* \quad \text{in } C(\Omega; H^1_0(D) \cap H^2(D)), \tag{3.2.4b}$$
$$\mu^{\gamma_n} \overset{*}{\rightharpoonup} \rho^* \quad \text{in } L^1_\pi(\Xi)^* \cong \mathcal{M}(\overline{\Xi}), \quad \text{i.e. } \rho \in \mathcal{M}(\overline{\Xi}), \tag{3.2.4c}$$
$$\mathbb{E}_{\mathbb{P}}\left[B_i^* \lambda_i^{\gamma_n}\right] \rightharpoonup \Lambda_i^* \quad \text{in } L^2(D), \tag{3.2.4d}$$
$$\eta_i^{\gamma_n} \rightharpoonup \eta_i^* \quad \text{in } L^2(D). \tag{3.2.4e}$$

Moreover, the limit point satisfies

$$z^* \in \mathcal{Z}_{\text{ad}} \tag{3.2.5a}$$

$$u^* = S(z_i^*, z_{-i}^*) + u_f \text{ and } u^* \geq \psi \tag{3.2.5b}$$

$$(\Lambda_i^*, \varphi) = \left(\mathbb{E}_{\mathbb{P}}\left[\boldsymbol{B}_i^* \boldsymbol{A}^{-*} T_i^*(T_i(u^* + u_f) - u_d^i)\right], \varphi\right) + \int_{\Xi} \boldsymbol{A}^{-1}(\omega)\boldsymbol{B}_i(\omega)\varphi \, d\rho^*(x,\omega) \tag{3.2.5c}$$

$$0 = (\Lambda_i^*, \varphi) + \nu_i (z_i^*, \varphi) + (\eta_i^*, \varphi) \text{ and } \eta_i^* \in \mathcal{N}_{Z_{\text{ad}}^i}(z_i^*) \tag{3.2.5d}$$

for an arbitrary test function $\varphi \in L^2(D)$*. Finally,* ρ^* *satisfies*

$$\langle \phi, \rho^* \rangle \leq 0, \qquad \text{for all } \phi \in C(\overline{\Xi}) \text{ such that } \phi \geq 0. \tag{3.2.6a}$$

$$\langle \psi - (u^* + u_f), \rho^* \rangle = 0 \tag{3.2.6b}$$

Note that (3.2.5c) and (3.2.5d) correspond to the subdifferential inclusion in Theorem 3.2. For ease of reading, we split the proof over the partial results.

Lemma 3.6. *Under the assumptions of Theorem 3.5, there exists a sequence of MY parameters* $\gamma_k \to \infty$ *such that the associated sequence of Nash equilibria* $\{z^{\gamma_k}\}_{k\to\infty}$ *converges weakly to a feasible strategy of the GNEP, i.e.* (3.2.5a) *and* (3.2.5b) *hold.*

Proof. Fix a sequence $\gamma_n \to \infty$ for $n \to \infty$. Since $\mathcal{Z}_{\text{ad}}$ is weakly compact in $L^2(D)^N$ and $z^{\gamma_n} \in \mathcal{Z}_{\text{ad}}$ for all γ^n, there exists a subsequence, denoted by $\gamma_k := \gamma_{n_k}$ and some element $z^* \in \mathcal{Z}_{\text{ad}}$ such that $z^{\gamma_k} \rightharpoonup z^*$ in $L^2(D)^N$. According to SUFR, there exists an $\varepsilon > 0$ and a sequence $\{v^{\gamma_k}\}_{k\to\infty} \subset \mathcal{Z}_{\text{ad}}$ such that $S(v_i^{\gamma_k}, z_{-i}^{\gamma_k}) + u_f \geq \psi + \varepsilon$ Ξ-a.s. for all $i = 1, \dots, N$. By definition of $\mathcal{Z}_{\text{ad}}$, $\{v^{\gamma_k}\}_{k\to\infty}$ is uniformly bounded in $L^2(D)^N$. Then for all γ_k, the non-negativity of the MY-term gives us the lower bound:

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\left[\frac{1}{2}\|T_i(S(z_i^{\gamma_k}, z_{-i}^{\gamma_k}) + u_f) - u_d^i\|_{L^2(D)}^2\right] + \frac{\nu_i}{2}\|z_i^{\gamma_k}\|_{L^2(D)}^2 \\
&\quad\leq \mathbb{E}_{\mathbb{P}}\left[\frac{1}{2}\|T_i(S(z_i^{\gamma_k}, z_{-i}^{\gamma_k}) + u_f) - u_d^i\|_{L^2(D)}^2\right] + \frac{\nu_i}{2}\|z_i^{\gamma_k}\|_{L^2(D)}^2 \\
&\qquad + \mathbb{E}_{\mathbb{P}}\left[\frac{\gamma_k}{2}\|(\psi - (S(z_i^{\gamma_k}, z_{-i}^{\gamma_k}) + u_f)_+\|_{L^2(D)}^2\right].
\end{aligned}$$

Since $S(v_i^{\gamma_k}, z_{-i}^{\gamma_k}) + u_f \geq \psi + \varepsilon$, it holds that

$$\mathbb{E}_{\mathbb{P}}\left[\frac{\gamma_k}{2}\|(\psi - (S(v_i^{\gamma_k}, z_{-i}^{\gamma_k}) + u_f)_+\|_{L^2(D)}^2\right] = 0.$$

Furthermore, by definition of a Nash equilibrium we have the simple upper bound

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\|T_i(S(z_i^{\gamma_k}, z_{-i}^{\gamma_k}) + u_f) - u_d^i\|_{L^2(D)}^2\right] + \frac{\nu_i}{2}\|z_i^{\gamma_k}\|_{L^2(D)}^2 \\
&+ \mathbb{E}_{\mathbb{P}}\left[\frac{\gamma_k}{2}\|(\psi - (S(z_i^{\gamma_k}, z_{-i}^{\gamma_k}) + u_f)_+\|_{L^2(D)}^2\right] \\
&\quad\leq \mathbb{E}_{\mathbb{P}}\left[\|\frac{1}{2}T_i(S(v_i^{\gamma_k}, z_{-i}^{\gamma_k}) + u_f) - u_d^i\|_{L^2(D)}^2\right] + \frac{\nu_i}{2}\|v_i^{\gamma_k}\|_{L^2(D)}^2
\end{aligned}$$

Using the fact that S is completely continuous into $L^2_\pi(\Xi)$ and each individual feasible set is bounded, we deduce the existence of a constant M independent of i, γ_k such that

$$\mathbb{E}_\mathbb{P}\left[\|\frac{1}{2}T_i(S(v_i^{\gamma_k}, z_{-i}^{\gamma_k}) + u_f) - u_d^i\|^2_{L^2(D)}\right] + \frac{\nu_i}{2}\|v_i^{\gamma_k}\|^2_{L^2(D)} \leq M.$$

Combining these observations yields

$$\mathbb{E}_\mathbb{P}\left[\frac{1}{2}\|T_i(S(z_i^{\gamma_k}, z_{-i}^{\gamma_k}) + u_f) - u_d^i\|^2_{L^2(D)}\right] + \frac{\nu_i}{2}\|z_i^{\gamma_k}\|^2_{L^2(D)} \leq M.$$

Using the weak lower semicontinuity of the objective functions, it follows that the bound also holds for the limit

$$\begin{aligned}&\mathbb{E}_\mathbb{P}\left[\frac{1}{2}\|T_i(S(z_i^*, z_{-i}^*) + u_f) - u_d^i\|^2_{L^2(D)}\right] + \frac{\nu_i}{2}\|z_i^*\|^2_{L^2(D)} \\ &\leq \liminf_{k\to\infty}\left(\mathbb{E}_\mathbb{P}\left[\frac{1}{2}\|T_i(S(z_i^{\gamma_k}, z_{-i}^{\gamma_k}) + u_f) - u_d^i\|^2_{L^2(D)}\right] + \frac{\nu_i}{2}\|z_i^{\gamma_k}\|^2_{L^2(D)}\right) \leq M.\end{aligned}$$

As a result, $\mathbb{E}_\mathbb{P}\left[\frac{\gamma_k}{2}\|(\psi - (S(z_i^{\gamma_k}, z_{-i}^{\gamma_k}) + u_f)_+\|^2_{L^2(D)}\right]$ is bounded. This can only hold if

$$\mathbb{E}_\mathbb{P}\left[\|(\psi - (S(z_i^{\gamma_k}, z_{-i}^{\gamma_k}) + u_f)_+\|^2_{L^2(D)}\right] \to 0,$$

since $\gamma_k \to \infty$. Since $S(z_i^{\gamma_k}, z_{-i}^{\gamma_k})$ converges strongly to $S(z_i^*, z_{-i}^*)$ in $L^2_\pi(\Xi)$, we also have

$$\mathbb{E}_\mathbb{P}\left[\|(\psi - (S(z_i^{\gamma_k}, z_{-i}^{\gamma_k}) + u_f)_+\|^2_{L^2(D)}\right] \to \mathbb{E}_\mathbb{P}\left[\|(\psi - (S(z_i^*, z_{-i}^*) + u_f)_+\|^2_{L^2(D)}\right].$$

We can conclude, that

$$\mathbb{E}_\mathbb{P}\left[\|(\psi - (S(z_i^*, z_{-i}^*) + u_f)_+\|^2_{L^2(D)}\right] = 0.$$

Thus, $z^* \in \mathcal{Z}_{\text{ad}}$ such that $S(z_i^*, z_{-i}^*) + u_f \geq \psi$ π-a.e., i.e. z^* is a feasible strategy vector for the GNEP. □

We note that for feasibility of z^*, it is not necessary for ε to be positive in the SUFR condition. In fact, even $\varepsilon = 0$ would lead to feasibility. We start by showing that z^* is also a generalized Nash equilibrium.

Lemma 3.7. *Suppose the assumptions of Theorem 3.5 hold. Let $\{\gamma_k\}$ be the sequence of MY parameters from the proof of Lemma 3.6. Then there exists a subsequence $\{\gamma_l\}$ with $\gamma_l := \gamma_{k_l} \to +\infty$ such that the weak limit point z^* is a generalized Nash equilibrium.*

Proof. Define $X_i = \{v_i \in \mathcal{Z}^i_{\text{ad}} \,|\, S(v_i, z^*_{-i}) + u_f \geq \psi \, \pi\text{-a.s}\}$. Due to the SUFR condition, X_i is non-empty. Since for all γ_k the associated z^{γ_k} is a Nash equilibrium, it holds that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\left[\|T_i(S(z_i^{\gamma_k}, z_{-i}^{\gamma_k}) + u_f) - u_d^i\|^2_{L^2(D)}\right] + \frac{\nu_i}{2}\|z_i^{\gamma_k}\|^2_{L^2(D)} \\
&+ \mathbb{E}_{\mathbb{P}}\left[\frac{\gamma_k}{2}\|(\psi - (S(z_i^{\gamma_k}, z_{-i}^{\gamma_k}) + u_f)_+\|^2_{L^2(D)}\right] \\
&\quad \leq \frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\|T_i(S(v_i, z_{-i}^{\gamma_k}) + u_f) - u_d^i\|^2_{L^2(D)}\right] + \frac{\nu_i}{2}\|v_i\|^2_{L^2(D)} \\
&\quad + \mathbb{E}_{\mathbb{P}}\left[\frac{\gamma_k}{2}\|(\psi - (S(v_i, z_{-i}^{\gamma_k}) + u_f)_+\|^2_{L^2(D)}\right]
\end{aligned}$$

for all $v_i \in X_i$. For any $v_i \in X_i$, we want to construct a strongly convergent sequence $\{v^{\gamma_k}\}_{k\to\infty}$ so such $v^{\gamma_k} \to v_i$ in $L^2(D)$ and $S(v_i^{\gamma_k}, z_{-i}^{\gamma_k}) + u_f \geq \psi$.

Due to the SUFR condition, there exists an $\varepsilon > 0$ and for all k, a $v_i^k \in Z^i_{\text{ad}}$ such that $S(v_i^k, z_{-i}^{\gamma_k}) + u_f \geq \psi + \varepsilon$, π-a.s. Clearly, $\{v_i^k\}_{k\to\infty}$ is uniformly bounded in $L^2(D)$. Since every admissible set of each player is convex, we have that

$$v_i^k(t) = tv_i^k + (1-t)v_i \tag{3.2.7}$$

lies in $\mathcal{Z}^i_{\text{ad}}$ for all $t \in (0,1)$. Due to the linearity of the operator $\mathbf{A}$ and $\mathbf{B}$, it holds that

$$\begin{aligned}
S(v_i^k(t), z_{-i}^{\gamma_k}) + u_f &= S(tv_i^k + (1-t)v_i, z_{-i}^{\gamma_k}) + u_f \\
&= t(S(v_i^k, z_{-i}^{\gamma_k}) + u_f) + (1-t)(S(v_i, z_{-i}^{\gamma_k}) + u_f) \\
&\geq t(\psi + \varepsilon) + (1-t)(S(v_i, z_{-i}^{\gamma_k}) + u_f).
\end{aligned} \tag{3.2.8}$$

We know, that for $\mathbb{P}$-a.e. $\omega \in \Omega$ the solution operator $S(v_i, \cdot)(\omega) + u_f(\omega)$ maps continuously from $L^2(D)^{N-1}$ into $H_0^1(D) \cap H^2(D)$. Due to the Sobolev and Rellich-Kondrachov theorem, the solution of the state equation can be continuously and compactly embedded into the space of continuous functions over $\bar{D}$ $\mathbb{P}$-a.s. Thus, $S(v_i, \cdot)(\omega) + u_f(\omega)$ maps from $L^2(D)^{N-1}$ into $C(\bar{D})$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Combining this with the regularity assumption on the solution of the state equation, we have $S(v_i, z_{-i}^{\gamma_k}) + u_f \to S(v_i, z^*_{-i}) + u_f$ in $C(\Omega; C(\bar{D}))$. Then by virtue of the nature of convergence in the $C(\Omega; C(\bar{D}))$-norm, we deduce the existence of a subsequence γ_{k_l}, denoted by γ_l, such that

$$S(v_i, z_{-i}^{\gamma_l}) + u_f \geq \psi - {}^1\!/_{2^l}$$

on D for all k. Now, setting

$$t_l = ({}^1\!/_{2^l})/(\varepsilon + {}^1\!/_{2^l}), \tag{3.2.9}$$

then $t_l \to 0$ and $t_l \in (0,1)$ for all l. Moreover, substituting (3.2.9) in (3.2.7) and due to (3.2.8), we have

$$\begin{aligned} S(v_i(t_l), z_{-i}^{\gamma_l}) + u_f &\geq t_l(\psi + \varepsilon) + (1-t_l)(S(v_i, z_{-i}^{\gamma_l}) + u_f) \\ &\geq t_l(\psi + \varepsilon) + (1-t_l)(\psi - 1/2^l) \\ &= \psi + t_l(\varepsilon + 1/2^l) - 1/2^l \\ &= \psi. \end{aligned}$$

Thus, $S(v_i(t_l), z_{-i}^{\gamma_l}) + u_f \geq \psi$ for all l. And finally, since

$$\begin{aligned} \|v_i(t_l) - v_i\|_{L^2(D)} &= \|t_l v_i^l + (1-t_l)v_i - v_i\|_{L^2(D)} \\ &= |t_l| \|v_i^l - v_i\|_{L^2(D)} \\ &\leq |t_l| \left(\|v_i^l\|_{L^2(D)} + \|v_i\|_{L^2(D)} \right), \end{aligned}$$

passing to the limit as $l \to \infty$ yields $|t_l| \left(\|v_i^l\|_{L^2(D)} + \|v_i\|_{L^2(D)} \right) \to 0$ due to the boundedness of $\{v_i(t_l)\}_{l\to\infty}$ and that $\{t_l\}_{l\to\infty}$ is a null sequence. Thus, we have constructed a sequence $\{v_i(t_l)\}_{l\to\infty}$ such that $v_i(t_l) \to v_i$ in $L^2(D)$ and $S(v_i(t_l), z_{-i}^{\gamma_l}) + u_f \geq \psi$. Note that $v_i \in X_i$ was arbitrary.

Finally, by substitution, we have

$$\begin{aligned} &\frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\|T_i(S(z_i^{\gamma_l}, z_{-i}^{\gamma_l}) + u_f) - u_d^i\|_{L^2(D)}^2\right] + \frac{\nu_i}{2}\|z_i^{\gamma_l}\|_{L^2(D)}^2 \\ &+ \mathbb{E}_{\mathbb{P}}\left[\frac{\gamma_l}{2}\|(\psi - (S(z_i^{\gamma_l}, z_{-i}^{\gamma_l}) + u_f)_+\|_{L^2(D)}^2\right] \\ &\quad \leq \frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\|T_i(S(v_i(t_l), z_{-i}^{\gamma_l}) + u_f) - u_d^i\|_{L^2(D)}^2\right] + \frac{\nu_i}{2}\|v_i(t_l)\|_{L^2(D)}^2 \\ &\qquad + \mathbb{E}_{\mathbb{P}}\left[\frac{\gamma_l}{2}\|(\psi - (S(v_i(t_l), z_{-i}^{\gamma_l}) + u_f)_+\|_{L^2(D)}^2\right]. \end{aligned}$$

For all $i = 1, \ldots, N$, passing to the limit inferior yields the following inequality

$$\begin{aligned} &\frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\|T_i(S(z_i^*, z_{-i}^*) + u_f) - u_d^i\|_{L^2(D)}^2\right] + \frac{\nu_i}{2}\|z_i^*\|_{L^2(D)}^2 \\ &\leq \frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\|T_i(S(v_i, z_{-i}^*) + u_f) - u_d^i\|_{L^2(D)}^2\right] + \frac{\nu_i}{2}\|v_i\|_{L^2(D)}^2 \end{aligned}$$

for all $v_i \in X_i$. Thus, (z_i^*, z_{-i}^*) is a generalized Nash equilibrium. □

Here, we see that the uniformity in the SUFR condition is crucial to prove that z^* is in fact a Nash equilibrium. In the following result, we obtain a stronger form of convergence to z^* as weak convergence. This is necessary to derive the adjoint equation in the limit.

Lemma 3.8. *Under the assumptions of Theorem 3.5,* (3.2.4a) *holds.*

Proof. First, we choose $z_i^* \in X_i$ in the construction of (3.2.7) with $t = t_l$ as in (3.2.9), then we have

$$v_i^*(t_l) = t_l v_i^k + (1 - t_l) z_i^*.$$

Recall that $v_i^*(t_l) \to z_i^*$ in $L^2(D)$ and $S(v_i^*(t_l), z_{-i}^{\gamma_l}) + u_f \geq \psi$ for all $l \in \mathbb{N}$. Then it holds that

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\|T_i(S(z_i^{\gamma_l}, z_{-i}^{\gamma_l}) + u_f) - u_d^i\|_{L^2(D)}^2\right] + \frac{\nu_i}{2}\|z_i^{\gamma_l}\|_{L^2(D)}^2 \\
&\leq \frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\|T_i(S(z_i^{\gamma_l}, z_{-i}^{\gamma_l}) + u_f) - u_d^i\|_{L^2(D)}^2\right] + \frac{\nu_i}{2}\|z_i^{\gamma_l}\|_{L^2(D)}^2 \\
&\quad + \mathbb{E}_{\mathbb{P}}\left[\frac{\gamma_l}{2}\|(\psi - (S(z_i^{\gamma_l}, z_{-i}^{\gamma_l}) + u_f)_+\|_{L^2(D)}^2\right] \\
&\leq \frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\|T_i(S(v_i^*(t_l), z_{-i}^{\gamma_l}) + u_f) - u_d^i\|_{L^2(D)}^2\right] + \frac{\nu_i}{2}\|v_i^*(t_l)\|_{L^2(D)}^2.
\end{aligned}$$

Passing to the limit superior yields

$$\begin{aligned}
&\limsup_{l\to\infty} \frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\|T_i(S(z_i^{\gamma_l}, z_{-i}^{\gamma_l}) + u_f) - u_d^i\|_{L^2(D)}^2\right] + \frac{\nu_i}{2}\|z_i^{\gamma_l}\|_{L^2(D)}^2 \\
&\leq \limsup_{l\to\infty} \frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\|T_i(S(v_i^*(t_l), z_{-i}^{\gamma_l}) + u_f) - u_d^i\|_{L^2(D)}^2\right] + \frac{\nu_i}{2}\|v_i^*(t_l)\|_{L^2(D)}^2.
\end{aligned} \tag{3.2.10}$$

Due to the completely continuity of S, we have

$$S(z_i^{\gamma_l}, z_{-i}^{\gamma_l}) \to S(z_i^*, z_{-i}^*) \quad \text{and} \quad S(v_i^*(t_l), z_{-i}^{\gamma_l}) \to S(z_i^*, z_{-i}^*).$$

Then (3.2.10) reads as

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\|T_i(S(z_i^*, z_{-i}^*) + u_f) - u_d^i\|_{L^2(D)}^2\right] + \frac{\nu_i}{2}\limsup_{l\to\infty}\|z_i^{\gamma_l}\|_{L^2(D)}^2 \\
&\leq \frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\|T_i(S(z_i^*, z_{-i}^*) + u_f) - u_d^i\|_{L^2(D)}^2\right] + \frac{\nu_i}{2}\|z_i^*\|_{L^2(D)}^2.
\end{aligned} \tag{3.2.11}$$

This implies that

$$\limsup_{l\to\infty}\|z_i^{\gamma_l}\|_{L^2(D)}^2 \leq \|z_i^*\|_{L^2(D)}^2.$$

Due to the weak convergence of $\{z_i^{\gamma_l}\}_{l\in\mathbb{N}}$, it holds that $\liminf_{l\to\infty}\|z_i^{\gamma_l}\|_{L^2(D)}^2 \geq \|z_i^*\|_{L^2(D)}^2$. This implies that

$$\lim_{l\to\infty}\|z_i^{\gamma_l}\|_{L^2(D)}^2 = \|z_i^*\|_{L^2(D)}^2.$$

Together with the weak convergence, the assertion follows. □

In what follows, we discuss the convergence of the stationary points individually. We proceed with the sequence of the state variables.

Lemma 3.9. *Under the assumptions of Theorem 3.5,* (3.2.4b) *and* (3.2.5b) *hold.*

Proof. This directly follows from the assumption, that $S(\cdot,\cdot) + u_f : L^2(D)^N \to C(\Omega; H_0^1(D) \cap H^2(D))$ is continuous and the fact the sequences $\{z_i^{\gamma_l}\}_{l\in\mathbb{N}}$ converges strongly in $L^2(D)$ for all $i = 1, \dots, N$. □

We note that the continuity in Ω is not really needed to prove a norm convergence result. Indeed since $\{z_i^{\gamma_l}\}_{l\in\mathbb{N}}$ is bounded, we still have that $\{u^{\gamma_l}\}_{l\in\mathbb{N}}$ is bounded in $L^2_{\mathbb{P}}(\Omega; H_0^1(D)\cap H^2(D))$. Then $u^{\gamma_{l_n}} \rightharpoonup u^*$ in $L^2_{\mathbb{P}}(\Omega; H_0^1(D)\cap H^2(D))$. By Corollary 2.25 we even know that $u^{\gamma_{l_n}} \to u^*$ in $L^\infty_{\mathbb{P}}(\Omega; H_0^1(D) \cap H^2(D))$ holds.

Next, we turn our attention to the sequence of the multipliers μ^γ for the state constraint. We will observe that the Slater condition is enough to obtain a bound on μ^γ. Recall that $\mu^\gamma = -\gamma(\psi - (S(\bar{z}_i^\gamma, \bar{z}_{-i}^\gamma) + u_f))_+$.

Lemma 3.10. *Suppose the assumptions of Theorem 3.5 hold. In particular,* (4.1.1) *is fulfilled. Then we have* (3.2.4c)*.*

Proof. We now prove the existence of a constant $c_0 > 0$ such that

$$|(\mu^\gamma, z)| \le c_0 \tag{3.2.12}$$

for any $z \in \mathbb{B}_\varepsilon(0) \subset L^\infty_\pi(\Xi)$ and some fixed $\varepsilon > 0$. For the sake of readability, we set $\beta : L^2_\pi(\Xi) \to \mathbb{R}_+$ such that

$$u \mapsto \beta(u) = \mathbb{E}\left[\frac{1}{2}\|(\psi - (u + u_f))_+\|^2_{L^2(D)}\right].$$

Unless otherwise noted, $(\cdot,\cdot)$ denotes the inner product on $L^2_\pi(\Xi)$ throughout the proof.

One readily shows that β is convex and continuously differentiable and therefore, $\mu^\gamma = \gamma\beta'(u^\gamma)$. Since β is convex, differentiable, and nonnegative, we obtain for any

$$y \in \left\{w \in L^2_\pi(\Xi) \mid w \ge \psi - u_f \ \pi\text{-a.e.}\right\}$$

the inequalities

$$0 = \gamma\beta(y) \ge \gamma\beta(u^\gamma) + (\mu^\gamma, y - u^\gamma) \ge (\mu^\gamma, y - u^\gamma). \tag{3.2.13}$$

By the assumption (3.1.1) there exists an $\varepsilon > 0$ and $z^0 \in \mathcal{Z}_{\text{ad}}$ such that for all $v \in \mathbb{B}_\varepsilon(0) \subset L^\infty_\pi(\Xi)$, we have

$$Sz^0 + u_f - \psi + v \ge 0.$$

Since $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and the spatial domain D is bounded, the Lebesgue spaces are nested, and it holds that $v \in L^2_\pi(\Xi)$. Furthermore, $Sz^0 + u_f \in L^2_\pi(\Xi)$. Fixing an arbitrary $v \in \mathbb{B}_\varepsilon(0)$, we have

$$(\mu^\gamma, v) = \left(\mu^\gamma, Sz^0 + v - Sz^\gamma\right) + \left(\mu^\gamma, Sz^\gamma - Sz^0\right).$$

Due to (3.2.13), we have

$$(\mu^\gamma, v) \leq \left(\mu^\gamma, Sz^\gamma - Sz^0\right) = \left(\mu^\gamma, \boldsymbol{A}^{-1}\boldsymbol{B}(z^\gamma - z^0)\right).$$

The definition of the multiplier μ^γ and the operator $\boldsymbol{B}$ yield

$$\begin{aligned}
&(\mu^\gamma, \boldsymbol{A}^{-1}\boldsymbol{B}(z^\gamma - z^0)) \\
&= \sum_{i=1}^{N} (T_i^* \left(T_i u^\gamma + T_i u_f - u_d^i\right) + \mu^\gamma - T_i^* \left(T_i u^\gamma + T_i u_f - u_d^i\right), \boldsymbol{A}^{-1}\boldsymbol{B}_i(z_i^\gamma - z_i^0)) \\
&= \sum_{i=1}^{N} (\boldsymbol{A}^* \lambda_i^\gamma - T_i^* T_i u^\gamma - T_i^* T_i u_f + T_i^* u_d^i, \boldsymbol{A}^{-1}\boldsymbol{B}_i(z_i^\gamma - z_i^0)).
\end{aligned}$$

Substituting the adjoint equation and applying the adjoint operator $\boldsymbol{B}^*\boldsymbol{A}^{-*}$ yield

$$\begin{aligned}
(\mu^\gamma, v) &\leq \sum_{i=1}^{N} (\boldsymbol{A}^* \lambda_i^\gamma - T_i^* T_i u^\gamma - T_i^* T_i u_f + T_i^* u_d^i, \boldsymbol{A}^{-1}\boldsymbol{B}_i(z_i^\gamma - z_i^0)) \\
&= \sum_{i=1}^{N} \mathbb{E}_{\mathbb{P}} \left[(\boldsymbol{B}_i^* \lambda_i^\gamma - \boldsymbol{B}_i^* \boldsymbol{A}^{-*}(T_i^* T_i u^\gamma + T_i^* T_i u_f - T_i^* u_d^i), z_i^\gamma - z_i^0)_{L^2(D)} \right] \\
&= \sum_{i=1}^{N} \mathbb{E}_{\mathbb{P}} \left[(\boldsymbol{B}_i^* \lambda_i^\gamma, z_i^\gamma - z_i^0)_{L^2(D)} \right] \\
&\quad - \mathbb{E}_{\mathbb{P}} \left[\left(\boldsymbol{B}_i^* \boldsymbol{A}^{-*}(T_i^* T_i u^\gamma + T_i^* T_i u_f - T_i^* u_d^i), z_i^\gamma - z_i^0 \right)_{L^2(D)} \right].
\end{aligned}$$

Applying [44, Thm. 3.7.12] yields

$$\begin{aligned}
(\mu^\gamma, z) &\leq \sum_{i=1}^{N} (\mathbb{E}_{\mathbb{P}} \left[\boldsymbol{B}_i^* \lambda_i^\gamma \right], z_i^\gamma - z_i^0)_{L^2(D)} \\
&\quad - \mathbb{E}_{\mathbb{P}} \left[\left(\boldsymbol{B}_i^* \boldsymbol{A}^{-*}(T_i^* T_i u^\gamma + T_i^* T_i u_f - T_i^* u_d^i), z_i^\gamma - z_i^0 \right)_{L^2(D)} \right].
\end{aligned}$$

Using $0 = \nu_i z_i^\gamma + \mathbb{E}_\mathbb{P}[\boldsymbol{B}_i^* \lambda_i^\gamma] + \eta_i^\gamma$ and the fact, that $z_i^0 \in Z_{\text{ad}}^i$ yields

$$
\begin{aligned}
&\sum_{i=1}^N \left(\mathbb{E}_\mathbb{P}[\boldsymbol{B}_i^* \lambda_i^\gamma], z_i^\gamma - z_i^0 \right)_{L^2(D)} \\
&- \sum_{i=1}^N \mathbb{E}_\mathbb{P} \left[\left(\boldsymbol{B}_i^* \boldsymbol{A}^{-*}(T_i^* T_i u^\gamma + T_i^* T_i u_f - T_i^* u_d^i), z_i^\gamma - z_i^0 \right)_{L^2(D)} \right] \\
&\quad = \sum_{i=1}^N \left(-\nu_i z_i^\gamma - \eta_i^\gamma, z_i^\gamma - z_i^0 \right)_{L^2(D)} \\
&\qquad - \sum_{i=1}^N \mathbb{E}_\mathbb{P} \left[\left(\boldsymbol{B}_i^* \boldsymbol{A}^{-*}(T_i^* T_i u^\gamma + T_i^* T_i u_f - T_i^* u_d^i), z_i^\gamma - z_i^0 \right)_{L^2(D)} \right] \\
&\quad \leq \sum_{i=1}^N \left(-\nu_i z_i^\gamma, z_i^\gamma - z_i^0 \right)_{L^2(D)} \\
&\qquad - \sum_{i=1}^N \mathbb{E}_\mathbb{P} \left[\left(\boldsymbol{B}_i^* \boldsymbol{A}^{-*}(T_i^* T_i u^\gamma + T_i^* T_i u_f - T_i^* u_d^i), z_i^\gamma - z_i^0 \right)_{L^2(D)} \right] \\
&\quad \leq c_0 < \infty.
\end{aligned}
$$

Here, the existence of c_0 is guaranteed, since the mappings

$$
u(z) \mapsto \mathbb{E}_\mathbb{P}\left[\frac{1}{2} \| T_i u(z) + T_i u_f - u_d^i \|_{L^2(D)}^2 \right] \quad \text{and} \quad z_i \mapsto \frac{\nu_i}{2} \|z_i\|_{L^2(D)}^2
$$

are continuously differentiable with uniformly bounded gradients on $\mathcal{Z}_{\text{ad}}$ for all $i = 1, \dots, N$. This proves (3.2.12), since z was arbitrary. Using the fact that the L^1-norm is positively homogeneous, subadditive and continuous, it follows from the Fenchel-Moreau theorem that the L^1-norm is equivalent to the bidual norm

$$
\begin{aligned}
\|\mu^\gamma\|_{L_\pi^1(\Xi)} &= \frac{1}{\varepsilon} \sup \left\{ \langle \mu^\gamma, z \rangle_{L_\pi^1(\Xi) \times L_\pi^\infty(\Xi)} \mid z \in \mathbb{B}_\varepsilon(0) \right\} \\
&= \frac{1}{\varepsilon} \sup \left\{ (\mu^\gamma, z)_{L_\pi^2(\Xi)} \mid z \in \mathbb{B}_\varepsilon(0) \right\} \\
&\leq \frac{1}{\varepsilon} c_0 < \infty.
\end{aligned}
$$

It follows that the sequence $\{\mu^\gamma\}_{\gamma \to \infty}$ is bounded in $L_\pi^1(\Xi)$. Therefore, by [21, Theorem IV.6.2] or [9, Corollary 2.4.3], we can extract a subsequence $\{\mu^{\gamma_l}\}_{l \in \mathbb{N}}$ which is weak* convergent to some regular countably additive Borel measure $\rho \in \mathcal{M}(\overline{\Xi})$. □

Next, we discuss the limit of the adjoint equation. We start by investigating the behavior of the expectation of the adjoint states. This leads to the derivation of a limiting adjoint state Λ^*.

Lemma 3.11. *Under the assumptions of Theorem 3.5, for all $i = 1, \dots, N$, (3.2.4d) holds.*

Proof. We start by constructing a specific test function. Let ϕ be the solution of the operator equation

$$\boldsymbol{A}(\omega)\phi = \boldsymbol{B}_i(\omega)\varphi \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega$$

for $\varphi \in L^2(D)$. Then by the assumptions, $\phi \in L^\infty_{\mathbb{P}}(\Omega; H^1_0(D) \cap H^2(D))$ and, by Assumption 2.12, $\phi \in C(\Omega; H^1_0(D) \cap H^2(D))$ holds. Using the adjoint state as a test function, we have

$$\begin{aligned}(\boldsymbol{A}^*(\omega)\lambda_i^{\gamma_l}(\omega), \phi(\omega)) &= (\lambda_i^{\gamma_l}(\omega), \boldsymbol{A}(\omega)\phi(\omega)) \\ &= (\lambda_i^{\gamma_l}(\omega), \boldsymbol{B}_i(\omega)\varphi) \\ &= (\boldsymbol{B}_i(\omega)^*\lambda_i^{\gamma_l}(\omega), \varphi)\,.\end{aligned} \tag{3.2.14}$$

Then due to the Cauchy-Schwarz inequality and Hölder-inequality, respectively, we obtain

$$\begin{aligned}&(\boldsymbol{A}^*(\omega)\lambda_i^{\gamma}(\omega), \phi(\omega)) \\ &= \left(T_i^*T_i(u^\gamma(\omega) + u_f(\omega)) - T_i^*u_d^i, \phi(\omega)\right) + (\mu^\gamma(\omega), \phi(\omega))_{L^2(D)} \\ &= \left(T_i^*T_i(u^\gamma(\omega) + u_f(\omega)) - T_i^*u_d^i, \phi(\omega)\right) + \langle\mu^\gamma(\omega), \phi(\omega)\rangle_{L^1(D)\times L^\infty(D)} \\ &\le \|T_i^*T_i(u^\gamma(\omega) + u_f(\omega)) - T_i^*u_d^i\|_{L^2(D)}\|\phi(\omega)\|_{L^2(D)} + \|\mu^\gamma(\omega)\|_{L^1(D)}\|\phi(\omega)\|_{L^\infty(D)}.\end{aligned}$$

Due to the continuous embedding of $H^2(D) \cap H^1_0(D)$ into $L^2(D)$ and $L^\infty(D)$, respectively, we have

$$\begin{aligned}&(\boldsymbol{A}^*(\omega)\lambda_i^{\gamma}(\omega), \phi(\omega)) \\ &\le C_1\|\phi(\omega)\|_{H^2(D)\cap H^1_0(D)}\left(\|T_i^*T_i(u^\gamma(\omega) + u_f(\omega)) - T_i^*u_d^i\|_{L^2(D)} + \|\mu^\gamma(\omega)\|_{L^1(D)}\right).\end{aligned} \tag{3.2.15}$$

By the assumptions on the operators $\mathbf{A}$ and $\mathbf{B}$, there exists $C_2 \in L^\infty_{\mathbb{P}}(\Omega)$ such that

$$\|\phi(\omega)\|_{H^2(D)\cap H^1_0(D)} \le C_2(\omega)\|\varphi\|_{L^2(D)}.$$

Now, combining the latter with (3.2.15) and (3.2.14), we obtain

$$\begin{aligned}&(\boldsymbol{B}_i(\omega)^*\lambda_i^{\gamma}(\omega), \varphi) \\ &\le C_1\|\phi(\omega)\|_{H^2(D)\cap H^1_0(D)}\left(\|T_i^*T_i(u^\gamma(\omega) + u_f(\omega)) - T_i^*u_d^i\|_{L^2(D)} + \|\mu^\gamma(\omega)\|_{L^1(D)}\right) \\ &\le C_1C_2(\omega)\|\varphi\|_{L^2(D)}\left(\|T_i^*T_i(u^\gamma(\omega) + u_f(\omega)) - T_i^*u_d^i\|_{L^2(D)} + \|\mu^\gamma(\omega)\|_{L^1(D)}\right) \\ &\le C_1C_2(\omega)\|\varphi\|_{L^2(D)}\left(C_3(\omega) + C_4(\omega)\right)\end{aligned}$$

for all $\varphi \in L^2(D)$. Here, $C_3 \in L^\infty_{\mathbb{P}}(\Omega)$ and $C_4 \in L^1_{\mathbb{P}}(\Omega)$. The existence of C_4 follows from the uniform bound on μ^γ in the $L^1_\pi(\Xi)$-norm. Taking the expectation and applying Fubini's theorem yield

$$(\mathbb{E}_{\mathbb{P}}[\boldsymbol{B}_i^*\lambda_i^\gamma], \varphi) \le \mathbb{E}_{\mathbb{P}}[C_1C_2C_3 + C_1C_2C_4]\,\|\varphi\|_{L^2(D)} < \infty$$

for all $\varphi \in L^2(D)$. In other words, the sequence $\{\mathbb{E}_{\mathbb{P}}[\boldsymbol{B}_i^*\lambda_i^\gamma]\}_{\gamma\to\infty}$ is bounded in $L^2(D)$. Thus, there exists a weakly convergent subsequence $\{\mathbb{E}_{\mathbb{P}}[\boldsymbol{B}_i^*\lambda_i^{\gamma_l}]\}_{l\in\mathbb{N}}$ and a $\Lambda_i^* \in L^2(D)$ such that $\mathbb{E}_{\mathbb{P}}[\boldsymbol{B}_i^*\lambda_i^{\gamma_l}] \rightharpoonup \Lambda_i^*$ in $L^2(D)$. $\square$

Next, we turn our attention on the adjoint equation in the limit.

Lemma 3.12. *Under the assumptions of Theorem 3.5,* (3.2.5c) *holds.*

Proof. As in the previous proof, we start by constructing a specific test function. In this case, let w be the solution of the operator equation

$$\boldsymbol{A}(\omega)w = \boldsymbol{B}_i(\omega)\varphi \quad \mathbb{P}\text{-a.e. } \omega \in \Omega$$

for all $\varphi \in L^2(D)$, then we know that $w \in C(\Omega; H_0^1(D) \cap H^2(D))$ holds. It follows that

$$(\boldsymbol{A}^*(\omega)\lambda_i^{\gamma_l}(\omega), w(\omega)) = (\boldsymbol{B}_i(\omega)^*\lambda_i^{\gamma_l}(\omega), \varphi) \,.$$

Taking the expectation on both sides yields

$$(\mathbb{E}_{\mathbb{P}}\left[\boldsymbol{B}_i^*\lambda_i^{\gamma_l}\right], \varphi) = \mathbb{E}_{\mathbb{P}}\left[\Big(T_i^*T_i(u^{\gamma_l} + u_f) - T_i^*u_d^i, w\Big)\right] + \mathbb{E}_{\mathbb{P}}\left[(\mu^{\gamma_l}, w)\right].$$

We know that $\mu^{\gamma_l} \rightharpoonup^* \rho^*$ in $\mathcal{M}(\overline{\Xi})$. The right hand side reads as

$$\mathbb{E}_{\mathbb{P}}\left[\Big(T_i^*T_i(u^{\gamma_l} + u_f) - T_i^*u_d^i, w\Big)\right] + \langle \mu^{\gamma_l}, w\rangle_{\mathcal{M}(\overline{\Xi}),C(\overline{\Xi})} \,.$$

Passing to the limit $l \to \infty$ yields

$$\begin{aligned}(\Lambda_i^*, \varphi) &= \mathbb{E}_{\mathbb{P}}\left[\Big(T_i^*T_i(u^* + u_f) - T_i^*u_d^i, w\Big)\right] + \int_{\Xi} w(x,\omega)\, d\rho^*(x,\omega)\\ &= \mathbb{E}_{\mathbb{P}}\left[\Big(T_i^*T_i(u^* + u_f) - T_i^*u_d^i, \boldsymbol{A}^{-1}\boldsymbol{B}_i\varphi\Big)\right] + \int_{\Xi} \boldsymbol{A}^{-1}(\omega)\boldsymbol{B}_i(\omega)\varphi\, d\rho^*(x,\omega)\\ &= \mathbb{E}_{\mathbb{P}}\left[\Big(\boldsymbol{B}_i^*\boldsymbol{A}^{-*}T_i^*(T_i(u^* + u_f) - u_d^i), \varphi\Big)\right] + \int_{\Xi} \boldsymbol{A}^{-1}(\omega)\boldsymbol{B}_i(\omega)\varphi\, d\rho^*(x,\omega)\end{aligned}$$

for all $\varphi \in L^2(D)$. □

Next, we turn to the sequence $\{\eta^\gamma\}_{\gamma\to\infty} \subset \mathcal{N}_{Z_{\mathrm{ad}}^i}(z_i^\gamma)$.

Lemma 3.13. *Under the assumptions of Theorem 3.5,* (3.2.4e) *and* (3.2.5d) *hold.*

Proof. Due (3.2.2), we can write

$$\eta_i^{\gamma_l} = -\mathbb{E}_{\mathbb{P}}\left[\boldsymbol{B}_i^*\lambda_i^{\gamma_l}\right] - \nu_i z_i^{\gamma_l}.$$

Then the boundedness of the sequence $\{\eta_i^{\gamma_l}\}_{l\in\mathbb{N}}$ in $L^2(D)$ directly follows from

$$\begin{aligned}\|\eta_i^{\gamma_l}\|_{L^2(D)} &= \| - \mathbb{E}_{\mathbb{P}}\left[\boldsymbol{B}_i^*\lambda_i^{\gamma_l}\right] - \nu_i z_i^{\gamma_l}\|_{L^2(D)}\\ &\le \|\mathbb{E}_{\mathbb{P}}\left[\boldsymbol{B}_i^*\lambda_i^{\gamma_l}\right]\|_{L^2(D)} + \|\nu_i z_i^{\gamma_l}\|_{L^2(D)} < \infty.\end{aligned}$$

Thus, there exists an $\eta_i^* \in L^2(D)$ and a subsequence $\{\eta_i^{\gamma_{l_n}}\}_{n\in\mathbb{N}}$ such that the assertion holds. □

Finally, we derive the complementarity system for the multiplier ρ^*.

Lemma 3.14. *Under the assumptions of Theorem 3.5,* (3.2.6a) *and* (3.2.6b) *hold.*

Proof. As used several times above, there exists a subsequence of MY parameters $\gamma_l \to +\infty$ along which the multipliers $\{\mu^{\gamma_l}\}$ converge weak* in $\mathcal{M}(\overline{\Xi})$ to some $\rho^* \in \mathcal{M}(\overline{\Xi})$. For each fixed l we have π-a.s.

$$\mu^{\gamma_l} = -\gamma_l(\psi - (S(\bar{z}_i^{\gamma_l}, \bar{z}_{-i}^{\gamma_l}) + u_f))_+ \leq 0.$$

Therefore, for any nonnegative test function $\phi \in C(\overline{\Xi})$, we have

$$\langle \phi, \mu^{\gamma_l} \rangle = \int_{\Xi} \phi \mu^{\gamma_l} \mathrm{d}\pi \leq 0.$$

By definition, $\langle \phi, \mu^{\gamma_l} \rangle \to \langle \phi, \rho^* \rangle$ as $l \to +\infty$. Hence, ρ^* is a negatively signed measure. Moreover, setting

$$\phi_l = \psi - (S(\bar{z}_i^{\gamma_l}, \bar{z}_{-i}^{\gamma_l}) + u_f),$$

which is continuous and converges strongly in $C(\overline{\Xi})$ (by assumption) to

$$\phi^* = \psi - (S(\bar{z}_i^*, \bar{z}_{-i}^*) + u_f) \leq 0,$$

we have

$$\langle \phi^*, \rho^* \rangle \geq 0.$$

Furthermore, for each l, we have $\langle \phi_l, \mu^{\gamma_l} \rangle \leq 0$ and $\langle \phi_l, \mu^{\gamma_l} \rangle \to \langle \phi^*, \rho^* \rangle$. Whence we have the complementarity condition.

□

Together the previous lemmas yield the derivation of Theorem 3.5. It is remarkable, that passing to the limit in the smoothing parameter γ, returns stationary points of the original problem whose multiplier for the state constraint is a Borel measure. Even though the theory for convex optimization problems can only ensure a bounded additive measure.

Chapter 4

Risk-Neutral PDE-Constrained Optimization Problem

In this section, we consider risk-neutral PDE-constrained optimization problems with state constraints since the literature is rather scarce on this treatment. Nevertheless we can refer to [26, 35] and the recent preprint [33]. Most of this section can be found in the preprint [29, Section 4.4]. The problem is in accordance with the individual player's problems in the PDE-constrained generalized Nash equilibrium problem of Chapter 3. Hence, throughout the section we assume that Assumption 2.1 and Assumption 2.7 hold, then the PDE-constrained optimization problems written as reduced space problem reads as

$$\min_{z\in Z_{\mathrm{ad}}} \left\{ \mathbb{E}_{\mathbb{P}}\left[\frac{1}{2}\|Tu-u_d\|^2_{L^2(D)}\right] + \frac{\nu}{2}\|z\|^2_{L^2(D)} \mid S(z)+u_f \geq \psi \text{ for } \mathcal{B}\text{-a.e. } (x,\omega)\in\Xi \right\}, \tag{4.0.1}$$

where the random field state $S(z)+u_f$ is $\mathbb{P}$-pointwise given as

$$S(z)(\omega)+u_f(\omega) = \boldsymbol{A}^{-1}(\omega)\boldsymbol{B}(\omega)z + \boldsymbol{A}^{-1}(\omega)\boldsymbol{B}(\omega)f(\omega).$$

This basically corresponds to the definition in (2.3.5) and (2.3.6) for $N=1$.

The treatment of state constraints in the form of

$$S(z)+u_f \geq \psi \text{ for } \mathcal{B}\text{-a.e. } (x,\omega)\in\Xi \tag{4.0.2}$$

for a given $\psi \in C(\overline{\Omega\times D})$ for which there exists $\varepsilon>0$ such that $\psi|_{\partial D}(\omega) \leq -\varepsilon$ $\mathbb{P}$-a.s. has not been considered in PDE-constrained optimization to the best of my knowledgement. Note that e.g. the abstract results in [34] can be used for (4.0.2). However, these results require a different kind of constraint qualification.

An alternative interpretation of (4.0.2) would be the formulation as probability constraints. We could either consider a pointwise probability constraint

$$\mathbb{P}(S(z)(x,\cdot)+u_f(x,\cdot)-\psi(x,\cdot)\geq 0) = 1 \quad \text{for a.e. } x\in D$$

or probability constraint given as joint form

$$\mathbb{P}(S(z)(x,\cdot) + u_f(x,\cdot) - \psi(x,\cdot) \geq 0 \text{ for a.e. } x \in D) = 1.$$

From the perspective of stochastic programming, this is rather restrictive. Since in general settings, e.g. beyond PDE-constrained optimization, this may lead to empty feasible sets. One ansatz to remedies this disadvantage, but also soften the constraint, could be by selecting a minimum probability level $\alpha \in (0, 1)$ and considering instead:

$$\mathbb{P}(S(z)(x,\cdot) + u_f(x,\cdot) - \psi(x,\cdot) \geq 0 \text{ for a.e. } x \in D) \geq \alpha. \tag{4.0.3}$$

Several recent studies have considered this perspective, see [26, 35].

However, these approaches do not circumvent the fundamental difficulties in connection with state constraints, namely the low multiplier regularity and the mesh-independence in numerical approaches. In addition, the resulting functional

$$\phi(z) = \mathbb{P}(S(z)(x,\cdot) + u_f(x,\cdot) - \psi(x,\cdot) \geq 0 \text{ for a.e. } x \in D)$$

is neither nontrivial to analyze nor to use in numerical algorithms e.g. due to the lack of point evaluation and the nonsmoothness.

This usually requires $\mathbb{P}$ to have nice properties, e.g. it has to admit a log-concave density. Plus the solution operator $S(z)(x, \omega)$ needs a very specific structure with respect to ω. For more on probability constraints, we refer the reader to [72, 84] and the related references therein. In the following, we will prove existence of optimal solutions and derive optimality conditions for (4.0.1). Further, we link the Moreau-Yosida regularization technique and probability constraints.

4.1 Existence and Optimality Conditions

In this section, we prove existence of optimal solutions of (4.0.1) and provide optimality conditions. As in Section 3.1, we will appeal to the general Lagrangian formalism in [13, Chap. 3] in order to derive first order optimality conditions for (4.0.1).

We rewrite

$$\min_{z \in Z_{\mathrm{ad}}} \left\{ \mathbb{E}_{\mathbb{P}} \left[J(\boldsymbol{A}^{-1}\boldsymbol{B}z + u_f, z) \right] \mid G(z) \in K \right\}$$

and we set

$$G(z) = \iota \boldsymbol{A}^{-1}\boldsymbol{B}z + \iota u_f - \psi \text{ and } K = L^\infty_\pi(\Xi)_+,$$

where

$$\iota : L^\infty_{\mathbb{P}}(\Omega; H^1_0(D) \cap H^2(D))) \to L^\infty_\pi(\Xi)$$

is the continuous embedding and K is the convex cone of all positive essentially bounded strongly $\mathcal{B}$-measurable functions. Note that we first use the continuous embedding of $L^\infty_\mathbb{P}(\Omega; H^1_0(D) \cap H^2(D))$ into $L^\infty_\mathbb{P}(\Omega; L^\infty(D))$ and then the continuous embedding of $L^\infty_\mathbb{P}(\Omega; L^\infty(D))$ into $L^\infty_\pi(\Xi)$ to define ι.

Since K has a nonempty interior and G is clearly convex with respect to the partial order induced by $-K$, (4.1.1) is equivalent to the constraint qualification $0 \in \operatorname{int}\{G(Z_{\text{ad}}) - K\}$ and therefore Robinson's CQ, cf. [13, Prop. 2.106]. Finally, the Lagrange function reads as

$$L(z, \mu) = F(z) + \langle G(z), \mu \rangle.$$

For the risk-neutral PDE-constrained optimization problems the existence and optimality conditions are formulated as follows.

Theorem 4.1. *Let Assumption 2.1 and Assumption 2.7 hold. Then* (4.0.1) *admits a solution $\bar{z}$. If $\nu > 0$, then $\bar{z}$ is unique. Moreover, if there exists a $z_0 \in Z_{\text{ad}}$ and a constant $\kappa > 0$ such that*

$$S(z_0) + u_f - \psi > \kappa \tag{4.1.1}$$

then there exists a measure $\bar{\mu} \in \mathbf{ba}(\Xi, \mathcal{B}, \pi)$ such that

*(i) (**Nonpositivity**) $\bar{\mu}$ satisfies*

$$\int_\Xi g(x, \omega)\, d\bar{\mu}(x, \omega) \le 0 \quad \text{for all } g \in L^\infty_\pi(\Xi)_+.$$

*(ii) (**Complementarity**) $\bar{\mu}$ fulfills*

$$\int_\Xi G(\bar{z})(x, \omega)\, d\bar{\mu}(x, \omega) = 0.$$

*(iii) (**Subgradient Condition**) The general inclusion holds*

$$0 \in \mathbb{E}_\mathbb{P}[\boldsymbol{B}^* \boldsymbol{A}^{-*} T^* (TS\bar{z} + Tu_f - u_d)] + \nu \bar{z} + \mathcal{N}_{Z_{ad}}(\bar{z}) + \boldsymbol{B}^* \boldsymbol{A}^{-*} \iota^* \bar{\mu}.$$

Here, the latter term must be understood

$$\langle \boldsymbol{B}^* \boldsymbol{A}^{-*} \iota^* \bar{\mu}, \delta z \rangle = \int_\Xi (\boldsymbol{A}^{-1}(\omega) \boldsymbol{B}(\omega) \delta z)(x)\, d\bar{\mu}(x, \omega)$$

for an arbitrary test function $\delta z \in L^2(D)$.

Conversely, if there exists a pair $(\bar{z}, \bar{\mu})$ such that (i)-(iii) hold, then $\bar{z}$ is an optimal solution of (4.0.1).

Proof. To prove existence, we need to argue that the feasible set is weakly sequentially closed and $F(z)$ defined as $F(z) = \mathbb{E}[J(S(z) + u_f, z)]$ is weakly sequentially lower semicontinuous on $L^2(D)$. Since the assumptions on J imply F is convex and the latter component of J is deterministic and continuous, we concentrate on the properties of S and their relation to the first argument of J.

By Assumption 2.1, (4.0.1) admits a feasible point and consequently a minimizing sequence $\{z_k\} \subset Z_{\mathrm{ad}}$ such that (4.0.2) holds. Since Z_{ad} is bounded, closed, and convex, $\{z_k\}$ admits a weakly convergent subsequence $\{z_{k_l}\}$. For each l, we have

$$S(z_{k_l}) + u_f \geq \psi \text{ for a.e. } x \in D,\ \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Since S is completely continuous as a mapping into $L^2_{\mathbb{P}}(\Omega; H^1_0(D))$, we have $S(z_{k_l}) \to S(\bar{z})$ strongly. Moreover, the Sobolev embedding theorem (see e.g. [2, 4.12 Theorem]) and the fact that $L^p(\Omega; X) \hookrightarrow L^q(\Omega; Y)$ if $X \hookrightarrow Y$ for $1 \leq q \leq p < \infty$ plus the equivalence of $L^1_{\mathbb{P}}(\Omega; L^1(D))$ and $L^1_\pi(\Xi)$ (see e.g. [54, Proposition 1.2.24]) imply that $S(z_{k_l}) \to S(\bar{z})$ in $L^1_\pi(\Xi)$. Therefore, there exists a subsequence $\{z_{k_{l_m}}\}$ such that $S(z_{k_{l_m}}) \to S(\bar{z})$ π-pointwise almost everywhere. It follows that

$$S(\bar{z}) + u_f \geq \psi \text{ for } \pi\text{-a.e. } (x, \omega) \in \Xi.$$

Continuing, the integrand J induces a superposition operator that is continuous from the product space $L^2_{\mathbb{P}}(\Omega; H^1_0(D)) \times L^2(D)$ to $L^1_{\mathbb{P}}(\Omega)$, see e.g., [63, Ex. 3.2]. Then by combining the properties of S with this continuity result, we deduce the weakly lower semicontinuity of F. It follows from the direct method that $\bar{z}$ is an optimal solution, which is of course unique if $\nu > 0$ as the objective would be strictly convex.

Due to convexity, it follows from [13, Thm. 3.6] that

$$0 \in \partial_z L(\bar{z}, \bar{\mu}) + \mathcal{N}_{Z_{\mathrm{ad}}}(\bar{z}) \text{ and } \bar{\mu} \in \mathcal{N}_K(G(\bar{z})),$$

are both necessary and sufficient for optimality. It remains to make the conditions more explicit.

Since K is a closed, convex cone, $\bar{\mu} \in \mathcal{N}_K(G(\bar{z}))$ yields assertions (i) and (ii). To obtain the form in (iii), we first note that

$$F'(\bar{z})(\delta z) = \mathbb{E}_{\mathbb{P}}[(T\boldsymbol{A}^{-1}\boldsymbol{B}\bar{z} + Tu_f - u_d, T\boldsymbol{A}^{-1}\boldsymbol{B}\delta z)] + \nu(\bar{z}, \delta z)$$

and

$$\langle G(\cdot), \bar{\mu}\rangle'(\bar{z}; \delta z) = \langle \iota \boldsymbol{A}^{-1}\boldsymbol{B}(\delta z), \bar{\mu}\rangle.$$

For the objective function F, we can exploit the equivalence with the pointwise adjoints and write

$$\mathbb{E}_{\mathbb{P}}[(\boldsymbol{B}^*\boldsymbol{A}^{-*}T^*(T\boldsymbol{A}^{-1}\boldsymbol{B}\bar{z} + Tu_f - u_d), \delta z)] + \nu(\bar{z}, \delta z).$$

Furthermore, the uniform integrability of the operators $\boldsymbol{A}, \boldsymbol{B}$, i.e. $\boldsymbol{B}^*$, $\boldsymbol{A}^{-*}$ allows us to write via [44, Thm. 3.7.12]

$$F'(\bar{z})\delta z = (\mathbb{E}_{\mathbb{P}}[(\boldsymbol{B}^*\boldsymbol{A}^{-*}T^*(T\boldsymbol{A}^{-1}\boldsymbol{B}\bar{z} + Tu_f - u_d)] + \nu\bar{z}, \delta z). \tag{4.1.2}$$

This concludes the proof.

□

4.2 Probability Constraints and Moreau- Yosida Regularization

In this section, we wish to draw a theoretical link between Moreau-Yosida regularization and probability constraints for risk-neutral PDE-constrained optimization problems as formulated in (4.0.1). The main tools are basic concentration inequalities from probability theory.

The Moreau-Yosida regularization yields the following γ-dependent optimization problem:

$$\min_{z \in Z_{\mathrm{ad}}} \left\{ \mathbb{E}_{\mathbb{P}} \left[J(S(z) + u_f, z) + \frac{\gamma}{2} \|(\psi - (S(z) + u_f))_+\|^2_{L^2(D)} \right] \right\}, \tag{4.2.1}$$

where $\gamma > 0$. We recall that yet another way of formulating the original state constraint is given by

$$\mathbb{P}\left(\|(\psi - (S(z) + u_f))_+\|^2_{L^2(D)} \le 0 \right) = 1.$$

Since the $L^\infty(D)$-norm allows strong violation of the constraint on small subsets of positive measure for the weaker constraint, it would be ideally, we would use the $L^\infty(D)$-norm as opposed to the $L^2(D)$-norm

$$\mathbb{P}\left(\|(\psi - (S(z) + u_f))_+\|^2_{L^2(D)} \le \varepsilon \right) = 1,$$

for $\varepsilon > 0$, but arbitrarily small. In order to derive a result of the type in the following theorem with the $L^\infty(D)$-norm, we would need a careful analysis similar to [49]. This could be considered in future work.

Proposition 4.2. *Let z^γ be the unique minimizer of* (4.2.1). *Then for any $\varepsilon > 0$, we have*

$$\mathbb{P}\left(\|(\psi - (S(z^\gamma) + u_f))_+\|^2_{L^2(D)} < \varepsilon \right) \ge 1 - \frac{2M}{\gamma\varepsilon},$$

where $M = \mathbb{E}_{\mathbb{P}}[J(S(z), z)]$ and z is the unique minimizer of (4.0.1).

Proof. Using Markov's inequality, we have

$$\mathbb{P}\left(\|(\psi-(S(z^\gamma)+u_f))_+\|^2_{L^2(D)}\geq\varepsilon\right)\leq\frac{\mathbb{E}_\mathbb{P}\left[\|(\psi-(S(z^\gamma)+u_f))_+\|^2_{L^2(D)}\right]}{\varepsilon}.$$

We use z^γ to obtain a simpler upper bound. By definition of z^γ, it holds that

$$\begin{aligned}&\mathbb{E}_\mathbb{P}\left[J(S(z_\gamma)+u_f,z_\gamma)\right]+\frac{\gamma}{2}\mathbb{E}_\mathbb{P}\left[\|(\psi-(S(z_\gamma)+u_f))_+\|^2_{L^2(D)}\right]\\&\leq\mathbb{E}_\mathbb{P}\left[J(S(v)+u_f,v)\right]+\frac{\gamma}{2}\mathbb{E}_\mathbb{P}\left[\|(\psi-(S(v)+u_f))_+\|^2_{L^2(D)}\right].\end{aligned}$$

for all $v\in Z_{\mathrm{ad}}$. In particular, we obtain the bound

$$\mathbb{E}_\mathbb{P}\left[J(S(z_\gamma)+u_f,z_\gamma)\right]+\frac{\gamma}{2}\mathbb{E}_\mathbb{P}\left[\|(\psi-(S(z_\gamma)+u_f))_+\|^2_{L^2(D)}\right]\leq\mathbb{E}_\mathbb{P}\left[J(S(v)+u_f,v)\right]$$

for all $v\in Z_{\mathrm{ad}}$ such that $S(v)+u_f\geq\psi$ for π-a.e. $(x,\omega)\in\Xi$. Using the minimizer z of (4.0.1) leads to

$$\begin{aligned}\frac{\gamma}{2}\mathbb{E}_\mathbb{P}\left[\|(\psi-(S(z_\gamma)+u_f))_+\|^2_{L^2(D)}\right]&\leq\mathbb{E}_\mathbb{P}\left[J(S(z)+u_f,z)\right]-\mathbb{E}_\mathbb{P}\left[J(S(z_\gamma)+u_f,z_\gamma)\right]\\&\leq\mathbb{E}_\mathbb{P}\left[J(S(z)+u_f,z)\right]\end{aligned}$$

since $\mathbb{E}_\mathbb{P}\left[J(S(z_\gamma)+u_f,z_\gamma)\right]\geq 0$. Now we set

$$M=\mathbb{E}_\mathbb{P}\left[J(S(z)+u_f,z)\right]$$

From this we obtain $\mathbb{E}_\mathbb{P}\left[\|(\psi-(S(z_\gamma)+u_f))_+\|^2_{L^2(D)}\right]\leq\frac{2M}{\gamma}$. Then returning to Markov's inequality, we now have

$$\mathbb{P}\left(\|(\psi-(S(z^\gamma)+u_f))_+\|^2_{L^2(D)}\geq\varepsilon\right)\leq\frac{2M}{\gamma\varepsilon}.$$

Finally, the complementary event is given by

$$\mathbb{P}\left(\|(\psi-(S(z^\gamma)+u_f))_+\|^2_{L^2(D)}<\varepsilon\right)\geq 1-\frac{2M}{\gamma\varepsilon}.\tag{4.2.2}$$

□

Using the analysis from Section 3.2.3, we know that there exists a sequence $\gamma_n\to+\infty$ such that the random variable

$$X_n:=\|(\psi-(S(z^{\gamma_n})+u_f))_+\|^2_{L^2(D)}$$

converges strongly in $L^1(\Omega,\mathcal{F},\mathbb{P})$ to

$$X^*:=\|(\psi-(S(z^*)+u_f))_+\|^2_{L^2(D)}.$$

Since z^* is feasible, the state constraint holds and $X^* \equiv 0$. Therefore, there exists a subsequence $\gamma_k := \gamma_{n_k}$ along which $X_k := X_{n_k}$ converges almost surely to 0; and consequently in distribution as well. For each k, we can set $\varepsilon_k = 1/\sqrt{\gamma_k}$ and treat $Y_k := \varepsilon_k$ as a degenerate random variable, which clearly converges in distribution to 0. It follows from Slutsky's theorem that $X_k + Y_k$ converges in distribution to $\|(\psi - (S(z^*) + u_f))_+\|^2_{L^2(D)}$, i.e., 0 and since

$$\mathbb{P}\left(\|(\psi - (S(z^\gamma) + u_f))_+\|^2_{L^2(D)} < \varepsilon\right) \leq \mathbb{P}\left(\|(\psi - (S(z^\gamma) + u_f))_+\|^2_{L^2(D)} \leq \varepsilon\right),$$

the Portmanteau lemma yields

$$\mathbb{P}\left(\|(\psi - (S(z^*) + u_f))_+\|^2_{L^2(D)} \leq 0\right) \geq \limsup_{k\to+\infty} \mathbb{P}\left(\|(\psi - (S(z^{\gamma_k}) + u_f))_+\|^2_{L^2(D)} - \varepsilon_k \leq 0\right) \geq \limsup_{k\to\infty} 1 - \frac{2M}{\sqrt{\gamma_k}} = 1.$$

In this sense, Proposition 4.2 provides us with a probabilistic rate of convergence from Moreau-Yosida to feasibility for the original problem.

Similarly, we can obtain information about the tail of the distribution of $\|(\psi - (S(z^\gamma) + u_f))_+\|^2_{L^2(D)}$.

Theorem 4.3. *We assume that* $X_\gamma = \|(\psi - (S(z^\gamma) + u_f))_+\|^2_{L^2(D)}$ *is* $o(\gamma^{-\frac{1}{q}})$ *for* $q \in (0,1)$. *For* $k > 0$, *we have*

$$\mathbb{P}(\|(\psi - (S(z^\gamma) + u_f))_+\|^2_{L^2(D)} \geq k) \leq \exp\left(o(1)\right) \exp\left(\frac{\overline{X}_\gamma}{8\gamma} - \frac{k}{\gamma^{1/2}}\right)$$

where $\overline{X}_\gamma = \operatorname{ess\,sup}_{\omega\in\Omega} \|(\psi - (S(z^\gamma) + u_f))_+\|^2_{L^2(D)}$.

Proof. Recall that for a random variable $X : \Omega \to \mathbb{R}$ the following exponential Chebyshev inequality holds

$$\mathbb{P}(X \geq k) \leq \exp(-tk)\mathbb{E}_{\mathbb{P}}[\exp(tX)]$$

for $t > 0$. In our setting this reads

$$\mathbb{P}(\|(\psi - (S(z^\gamma) + u_f))_+\|^2_{L^2(D)} \geq k) \leq \exp(-tk)\mathbb{E}_{\mathbb{P}}[\exp(t\|(\psi - (S(z^\gamma) + u_f))_+\|^2_{L^2(D)})] \tag{4.2.3}$$

for $t > 0$. Set $X_\gamma = \|(\psi - (S(z^\gamma) + u_f))_+\|^2_{L^2(D)}$. There exists $\overline{X}_\gamma > 0$ such that $0 \leq X_\gamma \leq \overline{X}_\gamma$ a.s, i.e. X_γ is a bounded random variable. We can apply Hoeffding's lemma in order to provide an upper bound for the moment-generating function. We obtain

$$\mathbb{E}_{\mathbb{P}}[\exp(t\|(\psi - (S(z^\gamma) + u_f))_+\|^2_{L^2(D)})] \leq \exp\left(t\mathbb{E}_{\mathbb{P}}[\|(\psi - (S(z^\gamma) + u_f))_+\|^2_{L^2(D)})] + \frac{t^2\overline{X}_\gamma}{8}\right).$$

Together with (4.2.3), this reads as

$$\begin{aligned}&\mathbb{P}(\|(\psi-(S(z^\gamma)+u_f))_+\|^2_{L^2(D)}\geq k)\\&\quad\leq\exp\left(t\mathbb{E}_{\mathbb{P}}[\|(\psi-(S(z^\gamma)+u_f))_+\|^2_{L^2(D)})]+\frac{t^2\overline{X}_\gamma}{8}-tk\right).\end{aligned}$$

Since $X_\gamma=o(\gamma^{-\frac{1}{q}})$ for $q\in(0,1)$, we have

$$\mathbb{P}(\|(\psi-(S(z^\gamma)+u_f))_+\|^2_{L^2(D)}\geq k)\leq\exp\left(to(\gamma^{-\frac{1}{q}})+\frac{t^2\overline{X}_\gamma}{8}-tk\right).$$

Choosing $q=1/2$ and $t=\gamma^{-1/2}$ we have

$$\mathbb{P}(\|(\psi-(S(z^\gamma)+u_f))_+\|^2_{L^2(D)}\geq k)\leq\exp\left(o(1)+\frac{\overline{X}_\gamma}{8\gamma}-\frac{k}{\gamma^{1/2}}\right).$$

□

In what follows, we provide a bound on how the regularization term $\frac{\gamma}{2}E_{\mathbb{P}}\left[\|(\psi-(S(z^\gamma)+u_f))_+\|^2_{L^2(D)}\right]$ deviates from its expected value by using Chebychev's inequality.

Theorem 4.4. *For $\varepsilon>0$ and z_γ minimizing* (4.2.1)*, we have*

$$\begin{aligned}&\mathbb{P}\left(|\|(\psi-(S(z^\gamma)+u_f))_+\|^2_{L^2(D)}-\mathbb{E}_{\mathbb{P}}\left[\|(\psi-(S(z^\gamma)+u_f))_+\|^2_{L^2(D)}\right]|\geq\varepsilon\right)\\&\qquad\leq\frac{E_{\mathbb{P}}\left[\|(\psi-(S(z^\gamma)+u_f))_+\|^4_{L^2(D)}\right]}{\varepsilon^2}.\end{aligned}$$

Proof. Applying the Chebychev's inequality directly yields

$$\begin{aligned}&\mathbb{P}\left(|\|(\psi-(S(z^\gamma)+u_f))_+\|^2_{L^2(D)}-\mathbb{E}_{\mathbb{P}}\left[\|(\psi-(S(z^\gamma)+u_f))_+\|^2_{L^2(D)}\right]|\geq\varepsilon\right)\\&\qquad\leq\frac{\mathrm{Var}\left[\|(\psi-(S(z^\gamma)+u_f))_+\|^2_{L^2(D)}\right]}{\varepsilon^2}.\end{aligned}$$

Since for any $a\in\mathbb{R}$ we have $\mathbb{E}\left[(X-a)^2\right]=\mathrm{Var}[X]+(\mu-a)^2$. Thus, $\mathrm{Var}(X)\leq\mathbb{E}\left[(X-a)^2\right]$ and $\mathrm{Var}[X]\leq\mathbb{E}\left[X^2\right]$ for $a\in\mathbb{R}$ and $a=0$ respectively. Choosing $X=\|(\psi-(S(z^\gamma)+u_f))_+\|^2_{L^2(D)}$ and $a=0$ provide the claim. □

Chapter 5

Theory of a Numerical Approach for Risk-Neutral PDE-Constrained Optimization Problem and GNEPs

In this section, we provide a theoretical presentation of an algorithm to indicate how stochastic PDE-constrained optimization problems subject to pointwise state constraints and PDE-constrained GNEPs under uncertainty might be solved building on our theoretical results. The basic idea derives from the success of semismooth Newton methods for solving deterministic PDE-constrained optimization problems subject to state constraints using Moreau-Yosida regularization and path-following for the parameter updates; see e.g., [47, 48] combined with a Monte Carlo estimator in order to replace the underlying probability distribution with an associated empirical probability measure. The focus will be on the theoretically numerical approach of the individual optimization problems. For the GNEP, a Krasnoselskii-Mann-type alternating method is suggested in which the competing agents use the solver from the individual optimization problems.

5.1 Solving the individual Problem

We shortly present the suggested component of an algorithm, before we state the algorithm that is a verbatim quote from [29].

5.1.1 Monte Carlo Estimator

Typically there are either projection-based or sample-based discretizations for PDEs with uncertain inputs. For the primal formulation, we approximate the expected value in the risk-neutral objective function using a sample-based discretization, namely Monte Carlo estimator. Sample-based discretizations allow us to more easily reuse existing deterministic optimal control solvers.

Given an independent and identically distributed set of samples with size M and random field $v \in L^2_{\mathbb{P}}(\Omega; H^1_0(D))$ we estimate the expectation $\mathbb{E}_{\mathbb{P}}[v]$ by the Monte Carlo estimator

$$\frac{1}{M}\sum_{i=1}^{M} v(\omega_i) = \int_{\Omega} v(\omega)\, d\mathbb{P}_M(\omega),$$

where $\mathbb{P}_M$ is the associated empirical probability measure. Thus, we can consider the discretization of Ω as replacing the continuous probability measure $\mathbb{P}$ with a sum of point masses centered at the sample points. Finally, we consider the deterministic problem

$$\min_{z\in Z_{\mathrm{ad}}} \left\{ \frac{1}{M}\sum_{m=1}^{M} \Big[J(S(z)(\omega^m) + u_f(\omega^m), z) + \frac{\gamma}{2}\|(\psi(\omega^m) - (S(z)(\omega^m) + u_f(\omega^m)))_+\|^2_{L^2(D)} \Big] \right\} \tag{5.1.1}$$

This typical Monte Carlo approach achieves a convergence rate of $M^{1/2}$.

5.1.2 Semismooth Newton

In order to solve the deterministic regularized problem (5.1.1) using semismooth Newton method, we recall the first order optimality system. Note that we can either consider

$$A(\omega^m)u^\gamma_m = B(\omega^m)z^{\gamma,M} \tag{5.1.2}$$

$$A^*(\omega^m)\lambda^\gamma_m = T^*(Tu^\gamma_m + Tu_f(\omega^m) - u_d) - \gamma(\psi(\omega^m) - (u^\gamma_m + u_f(\omega^m)))_+ \tag{5.1.3}$$

or we rewrite the first order optimality system as a single nonsmooth equation

$$z^{\gamma,M} = \mathrm{Proj}_{Z_{ad}}\left[-\frac{1}{\nu M}\sum_{m=1}^{M} B^*(\omega^m)\lambda^\gamma_m \right], \tag{5.1.4}$$

where for each $m = 1, \ldots, M$, $\lambda^\gamma_m \in H^1_0(D)$ solves (5.1.3) and $u^\gamma_m = S(z^{\gamma,M})(\omega^m) \in H^1_0(D)$ solves (5.1.2). Using the pointwise max-operator of 0 and an arbitrary function $f : D \to \mathbb{R}$, denoted by $(f)_+$ and defined by

$$(f)_+ = \max\{0, f(\cdot)\} : D \to \mathbb{R},$$

we reformulate (5.1.4) as follows

$$z^{\gamma,M} = -\frac{1}{\nu M}\sum_{m=1}^{M} B^*(\omega^m)\lambda^\gamma_m - \left(-\frac{1}{\nu M}\sum_{m=1}^{M} B^*(\omega^m)\lambda^\gamma_m - b \right)_+ + \left(\frac{1}{\nu M}\sum_{m=1}^{M} B^*(\omega^m)\lambda^\gamma_m + a \right)_+ .$$

Now, we define a mapping $F_M^\gamma : H_0^1(D) \times H_0^1(D) \to H_0^1(D) \times H_0^1(D)$, for which we are interested to compute a root.

$$F_M^\gamma(u_i^\gamma, \lambda_i^\gamma) = \tag{5.1.5}$$
$$\begin{pmatrix} B(\omega^i)\Big(-\frac{1}{\nu M}\sum_{m=1}^M B^*(\omega^m)\lambda_m^\gamma - \Big(-\frac{1}{\nu M}\sum_{m=1}^M B^*(\omega^m)\lambda_m^\gamma - b\Big)_+ + \Big(\frac{1}{\nu M}\sum_{m=1}^M B^*(\omega^m)\lambda_m^\gamma + a\Big)_+\Big) - A(\omega^i)u_i^\gamma \\ T^*(Tu_i^\gamma + Tu_f(\omega^i) - u_d) - \gamma(\psi(\omega^i) - (u_i^\gamma + u_f(\omega^i)))_+ - A^*(\omega^i)\lambda_i^\gamma \end{pmatrix}$$

for $i = 1, \ldots, M$.

Since the function F_M^γ in (5.1.5) is nonsmooth, we will need a generalized derivative concept in oder to design an algorithm based on a (nonsmooth) Newton step.

Definition 5.1. A mapping $F : X \to Y$, with X and Y Banach spaces, is called Newton differentiable in an open set $U \subset X$ if there exists a family of generalized derivatives $G_F(x) \in \mathcal{L}(X, Y)$, $x \in U$, such that

$$\lim_{h \to 0} \frac{\|F(x+h) - F(x) - G_F(x+h)h\|_Y}{\|h\|_X} = 0 \tag{5.1.6}$$

for every $x \in U$.

Every Frechet differentiable function with continuous Frechet derivative is Newton differentiable and the Newton derivative is exactly the Frechet derivative. We outline some calculus rules, which can be shown similarly to those for Frechet derivatives.

Lemma 5.2. *Let X, $Y, Y_1, \ldots, Y_m$ and Z be Banach spaces.*

(i) If $F : X \to Y$ is continuously Frechet differentiable at $x \in X$, then F is also Newton differentiable in x with Newton derivative $D_N F(x+h) = F'(x)h$.

(ii) If $F, G : X \to Y$ are Newton differentiable at $x \in X$, then so is $F + G$. The Newton derivative is given by $D_N(F+G)(x+h) = D_N F(x+h) + D_N G(x+h)$.

(iii) Let $F_i : X \to Y_i$ be Newton differentiable with Newton derivative $D_N F_i$ for $1 \leq i \leq m$. Then,

$$F : X \to Y_1 \times \cdots \times Y_m,\ x \mapsto (F_1(x), \ldots, F_m(x))^T$$

is also Newton differentiable with Newton derivative $D_N F(x) = (D_N F_1(x), \ldots, D_N F_m(x))^T$.

(iv) Suppose that $H : D \subset X \to Y$ is continuously Frechet differentiable at $x \in D$ and $G : Y \to Z$ is Newton differentiable at $H(x)$ with Newton derivative $D_N G$. Then $F = G \circ H$ is Newton differentiable at x with Newton derivative $D_N F(x+h) = D_N G(x+h)H'(x+h) \in \mathcal{L}(X, Z)$ for h sufficiently small.

Proof. For the sum rule this is immediate, the proof of the chain rule can be found in [55, Lemma 8.15] for example. □

Note that the chain rule also holds for two Newton differentiable functions.

Due to the concept of semismoothness in finite dimensions it is becoming customary to refer to the Newton method in infinite dimensions, as a semismooth Newton method, if (5.1.6) holds. As usual the Newton method for finding $x^* \in X$ such that $F(x^*) = 0$ consists of the iterations:

Algorithm 1: Semismooth Newton method

1: Choose $x_0 \in X$.
2: **while** some stopping rule is violated **do**
3: perform the update step

$$x_{k+1} = x_k - G_F(x_k)^{-1} F(x_k) \quad \text{for } k = 0, 1, \ldots. \tag{5.1.7}$$

4: **end while**

Property (5.1.6) quantifies one of the essential ingredients for the Newton method to be locally superlinearly convergent. In fact, the iteration (5.1.7) is locally q-superlinearly convergent to x^* within a neighborhood $U(x^*)$, if

i) $x_0 \in U(x^*)$,

ii) $G_F(x)$ is nonsingular for all $x \in U(x^*)$ as well as

iii) $\{\|G_F(x)^{-1}\|_{\mathcal{L}(X,Y)} \mid x \in U(x^*)\}$ is bounded.

For more details on semismooth Newton method, we refer the reader to [45] and [46].

This raises questions about the Newton differentiability of the function F_M^γ in (5.1.5) and if so whether the Newton derivative is invertible within a neighborhood of the solution and whether the inverse is uniformly bounded.

Lemma 5.3. *The mapping* $(\cdot)_+ = \max(0, \cdot) : L^p(D) \to L^q(D)$ *with* $1 \leq q < p \leq \infty$ *is Newton-differentiable on* $L^p(D)$ *and the Newton derivative is given by*

$$G_+(y)(x) = \begin{cases} 1, & y(x) > 0 \\ \delta, & y(x) = 0 \\ 0, & y(x) < 0, \end{cases} \tag{5.1.8}$$

where $\delta \in \mathbb{R}$ *is arbitrary.*

For the proof we refer to [46, Proposition 4.1 and Appendix A].

According to the definition, it holds that $G_+(y) \in \mathcal{L}(L^p(D), L^q(D))$. Moreover, choosing $\delta = 0$, then the Newton derivative can be written as a multiplying operator

$$G_+(y)h = \chi_{\mathcal{A}(y)} \cdot h$$

where $\mathcal{A}(y) = \{x \in D \,|\, y(x) > 0\}$ and $\chi : D \to \{0,1\}$ is the characteristic function. Note that G_+ in (5.1.8) can in general not serve as Newton derivative for $(\cdot)_+ : L^p(D) \to L^p(D)$ with $1 \leq p \leq \infty$.

Due to the Sobolev Embedding, it holds that

$$u_i^\gamma, \lambda_i^\gamma \in H_0^1(D) \hookrightarrow L^p(D)$$

where $p > 2$, more precisely $p < \infty$ if $d = 2$ and $p \leq 6$ if $d = 3$, cf. [2, Theorem 5.4] for all $i = 1, \dots, M$. The Stammpacchia's Lemma ensures $(u_i^\gamma)_+$ and $(\lambda_i^\gamma)_+ \in H_0^1(D)$ respectively since $u_i^\gamma, \lambda_i^\gamma \in H_0^1(D)$. Hence,

$$\frac{1}{\nu M}\sum_{m=1}^{M} B^*(\omega^m)\lambda_m^\gamma + a \in L^{2+\sigma}(D) \quad \text{and} \quad -\frac{1}{\nu M}\sum_{m=1}^{M} B^*(\omega^m)\lambda_m^\gamma - b \in L^{2+\sigma}(D).$$

Thus, we are in the situation where $(\cdot)_+ : L^p(D) \to L^2(D)$ with $1 \leq 2 < p \leq \infty$.

Now, by using Lemma 5.3 and the chain rule, we obtain the Newton derivative of F_M^γ in (5.1.5) by setting

$$G_{F_M^\gamma}(u_i^\gamma, \lambda_i^\gamma) = \begin{pmatrix} G^{1,1}_{F_M^\gamma}(u_i^\gamma, \lambda_i^\gamma) & G^{1,2}_{F_M^\gamma}(u_i^\gamma, \lambda_i^\gamma) \\ G^{2,1}_{F_M^\gamma}(u_i^\gamma, \lambda_i^\gamma) & G^{2,2}_{F_M^\gamma}(u_i^\gamma, \lambda_i^\gamma) \end{pmatrix},$$

where

$$\begin{aligned}
G^{1,1}_{F_M^\gamma}(u_i^\gamma, \lambda_i^\gamma) &= -A(\omega^i), \\
G^{1,2}_{F_M^\gamma}(u_i^\gamma, \lambda_i^\gamma) &= -\frac{1}{\nu M} B(\omega^i)B(\omega^i)^* \\
&\quad + B(\omega^i)B(\omega^i)^* G_+\left(-\frac{1}{\nu M}\sum_{m=1}^{M} B^*(\omega^m)\lambda_m^\gamma - b\right) \\
&\quad + B(\omega^i)B(\omega^i)^* G_+\left(\frac{1}{\nu M}\sum_{m=1}^{M} B^*(\omega^m)\lambda_m^\gamma + a\right), \\
G^{2,1}_{F_M^\gamma}(u_i^\gamma, \lambda_i^\gamma)) &= T^*T + \gamma G_+(\psi(\omega^i) - (u_i^\gamma + u_f(\omega^i))), \\
G^{2,2}_{F_M^\gamma}(u_i^\gamma, \lambda_i^\gamma) &= -A^*(\omega^i).
\end{aligned}$$

Clearly, $G_{F_M^\gamma}(u_i^\gamma, \lambda_i^\gamma) : H_0^1(D) \times H_0^1(D) \to H^{-1}(D) \times H^{-1}(D)$ is a bounded linear operator. We define the following approximations of the active sets:

$$\begin{aligned}
\mathcal{A}^a(\lambda_i^\gamma) &= \left\{x \in D \,\middle|\, \frac{1}{\nu M}\sum_{m=1}^{M} B^*(\omega^m)\lambda_m^\gamma + a > 0\right\}, \\
\mathcal{A}^b(\lambda_i^\gamma) &= \left\{x \in D \,\middle|\, -\frac{1}{\nu M}\sum_{m=1}^{M} B^*(\omega^m)\lambda_m^\gamma - b > 0\right\}, \\
\mathcal{A}^{\psi(\omega_i)}(u_i^\gamma) &= \left\{x \in D \mid \psi(\omega^i) - (u_i^\gamma + u_f(\omega^i)) > 0\right\}
\end{aligned}$$

for $i = 1, \dots, M$. We define approximations for the inactive sets by

$$I(\lambda_i^\gamma) = D \setminus \left(\mathcal{A}^a(\lambda_i^\gamma) \cup \mathcal{A}^b(\lambda_i^\gamma)\right), \; J_i(u_i) = D \setminus \mathcal{A}^{\psi(\omega_i)}(u_i^\gamma)$$

for $i = 1, \dots, M$. Then the Newton derivative of F_M^γ in (5.1.5) can be written as

$$\begin{aligned} G_{F_M^\gamma}(u_i^\gamma, \lambda_i^\gamma) &= \begin{pmatrix} -A(\omega^i) & -\frac{1}{\nu M} B_i B_i^* \left(I - \chi_{\mathcal{A}^b(\lambda_i^\gamma)} - \chi_{\mathcal{A}^a(\lambda_i^\gamma)}\right) \\ I + \gamma \chi_{\mathcal{A}^{\psi(\omega_i)}(u_i^\gamma)} & -A^*(\omega^i) \end{pmatrix} \\ &= \begin{pmatrix} -A(\omega^i) & -\frac{1}{\nu M} B_i B_i^* \chi_{I(\lambda_i^\gamma)} \\ I + \gamma \chi_{\mathcal{A}^{\psi(\omega_i)}(u_i^\gamma)} & -A^*(\omega^i) \end{pmatrix}. \end{aligned}$$

Since a Newton derivative of F_M^γ is available, a generalized Newton step can be derived. The update step (5.1.7) is equivalent to first solving the following system

$$\begin{pmatrix} -A(\omega^i) & -\frac{1}{\nu M} B_i B_i^* \chi_{I(\lambda_i^\gamma)} \\ I + \gamma \chi_{\mathcal{A}^{\psi(\omega_i)}(u_i^\gamma)} & -A^*(\omega^i) \end{pmatrix} \begin{pmatrix} du_i^\gamma \\ d\lambda_i^\gamma \end{pmatrix} = -F_M^\gamma(u_i^\gamma, \lambda_i^\gamma) \tag{5.1.9}$$

and then setting $u_i^{\gamma,\text{new}} = u_i^\gamma + du_i^\gamma$ and $\lambda_i^{\gamma,\text{new}} = \lambda_i^\gamma + d\lambda_i^\gamma$.

5.1.3 Algorithm

For readability, we will restrict ourselves to the first order optimality system as a single nonsmooth equation in z in (5.1.4). Therefore, we denote the mapping $z \mapsto B^*\lambda^\gamma$ as $\Lambda(z)$ or $\Lambda(z, \omega)$ to indicate the dependence on ω. Moreover, we set

$$F_M^\gamma(z) := z - \text{Proj}_{Z_{ad}} \left[-\frac{1}{\nu M} \sum_{m=1}^M \Lambda(z, \omega^m) \right].$$

In the current setting, $F_M^\gamma : L^2(D) \to L^2(D)$ admits a Newton derivative $G_M^\gamma(z)$ of the form

$$G_M^\gamma(z)dz = \left[I + \frac{1}{\nu M} \sum_{i=1}^M \mathcal{G}\left[\Lambda(z, \omega^m)\right] \Lambda'(z, \omega^m) \right] dz,$$

where $\mathcal{G}$ is the Newton derivative of the projection operator. This allows us to apply a semismooth Newton method [46, 92], which is known to be locally superlinearly convergent for each M and $\gamma > 0$. However, since γ must be taken to $+\infty$, such an algorithm would not be computationally efficient if M were chosen large for comparatively small γ. If M were to remain fixed, then we could use a strategy as in [1,47,48]. On the other hand, M should be ideally as large as possible or also treated as a parameter going to $+\infty$. To remedy this issue, we suggest to set a maximum allowable sample size $M_{\max} > 0$ and penalty parameter $\gamma_{\max} > 0$ and, starting with $M_0 \in \mathbb{N}$ and $\gamma_0 > 0$, we add samples to M_k every time γ_k passes a certain threshold. The full algorithm is given in Algorithm 2. A few comments are in order. Having the previous subsections in mind, we know that the operator $G^{\gamma_k}_{M_k}(z^k_l)$ is not explicitly given. Thus, it is necessary to use an iterative method to solve for the Newton steps dz^k_l, for which we use the tolerance $\texttt{tol}^{\texttt{newt}} \geq 0$. Since we plan to use a semismooth Newton iteration for pointwise bound constraints, the components of dz^k_l are fixed on the estimated active sets for each l and we only need to solve the linear systems on the potentially smaller inactive set as described in Section 5.1.2. Here, it is important to note that each evaluation of $G^{\gamma_k}_{M_k}(z^k_l)dz^k_l$ requires the solution of the forward equation and two adjoint equations for every sample $m_k = 1, \ldots, M_k$. Similarly, the evaluation of the residual $F^{\gamma}_{M_k}(z^k_l)$ requires a forward and adjoint solve for each sample. We suggest a direct solver for the linear elliptic PDEs. Due to these facts, we suggest starting with a relatively small M_0 and increasing slowly with γ_k. Moreover, we suggest a relatively large $\texttt{tol}^{\text{res}}_0 > 0$ and ρ^{res} close to 1. In step 13 of Algorithm 2, we simply set $\gamma_{k+1} = \phi(\gamma_k) = \gamma_k + 1$. More aggressive strategies may be possible, but empirical evidence suggests that this is not necessary and may even cause the Newton iteration to cycle. Finally, in step 15 of Algorithm 2, we link the increases of the sample sizes

M_k to γ_k.

Algorithm 2: SSN for Stochastic PDE-Constrained Optimization with State Constraints

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Input (Data): $u_d \in L^2(D)$; $\nu > 0$; $a, b \in L^\infty(D)$ $a < b$; $\psi \in C(\overline{\Xi})$; $f \in L^\infty_{\mathbb{P}}(\Omega; L^2(D))$
Input (Parameters): $k := 0$, $\gamma_0 > 0$, $\gamma_{\max} \geq \gamma_0 > 0$, $M_0 \in \mathbb{N}$, $M_{\max} \geq M_0$, $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, $\mathtt{tol}_0^{\mathrm{res}} > 0$, $\rho^{\mathrm{res}} \in (0, 1)$, $\mathtt{tol}^{\mathrm{newt}} \geq 0$
Input (Initial Values): $z_0 \in L^2(D)$, $dz_0 \equiv 0 \in L^2(D)$
while $\gamma_k < \gamma_{\max}$ **do**
  Set $l := 0$
  Set $z_l^k := z_k$
  while $\|F_{M_k}^{\gamma_k}(z_l^k)\| > \mathtt{tol}_k^{\mathrm{res}}$ **do**
    Find $dz_l^k \in L^2(D)$ such that

$$\|G_{M_k}^{\gamma_k}(z_l^k) dz_l^k + F_{M_k}^{\gamma_k}(z_l^k)\|_{L^2(D)} \leq \mathtt{tol}^{\mathrm{newt}} \tag{5.1.10}$$

    Set $z_{l+1}^k := z_l^k + dz_l^k$
    Set $l := l + 1$
  end while
  Set $z_{k+1} = z_l^k$
  Set $\gamma_{k+1} = \phi(\gamma_k)$
  if "penalty-to-sample threshold' **then**
    Choose $M_{k+1} \geq \min(M_k, M_{\max})$
  else
    Set $M_{k+1} := M_k$
  end if
  Set $\mathtt{tol}_{k+1}^{res} = \rho \mathtt{tol}_k^{res}$
  Set $k := k + 1$
end while

Numerical experiments as well as a full convergence analysis linking sampling, approximation and smoothing error remains open for future project.

5.2 Solving the Risk-Netural PDE-Constrained GNEP

The algorithm is a represented result from [29]. Due to the preceded analysis, especially the fact, that we focused on variational equilibria, we suggest to employ a fixed point strategy to solve a two-player, risk-neutral PDE-constrained GNEP.

5.2.1 Algorithm

As mentioned in the beginning of this chapter, we suggest a fixed point iteration such as a a standard Krasnoselskii-Mann iteration. We introduce the mappings $T^i(z_j)$, $i \neq j$, where

$$T^1(z_2) := \underset{z_1 \in Z^1_{\mathrm{ad}}}{\operatorname{argmin}} \left\{ \mathbb{E}_{\mathbb{P}} \left[J_1(S(z_1, z_2) + u_f, (z_1, z_2)) \right] | \mathbb{P}(S(z_1, z_2) + u_f \geq \psi) = 1 \right\}.$$

and $T^2(z_1)$ is defined analogously. We wish to find a fixed point of this mapping T, where the recursion is based on the following outer iteration:

1. Given $(z_1^{\mathrm{old}}, z_2^{\mathrm{old}}) \in L^2(D) \times L^2(D)$.
2. The first player determines $\widehat{z}_1 = T^1(z_2^{\mathrm{old}})$ through

   $$\min_{z_1 \in Z^1_{\mathrm{ad}}} \left\{ \mathbb{E}_{\mathbb{P}} \left[J_1(S(z_1, z_2^{\mathrm{old}}) + u_f, (z_1, z_2^{\mathrm{old}})) \right] | \mathbb{P}(S(z_1, z_2^{\mathrm{old}}) + u_f \geq \psi) = 1 \right\}$$

   to obtain $\widehat{z}_1$. This is revealed to the second player.
3. The second player then determines $\widehat{z}_2 = T^2(\widehat{z}_1)$ through

   $$\min_{z_2 \in Z^2_{\mathrm{ad}}} \left\{ \mathbb{E}_{\mathbb{P}} \left[J_2(S(\widehat{z}_1, z_2) + u_f, (\widehat{z}_1, z_2)) \right] | \mathbb{P}(S(\widehat{z}_1, z_2) + u_f \geq \psi) = 1 \right\}$$

   to obtain $\widehat{z}_2$, which is revealed to the first player.
4. The first player now determines $w_1 = T^1(\widehat{z}_2)$ through

   $$\min_{z_1 \in Z^1_{\mathrm{ad}}} \left\{ \mathbb{E}_{\mathbb{P}} \left[J_1(S(z_1, \widehat{z}_2) + u_f, (z_1, \widehat{z}_2)) \right] | \mathbb{P}(S(z_1, \widehat{z}_2) + u_f \geq \psi) = 1 \right\}$$

   to obtain w_1.
5. Choosing $\lambda \in (0, 1]$, the first player now updates their strategy by setting

   $$z_1^{\mathrm{new}} := (1 - \lambda) z_1^{\mathrm{old}} + \lambda w_1.$$

   The second player is assumed to choose $z_2^{\mathrm{new}} = T^2(z_1^{\mathrm{new}})$.

Obviously, (1)-(5) represents an ideal setting as the state constraint needs to be treated by a Moreau-Yosida approximation. In this context, we denote the γ-dependent mapping in steps (2)-(4) by T^i_γ. The full suggestion of an algorithm is depicted in Algorithm 3. Note that $\lambda = 1$ would correspond to a nonlinear Gauss-Seidel iteration. As the evaluation of the T-mappings requires an iterative solver in practice, we note here that z_1^{old} is used in (2), z_2^{old} in (3), $\widehat{z}_2$ in (4), and $\widehat{z}_2$ in (5). Many of the inputs in Algorithm 3 are either self-explanatory or play the same role as in Algorithm 2. Here, we introduce $\mathtt{tol}_0^{\mathrm{km}} > 0$ and $\rho^{\mathrm{km}} \in (0, 1]$, which allow us to successively reduce the tolerance used in the Krasnoselkskii-Mann iteration as γ_k (and consequently M_k) increase. We suppress the fact that certain fixed data and parameter values need to be passed to the T_γ-operators throughout the inner iterations. Though the structure of Algorithm 3 is very similar to that of Algorithm 2, it is important to note that each evaluation of $T^i_{\gamma_k}$ is associated with a semismooth

Newton solve for the current γ_k and sample of size M_k.

Algorithm 3: A Fixed Point Iteration for a Stochastic PDE-Constrained GNEP

Input (Data): $(u_{d,1}, u_{d,2}) \in L^2(D)^2$, $(\nu_1, \nu_2) \in \mathbb{R}^2_{++}$, $(a_i, b_i) \in L^\infty(D)^2$ $a_i < b_i$ $i = 1, 2$, $\psi \in C(\overline{\Xi})$, $f \in L^\infty_{\mathbb{P}}(\Omega; L^2(D))$

Input (Parameters): $k := 0$, $\lambda \in (0, 1]$, $\gamma_0 > 0$, $\gamma_{\max} \geq \gamma_0 > 0$, $M_0 \in \mathbb{N}$, $M_{\max} \geq M_0$, $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, $\mathtt{tol}_0^{\mathrm{km}} > 0$, $\rho^{\mathrm{km}} \in (0, 1]$, $\mathtt{tol}_0^{\mathrm{nwt}} > 0$, $\rho^{\mathrm{nwt}} \in (0, 1]$, $\mathtt{tol}^{\mathrm{newt}} \geq 0$

Input (Initial Values): $(z_{0,1}, z_{0,2}) \in L^2(D)^2$, $(dz_{0,1}, dz_{0,2}) \equiv 0 \in L^2(D)^2$

(Initialize): Set

$$\widehat{z}_{1,1} = T^1_{\gamma_k}(z_{0,2}), \quad \widehat{z}_{1,2} = T^2_{\gamma_k}(\widehat{z}_{1,1}), \quad w_{1,1} = T^1_{\gamma_k}(\widehat{z}_{1,2})$$

Set $z_{1,1} = (1-\lambda) z_{0,1} + \lambda w_{1,1}$, $z_{1,2} = T^2_{\gamma_k}(z_{1,1})$, and $k := k + 1$

while $\gamma_k \leq \gamma_{\max}$ **do**

    Set $l := 0$

    Set $z^k_{l,1} := z_{k,1}$ and $z^k_{l,2} := z_{k,2}$

    Set

$$\widehat{z}^k_{l+1,1} = T^1_{\gamma_k}(z^k_{l,2}) \quad \widehat{z}^k_{l+1,2} = T^2_{\gamma_k}(\widehat{z}^k_{l+1,1}) \quad w^k_{l+1,1} = T^1_{\gamma_k}(\widehat{z}^k_{l+1,2})$$

    Set $z^k_{l+1,1} = (1-\lambda) z^k_{l,1} + \lambda w^k_{l+1,1}$, $z^k_{l+1,2} = T^2_{\gamma_k}(z^k_{l+1,1})$, and $l := l + 1$

    while $\|z^k_{l,1} - z^k_{l-1,1}\|_{L^2(D)} > \mathtt{tol}_k^{\mathrm{km}}$ **do**

        Set

$$\widehat{z}^k_{l+1,1} = T^1_{\gamma_k}(z^k_{l,2}) \quad \widehat{z}^k_{l+1,2} = T^2_{\gamma_k}(\widehat{z}^k_{l+1,1}) \quad w^k_{l+1,1} = T^1_{\gamma_k}(\widehat{z}^k_{l+1,2})$$

        Set $z^k_{l+1,1} = (1-\lambda) z^k_{l,1} + \lambda w^k_{l+1,1}$, $z^k_{l+1,2} = T^2_{\gamma_k}(z^k_{l+1,1})$, and $l := l + 1$

    end while

    Set $z_{k+1,1} = z^k_{l,1}$ and $z_{k+1,2} = z^k_{l,2}$

    Set $\gamma_{k+1} = \phi(\gamma_k)$

    if "penalty-to-sample threshold' **then**

        Choose $M_{k+1} \geq \min(M_k, M_{\max})$

    else

        Set $M_{k+1} := M_k$

    end if

    Set $\mathtt{tol}^{\mathrm{km}}_{k+1} = \rho^{\mathrm{km}} \mathtt{tol}^{\mathrm{km}}_k$

    Set $k := k + 1$

end while

Also in this case, numerical experiments remains open for future project. For each fixed γ_k and M_k, the algorithm is basically a Krasnoselskii-Mann iteration with inexact evaluations of the fixed point mapping. As such, convergence can be guaranteed if the latter can be shown to be nonexpansive.

Chapter 6

Risk-Averse PDE-Constrained NEPs

In this section, we formulate a PDE-constrained NEP with risk-averse players with the use of coherent risk measures as introduced in Section 2.2. In what follows, we prove existence of Nash equilibria and derive optimality conditions. Since coherent risk measure are generally nonsmooth, the resulting PDE-constrained NEP is nonsmooth. In the light of this, we follow the general smoothing approach as presented in [65]. We smooth the coherent risk measures to obtain a sequence of parameter-dependent problems using epigraphical analysis.

6.1 Problem Formulation

Throughout this chapter, we consider an N-player noncooperative game in which the ith player faces the general problem:

$$\begin{aligned} \min_{z_i \in Z_i} \quad & \mathcal{R}_i(J_i(S(z_i, z_{-i}))) + \wp_i(z_i) \\ \text{subject to} \quad & z_i \in Z_{ad}^i. \end{aligned} \tag{6.1.1}$$

Here, z_i denotes the control variable of the ith player. Z_i denotes their space of decision variable and $Z_{ad}^i \subset Z_i$ is their set of admissible decision variables. $S(z_i, z_{-i})$ denotes the random field PDE solution, J_i is their individual quantity of interest to be minimized, and $\wp_i : Z_i \to \mathbb{R}$ is their deterministic cost of the control. Via the random field $S(z_i, z_{-i})$ the uncertainty enters in the individual quantity of interest $J_i(S(z_i, z_{-i}))$, thus $J_i(S(z_i, z_{-i}))$ is a random field. Since there is no ordering on Lebesgue space, we replace it with the scalar quantity of interest $\mathcal{R}_i(J_i(S(z_i, z_{-i})))$, where $\mathcal{R}_i : \mathcal{X} \to \bar{\mathbb{R}}$ denotes a coherent risk measure and $\mathcal{X}$ is a space of random variables. For readability, we will denote the composition of J_i and S as $F_i = J_i \circ S$. Then $F_i : Z_i \times Z_{-i} \to \mathcal{X}$ denotes the individual random field quantity of interest.

We set forth the abstract setting for our NEP. First, we introduce assumptions on spaces and the risk measures. For $i = 1, \dots, N$, we assume the following:

Assumption 6.1. (Spaces and risk measure)

(i) $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and $\mathcal{X} = L^p(\Omega, \mathcal{F}, \mathbb{P})$ with $p \in [1, \infty)$. Whenever necessary, we explicitly state the integrability p.

(ii) $\mathcal{R}_i : \mathcal{X} \to \bar{\mathbb{R}}$ is a proper coherent risk measure.

(iii) $\mathcal{R}_i$ is either finite on $\mathcal{X}$ or int dom$(\mathcal{R}_i) \neq \emptyset$.

(iv) Z_i is a real reflexive and separable Banach space.

(v) The set of admissible decision variables $Z^i_{ad} \subset Z_i$ is nonempty, closed and convex.

Regarding the individual cost functions, we assume the following for $i = 1, \dots, N$.

Assumption 6.2. (Properties of the cost functions)

(i) $\wp_i$ is convex, closed, lower semicontinuous and proper.

(ii) $\wp_i$ is continuously Gateaux differentiable.

(iii) $\wp_i'(\cdot)$ is weakly continuous, i.e., $\wp_i'(u_i^n) \rightharpoonup \wp_i'(u_i)$ whenever $u_i^n \rightharpoonup u_i$.

(iv) $(\wp_i'(\cdot), \cdot)_{U_i}$ is convex and lower semicontinuous.

As a consequence, $\wp_i$ is continuous.

In order to ensure the well-posedness of (6.1.1), we make the following assumptions on the random field quantity of interest F_i, the composition of J_i and S. For $i = 1, \dots, N$, we assume the following.

Assumption 6.3. (Properties of the random field quantity of interest F_i)

(i) For all $w_{-i} \in Z^{-i}_{ad}$, $F_i(\cdot, w_{-i}) : Z_i \to \mathcal{X}$ is continuous and convex with respect to the order on $\mathcal{X}$.

(ii) For given $w_{-i} \in Z^{-i}_{ad}$, $F_i(\cdot, w_{-i})$ is continuously Frechet differentiable.

(iii) $F_i(\cdot, \cdot)$ and $\nabla_i F_i(\cdot, \cdot)$ are completely continuous.

The appropriate assumption on the scalar quantity of interest $\mathcal{R}_i \circ F_i$ for ensuring the existence of Nash equilibria reads as follows for $i = 1, \dots, N$.

Assumption 6.4. (Properties of scalar quantity of interest $\mathcal{R}_i \circ F_i$)

(i) For all $w_{-i} \in Z^{-i}_{ad}$, $(\mathcal{R}_i \circ F_i)(\cdot, w_{-i}) : Z_i \to \mathbb{R}$ is lower semicontinuous.

(ii) For every weak convergent sequence $\{z^k_{-i}\}_{k\in\mathbb{N}} \subset Z^{-i}_{ad}$, i.e., $z^k_{-i} \rightharpoonup z^*_{-i}$, we set $\{f^k_i\}_{k\in\mathbb{N}}$ with $f^k_i : Z^i_{ad} \to \mathbb{R} \cup \{\infty\}$ where

$$f^k_i(\cdot) = (\mathcal{R}_i \circ F_i)(\cdot, z^k_{-i}) + \wp(\cdot).$$

We require $f^k_i \xrightarrow{M} f_i$, here $\xrightarrow{M}$ denotes the convergence in the sense of Mosco-convergence, i.e.,

- for all $v \in Z_i$ and all sequences $\{v^k\}_{k\in\mathbb{N}}$ with $v^k \rightharpoonup v$ such that

$$f_i(v) \leq \liminf_{k\in\mathbb{N}} f_i^k(v^k), \tag{6.1.2}$$

- for all $v \in Z_i$ there exists a sequence $\{v^k\}_{k\in\mathbb{N}}$ with $v^k \to v$ such that

$$f_i(v) \geq \limsup_{k\in\mathbb{N}} f_i^k(v^k), \tag{6.1.3}$$

where $f_i(\cdot) = (\mathcal{R}_i \circ F_i)(\cdot, z_{-i}^*) + \wp(\cdot)$.

The outlined setting may occur very abstract at first. In fact, we have already seen a class of risk averse NEPs fulfilling all required assumptions in Section 2.3. The mother problem, we have in mind, is formulated in Eqs. (2.3.1) to (2.3.3), where the coherent risk measure is chosen to be CVaR_{β_i} for a confidence level $\beta_i \in (0,1)$ see (2.2.1) for $i = 1,\ldots,N$. Then, the individual ith player solves the following optimization problem

$$\begin{aligned} \min_{z_i\in L^2(D)} \quad & \text{CVaR}_{\beta_i}\left[\frac{1}{2}\|S(z_i,z_{-i}) + u_f - u_d^i\|^2_{L^2(D)}\right] + \frac{\nu_i}{2}\|z_i\|^2_{L^2(D)} \\ \text{s.t.} \quad & z_i \in Z_{ad}^i. \end{aligned}$$

for $i = 1,\ldots,N$. Recall that CVaR_{β_i} highlights rare and unlikely events, i.e, events with low probability, when $\beta_i \geqslant 0$. In financial mathematics it is common to optimize the conditional value-at-risk to determine risk-averse investment strategies, see e.g. [28,93]. In engineering fields, tail-probability events often correspond to failure of the engineered system. Even so, tail-probability events are by definition rare, when they occur, they have outsized consequences. As such, there is an increased interest in avoiding them. By minimizing CVaR it is possible to conservatively manage these risks.

6.2 Existence and Optimality Conditions

In this section, we prove the existence of Nash equilibria and derive first-order optimality conditions for (6.1.1). As before, we restrict ourselves to variational equilibria, so that, again the essential ingredient in order to prove the existence of Nash equilibria is the Kakutani-Fan-Glicksberg fixed point theorem. In order to do so, we make use of the potentially set-valued collective best-response function. For readability, we deviate consciously from the previous notation of the collective best-response function $\widehat{\mathcal{R}}$ and write Ψ instead to highlight the differences between the risk measure and the collective best-response function. Here, we set $\Psi : Z_{ad}^1 \times \cdots \times Z_{ad}^N \rightrightarrows Z_{ad}^1 \times \cdots \times Z_{ad}^N$ such that

$$\Psi(z) = (\psi_1(z_{-1}),\ldots,\psi_N(z_{-N})),$$

where for $i = 1, \ldots, N$ the best-response function for each player is given by

$$\begin{aligned}\psi_i : Z_{ad}^{-i} &\rightrightarrows Z_{ad}^i \\ z_{-i} &\mapsto \operatorname*{arg\,min}_{v_i \in Z_{ad}^i} \left\{ \mathcal{R}_i(F_i(v_i, z_{-i})) + \wp_i(v_i) \right\}.\end{aligned}$$

Next, we study the properties of the best-response functions.

Lemma 6.5. *For $i = 1, \ldots, N$ the set valued function ψ_i has nonempty and convex images.*

Proof. Let z_{-i} be given, the objective functional $\mathcal{R}_i(F_i(\cdot, z_{-i})) + \wp(\cdot)$ is weakly lower semicontinuous and Z_{ad}^i is bounded. The direct method of the calculus of variations (cf. [67, Theorem 1.1] and [67, Remark 1.3]) ensures that ψ_i has nonempty values. Clearly, since $(\mathcal{R}_i \circ F_i)(\cdot, z_{-i}) + \wp_i(\cdot)$ and Z_{ad}^i are convex, so is the set of minimizers and by this $\psi_i(z_{-i})$ is convex. □

We recall that a Nash equilibrium can be characterized as fixed point of Ψ. Namely, a strategy vector $\bar{z} \in Z_{ad}$ is a Nash equilibrium if and only if $\bar{z} \in \Psi(\bar{z})$.

Theorem 6.6. *Let Assumption 6.1 to Assumption 6.4 hold. Then the set of Nash equilibria of the NEP* (6.1.1) *is nonempty and weakly compact.*

Proof. We wish to apply the fixed point theorem of Kakutani-Fan-Glicksberg on Ψ. To ensure compactness, we recast the problem in the space X_i, where X_i is Z_i endowed with the weak topology. As a result of this and the standing assumptions, each set Z_{ad}^i is compact in X_i. Consequently, we take $Z_{ad} = Z_{ad}^1 \times \cdots \times Z_{ad}^N$ and $X = X_1 \times \cdots \times X_N$. Then $Z_{ad} \subset X$, the set Z_{ad} is also nonempty, convex and compact in X. Due to the properties of each ψ_i (cf. Lemma 6.5), Ψ has nonempty and convex images. In order to show that Ψ has a closed graph, let $\{z^k\}_{k \in \mathbb{N}} \subset Z_{ad}$ such that $u^k \to u^*$ in Z and let $v^k \in \Psi(u^k)$ such that $v^k \to v^*$ in X. Due to the Mosco-epi convergence, in particular (6.1.2) and the fact that $\{v_i^k\}_{n \in \mathbb{N}} \subset Z_{ad}^i$ we have

$$\mathcal{R}_i(F_i(v_i^*, z_{-i}^*)) + \wp_i(v_i^*) \leq \liminf_{k \to \infty} \mathcal{R}_i(F_i(v_i^k, z_{-i}^k)) + \wp_i(v_i^k).$$

Since every $v_i^k \in \psi(z_{-i}^k)$, we know

$$\liminf_{k \to \infty} \mathcal{R}_i(F_i(v_i^k, z_{-i}^k)) + \wp_i(v_i^k) \leq \liminf_{k \to \infty} \mathcal{R}_i(F_i(w_i, z_{-i}^k)) + \wp_i(w_i)$$

for all $w_i \in Z_{ad}^i$. Clearly, we have

$$\liminf_{k \to \infty} \mathcal{R}_i(F_i(w_i, z_{-i}^k)) + \wp_i(w_i) \leq \limsup_{k \to \infty} \mathcal{R}_i(F_i(w_i, z_{-i}^k)) + \wp_i(w_i).$$

Consequently, for the constant sequence $w_i \to w_i$ the Mosco-epi upper limit (6.1.3) implies

$$\limsup_{k \to \infty} \mathcal{R}_i(F_i(w_i, z_{-i}^k)) + \wp_i(w_i) \leq \mathcal{R}_i(F_i(w_i, z_{-i}^*)) + \wp_i(w_i).$$

Thus, $v_i^* \in \psi(z_{-i}^*)$ and finally $v^* \in \Psi(z^*)$. Since Z is separable, the concept of closedness and the sequential characterization of closedness of the set coincide. The fixed point theorem of Kakutani-Fan-Glicksberg implies that the set of Nash equilibria of the NEP (6.1.1) is nonempty and compact in X and by this weakly compact in Z. □

Using recent results [65] on optimality conditions for risk-averse PDE-constrained optimization under uncertainty, we can characterize Nash equilibria via the following variational inequality.

Theorem 6.7. *Let Assumption 6.1 to Assumption 6.4 hold. Then for any Nash equilibrium $\bar{z} \in Z_{ad}$ the following first-order optimality condition holds. For $i = 1, \ldots, N$ there exists $\bar{\vartheta}_i \in \partial R_i(F_i(\bar{z}_i, \bar{z}_{-i}))$ such that*

$$\mathbb{E}_{\mathbb{P}}\left[\bar{\vartheta} F_i'(\bar{z}_i, \bar{z}_{-i})(v_i - \bar{z}_i, 0_{-i})\right] + \wp_i'(\bar{z}_i; v_i - \bar{z}_i) \geq 0, \quad \textit{for all } v_i \in Z_{ad}^i.$$

Proof. Set $J_i(\bar{z}_i, \bar{z}_{-i}) = \mathcal{R}_i(F_i(\bar{z}_i, \bar{z}_{-i})) + \wp_i(\bar{z}_i)$. We build the differential quotient of J_i at $(\bar{z}_i, \bar{z}_{-i})$ and obtain

$$\frac{1}{t}\left(\mathcal{R}_i(F_i(\bar{z}_i, \bar{z}_{-i})) - \mathcal{R}_i(F_i(\bar{z}_i + th_i, \bar{z}_{-i}))\right)$$

and

$$\frac{1}{t}\left(\wp_i(\bar{z}_i) - \wp_i(\bar{z}_i + th_i)\right).$$

Passing to the limit $t \searrow 0$ yields

$$\frac{1}{t}\left(\wp_i(\bar{u}_i) - \wp_i(\bar{u}_i + th_i)\right) \longrightarrow \wp_i'(\bar{u}_i; h_i)$$

and

$$\begin{aligned} &\frac{1}{t}\left(\mathcal{R}_i(F_i(\bar{z}_i, \bar{z}_{-i})) - \mathcal{R}_i(F_i(\bar{z}_i + th_i, \bar{z}_{-i}))\right) \\ &\quad = \frac{1}{t}\left(\mathcal{R}_i(F_i(\bar{z}_i, \bar{z}_{-i})) - \mathcal{R}_i(F_i((\bar{z}_i, \bar{z}_{-i}) + t(h_i, 0_{-i}))\right) \\ &\quad \longrightarrow \mathcal{R}_i'(F_i(\bar{z}_i, \bar{z}_{-i}); F_i'(\bar{z}_i, \bar{z}_{-i})(h_i, 0_{-i})) \end{aligned}$$

since $\mathcal{R}_i$ is Hadamard directionally differentiable, see Lemma 2.15. The directional derivative is given by

$$\begin{aligned} \mathcal{R}_i'(F_i(\bar{z}_i, \bar{z}_{-i}); F_i'(\bar{z}_i, \bar{z}_{-i})(h_i, 0_{-i})) &= \sup_{\vartheta_i \in \partial \mathcal{R}(F_i(\bar{z}_i, \bar{z}_{-i}))} \langle \vartheta_i, F_i'(\bar{z}_i, \bar{z}_{-i})(h_i, 0_{-i}) \rangle \\ &= \sup_{\vartheta_i \in \partial \mathcal{R}(F_i(\bar{z}_i, \bar{z}_{-i}))} \mathbb{E}_{\mathbb{P}}\left[\vartheta_i F_i'(\bar{z}_i, \bar{z}_{-i})(h_i, 0_{-i})\right]. \end{aligned}$$

The first-order optimality condition then reads as

$$\sup_{\vartheta_i \in \partial R_i(F_i(\bar{z}_i, \bar{z}_{-i}))} \mathbb{E}_{\mathbb{P}}\left[\vartheta_i F_i'(\bar{z}_i, \bar{z}_{-i})(h_i, 0_{-i})\right] + \wp_i'(\bar{z}_i; h_i) \geq 0, \text{ for all } h_i \in T_{Z^i_{ad}}(\bar{z}_i), \quad (6.2.1)$$

where $T_{Z^i_{ad}}(\bar{z}_i)$ denotes the tangent cone of Z^i_{ad} at $\bar{z}_i$, which is defined as

$$T_{Z^i_{ad}}(\bar{z}_i) = \{d_i \in Z^i \mid \exists \tau_k \searrow 0, \exists d_i^k \to d_i \text{ in } Z^i : \bar{z}_i + \tau_k d_i^k \in Z^i_{ad} \forall k\}.$$

Since $\partial R_i(F_i(\bar{z}_i, \bar{z}_{-i}))$ is a weak* compact convex set, (6.2.1) can be replaced by

$$\mathbb{E}_{\mathbb{P}}\left[\bar{\vartheta}_i F_i'(\bar{z}_i, \bar{z}_{-i})(h_i, 0_{-i})\right] + \wp_i'(\bar{z}_i; h_i) \geq 0, \quad \text{for all } h_i \in T_{Z^i_{ad}}(\bar{z}_i),$$

for some $\bar{\vartheta}_i \in \partial R_i(F_i(\bar{z}_i, \bar{z}_{-i}))$. The convexity of Z^i_{ad} leads to the following variational inequality characterization: There exists $\bar{\vartheta}_i \in \partial R_i(F_i(\bar{z}_i, \bar{z}_{-i}))$ such that

$$\mathbb{E}_{\mathbb{P}}\left[\bar{\vartheta} F_i'(\bar{z}_i, \bar{z}_{-i})(v_i - \bar{z}_i, 0_{-i})\right] + \wp_i'(\bar{z}_i; v_i - \bar{z}_i) \geq 0, \quad \text{for all } v_i \in Z^i_{ad}.$$

□

Instead of stating the first-order optimality condition as variational inequality, we can write down an equivalent form termed generalized equation.

Theorem 6.8. *Let Assumption 6.1 to Assumption 6.4 hold. A strategy vector $\bar{z} \in Z_1 \times \cdots \times Z_N$ is a Nash equilibrium provided that $\bar{z} \in Z_{ad}$ and for all $i = 1, \dots, N$ we have*

$$0 \in \nabla_i F_i(\bar{z}_i, \bar{z}_{-i})^* \, \partial_i \mathcal{R}_i(F_i(\bar{z}_i, \bar{z}_{-i})) + \partial_i \wp_i(\bar{z}_i) + \mathcal{N}_{Z^i_{ad}}(\bar{z}_i). \quad (6.2.2)$$

Proof. $\bar{z} \in Z_{ad}$ is a Nash equilibrium if and only if for all $i = 1, \dots, N$

$$\bar{z}_i \in \arg\min\{\mathcal{R}_i(F_i(v_i, \bar{z}_{-i})) + \wp_i(v_i) + \mathcal{I}_{Z^i_{ad}}(v_i)\}.$$

By convexity, this is equivalent to

$$0 \in \partial_i \left[\mathcal{R}_i(F_i(\cdot, \bar{z}_{-i})) + \wp_i(\cdot) + \mathcal{I}_{Z^i_{ad}}(\cdot)\right](\bar{z}_i).$$

By continuity, we can apply the sum rule for convex subdifferential and obtain

$$0 \in \partial_i \left(\mathcal{R}_i(F_i(\cdot, \bar{z}_{-i}))\right)(\bar{z}_i) + \partial_i(\wp_i(\cdot))(\bar{z}_i) + \mathcal{N}_{Z^i_{ad}}(\bar{z}_i),$$

where $\mathcal{N}_{Z^i_{ad}}(\bar{z}_i)$ denotes the normal cone of the closed convex set Z^i_{ad} at $\bar{z}_i$. Due to the Frechet differentiability of $F_i(\cdot, \bar{z}_{-i})$, the chain rule for convex subdifferential yields

$$\partial_i \left(\mathcal{R}_i(F_i(\cdot, \bar{z}_{-i}))\right)(\bar{z}_i) = \nabla_i F_i(\bar{z}_i, \bar{z}_{-i})^* \, \partial_i \mathcal{R}_i(F_i(\bar{z}_i, \bar{z}_{-i})).$$

□

Recall that the random field quantity of interest F_i is the composition of J_i and S. When we write $\nabla_i F_i(\bar{z}_i, \bar{z}_{-i})^*$ in (6.2.2), then this is

$$\nabla_i F_i(\bar{z}_i, \bar{z}_{-i})^* = (\nabla_i S(\bar{z}_i, \bar{z}_{-i}))^* \nabla J_i(S(\bar{z}_i, \bar{z}_{-i})).$$

6.3 Epi- Regularization Technique

The long-term goal is to solve risk-averse PDE-Constrained NEPs numerically. The most promising idea at the moment is the ansatz presented in [65] for risk-averse optimization setting and make it reachable for a game setting. In order to handle the non-smoothness of the coherent risk measure we smooth the risk measures in (6.1.1) to obtain parameter-dependent problems using epigraphical analysis. After the general construction of epi-regularization, we will prove existence of Nash equilibria of the epi-regularized NEPs and prove consistency of the approximation.

6.3.1 General Construction

In this section we briefly recall the general smoothing approach presented in [65]. For further details concerning the analysis of the infimal convolution see [16, 88].

To simplify the presentation, we define as customary

$$\Gamma_o(\mathcal{X}) = \{f : \mathcal{X} \to \bar{\mathbb{R}} \mid f \text{ is proper, closed and convex } \}$$

before we define epi-regularized risk measures. Note that as a direct consequence, $f \in \Gamma_o(\mathcal{X})$ is lower semicontinuous. We use the concept of epigraphical addition (i.e. infimal convolution), denoted by '$\square$', and epigraphical multiplication, denoted by '$\star$', to generate epi-regularized risk measures.

Definition 6.9. Let $\mathcal{R} \in \Gamma_o(\mathcal{X})$ be a coherent risk measure and let $\Phi \in \Gamma_o(\mathcal{X})$. For $\varepsilon > 0$, we define the epi-regularized risk measure as

$$\begin{aligned}\mathcal{R}^{\Phi,\varepsilon}[X] &= (\mathcal{R} \,\square\, (\varepsilon \star \Phi))\,[X] \\ &= \inf_{Y \in \mathcal{X}} \{\mathcal{R}[X-Y] + \varepsilon\Phi[\varepsilon^{-1}Y]\} \\ &= \inf_{Y \in \mathcal{X}} \{\mathcal{R}[Y] + \varepsilon\Phi[\varepsilon^{-1}(X-Y)]\}.\end{aligned}$$

We recall two desirable properties for optimization of the epi-regularized risk measure $\mathcal{R}_i^{\Phi,\varepsilon}$ for $i = 1, \dots, N$.

Lemma 6.10. *(i) $\mathcal{R}_i^{\Phi,\varepsilon}$ is convex.*

(ii) $\arg\min_{X \in \mathcal{X}} \mathcal{R}_i[X] + \varepsilon \arg\min_{X \in \mathcal{X}} \Phi[X] \subset \arg\min_{X \in \mathcal{X}} \mathcal{R}_i^{\Phi,\varepsilon}[X]$.

Proof. (i) The infimal convolution of two convex functions is convex, due to the convexity of the epigraphs of $\mathcal{R}_i$ and Φ and the fact that the Minkowski scalar multiples and sum of two convex sets is again convex.

(ii) For proofs, we refer to [8, Theorem 2.3].

□

In order to ensure that $\mathcal{R}_i^{\Phi,\varepsilon} \in \Gamma_o(\mathcal{X})$, we make the following assumption.

Assumption 6.11. (i) The functionals $\mathcal{R}_i \in \Gamma_o(\mathcal{X})$ and $\Phi \in \Gamma_o(\mathcal{X})$ satisfy $\operatorname{dom} \mathcal{R}_i^* \subset \operatorname{dom} \Phi^*$ where $F^* : \mathcal{X} \to \bar{\mathbb{R}}$ denotes the Fenchel conjugate of the functional $F : \mathcal{X} \to \bar{\mathbb{R}}$ as defined in Definition 2.13(iii).

(ii) In addition, $\mathcal{R}_i$ and Φ satisfy the following qualification condition:

$$0 \in \operatorname{int}\{\operatorname{dom} \mathcal{R}_i^* - \operatorname{dom} \Phi^*\}.$$

Under Assumption 6.11 the Fenchel conjugate of $\mathcal{R}_i^{\Phi,\varepsilon}$ reads

$$\begin{aligned}(\mathcal{R}_i^{\Phi,\varepsilon})^*(\vartheta) &= (\mathcal{R}_i \,\square\, (\varepsilon \star \Phi))^* (\vartheta) = \mathcal{R}_i^*(\vartheta) + (\varepsilon \star \Phi)^* (\vartheta) \\ &= \mathcal{R}_i^*(\vartheta) + \varepsilon \Phi^*[\vartheta]\end{aligned}$$

(cf. [88, Theorem 3.4 b)]). Therefore, we have that

$$\operatorname{dom}(\mathcal{R}_i^{\Phi,\varepsilon})^* = \operatorname{dom} \mathcal{R}^* \cap \operatorname{dom} \Phi^* = \operatorname{dom} \mathcal{R}^*. \tag{6.3.1}$$

Moreover, Theorem 9.4.1 and 9.4.2 in [9] ensures that $\mathcal{R}_i^{\Phi,\varepsilon} \in \Gamma_o(\mathcal{X})$ and the minimum in the definition of $\mathcal{R}_i^{\Phi,\varepsilon}$ is attained. The effect of the epi-regularization on the axioms for a coherent risk measure is investigated in [65, section 3.3]. We recall that epi-regularized coherent risk measures are convex risk measures in the sense of Föllmer and Schied (cf [28]), according to the axiomatic definitions $\mathcal{R}_i^{\Phi,\varepsilon}$ satisfies the convexity, monotonicity and translation equivariance, i.e., $\mathcal{R}_i^{\Phi,\varepsilon}$ satisfies (R1) , (R2) and (R3). For $C > 0$, $\mathcal{R}_i^{\Phi,\varepsilon}$ is not positive homogeneous but satisfies $\mathcal{R}_i^{\Phi,\varepsilon}[CX] = C\mathcal{R}_i^{\Phi,\varepsilon/C}[X]$.

We sum up the differentiability properties of epi-regularized coherent risk measures, which we recall here for ease of reference, cf. [65, Corollary 2.10].

Corollary 6.12. *Let Assumption 6.1 and Assumption 6.11 hold with* $\mathcal{X} = L^1_{\mathbb{P}}(\Omega)$ *and suppose* $\mathcal{R}_i : \mathcal{X} \to \mathbb{R}$ *is a finite, coherent risk measure. Suppose* Φ *is strictly convex on* $\operatorname{dom} \Phi$ *and continuously Frechet differentiable on some set* $\mathcal{S} \supseteq \{\varepsilon^{-1}(X - \mu(X)) | \, X \in \mathcal{X}\}$ *that is open relative to* $\operatorname{dom} \Phi$. *Here* $\mu : \mathcal{X} \to \mathcal{X}$ *with* $\mu(X) = \arg\min_{Y \in \mathcal{X}} \{\mathcal{R}_i[Y] + \varepsilon \Phi[\varepsilon^{-1}(X - Y)]\}$. *Additionally, for* $i = 1, \ldots, N$ *assume there exists* $\vartheta_i^0 \in \operatorname{dom} \mathcal{R}_i^*$ *such that* $X \mapsto \{\Phi(X) - \mathbb{E}_{\mathbb{P}}[\vartheta_i^0 X]\}$ *is weakly inf-compact, i.e. the lower sets* $\{X \in \mathcal{X} : \Phi[X] - \mathbb{E}_{\mathbb{P}}[\vartheta_i^0 X \leq \gamma\}$ *for* $\gamma \in \mathbb{R}$ *are weakly compact. Finally, assume* Φ *satisfies*

$$X_k \rightharpoonup X \;\; in \; \operatorname{span}(\operatorname{dom} \Phi) \;\; and \; \Phi[X_k] \to \Phi[X] \implies X_k \to X \;\; in \; \mathcal{X}.$$

Then for each $i = 1, \ldots, N$, *the epi-regularized coherent risk measure* $\mathcal{R}_i^{\Phi,\varepsilon}$ *is finite, continuous, subdifferential and continuously Frechet differentiable at* X *with derivative*

$$\nabla \mathcal{R}_i^{\Phi,\varepsilon}[X] = \nabla \Phi[\varepsilon^{-1}(X - X^\star)],$$

where $X^\star = \mu(X)$ *(singleton since* Φ *is strictly convex).*

Proof. Since $\mathcal{R}_i$ is finite for $i = 1, \dots, N$, we have

$$\mathcal{R}_i^{\Phi,\varepsilon}[X] = \inf_{Y \in \mathcal{X}} \{\mathcal{R}_i[X - Y] + \varepsilon\Phi[\varepsilon^{-1}Y]\} \leq \mathcal{R}_i[X - Z] + \varepsilon\Phi[\varepsilon^{-1}Z]$$

for all $X \in \mathcal{X}$. Hence, $\mathcal{R}_i^{\Phi,\varepsilon}$ is finite valued. Proposition 6.6 in [86] ensures that $\mathcal{R}_i$ is continuous and subdifferential, since $\mathcal{R}_i$ is finite and monotone. In our setting $\mathcal{R}_i^{\Phi,\varepsilon}$ is also a finite, convex and monotone risk measure and hence it is also continuous and subdifferentiable. The fact that $\mathcal{R}_i^{\Phi,\varepsilon}$ is continuously Frechet differentiable is shown in Corollary 2 in [65] and [88, Theorem 3.7, Theorem 3.9]. □

6.3.2 Epi-Regularized Problem, Existence and Optimality Conditions

Let Assumption 6.1 to Assumption 6.4 and Assumption 6.11 hold with $\mathcal{X} = L^1_{\mathbb{P}}(\Omega)$. Let $\Gamma_o(\mathcal{X})$, then for $i = 1, \dots, N$, the epi-regularized Nash equilibrium problem reads as

$$\min_{z_i \in Z_i} \quad \mathcal{R}_i^{\Phi,\varepsilon}(F_i(z_i, z_{-i})) + \wp_i(z_i)$$
$$\text{subject to} \quad z_i \in Z^i_{ad}. \tag{6.3.2}$$

The best-response function for each player in the epi-regularized setting is given by

$$\psi_i^{\Phi,\varepsilon} : Z^{-i}_{ad} \rightrightarrows Z^i_{ad}$$
$$z_{-i} \mapsto \arg\min_{v_i \in Z^i_{ad}} \left\{\mathcal{R}_i^{\Phi,\varepsilon}(F_i(v_i, z_{-i})) + \wp_i(v_i)\right\}$$

for $i = 1, \dots, N$. Clearly, for $i = 1, \dots, N$ the set valued mapping $\psi_i^{\Phi,\varepsilon}$ has nonempty and convex images (cf. Lemma 6.5). Additionally, we define $\Psi^{\Phi,\varepsilon} : Z^1_{ad} \times \cdots \times Z^N_{ad} \rightrightarrows Z^1_{ad} \times \cdots \times Z^N_{ad}$ such that

$$\Psi^{\Phi,\varepsilon}(z) = \psi_1^{\Phi,\varepsilon}(z_{-1}) \times \cdots \times \psi_N^{\Phi,\varepsilon}(z_{-N}).$$

Before we state the existence of Nash equilibria of the parameter-dependent problems, we show that the Mosco-convergence in Assumption 6.4 is preserved.

Lemma 6.13. *Let Assumption 6.1 to Assumption 6.4 hold. For every weak convergence sequence* $\{z^k_{-i}\}_{k\in\mathbb{N}} \subset Z^{-i}_{ad}$, $z^k_{-i} \rightharpoonup z^*_{-i}$ *we set* $\{f^k_{\Phi,\varepsilon,i}\}_{k\in\mathbb{N}}$, $f^k_{\Phi,\varepsilon,i} : Z^i_{ad} \to \bar{\mathbb{R}}$ *where*

$$f^k_{\Phi,\varepsilon,i}(\cdot) = \mathcal{R}_i^{\Phi,\varepsilon}(F_i(\cdot, z^k_{-i})) + \wp_i(\cdot).$$

Then $f^k_{\Phi,\varepsilon,i} \xrightarrow{M} f_{\Phi,\varepsilon,i}$ *in the sense of Mosco-convergence, where* $f_{\Phi,\varepsilon,i}(\cdot) = \mathcal{R}_i^{\Phi,\varepsilon}(F_i(\cdot, z^*_{-i}))+ \wp(\cdot)$.

Proof. For every weak convergence sequence $\{z^k_{-i}\}_{k\in\mathbb{N}} \subset Z^{-i}_{ad}$, $z^k_{-i} \rightharpoonup z^*_{-i}$ we set $\{f^k_i\}_{k\in\mathbb{N}}$, $f^k_i : Z^i_{ad} \to \mathbb{R}\cup\{\infty\}$ where

$$f^k_i(\cdot) = (\mathcal{R}_i \circ F_i)(\cdot, z^k_{-i}) + \wp_i(\cdot).$$

Then $f^k_i \xrightarrow{M} f_i$ in the sense of Mosco-convergence, where $f_i(\cdot) = (\mathcal{R}_i \circ F_i)(\cdot, z^*_{-i}) + \wp_i(\cdot)$. In this case, $\widetilde{f}^k_i = (\mathcal{R}_i \circ F_i)(\cdot, z^k_{-i})$ converges also in the sense of Mosco to $\widetilde{f}_i(\cdot) = (\mathcal{R}_i \circ F_i)(\cdot, z^*_{-i})$. The corresponding sequences of function for the epi-regularized risk measure reads

$$f^{k,\varepsilon}_i(\cdot) = \mathcal{R}^{\Phi,\varepsilon}_i(F_i(\cdot, z^k_{-i})) + \wp_i(\cdot) \quad \text{and} \quad \widetilde{f}^{k,\varepsilon}_i(\cdot) = \mathcal{R}^{\Phi,\varepsilon}_i(F_i(\cdot, z^k_{-i})).$$

In order to show that $f^{k,\varepsilon}_i(\cdot) \xrightarrow{M} f^\varepsilon_i$, it is sufficient to show that $\widetilde{f}^{k,\varepsilon}_i(\cdot) \xrightarrow{M} \widetilde{f}^\varepsilon_i$ since the epi-regularization is not effective in $\wp_i$. The following equivalence is well known

$$\widetilde{f}^k_i(\cdot) \xrightarrow{M} \widetilde{f}_i(\cdot) \quad \Longleftrightarrow \quad \widetilde{f}^k_i(\cdot)^* \xrightarrow{M} \widetilde{f}_i(\cdot)^*.$$

Recall that

$$\widetilde{f}^{k,\varepsilon}_i(\cdot)^* = \widetilde{f}^k_i(\cdot)^* + \varepsilon\,\Phi(\cdot)^*.$$

Due to the assumption, we know that

$$\widetilde{f}^k_i(\cdot)^* + \varepsilon\,\Phi(\cdot)^* \xrightarrow{M} \widetilde{f}_i(\cdot)^* + \varepsilon\,\Phi(\cdot)^*.$$

which is identical to

$$\widetilde{f}^{k,\varepsilon}_i(\cdot)^* \xrightarrow{M} \widetilde{f}^\varepsilon_i(\cdot)^*.$$

Finally, this is equivalent to

$$\widetilde{f}^{k,\varepsilon}_i(\cdot) \xrightarrow{M} \widetilde{f}^\varepsilon_i(\cdot).$$

□

Then as before a Nash equilibria of the epi-regularized problem (6.3.2) can be characterized as fixed point of $\Psi^{\Phi,\varepsilon}$ by $\bar{z}^\varepsilon \in \Psi^{\Phi,\varepsilon}(\bar{z}^\varepsilon)$.

Corollary 6.14. *Let Assumption 6.1 to Assumption 6.4 and Assumption 6.11 hold with $\mathcal{X} = L^1_{\mathbb{P}}(\Omega)$. The set of Nash equilibria of the epi-regularized NEP is nonempty and weakly compact.*

The proof closely follows the proof of Theorem 6.6.

Similar to Theorem 6.7, we can characterize a Nash equilibrium by the following variational inequality.

Corollary 6.15. *Let Assumption 6.1 to Assumption 6.4 and Assumption 6.11 hold with $\mathcal{X} = L^1_{\mathbb{P}}(\Omega)$. Then for any Nash equilibrium $\bar{z}^\varepsilon \in Z_{ad}$ for $i = 1, \dots, N$ the following first-order optimality condition holds*

$$\left\langle \nabla_i F_i(\bar{z}^\varepsilon_i, \bar{z}^\varepsilon_{-i})^* \nabla_i \mathcal{R}^{\Phi,\varepsilon}_i(F_i(\bar{z}^\varepsilon_i, \bar{z}^\varepsilon_{-i})) + \wp'_i(\bar{z}^\varepsilon), v_i - \bar{z}^\varepsilon_i \right\rangle \geq 0, \text{ for all } v_i \in Z^i_{ad}. \quad (6.3.3)$$

6.3.3 Consistency of Approximation

In this section, we demonstrate that both, a sequence of Nash equlibria and a sequence of stationary points for the epi-regularized problem (6.3.2) have a weakly converging subsequence whose limit is a Nash equilibrium of the original problem (6.1.1). We closely follow the approach in [51, section 4.2].

First, we reformulate the optimality conditions by means of generalized equations. Let z be a Nash equilibrium of (6.1.1) and z^ε be a Nash equilibrium of (6.3.2). Note, that the characterization (6.2.1) and (6.3.3) can equivalently be seen as

$$z_i \in Z^i_{ad}, \tag{6.3.4}$$
$$0 = \mathbb{E}_{\mathbb{P}}\left[\vartheta_i \nabla_i F_i(z_i, z_{-i})\right] + \wp_i'(z_i) + \lambda_i, \tag{6.3.5}$$
$$\lambda_i \in \partial_i \mathcal{I}_{Z^i_{ad}}(u_i), \tag{6.3.6}$$
$$\vartheta_i \in \partial_i R_i(F_i(z_i, z_{-i})) \tag{6.3.7}$$

and

$$z_i^\varepsilon \in Z^i_{ad}, \tag{6.3.8}$$
$$0 = \mathbb{E}_{\mathbb{P}}\left[\vartheta_i^\varepsilon \nabla_i F_i(z_i^\varepsilon, z_{-i}^\varepsilon)\right] + \wp_i'(z_i^\varepsilon) + \lambda_i^\varepsilon, \tag{6.3.9}$$
$$\lambda_i^\varepsilon \in \partial_i \mathcal{I}_{Z^i_{ad}}(z_i^\varepsilon), \tag{6.3.10}$$
$$\vartheta_i^\varepsilon = \nabla_i \mathcal{R}_i^{\Phi,\varepsilon}(F_i(z_i^\varepsilon, z_{-i}^\varepsilon)) \tag{6.3.11}$$

for $i = 1, \ldots, N$.

Since the epi-regularized Nash equilibrium problem does not have a unique solution in general, we have to consider boundedness of selections of paths of stationary points of (6.3.2) in order to show the convergence of stationary points of the epi-regularized NEP to a Nash equilibrium of the original NEP.

Definition 6.16. For $\varepsilon > 0$, we denote by

$$S^\varepsilon \subset \prod_{i=1}^N Z_i \times \prod_{i=1}^N Z_i^* \times (\mathcal{X}^*)^N$$

the set of solutions of the optimality conditions given by (6.3.8), (6.3.9), (6.3.10) and (6.3.11) and set

$$\mathscr{P} = \left\{\{(z^\varepsilon, \lambda^\varepsilon, \vartheta^\varepsilon)\}_{\varepsilon>0} \,:\, \forall \varepsilon > 0 \; (z^\varepsilon, \lambda^\varepsilon, \vartheta^\varepsilon) \in S^\varepsilon\right\}.$$

We call every element $\mathcal{P} = \{(z^\varepsilon, \lambda^\varepsilon, \vartheta^\varepsilon)\}_{\varepsilon>0} \in \mathscr{P}$ a stationary path.

Note that the set of solutions of the optimality conditions S^ε is an cartesian product space, i.e. $S^\varepsilon = \prod_{i=1}^N S_i^\varepsilon = S_1^\varepsilon \times \cdots \times S_N^\varepsilon$, where again $S_j^\varepsilon = Z_j \times Z_j^* \times \mathcal{X}^*$ for $j = 1, \ldots, N$.

The following lemma provides the boundedness of a stationary path.

Lemma 6.17. *Let Assumption 6.1 to Assumption 6.4 and Assumption 6.11 hold with* $\mathcal{X} = L^1_{\mathbb{P}}(\Omega)$. *Then there exists a positive constant* $\rho < \infty$ *such that for all* $\varepsilon > 0$, $(z^\varepsilon, \lambda^\varepsilon, \vartheta^\varepsilon) \in S^\varepsilon$ *satisfies*

$$\|z^\varepsilon\|_{\prod_{i=1}^N Z_i} + \|\lambda^\varepsilon\|_{\prod_{i=1}^N Z_i^*} + \|\vartheta^\varepsilon\|_{(\mathcal{X}^*)^N} \leq \rho.$$

Proof. By the boundedness of every Z^i_{ad}, we have

$$\|z^\varepsilon\|_{\prod_{i=1}^N Z_i} \leq C_z$$

since $z_i^\varepsilon \in Z^i_{ad}$. Recall that $\vartheta_i^\varepsilon = \nabla_i(\mathcal{R}_i^{\Phi,\varepsilon} \circ F_i)(z_i, z_{-i})$ and due to (6.3.1), we have $\vartheta_i^\varepsilon \in \operatorname{dom} \mathcal{R}_i^*$. Since $\mathcal{R}_i$ is coherent, $\operatorname{dom} \mathcal{R}_i^*$ is nonempty, convex, bounded and weak* compact. This properties are a consequence of the equality $\operatorname{dom} \mathcal{R}_i^* = \partial \mathcal{R}(0)$, cf. Corollary 2.21. Since these properties are preserved when building a finite Cartesian product, we have

$$\|\vartheta^\varepsilon\|_{(\mathcal{X}^*)^N} \leq C_\vartheta.$$

Due to (6.3.9), we have

$$\lambda_i^\varepsilon = -\mathbb{E}_{\mathbb{P}}\left[\vartheta_i^\varepsilon \nabla_i F_i(z_i^\varepsilon, z_{-i}^\varepsilon)\right] - \wp_i'(z_i^\varepsilon)$$

Let $\phi \in Z$ with $\|\phi\| = 1$, then due to [43, Theorem 3.7.12] we have

$$\begin{aligned}
&|\langle -\mathbb{E}_{\mathbb{P}}\left[\vartheta_i^\varepsilon \nabla_i F_i(z_i^\varepsilon, z_{-i}^\varepsilon)\right] - \wp_i'(u_i^\varepsilon), \phi\rangle| \\
&\quad \leq |\langle \mathbb{E}_{\mathbb{P}}\left[\vartheta_i^\varepsilon \nabla_i F_i(z_i^\varepsilon, z_{-i}^\varepsilon)\right], \phi\rangle| + |\langle \wp_i'(z_i^\varepsilon), \phi\rangle| \\
&\quad = |\mathbb{E}_{\mathbb{P}}\left[\vartheta_i^\varepsilon \langle \nabla_i F_i(z_i^\varepsilon, z_{-i}^\varepsilon), \phi\rangle\right]| + |\langle \wp_i'(z_i^\varepsilon), \phi\rangle| \\
&\quad \leq \mathbb{E}_{\mathbb{P}}\left[|\vartheta_i^\varepsilon||\langle \nabla_i F_i(z_i^\varepsilon, z_{-i}^\varepsilon), \phi\rangle|\right] + |\langle \wp_i'(z_i^\varepsilon), \phi\rangle| \\
&\quad \leq \mathbb{E}_{\mathbb{P}}\left[|\vartheta_i^\varepsilon|\|\nabla_i F_i(z_i^\varepsilon, z_{-i}^\varepsilon)\|_{Z^*}\|\phi\|_Z\right] + \|\wp_i'(z_i^\varepsilon)\|_{Z^*}\|\phi\|_Z \\
&\quad = \mathbb{E}_{\mathbb{P}}\left[|\vartheta_i^\varepsilon|\|\nabla_i F_i(z_i^\varepsilon, z_{-i}^\varepsilon)\|_{Z^*}\right] + \|\wp_i'(z_i^\varepsilon)\|_{Z^*} \\
&\quad < C_{\lambda_i}.
\end{aligned}$$

Where the complete continuity of $\nabla_i F_i(\cdot,\cdot)$ ensures that $\nabla_i F_i(\cdot,\cdot)$ maps bounded sets into bounded sets. Moreover, $\mathbb{E}_{\mathbb{P}}$ is a linear, and since $\wp_i$ is continuously Gateaux differentiable, it holds that $\wp_i'(\cdot)$ is a linear bounded operator. Finally, for all $i = 1, \ldots, N$ we have

$$\|\lambda_i^\varepsilon\|_{Z_i^*} = \| - \mathbb{E}_{\mathbb{P}}\left[\vartheta_i^\varepsilon \nabla_i F_i(z_i^\varepsilon, z_{-i}^\varepsilon)\right] - \wp_i'(z_i^\varepsilon)\|_{Z_i^*} \leq C_{\lambda_i}.$$

□

The following main result of this section establishes consistency of stationary points for the epi-regularized risk-averse Nash equilibrium problem.

Theorem 6.18. *Let Assumption 6.1 to Assumption 6.4 and Assumption 6.11 hold with $\mathcal{X} = L^1_{\mathbb{P}}(\Omega)$. For every stationary path $\mathcal{P} \in \mathscr{P}$ there exists a subsequence $\varepsilon_m \to 0$ such that*

$$\begin{aligned} &z^{\varepsilon_m} \rightharpoonup z^* \text{ in } Z_1 \times \cdots \times Z_N, \\ &\lambda^{\varepsilon_m} \rightharpoonup \lambda^* \text{ in } Z_1^* \times \cdots \times Z_N^*, \\ &\vartheta^{\varepsilon_m} \overset{*}{\rightharpoonup} \vartheta^* \text{ in } (\mathcal{X}^*)^N. \end{aligned}$$

Moreover, the point $(z^*, \lambda^*, \vartheta^*)$ fulfills the first-order optimality conditions (6.3.4), (6.3.5), (6.3.6) *and* (6.3.7). *In particular, z^* is a Nash equilibrium of* (6.1.1).

Proof. Choose $\{\varepsilon_k\}_{k\to\infty}$ with $\varepsilon_k \searrow 0$, then z^{ε_k}, λ^{ε_k} and $\vartheta^{\varepsilon_k}$ satisfy (6.3.8) and (6.3.9), (6.3.10), and (6.3.11) respectively. Due to the weakly* sequentially compactness of $\operatorname{dom}\mathcal{R}_1^* \times \cdots \times \operatorname{dom}\mathcal{R}_N^*$ (cf. Corollary 2.21), the bounded set $\{\vartheta^{\varepsilon_k}\}_{k\to\infty}$ has a weakly* converging subsequence denoted by $\{\vartheta^{\varepsilon_l}\}_{l\to\infty}$ with

$$(\vartheta_1^{\varepsilon_l}, \ldots, \vartheta_N^{\varepsilon_l}) \rightharpoonup^* (\vartheta_1^*, \ldots, \vartheta_N^*) \text{ in } (\mathcal{X}^*)^N. \tag{6.3.12}$$

By the assumption that $\prod_{i=1}^N Z_i$ and $\prod_{i=1}^N Z_i^*$ are reflexive Banach spaces, the bounded sets $\{z^{\varepsilon_l}\}_{l\to\infty} \subset \prod_{i=1}^N Z_{ad}^i$ and $\{\lambda^{\varepsilon_l}\}_{l\to\infty}$ have a weakly converging subsequences

$$\begin{aligned} &(z_1^{\varepsilon_m}, \ldots, z_N^{\varepsilon_m}) \rightharpoonup (z_1^*, \ldots, z_N^*) \text{ in } Z_1 \times \cdots \times Z_N, \\ &(\lambda_1^{\varepsilon_m}, \ldots, \lambda_N^{\varepsilon_m}) \rightharpoonup (\lambda_1^*, \ldots, \lambda_N^*) \text{ in } Z_1^* \times \cdots \times Z_N^*. \end{aligned}$$

Then clearly $(z_1^*, \ldots, z_N^*) \in Z_{ad}^1 \times \cdots \times Z_{ad}^N$. Thus, z^* fulfills (6.3.4).
Due to (6.3.10), it holds that

$$\langle \lambda_i^{\varepsilon_m}, v_i - z_i^{\varepsilon_m} \rangle \le 0 \text{ for all } v_i \in Z_{ad}^i.$$

Since $\lambda_i^{\varepsilon_m}$ solves (6.3.9), substitution yields

$$\langle -\mathbb{E}_{\mathbb{P}} \left[\vartheta_i^{\varepsilon_m} \nabla_i F_i(z_i^{\varepsilon_m}, z_{-i}^{\varepsilon_m}) \right] - \wp_i'(z_i^{\varepsilon_m}), v_i - z_i^{\varepsilon_m} \rangle \le 0 \text{ for all } v_i \in Z_{ad}^i.$$

Applying [43, Theorem 3.7.12] yields

$$-\mathbb{E}_{\mathbb{P}} \left[\vartheta_i^{\varepsilon_m} \langle \nabla_i F_i(z_i^{\varepsilon_m}, z_{-i}^{\varepsilon_m}) - \wp_i'(z_i^{\varepsilon_m}), v_i - z_i^{\varepsilon_m} \rangle \right] \le 0 \text{ for all } v_i \in Z_{ad}^i.$$

In the following, we rewrite the left hand side of the above inequality

$$\begin{aligned} &- \mathbb{E}_{\mathbb{P}} \left[\vartheta_i^{\varepsilon_m} \langle \nabla_i F_i(u_i^{\varepsilon_m}, z_{-i}^{\varepsilon_m}), v_i - z_i^{\varepsilon_m} \rangle \right] - \wp_i'(z_i^{\varepsilon_m}), v_i - z_i^{\varepsilon_m} \rangle \\ &= -\mathbb{E}_{\mathbb{P}} \left[\vartheta_i^{\varepsilon_m} \langle \nabla_i F_i(z_i^*, z_{-i}^*), v_i - z_i^{\varepsilon_m} \rangle \right] \\ &\quad - \mathbb{E}_{\mathbb{P}} \left[\vartheta_i^{\varepsilon_m} \langle \nabla_i F_i(z_i^{\varepsilon_m}, z_{-i}^{\varepsilon_m}) - \nabla_i F_i(z_i^*, z_{-i}^*), v_i - z_i^{\varepsilon_m} \rangle \right] \\ &\quad - \langle \wp_i'(z_i^{\varepsilon_m}), v_i - z_i^{\varepsilon_m} \rangle. \end{aligned}$$

Passing to the limit inferior and combining the last to equation yields

$$\begin{aligned}
&-\limsup_{m\to\infty} \mathbb{E}_{\mathbb{P}}\left[\vartheta_i^{\varepsilon_m}\langle \nabla_i F_i(z_i^*, z_{-i}^*), v_i - z_i^{\varepsilon_m}\rangle\right] \\
&\quad -\limsup_{m\to\infty} \mathbb{E}_{\mathbb{P}}\left[\vartheta_i^{\varepsilon_m}\langle \nabla_i F_i(z_i^{\varepsilon_m}, z_{-i}^{\varepsilon_m}) - \nabla_i F_i(z_i^*, z_{-i}^*), v_i - z_i^{\varepsilon_m}\rangle\right] \\
&\qquad -\limsup_{m\to\infty} \langle \wp_i'(z_i^{\varepsilon_m}), v_i - z_i^{\varepsilon_m}\rangle \leq 0. \qquad (6.3.13)
\end{aligned}$$

Similar to the proof of Theorem 5 in Section 4.2 of [65] we know, that $\{z_i^{\varepsilon_m}\}_{m\in\mathbb{N}}$ is bounded as weakly converging sequence. Therefore,

$$|\langle \nabla_i F_i(z_i^*, z_{-i}^*), v_i - z_i^{\varepsilon_m}\rangle| \leq \|\nabla_i F_i(z_i^*, z_{-i}^*)\|_{Z_i^*}\|v_i - z_i^{\varepsilon_m}\|_{Z_i} \leq \|\nabla_i F_i(z_i^*, z_{-i}^*)\|_{Z_i^*} M_i$$

for $\mathbb{P}$-a.e., where $M_i \geq \|v_i - z_i^{\varepsilon_m}\|_{Z_i}$ for all m and

$$\langle \nabla_i F_i(z_i^*, z_{-i}^*), v_i - z_i^{\varepsilon_m}\rangle \to \langle \nabla_i F_i(z_i^*, z_{-i}^*), v_i - z_i^*\rangle$$

for $\mathbb{P}$-a.e.. This two facts are the sufficient conditions for the Lebesgue's dominated convergence theorem [27, Theorem 2.24] and imply strong convergence in $L^1_{\mathbb{P}}(\Omega)$. Since $\{\vartheta_i^{\varepsilon_m}\}_{m\in\mathbb{N}}$ is weakly* convergent, we obtain for the first expectation in (6.3.13) that

$$\mathbb{E}_{\mathbb{P}}\left[\vartheta_i^{\varepsilon_m}\langle \nabla_i F_i(z_i^*, z_{-i}^*), v_i - z_i^{\varepsilon_m}\rangle\right] \to \mathbb{E}_{\mathbb{P}}\left[\vartheta_i^*\langle \nabla_i F_i(z_i^*, z_{-i}^*), v_i - z_i^*\rangle\right]. \qquad (6.3.14)$$

In order to see, that

$$\mathbb{E}_{\mathbb{P}}\left[\vartheta_i^{\varepsilon_m}\langle \nabla_i F_i(z_i^{\varepsilon_m}, z_{-i}^{\varepsilon_m}) - \nabla_i F_i(z_i^*, z_{-i}^*), v_i - z_i^{\varepsilon_m}\rangle\right] \to 0 \qquad (6.3.15)$$

we first note that

$$\begin{aligned}
0 &\leq \mathbb{E}_{\mathbb{P}}\left[|\langle \nabla_i F_i(z_i^{\varepsilon_m}, z_{-i}^{\varepsilon_m}) - \nabla_i F_i(z_i^*, z_{-i}^*), v_i - z_i^{\varepsilon_m}\rangle|^p\right]^{1/p} \\
&\leq \|v_i - z_i^{\varepsilon_m}\|_{Z_i}\left(\mathbb{E}_{\mathbb{P}}\left[\|\nabla_i F_i(z_i^{\varepsilon_m}, z_{-i}^{\varepsilon_m}) - \nabla_i F_i(z_i^*, z_{-i}^*)\|_{Z_i^*}^p\right]^{1/p}\right) \\
&\leq M_i\|\nabla_i F_i(z_i^{\varepsilon_m}, z_{-i}^{\varepsilon_m}) - \nabla_i F_i(z_i^*, z_{-i}^*)\|_{L^p_{\mathbb{P}}(\Omega; Z_i^*)}.
\end{aligned}$$

Due to the fact that $\nabla_i F_i$ is complete continuous from $Z_i \times Z_{-i}$ into $L^p_{\mathbb{P}}(\Omega; Z_i^*)$, we have

$$\|\nabla_i F_i(z_i^{\varepsilon_m}, z_{-i}^{\varepsilon_m}) - \nabla_i F_i(z_i^*, z_{-i}^*)\|_{L^p_{\mathbb{P}}(\Omega; Z_i^*)} \to 0.$$

This implies

$$\langle \nabla_i F_i(z_i^{\varepsilon_m}, z_{-i}^{\varepsilon_m}) - \nabla_i F_i(z_i^*, z_{-i}^*), v_i - z_i^{\varepsilon_m}\rangle \to 0$$

in $L^1_{\mathbb{P}}(\Omega)$. Then (6.3.15) follows due to the weak* convergence of $\{\vartheta_i^{\varepsilon_m}\}_{m\in\mathbb{N}}$. For the third summand in (6.3.13), we have

$$\begin{aligned}
\limsup_{m\to\infty} \langle \wp_i'(z_i^{\varepsilon_m}), v_i - z_i^{\varepsilon_m}\rangle &= \limsup_{m\to\infty} \left(\langle \wp_i'(z_i^{\varepsilon_m}), v_i\rangle - \langle \wp_i'(z_i^{\varepsilon_m}), z_i^{\varepsilon_m}\rangle\right) \\
&= \limsup_{m\to\infty} \langle \wp_i'(z_i^{\varepsilon_m}), v_i\rangle - \liminf_{m\to\infty} \langle \wp_i'(z_i^{\varepsilon_m}), z_i^{\varepsilon_m}\rangle.
\end{aligned}$$

Due to the weak continuity of $\wp_i'(\cdot)$, we have

$$\limsup_{m\to\infty} \langle \wp_i'(z_i^{\varepsilon_m}), v_i\rangle - \liminf_{m\to\infty} \langle \wp_i'(z_i^{\varepsilon_m}), z_i^{\varepsilon_m}\rangle = \langle \wp_i'(z_i^*), v_i\rangle - \liminf_{m\to\infty} \langle \wp_i'(z_i^{\varepsilon_m}), z_i^{\varepsilon_m}\rangle.$$

The lower semicontinuity yields

$$\begin{aligned}\langle \wp_i'(z_i^*), v_i\rangle - \liminf_{m\to\infty} \langle \wp_i'(z_i^{\varepsilon_m}), z_i^{\varepsilon_m}\rangle &\le \langle \wp_i'(z_i^*), v_i\rangle - \langle \wp_i'(z_i^*), z_i^*\rangle \\ &= \langle \wp_i'(z_i^*), v_i - z_i^*\rangle. \end{aligned} \tag{6.3.16}$$

Combing (6.3.14), (6.3.15) and (6.3.16), then

$$\langle -\mathbb{E}_{\mathbb{P}}\left[\vartheta_i^* \nabla_i F_i(z_i^*, z_{-i}^*)\right] - \wp_i'(z_i^*), v_i - z_i^*\rangle \le 0 \quad \text{for all } v_i \in Z_{ad}^i$$

is an implication of (6.3.13). Now, we resubstitute $\lambda_i^* = -\mathbb{E}_{\mathbb{P}}\left[\vartheta_i^* \nabla_i F_i(z_i^*, z_{-i}^*)\right] - \wp_i'(z_i^*)$. Then the weak limit of $\{\lambda^{\varepsilon_m}\}_{m\to\infty}$ fulfills (6.3.6), i.e. $\lambda_i^* \in \partial_i \mathcal{I}_{Z_{ad}^i}(z_i^*)$. Clearly, since (6.3.12) holds, for any subsequence of $\{\vartheta^{\varepsilon_l}\}_{l\to\infty}$, we have $(\vartheta_1^{\varepsilon_m}, \dots, \vartheta_N^{\varepsilon_m}) \rightharpoonup^* (\vartheta_1^*, \dots, \vartheta_N^*)$ in $(\mathcal{X}^*)^N$. Since $\vartheta_i^{\varepsilon_{k_{l_m}}} \in \partial_i \mathcal{R}_i^{\Phi,\varepsilon_m}(F_i(z_i^{\varepsilon_m}, z_{-i}^{\varepsilon_m})) = \{\nabla_i \mathcal{R}_i^{\Phi,\varepsilon_m}(F_i(z_i^{\varepsilon_m}, z_{-i}^{\varepsilon_m}))\}$, we have

$$\mathcal{R}_i^{\Phi,\varepsilon_m}(Y) \ge \mathcal{R}_i^{\Phi,\varepsilon_m}(F_i(z_i^{\varepsilon_m}, z_{-i}^{\varepsilon_m})) + \mathbb{E}_{\mathbb{P}}[\vartheta_i^{\varepsilon_m}(Y - F_i(z_i^{\varepsilon_m}, z_{-i}^{\varepsilon_m}))]$$

for all $Y \in \mathcal{X}$. Proposition 1 in section 3.1 of [65] gives the bound

$$\mathcal{R}_i^{\Phi,\varepsilon_m}(Y) \le \mathcal{R}_i(Y) + \varepsilon_m \Phi[0].$$

Combining the last two inequalities yields

$$\mathcal{R}_i(Y) + \varepsilon_m \Phi[0] \ge \mathcal{R}_i^{\Phi,\varepsilon_m}(F_i(z_i^{\varepsilon_m}, z_{-i}^{\varepsilon_m})) + \mathbb{E}_{\mathbb{P}}[\vartheta_i^{\varepsilon_m}(Y - F_i(z_i^{\varepsilon_m}, z_{-i}^{\varepsilon_m}))]$$

for all $Y \in \mathcal{X}$.

Passing to the limit inferior on both side, the Mosco convergence of $\mathcal{R}_i^{\Phi,\varepsilon_m}$ to $\mathcal{R}_i$ (cf. [65, Theorem 3, Section 3.5]) and the complete continuity of F_i yields

$$\begin{aligned}\mathcal{R}_i(Y) &= \liminf_{m\to\infty} \mathcal{R}_i(Y) + \varepsilon_m \Phi[0] \\ &\ge \liminf_{m\to\infty} \mathcal{R}_i^{\Phi,\varepsilon_m}(F_i(z_i^{\varepsilon_m}, z_{-i}^{\varepsilon_m})) + \liminf_{m\to\infty} \mathbb{E}_{\mathbb{P}}[\vartheta_i^{\varepsilon_m}(Y - F_i(z_i^{\varepsilon_m}, z_{-i}^{\varepsilon_m}))] \\ &\ge \liminf_{m\to\infty} \mathcal{R}_i^{\Phi,\varepsilon_m}(F_i(z_i^{\varepsilon_m}, z_{-i}^{\varepsilon_m})) + \lim_{m\to\infty} \mathbb{E}_{\mathbb{P}}[\vartheta_i^{\varepsilon_m}(Y - F_i(z_i^{\varepsilon_m}, z_{-i}^{\varepsilon_m}))] \\ &\ge \mathcal{R}_i(F_i(u_i^*, u_{-i}^*)) + \mathbb{E}_{\mathbb{P}}[\vartheta_i^*(Y - F_i(u_i^*, u_{-i}^*))]. \end{aligned}$$

for all $Y \in \mathcal{X}$. Hence, for the weak* limit of $\{\vartheta^{\varepsilon_l}\}_{l\to\infty}$, it holds that $\vartheta_i^* \in \partial_i \mathcal{R}_i(F_i(u_i^*, u_{-i}^*))$ for $i = 1, \dots, N$, i.e. ϑ_i^* fulfills (6.3.7). □

In what follows, we change the perspective. We start with a sequence of Nash equilibria for the epi-regularized Nash equilibrium problem (6.3.2) and demonstrate that for $\varepsilon_k \searrow 0$ there exists a weakly converging subsequence whose limit is a Nash equilibrium of (6.1.1).

Theorem 6.19. *Let Assumption 6.1 to Assumption 6.4 and Assumption 6.11 hold with* $\mathcal{X} = L^1_{\mathbb{P}}(\Omega)$*. For any sequence* $\{\varepsilon_k\}_{k\to\infty}$ *with* $\varepsilon_k \searrow 0$*, suppose* z^k *is a Nash equilibrium of* (6.3.2) *with* $\varepsilon = \varepsilon_k$*. Then any weak limit point of* $\{z^k\}_{k\to\infty}$ *is a Nash equilibrium of* (6.1.1)*.*

Proof. Fix an arbitrary $k \in \mathbb{N}$. For $i = 1, \ldots, N$, by the definition of a Nash equilibrium, we have

$$\mathcal{R}_i^{\Phi,\varepsilon_k}(F_i(v_i, z^k_{-i})) + \wp_i(v_i) \geq \mathcal{R}_i^{\Phi,\varepsilon_k}(F_i(z^k_i, z^k_{-i})) + \wp_i(z^k_i)$$

for all $v_i \in Z^i_{ad}$. We recall the upper bound given in [65, Proposition 1, section 3.1]

$$\mathcal{R}_i^{\Phi,\varepsilon_k}(F_i(v_i, z^k_{-i})) \leq \mathcal{R}_i(F_i(v_i, z^k_{-i})) + \varepsilon_k \Phi(0).$$

Combining this inequalities yields

$$\mathcal{R}_i(F_i(v_i, z^k_{-i})) + \varepsilon_k \Phi(0) + \wp_i(v_i) \geq \mathcal{R}_i^{\Phi,\varepsilon_k}(F_i(z^k_i, z^k_{-i})) + \wp_i(z^k_i).$$

Now, by the complete continuity of F_i, if $z_i^{k_j} \rightharpoonup z_i^*$ in Z^i_{ad} for all $i = 1, \ldots, N$, then $F_i(z_i^{k_j}, z_{-i}^{k_j}) \to F_i(z^*_i, z^*_{-i})$ in $\mathcal{X}$. Passing to the limit inferior in the inequality above, due to the continuity of $\mathcal{R}_i$, the Mosco convergence of $\mathcal{R}_i^{\Phi,\varepsilon_{k_j}}$ and the weakly lower semicontinuity of $\wp_i$, we obtain

$$\begin{aligned} \mathcal{R}_i(F_i(v_i, z^*_{-i})) + \wp_i(v_i) &= \lim_{j\to\infty} \mathcal{R}_i(F_i(v_i, z^{k_j}_{-i})) + \varepsilon_{k_j}\Phi(0) + \wp_i(v_i) \\ &\geq \liminf_{j\to\infty} \mathcal{R}_i^{\Phi,\varepsilon_{k_j}}(F_i(z_i^{k_j}, z_{-i}^{k_j})) + \wp_i(z_i^{k_j}) \\ &\geq \mathcal{R}_i(F_i(z^*_i, z^*_{-i})) + \wp_i(z^*_i) \end{aligned}$$

for any $v_i \in Z^i_{ad}$. Hence,

$$z^*_i \in \operatorname*{arg\,min}_{v_i \in Z^i_{ad}} \left\{ \mathcal{R}_i(F_i(v_i, z^*_{-i})) + \wp_i(v_i) \right\}$$

for all $i = 1, \ldots, N$. □

Note that the proof of the convergence of Nash equilibria in optimization setting as treated in [65, Theorem 4, section 3.1] is based on the pointwise convergence of $\mathcal{R}_i^{\Phi,\varepsilon_k}$ to $\mathcal{R}_i$ for $\varepsilon_k \searrow 0$ cf. [65, Theorem 3, section 3.5]. In total, we can say, that epi-regularized Nash equilibrium problems provide a consistent approximation since firstly, a sequence of Nash equilibria for the epi-regularized NEP produces a weakly converging subsequence whose limit is a Nash equilibrium of (6.1.1) and secondly, we have a similar result for sequences of stationary point.

Chapter 7

Conclusion and Outlook

In this thesis we have discussed PDE-constrained equilibrium problems under uncertainty. In detail, we focused on a class of risk-neutral generalized Nash equilibrium problems under uncertainty and a class of risk-averse Nash equilibrium problems under uncertainty.

For the risk-neutral generalized Nash equilibrium problems under uncertainty, we have succeeded to generalize the findings in the deterministic setting in [50, 51] with appropriate adjustments. Note that our extension of the state of the art problem class concludes beside the stochasticity also pointwise state constraints. We proved existence of solutions/equilibria and derived optimality conditions for both stochastic PDE-constrained optimization and equilibrium problems subject to state constraints. The addition of inequality constraints on the PDE solution to both, the PDE-constrained NEP and optimization problem, leads to complications in the derivation of Lagrange multipliers. In order to make use of the existing optimality theory for convex optimization problems, we rely on higher regularity of the random states. The proof of the higher regularity of the random states was therefore crucial and is mainly based on priori estimates for deterministic elliptic PDE. In the case of GNEPs, we used a GNEP-specific constraint qualification in order to develop a regularization approach on which both the theory and our numerical methods could be built. We rigorously passed to the limit in the smoothing parameter, whereby the low regularity of the Lagrange multipliers made this venture highly nontrivial. In the case of PDE-constrained optimization problems subject to state constraints, we drew the link between Moreau-Yosida regularization and probability constraints using results on concentration inequalities and asymptotic statistics. Finally, we suggest two algorithms: The first for solving risk-neutral PDE-constrained optimization problems subject to state constraints and the second for the extension to GNEPs. A numerical study as well as a full convergence analysis will be future research directions. An idea for latter is to link sampling, adaptive finite elements, smoothing, and convergence of these algorithms. It still is a long way to prove convergence rates of the GNEP solver since the underlying fixed point mapping is nonexpansive, this make it much more delicate.

Further, we introduced a class of risk-averse Nash equilibrium problems under uncertainty using coherent risk measures. This approach clearly has the advantage that we can incorporate risk management concepts without the necessity of the addition of further constraints. We proved existence of equilibria and derived optimality conditions. With regards to the long-term goal; solving this risk-averse Nash equilibrium problems under uncertainty using coherent risk measures numerically, we have laid the essential foundation by applying a variational smoothing technique called epigraphical (epi-)regularization as presented in [65]. After smoothing the coherent risk measure, we proved existence of Nash equilibria of the epi-regularized NEPs and derived optimality conditions. Finally, we ended the investigation of this class with a rigorous proof of consistency of the approximation. We have succeeded to pass to the limit in the smoothing parameter to return to both, equilibria and stationary points of the original problem. Concluding our theoretical results with numerical experiments as well as a consequential convergence analysis is future work. For this it will be necessary to solve the parameter-dependent NEPs by extending existing algorithm to the game setting.

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