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Graviweak Theory in Bicomformal Space

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GRAVIWEAK THEORY IN BICOMFORMAL SPACE

by

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ABSTRACT

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With the inception of the standard model, physicists tried various techniques to fit gravity under the same underlying framework. These attempts were thwarted by the Coleman-Mandula theorem, showing that a combined Poincarè-SU(N) gauge theory leads to a consistent quantum field theory only as a direct product. While supersymmetry provides an escape from the Coleman-Mandula conclusion, we explore a different approach. In this work, we use the technique of biconformal gauge theory to bridge the gap between the electroweak model and gravity, yielding a single graviweak theory. The quotient of the conformal group of a space of dim n = p + q , with SO(p,q) metric by its homogeneous Weyl subgroup. This gives a principal fiber bundle with 2n-dim base manifold and Weyl fibers. The Cartan generalization to a curved 2n-dim geometry admits an action functional linear in the curvatures, and the field equations generically yield general relativity on the cotangent bundle of spacetime. However, in a subclass of cases the extra n dimensions can give a fibration by a non-Abelian Lie group, with the maximal case for n = 4 being the electroweak group. Thus, while the final Lorentz and electroweak symmetries are of the direct product form required by Coleman-Mandula, the model is predictive of the specific group. Our principal interest is to develop a spinor representation for the 4-dimensional case of this model in detail to see if further properties of the electroweak theory are predicted. In addition to the usual operators within Dirac theory, we find a new projection which might be interpreted as either isospin or as the splitting between the gravity and electroweak sectors. We discuss these possibilities. Also, we derive the field equations in the self-dual/anti-self-dual spinor representation.

(159 pages)

PUBLIC ABSTRACT

Graviweak Theory in Bicomformal Space

Mubarak Ukashat

There are four basic forces in nature: the electromagnetic force, which accounts for interactions of particles with charges; the weak force, which is responsible for radioactive decay; the strong force, which holds the particles inside a nucleus tightly bound together; and the gravitational force, which is resposible for keeping us on our beautiful planet, Earth and holding together our entire solar system. Physicists have been on the hunt for a theory that can single-handedly explain all these forces under the same underlying mathematical formulation. So far, physicists have suceeded in unifying the electromagnetic and weak forces in what is called the electroweak theory. Some ways are known to unify the electroweak and strong interactions using group theory, but the odd one out is really gravitational force. Gravity is explained successfully so far by Einstein's general theory of relativity but it has seen limited quantum mechanical explanation. One possible route to full unification is string theory but we take an alternative approach. In this dissertation, we attempt to unify gravity with the electroweak interaction. We propose a graviweak theory based on a gauge field theory approach by harnessing the plethora of mathematical techniques found in biconformal gauge field theory. In this special kind of field theory, not only can we readily and easily get gravity, we simulteneously have a dual space that can accommodate the electroweak theory within the same formulation. We see that certain surprising properties of the electroweak theory such as the existence of isospin or its preference for left-handedness over right-handedness may have a natural explanation within biconformal theory.

From the very depths of my heart, my consciousness, and my soul; I want to dedicate this work firstly to Almighty Allah, the essence of my life, my first love and the source of all the knowledge and energy that I have poured into this work, and secondly, to my parents whose unimaginable support has made it possible for me to get to this height in life. Lastly, to my lovely wife and daughters for changing my entire world.

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Mubarak S. Ukashat

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CHAPTER 1

INTRODUCTION

1.1 Background

The standard model has been the greatest and most successful form of a unified theory in modern physics. Apart from its success in explaining how the strong force works, it is also based on the same framework of gauge field theory that unifies the electromagnetic and weak interactions into a single electroweak interaction. Although the standard model does not answer all questions, no experimental findings have been found to violate its predictions till this day. Since the inception of the standard model, physicists have tried various techniques to fit gravity under the same underlying framework. In this research, we use the technique of biconformal gauge theory to to bridge the gap between the electroweak model and gravity, unifying them into a single graviweak theory. We begin our task by considering the conformal group of a space of dim $n = p + q$, with $SO(p, q)$ metric and taking the quotient of this group by its homogeneous Weyl subgroup. This gives a principal fiber bundle with $2n$-dim base manifold and Weyl fibers and the Cartan generalization to a curved $2n$-dim geometry admits an action functional linear in the curvatures. Because symmetry is maintained between the translations and the special conformal transformations in the construction, these spaces are called biconformal [1]. Biconformal geometry is a form of double field theory; general relativity with integrable local scale invariance arises from its field equations. It is notable that the field equations reduce all curvature components to dependence only on the solder form of an n-dim Lagrangian submanifold, despite the increased number of curvature components and doubled number of initial independent variables. Our principal interest is to see how 2n-dimensional geometry furnishes the platform for electroweak theory with the proper symmetry breaking being an inherent consequence of the biconformal structures.

The formulation of the electroweak theory is strongly founded in gauge field theory. In the early days of exploring theories of unification, Weyl was the first to attempt to unify gravity and electromagnetism. In 1918 he was able to develop an electromagnetic-gravitational theorem based on the assumption that we can treat lengths and directions on an equal footing during parallel transport [2]. Einstein was able to figure out a flaw this theory since if the assumptions were true, then when we parallel transport an electron around some closed path in an electromagnetic field in a curved spacetime, its size will not be the same. This is totally against observations since it violates the existence of chemical elements with spectral lines of definite frequency and for this assumption to hold, the relative frequency of two neighboring atoms of the same kind would be different in general [3]. In 1921 Theodor Kaluza tried to unify electromagnetism and gravity by adding an extra dimension to the usual four-dimensional spacetime metric [4]. This assumption did not work out so well until 1926 when Oscar Klein introduced new paramaters for the extra fifth dimension and required that it be compactified [5].

In 1929, Weyl developed the functional method of gauge theory for electromagnetism using U (1) symmetry. This gave rise to many insights in the search for a united theory. Yang and Mills were the first to gauge a nonabelian group in 1954 using SU(2), while they were studying the behavior of protons and neutrons [6], [7]. Utiyama was the first to gauge the Lorentz group in 1954 [8], while Kibble in 1961 gauged the full Poincare group [9]. Using the methods of fiber bundles and Cartan geometry, we easily find unifying relationships between these different gauge theories.

While electromagnetic interactions only affect particles with charge and the strong interaction only affects quarks, gravitation affects all forms of energy, and every particle feels the weak interaction. This suggests the possibility of a geometric graviweak model. Properties of the weak interaction include flavor change, isospin doublets and singlets, parity violation, charge conjugation violation and CP violation.

The theoretical development of the electroweak model had two major setbacks. The first is related to the masslessness of the W and Z bosons for the theory to hold. We know from experiments that they are indeed massive. The second setback was that the theory separated the behavior of right-handed particles from left-handed particles but had terms involving interactions between them. It was difficult to explain this interaction based on previous assumptions. This is where the Higgs mechanism of spontaneous symmetry breaking came into play. The Higgs mechanism was able to solve both of these mysteries and also introduced the existence of a new Higgs particle which was later discovered years after its prediction.

Biconformal theory has been a very successful tool in developing effective theories of locally scale covariant general relativity. The double-field-theory nature of this model lets us propose other underlying theories with results consistent with observation. We propose to formulate a graviweak theory from the gauge theory of the conformal group of a 4-dimensional Euclidean biconformal space. We start with the Euclidean metric on R^4, compactifying it to extend it to its $SO(5,1)$ conformal group. The quotient of $SO(5,1)$ by the product of SO(4) with dilitations is a fiber bundle with an 8-dimensional homogeneous base manifold and $SO(4) \times$ dilatations fibers. Changing the connection to curve the base manifold gives an 8-dimensional biconformal space (biconformal because it doubles the dimension of space). The biconformal space gives us a Kahler manifold, with the metric, symplectic form, and complex structures all arising naturally from the conformal group.

The general scale-invariant curvature-linear action on this biconformal space has been shown to reduce to a locally scale invariant Euclidean general relativity an 4-dimensional Lagrangian submanifold of the full space, with the remaining space fibrated by copies of an 4-dimensional Lie group. Generically, this Lie group is Abelian, so the full biconformal space is consistently identified as the co-tangent bundle, and the techniques of [10] show how this can give rise to Lorentzian general relativity. However, an additional class of solutions exists for which the extra 4 dimensions describe a fibration by a 4-dimensional *non-Abelian* Lie group, $G(4)$. This extends the fibers symmetry of the original bundle to $G(4) \times SO(4) \times D$. The aim of our investigation is to determine under what conditions $G(4)$ is the $SU(2)_L \times U(1)_Y$ symmetry of electroweak theory, and to study the resulting graviweak unification.

This underlying structure is expected to be able to show results of left and right-handedness of particles, and naturally exhibit symmetry breaking. We also will examine whether the existence of the Higgs alongside its mass prediction as an inherent part of the theory rather than an input as it is in the standard model. We also propose in this new theory to explore other known properties such as parity violation and charge conjugation. The general approach will involve the usual gauge theory formulation with differential forms. We will also extend our model to include spinor representations and ‘t Hooft matrices. At the moment, although our theory beautifully separates left-handed particles from right-handed particles naturally, we are not yet certain what features determine when our new group is a subgroup of $SO(4)$ and when it is a group contraction.

Choosing the appropriate spinor representation in biconformal space is one of the first and foremost steps we undertake in order for our formulation and structure to be compatible with the standard model. Spinors are the representations of matter in the standard model which makes them play a very important role in our theory. They are the representations for the grop of weakly interacting matter $SU(2)$. The right way to think about a spinor γ^{μ} is that, despite the fact that it has a vector index μ on it it is actually a matrix in spin space with a pair of spinor indices which are often left out in literature. They actually look like γ^{μ}_{ab}. It is a matrix because of the two spin indices ab, but at the same time it got a vector index μ in spacetime not spin space. So, the appropriate way to think about this object is a 4-component vector of matrices which happen to be 4 by 4 matrices for the 4D case. In 3D, the analogous thing was a three component vector of 2 by 2 matrices. These gammas are interesting because they live both in spin space and space time simultaneously. At this point it is a good idea to start thinking in terms of spacetime and spin space separately. They both have their separate index structures even if they are related since spin space is a way of describing spinors on a spacetime, but we cannot use normal coordinates. Technically speaking, the spinors exist on the spin bundle over the manifold. At each point we define a spin space and a collection of these spin spaces throughout the spacetime which results in the spin bundle. This is analogous to vectors living in the tangent space to any given point whose collection gives the tangent bundle of the manifold. The reason why it gets confusing with vectors is that the indices in tangent space can be chosen to match with the coordinates used for spacetime and this is called the coordinate basis of the tangent space. Although we do not have to do this because we can represent the vectors using the orthonormal basis in the tangent space in which case the spacetime indices and the tangent space would look sort of disconnected from the coordinates in the same way as the spinors do. There are cases where separating vectors from coordinates has been more efficient.

1.2 Fiber Bundles

Consider three differentiable manifolds E, $M$and F where E is the bundle space, M the base space and F the fiber space where E and M may each be of any dimension. The dimension of F is the sum of the dimensions of E and M. At each and every point of the base space we attach a copy of the fiber space and the whole new space formed is the bundle space with a dimension equal to the sum of the fiber space and base space dimensions.

We have some features that come with the formal definition of every fiber bundle and these include;

1. Projection: If we take any point in the entire bundle space and apply a projection π, it gives us the corresponding point in the base space. This is also true if we take any point in the fiber bundle space and apply a projection π, it gives us the corresponding point in the base space.

2. Lie group G. There also exist a Lie group also called the structure group which acts on F from the left

3. Open Cover. These are some sets of open neighborhoods $\{U_i\}$ of the base space M (such that all of u_i gives back M) with a diffeomorphism $\phi : U \times F \to \pi^{-1}(U_i)$ such that $\pi \odot \phi_i(p, f) = p \in M$

4. On every non-empty overlap of two subgroups $U_i \cap U_j$ we require G-valued transition functions $t_{ij} = \phi_i \odot \phi_j^{-1}$ such that $\phi_i = t_{ij}\phi_j$

If we can cover the entire base M with only one U_i or if all of the t_{ij} are trivial ($t_{ij} = I_{ij}$) then globally, the bundle E is also trivial ($E = M \times F$). For a nontrivial bundle this property is restricted to a local neighborhood. If we consider a base space $M = S^1$ which is just a circle and the fiber space is taken to be $F = \{z\} \in [-1, 1]$ which is all values of a line segment between -1 and $+1$ including -1 and $+1$. We have two possibilities for this case, the first being we can take this to be the trivial bundle $E = S^1 \times [-1, 1]$ which is the surface of a cylinder. The second case is a nontrivial one which involves breaking S^1 into two neighborhoods U_1 and U_2 with two overlaps on each end breaking the circle in half. In each of these I have a line segment with its fibers and on one overlap we use a trivial transition function $t_{ij} = I_{ij}$ which is just the identity and on the other we use $t_i : z \to -z$. This makes a mobius strip. So we call both a cylinder and a Mobius strip as line bundles over S^1 where the cylinder is the trivial line bundle and the Mobius strip is the nontrivial line bundle over S^1. Another example of a nontrivial bundle is a circle bundle over a circle called the Klein bundle. Trivial bundles allow us to do a lot of things easily but the more restricted or nontrivial a bundle space is the more specific they are in regards to details of what can be described in those spaces. For instance we know that the Mobius strip is a non-oreientable surface which means that if I want to describe some physics in which orientation is important then the Mobius strip will not be a good space for that representation. There are also spaces that will not allow us have spinors on that space etc.

There are lots of spaces which can be categorized as bundles but the three most common in physics are the following;

- When F is a vector space we have a vector bundle with dimensions different or similar to the base space M
- When F is a particular vector space called the tangent space to M and then we have the tangent bundle called TM. We already know that we can define a tangent space at each point for any manifold irrespective of the space being a bundle in itself or not.

In general relativity for instance if we want to talk about tensors or vectors in general we will be restricted to objects in tangent spaces.

- If $F = G$, in other words if the fiber is a Lie group, then we have a principal bundle. In other words, if we have a base space as spacetime and at each point our fiber is a $U(1)$, then we are talking about a $U(1)$ principal bundle describing electromagnetism. If it is an $SU(2)$ fiber then it will be describing weak interactions and if it is an $SU(3)$ fiber it will be for QCD or the strong interactions. Principal fiber bundles are very useful for gauge theories.

The idea of magnetic monopoles now depends on whether the $U(1)$ principal bundle is trivial or not. If it is trivial then there are no magnetic monopoles but if we can twist the bundle in an analogous way to making a Mobius strip then we do have magnetic monopoles. Half of Maxwell's equations, the ones with sources come from an action prnciple and the other half, the ones withpout the sources, are actually statements about the electromagnetic geometry or the $U(1)$ bundle and if we want to amend those equations to describe physics on a nontrivial bundle then it is exactly adding a monopole term.

Some other features worth mentioning include;

- Sections. A section is a map that takes a single point in the fiber to a corresponding point in the base space. It is like a curve mapping each section in the base to a single point in the fiber. These are important in physics when we start writing out Lagrangians and actions and equations of motion they are all in terms of sections of bundles for instance in electromagnetism when we write $A^{\mu}(x)$ for the gauge potential it is actually a section

- Connection. A connection has to do with movement along the fibers or tangent to them or both simultenously. This splits motion into verticality or horizontality. A mathematical apparatus or construction which formalizes this for us is defined in terms of a one-form connection A. In a principal bundle this one-form corresponds to the gauge field while in a tangent bundle it is the Christoffel connection Γ as in GR. These connections are what formalizes taking derivatives of objects in both GR and particle physics which depends on taking differences of points with direction dependence.

- Characteristic Classes. If we had a space and wanted to tell if it the tangent bundle of this space was a trivial bundle or not, then we can deduce the answers from knowing the characteristic classes which are objects built from the connections or transition functions of these fiber bundles with a variety of ways to compute them and different types in general. One particular one we want to talk about related to connection is called the Chern class.

$$C\left(E\right)=\det\left(1+\frac{i}{2\pi}F\right)$$

where F is defined in terms of the connection A as $F=dA+A\wedge A$. This becomes the curvature 2-form of the bundle. In a $U\left(1\right)$ electromagnetic principal bundle where A is the vector potential, then F is just the field strength tensor $F=F_{\mu\nu}$, for the tangent bundle to a manifold TM, then F is just the Ricci tensor $R_{\mu\nu}$ created from the Christoffel connection. These classes can be extended to a set of terms which are as follows

$$\begin{aligned}C_0\left(E\right) &= I\\ C_1\left(E\right) &= Tr\left(\frac{i}{2\pi}F\right)\\ C_2\left(E\right) &= ...\end{aligned}$$

$C_1\left(E\right)$ is the first Chern class is really important with respect to Calabi-Yau manifolds. The idea basically is that if we have a trivial space then all the Chern classes vanish except the zeroth class.

- Composite fiber bundles. When we gauge GR as a group of $SO\left(1,3\right)^{\uparrow}\times T^4$ one way to make this work is by encoding the translations as another bundle over the rotations. The rotations form a fiber bundle over Minkowski space. In other words we can have fiber bundles within other fiber bundles.

1.3 Differential Forms

Differential forms are one of the useful mathematical tools we shall be using for the most part of this work. They provide one method for constructing coordinate invariant expressions, simplify certain calculations (e.g. curvature tensors) and play a central role in differential topology.

A differential p-form $\mathbf{A}^{(p)}$ is simply a $(0, p)$ tensor that is completely antisymmetric. In terms of components, a 0-form ϕ (no indices) has zero components, a 1-form A_μ is a dual vector with four components A_0,A_1,A_2,A_3. A 2-form $B_{\mu\nu} = -B_{\nu\mu}$, $B_{\mu\mu} = 0$, has 6 components B_{01}, B_{02}, B_{03}, B_{12}, B_{13}, and B_{23}, a 3-form $C_{\mu\nu\lambda} = C_{\lambda\mu\nu} = C_{\nu\lambda\mu} = -C_{\nu\mu\lambda} = -C_{\mu\lambda\nu} = -C_{\lambda\nu\mu}$, $C_{\mu\mu\lambda} = 0$ which leaves it with only four components C_{012}, C_{013}, C_{023}, and C_{123}. In general, any p-form in D dimensions will have $\begin{pmatrix} D \\ p \end{pmatrix} = \frac{D!}{p!(D-p)!}$ independent components. More interesting things start to happen when we consider products of forms, derivatives of forms and integral of forms.

1.3.1 Wedge Products

We can multiply two forms to get another form as long as we are careful to preserve antisymmetry. $\mathbf{A}^{(p)} \wedge \mathbf{B}^{(q)} = \mathbf{C}^{(p+q)}$ in terms of components $C_{\mu_1 \ldots \mu_{p+q}} = (\mathbf{A} \wedge \mathbf{B})_{\mu_1 \ldots \mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1 \ldots \mu_p} B_{\mu_{p+1} \ldots \mu_{p+q}]}$

For example if $p = q = 1$,

$$
\begin{aligned}
(\mathbf{A} \wedge \mathbf{B})_{\mu\nu} &= \frac{2!}{1!1!} A_{[\mu} B_{\nu]} \\
&= 2.\frac{1}{2} (A_\mu B_\nu - A_\nu B_\mu) \\
&= A_\mu B_\nu - A_\nu B_\mu
\end{aligned}
$$

and if $p = 1$, $q = 2$,

$$
\begin{aligned}
(\mathbf{A}\wedge\mathbf{B})_{\mu\nu\lambda} &= \frac{3!}{1!2!}A_{[\mu}B_{\nu\lambda]} \\
&= 3.\frac{1}{6}\left(A_\mu B_{\nu\lambda} + A_\lambda B_{\mu\nu} + A_\nu B_{\lambda\mu} - A_\mu B_{\lambda\nu} - A_\lambda B_{\nu\mu} - A_\nu B_{\mu\lambda}\right) \\
&= \frac{1}{2}\left(A_\mu B_{\nu\lambda} + A_\lambda B_{\mu\nu} + A_\nu B_{\lambda\mu} - A_\mu B_{\lambda\nu} - A_\lambda B_{\nu\mu} - A_\nu B_{\mu\lambda}\right)
\end{aligned}
$$

We can see that $\mathbf{A}\wedge\mathbf{B} = (-1)^{pq}\,\mathbf{B}\wedge\mathbf{A}$

For example if $p = q = 1$,

$$
\begin{aligned}
(\mathbf{B}\wedge\mathbf{A})_{\mu\nu} &= \frac{2!}{1!1!}B_{[\mu}A_{\nu]} \\
&= 2.\frac{1}{2}\left(B_\mu A_\nu - B_\nu A_\mu\right) \\
&= -\left(A_\mu B_\nu - A_\nu B_\mu\right) \\
&= -\left(\mathbf{A}\wedge\mathbf{B}\right)_{\mu\nu}
\end{aligned}
$$

and if $p = 1$, $q = 2$,

$$
\begin{aligned}
(\mathbf{B}\wedge\mathbf{A})_{\mu\nu\lambda} &= \frac{3!}{1!2!}B_{[\mu}A_{\nu\lambda]} \\
&= 3\cdot\frac{1}{6}\left(B_\mu A_{\nu\lambda} + B_\lambda A_{\mu\nu} + B_\nu A_{\lambda\mu} - B_\mu A_{\lambda\nu} - B_\lambda A_{\nu\mu} - B_\nu A_{\mu\lambda}\right) \\
&= \frac{1}{2}\left(B_{\mu\nu} A_\lambda + B_{\lambda\mu} A_\nu + B_{\nu\lambda} A_\mu - B_{\lambda\nu} A_\mu - B_{\mu\lambda} A_\nu - B_{\nu\mu} A_\lambda\right) \\
&= \frac{1}{2}\left(A_\mu B_{\nu\lambda} + A_\lambda B_{\mu\nu} + A_\nu B_{\lambda\mu} - A_\mu B_{\lambda\nu} - A_\lambda B_{\nu\mu} - A_\nu B_{\mu\lambda}\right) \\
&= \left(\mathbf{A}\wedge\mathbf{B}\right)_{\mu\nu\lambda}
\end{aligned}
$$

1.3.2 Exterior derivative

The components of differential forms can vary over space and time. This means that when we write $B_{\mu\nu}$ we really mean $B_{\mu\nu}\left(x^\lambda\right)$. It is then useful to discuss the derivative of forms. Because of the nature of forms if we use the usual partial derivative $\frac{\partial}{\partial x^\mu}$ we do not get very useful results. On the other hand, if we are careful enough we can get a derivative such that the derivative of a form gives another form.

Consider a form $\mathbf{A}^{(p)}$. The exterior derivative $\mathbf{dA}$ is a $(p+1)$-form with components

$$(\mathbf{dA})_{\mu_1\ldots\mu_{p+1}} = \partial_{[\mu_1} A_{\mu_2\ldots\mu_{p+1}]}$$

with the following properties:

- It satisfies a modified Leibniz rule. For $\mathbf{A}^{(p)}$ and $\mathbf{B}^{(q)}$ we have

$$\mathbf{d}\left(\mathbf{A}\wedge\mathbf{B}\right) \quad = \quad \mathbf{dA}\wedge\mathbf{B} + (-1)^p\,\mathbf{A}\wedge\mathbf{dB}$$

- In an attempt to treat forms like tensors on a curved space or in curvilinear coordinates, one might think that we would need to use $\partial_\mu \to D_\mu = \partial_\mu + \Gamma...$ to get back a true tensor or in this case, a form. It turns out that due to the antisymmetrization including the Christoffel connections make no difference as they still end up cancelling out. This means that we have a meaningful tensorial derivative without requiring a metric which means that these derivatives are defined on and only depend on the topology of the spacetime!

- It squares to zero. This is the Poincarè lemma. Since we would have to antisymmetrize the indices on each ∂ but partial derivatives commute we have $\mathbf{d}^2\mathbf{A} \equiv 0$ for any p-form $\mathbf{A}$. This feature is one of the keys to how exterior calculus of differential forms leads to topological invariants.

1.3.3 Integration

It is very important here to remember that the components of a form only arise when we decompose the form onto a basis. This basis is defined as

$$A^{(p)} = \frac{1}{p!} A_{\mu_1\ldots\mu_p} \mathbf{d}x^{\mu_1} \wedge \mathbf{d}x^{\mu_2} \wedge \ldots \wedge \mathbf{d}x^{\mu_p}$$

where the differentials anticommute, i.e. $\mathbf{d}x^\mu \wedge \mathbf{d}x^\nu = -\mathbf{d}x^\nu \wedge \mathbf{d}x^\mu$. Since coordinates are functions, their exterior derivatives are 1-forms, hence anticommuting.

Proof: Consider

$$\begin{aligned}\mathbf{A}^{(2)} &= \frac{1}{2}A_{\mu\nu}\mathbf{d}x^\mu \wedge \mathbf{d}x^\nu \\ &= \frac{1}{2}A_{\nu\mu}\mathbf{d}x^\nu \wedge \mathbf{d}x^\mu\end{aligned}$$

since $A_{\mu\nu} = -A_{\nu\mu}$ and $\mathbf{d}x^\mu \wedge \mathbf{d}x^\nu = -\mathbf{d}x^\nu \wedge \mathbf{d}x^\mu$ and in fact we could consider this as $\mathbf{d}x^{\mu_1} \wedge \mathbf{d}x^{\mu_2} \wedge ... \wedge \mathbf{d}x^{\mu_p}$ with $p = 2$.

To enable us comprehend the usefulness of a set of anticommuting differentials let us consider $dxdy$ and transform to $x'(x,y), y'(x,y)$. Then:

$$\begin{aligned}dx \to dx'dy' &= \left(\frac{\partial x'}{\partial x}dx + \frac{\partial x'}{\partial y}dy\right)\left(\frac{\partial y'}{\partial x}dx + \frac{\partial y'}{\partial y}dy\right) \\ &= \frac{\partial x'}{\partial x}dx\frac{\partial y'}{\partial x}dx + \frac{\partial x'}{\partial x}dx\frac{\partial y'}{\partial y}dy + \frac{\partial x'}{\partial y}dy\frac{\partial y'}{\partial x}dx + \frac{\partial x'}{\partial y}dy\frac{\partial y'}{\partial y}dy\end{aligned}$$

However, we know that $dxdy$ should transform with the Jacobian, i.e.

$$dx'dy' = \left(\frac{\partial x'}{\partial x}\frac{\partial y'}{\partial y} - \frac{\partial x'}{\partial y}\frac{\partial y'}{\partial x}\right)dxdy$$

which is exactly what we get if we use the wedge product, $\mathbf{d}x^i \wedge \mathbf{d}x^j = -\mathbf{d}x^j \wedge \mathbf{d}x^i \implies \mathbf{d}x \wedge \mathbf{d}x = \mathbf{d}y \wedge \mathbf{d}y = 0$.

We can now see that the basis of a p-form is actually an integration measure over a p-dimensional (oriented) volume. This means that an expression like $\int_{\sum_p} \mathbf{A}^{(p)}$ is perfectly well defined and coordinate invariant. The physical significance of this is that there is a natural coupling between p-form fields $\mathbf{A}^{(p)}$ and the p-dimensional world-surfaces swept out by $(p-1)$-dimensional objects. For instance

- $p = 1$ $\int_{\sum_1} \mathbf{A}^{(1)}$ is the natural coupling of a 1-form A_μ to a particle's worldline.

- $p = 2$ $\int_{\sum_2} \mathbf{B}^{(2)}$ is the natural coupling of a 2-form $B_{\mu\nu}$ to a string's worldsheet.

and in general

- $\int_{\sum_p} \mathbf{B}^{(p)}$ is the natural coupling of a p-form to a $(p-1)$-brane's world volume

1.3.4 Hodge Dual

In 3-dim we can define an invertible mapping between a 1-form and a 2-form because they both have three independent components. This is also applicable to a 0-form and a 3-form since they both have one component in 3-dim. This leads us to introduce the concept of a Hodge dual, or star, operator. In 3-dim and Cartesian coordinates set

$$\begin{aligned}
{}^*(\mathbf{d}x \wedge \mathbf{d}y) &= \mathbf{d}z \\
{}^*(\mathbf{d}y \wedge \mathbf{d}z) &= \mathbf{d}x \\
{}^*(\mathbf{d}z \wedge \mathbf{d}x) &= \mathbf{d}y \\
{}^*(\mathbf{d}x \wedge \mathbf{d}y \wedge \mathbf{d}z) &= 1
\end{aligned}$$

and further require the star to be its own inverse

$$^{**} = 1$$

With these rules we can compute the Hodge dual of any form in 3-dim.

These rules may be summarized using the Levi-Civita tensor. Still in 3-dim and Cartesian coordinates, the dual of any 1-form $\mathbf{A}$ is

$$\begin{aligned}
{}^*\mathbf{A} &= A_i g^{ij} \varepsilon_{jkl} \mathbf{d}x^k \wedge \mathbf{d}x^l \\
&= A^j \varepsilon_{jkl} \mathbf{d}x^k \wedge \mathbf{d}x^l
\end{aligned}$$

For example we can show that the dual of a general 1-form $\mathbf{A} = A_i \mathbf{d}x^i$ is the 2-form $S = A_z \mathbf{d}x \wedge \mathbf{d}y + A_y \mathbf{d}z \wedge \mathbf{d}x + A_x \mathbf{d}y \wedge \mathbf{d}z$.

$$\begin{aligned}
\mathbf{A} &= A_i \mathbf{d}x^i \\
&= A_1 \mathbf{d}x^1 + A_2 \mathbf{d}x^2 + A_3 \mathbf{d}x^3 \\
*\mathbf{A} &= *\left(A_i \mathbf{d}x^i\right) \\
&= *\left(A_1 \mathbf{d}x^1 + A_2 \mathbf{d}x^2 + A_3 \mathbf{d}x^3\right) \\
&= A_1 * \mathbf{d}x^1 + A_2 * \mathbf{d}x^2 + A_3 * \mathbf{d}x^3
\end{aligned}$$

Now, using the fact that a star is its own inverse, we can deduce that

$$\begin{aligned}
{}^{**}(\mathbf{d}x \wedge \mathbf{d}y) &= {}^{*}\mathbf{d}z \\
{}^{**}(\mathbf{d}y \wedge \mathbf{d}z) &= {}^{*}\mathbf{d}x \\
{}^{**}(\mathbf{d}z \wedge \mathbf{d}x) &= {}^{*}\mathbf{d}y \\
{}^{**}(\mathbf{d}x \wedge \mathbf{d}y \wedge \mathbf{d}z) &= {}^{*}1
\end{aligned}$$

which means that

$$\begin{aligned}
{}^{*}\mathbf{d}z &= (\mathbf{d}x \wedge \mathbf{d}y) \\
{}^{*}\mathbf{d}x &= (\mathbf{d}y \wedge \mathbf{d}z) \\
{}^{*}\mathbf{d}y &= (\mathbf{d}z \wedge \mathbf{d}x) \\
{}^{*}1 &= (\mathbf{d}x \wedge \mathbf{d}y \wedge \mathbf{d}z)
\end{aligned}$$

Now taking

$$\begin{aligned} A_1 \mathbf{d}x^1 &= A_x \mathbf{d}x \\ A_2 \mathbf{d}x^2 &= A_y \mathbf{d}y \\ A_3 \mathbf{d}x^3 &= A_z \mathbf{d}z \end{aligned}$$

we can now continue our derivation as follows

$$\begin{aligned} {}^*\mathbf{A} &= A_1{}^*\mathbf{d}x^1 + A_2{}^*\mathbf{d}x^2 + A_3{}^*\mathbf{d}x^3 \\ &= A_x{}^*\mathbf{d}x + A_y{}^*\mathbf{d}y + A_z{}^*\mathbf{d}z \\ &= A_x\left(\mathbf{d}y \wedge \mathbf{d}z\right) + A_y\left(\mathbf{d}z \wedge \mathbf{d}x\right) + A_z\left(\mathbf{d}x \wedge \mathbf{d}y\right) \end{aligned}$$

In general, we can see that the Hodge operator is a map for any general m-form, in n dimensions such that $* : \Lambda_p^m\left(R^n\right) \longrightarrow \Lambda_p^{n-m}\left(R^n\right)$ is linear and satisfies the following properties

$$*\Big(\mathbf{d}x^{i_1} \wedge \mathbf{d}x^{i_2} ... \wedge \mathbf{d}x^{i_m}\Big) = \frac{1}{(n-m)!}\epsilon^{i_1 ... i_m}{}_{i_{m+1} ... i_n} \mathbf{d}x^{i_{m+1}} \wedge ... \wedge \mathbf{d}x^{i_n}$$

where $\frac{1}{(n-m)!}\epsilon^{i_1 ... i_m}{}_{i_{m+1} ... i_n}$ is just a number.

If we consider $n = 3$ and $m = 1$, we can work backwards to derive our findings as before.

$$
\begin{aligned}
* : \Lambda^1_p\left(R^3\right) &\longrightarrow \Lambda^2_p\left(R^3\right) \\
\Longrightarrow *\mathbf{d}x^1 &= \frac{1}{2!}\epsilon^1{}_{jk}\mathbf{d}x^j \wedge \mathbf{d}x^k \\
&= \frac{1}{2!}\left(\epsilon^1{}_{23}\mathbf{d}x^2 \wedge \mathbf{d}x^3 + \epsilon^1{}_{32}\mathbf{d}x^3 \wedge \mathbf{d}x^2\right) \\
&= \frac{1}{2!}\left(\mathbf{d}x^2 \wedge \mathbf{d}x^3 - \mathbf{d}x^3 \wedge \mathbf{d}x^2\right) \\
&= \frac{1}{2!}\left(\mathbf{d}x^2 \wedge \mathbf{d}x^3 - \mathbf{d}x^3 \wedge \mathbf{d}x^2\right) \\
&= \mathbf{d}x^2 \wedge \mathbf{d}x^3
\end{aligned}
$$

We can similarly show that

$$
\begin{aligned}
*\mathbf{d}x^2 &= -\mathbf{d}x^1 \wedge \mathbf{d}x^3 \\
*\mathbf{d}x^3 &= \mathbf{d}x^1 \wedge \mathbf{d}x^2
\end{aligned}
$$

In general, the Hodge dual maps basis from one space to another linearly.

1.4 A biconformal model of graviweak interactions

The possibility for a unified graviweak theory based in biconformal geometry was recently established in [1]. These geometries arise as gauge theories of the conformal group $\mathcal{C}$ of spaces of dim $n = p+q$, with $SO\left(p,q\right)$ metric, where $\mathcal{C}$ may be written as $SO\left(p+1,q+1\right)$ or the corresponding spin group, $Spin\left(p+1,q+1\right)$. The quotient of this conformal group by its homogeneous Weyl subgroup gives a principal fiber bundle with 2n-dim base manifold and Weyl fibers. The Cartan generalization to a curved $2n$-dim geometry admits an action functional linear in the curvatures. Because symmetry is maintained between the translations and the special conformal transformations in the construction, these spaces are called biconformal; this same symmetry gives biconformal spaces overlapping structures with Kähler manifolds and with double field theories, including manifest T-duality. Because of the manifest duality between translations and special conformal transformations, biconformal space admit a gravity action linear in the curvatures [11]. In [1] it is established that the field equations arising from the linear action lead to n-dimensional general relativity with integrable local scale. It is notable that the field equations reduce all curvature components to dependence only on the solder form of an n-dim Lagrangian submanifold, despite the increased number of curvature components and doubled number of initial independent variables. The reduction occurs without need for a section condition.

Here we focus on one result presented in [1], that the torsion-free solutions are foliated by copies of an n-dim Lie group G. These Lie groups fall into two classes:

1. Generically, the Lie group G is abelian. Thus, the corresponding torsion-free solutions generically describe locally scale-covariant general relativity with symmetric, divergence-free sources with G representing either (a) the co-tangent bundle of n-dim (p, q)-spacetime or (b) the torus of double field theory.

2. The solutions admit a subclass of spacetimes with n-dim non-abelian Lie symmetry. The group G must be acted upon by the original $SO((p, q)$ or $Spin\,(p, q)$ fiber symmetry. As suggested in [1] these latter cases include the possibility of a unification of gravity and the electroweak interaction.

To realize a graviweak theory, we study the $SO\,(4)$ case.

Starting with a compactified Euclidean 4-space, we choose a spinor representation so that the conformal symmetry is $\mathcal{C} = Spin\,(5, 1)$. The homogeneous Weyl group then consists of dilatations together with $SO\,(4) = SU\,(2) \times SU\,(2)$. The non-abelian group may be a 4-dimensional subgroup, and the electroweak symmetry $SU\,(2) \times U\,(1)$ is an obvious possibility

In order for us to realize this possiblity we need the following background understanding:

1. Explore explicitly the electroweak theory accoording to the standard model.

2. Understand details of spinor representations.

3. Develop and understand the *spinor representation* of the biconformal gauge theory of $SU(2) \times SU(2)$. This involves new research since previous biconformal models have been based on orthogonal groups. Although the Cartan structure equations are the same for spin and orthogonal groups, differences arise when we implement the $SU(2) \times SU(2)$ product manifestly.

4. Study known methods for obtaining Lorentzian gravity from $SO(4)$ Euclidean symmety. This can be accomplished either according to [12] or using the method of [13]. Either way, this is doable. The former method gives us extra fields, some of which might cast light on the origin of the Higgs field.

In Chapter 2 we present a review of the Weinberg-Salam model of the electroweak interactions and Chapter 3 reviews basic properties of biconformal spaces.

In Chapter 4 we explore many details of spinor representations. Since the whole structure of the standard model ranging from the electroweak interactions to the strong interactions are based on spinor representations which are used to explain all matter as we know it till date. The importance of spinors therefore, cannot be overemphasized.

Following these introductory Chapters we present our two principal results:

- Develop a full spinor representation for the biconformal field equations that makes the $SU(2) \times SU(2)$ product structure manifest.

- Explore details of the breakdown from $SU(2) \times SU(2)$ to $SU(2) \times U(1)$. As a likely possibility, we show that the subgroup of $SU(2) \times SU(2)$ transformations that preserve the form of the Weyl vector is the electroweak symmetry. To do this, we expressed the real, 8-component Weyl vector in terms of spinors and studied its transformations and gauge properties.

The derivation of these results is presented in the final Chapters. Future work will begin with solutions to the field equations, using the splitting consistent with the Weyl vector to dictate the group breakdown.

CHAPTER 2
THE ELECTROWEAK MODEL

2.1 Gauge Theory and a Glance at the Electroweak Model

The procedure of gauge field theory, with the gauging of electromagnetism using the $U(1)$ group as an example, can be summarized in three stages; Firstly, we write down the Lagrangian which in our case will be the Dirac Lagrangian for a spin-half matter field.

$$\mathcal{L} = i\hbar c\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - mc^2\bar{\psi}\psi$$

It is easy to see that the Lagrangian in (1) preserves global $U(1)$ transformations.

$$\begin{aligned} \psi &\rightarrow \psi' = e^{iq\theta}\psi \\ \bar{\psi} &\rightarrow \bar{\psi}' = e^{-iq\theta}\bar{\psi} \end{aligned}$$

$$\Longrightarrow \mathcal{L}_{\frac{1}{2}} \rightarrow \mathcal{L}'_{\frac{1}{2}} = i\hbar c\bar{\psi}e^{-iq\theta}\gamma^{\mu}\partial_{\mu}e^{iq\theta}\psi - mc^2\bar{\psi}e^{-iq\theta}e^{iq\theta}\psi = \mathcal{L}_{\frac{1}{2}}$$

If we try to replace the global symmetry by a local $U(1)$ gauge transformation, we find out that this is not a symmetry of the Lagrangian.

$$\mathcal{L}_{\frac{1}{2}} \rightarrow \mathcal{L}'_{\frac{1}{2}} = i\hbar c\bar{\psi}e^{-iq\theta(x)}\gamma^{\mu}\partial_{\mu}e^{iq\theta(x)}\psi - mc^2\bar{\psi}e^{-iq\theta(x)}e^{iq\theta(x)}\psi \neq \mathcal{L}_{\frac{1}{2}}$$

This is due to the fact that the partial derivatives do not transform homogenously,

$$\partial_{\mu}\left(e^{iq\theta(x)}\psi\right) \neq e^{iq\theta(x)}\partial_{\mu}\left(\psi\right)$$

The next step is to promote the global symmetry to a local symmetry by replacing the partial derivatives with covariant derivatives

$$\partial_\mu \rightarrow D_\mu = \partial_\mu + iqA_\mu \tag{2.1}$$

where A_μ is a gauge field and q is a scalar. Defining the covariant derivative as given in Eq(2.1) guarantees that the new derivative transforms homogenously.

$$\widetilde{D}_\mu \psi = e^{i\theta(x)} D_\mu \psi \equiv \widetilde{D_\mu \psi}$$

with the transformation condition for the newly introduced gauge field as

$$iA_\mu \rightarrow iA'_\mu = iA_\mu - i\partial_\mu \theta\left(x\right)$$

The new Lagrangian under this new local gauging is

$$\begin{aligned} \mathcal{L}_{local} &= i\hbar c\bar{\psi}\gamma^\mu D_\mu \psi - mc^2\bar{\psi}\psi \\ &= \mathcal{L}_{\frac{1}{2}} - q\hbar c\gamma^\mu \bar{\psi} A_\mu \psi \end{aligned}$$

where $q\hbar c\gamma^\mu \bar{\psi} A_\mu \psi$ is the interaction term. The new Lagrangian transforms as follows:

$$\begin{aligned} \mathcal{L}_{local} \rightarrow \mathcal{L}'_{local} &= i\hbar c\bar{\psi}e^{-iq\theta(x)}\gamma^\mu D_\mu e^{iq\theta(x)}\psi - mc^2\bar{\psi}e^{-iq\theta(x)}e^{iq\theta(x)}\psi \\ &= i\hbar c\bar{\psi}e^{-iq\theta(x)}\gamma^\mu e^{iq\theta(x)} D_\mu \psi - mc^2\bar{\psi}\psi \\ &= \mathcal{L}_{local} \end{aligned}$$

The last step is to introduce a kinetic term for the propagation of the gauge field A_μ. This gauge field will be our photon and we will require a kinetic term for for a spin-1 particle to propagate it. The starting point is the Proca Lagrangian given as

$$\mathcal{L}_1 = \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} + \frac{1}{8} \left(\frac{mc}{\hbar}\right)^2 A^\mu A_\mu$$

where $F_{\mu\nu}$ is the electromagnetic field strength defined as

$$F_{\mu\nu} \;=\; \frac{i}{q} \left[D_\mu, D_\nu\right] = \partial_\nu A_\mu - \partial_\mu A_\nu$$

The mass term of the Proca Lagrangian for our gauge field must vanish for local gauging to be a symmetry. Then we can interpret the gauge field as the photon since it is massless, and the new Lagrangian becomes

$$\mathcal{L}_{\frac{1}{2}} \rightarrow \mathcal{L}'_{\frac{1}{2}} = \mathcal{L}_{\frac{1}{2}} - q\hbar c \gamma^\mu \bar{\psi} A_\mu \psi + \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}$$

where q is the coupling strength of the fermions described by ψ to the electromagnetic field A_μ, and this is what we define to be the electric charge.

2.1.1 Properties of the Electroweak Interactions

Electromagnetic interactions only affect particles with charge and the strong interaction only affects quarks. Gravitation effects all forms of energy, and every particle feels the weak interaction. Properties of the weak interaction include the following;

1. Flavor change
2. Parity Violation
3. Violation of Charge Conjugation
4. CP Violation
5. Isospin Doublets and Singlets

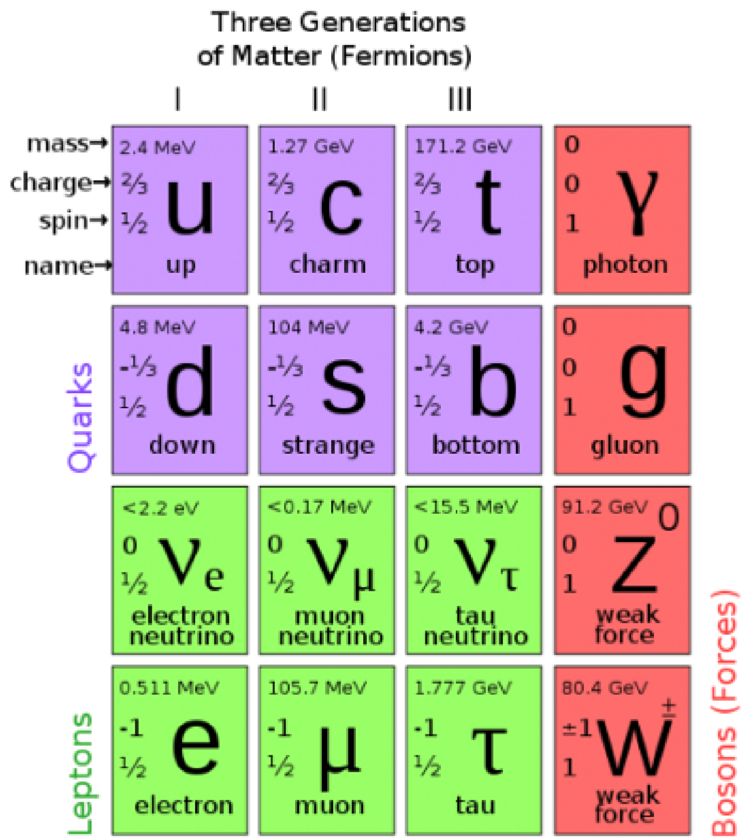

Fig. 2.1: Particles and their interactions according to the Standard Model [14]

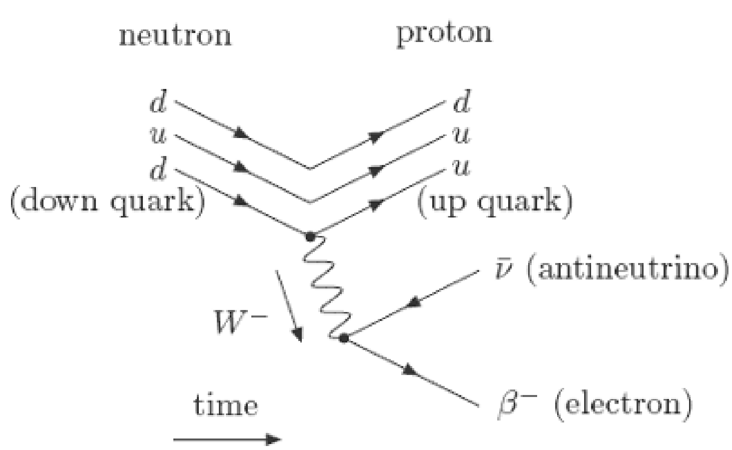

Fig. 2.2: A down quark turning into an up quark as an example of Flavor Changing in Weak Interactions [15]

2.1.2 Flavor change

In figure (2.1), the fermions in purple and the quarks in green are paired in flavors in three groups of up and down. These pairs are called flavors and the weak interaction can change from one flavor to another. To simplify this concept let us consider the beta decay of the Cobalt-60 atom which is an example of a weak interaction. The decay process is given by

$$^{60}_{27}Co \longrightarrow ^{60}_{28} Ni + e^- + \bar{\nu}_e$$

which can be broken down to the decay of a single neutron into a proton,

$$\Longrightarrow n \longrightarrow p + e^- + \bar{\nu}_e$$

This can be further broken down into the flavor change of a down quark into an up quark

$$(udd) \longrightarrow (uud) + e^- + \bar{\nu}_e$$
$$\Longrightarrow d \longrightarrow u + e^- + \bar{\nu}_e$$

This flavor change can be visualized in the figure (2.2). If we compare the result from figure (2.2) with those from figure (2.3), we find out that unlike the weak interactions, the strong and electromagnetic interactions do not change flavor. In other words the same particle that comes into any vertex as an incoming particle, comes out from the same vertex as an outgoing particle

2.1.3 Parity Violation

So far, we have seen how continuous symmetries are satisfied by the standard model but there are certain discrete symmetries that are violated within the standard model by the weak interaction. Parity refers to a spatial mirror inversion with respect to some given

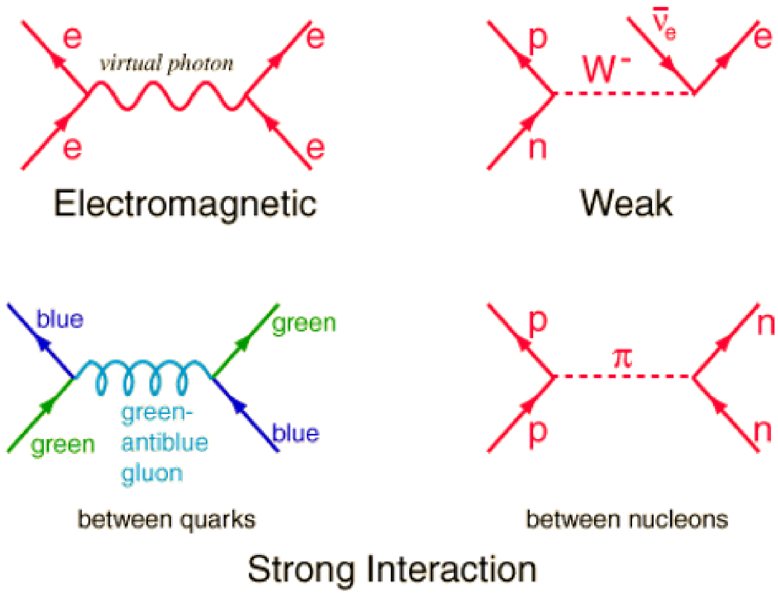

Fig. 2.3: Comparing the Strong and electromagnetic interactions which do not change flavor with the weak interactions that does change flavor [16]

direction. There are two ways of understanding parity violation in the standard model. The first method has to do with the concept of helicity.

A particle whose spin is in the same direction as its momentum is called a right-handed particle and is given a positive quantum number of unity known as its helicity while a particle whose spin is in the opposite direction as its momentum is called a left-handed particle. The later particle is given a negative one quantum number for its helicity. Parity operations on particles change right-handed particles into left-handed particles and vice versa.

In 1957, C. S. Wu alongside other physicists was the first to observe this parity violation in beta decay [17]. Considering the same beta decay for Cobalt-60 nucleus, one possible configuration is that a right-handed antineutrino and a left-handed electron be ejected as shown as spin configuration 1 in the figure (2.5) below.

If we act on spin configuration 1 with the parity operator, we expect that the left-handed electron becomes right-handed and the right-handed antineutrino become left-handed as shown in spin configuration 2. If the process was to preserve parity, then both configurations should be equally observed in experiments. It turns out that only the first configuration is

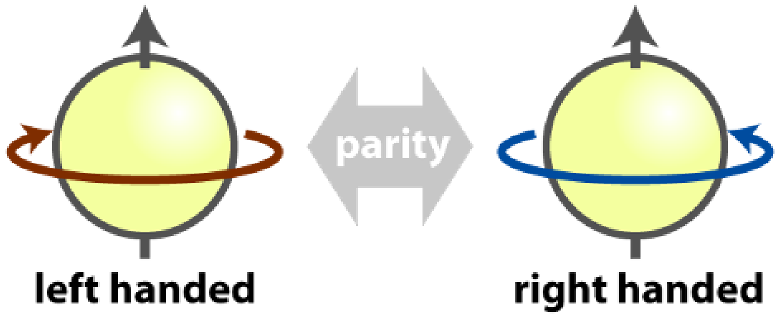

Fig. 2.4: A visualization of how the Parity operation changes a left-handed particle into a right-handed particle and vice versa [18]

observed as we do not have left-handed antineutrinos in nature.

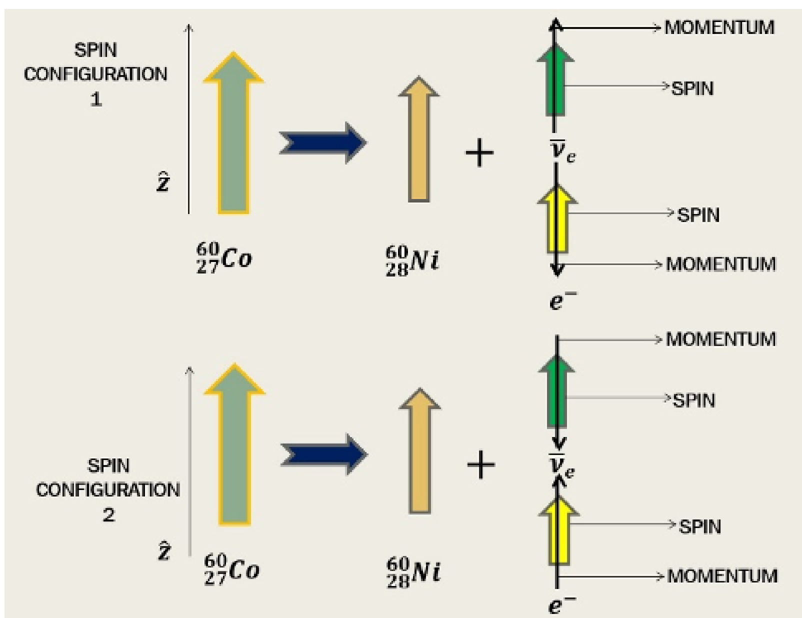

Fig. 2.5: Parity Violation Experiment based on Helicity test

This is what we mean by parity violation in the context of helicity. Another way to comprehend the concept of parity violation is the following;

Quantum mechanically, if we separate a wavefunction as follows;

$$\psi\left(r,\theta,\phi\right) = R\left(r\right)Y_l^m\left(\theta,\phi\right)$$

Under parity, the coordinates change as follows;

$$\left(r,\theta,\phi\right) \longrightarrow \left(-r,\pi-\theta,\pi+\phi\right)$$

The spherical harmonics ordinarily can be written as proportionality

$$Y_l^m\left(\theta,\phi\right) \propto P_l^m\left(\cos\theta\right)\cdot e^{(im\phi)}$$

Under parity, the individual components above transform as

$$\begin{aligned}
e^{(im\phi)} &\longrightarrow e^{im(\phi+\pi)} \\
&= \left(e^{i\pi}\right)^m e^{im\phi} \\
&= (-1)^m e^{im\phi} \\
\Longrightarrow P_l^m\left(\cos\theta\right) &\longrightarrow P_l^m\left(\cos\left(\pi-\theta\right)\right) \\
&= (-1)^{l+m} P_l^m\left(\cos\theta\right)
\end{aligned}$$

The last expression is deduced from the behavior of the associated Legendre polynomials when θ changes to $(\pi-\theta)$. Therefore, under parity. Therefore, under parity

$$\begin{aligned} Y_l^m(\theta, \phi) &\longrightarrow Y_l^m(\pi - \theta, \pi + \phi) \\ &= (-1)^{l+m} \cdot P_l^m(\cos\theta) \cdot (-1)^m . e^{im\phi} \\ &= (-1)^l Y_l^m(\theta, \phi) \end{aligned}$$

Therefore, parity of the spherical harmonics depends on the evenness or oddness of l.

To every particle we attribute a parity quantum number: plus one for scalars and pseudo-vectors and minus one for pseudo-scalars and vectors. Bosons will have the same parity quantum number as antibosons while fermions will have opposite parity values as antifermions. The pions or pi-mesons would then have a negative parity since they are classified as pseudo-scalars. Consider the weak decays below;

$$\begin{aligned} \theta^+ &\longrightarrow \pi^+ + \pi^0 \\ \tau^+ &\longrightarrow \pi^+ + \pi^0 + \pi^0 \end{aligned}$$

We multiply parities of products to get the parity of the parent. This means that θ^+ will have positive parity while τ^+ will have negative parity. When parity was considered a symmetry of the standard model in the early part of elementary particle physics, θ^+ and τ^+ were thought to be different particles even though other than this parity discrepancy they were identical in every other way. After the discovery of parity violation, they were then understood to be the same particle called K^+. Since, the later process preserves parity, it was attributed to any of the electromagnetic, strong, or weak interactions, while the former process was attributed to weak interactions alone because it violates parity.

2.1.4 Violation of Charge Conjugation:

Charge conjugation is an operation that takes particles into antiparticles. It is more than just changing the charges of the particles because it can affect even neutral particles.

For example, operating on a proton and a neutron with the charge conjugation operator will give us an antiproton and an antineutron as follows;

$$\begin{aligned} C\left|p\right\rangle &= C\left|uud\right\rangle \\ &= \left|\bar{u}\bar{u}\bar{d}\right\rangle \\ &= \left|\bar{p}\right\rangle \\ C\left|n\right\rangle &= C\left|udd\right\rangle \\ &= \left|\bar{u}\bar{d}\bar{d}\right\rangle \\ &= \left|\bar{n}\right\rangle \end{aligned}$$

Consider the decay below

$$\pi^{+} \longrightarrow \mu^{+} + \nu_{\mu} \tag{2.2}$$

The products yielded are both left-handed. If we then transform this decay under charge conjugation we get

$$\pi^{-} \longrightarrow \mu^{-} + \bar{\nu}_{\mu} \tag{2.3}$$

These products are just antiparticles of the previous ones due to the nature of charge conjugation but remain left-handed. But we do not have left-handed antineutrinos in nature, so, the later process is not valid and as a result the weak interactions do not preserve charge conjugation.

2.1.5 CP Violation

CP refers to a combination of charge conjugation followed by a parity operation. People thought that this might be a symmetry of the standard model and as it turns out, after charge conjugating Eq.(2.2) to get Eq.(2.3), if we further apply the parity operator on Eq.(2.3) we get back the right-handed antineutrinos as expected. It turns out that this symmetry is still broken but to a very small scale compared to that of parity and charge conjugation. This is known as minimal symmetry breaking for the case of CP violation. When we further act on the CP by a time reversal operator it happens that we get a symmetry for the standard model due to the CPT theorem. According to the CPT model, since CP is not a symmetry of the standard model, the CPT combination is a symmetry if and only if time reversal is also not a symmetry.

2.1.6 Isospin of Doublets and Singlets

We have left-handed electrons, muons and tau particles with corresponding left-handed neutrinos respectively. We do also have right-handed electrons, muons and tau particles but no corresponding right-handed neutrinos respectively. This is because they have not been found experimentally but have found right-handed anti neutrinos in beta decays and other weak interactions. For the quarks we have both right-handed and left-handed up, down, charm, strange, top, and bottom quarks respectively. From observations also, only left-handed particles take part in weak interactions.

The gauge group for the weak interaction is the $SU(2)_L$. Under this group we can consider the fields or wavefunctions corresponding to the left-handed particles and their neutrinos as a doublet

$$\begin{pmatrix} \psi_{\nu_{e_L}} \\ \psi_{e_L} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \psi_{\nu_e} \\ \psi_e \end{pmatrix}_L$$

while the right-handed particles have singlet fields (do not take part in the weak interaction) since they have no partners

$$\psi_{e_R} \longleftrightarrow (\psi_e)_R$$

Just as we assume the isospin in strong interactions for the nucleon is a composite of protons (isospin $+\frac{1}{2}$) and neutrons (isospin $-\frac{1}{2}$), we can also make such a formulation of isospin for the $SU(2)_L$. The field corresponding to the doublet $\begin{pmatrix} \psi_{\nu_e} \\ \psi_e \end{pmatrix}_L$ has isospin $I = \frac{1}{2}$, with isospin $I_3 = +\frac{1}{2}$ for ν_{e_L} and isospin $I_3 = -\frac{1}{2}$ for e_L . The isospin for the singlet is zero. The total doublets and singlets for both leptons and quarks in the weak interaction are

$$\begin{pmatrix} \psi_{\nu_\mu} \\ \psi_\mu \end{pmatrix}_L, \begin{pmatrix} \psi_{\nu_\tau} \\ \psi_\tau \end{pmatrix}_L, \begin{pmatrix} \psi_{\nu_e} \\ \psi_e \end{pmatrix}_L, \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}_L, \begin{pmatrix} \psi_c \\ \psi_s \end{pmatrix}_L, \begin{pmatrix} \psi_t \\ \psi_b \end{pmatrix}_L,$$
$$(\psi_e)_R, (\psi_\mu)_R, (\psi_\tau)_R, (\psi_u)_R, (\psi_d)_R, (\psi_c)_R, (\psi_s)_R, (\psi_t)_R, (\psi_b)_R$$

The gauge group corresponding to the electroweak interaction is the combination of the $SU(2)_L$ isospin and the $U(1)_Y$ hypercharge given as $SU(2)_L \times U(1)_Y$ The relationship between the charges is given by

$$Q = I_3 + \frac{Y}{2}$$

Q is the electromagnetic charge, I_3 is the third projection of isospin which is the charge corresponding to $SU(2)_L$, and Y is the hypercharge corresponding to $U(1)_Y$.

Now consider the doublet

$$\begin{pmatrix} \psi_{\nu_{e_L}} \\ \psi_{e_L} \end{pmatrix}$$

$Q_{\nu_{e_L}} = 0$ since it's a neutral particle, $I_{3\nu_{e_L}} = +\frac{1}{2}$, therefore, these fix $Y_{\nu_{e_L}} = -1$.

$Q_{e_L} = -1$ since it's a negative particle, $I_{3e_L} = -\frac{1}{2}$, therefore, these fix $Y_{e_L} = -1$.

$Q_{e_R} = -1$ since it's a negative particle, $I_{3e_R} = 0$, therefore, these fix $Y_{e_R} = -2$.

The isospin and hypercharge are two quantum numbers that help us distinguish between left-handed and right-handed particles. Since both left-handed and right-handed electrons have the same charge -1, they are indistinguishable in electromagnetic interactions but distinguishable in weak interactions.

2.2 Weinberg-Salam-Glashow (WSG) Model and L-R Symmetry Breaking

The WSG model unifies the electromagnetic and weak forces into the electroweak theory at high energies. The standard model at high energies is summarized by the interaction between elements of the group $SU(3) \times SU(2)_L \times U(1)_Y$ where L stands for left-handedness and Y for hypercharge in order to differentiate it from the electromagnetic charge. The left-handed particles exist as a doublet of two flavors which can be interchanged under a weak interaction. This is a special feature of weak interactions which is not obtainable from any other interactions. The right-handed particles are singlets. In the simplest of assumptions of this model, we write the Dirac Lagrangian as follows;

$$\begin{aligned}
\mathcal{L} &= i\hbar c\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - mc^2\bar{\psi}\psi \\
\psi &= \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \\
&= \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}
\end{aligned}$$

with projection operators defined as

$$\begin{aligned}
P_{\pm}\psi &= \frac{\left(1 \pm \gamma^5\right)}{2}\psi \\
\psi_R &= \frac{\left(1 + \gamma^5\right)}{2}\psi \\
\psi_L &= \frac{\left(1 - \gamma^5\right)}{2}\psi
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\bar{\psi}_L\gamma^\mu\psi_R &= 0 \\
\bar{\psi}_R\gamma^\mu\psi_L &= 0 \\
\bar{\psi}_R\psi_R &= 0 \\
\bar{\psi}_L\psi_L &= 0
\end{aligned}$$

Using these conditions, we restate our Lagrangian as follows in terms of the left-handed and right-handed spinors;

$$\mathcal{L} = i\hbar c\left(\bar{\psi}_R\gamma^\mu\partial_\mu\psi_R + \bar{\psi}_L\gamma^\mu\partial_\mu\psi_L\right) - mc^2\left(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R\right)$$

Now, using an electron as a specific example, we have

$$\mathcal{L} = i\hbar c\left(\bar{e}_R\gamma^\mu\partial_\mu e_R + \begin{pmatrix} \bar{\nu_e} \\ \bar{e} \end{pmatrix}_L \gamma^\mu\partial_\mu \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L\right) - mc^2\left(\bar{\psi}_R\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L + \begin{pmatrix} \bar{\nu_e} \\ \bar{e} \end{pmatrix}_L \psi_R\right)$$

An $SU(2)_L$ transformation acts on the doublets as follows using an arbitrary χ_L;

$$\begin{aligned}
\chi_L^{'} &\rightarrow e^{-\frac{ig}{2}\overrightarrow{\sigma}\overrightarrow{\theta}}\chi_L \\
&= e^{-\frac{ig}{2}\overrightarrow{\sigma}\overrightarrow{\theta}}\begin{pmatrix}\nu_e \\ e\end{pmatrix}_L \\
\overline{\chi}_L^{'} &\rightarrow e^{\frac{ig}{2}\overrightarrow{\sigma}\overrightarrow{\theta}}\overline{\chi}_L \\
&= e^{\frac{ig}{2}\overrightarrow{\sigma}\overrightarrow{\theta}}\begin{pmatrix}\bar{\nu_e} \\ \bar{e}\end{pmatrix}_L
\end{aligned}$$

where $\overrightarrow{\sigma}$ are the Pauli marices. Simulteneously we do a $U\,(1)_Y$ transformation on both the left-handed electron doublet and right-handed singlet in the form:

$$\begin{aligned}
\chi_L^{''} &\rightarrow e^{-g^{'}Y_{\chi_L}\phi}\chi_L \\
&= e^{-g^{'}Y_{\chi_L}\phi}\begin{pmatrix}\nu_e \\ e\end{pmatrix}_L \\
\overline{\chi}_L^{''} &\rightarrow e^{g^{'}Y_{\chi_L}\phi}\overline{\chi}_L \\
&= e^{g^{'}Y_{\chi_L}\phi}\begin{pmatrix}\bar{\nu_e} \\ \bar{e}\end{pmatrix}_L
\end{aligned}$$

and

$$\begin{aligned}
e_R^{''} &\rightarrow e^{-g^{'}Y_{\chi_L}\phi}e_R \\
&= e^{-g^{'}Y_{\chi_L}\phi}\,(e)_R \\
\overline{e}_R^{''} &\rightarrow e^{g^{'}Y_{\chi_L}\phi}\overline{e}_R \\
&= e^{g^{'}Y_{\chi_L}\phi}\,(\overline{e})_R
\end{aligned}$$

To make our lagrangian locally symmetric we do the following transformations for both the doublets and singlets:

$$
\begin{aligned}
\partial_\mu \chi_L &\longrightarrow D_\mu \chi_L \\
&= \partial_\mu \chi_L + \frac{ig}{2}\vec{\sigma}.\vec{W}_\mu \chi_L + ig^{'} B_\mu \chi_L \\
\partial_\mu e_R &\longrightarrow D_\mu e_R \\
&= \partial_\mu e_R + ig^{'} Y_{e_R} B_\mu e_R
\end{aligned}
$$

We require three gauge fields for $\vec{W}_\mu$ corresponding to the three generators of $SU(2)_L$ (one for each Pauli matrix), and one gauge field for B_μ corresponding to the single generators of $U(1)_Y$. To preserve our local symmetry these gauge fields will also transform as follows:

$$
\begin{aligned}
\vec{\sigma}.\vec{W}^{'}_\mu &\longrightarrow e^{-\frac{ig}{2}\vec{\sigma}.\vec{\theta}}\vec{\sigma}.\vec{W}_\mu e^{\frac{ig}{2}\vec{\sigma}.\vec{\theta}} + \frac{i}{g}\partial_\mu\left(e^{-\frac{ig}{2}\vec{\sigma}.\vec{\theta}}\right)e^{\frac{ig}{2}\vec{\sigma}.\vec{\theta}} \\
B^{'}_\mu &\longrightarrow B_\mu + \partial_\mu \phi
\end{aligned}
$$

The next step is to allow these new gauge fields propagate by giving them kinetic terms which we generally build by constructing

$$
F_{\mu\nu} = -\frac{i}{g}\left[D_\mu, D_\nu\right]
$$

For $U(1)$ this expression takes the form

$$
F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu
$$

while for $SU(2)$ which is non-abelian, we add an extra term and it takes the form

$$F^a_{\mu\nu} = \partial_\mu W^a_{\ \nu} - \partial_\nu W^{^a}_{\ \mu} - g\varepsilon^a_{\ bc} W^b_{\ \mu} W^c_{\ \nu}$$

2.3 The Higgs Mechanism

There were two major problems with the $SU(2)_L \times U(1)_Y$ electroweak model. The first problem was the fact that from experiments the weak gauge bosons are found to be massive while the Proca mass term $\left(\frac{mc}{\hbar}\right)^2 W^a_{\ \mu} W^{\mu a}$ is not locally gauge invariant which makes it vanish from our Lagrangian. Secondly, we recall that to have mass terms for spinors requires both the left and right-hand parts of ψ to combine as in $mc^2 \left(\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R\right)$, however we have just constructed a gauge theory where the left and right parts transform differently and as such we do not expect terms like $mc^2 \left(\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R\right)$ to be gauge invariant. We resolve both of these problems using the Higgs mechanism for mass generation.

A crucial part of this process is the breaking of $SU(2)_L \times U(1)_Y \longrightarrow U(1)_{EM}$. The electroweak group,$SU(2)_L \times U(1)_Y$, has four generators; $W^3_{\ \mu}, W^{\pm}_{\ \mu}, B_\mu$. After symmetry breaking to $U(1)_{EM}$, one would expect only one of these four symmetry generators to survive but this is not what happens. In actualty, the B_μ mixes with the neutral $W^3_{\ \mu}$ from $SU(2)_L$ to form two orthogonal states

$$A_\mu = B_\mu cos\theta_W + W^3_{\ \mu} sin\theta_W$$

representing the photon for $U(1)_{EM}$ and

$$Z_\mu = -B_\mu sin\theta_W + W^3_{\ \mu} cos\theta_W$$

for the massive neutral Z^0 boson of the weak interactions. The angle θ_W is the Weinberg mixing angle.

Another important point is that the original unified gauge group $SU(2)_L \times U(1)_Y$ will not be truly unified if the $SU(2)_L$ and $U(1)_Y$ factors had completely independent couplings g and g' respectively. Since we experience the broken version of this theory, it is useful to know how the couplings $W^{\pm}, Z^0$ and γ are related. It turns out that

$$\begin{aligned} g sin\theta_W &= g' cos\theta_W \\ &= g_\gamma \\ g &= g_{W^\pm} \\ g_Z &= \frac{g_\gamma}{sin\theta_W cos\theta_W} \end{aligned}$$

To understand how the Higgs mechanism works let us consider a simple construction of the model based on a $U(1)$ theory. Consider the following Lagrangian for a spin-zero scalar field

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi^*\right)\left(\partial^\mu \phi\right) - \frac{1}{2}\mu^2 \phi^* \phi + \frac{1}{4}\lambda^2 \left(\phi^* \phi\right)^2$$

When promoted to a local gauge symmetry and adding kinetic terms this becomes

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu + \frac{iq}{\hbar c}A_\mu\right)\phi^* \left(\partial_\mu + \frac{iq}{\hbar c}A_\mu\right)\phi - \frac{1}{2}\mu^2 \phi^* \phi + \frac{1}{4}\lambda^2 \left(\phi^* \phi\right)^2 + \frac{1}{16}F^{\mu\nu}F_{\mu\nu}$$

where the fields in terms of some arbitrary background configuration can be written as

$$\begin{aligned} \phi\left(x^\mu\right) &= \phi_0\left(x^\mu\right) + \delta\phi\left(x^\mu\right) \\ A_\mu\left(x^\mu\right) &= A_{\mu 0}\left(x^\mu\right) + \delta A_\mu\left(x^\mu\right) \end{aligned}$$

and $\phi_0(x^\mu)$ describes the constant background configuration of the field and $\delta\phi(x^\mu)$ represents fluctuations relative to this background. These fluctuations are interpreted as particles. We then proceed to find solutions for the background by first setting $\partial\phi$ to zero. This lets us solve for the first term in our Lagrangian. Secondly we vary the action. Finding $\frac{\partial\mathcal{L}}{\partial\phi^*}=0$ for the equations of motion, we have,

$$\begin{aligned} 0=\frac{\partial\mathcal{L}}{\partial\phi^*} &= \frac{\partial}{\partial\phi^*}\left(-\frac{1}{2}\mu^2\phi^*\phi+\frac{1}{4}\lambda^2\left(\phi^*\phi\right)^2\right) \\ &= -\frac{1}{2}\mu^2\phi+\frac{1}{2}\lambda^2\left(\phi^*\phi\right)\phi \\ &= -\frac{1}{2}\mu^2\phi+\frac{1}{2}\lambda^2\left|\phi^2\right|\phi \end{aligned}$$

One solution is taking $A_\mu=0$, $\phi=0$. Using this solution and studying

$$\begin{aligned} \phi\left(x^\mu\right) &= 0+\delta\phi\left(x^\mu\right) \\ A_\mu\left(x^\mu\right) &= 0+\delta A_\mu\left(x^\mu\right) \end{aligned}$$

we get

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu+\frac{iq}{\hbar c}\delta A_\mu\right)\delta\phi^*\left(\partial_\mu+\frac{iq}{\hbar c}\delta A_\mu\right)\delta\phi-\frac{1}{2}\mu^2\delta\phi^*\delta\phi+\frac{1}{4}\lambda^2\left(\delta\phi^*\delta\phi\right)^2+\frac{1}{16}F^{\mu\nu}F_{\mu\nu}$$

with $F_{\mu\nu}=\partial_\mu\delta A_\nu-\partial_\nu\delta A_\mu$ in this case. The result is exactly like the original Lagrangian with $\phi\to\delta\phi, \phi^*\to\delta\phi^*, A_\mu\to\delta A_\mu$.

But another solution is $A_\mu=0, \phi=\phi_0$ where $\phi_0=\phi_1+i\phi_2$ and

$$\frac{\partial\mathcal{L}}{\partial\phi^*}=-\frac{1}{2}\mu^2\phi_0+\frac{1}{2}\lambda^2\left|\phi_0^2\right|\phi_0=0$$

requires $|\phi_0|^2 = \frac{\mu^2}{\lambda^2} = \phi_{10}^2 + \phi_{20}^2$. Choosing the phase so that $\phi_{10} = \frac{\mu}{\lambda}, \phi_{20} = 0, A_\mu = 0$ and perturbing about this solution,

$$\begin{aligned}
\phi_1\left(x^\mu\right) &= \frac{\mu}{\lambda} + \delta\phi_1\left(x^\mu\right) \\
&\equiv \frac{\mu}{\lambda} + \eta\left(x^\mu\right) \\
A_\mu\left(x^\mu\right) &= 0 + \delta A_\mu\left(x^\mu\right) \\
&\equiv \beta\left(x^\mu\right) \\
A_\mu\left(x^\mu\right) &= 0 + \delta A_\mu\left(x^\mu\right) \\
&\equiv A_\mu\left(x^\mu\right)
\end{aligned}$$

we find

$$\begin{aligned}
\mathcal{L} = & \left[\frac{1}{2}\left(\partial_\mu\eta\right)\left(\partial^\mu\eta\right) + \mu^2\eta^2\right] + \left[\frac{1}{2}\left(\partial_\mu\beta\right)\left(\partial^\mu\beta\right)\right] \\
& + \left[\frac{1}{16}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\left(\frac{q}{\hbar c}\frac{\mu}{\lambda}\right)^2 A_\mu A^\mu\right] + \left(\frac{\mu}{\lambda}\frac{q}{\hbar c}\right)\left(\partial_\mu\beta\right)A^\mu - \left(\frac{\mu^2}{2\lambda}\right)^2 \\
& + \left\{\frac{q}{\hbar c}\left[\eta\left(\partial_\mu\beta\right) - \beta\left(\partial_\mu\eta\right)\right]A^\mu + \frac{\mu}{\lambda}\left(\frac{q}{\hbar c}\right)^2\eta\left(A_\mu A^\mu\right) + \frac{1}{2}\left(\frac{q}{\hbar c}\right)^2\left(\beta^2+\eta^2\right)A_\mu A^\mu\right\} \\
& + \left\{\lambda\mu\left(\eta^3 + \eta\beta^2\right) + \frac{1}{4}\lambda^2\left(\eta^4 + 2\eta^2\beta^2 + \beta^4\right)\right\}
\end{aligned}$$

Our new Lagrangian now describes a massive real scaler field η, a massive gauge field A_μ and a massless scalar β with a lot of complex interactions between them. ϕ in our original Lagrangian is the equivalent of the Higgs field and the procedure described so far is how it generates mass in theory [4], [5].

CHAPTER 3
BICONFORMAL GRAVITY

In order to have a basic understanding about how biconformal gauge theory works let us first consider the Poincarè group P and its Lorentz subgroup L . If we take the quotient P/L we can use this new quotient structure to immediately build a fiber bundle such that each point of the new manifold so formed is a coset isomorphic to L in P. This means that there is a one-to-one correspondence between points in this coset and elements of L which essentially makes it a Lorentz fiber bundle over a 4-dimensional manifold. We define $\boldsymbol{\omega}^b{}_c$ as the spin connection which defines the Lorentz part of the gauge or Lorentz fibers and $\boldsymbol{e^a}$ as the solder form which defines the set of orthonormal frame fields that span the cotangent spaces to the manifold. The pure-gauge spin connection may be gauged to zero so the solder forms become exact differentials of some coordinates, $\mathbf{e}^a = \delta^a_\alpha \mathbf{d}x^\alpha$ which makes the manifold a flat spacetime or Minkowski space.

$$\left\langle \mathbf{e}^a, \mathbf{e}^b \right\rangle = \eta^{ab}$$

The Lie algebra is

$$\begin{aligned}
[M^a_{\ b}, M^c_{\ d}] &= -\frac{1}{2}\left(\eta^b_{\ c} M^a_{\ d} - \eta^b_{\ d} M^a_{\ c} - \eta^a_{\ c} M^b_{\ d} - \eta^a_{\ d} M^b_{\ c}\right) \\
[M^a_{\ b}, P_c] &= \frac{1}{2}\left(\eta_{bc}\eta^{ad} - \delta^a_{\ c}\delta^d_{\ b}\right) P_d \\
[P_a, P_b] &= 0
\end{aligned}$$

It follows that the Maurer-Cartan equations are

$$\begin{aligned}
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^b \wedge \boldsymbol{\omega}^a{}_b \\
\mathbf{d}\boldsymbol{\omega}^a{}_b &= \boldsymbol{\omega}^c{}_b \wedge \boldsymbol{\omega}^a{}_c
\end{aligned}$$

The generalized connections become:

$$\begin{aligned}
\boldsymbol{\omega}^a_{\ b} &\longrightarrow \tilde{\boldsymbol{\omega}}^a_{\ b} \\
\mathbf{e}^a &\longrightarrow \tilde{\mathbf{e}}^a
\end{aligned}$$

We add the curvatures to generalize these equations in order to build a more general class of spacetimes and the Maurer-Cartan equations just become Cartan equations. We require the curvatures to be horizontal and the modified structure equations integrable. Horizontality means that Lorentz transformation leaves the closed loop integrals of the connections invariant.

$$\begin{aligned}
\mathbf{d}\tilde{\mathbf{e}}^a &= \tilde{\mathbf{e}}^b \wedge \tilde{\boldsymbol{\omega}}^a_{\ b} + \tilde{\mathbf{T}}^a \\
\mathbf{d}\tilde{\boldsymbol{\omega}}^a_{\ b} &= \tilde{\boldsymbol{\omega}}^c_{\ b} \wedge \tilde{\boldsymbol{\omega}}^a_{\ c} + \tilde{\mathbf{R}}^a_{\ b}
\end{aligned}$$

A general 2-form on this group manifold will have the form

$$\boldsymbol{\Omega} = \frac{1}{2}\Omega_{ab}\mathbf{e}^a \wedge \mathbf{e}^b + \Omega_{ab}{}^c\mathbf{e}^a \wedge \boldsymbol{\omega}^b{}_c + \frac{1}{2}\Omega^a{}_b{}^c{}_d\boldsymbol{\omega}^b{}_a \wedge \boldsymbol{\omega}^d{}_c$$

but because we do not want the curvature 2-forms $\mathbf{R}^a{}_b$ and $\mathbf{T}^a$ (the Riemann curvature and torsion respectively) to change when we change the Lorentz part of the gauge, we forbid the last two terms. In other words if we have our base manifold with Lorentz fibers and integrate the connection around any closed path, we do not want the result to depend on whether we raise or lower that path. Therefore if we do a Lorentz transformation locally $\boldsymbol{\Omega}$ will transform and that means we will get something different around two curves that differ only by a Lorentz transformation unless we ignore the last two terms. This condition is called horizontality. As a result, we express the curvature 2-forms in terms of only the horizontal basis which is the solder form in this case. By doing this we find that the resulting curvatures ($\mathbf{R}^{a}{}_{b}$ and $\mathbf{T}^{a}$ in this case) describe only the curvature of the base manifold, maintaining the underlying Lorentz symmetry. Therefore

$$\begin{aligned}
\mathbf{R}^{a}{}_{b} &= \frac{1}{2} R^{a}{}_{bcd} \mathbf{e}^{c} \wedge \mathbf{e}^{d} \\
\mathbf{T}^{a} &= \frac{1}{2} T^{a}{}_{bc} \mathbf{e}^{b} \wedge \mathbf{e}^{c}
\end{aligned}$$

and integrability implies:

$$\begin{aligned}
\mathbf{d}^2 \tilde{\boldsymbol{\omega}}^a_b &= 0 \\
\Longrightarrow 0 &= \mathbf{d} \left(\mathbf{d} \tilde{\boldsymbol{\omega}}^a_b \right) \\
&= \mathbf{d} \left(\tilde{\boldsymbol{\omega}}^c_b \wedge \tilde{\boldsymbol{\omega}}^a_c + \tilde{\mathbf{R}}^a_b \right) \\
&= \mathbf{d} \tilde{\boldsymbol{\omega}}^c_b \wedge \tilde{\boldsymbol{\omega}}^a_c - \tilde{\boldsymbol{\omega}}^c_b \wedge \mathbf{d} \tilde{\boldsymbol{\omega}}^a_c + \mathbf{d} \tilde{\mathbf{R}}^a_b \\
&= \mathbf{d} \tilde{\boldsymbol{\omega}}^c_b \wedge \tilde{\boldsymbol{\omega}}^a_c - \tilde{\boldsymbol{\omega}}^c_b \wedge \mathbf{d} \tilde{\boldsymbol{\omega}}^a_c + \mathbf{d} \tilde{\mathbf{R}}^a_b \\
&= \left(\tilde{\boldsymbol{\omega}}^e_b \wedge \tilde{\boldsymbol{\omega}}^c_e + \tilde{\mathbf{R}}^c_b \right) \wedge \tilde{\boldsymbol{\omega}}^a_c - \tilde{\boldsymbol{\omega}}^c_b \wedge \left(\tilde{\boldsymbol{\omega}}^e_c \wedge \tilde{\boldsymbol{\omega}}^a_e + \tilde{\mathbf{R}}^a_c \right) + \mathbf{d} \tilde{\mathbf{R}}^a_b \\
&= \left(\tilde{\boldsymbol{\omega}}^e_b \wedge \tilde{\boldsymbol{\omega}}^c_e \wedge \tilde{\boldsymbol{\omega}}^a_c + \tilde{\mathbf{R}}^c_b \wedge \tilde{\boldsymbol{\omega}}^a_c \right) - \left(\tilde{\boldsymbol{\omega}}^c_b \wedge \tilde{\boldsymbol{\omega}}^e_c \wedge \tilde{\boldsymbol{\omega}}^a_e + \tilde{\boldsymbol{\omega}}^c_b \wedge \tilde{\mathbf{R}}^a_c \right) + \mathbf{d} \tilde{\mathbf{R}}^a_b \\
&= \tilde{\boldsymbol{\omega}}^e_b \wedge \tilde{\boldsymbol{\omega}}^c_e \wedge \tilde{\boldsymbol{\omega}}^a_c + \tilde{\mathbf{R}}^c_b \wedge \tilde{\boldsymbol{\omega}}^a_c - \tilde{\boldsymbol{\omega}}^c_b \wedge \tilde{\boldsymbol{\omega}}^e_c \wedge \tilde{\boldsymbol{\omega}}^a_e - \tilde{\boldsymbol{\omega}}^c_b \wedge \tilde{\mathbf{R}}^a_c + \mathbf{d} \tilde{\mathbf{R}}^a_b \\
&= \tilde{\mathbf{R}}^c_b \wedge \tilde{\boldsymbol{\omega}}^a_c - \tilde{\boldsymbol{\omega}}^c_b \wedge \tilde{\mathbf{R}}^a_c + \mathbf{d} \tilde{\mathbf{R}}^a_b \\
&= \mathbf{D} \tilde{\mathbf{R}}^a_b \\
\Longrightarrow \mathbf{d}^2 \tilde{\boldsymbol{\omega}}^a_b &= 0 \\
&= \mathbf{D} \tilde{\mathbf{R}}^a_b
\end{aligned}$$

Similarly

$$
\begin{aligned}
\mathbf{d}^2\tilde{\mathbf{e}}^a &= 0 \\
\Longrightarrow 0 &= \mathbf{d}^2\tilde{e}^a \\
&= \mathbf{d}\left(\mathbf{d}\tilde{\mathbf{e}}^a\right) \\
&= \mathbf{d}\left(\tilde{\mathbf{e}}^b \wedge \tilde{\boldsymbol{\omega}}^a_{\ b} + \tilde{\mathbf{T}}^a\right) \\
&= \mathbf{d}\tilde{\mathbf{e}}^b \wedge \tilde{\boldsymbol{\omega}}^a_{\ b} - \tilde{\mathbf{e}}^b \wedge \mathbf{d}\tilde{\boldsymbol{\omega}}^a_{\ b} + \mathbf{d}\tilde{\mathbf{T}}^a \\
&= \left(\tilde{\mathbf{e}}^c \wedge \tilde{\boldsymbol{\omega}}^b_{\ c} + \tilde{\mathbf{T}}^b\right) \wedge \tilde{\boldsymbol{\omega}}^a_{\ b} - \tilde{\mathbf{e}}^b \wedge \left(\tilde{\boldsymbol{\omega}}^c_{\ b} \wedge \tilde{\boldsymbol{\omega}}^a_{\ c} + \tilde{\mathbf{R}}^a_{\ b}\right) + \mathbf{d}\tilde{\mathbf{T}}^a \\
&= \left(\tilde{\mathbf{e}}^c \wedge \tilde{\boldsymbol{\omega}}^b_{\ c} \wedge \tilde{\boldsymbol{\omega}}^a_{\ b} + \tilde{\mathbf{T}}^b \wedge \tilde{\boldsymbol{\omega}}^a_{\ b}\right) - \left(\tilde{\mathbf{e}}^b \wedge \tilde{\boldsymbol{\omega}}^c_{\ b} \wedge \tilde{\boldsymbol{\omega}}^a_{\ c} + \tilde{\mathbf{e}}^b \wedge \tilde{\mathbf{R}}^a_{\ b}\right) + \mathbf{d}\tilde{\mathbf{T}}^a \\
&= \tilde{\mathbf{e}}^c \wedge \tilde{\boldsymbol{\omega}}^b_{\ c} \wedge \tilde{\boldsymbol{\omega}}^a_{\ b} + \tilde{\mathbf{T}}^b \wedge \tilde{\boldsymbol{\omega}}^a_{\ b} - \tilde{\mathbf{e}}^b \wedge \tilde{\boldsymbol{\omega}}^c_{\ b} \wedge \tilde{\boldsymbol{\omega}}^a_{\ c} - \tilde{\mathbf{e}}^b \wedge \tilde{\mathbf{R}}^a_{\ b} + \mathbf{d}\tilde{\mathbf{T}}^a \\
&= \tilde{\mathbf{T}}^b \wedge \tilde{\boldsymbol{\omega}}^a_{\ b} - \tilde{\mathbf{e}}^b \wedge \tilde{\mathbf{R}}^a_{\ b} + \mathbf{d}\tilde{\mathbf{T}}^a \\
&= \mathbf{D}\tilde{\mathbf{T}}^a - \tilde{\mathbf{e}}^b \wedge \tilde{\mathbf{R}}^a_{\ b} \\
\Longrightarrow \mathbf{D}\tilde{\mathbf{T}}^a &= \tilde{\mathbf{e}}^b \wedge \tilde{\mathbf{R}}^a_{\ b}
\end{aligned}
$$

This is the first and second Bianchi identity equivalent to what we have in GR

$$
\begin{aligned}
R^a_{\ b[cd;e]} &= 0 \\
R^a_{\ [bcd]} &= T^a_{\ [bc;d]}
\end{aligned}
$$

If the torsion $T^a = 0$, then we get a description equivalent to the Riemannian geometry in GR. $\mathbf{R}^a_{\ b}$ is the Riemann curvature tensor $R^a_{\ bcd}$ as a 2-form.

Now we intend to build a theory from these Lorentz tensors T^a and $R^a_{\ b}$ by writing the most general action linear in these curvatures

$$
\mathcal{S} = \int \mathbf{R}^{ab} \wedge \mathbf{e}^c \wedge \mathbf{e}^d \varepsilon_{abcd}
$$

This is the integral of a 4-form. We first of all vary this action with respect to the spin connection $\delta_\omega \mathcal{S} = 0$ and this gives us field equations indicating vanishing torsion. Next we vary the action with respect to the solder form $\mathbf{e}$ and this gives us the following field equations

$$\begin{aligned} \left(\mathbf{R}^{ab} \wedge \mathbf{e}^{d}\right) \varepsilon_{abcd} &= 0 \\ \Longrightarrow \frac{1}{2} R^{ab}{}_{ef} \mathbf{e}^{e} \wedge \mathbf{e}^{f} \wedge \mathbf{e}^{d} \varepsilon_{abcd} &= 0 \end{aligned}$$

Reducing this further gives the Einstein Equation as expected in GR. The Poincarè fiber bundle now describes a general Einstein-Cartan (ECSK) geometry. In summary what we have done after taking the quotient of the Poincare group by the Lorentz group is creating a homogenous manifold which gives spacetime. We then generalized that into a curved space by changing the connection giving us the Cartan equations.

In this research we propose to consider a similar construction for the conformal group. Let the connection forms dual to the generators of the Lie algebra be written as spin connection $\boldsymbol{\omega}^{a}{}_{b}$ ($SO(p,q)$ transformations), solder form $\mathbf{e}^a$ (translations), co-solder form $\mathbf{f}_a$ (special conformal transformations, called co-translations in the context of these biconformal geometries), and Weyl vector $\boldsymbol{\omega}$ (dilatations). We take the quotient of the conformal group by its homogenous Weyl subgroup (which is just Lorentz transformations with dilatations). The resulting equations, with curvatures, are the Cartan equations which gives us the forms of the curvatures in terms of the connection. We then define our action (theory) as the Wheeler-Whener action. Generic solutions give GR in n-dimensions.

So far, we are confident in the biconformal gauge theory formulation. We found the structure equations from the commutator relations of our Lie group in a spinor representation, from which we wrote down the Maurer-Cartan equation. We then take the quotient of this group by a Lie subgroup to form our fiber bundle. We checked that the spinor representation reproduces the previous vector equations.

From the work done in [1] a space of dimension $n = p + q$, with an $SO(p,q)$-symmetric orthonormal metric η can be compactified with appropriate null cones at infinity to permit the inversions that give the space a well-defined conformal symmetry, $\mathcal{C} = SO(p+1, q+1)$. We take the quotient by $\mathcal{W}$, where $\mathcal{W} = SO(p,q) \times SO(1,1) \subset \mathcal{C}$ is the homogeneous Weyl subgroup which consists of the pseudo-rotations and dilatations. The quotient $\mathcal{C}/\mathcal{W}$ is a $2n$-dimensional homogeneous manifold from which we immediately have a principal fiber bundle with fiber symmetry $\mathcal{W}$. We take the local structure of this bundle as a model for a curved space à la Cartan, modifying the manifold and altering the connection subject to the two conditions:

1. The resulting curvature 2-forms must be horizontal.
2. The resulting Cartan structure equations satisfy their integrability conditions (generalized Bianchi identities).

In a vector representation, the Cartan structure equations are:

$$\mathbf{d}\boldsymbol{\omega}^a_b = \boldsymbol{\omega}^c_{\ b} \wedge \boldsymbol{\omega}^a_{\ c} + 2\Delta^{ad}_{cb}\mathbf{f}_d \wedge \mathbf{e}^c + \boldsymbol{\Omega}^a_{\ b} \tag{3.1}$$

$$\mathbf{d}\mathbf{e}^a = \mathbf{e}^b \wedge \boldsymbol{\omega}^a_{\ b} + \boldsymbol{\omega} \wedge \mathbf{e}^b + \mathbf{T}^a \tag{3.2}$$

$$\mathbf{d}\mathbf{f}_a = \boldsymbol{\omega}^b_{\ a} \wedge \mathbf{f}_b + \mathbf{f}_a \wedge \boldsymbol{\omega} + \mathbf{S}_a \tag{3.3}$$

$$\mathbf{d}\boldsymbol{\omega} = \mathbf{e}^a \wedge \mathbf{f}_a + \boldsymbol{\Omega} \tag{3.4}$$

Horizontality requires the curvature to be expanded in the $(\mathbf{e}^a, \mathbf{f}_b)$ basis, giving each of the components $(\boldsymbol{\Omega}^a_{\ b}, \mathbf{T}^a, \mathbf{S}_a, \boldsymbol{\Omega})$ the general form

$$\boldsymbol{\Omega}^A = \frac{1}{2}\Omega^A_{\ \ cd}\,\mathbf{e}^c \wedge \mathbf{e}^d + \Omega^{Ac}_{\ \ \ d}\,\mathbf{f}_c \wedge \mathbf{e}^d + \frac{1}{2}\Omega^{Acd}\,\mathbf{f}_c \wedge \mathbf{f}_d \tag{3.5}$$

and integrability follows from the Poincaré lemma, $\mathbf{d}^2 \equiv 0$.

The $\frac{(n-1)(n+2)}{2}$ curvature components $(\boldsymbol{\Omega}^a_{\ b}, \mathbf{T}^a, \mathbf{S}_a, \boldsymbol{\Omega})$ together comprise a single conformal curvature tensor. However, the local symmetries of the homogeneous Weyl symmetry of the biconformal bundle do not mix these four separate parts. Thereofore, we call the $SO(p,q)$ part of the full conformal curvature $\boldsymbol{\Omega}^a_{\ b}$ the *curvature*, the translational part of the curvature $\mathbf{T}^a$ the *torsion*, the special conformal part of the curvature the *co-torsion*, $\mathbf{S}_a$, and the dilatational portion $\mathbf{\Omega}$ the *dilatational curvature* or simply the *dilatation*.

Each of the curvatures each has three distinguishable parts, as seen in Eq.(3.5). We call the $\mathbf{e}^a \wedge \mathbf{e}^b$ term the *spacetime term*, the $\mathbf{f}_a \wedge \mathbf{e}^b$ term the *cross term,* and the $\mathbf{f}_a \wedge \mathbf{f}_b$ term the *momentum term*. While it may be somewhat abusive to call a signature (p,q) space "spacetime", for the gravitational applications we consider the name is ultimately appropriate. In the cases where the co-solder forms generate a nonabelian Lie group, the name "momentum" is not appropriate, and we will speak of the relevant group manifold.

To avoid introducing too many symbols, the symbols for the three parts of curvatures are distinguished purely by index position. Thus, $\Omega^a{}_b{}^c{}_d$ denotes the cross-term of the $SO(p,q)$ curvature and $\Omega^a{}_{bcd}$ the spacetime term of the $SO(p,q)$ curvature. *These are independent functions.* We therefore do not raise or lower indices unless, on some submanifold, there is no chance for ambiguity. Note also that the raised and lowered index positions indicate the conformal weights, $+1$ and -1 respectively, of all definite weight objects. Therefore, the torsion cross-term $T^{ab}{}_c$ has net conformal weight $+1$, the spacetime term of the co-torsion S_{abc} has conformal weight -3, and the full torsion 2-form $\mathbf{T}^a$ has conformal weight $+1$.

The generalized Bianchi identities are the integrability conditions for the Cartan equations which are found by applying the Poincaré lemma, $\mathbf{d}^2 \equiv 0$, to each structure equation, then using the structure equations again to eliminate all but curvature terms. Thus, for the $SO(p,q)$ curvature, we take the exterior derivative of the structure equations.(3.1),

$$\begin{aligned}
0 &\equiv \mathbf{d}^2\boldsymbol{\omega}^a{}_b \\
&= \mathbf{d}\boldsymbol{\omega}^c{}_b \wedge \boldsymbol{\omega}^a{}_c - \boldsymbol{\omega}^c{}_b \wedge \mathbf{d}\boldsymbol{\omega}^a{}_c + 2\Delta^{ac}_{db}\mathbf{d}\mathbf{f}_c \wedge \mathbf{e}^d - 2\Delta^{ac}_{db}\mathbf{f}_c \wedge \mathbf{d}\mathbf{e}^d + \mathbf{d}\mathbf{\Omega}^a{}_b \\
&= \mathbf{\Omega}^c{}_b \wedge \boldsymbol{\omega}^a{}_c - \boldsymbol{\omega}^c{}_b \wedge \mathbf{\Omega}^a{}_c + 2\Delta^{ac}_{db}\mathbf{S}_c \wedge \mathbf{e}^d - 2\Delta^{ac}_{db}\mathbf{f}_c \wedge \mathbf{T}^d + \mathbf{d}\mathbf{\Omega}^a{}_b \\
&= \mathbf{D}\mathbf{\Omega}^a{}_b + 2\Delta^{ac}_{db}\mathbf{S}_c \wedge \mathbf{e}^d - 2\Delta^{ac}_{db}\mathbf{f}_c \wedge \mathbf{T}^d
\end{aligned}$$

where we have identified the covariant exterior derivative, $\mathbf{D\Omega}^a{}_b = \mathbf{d\Omega}^a{}_b + \mathbf{\Omega}^c{}_b \wedge \boldsymbol{\omega}^a{}_c - \boldsymbol{\omega}^c{}_b \wedge \mathbf{\Omega}^a{}_c$. Proceeding through Eqs.(3.1) - (3.4), we find the full set of integrability conditions,

$$\begin{aligned}
\mathbf{D\Omega}^a{}_b + 2\Delta^{ad}_{cb}\left(\mathbf{S}_d \wedge \mathbf{e}^c - \mathbf{f}_d \wedge \mathbf{T}^c\right) &= 0 &(3.6)\\
\mathbf{DT}^a - \mathbf{e}^b \wedge \mathbf{\Omega}^a_b + \mathbf{\Omega} \wedge \mathbf{e}^a &= 0 &(3.7)\\
\mathbf{DS}_a + \mathbf{\Omega}^b{}_a \wedge \mathbf{f}_b - \mathbf{f}_a \wedge \mathbf{\Omega} &= 0 &(3.8)\\
\mathbf{D\Omega} + \mathbf{T}^a \wedge \mathbf{f}_a - \mathbf{e}^a \wedge \mathbf{S}_a &= 0 &(3.9)
\end{aligned}$$

where the covariant derivatives are given by

$$\begin{aligned}
\mathbf{D\Omega}^a{}_b &= \mathbf{d\Omega}^a{}_b + \mathbf{\Omega}^c{}_b \wedge \boldsymbol{\omega}^a_c - \boldsymbol{\omega}^c_b \wedge \mathbf{\Omega}^a{}_c\\
\mathbf{DT}^a &= \mathbf{dT}^a + \mathbf{T}^b \wedge \boldsymbol{\omega}^a_b - \boldsymbol{\omega} \wedge \mathbf{T}^a\\
\mathbf{DS}_a &= \mathbf{dS}_a - \boldsymbol{\omega}^b{}_a \wedge \mathbf{S}_b + \mathbf{S}_a \wedge \boldsymbol{\omega}\\
\mathbf{D\Omega} &= \mathbf{d\Omega} \qquad (3.10)
\end{aligned}$$

Since each Bianchi identity contains the covariant derivative of a curvature, it is typically difficult to use them to help find solutions to the field equations. They are simply the conditions on the curvatures that guarantee that a solution exists, and if we find a solution to the field equations, the Bianchi identities are necessarily satisfied. However, if one of the curvatures vanishes the relations become algebraic and can be extremely helpful.

Carrying out each of the connection variations on the Wheeler-Wehner action,

$$S = \int e_{ac\cdots d}{}^{be\cdots f}\left(\alpha\mathbf{\Omega}^a{}_b + \beta\delta^a_b\mathbf{\Omega} + \gamma\mathbf{e}^a \wedge \mathbf{f}_b\right) \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \qquad (3.11)$$

we arrive, in the vector representation, at the final field equations:

$$T^{ae}{}_{e} - T^{ea}{}_{e} - S_{e}{}^{ae} = 0 \tag{3.12}$$

$$T^{a}{}_{ca} + S_{c}{}^{a}{}_{a} - S_{a}{}^{a}{}_{c} = 0 \tag{3.13}$$

$$\alpha\Delta^{ar}_{sb}\left(T^{mb}{}_{a} - \delta^{m}_{a}T^{eb}{}_{e} - \delta^{m}_{a}S_{c}{}^{bc}\right) = 0 \tag{3.14}$$

$$\alpha\Delta^{ar}_{sb}\left(\delta^{b}_{c}T^{d}{}_{ad} + S_{c}{}^{b}{}_{a} - \delta^{b}_{c}S_{d}{}^{d}{}_{a}\right) = 0 \tag{3.15}$$

$$\alpha\left(\Omega^{a}{}_{e}{}^{e}{}_{b} - \Omega^{c}{}_{d}{}^{d}{}_{c}\delta^{a}{}_{b}\right) + \beta\left(\Omega^{a}{}_{b} - \Omega^{c}{}_{c}\delta^{a}{}_{b}\right) + \Lambda\delta^{a}{}_{b} = 0 \tag{3.16}$$

$$\alpha\Omega^{c}{}_{acb} + \beta\Omega_{ab} = 0 \tag{3.17}$$

$$\alpha\left(\Omega^{c}{}_{b}{}^{a}{}_{c} - \Omega^{c}{}_{e}{}^{e}{}_{c}\delta^{a}{}_{b}\right) + \beta\left(\Omega^{a}{}_{b} - \Omega^{c}{}_{c}\delta^{a}{}_{b}\right) + \Lambda\delta^{a}{}_{b} = 0 \tag{3.18}$$

$$\alpha\Omega^{a}{}_{c}{}^{cb} + \beta\Omega^{ab} = 0 \tag{3.19}$$

where the constant Λ is defined to be $\Lambda \equiv \left((n-1)\,\alpha - \beta + n^2\gamma\right)$.

When we gauge the conformal group of Euclidean space, the biconformal space still allows us the freedom to put a Lorentz connection on the spacetime and come up with GR. This can be done in different ways. Spencer and Wheeler [10] require orthogonality for the x and y spaces. They also require the restriction of the 2n-dimensional Killing metric to spacetime and the momentum space to be non-degenerate. These conditions are enough to force a Lorentzian metric onto spacetime even though we started with a Euclidean space. These conditions split the biconformal space in such way that we get emergence of time and therefore spacetime.

However, the Spencer-Wheeler approach forces the same Lorentzian metric on both spacetime and momentum space. We want the second 4-space to be a subgroup of $SO\,(4)$, hence of different signature. We can do this if we allow an angle between the momentum and configuration spaces. This allows different metrics on either lagrangian submanifoldof the biconformal space.

CHAPTER 4

EIGHT COMPONENT SPINORS

Spinors refer to the representations of spin groups on curves, spaces, or spacetimes. The whole structure of the standard model ranging from the electroweak interactions to the strong interactions are based on spinor representations which are used to explain all matter as we know it till date. The importance of spinors therefore, cannot be overemphasized. In *The Classical Groups. . .* [19], Herman Weyl writes :

> ". . . only with spinors do we strike that level in the theory of its representations on which Euclid himself, flourishing ruler and compass, so deftly moves in the realm of geometric figures. In some way Euclid's geometry must be deeply connected with the existence of the spin representation."

A $Spin\,(n)$ group is a double cover of the special orthogonal group $SO\,(n)$, and similarly the $Spin\,(p,q)$ group is a double cover of the Special orthogonal group $SO\,(p,q)$. The $Spin\,(1)$ group is the orthogonal group $O\,(1)$, the $Spin\,(2)$ group is the unitary group $U\,(1)$, the $Spin\,(3)$ group is the special unitary group $SU\,(2)$, and the $Spin\,(4)$ group is the product of two unitary groups $SU\,(2) \times SU\,(2)$. The general linear group $GL\,(1,R)$ is the double cover for the special orthogonal group $SO\,(1,1)$, the special linear group $SL\,(2,R)$ is the double cover for the special orthogonal group $SO\,(1,2)$, and the special linear group $SL\,(2,C)$ is the double cover for the special orthogonal group $SO\,(1,3)$.

4.1 Spin(4): 4-dimensional representation

The covering group, $Spin\,(4)$, of the $SO\,(4)$ symmetry of a 4-dimensional Euclidean space gives a representation of the conformal group. The Dirac matrices for $Spin\,(4)$ must satisfy the Clifford algebra

$$\left\{\gamma^a, \gamma^b\right\} = 2\delta^{ab}$$

where $a, b, \ldots = 1, 2, 3, 4$. We may choose the Dirac matrices to be the 4×4 matrices

$$\begin{aligned} \gamma^k &= \begin{pmatrix} & -i\sigma^k \\ i\sigma^k & \end{pmatrix} = \begin{pmatrix} & \tau^k \\ \bar{\tau}^k & \end{pmatrix} \\ \gamma^4 &= \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \end{aligned}$$

where $\sigma^k, k = 1, 2, 3$ are the Pauli matrices. These are chosen so that linear combinations $Q = q_a \gamma^a$ take the form

$$Q = \begin{pmatrix} & \bar{q} \\ q & \end{pmatrix}$$

where $q = q_4 + i\mathbf{q} \cdot \boldsymbol{\sigma}$ is a quaternion and $\bar{q}$ its conjugate.

In the usual way we build the additional matrices

$$\begin{aligned} \sigma^{ab} &= \left[\gamma^a, \gamma^b\right] \\ \gamma_5 &= \gamma^4\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

and $\gamma_5\gamma^a$. The 16 matrices $\Gamma^A = \left\{1, \gamma^a, \sigma^{ab}, \gamma_5\gamma^a, \gamma_5\right\}$ form a basis for all complex 4×4 matrices.

It is a coincidence that Γ^A, taken together, give a representation of the conformal Lie algebra,

$$\begin{aligned}
\left[\sigma^{ab}, \sigma^{cd}\right] &= 4\delta^{bc}\sigma^{ad} - 4\delta^{ac}\sigma^{bd} - 4\delta^{bd}\sigma^{ac} + 4\delta^{ad}\sigma^{bc} \\
\left[\sigma^{ab}, T^c_\pm\right] &= 4\left(\delta^{bc}T^a_\pm - \delta^{ac}T^b_\pm\right) \\
\left[T^a_+, T^b_-\right] &= \frac{1}{2}\left(\sigma^{ab} + 4\delta^{ab}D\right) \\
\left[D, T^a_\pm\right] &= \pm T^a_\pm \\
\left[D, \sigma^{ab}\right] &= 0 \\
\left[T^a_\pm, T^b_\pm\right] &= 0
\end{aligned}$$

where

$$\begin{aligned}
T^a_\pm &= \frac{1}{2}\left(1 \pm \gamma_5\right)\gamma^a \\
\sigma^{ab} &= \left[\gamma^a, \gamma^b\right] \\
D &= \frac{1}{2}\gamma_5
\end{aligned}$$

are identified as the generators of translations T^a_+, special conformal transformations T^a_-, $Spin\,(4)$ rotations σ^{ab}, and dilatations D.

Because $\gamma_5^2 = 1$, we have projection operators

$$P_\pm = \frac{1}{2}\left(1 \pm \gamma_5\right)$$

These project to self-dual and anti-self-dual subspaces. The matrices Γ^A generate conformal transformations on 4-component spinors,

$$\psi = \begin{pmatrix} \chi \\ \xi \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \mu \\ \nu \end{pmatrix} = \begin{pmatrix} a + bi \\ c + di \\ e + fi \\ g + hi \end{pmatrix}$$

4.1.1 Vectors from spinors

The standard way to form a real 4-vector is to write

$$u^a = \frac{1}{2}\psi^\dagger\gamma^a\psi$$

A direct check shows that

$$u^a u_a = (\bar{\chi}\chi)\left(\bar{\xi}\xi\right) \tag{4.1}$$

We can also form a pure-imaginary pseudo-vector ("pseudo" simply meaning these will have opposite properties from vector under spatial inversion). Multiplying by i gives a real vector,

$$v^a = \frac{i}{2}\bar{\psi}\gamma_5\gamma^a\psi$$

Also, if we combine these we form a complex 4-vector,

$$\begin{aligned} z^a &= \frac{1}{2}\left(\psi^\dagger\gamma^a\psi + \bar{\psi}\gamma_5\gamma^a\psi\right) \\ &= \psi^\dagger P_+\gamma^a\psi \end{aligned}$$

with complex conjugate

$$(z^a)^* = \psi^\dagger P_-\gamma^a\psi$$

We have two interesting limits:

$$\begin{aligned} u^a &= \left[\frac{1}{2}\psi^\dagger\gamma^a\psi\right]_{\xi_0} \\ v^a &= \left[\frac{1}{2}\psi^\dagger\gamma^a\psi\right]_{\chi_0} \end{aligned}$$

where

$$\begin{aligned}\xi_0 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \chi_0 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}\end{aligned}$$

Each of these is a general real 4-vector, but we cannot do both at once.

4.1.2 Vectors from projected spinors

Another way to form real 4-vectors is from projected spinors,

$$\begin{aligned}\psi_+ = \begin{pmatrix} \chi \\ 0 \end{pmatrix} &= \frac{1}{2}(1+\gamma_5)\,\psi \\ \psi_- = \begin{pmatrix} 0 \\ \xi \end{pmatrix} &= \frac{1}{2}(1-\gamma_5)\,\psi\end{aligned}$$

The only way these give us a vector from $\frac{1}{2}\psi^\dagger\gamma^a\psi$ is if we mix them because $\gamma^a P_+ = P_-\gamma^a$:

$$\begin{aligned}\frac{1}{2}\psi^\dagger\gamma^a P_+\psi &= \frac{1}{2}\psi^\dagger\gamma^a P_+ P_+\psi \\ &= \frac{1}{2}\psi^\dagger P_-\gamma^a P_+\psi \\ &= \frac{1}{2}\psi_-^\dagger\gamma^a\psi_+\end{aligned}$$

But

$$\begin{aligned}\frac{1}{2}\psi^\dagger P_+\gamma^a P_+\psi &= 0 \\ \frac{1}{2}\psi^\dagger P_-\gamma^a P_-\psi &= 0\end{aligned}$$

This means that the whole vector must vanish if either ξ or χ vanishes, in agreement with the norm, Eq.(4.1).

4.1.3 Vectors from outer products

Alternatively, we may take outer products of vectors or projected vectors:

$$\begin{aligned}
\psi \otimes \psi^{\dagger} &= (\chi, \xi) \otimes \begin{pmatrix} \chi^{\dagger} \\ \xi^{\dagger} \end{pmatrix} \\
&= \begin{pmatrix} \chi \otimes \chi^{\dagger} & \chi \otimes \xi^{\dagger} \\ \xi \otimes \chi^{\dagger} & \xi \otimes \xi^{\dagger} \end{pmatrix}
\end{aligned}$$

where the 2-component products are

$$\begin{aligned}
\chi^{\dagger} \otimes \chi &= (\alpha^*, \beta^*) \otimes \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
&= \begin{pmatrix} \alpha^* \alpha & \beta^* \alpha \\ \alpha^* \beta & \beta^* \beta \end{pmatrix} = u^a \sigma_a = u^4 1 + u^i \sigma_i
\end{aligned}$$

where α, β are complex numbers. The other outer products $\xi^{\dagger} \otimes \chi, \chi^{\dagger} \otimes \xi$, and $\xi^{\dagger} \otimes \xi$ are similar. The resulting matrices are Hermitian,

$$\begin{pmatrix} \alpha^* \alpha & \beta^* \alpha \\ \alpha^* \beta & \beta^* \beta \end{pmatrix}^{\dagger} = \begin{pmatrix} \alpha^* \alpha & \beta^* \alpha \\ \alpha^* \beta & \beta^* \beta \end{pmatrix}$$

and so is the full matrix $\psi \otimes \psi^{\dagger}$. Since we may write any Hermitian 2×2 as a linear combination of

$$\sigma_a = (1, \sigma_k)$$

we can set

$$\chi^{\dagger} \otimes \chi = u^a \sigma_a = \frac{1}{2} (\alpha^* \alpha + \beta^* \beta) 1 + \frac{1}{2} (\alpha^* \alpha - \beta^* \beta) \sigma_3 + \frac{1}{2} (\alpha^* \beta + \beta^* \alpha) \sigma_1 - \frac{i}{2} (\alpha^* \beta - \beta^* \alpha) \sigma_2$$

thereby defining a real 4-vector u^a.

These are not general 4-vectors because they are always "null" in the sense that

$$\begin{aligned}
\left(u^1\right)^2 + \left(u^2\right)^2 + \left(u^3\right)^2 &= \frac{1}{4}\left(\alpha^*\beta + \beta^*\alpha\right)^2 - \frac{1}{4}\left(\alpha^*\beta - \beta^*\alpha\right)^2 + \frac{1}{4}\left(\alpha^*\alpha - \beta^*\beta\right)^2 \\
&= \frac{1}{4}\left(\alpha^*\beta\alpha^*\beta + \alpha^*\beta\beta^*\alpha + \beta^*\alpha\alpha^*\beta + \beta^*\alpha\beta^*\alpha\right) \\
&\quad -\frac{1}{4}\left(\alpha^*\beta\alpha^*\beta - \alpha^*\beta\beta^*\alpha - \beta^*\alpha\alpha^*\beta + \beta^*\alpha\beta^*\alpha\right) \\
&\quad +\frac{1}{4}\left(\alpha^*\alpha\alpha^*\alpha - \alpha^*\alpha\beta^*\beta - \beta^*\beta\alpha^*\alpha + \beta^*\beta\beta^*\beta\right) \\
&= \frac{1}{4}\left(\alpha^*\alpha\alpha^*\alpha + 2\alpha\alpha^*\beta\beta^* + \beta^*\beta\beta^*\beta\right) \\
&= \frac{1}{4}\left(\alpha^*\alpha + \beta^*\beta\right)^2 \\
&= \left(u^4\right)^2
\end{aligned}$$

This form is therefore too restrictive for our purposes.

4.2 Spin(5,1)

While the conformal representation built from the $Spin\,(4)$ Dirac matrices given above is concise and easier to work with, the conformal group of Euclidean 4-space in a vector representation is $SO\,(5,1)$, with covering group $Spin\,(5,1)$. The 8-dimensional representation means calculations here get longer, so we will use the computational software for some of our calculations.

We now turn to an explicit representation for $Spin\,(5,1)$.

4.2.1 A convenient Clifford basis for the spin representation

A basis for the Clifford algebra $\mathfrak{spin}\,(5,1)$ is given (up to $GL\,(n,\mathbb{C})$ similarity transformation) by six matrices γ^A satisfying

$$\left\{\gamma^A, \gamma^B\right\} = 2\eta^{AB} \tag{4.2}$$

where

$$\eta^{AB} = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & -1 \end{pmatrix} \tag{4.3}$$

is the $SO(5,1)$ metric.

A rotation of the coordinates in the last two dimensions gives the form convenient for the $SO(5,1)$ vector representation of the conformal group.

$$\tilde{\eta}^{AB} = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 0 & -1 \\ & & & & -1 & 0 \end{pmatrix} \tag{4.4}$$

Matrices preserving $\tilde{\eta}^{AB}$ directly give the usual vector representation of $SO(4)$ rotations, translations, special conformal transformations, and dilatations, so may help to see how the usual rotation algebra relates to the conformal algebra. However, $\tilde{\eta}^{AB}$ will not give the canonical form of a Clifford algebra, for which the basis is (p,q)-diagonal.

Returning to Eq.(4.3), the γ^A matrices must be at least 8×8, so we will have 8-component spinors. We make a convenient choice of the gamma matrices,

$$
\begin{aligned}
\gamma^A &= \sigma^k \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}, \\
& \begin{pmatrix} & & & -i1 \\ & & i1 & \\ & -i1 & & \\ i1 & & & \end{pmatrix}, \begin{pmatrix} & & -i1 & \\ & & & -i1 \\ i1 & & & \\ & i1 & & \end{pmatrix}, \begin{pmatrix} & & -i1 & \\ & & & i1 \\ -i1 & & & \\ & i1 & & \end{pmatrix}
\end{aligned} \tag{4.5}
$$

where $k = 1, 2, 3$ and $A = 1, 2, 3, 4, 5, 6$. Here each component is a 2×2 matrix. These γ^A satisfy the Clifford anticommutation algebra, Eq.(4.2). Of course, no essential results should depend on any particular choice of the gamma matrices, but this basis gives the left- and right-handed projections a simple form.

The commutators of the γ^A,

$$\sigma^{AB} = \left[\gamma^A, \gamma^B\right]$$

generate the conformal group $Spin\,(5,1)$ with

$$g\left(w_{AB}\right) = \exp\left(\frac{1}{2} w_{AB} \sigma^{AB}\right)$$

where $w_{AB} = -w_{BA}$ depend on 15 real parameters.

In place of γ_5 of the $\mathfrak{spin}\,(4)$ algebra, we define $\gamma_V \equiv \gamma^1\gamma^2\gamma^3\gamma^4\gamma^5\gamma^6$, satisfying $\gamma_V^2 = 1$ and $\gamma_V^\dagger = \gamma_V$. We also have $\left\{\gamma^A, \gamma_V\right\} = 0$ as usual.

In the basis Eq.(4.5), γ_V takes the simple form

$$\gamma_V = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{4.6}$$

where 1 represents the 2×2 identity. This form makes subsequent calculations more transparent.

4.2.2 Real 6-vectors

Real 6-vectors may be written as

$$V^A = \Psi^\dagger \gamma^6 \gamma^A \Psi$$

These vectors are central to our current investigation. In particular, we show that the subset of conformal transformations on spinors Ψ that preserve any given vector V^A is the electroweak symmetry. Since the biconformal geometry provides a geometric vector, the Weyl vector, it is natural to find effects of $SU(2) \times U(1)$ playing a role as well.

Before demonstrating this invariance, we discuss various properties of these 6-vectors and 8-spinors.

Conformal inner product

To achieve the usual conformal vector product we may define

$$\begin{aligned} \tilde{\gamma}^5 &= \frac{1}{\sqrt{2}}\left(\gamma^6 + \gamma^5\right) \\ \tilde{\gamma}^6 &= \frac{1}{\sqrt{2}}\left(\gamma^6 - \gamma^5\right) \end{aligned}$$

and conversely,

$$\begin{aligned}
\gamma^5 &= \frac{1}{\sqrt{2}}\left(\tilde{\gamma}^5 - \tilde{\gamma}^6\right) \\
\gamma^6 &= \frac{1}{\sqrt{2}}\left(\tilde{\gamma}^5 + \tilde{\gamma}^6\right)
\end{aligned}$$

Then let

$$\tilde{V}^A = \Psi^\dagger \tilde{\gamma}^6 \tilde{\gamma}^A \Psi$$

so the components become

$$\begin{aligned}
\widetilde{V}^A &\equiv \frac{1}{\sqrt{2}}\Psi^\dagger\left(\tilde{\gamma}^5 + \tilde{\gamma}^6\right)\tilde{\gamma}^A\Psi \\
\widetilde{V}^a &= \Psi^\dagger\gamma^6\gamma^a\Psi = V^a \\
\widetilde{V}^5 &= \Psi^\dagger\left(\tilde{\gamma}^5 + \tilde{\gamma}^6\right)\tilde{\gamma}^5\Psi \\
&= \Psi^\dagger\gamma^6\frac{1}{\sqrt{2}}\left(\gamma^6 + \gamma^5\right)\Psi \\
&= \frac{1}{\sqrt{2}}\left(V^6 + V^5\right) \\
\widetilde{V}^6 &= \Psi^\dagger\gamma^6\tilde{\gamma}^6\Psi\Psi \\
&= \Psi^\dagger\gamma^6\frac{1}{\sqrt{2}}\left(\gamma^6 - \gamma^5\right)\Psi \\
&= \frac{1}{\sqrt{2}}\left(V^6 - V^5\right)
\end{aligned}$$

Summarizing,

$$\begin{aligned}
\widetilde{V}^a &= \Psi^\dagger\gamma^6\gamma^a\Psi = V^a \\
\widetilde{V}^5 &= \frac{1}{\sqrt{2}}\left(V^6 + V^5\right) \\
\widetilde{V}^6 &= \frac{1}{\sqrt{2}}\left(V^6 - V^5\right)
\end{aligned}$$

These are clearly still real. The inner product is

$$\begin{aligned}
\tilde{\eta}_{AB}\tilde{V}^A\tilde{V}^B &= \tilde{V}^a\tilde{V}_a - 2\tilde{V}^5\tilde{V}^6 \\
&= V^aV_a - 2\frac{1}{\sqrt{2}}\left(V^6 + V^5\right)\frac{1}{\sqrt{2}}\left(V^6 - V^5\right) \\
&= V^aV_a - \left(V^6V^6 - V^5V^5\right) \\
&= V^aV_a + V^5V^5 - V^6V^6 \\
&= \eta_{AB}V^AV^B
\end{aligned}$$

as required.

4-vectors from conformal 6-vectors

From a real conformal 6-vector we can map to the original compactified 4-space to get a vector at the origin and a vector at the point at infinity

$$\begin{aligned}
v^a &= \frac{\tilde{V}^a}{\tilde{V}^5} \\
w^a &= \frac{\tilde{V}^a}{2\tilde{V}^6}
\end{aligned}$$

These are proportional away from these the origin and infinity, and when not orthogonal satisfy

$$v^a w_a = \frac{\tilde{V}^a\tilde{V}_a}{2\tilde{V}^5\tilde{V}^6} = 1$$

Because Ψ has 8 complex components, we have 16 real degrees of freedom. We seek a way to use these degrees of freedom to form two independent 4-vectors.

4.2.3 Self-dual and anti-self-dual projections

We may use γ_V to form a pair of complementary projections in $Spin\,(5,1)$, analogous to $\frac{1}{2}\left(1 \pm \gamma_5\right)$ within $Spin\,(4)$. Compatibility with the conformal group is guaranteed because the projections

$$P_{\pm} \equiv \frac{1}{2}\left(1 \pm \gamma_V\right) \tag{4.7}$$

commute with the conformal generators,

$$\left[P_{\pm}, \sigma^{AB}\right] = 0$$

This also guarantees that the projections preserve the reality of 6-vectors,

$$\left(\Psi^{\dagger} P_{\pm}\right) \gamma^6 \gamma^Q \left(P_{\pm}\Psi\right) = \Psi^{\dagger}\gamma^6\gamma^Q P_{\pm}\Psi$$

Let an 8-component spinor Ψ be written as a pair of Dirac spinors, or a quartet of 2-component spinors,

$$\Psi = \begin{pmatrix} \psi \\ \phi \end{pmatrix} = \begin{pmatrix} \chi \\ \xi \\ \alpha \\ \beta \end{pmatrix}$$

Then $P_{\pm}$ project into the upper or lower pairs of spinors.

$$\Psi_+ = P_+\Psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix} = \begin{pmatrix} \chi \\ \xi \\ 0 \\ 0 \end{pmatrix}$$

$$\Psi_- = P_-\Psi = \begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \alpha \\ \beta \end{pmatrix}$$

This basis has been adapted to give these projections their simplest form.

4.2.4 Vector components

Now compute the components of 6-vectors. We need the products $\gamma^6\gamma^A$,

$$
\begin{aligned}
\gamma^6\gamma^k &= \begin{pmatrix} & & -i1 & \\ & & & i1 \\ -i1 & & & \\ & i1 & & \end{pmatrix}\begin{pmatrix} & & & \sigma^k \\ & & \sigma^k & \\ & \sigma^k & & \\ \sigma^k & & & \end{pmatrix} = \begin{pmatrix} 0 & -i\sigma^k & & \\ i\sigma^k & 0 & & \\ & & 0 & -i\sigma^k \\ & & i\sigma^k & 0 \end{pmatrix} \\
\gamma^6\gamma^4 &= \begin{pmatrix} & & -i1 & \\ & & & i1 \\ -i1 & & & \\ & i1 & & \end{pmatrix}\begin{pmatrix} & & & -i1 \\ & & i1 & \\ & -i1 & & \\ i1 & & & \end{pmatrix} = \begin{pmatrix} 0 & -1 & & \\ -1 & 0 & & \\ & & 0 & -1 \\ & & -1 & 0 \end{pmatrix} \\
\gamma^6\gamma^5 &= \begin{pmatrix} & & -i1 & \\ & & & i1 \\ -i1 & & & \\ & i1 & & \end{pmatrix}\begin{pmatrix} & & -i1 & \\ & & & -i1 \\ i1 & & & \\ & i1 & & \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \\
\gamma^6\gamma^6 &= \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}
\end{aligned} \tag{4.8}
$$

The components of real 6-vectors are

$$V^A = \Psi^\dagger\gamma^6\gamma^A\Psi$$

In terms of 2-component spinors, $\Psi = (\chi, \xi, \alpha, \beta)$, the components of V^A are

$$\begin{aligned}
\Psi^\dagger\gamma^6\gamma^k\Psi &= \left(\chi^\dagger, \xi^\dagger, \alpha^\dagger, \beta^\dagger\right) \begin{pmatrix} 0 & -i\sigma^k & & \\ i\sigma^k & 0 & & \\ & & 0 & -i\sigma^k \\ & & i\sigma^k & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \xi \\ \alpha \\ \beta \end{pmatrix} \\
&= \left(\chi^\dagger, \xi^\dagger, \alpha^\dagger, \beta^\dagger\right) \begin{pmatrix} -i\sigma^k\xi \\ i\sigma^k\chi \\ -i\sigma^k\beta \\ i\sigma^k\alpha \end{pmatrix} \\
&= i\left(\xi^\dagger\sigma^k\chi - \chi^\dagger\sigma^k\xi\right) + i\left(\beta^\dagger\sigma^k\alpha - \alpha^\dagger\sigma^k\beta\right) \\
\Psi^\dagger\gamma^6\gamma^4\Psi &= \left(\chi^\dagger, \xi^\dagger, \alpha^\dagger, \beta^\dagger\right) \begin{pmatrix} 0 & -1 & & \\ -1 & 0 & & \\ & & 0 & -1 \\ & & -1 & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \xi \\ \alpha \\ \beta \end{pmatrix} \\
&= -\chi^\dagger\xi - \xi^\dagger\chi - \alpha^\dagger\beta - \beta^\dagger\alpha \\
\Psi^\dagger\gamma^6\gamma^5\Psi &= \left(\chi^\dagger, \xi^\dagger, \alpha^\dagger, \beta^\dagger\right) \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \chi \\ \xi \\ \alpha \\ \beta \end{pmatrix} \\
&= \chi^\dagger\chi - \xi^\dagger\xi - \alpha^\dagger\alpha + \beta^\dagger\beta \\
\Psi^\dagger\gamma^6\gamma^6\Psi &= \left(\chi^\dagger, \xi^\dagger, \alpha^\dagger, \beta^\dagger\right) \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} \chi \\ \xi \\ \alpha \\ \beta \end{pmatrix} \\
&= -\chi^\dagger\chi - \xi^\dagger\xi - \alpha^\dagger\alpha - \beta^\dagger\beta
\end{aligned}$$

In sum,

$$
\begin{aligned}
V^k = \Psi^\dagger\gamma^6\gamma^k\Psi &= i\left(\xi^\dagger\sigma^k\chi - \chi^\dagger\sigma^k\xi\right) + i\left(\beta^\dagger\sigma^k\alpha - \alpha^\dagger\sigma^k\beta\right) \\
V^4 = \Psi^\dagger\gamma^6\gamma^4\Psi &= -\chi^\dagger\xi - \xi^\dagger\chi - \alpha^\dagger\beta - \beta^\dagger\alpha \\
V^5 = \Psi^\dagger\gamma^6\gamma^5\Psi &= \chi^\dagger\chi - \xi^\dagger\xi - \alpha^\dagger\alpha + \beta^\dagger\beta \\
V^6 = \Psi^\dagger\gamma^6\gamma^6\Psi &= -\chi^\dagger\chi - \xi^\dagger\xi - \alpha^\dagger\alpha - \beta^\dagger\beta
\end{aligned} \tag{4.9}
$$

In the next subsection we show unlike $Spin\,(4)$, that the projected spinors can form general, independent 6-vectors.

4.2.5 Components of self-dual vectors

We would like to know if we can form arbitrary 4-vectors from self-dual and anti-self-dual spinors,

$$
\begin{aligned}
V^A &= \Psi_+^\dagger\gamma^6\gamma^A\Psi_+ \\
U^A &= \Psi_-^\dagger\gamma^6\gamma^A\Psi_-
\end{aligned}
$$

Unlike $Spin\,(4)$, this is not automatically zero because of the presence of γ^6.

Now compute components. The self-dual projection eliminates α, β, so Eqs.(4.9) for V^A reduce to

$$
\begin{aligned}
V^k &= i\left(\xi^\dagger\sigma^k\chi - \chi^\dagger\sigma^k\xi\right) \\
V^4 &= -\chi^\dagger\xi - \xi^\dagger\chi \\
V^5 &= \chi^\dagger\chi - \xi^\dagger\xi \\
V^6 &= -\chi^\dagger\chi - \xi^\dagger\xi
\end{aligned} \tag{4.10}
$$

Similarly, in the anti-self-dual case, we have $\chi = \xi = 0$ so that U^A becomes

$$\begin{aligned} U^k &= i\left(\beta^\dagger\sigma^k\alpha - \alpha^\dagger\sigma^k\beta\right) \\ U^4 &= -\alpha^\dagger\beta - \beta^\dagger\alpha \\ U^5 &= -\alpha^\dagger\alpha + \beta^\dagger\beta \\ U^6 &= -\alpha^\dagger\alpha - \beta^\dagger\beta \end{aligned}$$

The self-dual and anti-self-dual vectors V^A, U^A are clearly completely independent vectors.

It is straightforward to show that each also gives rise to a *general* 4-vector. For a generic pair of 2-component spinors, $\chi = \begin{pmatrix} \rho \\ \sigma \end{pmatrix}$ and $\xi = \begin{pmatrix} \mu \\ \nu \end{pmatrix}$, Eq.(4.10) expands to

$$\begin{aligned} V^1 &= i\left(\mu^*\sigma - \sigma^*\mu + \nu^*\rho - \rho^*\nu\right) \\ V^2 &= \mu^*\sigma + \sigma^*\mu - \nu^*\rho - \rho^*\nu \\ V^3 &= i\left(\mu^*\rho - \rho^*\mu + \sigma^*\nu - \nu^*\sigma\right) \\ V^4 &= -\left(\mu^*\rho + \nu^*\sigma + \rho^*\mu + \sigma^*\nu\right) \\ V^5 &= \rho^*\rho + \sigma^*\sigma - \mu^*\mu - \nu^*\nu \\ V^6 &= -\rho^*\rho - \sigma^*\sigma - \mu^*\mu - \nu^*\nu \end{aligned}$$

A direct check of the norm shows that

$$V^aV^a = 4\left(\bar{\xi}\xi\right)\left(\bar{\chi}\chi\right)$$

so we need both ξ and χ nonvanishing. However, if we restrict these expressions by setting $\chi = \begin{pmatrix} \rho \\ \sigma \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$ then

$$\begin{aligned} V^1 &= a\,Im\,(\nu) \\ V^2 &= -a\,Re\,(\nu) \\ V^3 &= a\,Im\,(\mu) \\ V^4 &= -a\,Re\,(\mu) \\ V^5 &= -\left(\mu^*\mu + \nu^*\nu - a^2\right) \\ V^6 &= -\left(\mu^*\mu + \nu^*\nu + a^2\right) \end{aligned}$$

which, setting $\lambda = \frac{a}{a^2 - (\mu^*\mu + \nu^*\nu)}$, already clearly gives rise to a fully general 4-vector,

$$v^a = \lambda\left(Im\,(\nu)\,, -Re\,(\nu)\,, Im\,(\mu)\,, Re\,(\mu)\right)$$

Other choices for ρ, σ give considerable additional freedom. This additional freedom is central to our first principal result.

4.3 Projections

It is convenient to identify a complete set of independent projections for the spinor space. To independently specify each component of an 8-spinor we reqire three mutually compatible projections, each splitting the previous projection in half. In addition to the usual positive/negative energy and up/down spin projections for 4-spinors, we emply the self-dual/anti-self-dual projection provided by γ_V.

4.3.1 Self-dual and anti-self-dual projections

We have already identified projections using γ_V, Eqs.(4.7) and shown that the resulting self-dual and anti-self-dual spinors produce general, independent 6-vectors. The use of γ_V

divides 8-spinors into a pair of 4-spinors.

$$\begin{aligned}
\psi_+ &= \frac{1}{2}\left(1+\gamma_V\right)\psi = \begin{pmatrix} \chi \\ \xi \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \chi \\ \xi \end{pmatrix} \\
\psi_- &= \frac{1}{2}\left(1-\gamma_V\right)\psi = \begin{pmatrix} 0 \\ 0 \\ \alpha \\ \beta \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix}
\end{aligned}$$

and these may be projected in the usual way into particle/antiparticle and spin-up/spin-down components. The projections that accomplish this may be identified by first writing the three mutually commuting commutators $\sigma^{12}, \sigma^{34}, \sigma^{56}$. These are not all independent because their product is proportional to γ_V, but all are diagonal. Forming normalized combinations, these take the forms:

$$\begin{aligned}
\frac{1}{2i}\sigma^{12} &= \begin{pmatrix} \sigma^3 & & & \\ & \sigma^3 & & \\ & & \sigma^3 & \\ & & & \sigma^3 \end{pmatrix} \\
\frac{1}{2i}\sigma^{34} &= \begin{pmatrix} \sigma^3 & & & \\ & -\sigma^3 & & \\ & & \sigma^3 & \\ & & & -\sigma^3 \end{pmatrix} \\
\frac{1}{2}\sigma^{56} &= \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}
\end{aligned}$$

We take σ^{56} and σ^{34} for our discussion.

It is also useful to consider the product $\sigma^{12}\sigma^{34}$,

$$-\frac{1}{4}\sigma^{12}\sigma^{34} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

4.3.2 Projections analogous to quantum field theory

The projections in quantum field theory are given physical meaning only when applied to solutions of the Dirac equation. Here we are starting in Euclidean space–only later to develop spacetime signature spontaneously. Therefore, we need the Euclidean version of the Dirac equation–the 4-dimensional Helmholz equation–applied to the self-dual or anti-self-dual 4-spinors.

Reduced gamma matrices

We begin by projecting the modified gamma matrices, $\gamma^6\gamma^A$ of Eq.(4.8), into the self-dual and anti-self-dual subspaces,

$$\gamma^A_\pm \equiv iP_\pm\gamma^6\gamma^A$$

For the self-dual subspace in the current basis the explicit form is

$$\gamma^A_+ = \begin{pmatrix} 0 & \sigma^k & & \\ -\sigma^k & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i & & \\ -i & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \begin{pmatrix} i & & & \\ & -i & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \begin{pmatrix} -i & & & \\ & -i & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

where $\gamma^k, -i\gamma^5$ form a common choice of basis for $Spin\,(3,1)$. Here we choose the truncated anti-hermitian foursome,

$$\tilde{\gamma}_+^a \in \left\{\gamma_+^k, \gamma_+^4\right\} = \left\{\begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \begin{pmatrix} & -i \\ -i & \end{pmatrix}\right\} \tag{4.11}$$

satisfy a Euclidean Clifford subalgebra

$$\left\{\tilde{\gamma}_+^a, \tilde{\gamma}_+^b\right\} = -2\delta^{ab}1 \tag{4.12}$$

where $a, b = 1, 2, 3, 4$. The remaining matrices are proportional to $\tilde{\gamma}_5$ and the identity, respectively.

$$\begin{aligned} i\tilde{\gamma}_+^5 &= \tilde{\gamma}_+^1\tilde{\gamma}_+^2\tilde{\gamma}_+^3\tilde{\gamma}_+^4 = \tilde{\gamma}_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ \tilde{\gamma}_+^6 &= -i1 \end{aligned}$$

Similar expressions hold in the lower quadrant for γ_-^A.

$$\gamma_-^A \equiv iP_-\gamma^6\gamma^A$$

These anti-self-dual matrices take the explicit form

$$\gamma_-^A = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \sigma^k \\ & & -\sigma^k & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & -i \\ & & -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & -i & \\ & & & i \end{pmatrix}, \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & -i & \\ & & & -i \end{pmatrix}$$

Truncating as before gives the same Clifford algebra,

$$\tilde{\gamma}_-^a \in \left\{\gamma_-^k, \gamma_-^4\right\} = \left\{\begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \begin{pmatrix} & -i \\ -i & \end{pmatrix}\right\} \tag{4.13}$$

satisfying

$$\left\{\tilde{\gamma}_-^a, \tilde{\gamma}_-^b\right\} = -2\delta^{ab}1$$

where $a, b = 1, 2, 3, 4$. This time

$$\gamma_5 = \tilde{\gamma}_-^1 \tilde{\gamma}_-^2 \tilde{\gamma}_-^3 \tilde{\gamma}_-^4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -i\tilde{\gamma}_-^5$$

with $\tilde{\gamma}_+^6 = -i1$.

The Euclidean-Dirac equation

To be certain of signs, we repeat here the familiar derivation of the Dirac equation and solutions, but using the Euclidean inner product. The starting point is the 4-dimensional Helmholz equation,

$$\delta^{ab}\partial_a\partial_b\phi + m^2\phi = 0$$

This has plane-wave solutions $\phi \sim e^{\pm ik_a x^a}$. We proceed to look for a "square root".

Let $\left(a\tilde{\gamma}_+^a\partial_a + bm\right)\psi = 0$ and apply the conjugate, remembering that $\tilde{\gamma}_+^{b\dagger} = -\tilde{\gamma}_+^b$:

$$\begin{aligned} 0 &= \left(-\bar{a}\tilde{\gamma}_+^b\partial_b + \bar{b}m\right)\left(a\tilde{\gamma}_+^a\partial_a + bm\right)\psi \\ &= \left(-|a|^2\tilde{\gamma}_+^a\tilde{\gamma}_+^b\partial_b\partial_a - bm\bar{a}\tilde{\gamma}_+^b\partial_b + \bar{b}ma\tilde{\gamma}_+^a\partial_a + |b|^2 m^2\right)\psi \\ &= \left(|a|^2\delta^{ab}\partial_b\partial_a + |b|^2 m^2 + \left(\bar{b}a - b\bar{a}\right)m\tilde{\gamma}_+^a\partial_a\right)\psi \end{aligned}$$

where the Clifford algebra reduces the first term to $-|a|^2\delta^{ab}\partial_a\partial_b$. To reproduce the Helmholz equation we therefore need

$$\begin{aligned} |a|^2 &= |b|^2 \\ b\bar{a} - a\bar{b} &= 0 \end{aligned}$$

We satisfy the first by setting $a = a_0 e^{i\varphi}$ and $b = a_0 e^{i\theta}$. Then the second requires

$$e^{i(\theta-\varphi)} - e^{-i(\theta-\varphi)} = 2i \sin(\varphi - \theta) = 0$$

so that $\varphi = \theta + n\pi$. Thus,

$$\begin{aligned} a &= z \\ b &= \pm z \end{aligned}$$

for any complex number $z = a_0 e^{i\varphi}$. The normalization uses the real part of z, leaving the usual $U(1)$ invariance.

Dropping the overall factor, but maintaining the optional sign by setting $\lambda = \pm 1$, we require

$$\left(\tilde{\gamma}^a_+ \partial_a + \lambda m\right) \psi = 0 \tag{4.14}$$

To extract pseudo-physical information, we now solve the resulting Eucidean-Dirac equation by setting

$$\psi = w(k^a) e^{\pm i k_a x^a}$$

where w is a pair of 2-component spinors, $w = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. This yields the algebraic form

$$\left(\pm i \tilde{\gamma}^a_+ k_a + \lambda m\right) w = 0 \tag{4.15}$$

Expanding using Eq.(4.11)

$$\begin{aligned} \left[\pm i \left(\begin{pmatrix} 0 & \sigma^i k_i \\ -\sigma^i k_i & 0 \end{pmatrix} + \begin{pmatrix} & -ik_4 \\ -ik_4 & \end{pmatrix} \right) + \lambda m \right] \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= 0 \\ \begin{pmatrix} \mp i\lambda m & \sigma^i k_i - ik_4 \\ -\sigma^i k_i - ik_4 & \mp i\lambda m \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= 0 \end{aligned}$$

to give the algebraic Dirac equations. Writing this as a pair of equations

$$\mp i\lambda m\alpha + \left(\sigma^i k_i - ik_4\right)\beta = 0 \tag{4.16}$$

$$\left(-\sigma^i k_i - ik_4\right)\alpha \mp i\lambda m\beta = 0 \tag{4.17}$$

we proceed to solve.

An orthonormal basis

Solving the first equation for α,

$$\alpha = \pm\left(\frac{\sigma^i k_i - ik_4}{i\lambda m}\right)\beta$$

and substituting,

$$\begin{aligned}
\pm\frac{\left(-\sigma^j k_j - ik_4\right)\left(\sigma^i k_i - ik_4\right)}{i\lambda m}\beta \mp i\lambda m\beta &= 0\\
\pm\frac{-\sigma^j k_j \sigma^i k_i - k_4^2}{i\lambda m}\beta \mp i\lambda m\beta &= 0\\
\pm\left(-\delta^{ij}k_i k_j - k_4^2\right)\beta \pm \lambda^2 m^2\beta &= 0\\
\left(-\delta^{ij}k_i k_j - k_4^2 + m^2\right)\beta &= 0
\end{aligned}$$

and therefore

$$k^a k_a = m^2$$

as expected. The solution for the spinors is

$$v = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \pm\left(\frac{\sigma^i k_i - ik_4}{i\lambda m}\right)\beta \\ \beta \end{pmatrix} \tag{4.18}$$

If instead we solve Eq.(4.17 for β first,

$$\beta \;=\; \mp\left(\frac{\sigma^i k_i + ik_4}{i\lambda m}\right)\alpha$$

Eq.(4.16) becomes

$$\begin{aligned}
\mp i\lambda m\alpha + \left(\sigma^j k_j - ik_4\right)\left(\mp\left(\frac{\sigma^i k_i + ik_4}{i\lambda m}\right)\alpha\right) &= 0 \\
\left(i\lambda m + \frac{\left(\sigma^j k_j - ik_4\right)\left(\sigma^i k_i + ik_4\right)}{i\lambda m}\right)\alpha &= 0 \\
\left(-m^2 + \mathbf{k}^2 + k_4^2\right)\alpha &= 0
\end{aligned}$$

and the spinor form is

$$u = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \mp\left(\frac{\sigma^i k_i + ik_4}{i\lambda m}\right)\alpha \end{pmatrix} \tag{4.19}$$

From these various solutions we choose the orthonormal set. Let the first be Eq.(4.19) with $\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the spatial dependence $e^{-ik_a x^a}$ (bottom sign).

$$u_1 \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ \frac{k_3 + ik_4}{i\lambda m} \\ \frac{k_1 + ik_2}{i\lambda m} \end{pmatrix}$$

Then with $\alpha = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and the same spatial dependence we find we can choose

$$u_2 \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ \frac{k_1 - ik_2}{i\lambda m} \\ -\frac{k_3 - ik_4}{i\lambda m} \end{pmatrix}$$

Checking orthogonality,

$$
\begin{aligned}
\bar{u}_1 u_2 &= \frac{1}{2}\left(1, 0, \left(\frac{k_3 - ik_4}{-i\lambda m}\right), \left(\frac{k_1 - ik_2}{-i\lambda m}\right)\right)\begin{pmatrix} 0 \\ 1 \\ \frac{k_1 - ik_2}{i\lambda m} \\ -\frac{k_3 - ik_4}{i\lambda m} \end{pmatrix} \\
&= \frac{1}{2m^2}\left(-\left(k_3 - ik_4\right)\left(k_1 - ik_2\right) + \left(k_1 - ik_2\right)\left(k_3 - ik_4\right)\right) \\
&= 0
\end{aligned}
$$

For the $e^{ik_a x^a}$ modes (top sign) we have

$$
v = \begin{pmatrix} \frac{\sigma^i k_i - ik_4}{i\lambda m}\beta \\ \beta \end{pmatrix}
$$

Then for $\beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$
v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{k_3 - ik_4}{i\lambda m} \\ \frac{k_1 + ik_2}{i\lambda m} \\ 1 \\ 0 \end{pmatrix}
$$

and with $\beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$
v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{k_1 - ik_2}{i\lambda m} \\ \frac{-k_3 - ik_4}{i\lambda m} \\ 0 \\ 1 \end{pmatrix}
$$

These are easily seen to be orthonormal. then checking the remaining orthogonality

Checking orthogonality

$$
\begin{aligned}
\bar{u}_1 v_2 &= \frac{1}{2}\left(1, 0, \left(\frac{k_3 - ik_4}{-i\lambda m}\right), \left(\frac{k_1 - ik_2}{-i\lambda m}\right)\right)\begin{pmatrix} \frac{k_3-ik_4}{i\lambda m} \\ \frac{k_1+ik_2}{i\lambda m} \\ 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{2}\left(\frac{k_3 - ik_4}{i\lambda m} + \frac{k_3 - ik_4}{-i\lambda m}\right) \\
&= 0 \\
\bar{u}_1 v_1 &= \frac{1}{2}\left(1, 0, \left(\frac{k_3 - ik_4}{-i\lambda m}\right), \left(\frac{k_1 - ik_2}{-i\lambda m}\right)\right)\begin{pmatrix} \frac{k_1-ik_2}{i\lambda m} \\ \frac{-k_3-ik_4}{i\lambda m} \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{2}\left(\frac{k_1 - ik_2}{i\lambda m} + \frac{k_1 - ik_2}{-i\lambda m}\right) \\
&= 0
\end{aligned}
$$

and for u_2,

$$\begin{aligned}
\bar{u}_2 v_2 &= \frac{1}{2}\left(0, 1, \frac{k_1+ik_2}{-i\lambda m}, \frac{k_3+ik_4}{i\lambda m}\right)\begin{pmatrix} \frac{k_3-ik_4}{i\lambda m} \\ \frac{k_1+ik_2}{i\lambda m} \\ 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{2}\left(\frac{k_1+ik_2}{i\lambda m} + \frac{k_1+ik_2}{-i\lambda m}\right) \\
&= 0 \\
\bar{u}_2 v_1 &= \frac{1}{2}\left(0, 1, \frac{k_1+ik_2}{-i\lambda m}, \frac{k_3+ik_4}{i\lambda m}\right)\begin{pmatrix} \frac{k_1-ik_2}{i\lambda m} \\ \frac{-k_3-ik_4}{i\lambda m} \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{2}\left(\frac{-k_3-ik_4}{i\lambda m} + \frac{k_3+ik_4}{i\lambda m}\right) \\
&= 0
\end{aligned}$$

The choice of λ plays no role, so we set $\lambda = 1$. Collecting solutions,

$$
\begin{aligned}
u_1 e^{-ik_a x^a} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ \frac{k_3+ik_4}{im} \\ \frac{k_1+ik_2}{im} \end{pmatrix} e^{-ik_a x^a} \\
u_2 e^{-ik_a x^a} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \frac{k_1-ik_2}{im} \\ -\frac{k_3-ik_4}{im} \end{pmatrix} e^{-ik_a x^a} \\
v_2 e^{ik_a x^a} &= \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{k_3-ik_4}{im} \\ \frac{k_1+ik_2}{im} \\ 1 \\ 0 \end{pmatrix} e^{ik_a x^a} \\
v_1 e^{ik_a x^a} &= \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{k_1-ik_2}{im} \\ \frac{-k_3-ik_4}{im} \\ 0 \\ 1 \end{pmatrix} e^{ik_a x^a}
\end{aligned}
$$

Energy projections

Returning to the general algebraic form of the Dirac equation, Eq.(4.15), we recognize it as an eigenvalue equation

$$i\tilde{\gamma}_+^a k_a w = \mp m w$$

with eigenvectors $u_{1,2}$ for the upper sign and $v_{1,2}$ for the lower. This lets us construct projection operators.

Therefore, let

$$
\begin{aligned}
\Pi_{\pm} &= \frac{1}{2}\left(1 \mp \frac{i}{m}\tilde{\gamma}_{+}^{a} k_a\right) \\
\Pi_{-}\Pi_{-} &= \frac{1}{4}\left(1 + \frac{i}{m}\tilde{\gamma}_{+}^{a} k_a\right)\left(1 + \frac{i}{m}\tilde{\gamma}_{+}^{b} k_b\right) \\
&= \frac{1}{4}\left(1 + \frac{2i}{m}\tilde{\gamma}_{+}^{b} k_b - \frac{1}{2m^2}\left\{\tilde{\gamma}_{+}^{b}\tilde{\gamma}_{+}^{a}\right\} k_a k_b\right) \\
&= \frac{1}{4}\left(1 + \frac{2i}{m}\tilde{\gamma}_{+}^{b} k_b + \frac{1}{m^2}\delta^{ab} k_a k_b\right) \\
&= \frac{1}{2}\left(1 + \frac{i}{m}\tilde{\gamma}_{+}^{b} k_b\right) \\
&= \Pi_{-}
\end{aligned}
$$

and similarly for Π_{+}. For the mixed product,

$$
\begin{aligned}
\Pi_{-}\Pi_{+} &= \frac{1}{4}\left(1 + \frac{i}{m}\tilde{\gamma}_{+}^{a} k_a\right)\left(1 - \frac{i}{m}\tilde{\gamma}_{+}^{b} k_b\right) \\
&= \frac{1}{4}\left(1 + \frac{1}{2m^2}\left\{\tilde{\gamma}_{+}^{b}\tilde{\gamma}_{+}^{a}\right\} k_a k_b\right) \\
&= \frac{1}{4}\left(1 - \frac{1}{m^2}\delta^{ab} k_a k_b\right) \\
&= 0
\end{aligned}
$$

The form of the projections depends on

$$
\frac{i}{m}\tilde{\gamma}_{+}^{a} k_a = \frac{i}{m}\begin{pmatrix} 0 & k_i\sigma^i - ik_4 \\ -k_i\sigma^i - ik_4 & 0 \end{pmatrix}
$$

Then checking explicitly,

$$\begin{aligned}
\Pi_+ &= \frac{1}{2}\left(1 - \frac{i}{m}\tilde{\gamma}^a_+ k_a\right) \\
&= \frac{1}{2m}\begin{pmatrix} m & -ik_i\sigma^i - k_4 \\ ik_i\sigma^i - k_4 & m \end{pmatrix} \\
\Pi_- &= \frac{1}{2}\left(1 + \frac{i}{m}\tilde{\gamma}^a_+ k_a\right) \\
&= \frac{1}{2m}\begin{pmatrix} m & ik_i\sigma^i + k_4 \\ -ik_i\sigma^i + k_4 & m \end{pmatrix}
\end{aligned}$$

For the action of $\Pi_\pm$ on general states,

$$
\begin{aligned}
\Pi_+ u &= \frac{1}{\sqrt{2}}\frac{1}{2m}\begin{pmatrix} m & -ik_i\sigma^i - k_4 \\ ik_i\sigma^i - k_4 & m \end{pmatrix}\begin{pmatrix} \alpha \\ \frac{\sigma^i k_i + ik_4}{i\lambda m}\alpha \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\frac{1}{2m}\begin{pmatrix} m\alpha - \left(ik_i\sigma^i + k_4\right)\left(\frac{\sigma^i k_i + ik_4}{i\lambda m}\alpha\right) \\ \left(ik_i\sigma^i - k_4\right)\alpha + m\frac{\sigma^i k_i + ik_4}{i\lambda m}\alpha \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ \frac{k_3+ik_4}{im} \\ \frac{k_1+ik_2}{im} \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\frac{1}{2m}\begin{pmatrix} \frac{1}{m}\left(m^2 - k^2\right)\alpha \\ \left(ik_i\sigma^i - k_4 - i\sigma^i k_i + k_4\right)\alpha \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ \frac{k_3+ik_4}{im} \\ \frac{k_1+ik_2}{im} \end{pmatrix} \\
&= 0 \\
\Pi_+ v &= \frac{1}{\sqrt{2}}\frac{1}{2m}\begin{pmatrix} m & -ik_i\sigma^i - k_4 \\ ik_i\sigma^i - k_4 & m \end{pmatrix}\begin{pmatrix} \frac{\sigma^i k_i - ik_4}{i\lambda m}\beta \\ \beta \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\frac{1}{2m}\begin{pmatrix} m\frac{\sigma^i k_i - ik_4}{im}\beta - \left(ik_i\sigma^i + k_4\right)\beta \\ \left(\frac{\left(ik_i\sigma^i - k_4\right)\left(\sigma^i k_i - ik_4\right)}{im} + m\right)\beta \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\frac{1}{2m}\begin{pmatrix} \left(-2ik_i\sigma^i - 2k_4\right)\beta \\ \frac{1}{m}\left(k^2 + m^2\right)\beta \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{k_i\sigma^i - ik_4}{im}\beta \\ \beta \end{pmatrix} \\
&= v
\end{aligned}
$$

and for Π_-,

$$\begin{aligned}
\Pi_- u &= \frac{1}{\sqrt{2}}\frac{1}{2m}\begin{pmatrix} m & ik_i\sigma^i + k_4 \\ -ik_i\sigma^i + k_4 & m \end{pmatrix}\begin{pmatrix} \alpha \\ \frac{\sigma^i k_i + ik_4}{im}\alpha \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\frac{1}{2m}\begin{pmatrix} m\alpha + \frac{\left(ik_i\sigma^i + k_4\right)\left(\sigma^i k_i + ik_4\right)}{im}\alpha \\ \left(-ik_i\sigma^i + k_4\right)\alpha + m\frac{\sigma^i k_i + ik_4}{im}\alpha \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\frac{1}{2m}\begin{pmatrix} m\left(1 + \frac{k^2}{m^2}\right)\alpha \\ 2\left(-i\sigma^i k_i + k_4\right)\alpha \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha \\ \frac{1}{im}\left(\sigma^i k_i + ik_4\right)\alpha \end{pmatrix} \\
&= u \\
\Pi_- v &= \frac{1}{\sqrt{2}}\frac{1}{2m}\begin{pmatrix} m & ik_i\sigma^i + k_4 \\ -ik_i\sigma^i + k_4 & m \end{pmatrix}\begin{pmatrix} \frac{\sigma^i k_i - ik_4}{im}\beta \\ \beta \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\frac{1}{2m}\begin{pmatrix} m\frac{\sigma^i k_i - ik_4}{im}\beta + \left(ik_i\sigma^i + k_4\right)\beta \\ \left(-ik_i\sigma^i + k_4\right)\frac{\sigma^i k_i - ik_4}{im}\beta + m\beta \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\frac{1}{2m}\begin{pmatrix} \left(-i\sigma^i k_i - k_4 + ik_i\sigma^i + k_4\right)\beta \\ \left(\frac{\left(-ik_i^2 - k_4 k_i\sigma^i + k_4\sigma^i k_i - ik_4^2\right)}{im} + m\right)\beta \end{pmatrix} \\
&= 0
\end{aligned}$$

In short, $\Pi_\pm$ separate u_1, u_2 from v_1, v_2:

$$\begin{aligned}
\Pi_+ u_i &= 0 \\
\Pi_+ v_i &= v_i \\
\Pi_- u_i &= u_i \\
\Pi_- v_i &= 0
\end{aligned}$$

Since $\Pi_\pm$ are descendend from 8×8 matrices, on the full space they have the form

$$\Pi_\pm = \begin{pmatrix} \tilde{\Pi}_\pm & 0 \\ 0 & 0 \end{pmatrix}$$

and therefore commute with the duality projections,

$$[P_\pm, \Pi_\pm] = 0$$

and similarly for the anti-self-dual sector.

Spin projections

Finally, we define the projections on the

$$\begin{aligned}
\Sigma_\pm &= \frac{1}{2}\left(1 \pm s_a \tilde{\gamma}_5 \tilde{\gamma}_+^a\right) \\
\Sigma_+^2 &= \frac{1}{4}\left(1 + s_a \tilde{\gamma}_5 \tilde{\gamma}_+^a\right)\left(1 + s_b \tilde{\gamma}_5 \tilde{\gamma}_+^a\right) \\
&= \frac{1}{4}\left(1 + 2 s_a \tilde{\gamma}_5 \tilde{\gamma}_+^a - \frac{1}{2} s_a s_b \left\{\tilde{\gamma}_+^a, \tilde{\gamma}_+^b\right\}\right) \\
&= \frac{1}{4}\left(1 + 2 s_a \tilde{\gamma}_5 \tilde{\gamma}_+^a + \delta^{ab} s_a s_b\right) \\
&= \frac{1}{2}\left(\frac{1}{2}\left(1 + \delta^{ab} s_a s_b\right) + s_a \tilde{\gamma}_5 \tilde{\gamma}_+^a\right)
\end{aligned}$$

Therefore, these are projections provided

$$s^2 = 1$$

In addition, they commute with $\Pi_\pm$ if

$$
\begin{aligned}
0 &= [\Sigma_\pm, \Pi_\pm] \\
&= \left[\frac{1}{2}\left(1+\lambda s_a \tilde{\gamma}_5 \tilde{\gamma}_+^a\right), \frac{1}{2}\left(1 \mp \frac{i}{m} k_b \tilde{\gamma}_+^b\right)\right] \\
&= \frac{1}{4}\left(1+\lambda s_a \tilde{\gamma}_5 \tilde{\gamma}_+^a\right)\left(1 \mp \frac{i}{m} k_b \tilde{\gamma}_+^b\right) - \frac{1}{4}\left(1 \mp \frac{i}{m} k_b \tilde{\gamma}_+^b\right)\left(1+\lambda s_a \tilde{\gamma}_5 \tilde{\gamma}_+^a\right) \\
&= \frac{1}{4}\left(1 \mp \frac{i}{m} \tilde{\gamma}_+^a k_a + \lambda s_a \tilde{\gamma}_5 \tilde{\gamma}_+^a \mp \frac{i}{m} \lambda \tilde{\gamma}_5 \tilde{\gamma}_+^a \tilde{\gamma}_+^b s_a k_b\right) \\
&\quad -\frac{1}{4}\left(1+\lambda s_a \tilde{\gamma}_5 \tilde{\gamma}_+^a \mp \frac{i}{m} k_b \tilde{\gamma}_+^b \mp \lambda s_a \frac{i}{m} k_b \tilde{\gamma}_+^b \tilde{\gamma}_5 \tilde{\gamma}_+^a\right) \\
&= \mp \frac{1}{4} \frac{i\lambda}{m}\left(\tilde{\gamma}_5 \tilde{\gamma}_+^a \tilde{\gamma}_+^b s_a k_b - s_a k_b \tilde{\gamma}_+^b \tilde{\gamma}_5 \tilde{\gamma}_+^a\right) \\
&= \mp \frac{1}{4} \frac{i\lambda}{m} \tilde{\gamma}_5 s_a k_b \left(\left\{\tilde{\gamma}_+^a, \tilde{\gamma}_+^b\right\}\right) \\
&= \pm \frac{1}{2} \frac{i\lambda}{m} \tilde{\gamma}_5 \delta^{ab} s_a k_b
\end{aligned}
$$

and therefore we demand

$$
s^a k_a = 0
$$

In our basis the form of $\Sigma_\pm$ becomes

$$
\begin{aligned}
\Sigma_\pm &= \frac{1}{2}\left(1 \pm s_a \tilde{\gamma}_+^a \tilde{\gamma}_5\right) \\
&= \frac{1}{2}\left(1 \pm \begin{pmatrix} 0 & s_i\sigma^i - i s_4 \\ -s_i\sigma^i - i s_4 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&= \frac{1}{2}\begin{pmatrix} 1 & \pm\left(s_i\sigma^i - i s_4\right) \\ \pm\left(s_i\sigma^i + i s_4\right) & 1 \end{pmatrix}
\end{aligned}
$$

Let $s_a = (0,0,1,0)$ and $k_a = (0,0,0,m)$ and expand the matrices,

$$
\begin{aligned}
\Sigma_+ &= \frac{1}{2}\begin{pmatrix} 1 & \sigma^3 \\ \sigma^3 & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & & 1 & \\ & 1 & & -1 \\ 1 & & 1 & \\ & -1 & & 1 \end{pmatrix} \\
\Sigma_- &= \frac{1}{2}\begin{pmatrix} 1 & -\sigma^3 \\ -\sigma^3 & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & & -1 & \\ & 1 & & 1 \\ -1 & & 1 & \\ & 1 & & 1 \end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
u_1 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
u_2 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \\
v_2 &= \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
v_1 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}
\end{aligned}
$$

so the action of Σ_+ is

$$\begin{aligned}
\Sigma_+ u_1 &= \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & & 1 & \\ & 1 & & -1 \\ 1 & & 1 & \\ & -1 & & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
&= u_1 \\
\Sigma_+ u_2 &= \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & & 1 & \\ & 1 & & -1 \\ 1 & & 1 & \\ & -1 & & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \\
&= 0 \\
\Sigma_+ v_2 &= \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & & 1 & \\ & 1 & & -1 \\ 1 & & 1 & \\ & -1 & & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
&= 0 \\
\Sigma_+ v_1 &= \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & & 1 & \\ & 1 & & -1 \\ 1 & & 1 & \\ & -1 & & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \\
&= v_1
\end{aligned}$$

while for Σ_-,

$$
\begin{aligned}
\Sigma_- u_1 &= \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & & -1 & \\ & 1 & & 1 \\ -1 & & 1 & \\ & 1 & & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
&= 0 \\
\Sigma_- u_2 &= \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & & -1 & \\ & 1 & & 1 \\ -1 & & 1 & \\ & 1 & & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \\
&= u_2 \\
\Sigma_- v_2 &= \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & & -1 & \\ & 1 & & 1 \\ -1 & & 1 & \\ & 1 & & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
&= v_2 \\
\Sigma_- v_1 &= \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & & -1 & \\ & 1 & & 1 \\ -1 & & 1 & \\ & 1 & & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \\
&= 0
\end{aligned}
$$

Summarizing, $\Sigma_\pm$ separate u_1, v_1 from u_2, v_2. These are spin-up/spin-down projections.

The three projections in combination can project the components of any 8-spinor into any single component along three independent axes:

- Self-dual/anti-self-dual
- Energy eigenvalues, $\pm m$
- Spin-up/spin-down

Here we are principally interested in the self-dual/anti-self-dual projections, which allow us to partition any eight component spinor into two independent 4-spinors with thier corresponding 4-dimensional Clifford algebras.

4.4 Self-duality in 4-dimensions

4.4.1 Self-dual and anti-self-dual projections

The projections $P_{V_\pm} = \frac{1}{2}\left(1 \pm \gamma_V\right)$ produce two 4-dimensional subspaces, each described by its own representation and Clifford algebra of $SO(4)$ symmetry. Because $SO(4) \cong SU(2) \times SU(2)$, there is a further self-duality projection. Dropping the $\tilde{\gamma}^a_\pm$ notation in favor of simply γ^a and with $\gamma_5 = \gamma^1\gamma^2\gamma^3\gamma^4$, we define projections

$$\mathcal{P}_\pm \equiv \frac{1}{2}\left(1 \pm \gamma_5\right)$$

In the basis of Eqs.(4.11) and (4.12)

$$\gamma^a = \left\{ \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \begin{pmatrix} & -i \\ -i & \end{pmatrix} \right\}$$

satisfying $\left\{\tilde{\gamma}^a_+, \tilde{\gamma}^b_+\right\} = -2\delta^{ab}1$ and

$$\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The projections therefore take the form

$$\mathcal{P}_+ = \frac{1}{2}\left(1 + \gamma_5\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{P}_- = \frac{1}{2}\left(1 - \gamma_5\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Unlike $P_{\pm}$ the 2-component spinors resulting from these projection cannot build fully general vectors. Indeed, defining $\psi_{+} = \mathcal{P}_{+}\psi$, we would have

$$\begin{aligned} v^{a} &= \psi_{+}^{\dagger}\gamma^{a}\psi_{+} \\ &= \psi^{\dagger}\mathcal{P}_{+}\gamma^{a}\mathcal{P}_{+}\psi \\ &= 0 \end{aligned}$$

The effect of $\mathcal{P}_{\pm}$ is to separate spinors transforming under $SU(2)_{L}$ from those transforming under $SU(2)_{R}$. To see this, consider the $SO(4)$ rotation generators,

$$\sigma^{ab} = \frac{1}{2}\left[\gamma^{a}, \gamma^{b}\right]$$

Under the action of $\mathcal{P}_{\pm}$ we define

$$\begin{aligned} \sigma_{\pm}^{ab} &= \mathcal{P}_{\pm}\sigma^{ab}\mathcal{P}_{\pm}^{-1} \\ &= \mathcal{P}_{\pm}\sigma^{ab} \end{aligned} \tag{4.20}$$

where the generators take the form

$$
\begin{aligned}
\sigma^{ij} &= \frac{1}{2}\left[\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\right] \\
&= \frac{1}{2}\begin{pmatrix} -\left[\sigma^i, \sigma^j\right] & \\ & -\left[\sigma^i, \sigma^j\right] \end{pmatrix} \\
&= -i\varepsilon^{ij}{}_k\sigma^k \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \\
\sigma^{4i} &= \frac{1}{2}\left[\begin{pmatrix} & -i \\ -i & \end{pmatrix}, \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\right] \\
&= \frac{1}{2}\begin{pmatrix} & -i \\ -i & \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} & -i \\ -i & \end{pmatrix} \\
&= \begin{pmatrix} i\sigma^i & \\ & -i\sigma^i \end{pmatrix}
\end{aligned}
$$

so that the group generators are

$$
\Delta = \frac{1}{2}w_{ab}\sigma^{ab} \quad = \quad w_{4i}\sigma^{4i} + \frac{1}{2}w_{ij}\sigma^{ij}
$$

Using Eq.(4.20)

$$
\begin{aligned}
\sigma_+^{ij} &= -i\varepsilon^{ij}{}_k\sigma^k \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} \\
\sigma_+^{4i} &= i\sigma^k \begin{pmatrix} 0 & \\ & -1 \end{pmatrix} \\
\sigma_-^{ij} &= -i\varepsilon^{ij}{}_k\sigma^k \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \\
\sigma_-^{4i} &= i\sigma^k \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}
\end{aligned}
$$

we may write

$$
\begin{aligned}
\Delta_+ &= \frac{1}{2}w_{ab}\sigma_+^{ab} \\
&= w_{4i}\sigma^{4i} + \frac{1}{2}w_{ij}\sigma^{ij} \\
&= \left(w_{4k} - \frac{1}{2}w_{ij}\varepsilon^{ij}{}_k\right)i\sigma^k \\
&= u_k i\sigma^k \\
\Delta_- &= \frac{1}{2}w_{ab}\sigma_-^{ab} \\
&= -iw_{4i}\sigma^i - \frac{i}{2}w_{ij}\varepsilon^{ij}{}_k\sigma^k \\
&= \left(-w_{4k} - \frac{1}{2}w_{ij}\varepsilon^{ij}{}_k\right)i\sigma^k \\
&= v_k i\sigma^k
\end{aligned}
$$

Exponentiating $P_+\Delta_+ + P_-\Delta_-$

$$
g = \begin{pmatrix} e^{iv_k\sigma^k} & \\ & e^{iu_k\sigma^k} \end{pmatrix} \in SU(2) \times SU(2)
$$

and clearly $\mathcal{P}_\pm$ project into the two independent copies of $SU(2)$.

4.4.2 The 't Hooft matrices

It proves convenient to distinguish labels $A, B, \ldots = 1, 2, 3$ in $\mathfrak{su}(2)$ from 4-dimensional vector labels $a, b, \ldots = 1, 2, 3, 4$ under $SO(4)$, and we follow this convention in all subsequent expressions. Also, let $\mathcal{A}_2$ be the space of antisymmetric, rank 2 tensors, $\frac{1}{2}(T_{ab} - T_{ba}) \in \mathcal{A}_2$ and define the identity and dual mappings, $I : \mathcal{A}_2 \to \mathcal{A}_2$ and $E : \mathcal{A}_2 \to \mathcal{A}_2$

$$\begin{aligned} I^{cd}{}_{ab} &= \frac{1}{2}\left(\delta^c_a \delta^d_b - \delta^d_a \delta^c_b\right) \\ E_{ab}{}^{cd} &= \frac{1}{2}\varepsilon_{ab}{}^{cd} \end{aligned}$$

Clearly, $I^{cd}{}_{ab} T^{ab} = T^{ab}$ for all $T^{ab} \in \mathcal{A}_2$.

We may now write the 't Hooft matrices in a more symmetric form by defining

$$\varepsilon_{abc} \equiv \varepsilon_{ABC} \delta^A_a \delta^B_b \delta^C_c = \varepsilon_{abc4}$$

That is, ε_{abc} vanishes if any of a, b, c equals 4, and gives ε_{ABC} otherwise, where the antisymmetric components ε_{ABC} are the structure constants of $SU(2)$. Then

$$\begin{aligned} \Delta_+ &= \frac{1}{2} u_{ab} \left(\delta^a_A \delta^b_4 - \delta^a_4 \delta^b_A + \varepsilon^{ab}{}_A\right) i\sigma^A \\ &= u_{ab}\left(I^{ab}{}_{A4} + E^{ab}{}_{A4}\right) i\sigma^A \\ \Delta_- &= u_{ab}\left(I^{ab}{}_{A4} - E^{ab}{}_{A4}\right) i\sigma^A \end{aligned}$$

where the coefficients,

$$\begin{aligned} \eta^A_{+ab} &= \frac{1}{2}\left(\delta^A_a \delta^4_b - \delta^4_a \delta^A_b + \varepsilon_{ab}{}^A\right) = I^{A4}{}_{ab} + E^A{}_{ab4} \\ \eta^A_{-ab} &= \frac{1}{2}\left(\delta^A_a \delta^4_b - \delta^4_a \delta^A_b - \varepsilon_{ab}{}^A\right) = I^{A4}{}_{ab} - E^A{}_{ab4} \end{aligned}$$

are the 't Hooft matrices [20]. These provide a mapping from any antisymmetric, rank 2 tensor to $\mathfrak{su}(2)_+$ or $\mathfrak{su}(2)_-$.

Combining pairs of 't Hooft matrices gives us the usual self-dual and anti-self-dual projections for antisymmetric rank 2 tensors. We can raise and lower indices freely, since both metrics δ_{AB} and δ_{ab} are Euclidean. Computing

$$\begin{aligned}
\eta^A_{+ab}\eta^{Acd}_{+} &= \frac{1}{2}\left(\delta^A_a\delta^4_b - \delta^4_a\delta^A_b + \varepsilon_{ab}{}^{A}\right)\frac{1}{2}\left(\delta^{Ac}\delta^{4d} - \delta^{4c}\delta^{Ad} + \varepsilon^{cdA}\right) \\
&= \frac{1}{4}\left(\delta^A_a\delta^4_b\delta^{Ac}\delta^{4d} - \delta^A_a\delta^4_b\delta^{4c}\delta^{Ad} - \delta^4_a\delta^A_b\delta^{Ac}\delta^{4d} + \delta^4_a\delta^A_b\delta^{4c}\delta^{Ad} + \varepsilon_{ab}{}^{A}\varepsilon^{cdA}\right) \\
&\quad + \frac{1}{4}\left(\delta^A_a\delta^4_b\varepsilon^{cdA} - \delta^4_a\delta^A_b\varepsilon^{cdA} + \varepsilon_{ab}{}^{A}\delta^{Ac}\delta^{4d} - \varepsilon_{ab}{}^{A}\delta^{4c}\delta^{Ad}\right) \\
&= \frac{1}{4}\left(\left(\delta^c_a - \delta^4_a\delta^{4c}\right)\delta^4_b\delta^{4d} - \left(\delta^d_a - \delta^4_a\delta^{4d}\right)\delta^4_b\delta^{4c} - \delta^4_a\left(\delta^c_b - \delta^4_b\delta^{4c}\right)\delta^{4d} + \delta^4_a\left(\delta^d_b - \delta^4_b\delta^{4d}\right)\delta^{4c}\right) \\
&\quad + \frac{1}{4}\left(\left(\delta^c_a - \delta^c_4\delta^4_a\right)\left(\delta^d_b - \delta^d_4\delta^4_b\right) - \left(\delta^d_a - \delta^d_4\delta^4_a\right)\left(\delta^c_b - \delta^c_4\delta^4_b\right)\right) \\
&\quad + \frac{1}{4}\varepsilon_{ab}{}^{cd} \\
&= \frac{1}{4}\left(\delta^c_a\delta^4_b\delta^{4d} - \delta^4_a\delta^{4c}\delta^4_b\delta^{4d} - \delta^d_a\delta^4_b\delta^{4c} + \delta^4_a\delta^{4d}\delta^4_b\delta^{4c}\right) \\
&\quad + \frac{1}{4}\left(-\delta^c_b\delta^4_a\delta^{4d} + \delta^4_b\delta^{4c}\delta^4_a\delta^{4d} + \delta^d_b\delta^4_a\delta^{4c} - \delta^4_b\delta^{4d}\delta^4_a\delta^{4c}\right) \\
&\quad + \frac{1}{4}\left(\delta^c_a\delta^d_b - \delta^c_a\delta^d_4\delta^4_b - \delta^c_4\delta^4_a\delta^d_b + \delta^c_4\delta^4_a\delta^d_4\delta^4_b - \delta^d_a\delta^c_b + \delta^d_a\delta^c_4\delta^4_b + \delta^d_4\delta^4_a\delta^c_b - \delta^d_4\delta^4_a\delta^c_4\delta^4_b\right) \\
&\quad + \frac{1}{4}\varepsilon_{ab}{}^{cd} \\
&= \frac{1}{4}\left(\delta^c_a\delta^d_b - \delta^d_a\delta^c_b + \varepsilon_{ab}{}^{cd}\right) \\
&\quad + \frac{1}{4}\left(\delta^c_a\delta^4_b\delta^{4d} - \delta^c_a\delta^d_4\delta^4_b + \delta^d_b\delta^4_a\delta^{4c} - \delta^d_b\delta^c_4\delta^4_a + \delta^d_a\delta^c_4\delta^4_b - \delta^d_a\delta^4_b\delta^{4c} + \delta^c_b\delta^d_4\delta^4_a - \delta^c_b\delta^4_a\delta^{4d}\right) \\
&= \frac{1}{4}\left(\delta^c_a\delta^d_b - \delta^d_a\delta^c_b + \varepsilon_{ab}{}^{cd}\right)
\end{aligned}$$

so that

$$\eta^A_{+ab}\eta^{Acd}_{+} = \frac{1}{2}\left(I^{cd}{}_{ab} + E^{cd}{}_{ab}\right) = P^{cd}_{+}{}_{ab}$$

and similarly

$$\begin{aligned}
\eta^A_{-ab}\eta^{Acd}_{-} &= \frac{1}{4}\left(\varepsilon_{ab}{}^{A} - \left(\delta^A_a\delta^4_b - \delta^4_a\delta^A_b\right)\right)\left(\varepsilon^{cd}{}_{A} - \left(\delta^c_A\delta^d_4 - \delta^d_A\delta^c_4\right)\right) \\
&= \frac{1}{4}\left(\left(\delta^c_a\delta^d_b - \delta^d_a\delta^c_b\right) - \varepsilon_{ab}{}^{cd}\right) \\
&= \frac{1}{2}\left(I^{cd}{}_{ab} - E^{cd}{}_{ab}\right) \\
&= P^{cd}_{-}{}_{ab}
\end{aligned}$$

The projections $P_{\pm\ cd}^{ab}$ map the space $\mathcal{A}_2$ into self-dual and anti-self-dual parts.

$$P_{\pm} : \mathcal{A}_2 \to \mathcal{A}_2^{\pm}$$

The Hodge dual allows us to do the same on elements of $\mathcal{A}_2$ expressed as 2-forms, by taking them to their dual.

$$\begin{aligned}
\boldsymbol{\omega} &= \frac{1}{2}\omega_{ab}\mathbf{e}^a \wedge \mathbf{e}^b \\
{}^*\boldsymbol{\omega} &= \frac{1}{4}\omega_{ab}\varepsilon^{ab}{}_{cd}\mathbf{e}^c \wedge \mathbf{e}^d \\
&= \frac{1}{2}\left(E^{ab}{}_{cd}\omega_{ab}\right)\mathbf{e}^c \wedge \mathbf{e}^d
\end{aligned}$$

The self-dual and anti-self-dual 2-forms $\boldsymbol{\omega}_{\pm}$ may be written as

$$\begin{aligned}
\boldsymbol{\omega}_{\pm} \equiv \frac{1}{2}\left(\boldsymbol{\omega} \pm {}^*\boldsymbol{\omega}\right) &= \frac{1}{4}\left(I^{ab}{}_{cd} \pm E^{ab}{}_{cd}\right)\omega_{ab}\mathbf{e}^c \wedge \mathbf{e}^d \\
&= \frac{1}{2}P^{ab}{}_{cd}\omega_{ab}\mathbf{e}^c \wedge \mathbf{e}^d \\
&= \frac{1}{2}\omega^{\pm}_{cd}\mathbf{e}^c \wedge \mathbf{e}^d
\end{aligned}$$

Therefore the Hodge dual gives the same components as projection by $P_{\pm}{}^{ab}{}_{cd}$.

4.4.3 Identities with 't Hooft matrices

The 't Hooft matrices are

$$\begin{aligned}
\eta^A_{+ab} &= \frac{1}{2}\left(\delta^A_a\delta^4_b - \delta^4_a\delta^A_b + \varepsilon_{ab}{}^{A}\right) = I^{A4}{}_{ab} + E^{A}{}_{ab4} \\
\eta^A_{-ab} &= \frac{1}{2}\left(\delta^A_a\delta^4_b - \delta^4_a\delta^A_b - \varepsilon_{ab}{}^{A}\right) = I^{A4}{}_{ab} - E^{A}{}_{ab4}
\end{aligned}$$

where half the sum gives the identity and half the difference gives the $SU(2)$ structure constants.

Compute the products

$$
\begin{aligned}
I^{A4}{}_{ab}I^{B4bc} &= \frac{1}{4}\left(\delta^A_a\delta^4_b - \delta^4_a\delta^A_b\right)\left(\delta^{Bb}\delta^{4c} - \delta^{4b}\delta^{Bc}\right) \\
&= \frac{1}{4}\left(\delta^A_a\delta^4_b\delta^{Bb}\delta^{4c} - \delta^A_a\delta^4_b\delta^{4b}\delta^{Bc} - \delta^4_a\delta^A_b\delta^{Bb}\delta^{4c} + \delta^4_a\delta^A_b\delta^{4b}\delta^{Bc}\right) \\
&= -\frac{1}{4}\left(\delta^A_a\delta^{Bc} + \delta^4_a\delta^{4c}\delta^{AB}\right) \\
I^{A4}{}_{ab}E^{B4bc} &= \frac{1}{4}\left(\delta^A_a\delta^4_b - \delta^4_a\delta^A_b\right)\varepsilon^{Bbc} \\
&= \frac{1}{4}\varepsilon^{ABc}\delta^4_a \\
E^{A4}{}_{ab}I^{B4bc} &= \frac{1}{4}\varepsilon^A{}_{ab}\left(\delta^{Bb}\delta^{4c} - \delta^{Bc}\delta^{4b}\right) \\
&= -\frac{1}{4}\varepsilon^{AB}{}_a\delta^{4c} \\
E^{A4}{}_{ab}E^{B4bc} &= \frac{1}{4}\varepsilon^A{}_{ab}\varepsilon^{Bbc} \\
&= -\frac{1}{4}\left(\delta^{AB}\left(\delta^c_a - \delta^c_4\delta^4_a\right) - \delta^{Ac}\delta^B_a\right) \\
&= -\frac{1}{4}\left(\delta^{AB}\delta^c_a - \delta^{Ac}\delta^B_a - \delta^{AB}\delta^c_4\delta^4_a\right)
\end{aligned}
$$

Now consider further products.

$$
\begin{aligned}
\eta^A_{+ab}\eta^{Bbc}_+ &= \left(I^{A4}{}_{ab} + E^{A4}{}_{ab}\right)\left(I^{B4bc} + E^{B4bc}\right) \\
&= I^{A4}{}_{ab}I^{B4bc} + I^{A4}{}_{ab}E^{B4bc} + E^{A4}{}_{ab}I^{B4bc} + E^{A4}{}_{ab}E^{B4bc} \\
&= -\frac{1}{4}\left(\delta^A_a\delta^{Bc} + \delta^4_a\delta^{4c}\delta^{AB}\right) + \frac{1}{4}\varepsilon^{ABc}\delta^4_a \\
&\quad -\frac{1}{4}\varepsilon^{AB}{}_a\delta^{4c} - \frac{1}{4}\left(\delta^{AB}\delta^c_a - \delta^{Ac}\delta^B_a - \delta^{AB}\delta^c_4\delta^4_a\right) \\
&= \frac{1}{4}\left(-\delta^A_a\delta^{Bc} - \delta^4_a\delta^{4c}\delta^{AB} + \varepsilon^{ABc}\delta^4_a\right) \\
&\quad +\frac{1}{4}\left(-\varepsilon^{AB}{}_a\delta^{4c} - \delta^{AB}\delta^c_a + \delta^{Ac}\delta^B_a + \delta^{AB}\delta^c_4\delta^4_a\right) \\
&= \frac{1}{4}\left(\delta^{Ac}\delta^B_a - \delta^A_a\delta^{Bc} - \delta^{AB}\delta^c_a + \varepsilon^{ABc}\delta^4_a - \varepsilon^{AB}{}_a\delta^{4c}\right)
\end{aligned}
$$

This has traces,

$$\begin{aligned}
\eta^{A}_{+ab}\eta^{Bba}_{+} &= -\delta^{AB} \\
\eta_{+Bab}\eta^{Bbc}_{+} &= -\frac{3}{4}\delta^{c}_{a} \\
\eta_{+Bab}\eta^{Bba}_{+} &= -3
\end{aligned}$$

For $+/-$,

$$\begin{aligned}
\eta^{A}_{+ab}\eta^{Bbc}_{-} &= \left(I^{A4}{}_{ab} + E^{A4}{}_{ab}\right)\left(I^{B4bc} - E^{B4bc}\right) \\
&= I^{A4}{}_{ab}I^{B4bc} - I^{A4}{}_{ab}E^{B4bc} + E^{A4}{}_{ab}I^{B4bc} - E^{A4}{}_{ab}E^{B4bc} \\
&= -\frac{1}{4}\left(\delta^{A}_{a}\delta^{Bc} + \delta^{4}_{a}\delta^{4c}\delta^{AB}\right) - \frac{1}{4}\varepsilon^{ABc}\delta^{4}_{a} - \frac{1}{4}\varepsilon^{AB}{}_{a}\delta^{4c} \\
&\quad + \frac{1}{4}\left(\delta^{AB}\delta^{c}_{a} - \delta^{Ac}\delta^{B}_{a} - \delta^{AB}\delta^{c}_{4}\delta^{4}_{a}\right) \\
&= \frac{1}{4}\left(-\delta^{A}_{a}\delta^{Bc} - \delta^{4}_{a}\delta^{4c}\delta^{AB} - \varepsilon^{ABc}\delta^{4}_{a} - \varepsilon^{AB}{}_{a}\delta^{4c} + \delta^{AB}\delta^{c}_{a} - \delta^{Ac}\delta^{B}_{a} - \delta^{AB}\delta^{c}_{4}\delta^{4}_{a}\right) \\
&= \frac{1}{4}\left(-\delta^{Ac}\delta^{B}_{a} - \delta^{A}_{a}\delta^{Bc} + \delta^{AB}\left(\delta^{c}_{a} - 2\delta^{c}_{4}\delta^{4}_{a}\right) - \varepsilon^{ABc}\delta^{4}_{a} - \varepsilon^{AB}{}_{a}\delta^{4c}\right)
\end{aligned}$$

with traces

$$\begin{aligned}
\eta^{A}_{+cb}\eta^{Bbc}_{-} &= 0 \\
\eta_{+Bab}\eta^{Bbc}_{-} &= \frac{1}{4}\left(-2\left(\delta^{c}_{a} - \delta^{c}_{4}\delta^{4}_{a}\right) + 3\left(\delta^{c}_{a} - 2\delta^{c}_{4}\delta^{4}_{a}\right)\right) \\
&= \frac{1}{4}\left(\delta^{c}_{a} - 4\delta^{c}_{4}\delta^{4}_{a}\right)
\end{aligned}$$

Finally, for $-/-$,

$$
\begin{aligned}
\eta^{A}_{-ab}\eta^{Bbc}_{-} &= \left(I^{A4}{}_{ab} - E^{A4}{}_{ab}\right)\left(I^{B4bc} - E^{B4bc}\right) \\
&= I^{A4}{}_{ab}I^{B4bc} - I^{A4}{}_{ab}E^{B4bc} - E^{A4}{}_{ab}I^{B4bc} + E^{A4}{}_{ab}E^{B4bc} \\
&= -\frac{1}{4}\left(\delta^{A}_{a}\delta^{Bc} + \delta^{4}_{a}\delta^{4c}\delta^{AB}\right) - \frac{1}{4}\varepsilon^{ABc}\delta^{4}_{a} + \frac{1}{4}\varepsilon^{AB}{}_{a}\delta^{4c} \\
&\quad -\frac{1}{4}\left(\delta^{AB}\delta^{c}_{a} - \delta^{Ac}\delta^{B}_{a} - \delta^{AB}\delta^{c}_{4}\delta^{4}_{a}\right) \\
&= -\frac{1}{4}\left(\delta^{A}_{a}\delta^{Bc} + \delta^{4}_{a}\delta^{4c}\delta^{AB} + \varepsilon^{ABc}\delta^{4}_{a} - \varepsilon^{AB}{}_{a}\delta^{4c} + \delta^{AB}\delta^{c}_{a} - \delta^{Ac}\delta^{B}_{a} - \delta^{AB}\delta^{c}_{4}\delta^{4}_{a}\right) \\
&= -\frac{1}{4}\left(\delta^{AB}\delta^{c}_{a} + \delta^{A}_{a}\delta^{Bc} - \delta^{Ac}\delta^{B}_{a} + \varepsilon^{ABc}\delta^{4}_{a} - \varepsilon^{AB}{}_{a}\delta^{4c}\right)
\end{aligned}
$$

with further contractions

$$
\begin{aligned}
\eta^{A}_{-cb}\eta^{Bbc}_{-} &= -\delta^{AB} \\
\eta_{-Bab}\eta^{Bbc}_{-} &= -\frac{3}{4}\delta^{c}_{a} \\
\eta_{-Bcb}\eta^{Bbc}_{-} &= -3
\end{aligned}
$$

4.4.4 Summary of 4-dim self-dual and anti-self-dual projections

We make use of several forms of projections. $P_{V_\pm}$ divide the 8-component spinors into two sets of 4-component spinors. Then Π and Σ further subdivide the 4-component spinors into the usual particle/antiparticle, and spin-up/spin-down sectors. Each of the classes of 4-component spinors can produce an independent space of real 4-vectors spanning $\mathbb{R}^4$.

The overarching projections from 8-dimensional spinors to 4-dimensional subspaces are given by

$$
\begin{aligned}
P_{V_+} &= \frac{1}{2}\left(1 + \gamma_V\right) \\
P_{V_-} &= \frac{1}{2}\left(1 - \gamma_V\right)
\end{aligned}
$$

To describe the 4-component spinors, we take upper case Latin letters to be $\mathfrak{su}(2)$ indices, $A, B, \ldots = 1, 2, 3$ while 4-space indices are denoted by lower case Latin indices

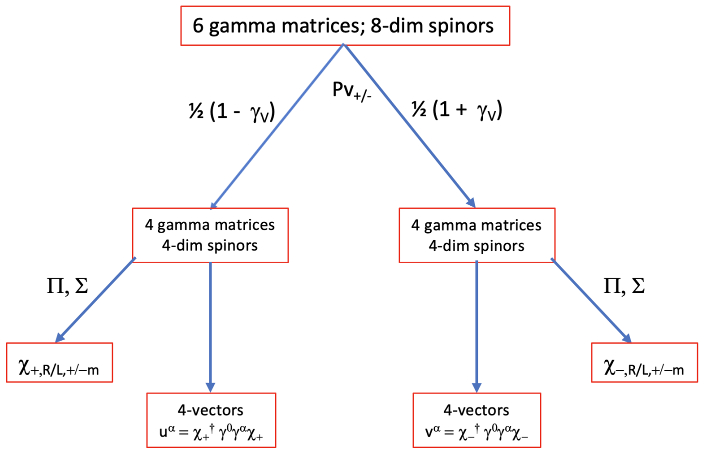

Fig. 4.1: Projections from 8-dimensional spinors

running $a, b, \ldots = 1, 2, 3, 4$. Then we have various equivalent ways to form self-dual and anti-self-dual projections

1. Direct self-dual and anti-self-dual projection tensors,

$$P_{V\pm}{}^{ab}{}_{cd} \;=\; \frac{1}{4}\left(\delta^a_c\delta^b_d - \delta^a_d\delta^b_c \pm \varepsilon^{ab}{}_{cd}\right)$$

These are idempotent orthogonal, and complete on the space of antisymmetric $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensors:

$$\begin{aligned} P_{V\pm}{}^{ab}{}_{cd}P_{V\pm}{}^{cd}{}_{ef} &= P_{V\pm}{}^{ab}{}_{ef} \\ P_{V+}{}^{ab}{}_{cd}P_{V-}{}^{cd}{}_{ef} &= 0 \\ P_{V+}{}^{ab}{}_{cd} + P_{V-}{}^{ab}{}_{cd} &= \frac{1}{2}\left(\delta^a_c\delta^b_d - \delta^a_d\delta^b_c\right) \end{aligned}$$

where $I^{ab}{}_{cd} = \frac{1}{2}\left(\delta^a_c\delta^b_d - \delta^a_d\delta^b_c\right)$ is the identity operation on antisymmetric $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensors, $\mathcal{A}_2$. These divide any antisymmetric $SO(4)$ tensor into self-dual and anti-self-dual parts.

2. The Hodge dual also maps antisymmetric $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensors, expressed as 2-forms, by taking them to their dual.

$$\begin{aligned} \boldsymbol{\omega} &= \frac{1}{2}\omega_{cd}\mathbf{e}^c \wedge \mathbf{e}^d \\ {}^*\boldsymbol{\omega} &= \frac{1}{4}\omega_{ab}\varepsilon^{ab}{}_{cd}\mathbf{e}^c \wedge \mathbf{e}^d \end{aligned}$$

3. Left and right handed spinor projections,

$$\begin{aligned} P_{right} &= \frac{1}{2}\left(1+\gamma_5\right) \\ P_{left} &= \frac{1}{2}\left(1-\gamma_5\right) \end{aligned}$$

Right-handed projections correspond to *self-dual* projections; *left-handed* projections correspond to *anti-self-dual* projections.

Each of these projections maps spinors to subspaces acted on by $SU(2)_+$ or $SU(2)_-$, respectively. These same projections take antisymmetric tensors into the two representations of $SU(2)$. The 't Hooft matrices map the projected tensors $\mathcal{A}_2^{\pm}$ to elements of $\mathfrak{su}(2)_{\pm}$.

$$\eta^A_{\pm ab} : \mathcal{A}_2 \to \mathfrak{su}(2)_{\pm}$$

The matrices are given by

$$\begin{aligned} \eta^A_{+ab} &= \frac{1}{2}\left(\delta^A_a\delta^4_b - \delta^4_a\delta^A_b + \varepsilon_{ab}{}^A\right) = I^{A4}{}_{ab} + E^A{}_{ab4} \\ \eta^A_{-ab} &= \frac{1}{2}\left(\delta^A_a\delta^4_b - \delta^4_a\delta^A_b - \varepsilon_{ab}{}^A\right) = I^{A4}{}_{ab} - E^A{}_{ab4} \end{aligned}$$

4.5 Preserving the Weyl Vector as Symmetry Breaking

We now come to our first main result.

From the quotient $Spin(5,1)/Weyl(4)$ we will develop a biconformal space which, in addition to the usual solder form, spin connection and curvature will have a gauge vector for dilatations, the Weyl vector. As a 1-form, the Weyl vector is $\boldsymbol{\omega} = W_\alpha \mathbf{d}x^\alpha + W^\alpha \mathbf{d}y_\alpha$ with gauge freedom $\boldsymbol{\omega}' = \boldsymbol{\omega} + \mathbf{d}f$ expressed through the dilatational part of the $Weyl\,(4)$ fibers.

It is known that the extra 4 dimensions of 8 dimensional biconformal spaces are fibrated by a Lie group. Generically, this Lie group is abelian but for a subclass of cases it may be non-abelian. Pursuing those cases of biconformal gravity solutions which permit a non-abelian subgroup, we expect half of the biconformal space to become an additional fiber symmetry. We know this symmetry must be a 4-dimensional Lie group, and a subgroup of the $SO\,(4)$ fiber symmetry. It is natural to suppose that since $SO\,(4) = SU\,(2) \times SU\,(2)$, this 4-dimensional group will be the electroweak group, $SU\,(2) \times U\,(1)$, but we still desire a derivation of this particular reduction.

Start with the spinor representation, $\Psi \in V^{(8)}\,(\mathbb{C})$ for $Spin\,(5,1)$. From each element Ψ we form a real vector

$$V^A = \Psi^\dagger \gamma^6 \gamma^A \Psi$$

which may be partitioned into two independent 4-vectors by first partitioning $\Psi \to \Psi_+, \Psi_-$.

$$\begin{aligned} U^a &= \Psi_+^\dagger \gamma^6 \gamma^a \Psi_+ \\ V^a &= \Psi_-^\dagger \gamma^6 \gamma^a \Psi_- \end{aligned}$$

The vector V^A has 8 degrees of freedom, while the spinor Ψ has 16. Therefore, there may be more than one Ψ that gives rise to any given vector. Our goal at this point is twofold:

- For a fixed V^A, identify the class of spinors such that $V^A = \Psi^\dagger \gamma^6 \gamma^A \Psi$.
- Demonstrate that fixing V^A as the Weyl vector is sufficient to identify the non-Abelian subgroup.
- Show that the subgroup is the electroweak group.

To gain insight into these goals, we first worked with some special cases, implementing the calculations using Maple.

Looking at the components of general V^A we noted that a simple k, o vector simplifies Φ to the form

$$\Phi = \begin{pmatrix} 0 \\ k \\ 0 \\ -o \\ 0 \\ k \\ 0 \\ o \end{pmatrix}$$

which is sufficient to find a nonvanishing vector. Indeed, we find

$$V^a = (0, 0, 0, 4ko)$$

and the Weyl vector is preserved. Moreover, it is clear that there exists a 3-dimensional rotation subgroup that will preserve this form, since the vector lies purely in the V^4 direction. The rotations will be implemented with the group $SU(2)$ acting on Ψ, with an additional phase freedom providing a $U(1)$ symmetry.

Explicitly, we found that the transformations that work have the form

$$\begin{aligned}\tilde{\Phi} &= \left(1 + \frac{1}{2}\varepsilon_{abcd}U^a V^b \sigma^{cd}\right)\Phi \\ &= \left(1 + 2ko\varepsilon_{a4cd}U^a \sigma^{cd}\right)\Phi \\ &= \left(1 - 2ko\varepsilon_{ijk}U^i \sigma^{jk}\right)\Phi\end{aligned}$$

where $i, j, k = 1, 2, 3$. We looked at the cases

$$\begin{aligned}U^i &= (1, 0, 0) \\ U^i &= (0, 1, 0) \\ U^i &= (0, 0, 1)\end{aligned}$$

which give

$$\sigma^{23}, \sigma^{31}, \sigma^{12}$$

as three independent generators leaving V^a invariant.

In a similar scenario we had used even a simpler 8-component spinor

$$\Phi = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and the Weyl vector is also preserved.

Further investigations with Maple prompted a more general approach.

Suppose a spinor Ψ gives us the Weyl vector

$$W^A = \Psi^\dagger \gamma^6 \gamma^A \Psi = (a, b, c, d, e, f)$$

where the 4-vector part is

$$W^a = (a, b, c, d)$$

Rotations of W^A are induced by the action of $SO(4)$ on Ψ. We want to find the rotations that leave this particular 4-vector unchanged. Let U^a be any other vector. Note that the remaining components, V^5, V^6 only rescale W^a so they will not affect its direction.

Generalizing the first simple examples, suppose we first rotate to a frame in which W^a takes the form $W^a = (w, 0, 0, 0)$. Then rotations in the $23, 34, 24$ planes clearly have no effect, and this corresponds to the induced action of $SU(2)$. In addition, a phase transformation of Ψ has no effect on W^a, providing an additional $U(1)$ symmetry. We would like to make these observations concrete by constructing a general form of the transformations.

The antisymmetric product

$$V^a U^b \quad - \quad V^b U^a$$

is an area in the plane spanned by U^a and V^a, just like the components of the cross product but in higher dimension. Among the rotations we want are those in the complementary plane, i.e. the plane dual to this one. That's what the Hodge dual gives us, and we can find it with the Levi-Civita tensor,

$$\Sigma_{cd} \quad = \quad \varepsilon_{abcd} U^a V^b$$

This is an area element in the space perpendicular to both U^a and V^a.

A rotation of this plane is generated by

$$\frac{1}{2} \Sigma_{cd} \sigma^{cd}$$

where σ^{cd} are our 6 "Lorentz" generators. The transformation is the exponential of this, or for an infinitesimal transformation

$$1 + \frac{\varepsilon}{2} \Sigma_{cd} \sigma^{cd} \quad = \quad 1 + \frac{\varepsilon}{2} \varepsilon_{abcd} U^a V^b \sigma^{cd}$$

This works for any 4-vector U^a but gives zero if U^a is parallel to V^a. This means that varying U^a will give us a 3-parameter family of transformations leaving V^a invariant.

To make the group more explicit let

$$\begin{aligned} U_{\perp}^{a} &= U^{a} - \frac{\delta_{bc} U^{b} V^{c}}{\delta_{de} V^{d} V^{e}} V^{a} \\ &= P^{a}{}_{b} U^{b} \end{aligned}$$

so that $U_{\perp}^{a} V_{a} = 0$. The projection $P^{a}{}_{b} = \delta_{b}^{a} - \frac{1}{|V|^{2}} V^{a} V_{b}$ produces a 3-dimensional vector subspace $\mathcal{V}^{(3)}$ since

$$\alpha \left(P^{a}{}_{b} U_{1}^{b} \right) + \beta \left(P^{a}{}_{b} U_{2}^{b} \right) = P^{a}{}_{b} \left(\alpha U_{1}^{b} + \beta U_{2}^{b} \right)$$

General linear transformations of this subspace $GL(3, \mathbb{R})$ do not affect V^{a}. Since the non-abelian group must be a subgroup of $SO(4) = SU(2) \times SU(2)$, we are restricted to $SU(2)$.

We implemented this procedure in Maple, starting with a fully general spinor,

$$\Phi = \begin{pmatrix} a + bI \\ c + dI \\ e + fI \\ g + hI \\ i + jI \\ k + lI \\ m + nI \\ o + pI \end{pmatrix}$$

We construct the vector

$$V^{a} = \Phi^{\dagger} \gamma^{6} \gamma^{a} \Phi$$

Next, rotate Φ by any transformation of the form $1 + \frac{\varepsilon}{2} \varepsilon_{abcd} U^{a} V^{b} \sigma^{cd}$,

$$\tilde{\Phi} = \left(1 + \frac{\varepsilon}{2}\varepsilon_{abcd}U^{a}V^{b}\sigma^{cd}\right)\Phi \quad (4.21)$$

and construct the new vector

$$\tilde{V}^{a} = \tilde{\Phi}^{\dagger}\gamma^{6}\gamma^{a}\tilde{\Phi}$$

and check that it is the same no matter how we change U^a. We can let U^a depend on some parameters, say $U^a = (u, v, w, x)$, and show that V'^a is independent of them.

4.6 Isospin

Begin with a vector representation of $SU(2) \times SU(2)$, that is, a $Spin(5,1)$ spinor. The $Spin(5,1)$ transformations have generators

$$\sigma^{AB} = \left[\gamma^{A}, \gamma^{B}\right]$$

These are 8×8 matrices. The parity matrix, $\gamma_V = \gamma^1\gamma^2\gamma^3\gamma^4\gamma^5\gamma^6$ gives a projection of all initial spinors into pairs of 4-spinors. We identify this splitting with isospin and may label 4-spinors as up or down:

$$\begin{pmatrix} \chi \\ \xi \\ \alpha \\ \beta \end{pmatrix} \underset{P_V}{\Rightarrow} \begin{pmatrix} \chi_u \\ \xi_u \\ \alpha_d \\ \beta_d \end{pmatrix}$$

where $P_V^{\pm} = \frac{1}{2}(1 \pm \gamma_V)$. Then γ_V alone is proportional to the z-component of isospin operator:

$$I_3 \equiv \frac{1}{2}\gamma_V \Psi = \frac{1}{2}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} \chi \\ \xi \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\chi \\ \frac{1}{2}\xi \\ -\frac{1}{2}\alpha \\ -\frac{1}{2}\beta \end{pmatrix}$$

The weak interaction mixes isospin doublets. The $SU(2)$ symmetry rotates isospin pairs,

$$\begin{aligned} \Psi &= \begin{pmatrix} \Phi \\ \Xi \end{pmatrix} \\ e^{\frac{i\varphi}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}\Psi &= e^{\frac{i\varphi}{2}\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}\begin{pmatrix} \Phi \\ \Xi \end{pmatrix} \end{aligned}$$

How does this related to the y-space Lie group?

The spaces of 4-spinors have their own projection, $P_5^{\pm} = \frac{1}{2}(1 \pm \gamma_5)$ based on $\tilde{\gamma}_5 = \tilde{\gamma}^1\tilde{\gamma}^2\tilde{\gamma}^3\tilde{\gamma}^4$. This breaks each 4-spinor into left and right handed parts,

$$\begin{pmatrix} \chi_u \\ \xi_u \\ \alpha_d \\ \beta_d \end{pmatrix} \underset{P_5}{\Rightarrow} \begin{pmatrix} \chi_{u,l} \\ \xi_{u,l} \\ \alpha_{d,l} \\ \beta_{d,l} \end{pmatrix} + \begin{pmatrix} \chi_{u,r} \\ \xi_{u,r} \\ \alpha_{d,r} \\ \beta_{d,r} \end{pmatrix}$$

where

$$P_5^{\pm} = \begin{pmatrix} \frac{1}{2}(1 \pm \gamma_5) & 0 \\ 0 & \frac{1}{2}(1 \pm \gamma_5) \end{pmatrix}$$

These commute with I_3, so that isospin doublets may be made left handed or right handed.

Any operator of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

will commute with I_3, and we may write the particle/antiparticle and spin-up/spin-down projections this way.

We know from experiments however, that right handed particles have zero value for the third component of isospin. This ooses a difficulty from our represeb=ntation above. In order to fit our model with experiments we will have to choose a diferent basis for our gamma mattrices such that we get I_3 in the form

$$I_3 \equiv \frac{1}{2}\gamma_V \Psi \quad = \tfrac{1}{2} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} \chi \\ \xi \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\chi \\ \frac{1}{-2}\xi \\ \frac{1}{2}\alpha \\ -\frac{1}{2}\beta \end{pmatrix}$$

So that $P_5^{\pm} = \frac{1}{2}\left(1 \pm \gamma_5\right)$ based on $\tilde{\gamma}_5 = \tilde{\gamma}^1\tilde{\gamma}^2\tilde{\gamma}^3\tilde{\gamma}^4$ then breaks each 4-spinor into left and right handed parts,

$$\begin{pmatrix} \chi_u \\ \xi_d \\ \alpha_u \\ \beta_d \end{pmatrix} \underset{P_5}{\Rightarrow} \begin{pmatrix} \chi_{u,l} \\ \xi_{d,l} \\ \alpha_{u,l} \\ \beta_{d,l} \end{pmatrix} + \begin{pmatrix} \chi_{u,r} \\ \xi_{d,r} \\ \alpha_{u,r} \\ \beta_{d,r} \end{pmatrix}$$

where

$$P_5^{\pm} = \begin{pmatrix} \frac{1}{2}\left(1 \pm \gamma_5\right) & 0 \\ 0 & \frac{1}{2}\left(1 \pm \gamma_5\right) \end{pmatrix}$$

These commute with I_3, so that isospin doublets may be made left handed or right handed.

Any operator of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

will commute with I_3, and we may write the particle/antiparticle and spin-up/spin-down projections this way.

$$\begin{pmatrix} \chi_u \\ \xi_d \\ \alpha_u \\ \beta_d \end{pmatrix} \underset{P_5}{\Rightarrow} \begin{pmatrix} \chi_{u,l} & 0 \\ \xi_{d,l} & 0 \\ 0 & \alpha_{u,r} \\ 0 & \beta_{d,r} \end{pmatrix}$$

We already klnow that fixing a spinor preserves the Weyl vector. In this case what this allows us to do is that we can choose a spinor say

$$\phi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

or

$$\phi = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

which breaks the right handed doublet into a singlet. (It is not clear how this gives rise to zero isospin for the singlet). Projections using the $P_{V_\pm}$ give us two sets of 4 gamma matrices and with the 4-dimensional gamma matrices we can still build the conformal group which still gives us a representation of $SO(5,1)$. One interpretation of this projection is in terms of Isospin up and down (after choosing the correct basis) and this gives us the first picture of interpreting the $P_{V_{\pm}}$ projection as illustrated in more details in the diagram below;

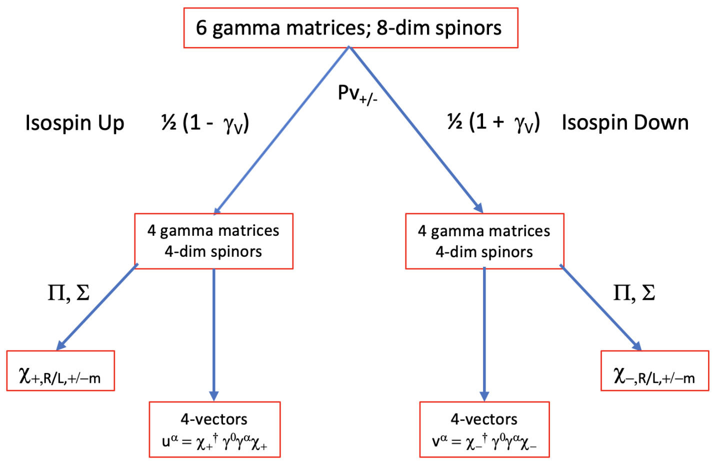

Fig. 4.2: Isospin Interpretation of Projections

Another interpretation is to say that one of $P_{V_{\pm}}$ sets applies to the x-space and generates the spacetime symmetry while the other applies to the y-space and generates the electroweak symmetry since any of these outcomes are possible within a conformal group representation. This is also illustrated in more details in the diagram below:

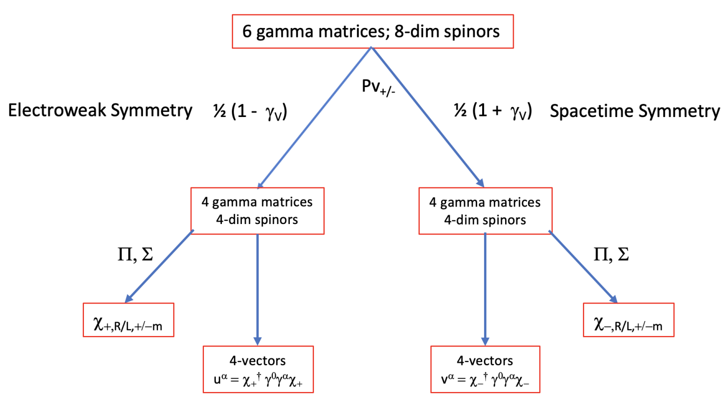

Fig. 4.3: Graviweak symmetry interpretation of the projections

We anticipate that further study in the context of solutions to the field equations will clarify which interpretation is most appropriate.

CHAPTER 5

SELF-DUAL AND ANTI-SELF-DUAL CONNECTIONS

Using the projections (see Appendix A)

$$
\begin{aligned}
P_{+\ cd}^{ab} &= \frac{1}{4}\left(\delta_c^a \delta_d^b - \delta_d^a \delta_c^b \pm \varepsilon^{ab}{}_{cd}\right) \\
P_{+} &= \frac{1}{2}\left(1-\gamma_5\right) \\
\eta_{Aab} &= \varepsilon_{Aab4} + \delta_{Aa}\delta_{b4} - \delta_{Ab}\delta_{a4} \\
\mathbf{F}_{+} &= \frac{1}{2}\left(\mathbf{F} + {}^{*}\mathbf{F}\right)
\end{aligned}
$$

and their complements, we can form the self-dual parts of the connection and curvature in a variety of ways.

Begin with the spin connection,

$$
\begin{aligned}
\boldsymbol{\omega}^{ab} &= \boldsymbol{\omega}_{+}^{ab} + \boldsymbol{\omega}_{-}^{ab} \\
&\equiv P_{+\ cd}^{ab}\boldsymbol{\omega}^{cd} + P_{-\ cd}^{ab}\boldsymbol{\omega}^{cd}
\end{aligned} \tag{5.1}
$$

where

$$
\begin{aligned}
P_{\pm\ cd}^{ab}\boldsymbol{\omega}^{cd} &= \frac{1}{4}\left(\left(\delta_c^a \delta_d^b - \delta_d^a \delta_c^b\right)\boldsymbol{\omega}^{cd} \pm \varepsilon^{ab}{}_{cd}\boldsymbol{\omega}^{cd}\right) \\
&= \frac{1}{2}\left(\boldsymbol{\omega}^{ab} \pm \frac{1}{2}\varepsilon^{ab}{}_{cd}\boldsymbol{\omega}^{cd}\right)
\end{aligned}
$$

5.1 Identities

In terms of the 't Hooft matrices we can get equivalent pieces,

$$\begin{aligned}
\boldsymbol{\omega}_+^A &= \frac{1}{2}\eta^A{}_{ab}\boldsymbol{\omega}^{ab} \\
\boldsymbol{\omega}_-^A &= \frac{1}{2}\bar{\eta}^A{}_{ab}\boldsymbol{\omega}^{ab}
\end{aligned}$$

and from the relations

$$\begin{aligned}
\eta_{Aab} &= \frac{1}{2}\varepsilon_{abcd}\eta_{Acd} \\
\bar{\eta}_{Aab} &= -\frac{1}{2}\varepsilon_{abcd}\bar{\eta}_{Acd}
\end{aligned}$$

and

$$\begin{aligned}
P_{+\ \ cd}^{ab}\eta^{Acd} &= \eta^{Aab} \\
P_{-\ \ cd}^{ab}\eta^{Acd} &= 0 \\
P_{+\ \ cd}^{ab}\bar{\eta}^{Acd} &= 0 \\
P_{-\ \ cd}^{ab}\bar{\eta}^{Acd} &= \bar{\eta}^{Aab}
\end{aligned}$$

we confirm the consistency,

$$\begin{aligned}
\boldsymbol{\omega}_+^A &= \frac{1}{2}\eta^A{}_{ab}\boldsymbol{\omega}^{ab} \\
&= \frac{1}{2}\eta^A{}_{ab}P_{+\quad cd}^{ab}\boldsymbol{\omega}^{cd} \\
&= \frac{1}{2}\eta^A{}_{ab}\boldsymbol{\omega}_+^{cd}
\end{aligned}$$

To invert this, act again,

$$\begin{aligned}
\frac{1}{2}\boldsymbol{\omega}_+^A\eta_{Aab} &= \frac{1}{4}\eta_{Aab}\eta^A{}_{cd}\boldsymbol{\omega}^{cd} \\
&= P_{abcd}\boldsymbol{\omega}^{cd} \\
&= \boldsymbol{\omega}_+^{cd}
\end{aligned} \tag{5.2}$$

In this way the 't Hooft matrices give a bijection,

$$\begin{array}{rcl} \boldsymbol{\omega}_{+}^{A} & \leftrightarrow & \boldsymbol{\omega}_{+}^{ab} \\ \boldsymbol{\omega}_{-}^{A} & \leftrightarrow & \boldsymbol{\omega}_{-}^{ab} \end{array}$$

We can reconstruct the tensor projections as

$$\begin{array}{rcl} \frac{1}{4}\eta_{Aab}\eta^{A}{}_{cd} & = & P^{+}_{abcd} \\ \frac{1}{4}\bar{\eta}_{Aab}\bar{\eta}^{A}{}_{cd} & = & P^{-}_{abcd} \end{array}$$

and

$$\frac{1}{4}\left(\eta^{Aab}\eta_{Acd} + \bar{\eta}^{Aab}\bar{\eta}_{Acd}\right) = \frac{1}{2}\left(\delta^a_c\delta^b_d - \delta^b_c\delta^a_d\right)$$

We also have projections in a spinor representation. Let

$$\left\langle \boldsymbol{\omega}^{ab}, \sigma_{cd} \right\rangle = \delta^a_c\delta^b_d - \delta^a_d\delta^b_c$$

Then we can project with any of the projections,

$$\begin{array}{rcl} \sigma^{+}_{cd} & = & P_{+}\sigma_{cd} \\ & = & \frac{1}{2}\left(1-\gamma_5\right)\sigma_{cd} \\ & = & P^{ab}{}_{cd}\sigma_{ab} \\ \sigma^{A}_{+} & = & \frac{1}{2}\eta^{Aab}\sigma_{ab} \\ \frac{1}{2}\eta_{A}{}^{cd}\sigma^{A}_{+} & = & \frac{1}{4}\eta_{A}{}^{cd}\eta^{Aab}\sigma_{ab} \\ & = & \frac{1}{4}\left(4P^{abcd}\right)\sigma_{ab} \\ & = & P^{abcd}\sigma_{ab} \\ \sigma^{A}_{-} & = & \frac{1}{2}\bar{\eta}^{Aab}\sigma_{ab} \end{array}$$

Each of the last two generates an independent $SU(2)$.

5.2 Connection and curvature

Now compute the curvature by expanding

$$\begin{aligned}
\boldsymbol{\omega}_{ab} &= \frac{1}{2}\boldsymbol{\omega}_{+}^{A}\eta_{Aab} + \frac{1}{2}\boldsymbol{\omega}_{-}^{A}\bar{\eta}_{Aab} \\
\boldsymbol{\omega}^{a}{}_{b} &= \frac{1}{2}\boldsymbol{\omega}_{+}^{A}\eta_{A}{}^{a}{}_{b} + \frac{1}{2}\boldsymbol{\omega}_{-}^{A}\bar{\eta}_{A}{}^{a}{}_{b}
\end{aligned}$$

and similarly for the curvature,

$$\begin{aligned}
\boldsymbol{\Omega}_{ab} &= \frac{1}{2}\boldsymbol{\Omega}_{+}^{A}\eta_{Aab} + \frac{1}{2}\boldsymbol{\Omega}_{-}^{A}\bar{\eta}_{Aab} \\
\boldsymbol{\Omega}^{a}{}_{b} &= \frac{1}{2}\boldsymbol{\Omega}_{+}^{A}\eta_{A}{}^{a}{}_{b} + \frac{1}{2}\boldsymbol{\Omega}_{-}^{A}\bar{\eta}_{A}{}^{a}{}_{b}
\end{aligned}$$

To invert these,

$$\begin{aligned}
\frac{1}{2}\boldsymbol{\Omega}_{+}^{A}\eta_{A}{}^{a}{}_{b} &= \frac{1}{2}\left(\frac{1}{2}\eta^{A}{}_{cd}\boldsymbol{\Omega}^{cd}\right)\eta_{A}{}^{a}{}_{b} \\
&= \frac{1}{4}\eta^{A}{}_{cd}\eta_{A}{}^{a}{}_{b}\boldsymbol{\Omega}^{cd} \\
&= P^{a}_{+bcd}\boldsymbol{\Omega}^{cd}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\Omega}^{a}{}_{b} &= \frac{1}{2}\boldsymbol{\Omega}_{+}^{A}\eta_{A}{}^{a}{}_{b} + \frac{1}{2}\boldsymbol{\Omega}_{-}^{A}\bar{\eta}_{A}{}^{a}{}_{b} \\
\boldsymbol{\Omega}^{a}{}_{b}\eta_{B}{}^{b}{}_{a} &= \frac{1}{2}\boldsymbol{\Omega}_{+}^{A}\eta_{A}{}^{a}{}_{b}\eta_{B}{}^{b}{}_{a} + \frac{1}{2}\boldsymbol{\Omega}_{-}^{A}\bar{\eta}_{A}{}^{a}{}_{b}\eta_{B}{}^{b}{}_{a} \\
&= 2\boldsymbol{\Omega}_{+}^{A}\delta_{AB}
\end{aligned}$$

and therefore,

$$\begin{aligned}
\boldsymbol{\Omega}_{+}^{A} &= \frac{1}{2}\eta^{Ab}{}_{a}\boldsymbol{\Omega}^{a}{}_{b} \\
\boldsymbol{\omega}_{+}^{A} &= \frac{1}{2}\eta^{Ab}{}_{a}\boldsymbol{\omega}^{a}{}_{b}
\end{aligned}$$

We may also expand the antisymmetric identity, Δ^{ac}_{db}

$$\begin{aligned}
\delta^a_d\delta^c_b - \delta^c_d\delta^a_b &= P_+^{ac}{}_{db} + P_-^{ac}{}_{db} \\
&= \frac{1}{4}\left(\eta^{Aab}\eta_{Acd} + \bar\eta^{Aab}\bar\eta_{Acd}\right)
\end{aligned}$$

5.3 Splitting the curvature structure equation

Substituting into

$$\mathbf{d}\boldsymbol{\omega}^a{}_b = \boldsymbol{\omega}^c{}_b \wedge \boldsymbol{\omega}^a{}_c + 2\Delta^{ac}_{db}\mathbf{f}_c \wedge \mathbf{e}^d + \boldsymbol{\Omega}^a{}_b$$

we have

$$\begin{aligned}
\frac{1}{2}\mathbf{d}\boldsymbol{\omega}_+^A\eta_A{}^a{}_b + \frac{1}{2}\mathbf{d}\boldsymbol{\omega}_-^A\bar\eta_A{}^a{}_b &= \left(\frac{1}{2}\boldsymbol{\omega}_+^A\eta_A{}^c{}_b + \frac{1}{2}\boldsymbol{\omega}_-^A\bar\eta_A{}^c{}_b\right)\wedge\left(\frac{1}{2}\boldsymbol{\omega}_+^A\eta_A{}^a{}_c + \frac{1}{2}\boldsymbol{\omega}_-^A\bar\eta_A{}^a{}_c\right) \\
&\quad +2\Delta^{ac}_{db}\mathbf{f}_c\wedge\mathbf{e}^d + \frac{1}{2}\boldsymbol{\Omega}_+^A\eta_A{}^a{}_b + \frac{1}{2}\boldsymbol{\Omega}_-^A\bar\eta_A{}^a{}_b \\
&= \left(\frac{1}{2}\boldsymbol{\omega}_+^A\eta_A{}^c{}_b + \frac{1}{2}\boldsymbol{\omega}_-^A\bar\eta_A{}^c{}_b\right)\wedge\left(\frac{1}{2}\boldsymbol{\omega}_+^A\eta_A{}^a{}_c + \frac{1}{2}\boldsymbol{\omega}_-^A\bar\eta_A{}^a{}_c\right) \\
&\quad +2\frac{1}{4}\left(\eta^{Aab}\eta_{Acd} + \bar\eta^{Aab}\bar\eta_{Acd}\right)\mathbf{f}_c\wedge\mathbf{e}^d + \frac{1}{2}\boldsymbol{\Omega}_+^A\eta_A{}^a{}_b + \frac{1}{2}\boldsymbol{\Omega}_-^A\bar\eta_A{}^a{}_b
\end{aligned}$$

Separating like terms,

$$\begin{aligned}
0 &= -\frac{1}{2}\mathbf{d}\boldsymbol{\omega}_+^A\eta_A{}^a{}_b + \frac{1}{2}\boldsymbol{\omega}_+^A\wedge\frac{1}{2}\boldsymbol{\omega}_+^B\eta_A{}^c{}_b\eta_B{}^a{}_c + \frac{1}{2}\eta_A{}^a{}_b\eta^A{}_{cd}\mathbf{f}_c\wedge\mathbf{e}^d + \frac{1}{2}\boldsymbol{\Omega}_+^A\eta_A{}^a{}_b \\
&\quad +\frac{1}{4}\boldsymbol{\omega}_-^A\wedge\boldsymbol{\omega}_+^B\left(\bar\eta_A{}^c{}_b\eta_B{}^a{}_c - \bar\eta_A{}^a{}_c\eta_B{}^c{}_b\right) \\
&\quad -\frac{1}{2}\mathbf{d}\boldsymbol{\omega}_-^A\bar\eta_A{}^a{}_b + \frac{1}{4}\boldsymbol{\omega}_-^A\bar\eta_A{}^c{}_b\wedge\boldsymbol{\omega}_-^A\bar\eta_A{}^a{}_c + \frac{1}{2}\bar\eta_A{}^a{}_b\bar\eta^A{}_{cd}\mathbf{f}_c\wedge\mathbf{e}^d + \frac{1}{2}\boldsymbol{\Omega}_-^A\bar\eta_A{}^a{}_b
\end{aligned} \tag{5.3}$$

For the cross-terms,

$$
\begin{aligned}
\bar{\eta}_A{}^c{}_b \eta_B{}^a{}_c - \bar{\eta}_A{}^a{}_c \eta_B{}^c{}_b &= \eta_B{}^a{}_c \bar{\eta}_A{}^c{}_b - \bar{\eta}_A{}^a{}_c \eta_B{}^c{}_b \\
&= \left(\varepsilon_B{}^a{}_{c4} + \delta^a_B \delta_{c4} - \delta_{Bc}\delta^a_4\right) \bar{\eta}_A{}^c{}_b - \left(\varepsilon_B{}^a{}_{c4} - \delta^a_B \delta_{c4} + \delta_{Bc}\delta^a_4\right) \eta_B{}^c{}_b \\
&= \left(\varepsilon_B{}^a{}_{c4} + \delta^a_B \delta_{c4} - \delta_{Bc}\delta^a_4\right)\left(\varepsilon_A{}^c{}_{b4} - \delta^c_A \delta_{b4} + \delta_{Ab}\delta^c_4\right) \\
&\quad - \left(\varepsilon_A{}^a{}_{c4} - \delta^a_A \delta_{c4} + \delta_{Ac}\delta^a_4\right)\left(\varepsilon_B{}^c{}_{b4} + \delta^c_B \delta_{b4} - \delta_{Bb}\delta^c_4\right) \\
&= \varepsilon_B{}^{ac}{}_4 \varepsilon_{Acb4} - \varepsilon_B{}^a{}_{A4}\delta_{b4} + \delta^a_B \delta_{Ab} - \delta^a_4 \varepsilon_{ABb4} + \delta_{BA}\delta^a_4 \delta_{b4} \\
&\quad - \varepsilon_A{}^a{}_{c4} \varepsilon_B{}^c{}_{b4} - \varepsilon_A{}^a{}_{B4}\delta_{b4} - \delta^a_A \delta_{Bb} - \delta^a_4 \varepsilon_{BAb4} - \delta_{AB}\delta^a_4 \delta_{b4} \\
&= \varepsilon_B{}^{ac}{}_4 \varepsilon_{Acb4} - \varepsilon_{Bcb4}\varepsilon_A{}^{ac}{}_4 + \left(\delta^a_4 \varepsilon_{ABb4} - \delta^a_4 \varepsilon_{ABb4}\right) \\
&\quad + \left(\varepsilon_{AB}{}^a{}_4 \delta_{b4} - \varepsilon_{AB}{}^a{}_4 \delta_{b4}\right) \\
&\quad + \delta^a_B \delta_{Ab} - \delta^a_A \delta_{Bb} + \left(\delta_{BA}\delta^a_4 \delta_{b4} - \delta_{AB}\delta^a_4 \delta_{b4}\right)
\end{aligned}
$$

and then

$$
\begin{aligned}
\bar{\eta}_A{}^c{}_b \eta_B{}^a{}_c - \bar{\eta}_A{}^a{}_c \eta_B{}^c{}_b &= \varepsilon_B{}^{ac}{}_4 \varepsilon_{Acb4} - \varepsilon_{Bcb4}\varepsilon_A{}^{ac}{}_4 + \delta^a_B \delta_{Ab} - \delta^a_A \delta_{Bb} \\
&= \left(-\delta_{BA}\delta^a_b + \delta^a_A \delta_{Bb}\right) - \left(-\delta_{BA}\delta^a_b + \delta^a_B \delta_{Ab}\right) + \delta^a_B \delta_{Ab} - \delta^a_A \delta_{Bb} \\
&= \delta^a_A \delta_{Bb} - \delta^a_B \delta_{Ab} + \delta^a_B \delta_{Ab} - \delta^a_A \delta_{Bb} \\
&= 0
\end{aligned}
$$

Returning to the curvature,

$$
\begin{aligned}
0 \;=\; & -\mathbf{d}\boldsymbol{\omega}^A_+ \eta_A{}^a{}_b + \frac{1}{2}\boldsymbol{\omega}^A_+ \wedge \boldsymbol{\omega}^B_+ \eta_A{}^c{}_b \eta_B{}^a{}_c + \eta_A{}^a{}_b \eta^A{}_{cd} \mathbf{f}_c \wedge \mathbf{e}^d + \boldsymbol{\Omega}^A_+ \eta_A{}^a{}_b \\
& -\mathbf{d}\boldsymbol{\omega}^A_- \bar{\eta}_A{}^a{}_b + \frac{1}{2}\boldsymbol{\omega}^A_- \bar{\eta}_A{}^c{}_b \wedge \boldsymbol{\omega}^A_- \bar{\eta}_A{}^a{}_c + \bar{\eta}_A{}^a{}_b \bar{\eta}^A{}_{cd} \mathbf{f}_c \wedge \mathbf{e}^d + \boldsymbol{\Omega}^A_- \bar{\eta}_A{}^a{}_b
\end{aligned}
$$

Now we need

$$
\begin{aligned}
\eta_{Acb}\eta_B{}^{ca} - \eta_{Bcb}\eta_A{}^{ca} &= \left(\varepsilon_{Acb4} + \delta_{Ac}\delta_{b4} - \delta_{Ab}\delta_{4c}\right)\left(\varepsilon_B{}^{ca}{}_4 + \delta^c_B\delta^a_4 - \delta^a_B\delta^c_4\right) - (A \leftrightarrow B) \\
&= \varepsilon_{Abc4}\varepsilon_B{}^{ac}{}_4 + \varepsilon_{ABb4}\delta^a_4 + \delta_{b4}\varepsilon_{BA}{}^a{}_4 + \delta_{AB}\delta_{b4}\delta^a_4 + \delta_{Ab}\delta^a_B \\
&\quad -\varepsilon_{Bbc4}\varepsilon_A{}^{ac}{}_4 - \varepsilon_{BAb4}\delta^a_4 - \delta_{b4}\varepsilon_{AB}{}^a{}_4 - \delta_{BA}\delta_{b4}\delta^a_4 - \delta_{Bb}\delta^a_A \\
&= \left(\delta_{AB}\delta^a_b - \delta^a_A\delta_{bB}\right) - \left(\delta_{BA}\delta^a_b - \delta^a_B\delta_{bA}\right) \\
&\quad +\delta_{Ab}\delta^a_B - \delta_{Bb}\delta^a_A + 2\varepsilon_{ABb4}\delta^a_4 - 2\delta_{b4}\varepsilon_{AB}{}^a{}_4 \\
&= 2\left(\varepsilon_{bAB4}\delta^a_4 - \delta_{b4}\varepsilon^a{}_{AB4} - \delta^a_A\delta_{bB} + \delta_{Ab}\delta^a_B\right) \\
&= 2\left(\varepsilon_{bAB}\delta^a_4 - \delta_{b4}\varepsilon^a{}_{AB} - \varepsilon^a{}_{bc}\varepsilon_{AB}{}^c\right) \\
&= 2\varepsilon_{AB}{}^c\left(\delta_{bc}\delta^a_4 - \delta^a_c\delta_{b4} - \varepsilon^a{}_{bc}\right) \\
&= -2\varepsilon_{AB}{}^C\left(\varepsilon^a{}_{bC} - \delta_{bC}\delta^a_4 + \delta^a_C\delta_{b4}\right) \\
&= -2\varepsilon_{AB}{}^C\left(\varepsilon_C{}^a{}_b + \delta^a_C\delta_{b4} - \delta_{bC}\delta^a_4\right) \\
&= -2\varepsilon_{AB}{}^C\eta_C{}^a{}_b
\end{aligned}
$$

Then finally,

$$
\begin{aligned}
0 &= -\mathbf{d}\boldsymbol{\omega}^A_+\eta_A{}^a{}_b + \frac{1}{4}\boldsymbol{\omega}^A_+ \wedge \boldsymbol{\omega}^B_+\left(\eta_{Acb}\eta_B{}^{ca} - \eta_{Bcb}\eta_A{}^{ca}\right) + \eta_A{}^a{}_b\eta^A{}_{cd}\mathbf{f}_c \wedge \mathbf{e}^d + \boldsymbol{\Omega}^A_+\eta_A{}^a{}_b \\
&\quad -\mathbf{d}\boldsymbol{\omega}^A_-\bar{\eta}_A{}^a{}_b + \frac{1}{2}\boldsymbol{\omega}^A_- \wedge \boldsymbol{\omega}^B_-\left(\bar{\eta}_{Acb}\bar{\eta}_B{}^{ca} - \bar{\eta}_{Bcb}\bar{\eta}_A{}^{ca}\right) + \bar{\eta}_A{}^a{}_b\bar{\eta}^A{}_{cd}\mathbf{f}_c \wedge \mathbf{e}^d + \boldsymbol{\Omega}^A_-\bar{\eta}_A{}^a{}_b \\
&= \left(-\mathbf{d}\boldsymbol{\omega}^A_+ - \frac{1}{2}\boldsymbol{\omega}^B_+ \wedge \boldsymbol{\omega}^C_+\varepsilon_{BC}{}^A + \eta^A{}_{cd}\mathbf{f}_c \wedge \mathbf{e}^d + \boldsymbol{\Omega}^A_+\right)\eta_A{}^a{}_b \\
&\quad + \left(-\mathbf{d}\boldsymbol{\omega}^A_- - \frac{1}{2}\boldsymbol{\omega}^B_- \wedge \boldsymbol{\omega}^C_-\varepsilon_{BC}{}^A + \bar{\eta}^A{}_{cd}\mathbf{f}_c \wedge \mathbf{e}^d + \boldsymbol{\Omega}^A_-\right)\bar{\eta}_A{}^a{}_b
\end{aligned}
$$

and therefore the curvature projects cleanly into a pair of conformal $SU(2)$ curvatures:

$$
\begin{aligned}
\mathbf{d}\boldsymbol{\omega}^A_+ &= -\frac{1}{2}\varepsilon^A{}_{BC}\boldsymbol{\omega}^B_+ \wedge \boldsymbol{\omega}^C_+ + \eta^{Ac}{}_d\mathbf{f}_c \wedge \mathbf{e}^d + \boldsymbol{\Omega}^A_+ \\
\mathbf{d}\boldsymbol{\omega}^A_- &= -\frac{1}{2}\varepsilon^A{}_{BC}\boldsymbol{\omega}^B_- \wedge \boldsymbol{\omega}^C_- + \bar{\eta}^{Ac}{}_d\mathbf{f}_c \wedge \mathbf{e}^d + \boldsymbol{\Omega}^A_-
\end{aligned}
$$

For the remaining connection forms,

$$\begin{aligned}
\mathbf{de}^a &= \mathbf{e}^b \wedge \boldsymbol{\omega}^a{}_b + \boldsymbol{\omega} \wedge \mathbf{e}^a + \mathbf{T}^a \\
&= \frac{1}{2}\mathbf{e}^b \wedge \boldsymbol{\omega}_+^A \eta_A{}^a{}_b + \frac{1}{2}\mathbf{e}^b \wedge \boldsymbol{\omega}_-^A \bar{\eta}_A{}^a{}_b + \boldsymbol{\omega} \wedge \mathbf{e}^a + \mathbf{T}^a \\
\mathbf{df}_a &= \boldsymbol{\omega}^b{}_a \wedge \mathbf{f}_b + \mathbf{f}_a \wedge \boldsymbol{\omega} + \mathbf{S}_a \\
&= \frac{1}{2}\boldsymbol{\omega}_+^A \wedge \mathbf{f}_b \eta_A{}^b{}_a + \frac{1}{2}\boldsymbol{\omega}_-^A \wedge \mathbf{f}_b \bar{\eta}_A{}^b{}_a + \mathbf{f}_a \wedge \boldsymbol{\omega} + \mathbf{S}_a \\
\mathbf{d}\boldsymbol{\omega} &= \mathbf{e}^a \wedge \mathbf{f}_a + \boldsymbol{\Omega}
\end{aligned}$$

$$\delta\boldsymbol{\omega}^a{}_b = A^a{}_{bc}\mathbf{e}^c + B^a{}_b{}^c\mathbf{f}_c$$

This depends on $\left[A_{[ab]c}\right] = 24$ plus $\left[B_{[ab]c}\right] = 24$ degrees of freedom, for a total of 48 equations. The variations preserving duality are

$$\delta\boldsymbol{\omega}_\pm^A = A_\pm^A{}_c\mathbf{e}^c + B_\pm^{Ac}\mathbf{f}_c$$

for a total of $12+12+12+12 = 48$ variations. The count is the same, so nothing is changed if the variation preserves duality.

5.4 Summary of the structure equations

We have

$$\begin{aligned}
\mathbf{d}\boldsymbol{\omega}_+^A &= -\frac{1}{2}\varepsilon^A{}_{BC}\boldsymbol{\omega}_+^B \wedge \boldsymbol{\omega}_+^C + \eta^{Ac}{}_d\mathbf{f}_c \wedge \mathbf{e}^d + \boldsymbol{\Omega}_+^A \\
\mathbf{d}\boldsymbol{\omega}_-^A &= -\frac{1}{2}\varepsilon^A{}_{BC}\boldsymbol{\omega}_-^B \wedge \boldsymbol{\omega}_-^C + \bar{\eta}^{Ac}{}_d\mathbf{f}_c \wedge \mathbf{e}^d + \boldsymbol{\Omega}_-^A \\
\mathbf{de}^a &= \frac{1}{2}\mathbf{e}^b \wedge \boldsymbol{\omega}_+^A \eta_A{}^a{}_b + \frac{1}{2}\mathbf{e}^b \wedge \boldsymbol{\omega}_-^A \bar{\eta}_A{}^a{}_b + \boldsymbol{\omega} \wedge \mathbf{e}^a + \mathbf{T}^a \\
&= \frac{1}{2}\mathbf{e}^b \wedge \boldsymbol{\omega}^a{}_b + \boldsymbol{\omega} \wedge \mathbf{e}^a + \mathbf{T}^a \\
\mathbf{df}_a &= \frac{1}{2}\boldsymbol{\omega}_+^A \wedge \mathbf{f}_b \eta_A{}^b{}_a + \frac{1}{2}\boldsymbol{\omega}_-^A \wedge \mathbf{f}_b \bar{\eta}_A{}^b{}_a + \mathbf{f}_a \wedge \boldsymbol{\omega} + \mathbf{S}_a \\
&= \boldsymbol{\omega}^b{}_a \wedge \mathbf{f}_b + \mathbf{f}_a \wedge \boldsymbol{\omega} + \mathbf{S}_a \\
\mathbf{d}\boldsymbol{\omega} &= \mathbf{e}^a \wedge \mathbf{f}_a + \boldsymbol{\Omega}
\end{aligned}$$

CHAPTER 6
ACTION AND FIELD EQUATIONS

6.1 The action in the spin basis

6.1.1 Introduction of the curvatures

The biconformal action,

$$S = \int \left(\alpha \mathbf{\Omega}^{a}{}_{b} + \beta \delta^{a}_{b} \mathbf{\Omega} + \gamma \mathbf{e}^{a} \wedge \mathbf{f}_{b}\right) \wedge \mathbf{f}_{e} \wedge \cdots \wedge \mathbf{f}_{f} \wedge \mathbf{e}^{c} \wedge \cdots \wedge \mathbf{e}^{d} e^{be\cdots f}{}_{ac\cdots d} \tag{6.1}$$

may now be written as

$$S = \int \left(\alpha \mathbf{\Omega}_{+}{}^{a}{}_{b} + \alpha \mathbf{\Omega}_{-}{}^{a}{}_{b} + \beta \delta^{a}_{b} \mathbf{\Omega} + \gamma \mathbf{e}^{a} \wedge \mathbf{f}_{b}\right) \wedge \mathbf{f}_{e} \wedge \cdots \wedge \mathbf{f}_{f} \wedge \mathbf{e}^{c} \wedge \cdots \wedge \mathbf{e}^{d} e^{be\cdots f}{}_{ac\cdots d}$$

We vary $\boldsymbol{\omega}_{+}{}^{a}{}_{b}, \boldsymbol{\omega}_{-}{}^{a}{}_{b}, \mathbf{e}^{a}, \mathbf{f}_{a}, \boldsymbol{\omega}$ independently.

However, the curvatures we have are

$$\mathbf{\Omega}^{M}_{+}, \mathbf{\Omega}^{M}_{-}, \mathbf{T}^{a}, \mathbf{S}_{a}, \mathbf{\Omega}$$

where $M = 1, 2, 3$ and $a = 1, 2, 3, 4$. To connect the curvatures to the basis forms, we use the 't Hooft matrices, $\eta_{M}{}^{a}{}_{b}, \bar{\eta}_{M}{}^{a}{}_{b}$. These connect the six antisymmetric pairs, $[a, b]$ to two sets of three $SU(2)$ indices, $M\pm$. With this notation, we may write

$$S = \int \left(\alpha \mathbf{\Omega}^{M}_{+} \bar{\eta}_{M}{}^{a}{}_{b} + \alpha \mathbf{\Omega}^{M}_{-} \eta_{M}{}^{a}{}_{b} + \beta \delta^{a}_{b} \mathbf{\Omega} + \gamma \mathbf{e}^{a} \wedge \mathbf{f}_{b}\right) \wedge \mathbf{f}_{e} \wedge \cdots \wedge \mathbf{f}_{f} \wedge \mathbf{e}^{c} \wedge \cdots \wedge \mathbf{e}^{d} e^{be\cdots f}{}_{ac\cdots d} \tag{6.2}$$

We vary $\boldsymbol{\omega}^{m}_{+}, \boldsymbol{\omega}^{m}_{-}, \mathbf{e}^{A}, \mathbf{f}_{A}, \boldsymbol{\omega}$.

6.2 Variation of the action

We vary

$$S=\int\left(\alpha\boldsymbol{\Omega}_{+}^{A}\bar{\eta}_{A}{}^{a}{}_{b}+\alpha\boldsymbol{\Omega}_{-}^{A}\eta_{A}{}^{a}{}_{b}+\beta\delta_{b}^{a}\boldsymbol{\Omega}+\gamma\mathbf{e}^{a}\wedge\mathbf{f}_{b}\right)\wedge\mathbf{f}_{e}\wedge\cdots\wedge\mathbf{f}_{f}\wedge\mathbf{e}^{c}\wedge\cdots\wedge\mathbf{e}^{d}e^{be\cdots f}{}_{ac\cdots d}$$

where the curvatures are

$$\begin{aligned}
\boldsymbol{\Omega}_{+}^{A} &= \mathbf{d}\boldsymbol{\omega}_{+}^{A}+\frac{1}{2}\varepsilon^{A}{}_{BC}\boldsymbol{\omega}_{+}^{B}\wedge\boldsymbol{\omega}_{+}^{C}-2\bar{\eta}^{Aa}{}_{b}\mathbf{f}_{a}\wedge\mathbf{e}^{b}\\
\boldsymbol{\Omega}_{-}^{A} &= \mathbf{d}\boldsymbol{\omega}_{-}^{A}+\frac{1}{2}\varepsilon^{A}{}_{BC}\boldsymbol{\omega}_{-}^{B}\wedge\boldsymbol{\omega}_{-}^{C}-2\eta^{Aa}{}_{b}\mathbf{f}_{a}\wedge\mathbf{e}^{b}\\
\mathbf{T}^{a} &= \mathbf{d}\mathbf{e}^{a}-\frac{1}{2}\bar{\eta}_{A}{}^{a}{}_{b}\boldsymbol{\omega}_{+}^{A}\wedge\mathbf{e}^{b}-\frac{1}{2}\eta_{A}{}^{a}{}_{b}\boldsymbol{\omega}_{-}^{A}\wedge\mathbf{e}^{b}-\boldsymbol{\omega}\wedge\mathbf{e}^{a}\\
\mathbf{S}_{a} &= \mathbf{d}\mathbf{f}_{a}-\frac{1}{2}\bar{\eta}_{Aa}{}^{b}\boldsymbol{\omega}_{+}^{A}\wedge\mathbf{f}_{b}-\frac{1}{2}\eta_{Aa}{}^{b}\boldsymbol{\omega}_{-}^{A}\wedge\mathbf{f}_{b}+\boldsymbol{\omega}\wedge\mathbf{f}_{a}\\
\boldsymbol{\Omega} &= \mathbf{d}\boldsymbol{\omega}-2\mathbf{e}^{a}\wedge\mathbf{f}_{a}
\end{aligned}$$

Upper case Latin indices refer to $SU(2)$, with $A,B,\ldots=1,2,3$, while lower case Latin indices refer to $SO(4)$ and run $a,b,\ldots=1,2,3,4$.

6.2.1 Solder form variation

Vary $\mathbf{e}^a$,

$$\begin{aligned}
\delta S &= \int \left(\alpha\delta_e \mathbf{\Omega}^A_+ \bar{\eta}_A{}^a{}_b + \alpha\delta_e \mathbf{\Omega}^A_- \eta_A{}^a{}_b\right) \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&\quad + \int \left(\beta\delta^a_b \delta_e \mathbf{\Omega} + \gamma\delta\mathbf{e}^a \wedge \mathbf{f}_b\right) \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&\quad + \int \left(\alpha \mathbf{\Omega}^A_+ \bar{\eta}_A{}^a{}_b + \alpha \mathbf{\Omega}^A_- \eta_A{}^a{}_b\right) \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge (n-1)\,\delta\mathbf{e}^c \wedge \mathbf{e}^g \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{acg\cdots d} \\
&\quad + \int \left(\beta\delta^a_b \mathbf{\Omega} + \gamma\mathbf{e}^a \wedge \mathbf{f}_b\right) \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge (n-1)\,\delta\mathbf{e}^c \wedge \mathbf{e}^g \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{acg\cdots d} \\
&= \int \delta\mathbf{e}^h \wedge \left(2\alpha\bar{\eta}^{Ag}{}_h \mathbf{f}_g \bar{\eta}_A{}^a{}_b\right) \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&\quad + \int \delta\mathbf{e}^h \wedge \left(2\alpha\eta^{Ag}{}_h \mathbf{f}_g \eta_A{}^a{}_b - 2\beta\delta^a_b \mathbf{f}_h + \gamma\delta^a_h \mathbf{f}_b\right) \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&\quad + (-1)^{n-1}(n-1) \int \delta\mathbf{e}^h \wedge \left(\alpha \mathbf{\Omega}^A_+ \bar{\eta}_A{}^a{}_b\right) \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^g \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ahg\cdots d} \\
&\quad + (-1)^{n-1}(n-1) \int \delta\mathbf{e}^h \wedge \left(\alpha \mathbf{\Omega}^A_- \eta_A{}^a{}_b\right) \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^g \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ahg\cdots d} \\
&\quad + (-1)^{n-1}(n-1) \int \delta\mathbf{e}^h \wedge \left(\beta\delta^a_b \mathbf{\Omega} + \gamma\mathbf{e}^a \wedge \mathbf{f}_b\right) \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^g \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ahg\cdots d}
\end{aligned}$$

Looking at just the variation of the Lagrangian density, and setting

$$\delta\mathbf{e}^h = A^h{}_k \mathbf{e}^k + B^{hk} \mathbf{f}_k$$

we find

$$
\begin{aligned}
\delta\mathcal{L} &= \left(A^h{}_k\mathbf{e}^k + B^{hk}\mathbf{f}_k\right)\wedge\left(2\alpha\bar{\eta}^{Ag}{}_h\bar{\eta}_A{}^a{}_b\right)\mathbf{f}_g\wedge\mathbf{f}_e\wedge\cdots\wedge\mathbf{f}_f\wedge\mathbf{e}^c\wedge\cdots\wedge\mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&\quad+\left(A^h{}_k\mathbf{e}^k + B^{hk}\mathbf{f}_k\right)\wedge\left(2\alpha\eta^{Ag}{}_h\eta_A{}^a{}_b\right)\mathbf{f}_g\wedge\mathbf{f}_e\wedge\cdots\wedge\mathbf{f}_f\wedge\mathbf{e}^c\wedge\cdots\wedge\mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&\quad-\left(A^h{}_k\mathbf{e}^k + B^{hk}\mathbf{f}_k\right)\wedge\left(2\beta\delta^a_b\delta^g_h+\gamma\delta^a_h\delta^g_b\right)\mathbf{f}_g\wedge\mathbf{f}_e\wedge\cdots\wedge\mathbf{f}_f\wedge\mathbf{e}^c\wedge\cdots\wedge\mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&\quad+(-1)^{n-1}(n-1)\left(A^h{}_k\mathbf{e}^k + B^{hk}\mathbf{f}_k\right)\wedge\left(\alpha\mathbf{\Omega}^A_+\bar{\eta}_A{}^a{}_b\right)\wedge\mathbf{f}_e\wedge\cdots\wedge\mathbf{f}_f\wedge\mathbf{e}^g\wedge\cdots\wedge\mathbf{e}^d e^{be\cdots f}{}_{ahg\cdots d} \\
&\quad+(-1)^{n-1}(n-1)\left(A^h{}_k\mathbf{e}^k + B^{hk}\mathbf{f}_k\right)\wedge\left(\alpha\mathbf{\Omega}^A_-\eta_A{}^a{}_b\right)\wedge\mathbf{f}_e\wedge\cdots\wedge\mathbf{f}_f\wedge\mathbf{e}^g\wedge\cdots\wedge\mathbf{e}^d e^{be\cdots f}{}_{ahg\cdots d} \\
&\quad+(-1)^{n-1}(n-1)\left(A^h{}_k\mathbf{e}^k + B^{hk}\mathbf{f}_k\right)\wedge\left(\beta\delta^a_b\mathbf{\Omega}+\gamma\mathbf{e}^a\wedge\mathbf{f}_b\right)\wedge\mathbf{f}_e\wedge\cdots\wedge\mathbf{f}_f\wedge\mathbf{e}^g\wedge\cdots\wedge\mathbf{e}^d e^{be\cdots f}{}_{ahg\cdots d} \\
&= (-1)^n A^h{}_k\left(2\alpha\bar{\eta}^{Ag}{}_h\bar{\eta}_A{}^a{}_b\right)\mathbf{f}_g\wedge\mathbf{f}_e\wedge\cdots\wedge\mathbf{f}_f\wedge\mathbf{e}^k\wedge\mathbf{e}^c\wedge\cdots\wedge\mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&\quad+(-1)^n A^h{}_k\left(2\alpha\eta^{Ag}{}_h\eta_A{}^a{}_b-2\beta\delta^a_b\delta^g_h+\gamma\delta^a_h\delta^g_b\right)\mathbf{f}_g\wedge\mathbf{f}_e\wedge\cdots\wedge\mathbf{f}_f\wedge\mathbf{e}^k\wedge\mathbf{e}^c\wedge\cdots\wedge\mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&\quad+(-1)^n(n-1)A^h{}_k\left(\alpha\Omega^{A\,m}_+{}_n\bar{\eta}_A{}^a{}_b\right)\mathbf{f}_m\wedge\mathbf{f}_e\wedge\cdots\wedge\mathbf{f}_f\wedge\mathbf{e}^k\wedge\mathbf{e}^n\wedge\mathbf{e}^g\wedge\cdots\wedge\mathbf{e}^d e^{be\cdots f}{}_{ahg\cdots d} \\
&\quad+(-1)^n(n-1)A^h{}_k\left(\alpha\Omega^{A\,m}_-{}_n\eta_A{}^a{}_b\right)\mathbf{f}_m\wedge\mathbf{f}_e\wedge\cdots\wedge\mathbf{f}_f\wedge\mathbf{e}^k\wedge\mathbf{e}^n\wedge\mathbf{e}^g\wedge\cdots\wedge\mathbf{e}^d e^{be\cdots f}{}_{ahg\cdots d} \\
&\quad+(-1)^n(n-1)A^h{}_k\left(\beta\delta^a_b\Omega^m{}_n-\gamma\delta^a_n\delta^m_b\right)\mathbf{f}_m\wedge\mathbf{f}_e\wedge\cdots\wedge\mathbf{f}_f\wedge\mathbf{e}^k\wedge\mathbf{e}^n\wedge\mathbf{e}^g\wedge\cdots\wedge\mathbf{e}^d e^{be\cdots f}{}_{ahg\cdots d} \\
&\quad+\frac{1}{2}(-1)^{n-1}(n-1)B^{hk}\left(\alpha\Omega^A_{+\,mn}\bar{\eta}_A{}^a{}_b\right)\wedge\mathbf{f}_k\wedge\mathbf{f}_e\wedge\cdots\wedge\mathbf{f}_f\wedge\mathbf{e}^m\wedge\mathbf{e}^n\wedge\mathbf{e}^g\wedge\cdots\wedge\mathbf{e}^d e^{be\cdots f}{}_{ahg\cdots d} \\
&\quad+\frac{1}{2}(-1)^{n-1}(n-1)B^{hk}\left(\alpha\Omega^A_{-\,mn}\eta_A{}^a{}_b\right)\wedge\mathbf{f}_k\wedge\mathbf{f}_e\wedge\cdots\wedge\mathbf{f}_f\wedge\mathbf{e}^m\wedge\mathbf{e}^n\wedge\mathbf{e}^g\wedge\cdots\wedge\mathbf{e}^d e^{be\cdots f}{}_{ahg\cdots d} \\
&\quad+\frac{1}{2}(-1)^{n-1}(n-1)B^{hk}\left(\beta\delta^a_b\Omega_{mn}\right)\wedge\mathbf{f}_k\wedge\mathbf{f}_e\wedge\cdots\wedge\mathbf{f}_f\wedge\mathbf{e}^m\wedge\mathbf{e}^n\wedge\mathbf{e}^g\wedge\cdots\wedge\mathbf{e}^d e^{be\cdots f}{}_{ahg\cdots d}
\end{aligned}
$$

Now replace the volume forms with

$$
\begin{aligned}
\mathbf{f}_{c\cdots d}\wedge\mathbf{e}^{e\cdots f} &= \frac{1}{\sqrt{K}}\varepsilon^{e\cdots f}{}_{c\cdots d}\mathbf{\Phi} \\
&= \bar{e}_{c\cdots d}{}^{e\cdots f}\mathbf{\Phi}
\end{aligned}
$$

to give

$$\begin{aligned}
\delta\mathcal{L} &= (-1)^n A^h{}_k \left(2\alpha\bar{\eta}^{Ag}{}_h\bar{\eta}_A{}^a{}_b + 2\alpha\eta^{Ag}{}_h\eta_A{}^a{}_b - 2\beta\delta^a_b\delta^g_h + \gamma\delta^a_h\delta^g_b\right)\bar{e}_{ge\cdots f}{}^{kc\cdots d}e^{be\cdots f}{}_{ac\cdots d}\mathbf{\Phi} \\
&+ (-1)^n (n-1) A^h{}_k \left(\alpha\Omega_+^{A\,m}{}_n\bar{\eta}_A{}^a{}_b + \alpha\Omega_-^{A\,m}{}_n\eta_A{}^a{}_b\right)\bar{e}_{me\cdots f}{}^{kng\cdots d}e^{be\cdots f}{}_{ahg\cdots d}\mathbf{\Phi} \\
&+ (-1)^n (n-1) A^h{}_k \left(\beta\delta^a_b\Omega^m{}_n - \gamma\delta^a_n\delta^m_b\right)\bar{e}_{me\cdots f}{}^{kng\cdots d}e^{be\cdots f}{}_{ahg\cdots d}\mathbf{\Phi} \\
&+\frac{1}{2}(-1)^{n-1}(n-1) B^{hk}\left(\alpha\Omega^A_{+\,mn}\bar{\eta}_A{}^a{}_b + \alpha\Omega^A_{-\,mn}\eta_A{}^a{}_b + \beta\delta^a_b\Omega_{mn}\right)\bar{e}_{ke\cdots f}{}^{mng\cdots d}e^{be\cdots f}{}_{ahg\cdots d}\mathbf{\Phi} \\
&= (n-1)!\,(n-1)!\,(-1)^n A^h{}_k \left(2\alpha\bar{\eta}^{Ag}{}_h\bar{\eta}_A{}^a{}_b + 2\alpha\eta^{Ag}{}_h\eta_A{}^a{}_b - 2\beta\delta^a_b\delta^g_h + \gamma\delta^a_h\delta^g_b\right)\delta^k_a\delta^b_g\mathbf{\Phi} \\
&+ (-1)^n (n-1)!\,(n-2)!\,(n-1) A^h{}_k \left(\alpha\Omega_+^{A\,m}{}_n\bar{\eta}_A{}^a{}_b + \alpha\Omega_-^{A\,m}{}_n\eta_A{}^a{}_b\right)\delta^b_m\left(\delta^k_a\delta^n_h - \delta^n_a\delta^k_h\right)\mathbf{\Phi} \\
&+ (-1)^n (n-1)!\,(n-2)!\,(n-1) A^h{}_k \left(\beta\delta^a_b\Omega^m{}_n - \gamma\delta^a_n\delta^m_b\right)\delta^b_m\left(\delta^k_a\delta^n_h - \delta^n_a\delta^k_h\right)\mathbf{\Phi} \\
&+\frac{1}{2}(-1)^{n-1}(n-1)!\,(n-2)!\,(n-1) B^{hk}\left(\alpha\Omega^A_{+\,mn}\bar{\eta}_A{}^a{}_b\right)\delta^b_k\left(\delta^m_a\delta^n_h - \delta^n_a\delta^m_h\right)\mathbf{\Phi} \\
&+\frac{1}{2}(-1)^{n-1}(n-1)!\,(n-2)!\,(n-1) B^{hk}\left(\alpha\Omega^A_{-\,mn}\eta_A{}^a{}_b + \beta\delta^a_b\Omega_{mn}\right)\delta^b_k\left(\delta^m_a\delta^n_h - \delta^n_a\delta^m_h\right)\mathbf{\Phi} \\
&= (n-1)!\,(n-1)!\,(-1)^n [A^h{}_a \left(2\alpha\bar{\eta}^{Ab}{}_h\bar{\eta}_A{}^a{}_b + 2\alpha\eta^{Ab}{}_h\eta_A{}^a{}_b - 2\beta\delta^a_b\delta^b_h + n\gamma\delta^a_h\right) \\
&+A^n{}_a\left(\alpha\Omega_+^{A\,b}{}_n\bar{\eta}_A{}^a{}_b + \alpha\Omega_-^{A\,b}{}_n\eta_A{}^a{}_b + \beta\Omega^a{}_n\right) \\
&-A^n{}_a\left(\delta^a_n\left(\alpha\Omega_+^{A\,b}{}_c\bar{\eta}_A{}^c{}_b + \alpha\Omega_-^{A\,b}{}_c\eta_A{}^c{}_b + \beta\Omega^c{}_c - n(n-1)\gamma\right)\right) \\
&-B^{hb}\left(\alpha\Omega^A_{+\,ah}\bar{\eta}_A{}^a{}_b + \alpha\Omega^A_{-\,ah}\eta_A{}^a{}_b + \beta\delta^a_b\Omega_{ah}\right)]\mathbf{\Phi}
\end{aligned}$$

Collecting, we use the identity (see Appendix)

$$\begin{aligned}
\eta^{Aab}\eta_{Acd} + \bar{\eta}^{Aab}\bar{\eta}_{Acd} &= 2\left(\delta^a_c\delta^b_d - \delta^b_c\delta^a_d\right) \\
\eta^{Aa}{}_b\eta_A{}^c{}_d + \bar{\eta}^{Aa}{}_b\bar{\eta}_A{}^c{}_d &= 2\left(\delta^{ac}\delta_{bd} - \delta^c_b\delta^a_d\right)
\end{aligned}$$

and definitions

$$\begin{aligned}
\Omega_+^{A\,c}{}_b\bar{\eta}_A{}^a{}_c &= \Omega^a_{+\,c}{}^c{}_b \\
\Omega_-^{A\,c}{}_b\bar{\eta}_A{}^a{}_c &= \Omega^a_{-\,c}{}^c{}_b
\end{aligned}$$

Then

$$
\begin{aligned}
\frac{(-1)^n}{(n-1)!\,(n-1)!}\delta\mathcal{L} &= [A^h{}_a\left(2\alpha\left(2\left(\delta^{ba}\delta_{hb}-\delta^a_h\delta^b_b\right)\right)-2\beta\delta^a_b\delta^b_h+n\gamma\delta^a_h\right)\\
&\quad +A^n{}_a\left(\alpha\Omega^a_{+\ b}{}^b{}_n+\alpha\Omega^a_{-\ b}{}^b{}_n+\beta\Omega^a{}_n\right)\\
&\quad -A^n{}_a\left(\delta^a_n\left(\alpha\Omega^{A\ b}_{+\ \ c}\bar{\eta}_A{}^c{}_b+\alpha\Omega^{A\ b}_{-\ \ c}\eta_A{}^c{}_b\right)\right)\\
&\quad -A^n{}_a\left(\delta^a_n\left(\beta\Omega^c{}_c-n\left(n-1\right)\gamma\right)\right)\\
&\quad -B^{hb}\left(\alpha\Omega^a_{+\ bah}+\alpha\Omega^a_{-\ bah}+\beta\delta^a_b\Omega_{ah}\right)]\mathbf{\Phi}\\
&= A^b{}_a\left(\alpha\Omega^a_{+\ c}{}^c{}_b+\alpha\Omega^a_{-\ c}{}^c{}_b+\beta\Omega^a{}_b\right)\mathbf{\Phi}\\
&\quad -A^b{}_a\left(\delta^a_b\left(\alpha\Omega^d_{+\ c}{}^c{}_d+\alpha\Omega^d_{-\ c}{}^c{}_d\right)\right)\mathbf{\Phi}\\
&\quad -A^b{}_a\left(\delta^a_b\left(\beta\Omega^c{}_c+4\left(n-1\right)\alpha+2\beta-n^2\gamma\right)\right)\mathbf{\Phi}\\
&\quad -B^{hb}\left(\alpha\Omega^a_{+\ bah}+\alpha\Omega^a_{-\ bah}+\beta\delta^a_b\Omega_{ah}\right)\mathbf{\Phi}
\end{aligned}
$$

$$
\begin{aligned}
\frac{(-1)^n}{(n-1)!\,(n-1)!}\delta\mathcal{L} &= A^b{}_a\left(\alpha\Omega^a_{+\ c}{}^c{}_b+\alpha\Omega^a_{-\ c}{}^c{}_b+\beta\Omega^a{}_b\right)\mathbf{\Phi}\\
&\quad -A^b{}_a\left(\delta^a_b\left(\alpha\Omega^d_{+\ c}{}^c{}_d+\alpha\Omega^d_{-\ c}{}^c{}_d\right)\right)\mathbf{\Phi}\\
&\quad -A^b{}_a\left(\delta^a_b\left(\beta\Omega^c{}_c+4\left(n-1\right)\alpha+2\beta-n^2\gamma\right)\right)\mathbf{\Phi}\\
&\quad -B^{ab}\left(\alpha\Omega^c_{+\ bca}+\alpha\Omega^c_{-\ bca}+\beta\Omega_{ba}\right)\mathbf{\Phi}
\end{aligned}
$$

6.2.2 Co-solder form variation

Now vary the co-solder form,

$$\begin{aligned}
\delta_f S &= \int \left(\alpha\delta_f \mathbf{\Omega}_+^A \bar{\eta}_A{}^a{}_b + \alpha\delta_f \mathbf{\Omega}_-^A \eta_A{}^a{}_b\right) \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&+ \int \left(\beta\delta^a_b \delta_f \mathbf{\Omega} + \gamma \mathbf{e}^a \wedge \delta_f \mathbf{f}_b\right) \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&+ \int \left(\alpha \mathbf{\Omega}_+^A \bar{\eta}_A{}^a{}_b\right) \wedge (n-1)\, \delta\mathbf{f}_e \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{beg\cdots f}{}_{ac\cdots d} \\
&+ \int \left(\alpha \mathbf{\Omega}_-^A \eta_A{}^a{}_b\right) \wedge (n-1)\, \delta\mathbf{f}_e \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{beg\cdots f}{}_{ac\cdots d} \\
&+ \int \left(\beta\delta^a_b \mathbf{\Omega} + \gamma \mathbf{e}^a \wedge \mathbf{f}_b\right) \wedge (n-1)\, \delta\mathbf{f}_e \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{beg\cdots f}{}_{ac\cdots d} \\
&= \int \left(\alpha\left(-2\bar{\eta}^{Am}{}_n \delta\mathbf{f}_m \wedge \mathbf{e}^n\right) \bar{\eta}_A{}^a{}_b\right) \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&+ \int \left(\alpha\left(-2\eta^{Am}{}_n \delta\mathbf{f}_m \wedge \mathbf{e}^n\right) \eta_A{}^a{}_b\right) \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&- \int \left(-2\beta\delta^a_b \mathbf{e}^m \wedge \delta\mathbf{f}_m + \gamma\delta^m_b \mathbf{e}^a \wedge \delta\mathbf{f}_m\right) \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&+ \int \left(\alpha \mathbf{\Omega}_+^A \bar{\eta}_A{}^a{}_b\right) \wedge (n-1)\, \delta\mathbf{f}_e \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{beg\cdots f}{}_{ac\cdots d} \\
&+ \int \left(\alpha \mathbf{\Omega}_-^A \eta_A{}^a{}_b\right) \wedge (n-1)\, \delta\mathbf{f}_e \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{beg\cdots f}{}_{ac\cdots d} \\
&+ \int \left(\beta\delta^a_b \mathbf{\Omega} + \gamma \mathbf{e}^a \wedge \mathbf{f}_b\right) \wedge (n-1)\, \delta\mathbf{f}_e \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{beg\cdots f}{}_{ac\cdots d} \\
\\
&= \int \delta\mathbf{f}_m \wedge \left(-2\alpha\bar{\eta}^{Am}{}_n \bar{\eta}_A{}^a{}_b\right) \mathbf{e}^n \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&+ \int \delta\mathbf{f}_m \wedge \left(-2\alpha\eta^{Am}{}_n \eta_A{}^a{}_b\right) \mathbf{e}^n \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&+ \int \delta\mathbf{f}_m \wedge \left(2\beta\delta^a_b\delta^m_n - \gamma\delta^m_b\delta^a_n\right) \mathbf{e}^n \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&+ \int (n-1)\, \delta\mathbf{f}_m \wedge \left(\alpha \mathbf{\Omega}_+^A \bar{\eta}_A{}^a{}_b\right) \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{bmg\cdots f}{}_{ac\cdots d} \\
&+ \int (n-1)\, \delta\mathbf{f}_m \wedge \left(\alpha \mathbf{\Omega}_-^A \eta_A{}^a{}_b\right) \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{bmg\cdots f}{}_{ac\cdots d} \\
&+ \int (n-1)\, \delta\mathbf{f}_m \wedge \left(\beta\delta^a_b \mathbf{\Omega} + \gamma \mathbf{e}^a \wedge \mathbf{f}_b\right) \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{bmg\cdots f}{}_{ac\cdots d}
\end{aligned}$$

Let

$$\delta\mathbf{f}_m = C_{mk}\mathbf{e}^k + D_m{}^k \mathbf{f}_k$$

Then

$$\begin{aligned}
\delta_f \mathcal{L} \;=\; & (-1)^{n-1} D_m{}^k \left(-2\alpha \bar{\eta}^{Am}{}_n \bar{\eta}_A{}^a{}_b\right) \mathbf{f}_k \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^n \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
& + (-1)^{n-1} D_m{}^k \left(-2\alpha \eta^{Am}{}_n \eta_A{}^a{}_b\right) \mathbf{f}_k \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^n \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
& + (-1)^{n-1} D_m{}^k \left(2\beta \delta^a_b \delta^m_n\right) \mathbf{f}_k \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^n \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
& + (-1)^{n-1} D_m{}^k \left(-\gamma \delta^m_b \delta^a_n\right) \mathbf{f}_k \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^n \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
& + \frac{1}{2} (n-1) (-1)^n C_{mk} \left(\alpha \Omega_+^{Ahi} \bar{\eta}_A{}^a{}_b\right) \mathbf{f}_h \wedge \mathbf{f}_i \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^k \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{bmg\cdots f}{}_{ac\cdots d} \\
& + \frac{1}{2} (n-1) (-1)^n C_{mk} \left(\alpha \Omega_-^{Ahi} \eta_A{}^a{}_b\right) \mathbf{f}_h \wedge \mathbf{f}_i \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^k \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{bmg\cdots f}{}_{ac\cdots d} \\
& + \frac{1}{2} (n-1) (-1)^n C_{mk} \left(\beta \delta_f \Omega^{hi}\right) \mathbf{f}_h \wedge \mathbf{f}_i \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^k \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{bmg\cdots f}{}_{ac\cdots d} \\
& + (n-1) (-1)^n D_m{}^k \left(\alpha \Omega_+^{Ah}{}_i \bar{\eta}_A{}^a{}_b\right) \mathbf{f}_k \wedge \mathbf{f}_h \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^i \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{bmg\cdots f}{}_{ac\cdots d} \\
& + (n-1) (-1)^n D_m{}^k \left(\alpha \Omega_-^{Ah}{}_i \eta_A{}^a{}_b\right) \mathbf{f}_k \wedge \mathbf{f}_h \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^i \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{bmg\cdots f}{}_{ac\cdots d} \\
& + (n-1) (-1)^n D_m{}^k \left(\beta \delta^a_b \Omega^h{}_i\right) \mathbf{f}_k \wedge \mathbf{f}_h \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^i \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{bmg\cdots f}{}_{ac\cdots d} \\
& + (n-1) (-1)^n D_m{}^k \left(-\gamma \delta^h_b \delta^a_i\right) \mathbf{f}_k \wedge \mathbf{f}_h \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^i \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{bmg\cdots f}{}_{ac\cdots d} \\
& + (n-1) (-1)^n D_m{}^k \left(\alpha \Omega_+^{Ah}{}_i \bar{\eta}_A{}^a{}_b\right) \mathbf{f}_k \wedge \mathbf{f}_h \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^i \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{bmg\cdots f}{}_{ac\cdots d} \\
& + (n-1) (-1)^n D_m{}^k \left(\alpha \Omega_-^{Ah}{}_i \eta_A{}^a{}_b\right) \mathbf{f}_k \wedge \mathbf{f}_h \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^i \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{bmg\cdots f}{}_{ac\cdots d} \\
& + (n-1) (-1)^n D_m{}^k \left(\beta \delta^a_b \Omega^h{}_i\right) \mathbf{f}_k \wedge \mathbf{f}_h \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^i \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{bmg\cdots f}{}_{ac\cdots d} \\
& + (n-1) (-1)^n D_m{}^k \left(-\gamma \delta^h_b \delta^a_i\right) \mathbf{f}_k \wedge \mathbf{f}_h \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^i \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{bmg\cdots f}{}_{ac\cdots d}
\end{aligned}$$

Now

$$\mathbf{f}_{c\cdots d} \wedge \mathbf{e}^{e\cdots f} \;=\; \bar{e}_{c\cdots d}{}^{e\cdots f} \mathbf{\Phi}$$

so

$$\begin{aligned}
\delta_f \mathcal{L} &= (-1)^{n-1} D_m{}^k \left(-2\alpha \bar{\eta}^{Am}{}_n \bar{\eta}_A{}^a{}_b\right) \bar{e}_{ke\cdots f}{}^{nc\cdots d} e^{be\cdots f}{}_{ac\cdots d} \mathbf{\Phi} \\
&+ (-1)^{n-1} D_m{}^k \left(-2\alpha \eta^{Am}{}_n \eta_A{}^a{}_b\right) \bar{e}_{ke\cdots f}{}^{nc\cdots d} e^{be\cdots f}{}_{ac\cdots d} \mathbf{\Phi} \\
&+ (-1)^{n-1} D_m{}^k \left(2\beta \delta^a_b \delta^m_n - \gamma \delta^m_b \delta^a_n\right) \bar{e}_{ke\cdots f}{}^{nc\cdots d} e^{be\cdots f}{}_{ac\cdots d} \mathbf{\Phi} \\
&+ \frac{1}{2}(n-1)(-1)^n C_{mk} \left(\alpha \Omega_+^{Ahi} \bar{\eta}_A{}^a{}_b\right) \bar{e}_{hig\cdots f}{}^{kc\cdots d} e^{bmg\cdots f}{}_{ac\cdots d} \mathbf{\Phi} \\
&+ \frac{1}{2}(n-1)(-1)^n C_{mk} \left(\alpha \Omega_-^{Ahi} \eta_A{}^a{}_b + \beta \delta_f \Omega^{hi}\right) \bar{e}_{hig\cdots f}{}^{kc\cdots d} e^{bmg\cdots f}{}_{ac\cdots d} \mathbf{\Phi} \\
&+ (n-1)(-1)^n D_m{}^k \left(\alpha \Omega_+^{Ah}{}_i \bar{\eta}_A{}^a{}_b\right) \bar{e}_{khg\cdots f}{}^{ic\cdots d} e^{bmg\cdots f}{}_{ac\cdots d} \mathbf{\Phi} \\
&+ (n-1)(-1)^n D_m{}^k \left(\alpha \Omega_-^{Ah}{}_i \eta_A{}^a{}_b\right) \bar{e}_{khg\cdots f}{}^{ic\cdots d} e^{bmg\cdots f}{}_{ac\cdots d} \mathbf{\Phi} \\
&+ (n-1)(-1)^n D_m{}^k \left(\beta \delta^a_b \Omega^h{}_i - \gamma \delta^h_b \delta^a_i\right) \bar{e}_{khg\cdots f}{}^{ic\cdots d} e^{bmg\cdots f}{}_{ac\cdots d} \mathbf{\Phi} \\
&= (n-1)!\,(n-2)!\,(n-1)(-1)^n D_m{}^k \left(\alpha \Omega_+^{Ah}{}_i \bar{\eta}_A{}^a{}_b\right) \delta^i_a \left(\delta^b_k \delta^m_h - \delta^m_k \delta^b_h\right) \mathbf{\Phi} \\
&+ (n-1)!\,(n-2)!\,(n-1)(-1)^n D_m{}^k \left(\alpha \Omega_-^{Ah}{}_i \eta_A{}^a{}_b\right) \delta^i_a \left(\delta^b_k \delta^m_h - \delta^m_k \delta^b_h\right) \mathbf{\Phi} \\
&+ (n-1)!\,(n-2)!\,(n-1)(-1)^n D_m{}^k \left(\beta \Omega^h{}_i - \gamma \delta^h_b \delta^a_i\right) \delta^i_a \left(\delta^b_k \delta^m_h - \delta^m_k \delta^b_h\right) \mathbf{\Phi} \\
&+ (-1)^{n-1}(n-1)!\,(n-1)! D_m{}^k \left(-2\alpha \bar{\eta}^{Am}{}_n \bar{\eta}_A{}^a{}_b - 2\alpha \eta^{Am}{}_n \eta_A{}^a{}_b\right) \delta^b_k \delta^n_a \mathbf{\Phi} \\
&+ (-1)^{n-1}(n-1)!\,(n-1)! D_m{}^k \left(2\beta \delta^a_b \delta^m_n - \gamma \delta^m_b \delta^a_n\right) \delta^b_k \delta^n_a \mathbf{\Phi} \\
&+ \frac{1}{2}(n-1)!\,(n-2)!\,(n-1)(-1)^n C_{mk} \left(\alpha \Omega_+^{Ahi} \bar{\eta}_A{}^a{}_b\right) \delta^k_a \left(\delta^b_h \delta^m_i - \delta^m_h \delta^b_i\right) \mathbf{\Phi} \\
&+ \frac{1}{2}(n-1)!\,(n-2)!\,(n-1)(-1)^n C_{mk} \left(\alpha \Omega_-^{Ahi} \eta_A{}^a{}_b\right) \delta^k_a \left(\delta^b_h \delta^m_i - \delta^m_h \delta^b_i\right) \mathbf{\Phi} \\
&+ \frac{1}{2}(n-1)!\,(n-2)!\,(n-1)(-1)^n C_{mk} \left(\beta \delta^a_b \Omega^{hi}\right) \delta^k_a \left(\delta^b_h \delta^m_i - \delta^m_h \delta^b_i\right) \mathbf{\Phi}
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\frac{(-1)^n}{(n-1)!\,(n-1)!}\delta_f \mathcal{L} &= D_m{}^k \left(\alpha \Omega_+^{Ah}{}_a \bar{\eta}_A{}^a{}_b + \alpha \Omega_-^{Ah}{}_a \eta_A{}^a{}_b + \beta \Omega^h{}_a - n\gamma \delta_b^h\right)\left(\delta_k^b \delta_h^m - \delta_k^m \delta_h^b\right)\mathbf{\Phi} \\
&\quad - D_m{}^k \left(-2\alpha \bar{\eta}^{Am}{}_a \bar{\eta}_A{}^a{}_k - 2\alpha \eta^{Am}{}_a \eta_A{}^a{}_k + 2\beta \delta_k^m - n\gamma \delta_k^m\right)\mathbf{\Phi} \\
&\quad + \frac{1}{2} C_{ma} \left(\alpha \Omega_+^{Ahi} \bar{\eta}_A{}^a{}_b + \alpha \Omega_-^{Ahi} \eta_A{}^a{}_b + \beta \delta_b^a \Omega^{hi}\right)\left(\delta_h^b \delta_i^m - \delta_h^m \delta_i^b\right)\mathbf{\Phi} \\
&= D_m{}^k \left(\alpha \left(\Omega_+^a{}_k{}^m{}_a - \delta_k^m \Omega_+^a{}_b{}^b{}_a\right) + \alpha \left(\Omega_-^a{}_k{}^m{}_a - \delta_k^m \Omega_-^a{}_b{}^b{}_a\right)\right)\mathbf{\Phi} \\
&\quad + D_m{}^k \left(\beta \left(\Omega^m{}_k - \Omega^a{}_a \delta_k^m\right) + n(n-1)\gamma \delta_k^m\right)\mathbf{\Phi} \\
&\quad - D_m{}^k \left(-4\alpha \left(\delta^{ac}\delta_{bd} - \delta_b^c \delta_d^a\right) + 2\beta \delta_k^m - n\gamma \delta_k^m\right)\mathbf{\Phi} \\
&\quad + C_{ma} \left(\alpha \Omega_+^a{}_b{}^{bm} + \alpha \Omega_+^a{}_b{}^{bm} + \beta \Omega^{bm}\right)\mathbf{\Phi}
\end{aligned}
$$

where again

$$
\begin{aligned}
\eta^{Aab}\eta_{Acd} + \bar{\eta}^{Aab}\bar{\eta}_{Acd} &= 2\left(\delta_c^a \delta_d^b - \delta_c^b \delta_d^a\right) \\
\eta^{Am}{}_a \eta_A{}^a{}_k + \bar{\eta}^{Am}{}_a \bar{\eta}_A{}^a{}_k &= -2(n-1)\delta_k^m
\end{aligned}
$$

so that finally

$$
\begin{aligned}
\frac{(-1)^n}{(n-1)!\,(n-1)!}\delta_f \mathcal{L} &= D_m{}^k \left(\alpha \left(\Omega_+^a{}_k{}^m{}_a - \delta_k^m \Omega_+^a{}_b{}^b{}_a\right) + \alpha \left(\Omega_-^a{}_k{}^m{}_a - \delta_k^m \Omega_-^a{}_b{}^b{}_a\right)\right)\mathbf{\Phi} \\
&\quad + D_m{}^k \left(\beta \left(\Omega^m{}_k - \Omega^a{}_a \delta_k^m\right) + \delta_k^m \left(n^2\gamma - 4\alpha(n-1) - 2\beta\right)\right)\mathbf{\Phi} \\
&\quad + C_{ma} \left(\alpha \Omega_+^a{}_b{}^{bm} + \alpha \Omega_+^a{}_b{}^{bm} + \beta \Omega^{bm}\right)\mathbf{\Phi}
\end{aligned}
$$

6.2.3 Left spin connection

Varying ω_+^A,

$$\begin{aligned}
\delta S &= \int \alpha \delta \mathbf{\Omega}_+^A \bar{\eta}_A{}^a{}_b \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&= \int \alpha \left(\mathbf{d}\delta\boldsymbol{\omega}_+^A + \frac{1}{2}\varepsilon^A{}_{BC}\delta\boldsymbol{\omega}_+^B \wedge \boldsymbol{\omega}_+^C \right) \bar{\eta}_A{}^a{}_b \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&\quad + \int \alpha \left(\frac{1}{2}\varepsilon^A{}_{BC}\boldsymbol{\omega}_+^B \wedge \delta\boldsymbol{\omega}_+^C \right) \bar{\eta}_A{}^a{}_b \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&= \int \alpha \left(\mathbf{d}\delta\boldsymbol{\omega}_+^A + \delta\boldsymbol{\omega}_+^B \wedge \left(\varepsilon^A{}_{BC}\boldsymbol{\omega}_+^C\right)\right) \bar{\eta}_A{}^a{}_b \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&= \int \alpha \left(\mathbf{D}_+ \delta\boldsymbol{\omega}_+^A \right) \bar{\eta}_A{}^a{}_b \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&= \int \alpha \mathbf{D}_+ \left(\delta\boldsymbol{\omega}_+^A \bar{\eta}_A{}^a{}_b \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d \right) e^{be\cdots f}{}_{ac\cdots d} \\
&= \int \alpha \delta\boldsymbol{\omega}_+^A \bar{\eta}_A{}^a{}_b \wedge \mathbf{D}_+ \left(\mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d \right) e^{be\cdots f}{}_{ac\cdots d} \\
&= (n-1) \int \alpha \delta\boldsymbol{\omega}_+^A \bar{\eta}_A{}^a{}_b \wedge \left(\left((\mathbf{D}_+ \mathbf{f}_e) \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \mathbf{e}^h \wedge \cdots \wedge \mathbf{e}^d \right) \right) e^{beg\cdots f}{}_{ach\cdots d} \\
&\quad + (n-1) \int \alpha \delta\boldsymbol{\omega}_+^A \bar{\eta}_A{}^a{}_b \wedge (-1) \left(\mathbf{f}_e \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{D}_+ \mathbf{e}^c \wedge \mathbf{e}^h \wedge \cdots \wedge \mathbf{e}^d \right) e^{beg\cdots f}{}_{ach\cdots d} \\
&= (n-1) \int \alpha \delta\boldsymbol{\omega}_+^A \bar{\eta}_A{}^a{}_b \wedge \left(\mathbf{S}_e^+ \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \mathbf{e}^h \wedge \cdots \wedge \mathbf{e}^d \right) e^{beg\cdots f}{}_{ach\cdots d} \\
&\quad + (n-1) \int \alpha \delta\boldsymbol{\omega}_+^A \bar{\eta}_A{}^a{}_b \wedge \left((-1)^{n-1} \left(\mathbf{f}_e \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{T}_+^c \wedge \mathbf{e}^h \wedge \cdots \wedge \mathbf{e}^d \right) \right) e^{beg\cdots f}{}_{ach\cdots d} \\
&= (n-1) \int \alpha \delta\boldsymbol{\omega}_+^A \bar{\eta}_A{}^a{}_b \wedge \left(\mathbf{S}_e^+ \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \mathbf{e}^h \wedge \cdots \wedge \mathbf{e}^d \right) e^{beg\cdots f}{}_{ach\cdots d} \\
&\quad + (n-1) \int \alpha \delta\boldsymbol{\omega}_+^A \bar{\eta}_A{}^a{}_b \wedge \left((-1)^{n-1} \left(\mathbf{f}_e \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{T}_+^c \wedge \mathbf{e}^h \wedge \cdots \wedge \mathbf{e}^d \right) \right) e^{beg\cdots f}{}_{ach\cdots d}
\end{aligned}$$

where we know (and have shown elsewhere) that the covariant integration by parts must yield tensors. Extracting the variation of the Lagrangian and expanding the connection as

$$\delta\boldsymbol{\omega}_+^A = A^{Am}\mathbf{e}^m + B^{Am}\mathbf{f}_m$$

then the torsion and co-torsion,

$$\begin{aligned}
\delta S &= (n-1)\int \alpha\delta\boldsymbol{\omega}_+^A \bar{\eta}_A{}^a{}_b \wedge \left(\mathbf{S}_e^+ \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \mathbf{e}^h \wedge \cdots \wedge \mathbf{e}^d\right) e^{beg\cdots f}{}_{ach\cdots d} \\
&\quad + (n-1)\int \alpha\delta\boldsymbol{\omega}_+^A \bar{\eta}_A{}^a{}_b \wedge \left((-1)^{n-1}\left(\mathbf{f}_e \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{T}_+^c \wedge \mathbf{e}^h \wedge \cdots \wedge \mathbf{e}^d\right)\right) e^{beg\cdots f}{}_{ach\cdots d} \\
&= (n-1)\int \alpha A^{Am}\bar{\eta}_A{}^a{}_b \mathbf{e}^m \wedge \left(\mathbf{S}_e^+ \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \mathbf{e}^h \wedge \cdots \wedge \mathbf{e}^d\right) e^{beg\cdots f}{}_{ach\cdots d} \\
&\quad + (n-1)\int \alpha A^{Am}\bar{\eta}_A{}^a{}_b \mathbf{e}^m \wedge \left((-1)^{n-1}\left(\mathbf{f}_e \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{T}_+^c \wedge \mathbf{e}^h \wedge \cdots \wedge \mathbf{e}^d\right)\right) e^{beg\cdots f}{}_{ach\cdots d} \\
&\quad + (n-1)\int \alpha B^{Am}\mathbf{f}_m \bar{\eta}_A{}^a{}_b \wedge \left(\mathbf{S}_e^+ \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \mathbf{e}^h \wedge \cdots \wedge \mathbf{e}^d\right) e^{beg\cdots f}{}_{ach\cdots d} \\
&\quad + (n-1)\int \alpha B^{Am}\mathbf{f}_m \bar{\eta}_A{}^a{}_b \wedge (-1)\left(\mathbf{f}_e \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{T}_+^c \wedge \mathbf{e}^h \wedge \cdots \wedge \mathbf{e}^d\right) e^{beg\cdots f}{}_{ach\cdots d} \\
&= (n-1)\int \alpha A^{Am}\bar{\eta}_A{}^a{}_b \mathbf{e}^m \wedge \left(\frac{1}{2}S_e^{+\ ij} \wedge \mathbf{f}_i \wedge \mathbf{f}_j \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \mathbf{e}^h \wedge \cdots \wedge \mathbf{e}^d\right) e^{beg\cdots f}{}_{ach\cdots d} \\
&\quad + (n-1)\int \alpha A^{Am}\bar{\eta}_A{}^a{}_b \mathbf{e}^m \wedge (-1)\left(\mathbf{f}_e \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge T_+^{ci}{}_j \mathbf{f}_i \wedge \mathbf{e}^j \wedge \mathbf{e}^h \wedge \cdots \wedge \mathbf{e}^d\right) e^{beg\cdots f}{}_{ach\cdots d} \\
&\quad + (n-1)\int \alpha B^{Am}\mathbf{f}_m \bar{\eta}_A{}^a{}_b \wedge \left(S_e^{+\,i}{}_j \mathbf{f}_i \wedge \mathbf{e}^j \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \mathbf{e}^h \wedge \cdots \wedge \mathbf{e}^d\right) e^{beg\cdots f}{}_{ach\cdots d} \\
&\quad + (n-1)\int \alpha B^{Am}\mathbf{f}_m \bar{\eta}_A{}^a{}_b \wedge (-1)\left(\mathbf{f}_e \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \frac{1}{2}T_{+\,ij}^c \mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^h \wedge \cdots \wedge \mathbf{e}^d\right) e^{beg\cdots f}{}_{ach\cdots d} \\
&= (n-1)\int \frac{\alpha}{2} A^{Am}\bar{\eta}_A{}^a{}_b (-1)^n S_e^{+\ ij} \wedge \mathbf{f}_i \wedge \mathbf{f}_j \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^m \wedge \mathbf{e}^c \wedge \mathbf{e}^h \wedge \cdots \wedge \mathbf{e}^d e^{beg\cdots f}{}_{ach\cdots d} \\
&\quad + (n-1)\int \alpha A^{Am}\bar{\eta}_A{}^a{}_b (-1) T_+^{ci}{}_j \left(\mathbf{f}_i \wedge \mathbf{f}_e \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^j \wedge \mathbf{e}^m \wedge \mathbf{e}^h \wedge \cdots \wedge \mathbf{e}^d\right) e^{beg\cdots f}{}_{ach\cdots d} \\
&\quad + (n-1)\int \alpha B^{Am}\bar{\eta}_A{}^a{}_b (-1) S_e^{+\,i}{}_j \mathbf{f}_m \wedge \mathbf{f}_i \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^j \wedge \mathbf{e}^c \wedge \mathbf{e}^h \wedge \cdots \wedge \mathbf{e}^d e^{beg\cdots f}{}_{ach\cdots d} \\
&\quad + (n-1)\int \alpha B^{Am}\bar{\eta}_A{}^a{}_b (-1)\frac{1}{2}T_{+\,ij}^c \left(\mathbf{f}_m \wedge \mathbf{f}_e \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^h \wedge \cdots \wedge \mathbf{e}^d\right) e^{beg\cdots f}{}_{ach\cdots d} \\
&= (n-1)(-1)^n \int \alpha A^{Am}\bar{\eta}_A{}^a{}_b \left(-T_+^{ci}{}_j \bar{e}_{ieg\cdots f}{}^{jmh\cdots d} e^{beg\cdots f}{}_{ach\cdots d}\right)\mathbf{\Phi} \\
&\quad + (n-1)(-1)^n \int \alpha A^{Am}\bar{\eta}_A{}^a{}_b \left(\frac{1}{2}S_e^{+\ ij}\bar{e}_{ijg\cdots f}{}^{mch\cdots d} e^{beg\cdots f}{}_{ach\cdots d}\right)\mathbf{\Phi} \\
&\quad + (n-1)(-1)^n \int \alpha B^{Am}\bar{\eta}_A{}^a{}_b \left(S_e^{+\,i}{}_j \bar{e}_{mig\cdots f}{}^{jch\cdots d} e^{beg\cdots f}{}_{ach\cdots d}\right)\mathbf{\Phi} \\
&\quad + (n-1)(-1)^n \int \alpha B^{Am}\bar{\eta}_A{}^a{}_b \left(-\frac{1}{2}T_{+\,ij}^c \bar{e}_{meg\cdots f}{}^{ijh\cdots d} e^{beg\cdots f}{}_{ach\cdots d}\right)\mathbf{\Phi}
\end{aligned}$$

Then

$$
\begin{aligned}
\delta\mathcal{L} &= (n-1)!\,(n-1)!\,(-1)^n \int \alpha A^{Am} \bar{\eta}_A{}^a{}_b \left(\frac{1}{2} S_e^{+\ ij} \delta_a^m \left(\delta_i^b \delta_j^e - \delta_i^e \delta_j^b\right)\right) \mathbf{\Phi} \\
&\quad + (n-1)!\,(n-1)!\,(-1)^n \int \alpha A^{Am} \bar{\eta}_A{}^a{}_b \left(-T_{+\ \ j}^{ci} \delta_i^b \left(\delta_a^j \delta_c^m - \delta_a^m \delta_c^j\right)\right) \mathbf{\Phi} \\
&\quad + (n-1)!\,(n-1)!\,(-1)^n \int \alpha B^{Am} \bar{\eta}_A{}^a{}_b \left(S_e^{+\ i}{}_j \delta_a^j \left(\delta_m^b \delta_i^e - \delta_m^e \delta_i^b\right)\right) \mathbf{\Phi} \\
&\quad + (n-1)!\,(n-1)!\,(-1)^n \int \alpha B^{Am} \bar{\eta}_A{}^a{}_b \left(-\frac{1}{2} T_{+\ ij}^c \delta_m^b \left(\delta_a^i \delta_c^j - \delta_a^j \delta_c^i\right)\right) \mathbf{\Phi}
\end{aligned}
$$

and finally

$$
\begin{aligned}
\frac{(-1)^n}{(n-1)!\,(n-1)!}\delta\mathcal{L} &= \int \alpha A^{Am} \left[\bar{\eta}_A{}^a{}_b \left(S_e^{+\ be} \delta_a^m - \left(T_{+\ \ a}^{mb} - T_{+\ \ c}^{cb} \delta_a^m\right)\right)\right] \mathbf{\Phi} \\
&\quad + \int \alpha B^{Am} \left[\bar{\eta}_A{}^a{}_b \left(\left(S_i^{+\ i}{}_a \delta_m^b - S_m^{+\ b}{}_a\right) - T_{+\ ac}^c \delta_m^b\right)\right] \mathbf{\Phi}
\end{aligned}
$$

Note that the projection is tighter than the previous Δ_{cd}^{ab}, and that the torsion and co-torsion are only the self-dual parts.

6.2.4 Right spin connection

Varying $\boldsymbol{\omega}_-^A$, is identical to varying $\boldsymbol{\omega}_-^A$ except for the presence of $\eta_A{}^a{}_b$ instead of $\bar{\eta}_A{}^a{}_b$ and minus instead of plus. We follow it through as a check.

$$
\begin{aligned}
\delta_- S &= \int \alpha \delta_- \mathbf{\Omega}_-^A \eta_A{}^a{}_b \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&= \int \alpha \left(\mathbf{d}\delta\boldsymbol{\omega}_-^A + \varepsilon^A{}_{BC} \delta\boldsymbol{\omega}_-^B \wedge \boldsymbol{\omega}_-^C\right) \eta_A{}^a{}_b \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&= \int \alpha \left(\mathbf{D}_- \left(\delta\boldsymbol{\omega}_-^A\right)\right) \eta_A{}^a{}_b \wedge \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&= (n-1) \int \alpha \delta\boldsymbol{\omega}_-^A \eta_A{}^a{}_b \wedge \left(\mathbf{D}_- \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d}\right) \\
&\quad + (n-1) \int \alpha \delta\boldsymbol{\omega}_-^A \eta_A{}^a{}_b \wedge \left((-1)^{n-1} \mathbf{f}_e \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{D}_- \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d}\right) \\
&= (n-1) \int \alpha \delta\boldsymbol{\omega}_-^A \eta_A{}^a{}_b \wedge \left(\mathbf{S}_e^- \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{e}^c \wedge \mathbf{e}^h \wedge \cdots \wedge \mathbf{e}^d e^{beg\cdots f}{}_{ach\cdots d}\right) \\
&\quad + (n-1) \int \alpha \delta\boldsymbol{\omega}_-^A \eta_A{}^a{}_b \wedge \left(\mathbf{f}_e \wedge \mathbf{f}_g \wedge \cdots \wedge \mathbf{f}_f \wedge \mathbf{T}_-^c \wedge \mathbf{e}^h \wedge \cdots \wedge \mathbf{e}^d e^{beg\cdots f}{}_{ach\cdots d}\right)
\end{aligned}
$$

With

$$\delta\boldsymbol{\omega}_{-}^{A} = A^{Am}\mathbf{e}^{m} + B^{Am}\mathbf{f}_{m}$$

the variation of the Lagrange density is

$$\begin{aligned}
\delta_{-}\mathcal{L} &= \alpha(n-1)A^{Am}\mathbf{e}^{m}\eta_{A}{}^{a}{}_{b}\wedge\left(\mathbf{S}_{e}^{-}\wedge\mathbf{f}_{g}\wedge\cdots\wedge\mathbf{f}_{f}\wedge\mathbf{e}^{c}\wedge\mathbf{e}^{h}\wedge\cdots\wedge\mathbf{e}^{d}e^{beg\cdots f}{}_{ach\cdots d}\right) \\
&\quad+\alpha(n-1)A^{Am}\mathbf{e}^{m}\eta_{A}{}^{a}{}_{b}\wedge(-1)\mathbf{f}_{e}\wedge\mathbf{f}_{g}\wedge\cdots\wedge\mathbf{f}_{f}\wedge\mathbf{T}_{-}^{c}\wedge\mathbf{e}^{h}\wedge\cdots\wedge\mathbf{e}^{d}e^{beg\cdots f}{}_{ach\cdots d} \\
&\quad+\alpha(n-1)B^{Am}\mathbf{f}_{m}\eta_{A}{}^{a}{}_{b}\wedge\left(\mathbf{S}_{e}^{-}\wedge\mathbf{f}_{g}\wedge\cdots\wedge\mathbf{f}_{f}\wedge\mathbf{e}^{c}\wedge\mathbf{e}^{h}\wedge\cdots\wedge\mathbf{e}^{d}e^{beg\cdots f}{}_{ach\cdots d}\right) \\
&\quad+\alpha(n-1)B^{Am}\mathbf{f}_{m}\eta_{A}{}^{a}{}_{b}\wedge(-1)\mathbf{f}_{e}\wedge\mathbf{f}_{g}\wedge\cdots\wedge\mathbf{f}_{f}\wedge\mathbf{T}_{-}^{c}\wedge\mathbf{e}^{h}\wedge\cdots\wedge\mathbf{e}^{d}e^{beg\cdots f}{}_{ach\cdots d} \\
&= \alpha(n-1)A^{Am}\eta_{A}{}^{a}{}_{b}(-1)\frac{1}{2}S_{e}^{-\,ij}\mathbf{f}_{i}\wedge\mathbf{f}_{j}\wedge\mathbf{f}_{g}\wedge\cdots\wedge\mathbf{f}_{f}\wedge\mathbf{e}^{m}\wedge\mathbf{e}^{c}\wedge\mathbf{e}^{h}\wedge\cdots\wedge\mathbf{e}^{d}e^{beg\cdots f}{}_{ach\cdots d} \\
&\quad+\alpha(n-1)A^{Am}\eta_{A}{}^{a}{}_{b}(-1)T_{-}^{ci}{}_{j}\mathbf{f}_{i}\wedge\mathbf{f}_{e}\wedge\mathbf{f}_{g}\wedge\cdots\wedge\mathbf{f}_{f}\wedge\mathbf{e}^{j}\wedge\mathbf{e}^{m}\wedge\mathbf{e}^{h}\wedge\cdots\wedge\mathbf{e}^{d}e^{beg\cdots f}{}_{ach\cdots d} \\
&\quad+\alpha(n-1)B^{Am}\eta_{A}{}^{a}{}_{b}(-1)S_{e}^{-\,i}{}_{j}\mathbf{f}_{m}\wedge\mathbf{f}_{i}\wedge\mathbf{f}_{g}\wedge\cdots\wedge\mathbf{f}_{f}\wedge\mathbf{e}^{j}\wedge\mathbf{e}^{c}\wedge\mathbf{e}^{h}\wedge\cdots\wedge\mathbf{e}^{d}e^{beg\cdots f}{}_{ach\cdots d} \\
&\quad+\frac{\alpha}{2}(n-1)B^{Am}\eta_{A}{}^{a}{}_{b}(-1)^{n-1}\mathbf{f}_{m}\wedge\mathbf{f}_{e}\wedge\mathbf{f}_{g}\wedge\cdots\wedge\mathbf{f}_{f}\wedge T_{-}^{c}{}_{ij}\mathbf{e}^{i}\wedge\mathbf{e}^{j}\wedge\mathbf{e}^{h}\wedge\cdots\wedge\mathbf{e}^{d}e^{beg\cdots f}{}_{ach\cdots d} \\
&= (-1)^{n}(n-1)\alpha A^{Am}\eta_{A}{}^{a}{}_{b}\left(\frac{1}{2}S_{e}^{-\,ij}\bar{e}_{ijg\cdots f}{}^{mch\cdots d}e^{beg\cdots f}{}_{ach\cdots d}\right)\mathbf{\Phi} \\
&\quad+(-1)^{n}(n-1)\alpha A^{Am}\eta_{A}{}^{a}{}_{b}\left(-T_{-}^{ci}{}_{j}\bar{e}_{ieg\cdots f}{}^{jmh\cdots d}e^{beg\cdots f}{}_{ach\cdots d}\right)\mathbf{\Phi} \\
&\quad+(-1)^{n}(n-1)\alpha B^{Am}\eta_{A}{}^{a}{}_{b}\left(S_{e}^{-\,i}{}_{j}\bar{e}_{mig\cdots f}{}^{jch\cdots d}e^{beg\cdots f}{}_{ach\cdots d}\right)\mathbf{\Phi} \\
&\quad+(-1)^{n}(n-1)\alpha B^{Am}\eta_{A}{}^{a}{}_{b}\left(-\frac{1}{2}T_{-}^{c}{}_{ij}\bar{e}_{meg\cdots f}{}^{ijh\cdots d}e^{beg\cdots f}{}_{ach\cdots d}\right)\mathbf{\Phi}
\end{aligned}$$

Next,

$$\begin{aligned}
(-1)^{n}\delta_{-}\mathcal{L} &= \alpha(n-1)!(n-1)!A^{Am}\eta_{A}{}^{a}{}_{b}\left(\frac{1}{2}S_{e}^{-\,ij}\delta_{a}^{m}\left(\delta_{i}^{b}\delta_{j}^{e}-\delta_{i}^{e}\delta_{j}^{b}\right)\right)\mathbf{\Phi} \\
&\quad+\alpha(n-1)!(n-1)!A^{Am}\eta_{A}{}^{a}{}_{b}\left(-T_{-}^{ci}{}_{j}\delta_{i}^{b}\left(\delta_{a}^{j}\delta_{c}^{m}-\delta_{a}^{m}\delta_{c}^{j}\right)\right)\mathbf{\Phi} \\
&\quad+\alpha(n-1)!(n-1)!B^{Am}\eta_{A}{}^{a}{}_{b}\left(S_{e}^{-\,i}{}_{j}\delta_{a}^{j}\left(\delta_{m}^{b}\delta_{i}^{e}-\delta_{m}^{e}\delta_{i}^{b}\right)\right)\mathbf{\Phi} \\
&\quad+\alpha(n-1)!(n-1)!B^{Am}\eta_{A}{}^{a}{}_{b}\left(-\frac{1}{2}T_{-}^{c}{}_{ij}\delta_{m}^{b}\left(\delta_{a}^{i}\delta_{c}^{j}-\delta_{a}^{j}\delta_{c}^{i}\right)\right)\mathbf{\Phi}
\end{aligned}$$

$$\begin{aligned}
\frac{(-1)^{n}}{(n-1)!(n-1)!}\delta_{-}\mathcal{L} &= \alpha A^{Am}\eta_{A}{}^{a}{}_{b}\left(S_{e}^{-\,be}\delta_{a}^{m}-\left(T_{-}^{mb}{}_{a}-T_{-}^{cb}{}_{c}\delta_{a}^{m}\right)\right)\mathbf{\Phi} \\
&\quad+\alpha B^{Am}\eta_{A}{}^{a}{}_{b}\left(\left(S_{e}^{-\,e}{}_{a}\delta_{m}^{b}-S_{m}^{-\,b}{}_{a}\right)-T_{-}^{c}{}_{ac}\delta_{m}^{b}\right)\mathbf{\Phi}
\end{aligned}$$

6.2.5 Weyl vector

Vary,

$$
\begin{aligned}
\delta_W S &= \int \beta\delta^a_b \mathbf{d}\delta\boldsymbol{\omega}\wedge\mathbf{f}_e\wedge\cdots\wedge\mathbf{f}_f\wedge\mathbf{e}^c\wedge\cdots\wedge\mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&= \int \beta\delta^a_b \mathbf{D}\delta\boldsymbol{\omega}\wedge\mathbf{f}_e\wedge\cdots\wedge\mathbf{f}_f\wedge\mathbf{e}^c\wedge\cdots\wedge\mathbf{e}^d e^{be\cdots f}{}_{ac\cdots d} \\
&= (n-1)\int \beta\delta^a_b\delta\boldsymbol{\omega}\wedge\left(\mathbf{D}\mathbf{f}_e\wedge\mathbf{f}_g\wedge\cdots\wedge\mathbf{f}_f\wedge\mathbf{e}^c\wedge\mathbf{e}^h\wedge\cdots\wedge\mathbf{e}^d\right)e^{beg\cdots f}{}_{ach\cdots d} \\
&\quad +(n-1)\int \beta\delta^a_b\delta\boldsymbol{\omega}\wedge\left((-1)^{n-1}\mathbf{f}_e\wedge\mathbf{f}_g\wedge\cdots\wedge\mathbf{f}_f\wedge\mathbf{D}\mathbf{e}^c\wedge\mathbf{e}^h\wedge\cdots\wedge\mathbf{e}^d\right)e^{beg\cdots f}{}_{ach\cdots d} \\
&= (n-1)\beta\int \delta^a_b A_m\mathbf{e}^m\wedge\left(\mathbf{S}_e\wedge\mathbf{f}_g\wedge\cdots\wedge\mathbf{f}_f\wedge\mathbf{e}^c\wedge\mathbf{e}^h\wedge\cdots\wedge\mathbf{e}^d\right)e^{beg\cdots f}{}_{ach\cdots d} \\
&\quad +(n-1)\beta\int \delta^a_b A_m\mathbf{e}^m\wedge\left((-1)^{n-1}\mathbf{f}_e\wedge\mathbf{f}_g\wedge\cdots\wedge\mathbf{f}_f\wedge\mathbf{T}^c\wedge\mathbf{e}^h\wedge\cdots\wedge\mathbf{e}^d\right)e^{beg\cdots f}{}_{ach\cdots d} \\
&\quad +(n-1)\beta\int \delta^a_b B^m\mathbf{f}_m\wedge\left(\mathbf{S}_e\wedge\mathbf{f}_g\wedge\cdots\wedge\mathbf{f}_f\wedge\mathbf{e}^c\wedge\mathbf{e}^h\wedge\cdots\wedge\mathbf{e}^d\right)e^{beg\cdots f}{}_{ach\cdots d} \\
&\quad +(n-1)\beta\int \delta^a_b B^m\mathbf{f}_m\wedge\left((-1)^{n-1}\mathbf{f}_e\wedge\mathbf{f}_g\wedge\cdots\wedge\mathbf{f}_f\wedge\mathbf{T}^c\wedge\mathbf{e}^h\wedge\cdots\wedge\mathbf{e}^d\right)e^{beg\cdots f}{}_{ach\cdots d} \\
\\
&= (-1)^n(n-1)\beta\int \delta^a_b A_m\left(\frac{1}{2}S_e{}^{ij}\bar{e}_{ijg\cdots f}{}^{mch\cdots d}\boldsymbol{\Phi}+T^{ci}{}_j\bar{e}_{ieg\cdots f}{}^{mjh\cdots d}\boldsymbol{\Phi}\right)e^{beg\cdots f}{}_{ach\cdots d} \\
&\quad +(-1)^n(n-1)\beta\int \delta^a_b B^m\left(S_e{}^i{}_j\bar{e}_{mig\cdots f}{}^{jch\cdots d}\boldsymbol{\Phi}-\frac{1}{2}T^c{}_{ij}\bar{e}_{meg\cdots f}{}^{ijh\cdots d}\boldsymbol{\Phi}\right)e^{beg\cdots f}{}_{ach\cdots d}
\end{aligned}
$$

so

$$
\begin{aligned}
\frac{(-1)^n}{(n-1)!\,(n-1)!}\delta\mathcal{L} &= \beta\delta^a_b A_m\left(\frac{1}{2}S_e{}^{ij}\delta^m_a\left(\delta^b_i\delta^e_j-\delta^e_i\delta^b_j\right)+T^{ci}{}_j\delta^b_i\left(\delta^m_a\delta^j_c-\delta^j_a\delta^m_c\right)\right)\boldsymbol{\Phi} \\
&\quad +\beta\delta^a_b B^m\left(S_e{}^i{}_j\delta^j_a\left(\delta^b_m\delta^e_i-\delta^e_m\delta^b_i\right)-\frac{1}{2}T^c{}_{ij}\delta^b_m\left(\delta^i_a\delta^j_c-\delta^j_a\delta^i_c\right)\right)\boldsymbol{\Phi} \\
\\
&= \beta\int A_m\left(S_e{}^{be}\delta^m_b+T^{cm}{}_c-T^{ma}{}_a\right)\boldsymbol{\Phi} \\
&\quad +\beta\int B^m\left(S_c{}^c{}_m-S_m{}^b{}_b-T^c{}_{mc}\right)\boldsymbol{\Phi}
\end{aligned}
$$

6.2.6 Collected field equations

The Lagrange density variations are

$$
\begin{aligned}
\frac{(-1)^n}{(n-1)!\,(n-1)!}\delta\mathcal{L} &= \int \alpha A^{Am}\left[\bar{\eta}_A{}^a{}_b\left(S^{+}_e{}^{be}\delta^m_a - \left(T^{mb}_{+}{}_a - T^{cb}_{+}{}_c\delta^m_a\right)\right)\right]\mathbf{\Phi} \\
&\quad + \int \alpha B^{Am}\left[\bar{\eta}_A{}^a{}_b\left(\left(S^{+}_i{}^i{}_a\delta^b_m - S^{+}_m{}^b{}_a\right) - T^c_{+}{}_{ac}\delta^b_m\right)\right]\mathbf{\Phi} \\
\frac{(-1)^n}{(n-1)!\,(n-1)!}\delta_{-}\mathcal{L} &= \alpha A^{Am}\eta_A{}^a{}_b\left(S^{-}_e{}^{be}\delta^m_a - \left(T^{mb}_{-}{}_a - T^{cb}_{-}{}_c\delta^m_a\right)\right)\mathbf{\Phi} \\
&\quad + \alpha B^{Am}\eta_A{}^a{}_b\left(\left(S^{-}_e{}^e{}_a\delta^b_m - S^{-}_m{}^b{}_a\right) - T^c_{-}{}_{ac}\delta^b_m\right)\mathbf{\Phi} \\
\frac{(-1)^n}{(n-1)!\,(n-1)!}\delta\mathcal{L} &= A^b{}_a\left(\alpha\left(\Omega^a_{+}{}_c{}^c{}_b - \delta^a_b\Omega^d_{+}{}_c{}^c{}_d\right) + \alpha\left(\Omega^a_{-}{}_c{}^c{}_b - \delta^a_b\Omega^d_{-}{}_c{}^c{}_d\right)\right)\mathbf{\Phi} \\
&\quad + A^b{}_a\left(\beta\left(\Omega^a{}_b - \delta^a_b\Omega^c{}_c\right) - \delta^a_b\left(4\left(n-1\right)\alpha \vec{+} 2\beta - n^2\gamma\right)\right)\mathbf{\Phi} \\
&\quad - B^{ab}\left(\alpha\Omega^c_{+}{}_{bca} + \alpha\Omega^c_{-}{}_{bca} + \beta\Omega_{ba}\right)\mathbf{\Phi} \\
\frac{(-1)^n}{(n-1)!\,(n-1)!}\delta_f\mathcal{L} &= D_m{}^k\left(\alpha\left(\Omega^a_{+}{}_k{}^m{}_a - \delta^m_k\Omega^a_{+}{}_b{}^b{}_a\right) + \alpha\left(\Omega^a_{-}{}_k{}^m{}_a - \delta^m_k\Omega^a_{-}{}_b{}^b{}_a\right)\right)\mathbf{\Phi} \\
&\quad + D_m{}^k\left(\beta\left(\Omega^m{}_k - \Omega^a{}_a\delta^m_k\right) - \delta^m_k\left(4\alpha\left(n-1\right) + 2\beta - n^2\gamma\right)\right)\mathbf{\Phi} \\
&\quad + C_{ma}\left(\alpha\Omega^a_{+}{}_b{}^{bm} + \alpha\Omega^a_{+}{}_b{}^{bm} + \beta\Omega^{bm}\right)\mathbf{\Phi} \\
\frac{(-1)^n}{(n-1)!\,(n-1)!}\delta\mathcal{L} &= \beta A_m\left(S_e{}^{be}\delta^m_b + T^{cm}{}_c - T^{ma}{}_a\right)\mathbf{\Phi} \\
&\quad + \beta B^m\left(S_c{}^c{}_m - S_m{}^b{}_b - T^c{}_{mc}\right)\mathbf{\Phi}
\end{aligned}
$$

so the field equations are

$$
\begin{aligned}
0 &= \alpha\bar{\eta}_A{}^a{}_b\left(S_e^{+\ be}\delta_a^m - T_+^{mb}{}_a + T_+^{cb}{}_c\delta_a^m\right) \\
0 &= \alpha\bar{\eta}_A{}^a{}_b\left(S_i^{+\ i}{}_a\delta_m^b - S_m^{+\ b}{}_a - T_+^c{}_{ac}\delta_m^b\right) \\
0 &= \alpha\eta_A{}^a{}_b\left(S_e^{-\ be}\delta_a^m - T_-^{mb}{}_a + T_-^{cb}{}_c\delta_a^m\right) \\
0 &= \alpha\eta_A{}^a{}_b\left(S_e^{-\ e}{}_a\delta_m^b - S_m^{-\ b}{}_a - T_-^c{}_{ac}\delta_m^b\right) \\
0 &= \beta\left(S_e{}^{be}\delta_b^m + T^{cm}{}_c - T^{ma}{}_a\right) \\
0 &= \beta\left(S_c{}^c{}_m - S_m{}^b{}_b - T^c{}_{mc}\right)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \alpha\left(\Omega_+^a{}_c{}^c{}_b - \delta_b^a\Omega_+^d{}_c{}^c{}_d\right) + \alpha\left(\Omega_-^a{}_c{}^c{}_b - \delta_b^a\Omega_-^d{}_c{}^c{}_d\right) + \beta\left(\Omega^a{}_b - \delta_b^a\Omega^c{}_c\right) \\
&\quad -\delta_b^a\left(4(n-1)\alpha \vec{+} 2\beta - n^2\gamma\right) \\
0 &= \alpha\left(\Omega_+^a{}_k{}^m{}_a - \delta_k^m\Omega_+^a{}_b{}^b{}_a\right) + \alpha\left(\Omega_-^a{}_k{}^m{}_a - \delta_k^m\Omega_-^a{}_b{}^b{}_a\right) + \beta\left(\Omega^m{}_k - \Omega^a{}_a\delta_k^m\right) \\
&\quad -\delta_k^m\left(4\alpha(n-1) + 2\beta - n^2\gamma\right) \\
0 &= \alpha\left(\Omega_+^c{}_{bca} + \Omega_-^c{}_{bca}\right) + \beta\Omega_{ba} \\
0 &= \alpha\left(\Omega_+^a{}_b{}^{bm} + \Omega_+^a{}_b{}^{bm}\right) + \beta\Omega^{bm}
\end{aligned}
$$

These equations comprise our second principal result.

6.3 Conclusion

We successfully constructed the biconformal gauge theory in 8-dimensional spinor representation of Spin $(5,1)$. The quotient of this conformal group by its homogeneous Weyl subgroup gives a principal fiber bundle with 8-dim base manifold and Weyl fibers. The Cartan generalization to a curved 8-dim geometry admits an action functional linear in the curvatures, and the field equations generically yield general relativity on the cotangent bundle of spacetime. We focussed on the subclass of cases when the extra 4 dimensions can give a fibration by a non-Abelian Lie group, where the maximal case is the electroweak group. Thus, while the final Lorentz and electroweak symmetries are of the direct product form required by Coleman-Mandula, the model is predictive of the specific group.

Satisfying the Coleman-Mandula theorem comes automatically because after taking the quotient, the Lie group on the 4-dimensional y-subspace effectively extends the bundle symmetry as a direct product of the fibers.

Our procedue in spinor representation includes projections that split left-handed particles from the right-handed ones, particles from antiparticles, and spin states of particles as known in quatum field theory. This procedure also separates our curvatures and field equations into self-dual and anti-delf-dual parts, making the field equations distinct from those of previous studies.

However, due to our choice of basis, the projection operator $P_{V_{\pm}} = \frac{1}{2}\left(1 \pm \gamma_V\right)$ also produces two 4-dimensional $\mathrm{SO}(5,1)$ subspaces, each described by its own representation and Clifford algebra of $SO\left(4\right)$ symmetry. This gives us a new, alternative opportunity to realize gravity other than the previous method described above. We conjecture that, if we apply $P_{V_{\pm}}$ on the group before taking the quotient by the homogenous Weyl group, we can generate the correct gravity and electroweak symmetries directly. Specifically, if we take the quotient of the first $\mathrm{SO}(5,1)$ partition say P_{V_+} by $SO\left(3,1\right)$, this leaves a spacetime signature on the fibers from which we get Einsteins equations for gravity leaving behind and an extra $SO\left(2\right)$ on the base manifold. Then if we take the quotient of the second $\mathrm{SO}(5,1)$ partition say P_{V_-} by $SO\left(4\right) = SU\left(2\right) \times SU\left(2\right)$, this leaves an electroweak signature on the fibers and leaving behind and an extra $SO\left(1,1\right)$ on the base manifold.

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APPENDIX

Identities with 't Hooft matrices

The t'Hooft matrices are

$$\begin{aligned}
\eta^{A}{}_{ab} &= \varepsilon^{A}{}_{ab} + \delta^{A}_{a}\delta^{4}_{b} - \delta^{A}_{b}\delta^{4}_{a} \\
\bar{\eta}^{A}{}_{ab} &= \varepsilon^{A}{}_{ab} - \delta^{A}_{a}\delta^{4}_{b} + \delta^{A}_{b}\delta^{4}_{a}
\end{aligned}$$

If we add these,

$$\begin{aligned}
\eta^{A}{}_{ab} + \bar{\eta}^{A}{}_{ab} &= \varepsilon^{A}{}_{ab} + \delta^{A}_{a}\delta^{4}_{b} - \delta^{A}_{b}\delta^{4}_{a} + \varepsilon^{A}{}_{ab} - \delta^{A}_{a}\delta^{4}_{b} + \delta^{A}_{b}\delta^{4}_{a} \\
&= 2\varepsilon^{A}{}_{ab}
\end{aligned}$$

we do not recover the identity. This is awkward, because we would like to recover the vector structure equations by adding the spinor ones.

Raise an index:

$$\begin{aligned}
\eta^{Aa}{}_{b} &= \varepsilon^{Aa}{}_{b} + \delta^{Aa}\delta^{4}_{b} - \delta^{4a}\delta^{A}_{b} \\
\bar{\eta}^{Aa}{}_{ab} &= \varepsilon^{Aa}{}_{b} - \delta^{Aa}\delta^{4}_{b} + \delta^{4a}\delta^{A}_{b}
\end{aligned}$$

The product is

$$\begin{aligned}
\eta^{Aa}{}_{b}\bar{\eta}^{Bb}{}_{c} &= \left(\varepsilon^{Aa}{}_{b} + \delta^{Aa}\delta^{4}_{b} - \delta^{4a}\delta^{A}_{b}\right)\left(\varepsilon^{Bb}{}_{c} - \delta^{Bb}\delta^{4}_{c} + \delta^{4b}\delta^{B}_{c}\right) \\
&= \varepsilon^{Aa}{}_{b}\varepsilon^{Bb}{}_{c} - \varepsilon^{AaB}\delta^{4}_{c} - \delta^{Aa}\delta^{B4}\delta^{4}_{c} + \delta^{Aa}\delta^{B}_{c} - \delta^{4a}\varepsilon^{BA}{}_{c} + \delta^{4a}\delta^{BA}\delta^{4}_{c} \\
&= -\delta^{AB}\delta^{a}_{K}\delta^{K}_{c} + \delta^{A}_{c}\delta^{Ba} - \varepsilon^{AaB}\delta^{4}_{c} - \delta^{4a}\varepsilon^{BA}{}_{c} - \delta^{Aa}\delta^{B4}\delta^{4}_{c} + \delta^{Aa}\delta^{B}_{c} + \delta^{4a}\delta^{BA}\delta^{4}_{c} \\
&= -\varepsilon^{AaB}\delta^{4}_{c} - \delta^{4a}\varepsilon^{BA}{}_{c} - \delta^{AB}\delta^{a}_{K}\delta^{K}_{c} + \delta^{A}_{c}\delta^{Ba} + \delta^{Aa}\delta^{B}_{c} + \delta^{4a}\delta^{BA}\delta^{4}_{c} - \delta^{Aa}\delta^{B4}\delta^{4}_{c}
\end{aligned}$$

Consider the special cases

$$\begin{aligned}
\eta^{AC}{}_{b}\bar{\eta}^{Bb}{}_{D} &= -\varepsilon^{ACB}\delta^4_D - \delta^{4C}\varepsilon^{BA}{}_{D} - \delta^{AB}\delta^C_K\delta^K_D + \delta^A_D\delta^{BC} + \delta^{AC}\delta^B_D + \delta^{4C}\delta^{BA}\delta^4_D - \delta^{AC}\delta^{B4}\delta^4_D \\
&= \delta^A_D\delta^{BC} - \delta^{AB}\delta^C_D + \delta^{AC}\delta^B_D
\end{aligned}$$

$$\begin{aligned}
\eta^{A4}{}_{b}\bar{\eta}^{Bb}{}_{D} &= -\varepsilon^{A4B}\delta^4_D - \delta^{44}\varepsilon^{BA}{}_{D} - \delta^{AB}\delta^4_K\delta^K_D + \delta^A_D\delta^{B4} + \delta^{A4}\delta^B_D + \delta^{44}\delta^{BA}\delta^4_D - \delta^{A4}\delta^{B4}\delta^4_D \\
&= -\varepsilon^{BA}{}_{D}
\end{aligned}$$

$$\begin{aligned}
\eta^{AC}{}_{b}\bar{\eta}^{Bb}{}_{4} &= -\varepsilon^{ACB}\delta^4_4 - \delta^{4C}\varepsilon^{BA}{}_{4} - \delta^{AB}\delta^C_K\delta^K_4 + \delta^A_4\delta^{BC} + \delta^{AC}\delta^B_4 + \delta^{4C}\delta^{BA}\delta^4_4 - \delta^{AC}\delta^{B4}\delta^4_4 \\
&= -\varepsilon^{ACB}
\end{aligned}$$

$$\eta^{A4}{}_{b}\bar{\eta}^{Bb}{}_{4} = \delta^{BA}$$

One further contraction gives

$$\begin{aligned}
\eta^{Aa}{}_{b}\bar{\eta}^{Bb}{}_{a} &= -\varepsilon^{AaB}\delta^4_a - \delta^{4a}\varepsilon^{BA}{}_{a} - \delta^{AB}\delta^a_K\delta^K_a + \delta^A_a\delta^{Ba} + \delta^{Aa}\delta^B_a + \delta^{4a}\delta^{BA}\delta^4_a - \delta^{Aa}\delta^{B4}\delta^4_a \\
&= -3\delta^{AB} + \delta^{AB} + \delta^{AB} + \delta^{BA} \\
&= 0
\end{aligned}$$

If we multiply the same one:

$$\begin{aligned}
\eta^{Aa}{}_{b}\eta^{Bb}{}_{c} &= \left(\varepsilon^{Aa}{}_{b} + \delta^{Aa}\delta^4_b - \delta^{4a}\delta^A_b\right)\left(\varepsilon^{Bb}{}_{c} + \delta^{Bb}\delta^4_c - \delta^{4b}\delta^B_c\right) \\
&= \varepsilon^{Aa}{}_{b}\varepsilon^{Bb}{}_{c} + \varepsilon^{Aa}{}_{b}\delta^{Bb}\delta^4_c - \varepsilon^{Aa}{}_{b}\delta^{4b}\delta^B_c + \delta^{Aa}\delta^4_b\varepsilon^{Bb}{}_{c} \\
&\quad + \delta^{Aa}\delta^4_b\delta^{Bb}\delta^4_c - \delta^{Aa}\delta^4_b\delta^{4b}\delta^B_c - \delta^{4a}\delta^A_b\varepsilon^{Bb}{}_{c} - \delta^{4a}\delta^A_b\delta^{Bb}\delta^4_c + \delta^{4a}\delta^A_b\delta^{4b}\delta^B_c \\
&= \delta^A_c\delta^{aB} - \delta^{AB}\delta^a_K\delta^K_c + \varepsilon^{AaB}\delta^4_c - \delta^{Aa}\delta^B_c - \delta^{4a}\varepsilon^{BA}{}_{c} - \delta^{4a}\delta^{AB}\delta^4_c \\
&= \varepsilon^{AaB}\delta^4_c - \delta^{4a}\varepsilon^{BA}{}_{c} + \delta^A_c\delta^{aB} - \delta^{AB}\delta^a_K\delta^K_c - \delta^{Aa}\delta^B_c - \delta^{4a}\delta^{AB}\delta^4_c
\end{aligned}$$

Special cases:

$$\begin{aligned}
\eta^{AC}{}_b\eta^{Bb}{}_D &= \varepsilon^{ACB}\delta^4_D - \delta^{4C}\varepsilon^{BA}{}_D + \delta^A_D\delta^{CB} - \delta^{AB}\delta^C_K\delta^K_D - \delta^{AC}\delta^B_D - \delta^{4C}\delta^{AB}\delta^4_D \\
&= \varepsilon^{ACB}\delta^4_D - \delta^{4C}\varepsilon^{BA}{}_D + \delta^A_D\delta^{CB} - \delta^{AB}\delta^C_D - \delta^{AC}\delta^B_D
\end{aligned}$$

$$\begin{aligned}
\eta^{A4}{}_b\eta^{Bb}{}_D &= \varepsilon^{A4B}\delta^4_D - \delta^{44}\varepsilon^{BA}{}_D + \delta^A_D\delta^{4B} - \delta^{AB}\delta^4_K\delta^K_D - \delta^{A4}\delta^B_D - \delta^{44}\delta^{AB}\delta^4_D \\
&= -\varepsilon^{BA}{}_D
\end{aligned}$$

$$\begin{aligned}
\eta^{AC}{}_b\eta^{Bb}{}_4 &= \varepsilon^{ACB}\delta^4_4 - \delta^{4C}\varepsilon^{BA}{}_4 + \delta^A_4\delta^{CB} - \delta^{AB}\delta^C_K\delta^K_4 - \delta^{AC}\delta^B_4 - \delta^{4C}\delta^{AB}\delta^4_4 \\
&= \varepsilon^{ACB}
\end{aligned}$$

$$\begin{aligned}
\eta^{A4}{}_b\eta^{Bb}{}_4 &= \varepsilon^{A4B}\delta^4_4 - \delta^{44}\varepsilon^{BA}{}_4 + \delta^A_4\delta^{4B} - \delta^{AB}\delta^4_K\delta^K_4 - \delta^{A4}\delta^B_4 - \delta^{44}\delta^{AB}\delta^4_4 \\
&= -\delta^{AB}
\end{aligned}$$

Contracting again,

$$\begin{aligned}
\eta^{Aa}{}_b\eta^{Bb}{}_a &= \varepsilon^{AaB}\delta^4_a - \delta^{4a}\varepsilon^{BA}{}_a + \delta^A_a\delta^{aB} - \delta^{AB}\delta^a_K\delta^K_a - \delta^{Aa}\delta^B_a - \delta^{4a}\delta^{AB}\delta^4_a \\
&= \delta^{AB} - 3\delta^{AB} - \delta^{AB} - \delta^{AB} \\
&= -4\delta^{AB}
\end{aligned}$$

Now the conjugates,

$$\begin{aligned}
\bar{\eta}^{Aa}{}_b\bar{\eta}^{Bb}{}_c &= \left(\varepsilon^{Aa}{}_b - \delta^{Aa}\delta^4_b + \delta^{4a}\delta^A_b\right)\left(\varepsilon^{Bb}{}_c - \delta^{Bb}\delta^4_c + \delta^{4b}\delta^B_c\right) \\
&= \varepsilon^{Aa}{}_b\varepsilon^{Bb}{}_c - \varepsilon^{AaB}\delta^4_c - \delta^{Aa}\delta^4_b\varepsilon^{Bb}{}_c - \delta^{Aa}\delta^B_c + \delta^{4a}\varepsilon^{BA}{}_c - \delta^{4a}\delta^{AB}\delta^4_c \\
&= -\delta^{AB}\delta^a_K\delta^K_c + \delta^A_c\delta^{Ba} - \varepsilon^{AaB}\delta^4_c - \delta^{Aa}\delta^B_c + \delta^{4a}\varepsilon^{BA}{}_c - \delta^{4a}\delta^{AB}\delta^4_c
\end{aligned}$$

Special cases:

$$\begin{aligned}
\bar{\eta}^{AC}{}_{b}\bar{\eta}^{Bb}{}_{D} &= -\delta^{AB}\delta^{C}_{K}\delta^{K}_{D} + \delta^{A}_{D}\delta^{BC} - \varepsilon^{ACB}\delta^{4}_{D} - \delta^{AC}\delta^{B}_{D} + \delta^{4C}\varepsilon^{BA}{}_{D} - \delta^{4C}\delta^{AB}\delta^{4}_{D} \\
&= -\delta^{AB}\delta^{C}_{D} + \delta^{A}_{D}\delta^{BC} - \delta^{AC}\delta^{B}_{D}
\end{aligned}$$

$$\begin{aligned}
\bar{\eta}^{A4}{}_{b}\bar{\eta}^{Bb}{}_{D} &= -\delta^{AB}\delta^{4}_{K}\delta^{K}_{D} + \delta^{A}_{D}\delta^{B4} - \varepsilon^{A4B}\delta^{4}_{D} - \delta^{A4}\delta^{B}_{D} + \delta^{44}\varepsilon^{BA}{}_{D} - \delta^{44}\delta^{AB}\delta^{4}_{D} \\
&= \varepsilon^{BA}{}_{D}
\end{aligned}$$

$$\begin{aligned}
\bar{\eta}^{AC}{}_{b}\bar{\eta}^{Bb}{}_{4} &= -\delta^{AB}\delta^{C}_{K}\delta^{K}_{4} + \delta^{A}_{4}\delta^{BC} - \varepsilon^{ACB}\delta^{4}_{4} - \delta^{AC}\delta^{B}_{4} + \delta^{4C}\varepsilon^{BA}{}_{4} - \delta^{4C}\delta^{AB}\delta^{4}_{4} \\
&= -\varepsilon^{ACB}
\end{aligned}$$

$$\begin{aligned}
\bar{\eta}^{A4}{}_{b}\bar{\eta}^{Bb}{}_{4} &= -\delta^{AB}\delta^{4}_{K}\delta^{K}_{4} + \delta^{A}_{4}\delta^{B4} - \varepsilon^{A4B}\delta^{4}_{4} - \delta^{A4}\delta^{B}_{4} + \delta^{44}\varepsilon^{BA}{}_{4} - \delta^{44}\delta^{AB}\delta^{4}_{4} \\
&= -\delta^{AB}
\end{aligned}$$

and another contraction,

$$\begin{aligned}
\bar{\eta}^{Aa}{}_{b}\bar{\eta}^{Bb}{}_{a} &= -\delta^{AB}\delta^{a}_{K}\delta^{K}_{a} + \delta^{A}_{a}\delta^{Ba} - \varepsilon^{AaB}\delta^{4}_{a} - \delta^{Aa}\delta^{B}_{a} + \delta^{4a}\varepsilon^{BA}{}_{a} - \delta^{4a}\delta^{AB}\delta^{4}_{a} \\
&= -3\delta^{AB} + \delta^{AB} - \delta^{AB} - \delta^{AB} \\
&= -4\delta^{AB}
\end{aligned}$$

CURRICULUM VITAE

Mubarak S. Ukashat

DOCTORAL RESEARCH

My research examined the use of biconformal gauge field theory in the creation of a new framework with a unique structure that allows the uni- fication of gravity and electroweak theory. The quotient of the conformal group of a space of dim n=p+q, with SO(p,q) metric by its homogeneous Weyl subgroup, gives a principal fiber bundle with 2n-dim base manifold and Weyl fibers. The Cartan generalization to a curved 2n-dim geometry admits an action functional linear in the curvatures, and the field equations generically yield general relativity on the cotangent bundle of spacetime. However, in a subclass of cases the extra n dimensions can give a fibration by a non-Abelian Lie group, with the maximal case for n = 4 being the electroweak group. Thus, while the final Lorentz and electroweak symmetries are of the direct product form required by Coleman-Mandula, the model is predictive of the specific group. We de- veloped a spinor representation for the 4-dimensional case of this model in detail to see if further properties of the electroweak theory are predicted. In addition to the usual operators within Dirac theory, we also found a new projection which might be interpreted as either isospin or as the splitting between the gravity and electroweak sectors.

WORK EXPERIENCE:

Berea College (CURRENT, AUG 2022)

Visiting Assistant Professor (Physics Department)

As part of this job requirements, I teach two intro to physics classes and the intermediate classical mechanics class. I also discuss research with students on general relativity, gauge field theories, the standard model and unification.

Salt Lake Community College (JUL 2019-JUNE 2022)

- Adjunct Faculty (Physics Department)

Taught first year intro physics and physics for scientists and engineers. Also taught introductory astronomy classes and a couple of physics labs

Utah State University (JUL 2019-JUNE 2022)

- Teaching Assistant (Physics Department)

This position involved helping with students recitations and teaching labs for first year undergraduates.

EDUCATION

- 2015 – 2022 - Doctor of Philosophy - Theoretical Physics - Utah State University.
- 2013-2016 - Master of Science (Distinction) - Theoretical Physics - King Fahd University of Petroleum and Minerals.
- 2007 – 2011 - Bachelor of Physics - Department of Physics - Federal University of Petroleum Resources, Nigeria.
- 2008 – 2011 - Diploma in Petroleum Engineering - Department of Petroleum Engineering - Petroleum Training Institute, Nigeria.

AWARDS

- 2011 - Best Graduating Student, Physics Department, Federal University of Petroleum Resources, Nigeria.
- 2011 Top Achiever Award, Federal University of Petroleum Resources, Nigeria.

COMPUTER SKILLS

- Java,
- MS DOS,
- Javascript,
- Python,
- HTML,
- CSS,
- Microsoft Windows,
- Data Mining & Machine Learning,
- Lyx, Maple, WEKA, LATEX

COMMUNICATION SKILLS

- Oral Presentation at the Annual USU Theoretical Physics Conference (2018 – 2019)

SKILLS

Goal Oriented

I strive for excellence and precision at all times in all posi- tions and circumstances, attaining professional distinction and proficiency and achieving goals. I believe in action over long-winded discussions. I listen to everyone's viewpoints and use my judgement to immediately act based on con- sensus to achieve goals quickly and efficiently.

Physical Dexterity

I know that learning is difficult, so I strive to make it re- warding. I think of it as fun, but mastery of difficult con- cepts fosters confidence and continued engagement, which means everyone can understand and follow along. My teach- ing philosophy is mainly based on the constructivist pedagogy. I am an educator that helps students search rather than follow, taking the role of mediator for the students and environments, not simply as a giver of information and manager of behavior. I encourage and accept student au- tonomy and initiative while utilizing raw data and primary sources, along with manipulative, interactive, and physical materials.

Passionate

I have been interested in theoretical physics such as quan- tum mechanics and relativity from an early age. My edu- cation and research have cemented this interest into a passion. I greatly enjoy carrying out fundamental physics re- search with potential practical applications.

PUBLICATIONS

A.A. Naqvi, F.Z. Khiari, M. Maslehuddin, M.A. Gondal, O.S.B Al-Amoudi, M.S. Ukashat, A.M. Ilyas, F.A. Liadi, A.A. Isab, Khateeb-ur Rehman, M. Raashid, M.A. Dastageer. (2015). Pulse height tests of a large diameter fast LaBr3:Ce scintillation detector. Applied Radiation and Isotopes, ISSN 0969-8043(104), 224-231.

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