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Joint Invariants of Primitive Homogenous Spaces

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JOINT INVARIANTS OF PRIMITIVE HOMOGENEOUS SPACES

by

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ABSTRACT

Joint Invariants of Primitive Homogeneous Spaces

by

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We develop a reduction technique which identifies joint invariants of homogeneous spaces with invariants of their corresponding isotropy action on a smaller space. The reduction technique is then applied to compute joint invariants for primitive homogenoeus spaces of affine type and minimal dimensional symmmetric type.

(155 pages)

PUBLIC ABSTRACT

Joint Invariants of Primitive Homogeneous Spaces

Illia Hayes

Joint invariants are motivated by the study of congruence problems in Euclidean geometry, where they provide necessary and sufficient conditions for congruence. More recently joint invariants have been used in computer image recognition problems. This thesis develops new methods to compute joint invariants by developing a reduction technique, and applies the reduction to a number of important examples.

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CHAPTER 1

Introduction

This thesis is devoted to the development of a reduction technique used to construct joint invariants for group actions on primitive homogeneous spaces. The study of joint invariants for homogeneous spaces is motivated by congruence problems in Euclidian geometry, in which case a complete set of joint invariants give necessary and sufficient conditions for two k point polygons to be congruent. Our methods are motivated by the algorithmic approach to classifying joint invariants developed in Olver [17], which use the theory of moving frames to find a direct method of determining the joint invariants for homogeneous space.

A practical application of joint invariants arises in image recognition. Two images in a homogeneous space can be related by the transformations of the transitive group action when their boundaries are congruent. A suitable collection of differential invariants evaluated on the boundary of an image parameterize a signature manifold, which is invariant under the action of the group on the original image, and classify the object up to these transformations. However, in most applications the differential invariants depend on derivatives of high order which are very sensitive to noise. Joint invariants can be used as a noise resistant alternative to parameterize a signature manifold, though the number of joint invariants required to classify image boundaries, and therefore the dimension of the corresponding signature manifold, is often quite large. There are two main ways to mediate the large number of joint invariants in applications. Olver [17] shows that using a small number of joint invariants and their derivatives can classify image boundary, and that the highest order of these joint differential invariants is smaller than the approach of using purely differential invariants.

Boutain [4] shows that one can use a collection of pure joint invariants along with a well chosen preferred set of points on the image boundary, called landmarks, to characterize the boundary with a lower dimensional signature manifold. For example consider the image boundaries in the left image of Figure 1.1. If the image boundary of one of the rabbits is sampled and ordered in a counterclockwise direction, $(x_i, y_i)_{0 \le i \le n}$, one can designate the first and last points in the collection as landmarks. Using the landmarks $(x_0, y_0), (x_n, y_n)$, define two functions by using the joint invariant of Euclidean transformations given by the distance between two points,

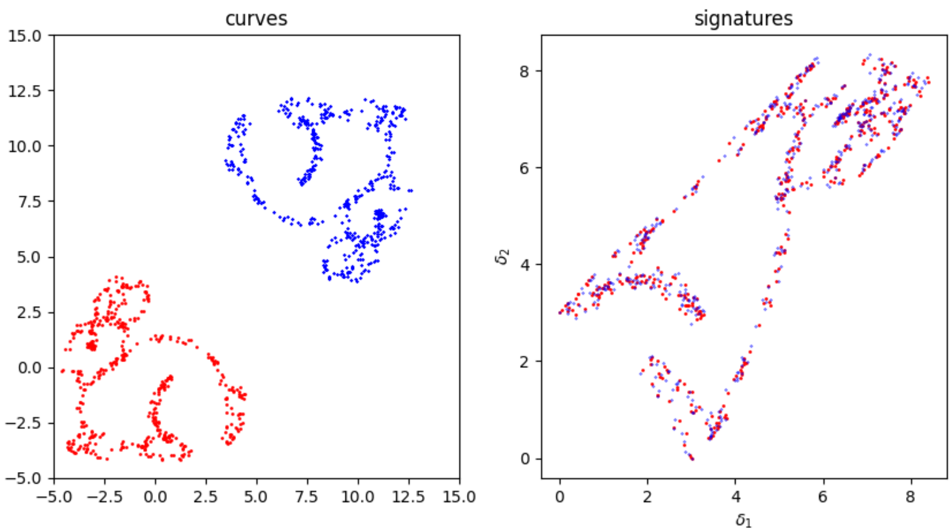
$$\delta_1(i) = \sqrt{(x_i - x_0)^2 + (y_i - y_0)^2}$$
$$\delta_2(i) = \sqrt{(x_i - x_n)^2 + (y_i - y_n)^2}.$$

That is $\delta_1(i)$ is the interpoint distance from the first point (x_0, y_0) to the i^{th} point (x_i, y_i), and $\delta_2(i)$ is the interpoint distance between (x_i, y_i) and (x_n, y_n). Plotting the points $(\delta_1(i), \delta_2(i))$ for $1 \le i \le n$ gives a curve which is invariant under Euclidean transformations of the original data points (x_i, y_i). By comparing the signatures of two image boundaries one can determine the "closeness" of the two images in a way that is invariant of Euclidean transformations.

Figure 1.1 shows two image boundaries in the left image and their corresponding signature curves on the right. The two images on the left are clearly related by a Euclidean transformation, which is reflected in the overlap of their corresponding their signature curves. The code used for generating these images is provided in Appendix A.

The example of $\mathbb{R}^2$ as a homogeneous space of the Euclidean group is an example of what is called a primitive homogeneous space. Primitive homogeneous spaces play a role in the theory similar to irreducible representations in representation theory. If G is a Lie group and H a closed subgroup, and the homogeneous space, G/H, admits a foliation by immersed submanifolds where the elements of the group G map each immersed submanifold to another immersed submanifold in the foliation, then the foliation is called invariant under the action of G. An invariant foliation of G/H defines an equivalence relation where the group G naturally acts on the quotient by this relation, and any joint invariant of the natural action of G on the quotient determines a joint invariant of the original space G/H.

Fig. 1.1: Joint invariant signatures example



A homogeneous space which admits no invariant foliations is called a primitive homogeneous space.

Primitive homogeneous spaces G/H of a Lie group G are classified by the closed Lie subgroups H of G such that H is not contained in any Lie subgroup of higher dimension. In the case that G is a Lie group with Lie algebra $\mathfrak{g}$ which is not simple, the subalgebras $\mathfrak{h}$ corresponding to a closed Lie subgroup H for which G/H is primitive are the maximal subalgebras of $\mathfrak{g}$. Morosoff [13] classified the possible Lie algebra subalgebra pairs which correspond to primitive homogeneous spaces in this case. The classification identifies two main types of Lie algebra subalgebra pair, which we call the affine and symmetric types. When G is a simple Lie group the possible subalgebras for the Lie subgroups H of G which correspond to primitive homogeneous spaces G/H are more complicated. It is still true that every closed subgroup H which has a maximal subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ corresponds to a primitive homogeneous space G/H but there are examples of subgroups H of G which do not have maximal subalgebras while the corresponding homogeneous spaces are primitive, see Golubitsky [10] and the example in Section 5.3.

This thesis builds upon the methods of Olver [17] by proving a reduction theorem which shows that the k point joint invariants of a homogeneous space G/H are determined by the $k-1$ point joint invariants of the isotropy subgroup H, where we identify a partial cross section to the orbits of G and use this to identify the orbits of G on $(G/H)^k$ with the orbits of H on $(G/H)^{k-1}$. In applying this reduction theorem to examples of primitive homogeneous spaces when G is not a simple Lie group we show that the invariants of the affine case are constructed from invariants of the isotropy subgroup representation on an abelian subalgebra of the Lie algebra for $\mathfrak{g}$, and that the two point joint invariants for examples of symmetric type are constructed from invariants of the action of the isotropy subgroup on itself by conjugation See Section 3.3, Theorem 4.4 and Chapter 7.

We organize this thesis as follows. Chapter 2 presents the background information and basic definitions needed for the proof of the reduction theorem in Chapter 3. In Section 3.1 of Chapter 3 we present the main results of the reduction, Theorem 3.1, Corollary 3.1, Corollary 3.2, and Lemma 3.1 give a method for evaluating the joint invariants using a partial cross section. In Section 3.3 we show how the reduction theorem can be applied to an example related to primitive homogeneous spaces of symmetric type by reducing the problem of classifying two point joint invariants to the classification of class functions. Chapter 3 concludes with Section 3.4 which gives some technical results about the reduction which are needed in later chapters. In Chapter 4 we proceed to give a detailed overview of primitive homogeneous spaces. Theorem 4.1 in Section 4.2 shows how to explicitly construct an invariant foliation on a homogeneous space G/H when the subgroup H is contained in a closed subgroup K of G with strictly higher dimension than H. The classification of primitive Lie algebra subalgebra pairs in the case where G is not simple is given in Theorem 4.3. Theorem 4.4 shows that the example investigated in Section 3.3 is a primitive homogeneous space. Chapter 5 gives a description of the possible two dimensional primitive spaces of the simple Lie group $\mathrm{PSL}(2,\mathbb{R})$ via the Adjoint representation, and in Theorem 5.1 identifies which of these homogeneous spaces are primitive. In Section 5.2 we explicitly construct the invariant foliations for the non primitive two dimensional homogeneous spaces and identify them with invariant foliations of the orbits in the Adjoint representation. Then to conclude the chapter Section 5.3 gives an example of a subgroup H of $\mathrm{SL}(2,\mathbb{R})$ such that $\mathrm{SL}(2,\mathbb{R})/H$

is a primitive homogeneous space, and where the isotropy subgroup H does not have a maximal subalgebra.

In Chapter 6 we consider two cases of primitive homogeneous spaces which are of Affine type, $\mathrm{A}(n)$ and $\mathrm{SA}(n)$. Section 6.1 shows that these are primitive homogeneous spaces, and Subsections 6.3 and 6.3 apply the reduction theorem of Chapter 3 to determine complete sets of local joint invariants for these spaces.

Finally Chapter 7 considers two examples of primitive homogeneous spaces of Symmetric type, $\mathrm{SU}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{R})$ which are primitive homogeneous spaces as described in Section 3.3 and Theorem 4.4. For $\mathrm{SU}(2,\mathbb{R})$ we construct a complete set of two and three point joint invariants in Theorem 7.1 and Theorem 7.2. The construction of three point joint invariants is then conducted in an alternative way in Section 7.1.3 which demonstrates that when a slice exists where the isotropy subgroups of every point of a cross section are conjugate, one can again reduce the action to a transitive group action on a product of homogeneous spaces. We then apply the reduction theorem of Chapter 3 again to construct the joint invariants. The last section of Chapter 7 constructs the two point joint invariants for $\mathrm{SL}(2,\mathbb{R})$ and demonstrates the complexity of classifying joint invariants even in the two point case where multiple distinct orbit types are present.

CHAPTER 2

Background

This chapter provides the basic definitions and results that are used in the thesis. For more information about this background material see Bredon [5], or Dummit and Foote [7]. Additionally the books by Boothby [3] Warner [19] and Helgason [11] contain more detailed discussion in the case of smooth actions of Lie groups on manifolds as described in Section 2.2.

2.1 Group Actions

Definition 2.1. Let G be a group and X a set. A map $\mu : G \times X \to X$ is called a (left) *group action* of G on X if μ satisfies the following conditions.

i) If $x \in X$ then $\mu(e, x) = x$.

ii) If $a, b \in G$ and $x \in X$ then

$$\mu(a, \mu(b, x)) = \mu(ab, x).$$

When the context is clear we will denote μ by juxtaposition or a dot,

$$\mu(a, x) = a \cdot x \quad \text{or} \quad \mu(a, x) = ax.$$

Remark 2.1. If $\mu : G \times X \to X$ satisfies part i) of Definition 2.1 but instead of part ii) the map μ satisfies

$$\mu(a, \mu(b, x)) = \mu(ba, x)$$

for all $a, b \in G$ and $x \in X$, then μ is called a right action. Every right action μ can be converted into a left action $\tilde{\mu} : G \times X \to X$ defined by

$$\tilde{\mu}(a, x) = \mu(a^{-1}, x).$$

Left actions of a group G on a set X are equivalent to group homomorphisms from G into permutations on X, denoted $\mathrm{Perm}(X)$.

Theorem 2.1. *Let $\mu : G \times X \to X$ be a map. Then the following conditions are equivalent.*

1) The map μ is a group action.

2) The map $\Phi_\mu : G \to \mathrm{Perm}(X)$ defined by $\Phi_\mu(a) = \mu_a$, where $\mu_a : X \to X$ is

$$\mu_a(x) = \mu(a, x),$$

is a homomorphism of groups.

Proof. Suppose that μ is a group action. Then we first show that Φ_μ is a well defined map. Let $a \in G$ and consider $\Phi_\mu(a) = \mu_a$. The map $\mu_{a^{-1}}$ is a two sided inverse of μ_a, $\mu_a \circ \mu_{a^{-1}} = \mu_{a^{-1}} \circ \mu_a = \mathrm{Id}$, and so $\mu_a \in \mathrm{Perm}(X)$. Now let $a, b \in G$ and fix $x \in X$. Consider

$$\begin{aligned} \Phi_\mu(a) \circ \Phi_\mu(b)[x] &= \mu_a \circ \mu_b(x) \\ &= \mu\big(a, \mu(b, x)\big) \\ &= \mu(ab, x) \\ &= \Phi_\mu(ab)[x]. \end{aligned}$$

So $\Phi_\mu(a) \circ \Phi_\mu(b) = \Phi_\mu(ab)$ which verifies that Φ is a homomorphism, and completes the proof that 1) implies 2).

On the other hand suppose that the map Φ_μ is a homomorphism. Consider $e \in G$ and fix $x \in X$. Then

$$\mu(e,x) = \mu_e(x) = \Phi_\mu(e)[x] = \mathrm{Id}[x] = x$$

since Φ_μ is a homomorphism, and μ satisfies part i) of Definition 2.1. Now let $a, b \in G$ and $x \in X$. Consider

$$\mu(a,\mu(b,x)) = \mu_a \circ \mu_b(x) = \Phi_\mu(a) \circ \Phi_\mu(b)[x] = \Phi_\mu(ab)[x] = \mu_{ab}(x) = \mu(ab,x)$$

which verifies part ii) of Definition 2.1. Hence μ is a group action verifying that 2) implies 1) and completing the proof. □

Remark 2.2. A similar argument to the proof of Theorem 2.1 shows that right actions $\mu : G \times X \to X$ are equivalent to "antihomomorphisms" of G into $\mathrm{Perm}(X)$. That is, maps $\Psi_\mu : G \to \mathrm{Perm}(X)$ such that $\Psi_\mu(ab) = \Psi_\mu(b) \circ \Psi_\mu(a)$. The process in Remark 2.1 for constructing a left action given a right action, μ, corresponds to saying every right action Ψ_μ can be converted into a left action $\Phi_{\tilde{\mu}}$ by precomposing Ψ_μ with the inversion map $\mathrm{inv} : G \to G$ given by $\mathrm{inv}(a) = a^{-1}$. That is $\Phi_{\tilde{\mu}} = \Psi_\mu \circ \mathrm{inv}$ is a group homomorphism, and therefore the corresponding action $\tilde{\mu}$ is a left group action.

Below we define two important kinds of group actions.

Definition 2.2. Let $\mu : G \times X \to X$ be a group action. If for every fixed $x \in X$

$$\mu(a,x) = x$$

implies $a = e$, where e is the identity element of G, then the action is called *free.*

See part 1) of Remark 2.5 for an alternative characterization of a free action.

Definition 2.3. A group action $\mu : G \times X \to X$ where

$$\mu(a, x) = x$$

for all $x \in X$ implies $a = e$ is called *effective* or *faithful*.

Note that an action μ is effective if and only if the map Φ_μ from part 2) of Theorem 2.1 has a trivial kernel, and so defines an isomorphism of G with a subgroup of the permutation group $\mathrm{Perm}(X)$. See Remark 2.7 for an equivalent characterization of effective actions.

A group action μ defines a relation on X,

$$R_\mu = \{(x_1, x_2) \in X \times X \,|\, x_2 = \mu(a, x_1),\ a \in G\} \subset X \times X. \tag{2.1}$$

We would like to determine necessary and sufficient conditions for when $(x_1, x_2) \in R_\mu$, which motivates the following definition.

Definition 2.4. Let $\mu : G \times X \to X$ be a group action. Two points $x_1, x_2 \in X$ are said to be *congruent* if there exists $a \in G$ such that $x_1 = \mu(a, x_2)$, and the set

$$[x_1]_\mu = \{x \in X \,|\, (x_1, x) \in R_\mu\}$$

is called the *congruence class* or *orbit* of x_1.

When the action is understood from context, the congruence class or orbit of x is sometimes called the G orbit of x and denoted by $[x]_G$, or we will use Gx for the orbits of a left action and xG for the orbits of a right action.

In this thesis we want to solve the congruence problem for an action μ, that is to find a set of necessary and sufficient conditions for when two points $x_1, x_2 \in X$ are congruent. Specifically we are interested in solving the congruence problem for a family of actions induced by a given μ as described below.

Whenever an action $\mu : G \times X \to X$ is fixed there is an induced action of the group G on products of X.

Definition 2.5. Let $\mu : G \times X \to X$ be an action of a group G on a set X. The *diagonal action* of G on the product set

$$X^k := \underbrace{(X \times X \times \cdots \times X)}_{k \text{ copies}}$$

is the map $\mu^k : G \times X^k \to X^k$ defined by

$$\mu^k \left(a, \left(x_1, x_2, \ldots, x_k\right)\right) = \left(\mu\left(a, x_1\right), \mu\left(a, x_2\right), \ldots, \mu\left(a, x_k\right)\right).$$

Subsets $U \subset X$ where the congruence class of any point $u \in U$ is a subset of U itself are useful because any point that u is congruent to must be an element of U. So for the purposes of solving the congruence problem, we can restrict our attention to the subset U instead of the whole set X which motivates the following definition.

Definition 2.6. Let $\mu : G \times X \to X$ be an action of a group G on a set X. If $U \subset X$ is a subset which satisfies

$$\mu(a, u) \in U \tag{2.2}$$

for all $u \in U$ and $a \in G$, then the set U is called an *invariant* subset of X with respect to μ.

If $U \subset X$ is an invariant subset then the restriction of the action map in the second argument, $\mu|_U : G \times U \to U$, is a well defined action of G on U. In most examples the solution to the congruence problem is solved on the invariant subsets of X with respect to μ.

Note that in particular the orbit of any point $x_0 \in X$

$$[x_0]_\mu = \{x \in X \mid (x_0, x) \in R_\mu\},$$

as in Definition 2.4 is an invariant subset of X. The next section shows that R_μ is an equivalence relation and therefore the set of all congruence classes form a partition of X.

2.1.1 Orbits

For a given action $\mu : G \times X \to X$ of a group G on a set X the relation R_μ is an equivalence relation, which we will denote by $x_1 \sim_\mu x_2$ if $(x_1, x_2) \in R_\mu$.

Remark 2.3. Let $\mu : G \times X \to X$ be a group action. The equivalence class of a point $x_0 \in X$ under the relation $\sim_\mu$ as given in Definition 2.4 is the orbit or congruence class of x_0 and is equal to the set

$$[x_0]_\mu = \{x \in X \,|\, x = \mu(a, x_0),\, a \in G\}\,.$$

Remark 2.3 motivates another way to define the orbits of μ. Fix the second argument of the action by picking some $x_0 \in X$, and considering the map $\mu_{x_0} : G \to X$ defined by

$$\mu_{x_0}(a) = \mu(a, x_0). \tag{2.3}$$

The orbit of x_0 with respect to μ is the image of μ_{x_0}, that is

$$[x_0]_\mu = \mu_{x_0}(G).$$

Since the orbits of μ partition X we can define the quotient of X as the set of equivalence classes of R_μ.

Definition 2.7. Let $\mu : G \times X \to X$ be a group action. We denote by X/μ, the set of $\sim_\mu$ equivalence classes, called the *orbit space* of X mod G. The notation X/G for the orbit space will also be used.

2.1.2 Invariant functions

Let X, be a set with an equivalence relation R and denote equivalence with respect to R by $x_1 \sim x_2$ if and only if $(x_1, x_2) \in R$. Functions $f : X \to Y$ which respect $\sim$ in the sense that they are constant on the equivalence classes of R are defined as follows.

Definition 2.8. Let R be an equivalence relation on a set X. A function $f : X \to Y$ such that $x_1 \sim x_2$ implies that $f(x_1) = f(x_2)$, is called an *invariant* of the equivalence relation R. Or when the relation is clear from context, f is called an invariant.

Let $[x] \subset X$ denote the equivalence classes of R in X. The quotient of X by R is the set of equivalence classes of R, denoted by $\tilde{X} = \{[x] \,|\, x \in X\}$. Let $\pi : X \to \tilde{X}$ be the map which takes each element $x \in X$ to its equivalence class $[x]$,

$$\pi(x) = [x].$$

The map π is called the quotient map with respect to R and is an invariant, in fact a stronger statement can be made, $\pi(x_1) = \pi(x_2)$ if and only if $x_1 \sim x_2$. This is immediate because x_1 and x_2 belong to the same equivalence class if and only if $x_1 \sim x_2$.

The next theorem shows that the invariant functions of R are in one to one correspondance with functions on the set of equivalence classes $\tilde{X}$.

Theorem 2.2 (Universal Property of Quotients)**.** *Let R be an equivalence relation on a set X, $f : X \to Y$ be a Y valued function on X, and $\pi : X \to \tilde{X}$ be the quotient map.*

The function f is an invariant if and only if there a exists a unique function $\tilde{f} : \tilde{X} \to Y$ such that $f = \tilde{f} \circ \pi$.

Proof. Suppose there exists a function $\tilde{f} : \tilde{X} \to Y$ such that $f = \tilde{f} \circ \pi$. Let $x_1, x_2 \in X$ be two points such that $x_1 \sim x_2$, and consider $f(x_1), f(x_2)$. Since $\pi(x_1) = \pi(x_2)$ then

$$f(x_1) = \tilde{f} \circ \pi(x_1) = \tilde{f} \circ \pi(x_2) = f(x_2),$$

which proves that f is an invariant of the equivalence relation R.

Conversely suppose that $f : X \to Y$ is an invariant of the equivalence relation R. We will define a function $\tilde{f} : \tilde{X} \to Y$ by the following logic. Let $[x] \in \tilde{X}$, and let $x \in X$ be any element such that $\pi(x) = [x]$. Then let the value of $\tilde{f}([x])$ be $\tilde{f}([x]) = f(x)$.

We now show that $\tilde{f}$ is well defined. Suppose to the contrary that there exists $[x] \in X$ and $y_1, y_2 \in Y$ with $y_1 \neq y_2$ such that $\tilde{f}([x]) = y_1$ and $\tilde{f}([x]) = y_2$. By the definition of $\tilde{f}$

there exist $x_1, x_2 \in X$ with $\pi(x_1) = [x]$ and $\pi(x_2) = [x]$ such that $f(x_1) = y_1$, and $f(x_2) = y_2$. But since $\pi(x_1) = \pi(x_2)$ then $x_1 \sim x_2$ and therefore $f(x_1) = f(x_2)$ which is a contradiction in assuming that $y_1 \neq y_2$. Hence the function $\tilde{f}$ is well defined. Moreover, the function $\tilde{f}$ satisfies $f = \tilde{f} \circ \pi$ by construction.

Now we show that $\tilde{f}$ is unique. Suppose there are two functions $\tilde{f}, \tilde{g}$ such that

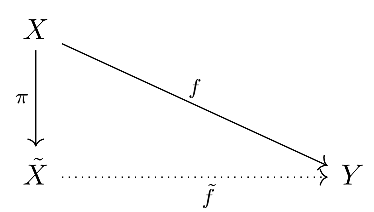
$$\tilde{f} \circ \pi = f = \tilde{g} \circ \pi.$$

Let $[x] \in \tilde{X}$. Then there exists an element $x \in X$ such that $\pi(x) = [x]$ and

$$\tilde{g}([x]) = f(x) = \tilde{f}([x])$$

so $\tilde{g} = \tilde{f}$ and the function $\tilde{f}$ is unique. □

The proof above is equivalent to the statement that there exists a unique function $\tilde{f}$ which makes the diagram



(2.4)

commute if and only if f is an invariant function.

Because an invariant $f : X \to Y$ is constant on the equivalence classes of R the condition $f(x_1) = f(x_2)$ is necessary for $x_1 \sim x_2$. Invariants with the stronger condition that $f(x_1) = f(x_2)$ if and only if $x_1 \sim x_2$ play a key role in understanding the equivalence classes of R, motivating the following definition.

Definition 2.9. Let X be a set and R be an equivalence relation on X. If $\{f^\alpha\}_{\alpha \in A}$ is a collection of functions, $f^\alpha : X \to Y$, such that $f^\alpha(x_1) = f^\alpha(x_2)$ for all $\alpha \in A$ if and only if $x_1 \sim x_2$ then we call $\{f^\alpha\}_{\alpha \in A}$ a *complete set of invariants* of the equivalence relation R. If the set $\{f^\alpha\}_{\alpha \in A}$ contains a single function, f, then f is called a complete invariant of R.

Given an action μ and a complete set of invariants, $\{f^\alpha\}_{\alpha\in A}$ for R_μ as defined in Equation (2.1), the collection $\{f^\alpha\}_{\alpha\in A}$ give necessary and sufficient conditions for two elements of X to be congruent.

The following corollary of Theorem 2.2 shows that the set of equivalence classes, $\tilde{X}$ is unique in the sense that if $f : X \to Y$ is a complete Y valued invariant of R which is surjective, then f induces a canonical bijection between $\tilde{X}$ and Y.

Corollary 2.1 (Uniqueness of Quotients)**.** *Let $f : X \to Y$ be a complete Y valued invariant of R which is also a surjective function. Then the unique map $\tilde{f} : \tilde{X} \to Y$ such that $f = \tilde{f} \circ \pi$, is a bijection.*

Proof. Fix $y \in Y$. Since f is surjective there exists an element $x \in X$ such that $f(x) = y$. Let $[x] = \pi(x)$ and consider

$$\tilde{f}([x]) = \tilde{f} \circ \pi(x) = f(x) = y$$

and $\tilde{f}$ is surjective. We now show that $\tilde{f}$ is injective. Let $[x]_1, [x]_2 \in \tilde{X}$ be such that $\tilde{f}([x]_1) = \tilde{f}([x]_2)$. Then there exist elements $x_1, x_2 \in X$ such that $\pi(x_1) = [x]_1$ and $\pi(x_2) = [x]_2$ with $\tilde{f}([x]_1) = f(x_1)$ and $\tilde{f}([x]_2) = f(x_2)$. So $f(x_1) = f(x_2)$, and $x_1 \sim x_2$ since f is a complete invariant. Hence $[x]_1 = [x]_2$ verifying that $\tilde{f}$ is injective. □

Corollary 2.1 shows that whenever one finds a complete surjective invariant $f : X \to Y$ of the relation R then the quotient of X by R can be replaced by Y without any loss of generality.

We will mostly be concerned with the relations R_μ from Equation (2.1) induced by group actions, which motivates the following definition.

Definition 2.10. Let $\mu : G \times X \to X$ be an action of G on X. Then a function $f : X \to Y$ which is invariant under the relation R_μ,

$$f\left(\mu(a, x)\right) = f(x)$$

for $a \in G$ and $x \in X$, is called a Y valued invariant of μ. Sometimes when the action μ is clear f is just called a Y valued invariant of G.

Let μ be an action of G on X and consider two a collections of k points in X, $(x_1, \dots, x_{k-1})$ and $(y_1, \dots, y_{k-1})$. The invariants of the diagonal action μ^k as in Definition 2.5 give necessary conditions for when $(x_1, \dots, x_{k-1})$ and $(y_1, \dots, y_{k-1})$ are congruent. These invariants are defined below.

Definition 2.11. Let $\mu : G \times X \to X$ be a group action and let $f : X^k \to Y$ be a map satisfying,

$$f\left(\mu(a, x_1), \dots, \mu(a, x_k)\right) = f\left(x_1, \dots, x_k\right) \quad \forall\, a \in G,$$

then f is called a *$k-$point joint invariant of μ.*

Note that the k point joint invariants of μ are the invariants of the diagonal action μ^k from Definition 2.5.

2.1.3 Equivariant functions

The homomorphisms between spaces with an action of G are defined as follows.

Definition 2.12. Let $\mu_X : G \times X \to X$ and $\mu_Y : G \times Y \to Y$ be actions of a group G on sets X and Y respectively. A function $\phi : X \to Y$ is said to be *equivariant* with respect to μ_X and μ_Y if

$$\phi\left(\mu_X(a, x)\right) = \mu_Y\big(a, f(x)\big) \tag{2.5}$$

for all $a \in G$, and $x \in X$.

If μ_X and μ_Y are actions of a group G, the set of equivariant functions respect the action of G on these sets. In particular the image of invariant subsets of X with respect to μ_X are invariant subsets of Y with respect to μ_Y.

Theorem 2.3. *Let $\mu_X : G \times X \to X$ and $\mu_Y : G \times Y \to Y$ be actions of a group G on sets X and Y respectively. Then if $\phi : X \to Y$ is an equivariant function and $U \subset X$ is an invariant subset with respect to μ_X then $\phi(U) \subset Y$ is an invariant subset with respect to μ_Y.*

Proof. Let $V = \phi(U)$. Fix some $v \in V$ and $a \in G$. Then consider $\mu_Y(a, v)$. Since $v \in \phi(U)$ then there exists some $u \in U$ with $\phi(u) = v$. Now using the equivariance of ϕ we have

$$\mu_Y(a, v) = \mu_Y(a, \phi(u)) = \phi(\mu_X(a, u)),$$

where $\mu_X(a, u) = u' \in U$ since U is an invariant subset of μ_X. Hence $\mu_Y(a, v) = \phi(u') \in V$ completing the proof. □

The orbits of an action μ_X are invariant subsets, so an equivariant function will map orbits of μ_X to orbits of μ_Y in Y.

If $\phi : X \to Y$ is equivariant and a bijection of the sets X and Y then ϕ induces a canonical bijection between the orbit spaces X/μ_X and Y/μ_Y. In fact one can make a more general statement that will be useful in proving the results of Chapter 3. If isomorphic groups G and K act on sets X and Y by μ and θ respectively, and $\Phi : X \to Y$ satisfies

$$\phi\left(\mu(a, x)\right) = \theta\left(\sigma(a), \phi(x)\right)$$

then Φ induces a canonical bijection between the orbit spaces X/μ and Y/θ. This claim is proved in the following theorem.

Theorem 2.4. *Let $\mu : G \times X \to X$ be an action of a group G on X, with $\pi_\mu : X \to X/G$ its quotient map, and let $\theta : K \times Y \to Y$ be an action of a group K on Y with $\pi_\theta : Y \to Y/K$ as its quotient map. Let $\sigma : G \to K$ be an isomorphism of groups. If $\phi : X \to Y$ is a bijection which satisfies the identity*

$$\phi\left(\mu(a, x)\right) = \theta\left(\sigma(a), \phi(x)\right) \tag{2.6}$$

then the map $\Phi : X/G \to Y/K$ *defined by*

$$\Phi\big([x]_G)\big) = \{\phi(x)\}_K,$$

where $[x]_G$ *is the orbit of* x *under* G *and* $\{\phi(x)\}_K$ *is the orbit of* $\phi(x)$ *under* K, *is a unique bijection between the orbit spaces such that* $\Phi \circ \pi_{\mu_G} = \pi_{\mu_K} \circ \phi$.

Proof. First we show that Φ is well defined. Suppose that we have two points $x_1, x_2 \in X$ such that $[x_1]_G = [x_2]_G$, then there exists $a \in G$ such that $\mu(a, x_1) = x_2$. So

$$\Phi\big([x_2]_G\big) = \{\phi(x_2)\}_K = \big\{\phi\left(\mu\big(a, x_1)\right)\big\}_K = \big\{\theta\left(\sigma(a), \phi\big(x_1)\right)\big\}_K = \big\{\phi\big(x_1\big)\big\}_K = \Phi\big([x_1]_G),$$

and Φ is well defined.

Now we show Φ is an injection. Let $[x_1]_G, [x_2]_G \in X/G$ be such that $\Phi([x_1]_G) = \Phi([x_2]_G)$. Then $\{\phi(x_1)\}_K = \{\phi(x_2)\}_K$ and there are representatives $\phi(x_1), \phi(x_2)$ of the K equivalence classes such that

$$\theta(k, \phi(x_1)) = \phi(x_2)$$

for some $k \in K$. Pick $a \in G$ such that $\sigma(a) = k$. From the identity in Equation (2.6)

$$\phi\left(\mu\big(a, x_1)\right) = \theta\left(\sigma(a), \phi\big(x_1)\right) = \phi(x_2)$$

and since ϕ is a bijection then $\mu(a, x_1) = x_2$, so that $[x_1]_G = [x_2]_G$ proving Φ is injective.

Now we show that Φ is surjective. Let $\{y\}_K \in Y/K$. Fix any representative y for $[y]_K$, and let x be the pre-image of y under ϕ. Then

$$\Phi([x]_G) = \{\phi(x)\}_K = \{y\}_K$$

and Φ is onto.

Now suppose that $\Gamma : X/G \to Y/K$ is a bijection which satisfies $\Gamma \circ \pi_\mu = \pi_\theta \circ \phi$. Fix some $[x]_G \in X/G$. If $x \in X$ is a representative for $[x]_G$ so that $p_\mu(x) = [x]_G$ then

$$\Gamma([x]_G) = \Gamma \circ \pi_\mu(x) = \pi_\theta \circ \phi(x)$$

and $\pi_\theta \circ \phi(x) = \{\phi(x)\}_K = \Phi([x]_G)$ verifying uniqueness. □

Whenever an equivariant bijection is present as in Theorem 2.4 then the canonical identification of orbit spaces X/μ_X and Y/μ_Y provides a canonical identification of the Z valued invariants of μ_Y with the Z valued invariants of μ_X.

Corollary 2.2. *Let $\mu_X : G \times X \to X$ and $\mu_Y : G \times Y \to Y$ be actions of a group G on the sets X and Y respectively. If $\phi : X \to Y$ is an equivariant bijection with respect to μ_X and μ_Y then the map $\phi^* : \mathcal{F}(Y, Z) \to \mathcal{F}(X, Z)$ given by $\phi^*(f) = f \circ \phi$ is a bijection between the set of μ_Y invariants and the set of μ_X invariants.*

The next definition will be used to provide an equivalent definition of both equivariant and invariant functions.

Definition 2.13. Let $\mu : G \times X \to X$ be an action of a group G on a set X. An element $x \in X$ such that

$$\mu(a, x) = x$$

for all $a \in G$ is called a *fixed point* of the action μ. The set of all fixed points of μ in X denoted X^μ or X^G, is the set

$$X^\mu = \{x \in X \,|\, \mu(a, x) = z,\ \forall a \in G\}.$$

Let $\mu_X : G \times X \to X$ and $\mu_Y : G \times Y \to Y$ be actions of G on X and Y. Then $\theta : G \times \mathcal{F}(X, Y) \to \mathcal{F}(X, Y)$ defined by

$$\mu(a, f)[x] = \mu_Y\left(a, f\big(\mu_X(a^{-1}, x)\big)\right) \tag{2.7}$$

is an action on the set of Y valued functions on X. The fixed point set of θ from Equation (2.7), is equal to the set of Y valued equivariant functions on X with respect to μ_X and μ_Y.

Theorem 2.5. *Let μ_X and μ_Y be actions of a group G on the sets X and Y. A function $f : X \to Y$ is equivariant if and only if f is in the set of fixed points,*

$$\mathcal{F}(X,Y)^\theta = \left\{ f : X \to Y \,\middle|\, \theta\left(a, f\right)\left(x\right) = f(x)\ \forall x \in X, a \in G \right\},$$

where θ is the action from Equation (2.7)*.*

Proof. First suppose that $f \in \mathcal{F}(X,Y)^\theta$. Fix some $x \in X$ and $a \in G$. Since f is a fixed point of θ then

$$\begin{aligned} \theta(a,f)(x) &= f(x) \\ \mu_Y\left(a, f\left(\mu\left(a^{-1}, x\right)\right)\right) &= f(x) \\ f\left(\mu_X\left(a^{-1}, x\right)\right) &= \mu_Y\left(a^{-1}, f(x)\right). \end{aligned}$$

So f satisfies Equation (2.5) and f is equivariant.

Conversely suppose that f is equivariant. Fix $x \in X$ and $a \in G$. Consider the value of

$$\theta(a,f)(x) = \mu_Y\left(a, f\left(\mu_X\left(a^{-1}, x\right)\right)\right) = \mu_Y\left(a, \mu_Y\left(a^{-1}, f(x)\right)\right)$$

by equivariance of f. Then using that μ_Y is an action $\theta(a,f)(x) = f(x)$ and $f \in \mathcal{F}(X,Y)^\theta$. □

If the action μ_Y in Equation (2.7) is trivial, that is

$$\mu_Y(a,y) = y,$$

then the equivariant functions with respect to μ_X and μ_Y are the Y valued invariant functions on X.

Corollary 2.3. *Let μ_Y be the trivial action $\mu_Y(a, y) = y$. Then the set of equivariant functions with respect to μ_X and μ_Y is equal to the set of Y valued invariant functions of μ_X.*

That is the invariant functions of an action μ_X are the fixed points of the induced action on functions, where the group G acts trivially on the codomain.

2.1.4 Commuting Actions

Now consider the situation where two groups H and K act on the same set X.

Definition 2.14. Let H and K act on a set X by $\mu_H : H \times X \to X$ and $\mu_K : K \times X \to X$. If

$$\mu_H\big(h, \mu_K(k, x)\big) = \mu_K\big(k, \mu_H(h, x)\big)$$

for all $h \in H$, $k \in K$, and $x \in X$ then the actions are said to *commute.*

The following lemma records some properties of commuting actions used in the thesis.

Lemma 2.1. *Let H and K be groups and X be a set. Suppose that $\mu_H : H \times X \to X$ and $\mu_K : K \times X \to X$ are actions of H and K on X respectively, and let $\pi_{\mu_H} : X \to X/\mu_H$ and $\pi_{\mu_K} : X \to X/\mu_K$ be the quotient maps. If the actions of H and K commute then:*

(i) the group K acts on X/μ_H by $\eta_K : K \times X/\mu_H \to X/\mu_H$ given by

$$\eta_K\left(k, \pi_{\mu_H}(x)\right) = \pi_{\mu_H}\left(\mu_K(k, x)\right) \quad x \in X, \tag{2.8}$$

(ii) and the group H acts on X/μ_K by $\eta_H : H \times X/\mu_K \to X/\mu_K$ given by

$$\eta_H\left(h, \pi_{\mu_K}(x)\right) = \pi_{\mu_K}\left(\mu_H(h, x)\right) \quad x \in X. \tag{2.9}$$

(iii) The projection maps $\pi_{\mu_H} : X \to X/\mu_H$ and $\pi_{\mu_K} : X \to X/\mu_K$ are equivariant with respect to the induced actions η_K and η_H respectively.

(iv) There is a canonical bijection, $\tau : (X/\mu_K)/\eta_H \to (X/\mu_H)/\eta_K$*, given by*

$$\tau\left(\pi_{\eta_H}\left(\pi_{\mu_K}(x)\right)\right) = \pi_{\eta_K}\left(\pi_{\mu_H}(x)\right), \quad x \in X.$$

Proof. Proof of part (i). First we show that the maps in Equation (2.8) are well defined. That is for each $k \in K$ that $\eta_K(k,\cdot) : X/\mu_H \to X/\mu_H$, is well defined. Suppose that $x, y \in X$ define the same H orbit. Then there exists $h \in H$ such that $\mu_H(h,x) = y$. Now consider

$$\begin{aligned}
\eta_K(k,\pi_H(y)) &= \pi_{\mu_H}\left(\mu_K(k,y)\right) \\
&= \pi_{\mu_H}\Bigg(\mu_K\Bigg(k,\mu_H\big(h,x\big)\Bigg)\Bigg) \\
&= \pi_{\mu_H}\Bigg(\mu_H\Bigg(h,\mu_K\big(k,x\big)\Bigg)\Bigg) \\
&= \pi_{\mu_H}\left(h,\mu_K\big(k,x\big)\right) \\
&= \eta_K\left(k,p_H(x)\right),
\end{aligned}$$

so $\eta_K(k,\cdot)$ is well defined. Showing that η_K satisfies Definition 2.1 is clear and will be omitted from the proof. The proof of part (ii) is similar to the argument for part (i).

Now we give the proof of part (iii). By construction the projection map π_H satisfies

$$\pi_{\mu_H}\left(\mu_K(k,x)\right) = \eta_K\left(k,\pi_{\mu_H}(x)\right)$$

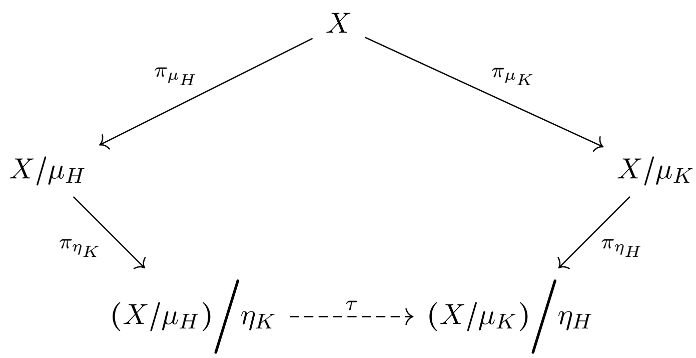
which proves that π_{μ_H} is equivariant, and the argument for π_{μ_K} is similar.

Finally for the proof of part (iv) we note that $\sigma : (X/\mu_H)\big/\eta_K \to (X/\mu_K)\big/\eta_H$ given by

$$\sigma\left(\pi_{\eta_K}\left(\pi_{\mu_H}(x)\right)\right) = \pi_{\eta_H}\left(\pi_{\mu_K}(x)\right),$$

satisfies $\tau \circ \sigma = \mathrm{Id}_{(X/\mu_H)/\eta_K}$ and $\sigma \circ \tau = \mathrm{Id}_{(X/\mu_K)/\eta_H}$ so $\sigma = \tau^{-1}$ and τ is a bijection which completes the proof. □

The proof above is equivalent to saying that the diagram



commutes.

By Lemma 2.8 if μ_G and μ_K are commuting actions on X then there is an induced action of K on X/μ_G. The following theorem, an extension of Corollary 2.1, shows that if K acts on a space Y and X covers Y by a complete invariant of μ_G which is also equivariant with respect to the action of K, then Y can be considered a unique relabeling of X/G which respects the actions of K.

Theorem 2.6 (Uniqueness of quotients by commuting actions)**.** *Let μ_G and μ_K be commuting actions on a set X and suppose that $\theta_K : K \times Y \to Y$ is an action of K on a set Y.*

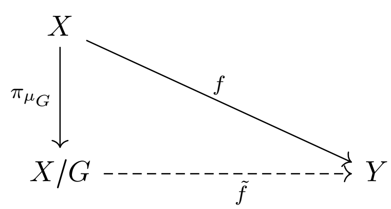
If $f : X \to Y$ is a complete surjective invariant and equivariant with respect to μ_K and θ_K then the unique map $\tilde{f} : X/\mu_K \to Y$ such that $\tilde{f} \circ \pi_{\mu_G}$ is an equivariant bijection with respect to the induced action of K on X/μ_G, η_K, and the action θ_K.

Proof. Corollary 2.1 shows that the map $\tilde{f} : X/\mu_G \to Y$ is a canonical bijection, so we will show that it is equivariant with respect to η_K and θ_K. Let $k \in K$ and $[x]_G \in X/\mu_G$. Pick any representative $x \in X$ such that $\pi_{\mu_G}(x) = [x]_G$, and consider

$$\begin{aligned}
\tilde{f}\left(\eta_K\left(k,[x]_G\right)\right) = \tilde{f}\left(\eta_K\left(k,\pi_{\mu_G}(x)\right)\right) &= \tilde{f}\left(\pi_{\mu_G}\left(\mu_K\left(k,x\right)\right)\right) \\
&= \tilde{f} \circ \pi_{\mu_G}\left(\mu_K\left(a,x\right)\right) \\
&= f\left(\mu_K(a,x)\right) \\
&= \theta_K\left(a, f(x)\right) \\
&= \theta_K\left(a, \tilde{f} \circ \pi_{\mu_G}(x)\right) \\
&= \theta_K\left(a, \tilde{f}\left([x]_G\right)\right),
\end{aligned}$$

which completes the proof. □

The theorem above verifies that the following diagram



commutes for a unique $\tilde{f}$ which is an equivariant bijection with respect to the induced action η_K of K on X/μ_G and θ_K on Y.

2.1.5 Stabilizers and Isotropy

Let $\mu : G \times X \to X$ be a left group action and let $x \in X$ be fixed. Consider the map $\mu_x : G \to X$ as defined in Equation 2.3,

$$\mu_x(a) = \mu(a,x),$$

where the image of μ_x is the orbit of the element $x \in X$. In general the map μ_x will not be injective, however if $\mu_x(a) = \mu_x(b)$, then a, b lie in the same left coset of a subgroup of G.

Definition 2.15. Let $\mu : G \times X \to X$ be a group action of G on a set X. Then for each $x \in X$ the set

$$G_x = \left\{a \in G \,|\, \mu_x(a) = x\right\}, \tag{2.10}$$

is a subgroup of G called the *isotropy* or *stabilizer* subgroup of x.

Lemma 2.2. *Let $\mu : G \times X \to X$ be a left group action. Then for each $x \in X$ there exist $a, b \in G$ such that $\mu_x(a) = \mu_x(b)$ if and only if $aG_x = bG_x$.*

Proof. Let $x \in X$ be fixed, and suppose that $a, b \in G$ are elements such that μ_x satisfies $\mu_x(a) = \mu_x(b)$. Then

$$\mu(a, x) = \mu(b, x)$$

which implies that $a^{-1}b \in G_x$, and therefore $bG_x = aG_x$. Conversely if $a, b \in G$ are such that $aG_x = bG_x$ then $a = bh$ for some $h \in G_x$. Hence,

$$\mu(b, x) = \mu(ah, x) = \mu(a, x)$$

and $\mu_x(a) = \mu_x(b)$ completing the proof. □

Remark 2.4. If μ is taken to be a right action in Lemma 2.2, then the condition

$$\mu(a, x) = \mu(b, x)$$

implies that a, b are in the same right coset, $G_x a = G_x b$, since $ab^{-1} \in G_x$.

Since G_x is a subgroup of G then the condition $a \sim_{G_x} b$ if and only if a and b are in the same coset is an equivalence relation, and the quotient of G by $\sim_{G_x}$ is the space of cosets G/G_x. Lemma 2.2 shows that the action map μ_x is a complete invariant of $\sim_{G_x}$ and therefore the image of μ_x, the orbit of x, is canonically bijective with the coset space G/G_x.

Moreover the induced bijection $\tilde{\mu}_x$ is equivariant with respect to the action of G on G/G_x, the space of right cosets, by left multiplication and μ on X.

Theorem 2.7. *Let G act on G/G_x by left multiplication and let $\mu : G \times X \to X$ be an action of G on X. If $x \in X$ then the map $\tilde{\mu}_x : G/G_x \to X$, given by*

$$\tilde{\mu}_x(aG_x) = \mu_x(a) = \mu(a, x), \tag{2.11}$$

is a well defined bijection of G/G_x with the orbit $[x]_\mu$ which is equivariant with respect to the standard action of G on G/G_x by left multiplication and the restriction of μ to $[x]_\mu$.

Proof. From Lemma 2.2 the map μ_x is a complete invariant of the equivalence relation on G given by the right cosets of G_x, and μ_x is a surjective function onto the orbit of x. Then Corollary 2.1 implies $\tilde{\mu}_x$ is a well defined bijection of G/G_x and $[x]_\mu$. So we show that $\tilde{\mu}_x$ is equivariant. Let $\alpha \in G$, and fix some $aG_x \in G/G_x$. If $a \in G$ is any representative of the coset aG_x then αa is a representative of $\alpha a G_x$, the image of aG_x under left multiplication by α. Now consider,

$$\tilde{\mu}_x(\alpha a G_x) = \mu_x(\alpha a) = \mu(\alpha a, x) = \mu(\alpha, \mu(a, x)) = \mu(\alpha, \mu_x(a)) = \mu(\alpha, \tilde{\mu}_x(aG_x)),$$

which verifies that $\tilde{\mu}_x$ is equivariant with respect to left multiplication by G on G/G_x and μ and completing the proof. □

Remark 2.5. let $\mu : G \times X \to X$ be an action.

1) The action μ is free as in Definition 2.2 if and only if for every $x \in X$ the stabilizer G_x is the trivial subgroup, $G_x = \{e\}$.

2) A point $x \in X$ is a fixed point of μ as in Definition 2.13 if and only if the stabilizer of x is the whole group, $G_x = G$.

Now we show that if two points in X lie on the same orbit their stabilizer subgroups are conjugate.

Lemma 2.3. *Let G act on a set X and let $x \in X$. If the point $y \in X$ satisfies $y = a \cdot x$ for some $a \in G$ then the isotropy G_y is conjugate to the isotropy G_x by the element a,*

$$G_y = aG_xa^{-1}.$$

Proof. Let $h \in G_x$ be an element of the isotropy for x, so that $h \cdot x = x$. We can substitute $a^{-1} \cdot y = x$ to get $h(a^{-1}y) = a^{-1}y$. Isolating y on the right hand side gives

$$\left(aha^{-1}\right) \cdot y = y,$$

so $aha^{-1} \in G_y$ and $aG_xa^{-1} \subset G_y$.

On the other hand let $k \in G_y$. Then $a \cdot x = y$ and $k \cdot (a \cdot x) = a \cdot x$ so $a^{-1}ka \in G_x$. Then $k \in aG_xa^{-1}$ and $G_y \subset aG_xa^{-1}$. □

Example 2.1.1. Let μ be an action of G on X. Then for any $x \in X$ by Theorem 2.7 the map $\tilde{\mu}_x : G/G_x \to X$ is a canonical bijection of G/G_x with the orbit of x. Suppose that $y \in [x]_\mu$ is some other point in the same orbit. Then since μ_x is surjective onto $[x]_\mu$ there exists $a \in G$ such that $y = a \cdot x$. By Lemma 2.3 the stabilizer of y is

$$G_y = aG_xa^{-1}.$$

Again by Theorem 2.7 the map $\tilde{\mu}_y : G/G_y \to X$,

$$\tilde{\mu}_y([b]_{G_y}) = b \cdot y$$

is a canonical bijection. The inverse $\tilde{\mu}_y^{-1} : X \to G/G_y$ is given by

$$\tilde{\mu}_y^{-1}(z) = \tilde{\mu}_y^{-1}(c \cdot y) = cG_y$$

for all $z \in X$. We know this map is well defined since if $c_1 \cdot x = c_2 \cdot x$ then $c_1G_x = c_2G_x$.

Then there is a canonical bijection between the two coset spaces, $\hat{\mu} : G/G_x \to G/G_y$ given by

$$\hat{\mu} = \tilde{\mu}_y^{-1} \circ \tilde{\mu}_x$$

explicitly this is

$$\begin{aligned}
\hat{\mu}\big(bG_x\big) &= \tilde{\mu}_y^{-1} \circ \tilde{\mu}_x(bG_x) \\
&= \tilde{\mu}_y^{-1}(b \cdot x) \\
&= \tilde{\mu}_y^{-1}(ba^{-1} \cdot y) \\
&= ba^{-1}[e]_{G_y} a^{-1} \\
&= bG_x a^{-1}
\end{aligned}$$

That is once a representative of the conjugacy class for G_x is fixed there is a canonical bijection between the coset spaces G/G_x and G/G_y for any y on the same orbit as x given by the mapping $G_x \mapsto G_x a^{-1}$.

When an action μ has only one orbit equal to the whole set X, then X is parameterized by G/G_x for any point $x \in X$. This motivates the definition below.

Definition 2.16. Let G be a group acting on a set X.

i) The group G acts transitively if for each $x, y \in X$ there exists $a \in G$ such that $a \cdot x = y$.

ii) If G acts on X transitively, we call X a homogeneous space of G and if $x \in X$ then the map $\tilde{\mu}_x$ given as in Theorem 2.7 is a canonical equivariant bijection of X and G/G_x.

Remark 2.6. Note that if H is a subgroup of a group G then the left coset space G/H is a homogeneous space of G when G acts on G/H by left multiplication,

$$a \cdot xH = axH. \tag{2.12}$$

Where the subgroup H is the isotropy subgroup of the identity coset $eH \in G/H$.

For the diagonal action μ^k on X^k induced by an action $\mu : G \times X \to X$ as in Definition 2.5, the stabilizer of a point $(x_0, \ldots, x_{k-1}) \in X^k$ must necessarily fix each element $x_i \in X$ simultaneously. The next lemma characterizes the stabilizer subgroups for points in X^k with respect to the diagonal action μ^k.

Lemma 2.4. *Let $\mu : G \times X \to X$ be a group action and $\mu^k : G \times X^k \to X^k$ the corresponding diagonal action of μ on k copies of X. Then the isotropy subgroup of $(x_0, \ldots, x_{k-1}) \in X^k$ with respect to μ^k,*

$$G_{(x_0,\ldots,x_{k-1})} = G_{x_0} \cap G_{x_1} \cap \cdots \cap G_{x_{k-1}}$$

where G_{x_i} is the isotropy subgroup of x_i with respect to μ.

The proof is straightforward and will be omitted.

2.1.6 Effectiveness

Definition 2.17. Let G act on a set X, then the subset $G_X^* \subset G$ given by

$$G_X^* = \left\{a \in G : a \cdot x = x,\ \forall x \in X\right\},$$

is a normal subgroup of G called the *global isotropy* of the action.

Remark 2.7. An action $\mu : G \times X \to X$ is effective as in Definition 2.3 if and only if the global isotropy subgroup $G_X^* = \{e\}$.

Theorem 2.8. *Let $\mu : G \times X \to X$ be an action. The normal subgroup which is the intersection over X of all stabilizer subgroups,*

$$\bigcap_{x \in X} G_x,$$

is equal to the global isotropy subgroup.

The proof is an immediate consequence of Lemma 2.4.

Theorem 2.9. *Let G be a group and H a subgroup of G. The action of G on the coset space G/H by left multiplication is effective if and only if H contains no non-trivial normal subgroups of G.*

Proof. First assume that the action of G is effective. By contradiction suppose that H contains a non-trivial normal subgroup N. Let $n \in N$ be any non trivial element, $n \neq e$. Then fix some $xH \in G/H$. Let $x \in G$ be a representative of xH. Since N is normal, there exists some $n' \in N$ such that $nx = xn'$ and

$$nxH = xn'H = xH$$

since $n' \in H$. This contradicts that G acts effectively so H cannot contain any normal subgroups of G when the action of G is effective.

Now suppose that H contains no non-trivial normal subgroups of G. Let G^* be the global isotropy subgroup of the action by G. Let $n \in G^*$ and consider

$$neH = nH = H$$

since $n \in G^*$, and hence n is in H, the stabilizer subgroup of the identity coset $eH \in G/H$. Then G^* is trivial since it is normal in G and the action is effective by Remark 2.7. □

Corollary 2.4. *Let G be a group and $Z(G)$ be the center of G,*

$$Z(G) = \left\{a \in G : aba^{-1} = b,\, b \in G\right\}.$$

If $Z(G) \cap H \neq \{e\}$ then the action of G on G/H by left multiplication is not effective.

2.2 Group Actions on Manifolds

When G is a Lie group acting on a smooth (C^∞) manifold, we can define the concept of a smooth action as in Boothby [3].

Definition 2.18. Let M be a C^∞ manifold, G a Lie group, and $\mu : G \times M \to M$ a group action. If μ is a smooth function then we say that the action μ is a *smooth group action.*

Theorem 2.1 shows that the action μ is equivalent to a homomorphism $\Phi_\mu : G \to \text{Perm}(M)$ by $\Phi_\mu(a) = \mu_a$ where $\mu_a : M \to M$ is defined by

$$\mu_a(p) = \mu(a,p), \quad a \in G, \quad p \in M. \tag{2.13}$$

If the set M has a smooth structure, G is a Lie group, and μ is a smooth action then the maps μ_a from Equation (2.13) are diffeomorphisms of M to itself.

Since the manifolds M and G are equipped with topologies and μ is a smooth action we can define the following local conditions which correspond to free and effective actions.

Definition 2.19. A Lie group G acting smoothly on a manifold M is said to be

1) *locally free* if at each point $m \in M$ the isotropy subgroup G_m is a discrete subgroup of G,

2) and *locally effective* if the global isotropy subgroup G_X^* is a discrete subgroup of G.

Let $\mu : G \times M \to M$ be a smooth action of an r−dimensional Lie group G on an d−dimensional smooth manifold M. In summary the action is:

1. Free if $G_m = \{e\}$ for all $m \in M$.

2. Locally free if G_m is discrete for all $m \in M$.

3. Effective if $G_M^* = \{e\}$.

4. Locally Effective if G_M^* is discrete.

5. Transitive if $[x]_\mu = M$ for all and hence any $x \in M$.

2.2.1 Homogeneous Spaces of Lie Groups

If $\mu : G \times M \to M$ is a smooth and transitive action of a Lie group G smooth manifold M then for any point $m \in M$ the map $\tilde{\mu}_m : G/G_m \to M$ as in Equation 2.11 is a canonical equivariant bijection. As shown in Example 2.1.1 the classification of the homogeneous spaces of G is equivalent to the classification of the subgroups of G up to conjugation. The next theorem shows that the coset spaces of a *closed* subgroup H of a Lie group G are manifolds, so the classification of the homogeneous spaces of G which are manifolds is equivalent to classifying the closed subgroups H of G up to conjugation.

Theorem 2.10. *Let H be a closed subgroup of a Lie group G, and let G/H be the set of left cosets modulo H. Let $\pi : G \to G/H$ denote the natural projection $\pi(a) = aH$. Then G/H has a unique (smooth) manifold structure such that*

(a) π *is* C^∞.

(b) There exist local smooth sections of G/H in G; that is, if $aH \in G/H$, there is an (open) neighborhood W of aH and a C^∞ map $\tau : W \to G$ such that $\pi \circ \tau = \mathrm{Id}$.

Theorem 2.11. *Let G be a Lie group and M a smooth manifold with a smooth transitive group action $\mu : G \times M \to M$. Then for any $m \in M$ the map $\tilde{\mu}_m : G/G_m \to M$ as defined in Theorem 2.7 is an equivariant diffeomorphism.*

For proofs of these theorems see Warner [19] or Boothby [3].

If $\mu : G \times M \to M$ is a smooth action which is free then we are guaranteed a sufficient number of independent local invariants to solve a number of local congruence problems, see Olver [16] [17]. When the action μ is not free there are two common ways of geometric significance to construct a corresponding free action from μ. The first is to consider the induced action of G on submanifolds of M and their derivatives, which is guaranteed to become free provided μ satisfies certain regularity conditions, see Adams and Olver [1] [2]. The other is to extend μ to the diagonal action μ^k since the stabilizers of an element in the product space $(x_0, \ldots, x_{k-1}) \in M^k$ is the intersection of the stabilizers for every point x_i by Lemma 2.4. There are examples where the product action does not become free as shown in Olver [17], however in most examples the product action will become free on a suitable invariant open dense subset of M^k.

In order to find the points where μ^k will eventually become free we introduce the following definition.

Definition 2.20. Let G/H be a homogeneous space. For any point $(x_0H, x_1H, \ldots, x_{k-1}H) \in (G/H)^k$ let $H_i = G_{x_iH}$ be the isotropy subgroup of the point x_iH. The point $(x_0H, x_1H, \ldots, x_{k-1}H)$ is said to be in *general position* provided the subgroups $G_k = \bigcap_{0\leq i\leq k} H_i$ satisfy $\dim(G_{k-1}) - \dim(G_k)$ is maximal.

A point $(x_0H, \ldots, x_{k-1}H) \in (G/H)^k$ is in general position provided that for each $0 \leq i \leq k-1$ the isotropy subgroup of $(x_0H, \ldots, x_{k-1}H)$ has minimal dimension among the points in $(G/H)^k$, so that the μ^i orbits of these points have maximal dimension.

CHAPTER 3

Reduction to the Isotropy

In this chapter we show that the k point joint invariants of G acting on G/H by left multiplication are determined by the $k-1$ point joint invariants of H on G/H, by left multiplication.

If $\mu : G \times X \to X$ is a group action, a subset $K \subset X$ with the property that for each $x \in X$ the intersection of the orbit of x with the subset K is a single point,

$$[x]_\mu \cap K = \{k\},$$

then K is called a cross section to the group action μ. The reduction formalizes the observation that $H \times (G/H)^{k-1}$ is a partial cross section to the orbits of μ_G^k. That is if $(x_0H, \ldots, x_{k-1}H) \in (G/H)^k$ then the μ_G^k orbit $[(x_0H, \ldots, x_{k-1}H)]_G$ intersects $H \times (G/H)^{k-1}$ by the subset

$$H \times [(z_1H, \ldots, z_{k-1}H)]_H,$$

where $[(z_1H, \ldots, z_{k-1}H)]_H$ is the μ_H^{k-1} orbit of the point $(x_0^{-1}x_1H, \ldots, x_0^{-1}x_{k-1}H)$ for any choice of representatives $x_i \in G$ of the cosets $x_iH \in (G/H)$. So a complete set of $k-1$ point joint invariants for the action of H give necessary and sufficient conditions for congruence in $(G/H)^k$. This construction is equivalent to considering two points $(x_0H, \ldots, x_{k-1}H)$, and $(y_0H, \ldots, y_{k-1}H)$ in $(G/H)^k$ and by the transitivity of the standard action of G on G/H by left multiplication, translate the first entry of each collection to the origin. Once centered at the origin we restrict to using transformations that fix the origin, which are the transformations in the isotropy subgroup H, in order to determine congruence of $(x_0^{-1}x_1H, \ldots, x_0^{-1}x_{k-1}H)$ and $(y_0^{-1}y_1H, \ldots, y_0^{-1}y_{k-1}H)$ in $(G/H)^{k-1}$.

In Section 3.3 we give an application of the reduction theorem where we consider the action of a group L on itself by left and right multiplication. Applying the reduction to this case yields Theorem 3.3 which shows that the two point joint invariants are determined by class functions on L. This important example will be further developed throughout the thesis, see Theorem 4.4 and Chapter 7.

This chapter is organized as follows, Section 3.1 gives an overview of the main results, and Section 3.2 provides the proofs. Section 3.3 develops the example described above. The chapter concludes with Section 3.4 which proves a theorem relating the isotropy subgroups of points in $(G/H)^k$ and $(G/H)^{k-1}$ which are related by the reduction.

3.1 Overview of Reduction to Isotropy Results

In this section we summarize our results on the reduction to isotropy method for joint invariants and its application to the congruence problem. Throughout let G be a group let H be a proper subgroup of G. Consider the standard transitive action of G on a homogeneous space G/H, and extend this action to $(G/H)^k$ by the diagonal action given in Definition 2.5. The following theorem demonstrates that these orbit spaces $(G/H)^k/G$ and $(G/H)^{k-1}/H$ are bijective.

Theorem 3.1. *Let G act on $(G/H)^k$ and H act on $(G/H)^{k-1}$ by the diagonal actions induced by left multiplication.*

The map $\Phi : (G/H)^k/G \to (G/H)^{k-1}/H$, given by

$$\Phi\left(\left[\ (x_0H, x_1H, \ldots, x_kH)\ \right]_G\right) = \left[\ \left(x_0^{-1}x_1H, \ldots, x_0^{-1}x_1H\right)\ \right]_H,$$

is a bijection of the orbit spaces, $(G/H)^k/G$ and $(G/H)^{k-1}/H$.

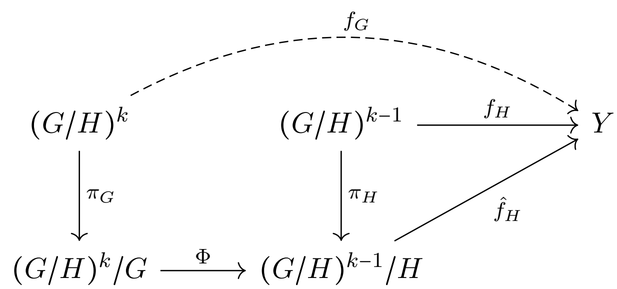
The map Φ above is not canonical since it involves a choice of one of the factors in $(G/H)^k$ to remove, but once this choice is made the map Φ is determined uniquely. Corollary 3.1 to Theorem 3.1 below identifies the invariants for the G action on $(G/H)^k$ with the invariants for the action of H on $(G/H)^{k-1}$.

Corollary 3.1. *The identity*

$$f_G(x_0H, x_1H, \ldots, x_{k-1}H) = f_H(x_0^{-1}x_1H, \ldots, x_0^{-1}x_{k-1}H), \tag{3.1}$$

defines a unique G invariant function $f_G : (G/H)^k \to Y$ for every H invariant function $f_H : (G/H)^{k-1} \to Y$ and conversely defines a unique H invariant function $f_H : (G/H)^{k-1} \to Y$ for each G invariant function $f_G : (G/H)^k \to Y$.

Corollary 3.1 verifies that the diagram,



(3.2)

commutes for a unique G invariant f_G for each H invariant f_H. Conversely if f_G is given there is a unique f_H given by the identity in Equation (3.1) such that the diagram commutes. In particular, Corollary 3.1 shows that the invariants of the diagonal action by G on $(G/H)^k$ are in one to one correspondence with the invariants of the diagonal action of H on $(G/H)^{k-1}$.

Remark 3.1. The reader may be inclined to introduce a map $\gamma : (G/H)^k \to (G/H)^{k-1}$ defined by

$$\gamma\,(x_0H, x_1H, \ldots, x_{k-1}H) = \left(x_0^{-1}x_1H, \ldots, x_0^{-1}x_{k-1}H\right)$$

to the diagram in Equation (3.2), but this map is not well defined.

While the map γ in Remark 3.1 is not well defined, in most examples one would like to complete the diagram from Equation 3.2 to compute the invariants $\tilde{f}$ without having a concrete description of the orbit spaces $(G/H)^k/G$ and $(G/H)^{k-1}/H$. Let $\rho : G/H \to G$ be any function satisfying the identity

$$\rho(xH) \cdot xH = eH, \tag{3.3}$$

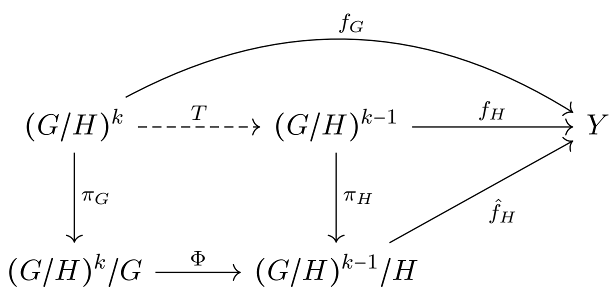
for example the map $\rho(xH) = x^{-1}$ for any representative x of the xH coset.

Lemma 3.1. *Let $\rho : G/H \to G$ be a map satisfying the identity in Equation* (3.3)*, and let $T : (G/H)^k \to (G/H)^{k-1}$ be defined by*

$$T(x_0H, x_1H, \ldots, x_{k-1}H) = (\rho(x_0H)x_1H, \ldots, \rho(x_0H)x_{k-1}H). \tag{3.4}$$

If $f_H : (G/H)^{k-1} \to Y$ is a Y valued H invariant then $f_H \circ T = f_G$ is a Y valued G invariant on $(G/H)^k$ which is independent of ρ and hence of T.

Lemma 3.1 verifies that the completed diagram,



(3.5)

commutes and f_G is independent of the T chosen. This will be the main result used in Chapters 6 and 7 to describe the joint invariants guaranteed by Corollary 3.1.

Finally, Theorem 3.1 also gives the following corollary about the congruence problem.

Corollary 3.2. *Let $X \in (G/H)^k$ be given by $X = (x_0H, \ldots, x_{k-1}H)$. The map $\zeta : (G/H)^k \to (G/H)^{k-1}/H$ given by*

$$\zeta\left(\left(x_0H, \ldots, x_{k-1}H\right)\right) = \left[\left(x_0^{-1}x_1H, \ldots, x_0^{-1}x_{k-1}\right)\right]_H,$$

is a complete invariant for the diagonal action of G on $(G/H)^k$.

Corollary 3.2 gives a solution to the congruence problem, if $X, Y \in (G/H)^k$ are chosen, and given by

$$X = (x_0H, \ldots, x_{k-1}H) \quad \text{and} \quad Y = (y_0H, \ldots, y_{k-1}H),$$

then $[X]_G = [Y]_G$ if and only if

$$\left[\left(x_0^{-1}x_1H, \ldots, x_0^{-1}x_{k-1}H\right)\right]_H = \left[\left(y_0^{-1}y_1H, \ldots, y_0^{-1}y_{k-1}H\right)\right]_H.$$

3.2 Proofs of Theorems.

The following section provides proofs of Theorem 3.1, Corollaries 3.1 and 3.2, and Lemma 3.1.

3.2.1 Notation Used in Proofs

We start by defining actions used throughout this section. For any positive integer k let $\mu_{H^k} : H^k \times G^k \to G^k$ defined by

$$\mu_{H^k}\left(\left(h_0, h_1, \ldots, h_{k-1}\right), \left(x_0, x_1, \ldots, x_{k-1}\right)\right) = (x_0h_0^{-1}, x_1h_1^{-1}, \ldots, x_{k-1}h_{k-1}^{-1}). \tag{3.6}$$

where $(x_0, \ldots, x_{k-1}) \in G^k$ and $(h_0, \ldots, h_{k-1}) \in H^k$. This is the action of H^k on G^k by right multiplication, where the inverse is added to ensure it is a left action.

Now let $\mu_G^k : G \times G^k \to G^k$ the diagonal action of G on G^k defined by

$$\mu_G^k(a, (x_0, x_1, \ldots, x_{k-1})) = (ax_0, ax_1, \ldots, ax_{k-1}) \tag{3.7}$$

for $a \in G$ and $(x_0, \dots, x_{k-1}) \in G^k$. Then let the restriction of μ_G^k to H in the first argument be, $\mu_H^k : H \times G^k \to G^k$, which is given by

$$\mu_H^k(h, (x_0, x_1, \dots, x_{k-1})) = (hx_0, hx_1, \dots, hx_{k-1}) \tag{3.8}$$

for $h \in H$ and $(x_0, \dots, x_{k-1}) \in G^k$. This is the diagonal action of H on G^k.

Let $\theta_{H^k} : H^k \times G^{k-1} \to G^{k-1}$ defined by

$$\theta_{H^k}\left(\left(h_0, h_1, \dots, h_{k-1}\right), \left(y_1, y_2, \dots, y_{k-1}\right)\right) = \left(h_0 y_1 h_1^{-1}, h_0 y_2 h_2^{-1}, \dots, h_0 y_{k-1} h_{k-1}^{-1}\right) \tag{3.9}$$

be an action of H^k on G^{k-1}.

Now we will denote the diagonal actions of G and H on $(G/H)^k$ by

$$\delta_G^k\left(a, \left(x_0 H, \dots, x_{k-1} H\right)\right) = (a x_0 H, \dots, a x_{k-1} H), \tag{3.10}$$

$$\delta_H^k\left(h, \left(x_0 H, \dots, x_{k-1} H\right)\right) = (h x_0 H, \dots, h x_{k-1} H) \tag{3.11}$$

where $a \in G$, $h \in H$, and $(x_0, \dots, x_{k-1}) \in (G/H)^k$.

Due to the large number of actions and groups present in the proofs we will denote the orbit spaces with a subscript representing which action generates the quotient, and the elements with square brackets and a subscript with the name of the action. For example the quotient of G^k by the action μ_{H^k} will be

$$G^k \big/ \mu_{H^k},$$

and the elements by $[(x_0, \dots, x_{k-1}]_{\mu_{H^k}}$.

Now since the actions μ_G^k in Equation (3.7) and μ_{H^k} in Equation (3.6) commute on G^k then there is an induced action of H^k on the quotient $G^k \big/ \mu_G^k$, denoted $\eta_{H^k} : H^k \times G^k / \mu_G^k \to$

G^k/μ_G^k given by

$$\eta_{H^k}\left((h_0,\ldots,h_{k-1}),\left[(x_0,\ldots,x_{k-1})\right]_{\mu_G^k}\right)=\left[(x_0h_0^{-1},\ldots,x_{k-1}h_{k-1}^{-1})\right]_{\mu_G^k} \tag{3.12}$$

And $\eta_G^k : G\times G^k/\mu_{H^k}\to G^k/\mu_{H^k}$ by

$$\eta_G^k\left(a,\left[(x_0,\ldots,x_{k-1})\right]_{\mu_{H^k}}\right)=\left[(ax_0,\ldots,ax_{k-1})\right]_{\mu_{H^k}}. \tag{3.13}$$

Finally we will denote the quotient maps by subscripts given by the action name as well. So $\pi_{\mu_{H^k}} : G^k\to G^k/\mu_{H^k}$ is the quotient

$$\pi_{\mu_{H^k}}\left((x_0,\ldots,x_{k-1})\right)=\left[(x_0,\ldots,x_{k-1})\right]_{\mu_{H^k}}.$$

3.2.2 Lemmas for Proofs of Main Results

The following lemma identifies G^k/μ_G^k with G^{k-1} through an equivariant bijection with respect to the actions η_{H^k} as in Equation (3.12) and θ_{H^k} as in Equation (3.9) and also identifies the double quotient $\left(G^k/\mu_G^k\right)/\eta_{H^k}$ with G^{k-1}/θ_{H^k}.

Lemma 3.2. *For any positive integer k*

i) the map $\phi_{\mu_G^k} : G^k/\mu_G^k\to G^{k-1}$ defined by

$$\phi_{\mu_G^k}\left(\left[(x_0,x_1,\ldots,x_{k-1})\right]_{\mu_G^k}\right)=(x_0^{-1}x_1,\ldots,x_0^{-1}x_{k-1})$$

is a bijection of G^k/μ_G^k and G^{k-1} which is equivariant with respect to η_{H^k} and θ_{H^k},

ii) and the map $\Phi_{\mu_G^k} : (G^k/\mu_G^k)/\eta_{H^k}\to G^{k-1}/\theta_{H^k}$ defined by

$$\Phi_{\mu_G^k}\left(\left[\left[(x_0,x_1,\ldots,x_{k-1})\right]_{\mu_G^k}\right]_{\eta_{H^k}}\right)=\left[(x_0^{-1}x_1,\ldots,x_0^{-1}x_{k-1}\right]_{\theta_{H^k}}$$

is the unique bijection of the orbit spaces which satisfies

$$\Phi_{\mu_G^k} \circ \pi_{\eta_{H^k}} = \pi_{\theta_{H^k}} \circ \phi_{\mu_G^k}.$$

Proof. Consider the map $f_{\mu_G^k} : G^k \to G^{k-1}$ given by

$$f_{\mu_G^k}(x_0, x_1, \dots, x_{k-1}) = (x_0^{-1}x_1, \dots, x_0^{-1}x_{k-1}) \tag{3.14}$$

The map $f_{\mu_G^k}$ is clearly surjective and equivariant with respect to the actions μ_{H^k} and θ_{H^k}. We show it is a complete invariant. Fix some $(x_0, x_1, \dots, x_{k-1}) \in G^k$. Then

$$f_{\mu_G^k}\Big(\mu_G^k\left(a, \left(x_0, x_1, \dots, x_{k-1}\right)\right)\Big) = f_{\mu_G^k}\left(ax_0, ax_1, \dots, ax_{k-1}\right) = \left(x_0^{-1}x_1, \dots, x_0^{-1}x_{k-1}\right),$$

so it is an invariant of the action μ_G^k.

Now suppose that $X = (x_0, x_1, \dots, x_{k-1})$ and $Y = (y_0, y_1, \dots, y_{k-1})$, are two points in G^k such that $f_{\mu_G^k}(X) = f_{\mu_G^k}(Y)$, that is

$$\left(x_0^{-1}x_1, \dots, x_0^{-1}x_{k-1}\right) = \left(y_0^{-1}y_1, \dots, y_0^{-1}y_{k-1}\right). \tag{3.15}$$

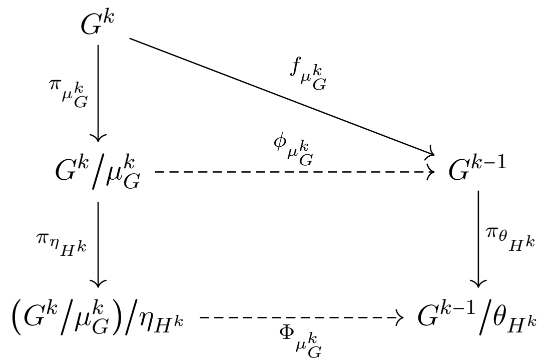
Then consider

$$\mu_G^k\left(y_0x_0^{-1}, \left(x_0, x_1, \dots, x_{k-1}\right)\right) = \left(y_0,\, y_0x_0^{-1}x_1, \dots, y_0x_0^{-1}x_{k-1}\right)$$

which is equal to Y by Equation (3.15). Now it is easily checked that $f_{\mu_G^k} = \phi_{\mu_G^k} \circ \pi_{\mu_G^k}$ and by Theorem 2.6 the map $\phi_{\mu_G^k}$ is the unique bijection which is equivariant with respect to η_{H^k} and θ_{H^k}.

From Theorem 2.4, with $\sigma = \mathrm{Id}$, the map $\phi_{\mu_G^k}$ induces the unique bijection $\Phi_{\mu_G^k}$ of orbit spaces as claimed in part ii). □

Lemma 3.2 above proves that the diagram,



commutes, where $f_{\mu_G^k}$ is as defined in Equation (3.14).

Remark 3.2. The map $f_{\mu_G^k}$ in Equation (3.14) is not a group homomorphism, which is reflected in the fact that the quotient of G^k by the diagonal action of G does not inherit a group structure through the equivariant bijection $\phi_{\mu_G^k}$.

Now we consider G^k/μ_{H^k} and $(G/H)^k$.

Lemma 3.3. *i) The map* $\phi_{\mu_{H^k}} : G^k/\mu_{H^k} \to (G/H)^k$ *defined by*

$$\phi_{\mu_{H^k}}\left(\left[(x_0, x_1, \ldots, x_{k-1})\right]_{\mu_{H^k}}\right) = (x_0H, x_1H, \ldots, x_{k-1}H)$$

is an equivariant bijection of G^k/μ_{H^k} *and* $(G/H)^k$ *with respect to* η_G^k *on* G^k/μ_H^k *given in Equation* (3.13) *and* δ_G^k *given in Equation* (3.10) *on* $(G/H)^k$.

ii) The map $\Phi_{\mu_{H^k}} : (G^k/\mu_{H^k})/\eta_G^k \to (G/H)^k/\delta_G^k$ *defined by*

$$\Phi_{\mu_{H^k}}\left(\left[\left[(x_0, x_1, \ldots, x_{k-1})\right]_{\mu_{H^k}}\right]_{\eta_G^k}\right) = \left[(x_0H, x_1H, \ldots, x_{k-1}H\right]_{\delta_G^k}$$

is the unique bijection which satisfies

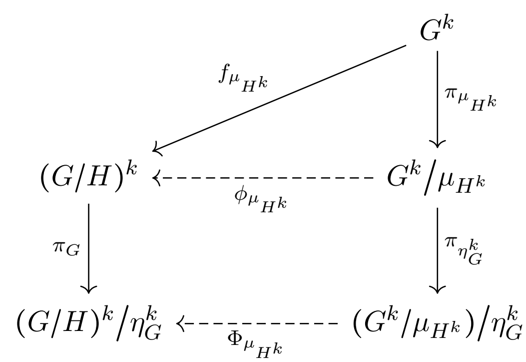
$$\Phi_{\mu_{H^k}} \circ \pi_{\eta_{H^k}} = \pi_{\delta_G^k} \circ \phi_{\mu_{H^k}}.$$

The proof is similar to that of Lemma 3.2 and follows from showing the map $f_{\mu_{H^k}}$: $G^k \to (G/H)^k$ defined by

$$f_{\mu_{H^k}}\left((x_0,\dots,x_{k-1})\right) = (x_0H,\dots,x_{k-1}H) \tag{3.16}$$

is a surjective function, a complete invariant of μ_H^k, and equivariant with respect to the actions μ_G^k and δ_G^k.

The unique bijections $\phi_{\mu_{H^k}}$ and $\Phi_{\mu_{H^k}}$ from Lemma 3.3 make the following diagram



commute, where $f_{\mu_{H^k}^k}$ is as defined in Equation (3.16). Note that $\Phi_{\mu_{H^k}}^{-1} : (G/H)^k/\delta_G^k \to (G^k/\mu_{H^k})/\eta_G^k$ is given by

$$\Phi_{\mu_{H^k}}^{-1}\left(\left[(x_0H, x_1H,\dots,x_{k-1}H\right]_{\delta_G^k}\right) = \left[\left[(x_0,x_1,\dots,x_{k-1})\right]_{\mu_{H^k}}\right]_{\eta_G^k} \tag{3.17}$$

Lemma 3.4. *The map* $\Gamma : G^{k-1}/\theta_{H^k} \to (G/H)^{k-1}/\delta_H^{k-1}$ *defined by*

$$\Gamma\left(\left[(y_1,y_2,\dots,y_{k-1})\right]_{\theta_{H^k}}\right) = \left[y_1H, y_2H,\dots,y_{k-1}H\right]_{\delta_H^{k-1}}$$

is the unique bijection on the orbit spaces which satisfies

$$\Gamma \circ \pi_{\theta_{H^k}} = \pi_{\delta_H^{k-1}} \circ f_{\mu_{H^{k-1}}}$$

where $f_{\mu_{H^{k-1}}}$ *is as defined in Equation* (3.16).

Proof. The composition $f_{\mu_{H^{k-1}}} \circ \pi_{\mu_{H^{k-1}}}$ is a surjective invariant of the action θ_{H^k} on G^{k-1}. Moreover if $X = (x_1, \ldots, x_{k-1})$ and $Y = (y_1, \ldots, y_{k-1})$ are two points in G^{k-1} such that $f_{\mu_{H^{k-1}}} \circ \pi_{\mu_{H^{k-1}}}(X) = f_{\mu_{H^{k-1}}} \circ \pi_{\mu_{H^{k-1}}}(Y)$ then

$$\big[(x_1H, \ldots, x_{k-1}H)\big]_{\delta_H^{k-1}} = \big[(y_1H, \ldots, y_{k-1}H)\big]_{\delta_H^{k-1}}.$$

Then there exists an $h_0 \in H$ and $(h_1, \ldots, h_{k-1}) \in H^{k-1}$ so that

$$\theta_{H^k}\big((h_0, h_1, \ldots, h_{k-1}), (x_1, \ldots, x_{k-1})\big) = \big(y_1, \ldots, y_{k-1}\big)$$

and $f_{\mu_{H^{k-1}}} \circ \pi_{\mu_{H^{k-1}}}$ is a complete surjective invariant. So applying Theorem 2.6 or Theorem 2.1 completes the proof. □

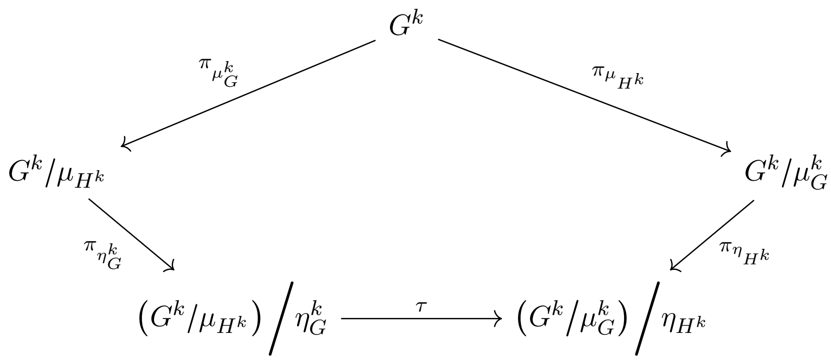
Lemma 3.4 proves that there exists a unique Γ such that the diagram

$$\begin{array}{ccc} G^{k-1} & \xrightarrow{f_{\mu_{H^{k-1}}}} & (G/H)^{k-1} \\ \downarrow{\scriptstyle \pi_{\theta_{H^k}}} & & \downarrow{\scriptstyle \pi_{\delta_H^{k-1}}} \\ G^{k-1}/\theta_{H^k} & \xrightarrow{\Gamma} & (G/H)^{k-1}/\delta_H^{k-1} \end{array}$$

commutes.

3.2.3 Proofs of the Main Results

Proof of Theorem 3.1. The actions μ_G^k and μ_{H^k} on G^k given by Equations (3.7) and (3.6) commute, so that by Lemma 2.1 the diagram



commutes, where $\tau : (G^k/\mu_{H^k})/\eta_G^k \to (G^k/\mu_G^k)/\eta_{H^k}$ given by

$$\tau\left(\left[\left[(x_0, x_1, \ldots, x_{k-1})\right]_{\mu_{H^k}}\right]_{\eta_G^k}\right) = \left[\left[(x_0, x_1, \ldots, x_{k-1})\right]_{\mu_G^k}\right]_{\eta_{H^k}}$$

as in Lemma 2.1 is a canonical bijection. Using the maps from Lemmas 3.2, 3.3, and 3.4,

$$\begin{aligned}
\Gamma \circ \Phi_{\mu_G^k} \circ \tau \circ \Phi_{\mu_{H^k}}^{-1}\left(\left[\left(x_0H, \ldots, x_{k-1}H\right)\right]_{\delta_G^k}\right) &= \Gamma \circ \Phi_{\mu_G^k} \circ \tau\left(\left[\left[(x_0, \ldots, x_{k-1})\right]_{\mu_{H^k}}\right]_{\eta_G^k}\right) \\
&= \Gamma \circ \Phi_{\mu_G^k}\left(\left[\left[(x_0, \ldots, x_{k-1})\right]_{\mu_G^k}\right]_{\eta_{H^k}}\right) \\
&= \Gamma\left(\left[\left(x_0^{-1}x_1, \ldots, x_0^{-1}x_{k-1}\right)\right]_{\theta_{H^k}}\right) \\
&= \left[\left(x_0^{-1}x_1, \ldots, x_0^{-1}x_{k-1}H\right)\right]_{\delta_H^{k-1}} \\
&= \Phi\left(\left[(x_0H, \ldots, x_{k-1}H)\right]_{\delta_G^k}\right)
\end{aligned}$$

So Φ is a composition of bijections and hence a bijection proving the claim. □

Note that Φ is uniquely determined up to the choice of $f_{\mu_G^k}$ which reflects the choice of which point to translate to the identity coset in the congruence problem.

We now give complete proof of Corollaries 3.1 and 3.2 to Theorem 3.1.

Proof of Corollary 3.1. First let f_H be an invariant of δ_H^{k-1} and let $\hat{f}_H : (G/H)^{k-1}/\delta_H^{k-1} \to Y$ be the unique map such that $\hat{f}_H \circ \pi_{\delta_H^{k-1}} = f_H$ from Theorem 2.2. Then define $f_G = \hat{f}_H \circ \Phi^{-1} \circ \pi_{\delta_G^k}$. We claim that f_G is the unique δ_G^k invariant that satisfies the identity,

$$f_G(x_0H, \dots, x_{k-1}H) = f_H(x_0^{-1}x_1H, \dots, x_0^{-1}x_{k-1}H). \tag{3.18}$$

Clearly f_G is an invariant of δ_G^k since $\pi_{\delta_G^k}$ is. Let $X = (x_0H, \dots, x_{k-1}H) \in (G/H)^k$. Then

$$\begin{aligned} f_G\left(x_0H, \dots, x_{k-1}H\right) &= \hat{f}_H \circ \Phi \circ \pi_{\delta_G^k}\left(x_0H, \dots, x_{k-1}H\right) \\ &= \hat{f}_H\left(\left[\left(x_0^{-1}x_1H, \dots, x_0^{-1}x_{k-1}H\right)\right]_{\delta_H^{k-1}}\right), \end{aligned}$$

and since $\left(x_0^{-1}x_1H, \dots, x_0^{-1}x_{k-1}H\right)$ is a representative of the orbit $\left[\left(x_0^{-1}x_1H, \dots, x_0^{-1}x_{k-1}H\right)\right]_{\delta_H^{k-1}}$ then

$$\begin{aligned} \hat{f}_H\left(\left[\left(x_0^{-1}x_1H, \dots, x_0^{-1}x_{k-1}H\right)\right]_{\delta_H^{k-1}}\right) &= \hat{f}_H \circ \pi_H\left(x_0^{-1}x_1H, \dots, x_0^{-1}x_{k-1}H\right) \\ &= f_H\left(x_0^{-1}x_1H, \dots, x_0^{-1}x_{k-1}H\right). \end{aligned}$$

Now suppose that F_G is defined by the identity in Equation (3.18). We claim F_G is a well defined invariant of δ_G^k, and that $F_G = f_G$. Let $X = (x_0H, \dots, x_{k-1}H) \in (G/H)^k$ and suppose $X = (x_0h_0H, \dots, x_{k-1}h_{k-1}H)$ is another representative. Then

$$F_G\left(x_0h_0H, \dots, x_{k-1}h_{k-1}H\right) = f_H\left(h_0^{-1}x_0^{-1}x_1H, \dots, h_0^{-1}x_0^{-1}x_{k-1}H\right) = f_H\left(x_0^{-1}x_1H, \dots, x_0^{-1}x_{k-1}H\right)$$

as f_H is an invariant of δ_H^{k-1}. Now consider

$$F_G\left(ax_0H, \dots, ax_{k-1}H\right) = f_H\left(x_0^{-1}x_1H, \dots, x_0^{-1}x_{k-1}H\right)$$

and F_G is an invariant of δ_G^k. Consider,

$$F_G\left(x_0H,\dots,x_{k-1}H\right) = f_H\left(x_0^{-1}x_1H,\dots,x_0^{-1}x_{k-1}H\right) = f_G\left(x_0H,\dots,x_{k-1}H\right),$$

so that $F_G = f_G$ completing the proof of this claim.

On the other hand let $f_G : (G/H)^k \to Y$ be an invariant of δ_G^k. Then there exists a unique map $\hat{f}_G : (G/H)^k/\delta_G^k \to Y$ such that $\hat{f}_G \circ \pi_{\delta_G^k} = f_G$ by Theorem 2.2. Define

$$f_H = \hat{f}_G \circ \Phi^{-1} \circ \pi_{\delta_H^{k-1}}.$$

We claim that f_H is the unique invariant of δ_H^{k-1} satisfying the identity in Equation (3.18).

Clearly f_H is an invariant. First let $(y_1H,\dots,y_{k-1}H) \in (G/H)^{k-1}$, and consider

$$\begin{aligned} f_H(y_1H,\dots,y_{k-1}H) &= \hat{f}_G \circ \Phi^{-1} \circ \pi_{\delta_H^{k-1}}(y_1H,\dots,y_{k-1}H) \\ &= \hat{f}_G \circ \Phi^{-1}\left(\left[(y_1H,\dots,y_{k-1}H)\right]_{\delta_H^{k-1}}\right) \\ &= \hat{f}_G\left(\left[(H,y_1H,\dots,y_{k-1}H)\right]_{\delta_G^k}\right) \\ &= f_G(H,y_1H,\dots,y_{k-1}H), \end{aligned}$$

for any representative $(H,y_1H,\dots,y_{k-1}H)$ of the orbit $\left[(H,y_1H,\dots,y_{k-1}H)\right]_{\delta_G^k}$.

Now Let $(x_0H,\dots,x_{k-1}H) \in (G/H)^k$ this is a representative of the orbit, $\left[(H,x_0^{-1}x_1H,\dots,x_0^{-1}x_{k-1}H)\right]_{\delta_G^k}$. So from the observation above

$$\begin{aligned} f_G(x_0H,\dots,x_{k-1}H) &= \hat{f}_G\left(\left[H,x_0^{-1}x_1H,\dots,x_0^{-1}x_{k-1}H\right]_{\delta_G^k}\right) \\ &= f_H\left(x_0^{-1}x_1H,\dots,x_0^{-1}x_{k-1}H\right), \end{aligned}$$

which verifies that f_H satisfies the identity in Equation (3.18).

Now we show uniqueness, Suppose that $F_H : (G/H)^{k-1} \to Y$ is given by the identity in Equation (3.18), that is,

$$F_H(y_1H, \dots, y_{k-1}H) = f_G(H, y_1H, \dots, y_{k-1}H),$$

which is clearly well defined. We show that F_H is an invariant of δ_H^{k-1}. Let $h \in H$ and consider

$$F_H(hy_1H, \dots, hy_{k-1}H) = f_G(H, hy_1H, \dots, hy_{k-1}H) = f_G(H, y_1H, \dots, y_{k-1}H)$$

since $h \in H$ and f_G is invariant under δ_G^k. Now consider

$$F_H(y_1H, \dots, y_{k-1}H) = f_G(H, y_1H, \dots, y_{k-1}H) = f_H(y_1H, \dots, y_{k-1}H)$$

which completes the proof. □

Proof of Lemma 3.1. We show the subdiagram,

$$\begin{array}{ccc} (G/H)^k & \overset{T}{\dashrightarrow} & (G/H)^{k-1} \\ \downarrow \pi_{\delta_G^k} & & \downarrow \pi_{\delta_G^{k-1}} \\ (G/H)^k/\delta_G^k & \xrightarrow{\Phi} & (G/H)^{k-1}/\delta_H^{k-1} \end{array}$$

of the diagram in Equation (3.5) commutes.

Consider $(x_0H, x_1H, \dots, x_{k-1}H) \in (G/H)^k$. The orbit of this point with respect to δ_G^k as defined in Equation (3.10) contains

$$(H, \rho(x_0H)x_1H, \dots, \rho(x_0H)x_{k-1}H)$$

so

$$\left[(x_0H, x_1H, \dots, x_{k-1}H)\right]_{\delta_G^k} = \left[(H, \rho(x_0H)x_1H, \dots, \rho(x_0H)x_{k-1}H)\right]_{\delta_G^k}.$$

Then using this equation we have

$$\begin{aligned}\Phi \circ \pi_{\delta_G^k}\left(\left(x_0H, x_1H, \dots, x_{k-1}H\right)\right) &= \Phi\left(\left[(H, \rho(x_0H)x_1H, \dots, \rho(x_0H)x_{k-1}H)\right]_{\delta_G^k}\right)\\ &= \left[(\rho(x_0H)x_1H, \dots, \rho(x_0H)x_{k-1}H)\right]_{\delta_H^{k-1}}\\ &= \pi_{\delta_H^{k-1}} \circ T\left(\left(x_0H, x_1H, \dots, x_{k-1}H\right)\right)\end{aligned}$$

and the diagram commutes.

We now show the second part of the lemma. Let $f : (G/H)^{k-1} \to Y$ be a Y valued H invariant. Then consider two choices ρ and ρ' satisfying Equation (3.3) which induce the corresponding T and T' as in Equation (3.4). Then consider $f \circ T$ and $f \circ T'$. Since the diagram in Equation (3.5) commutes then $\tilde{f} = f \circ T$ and $\tilde{f} = f \circ T'$. And since $\tilde{f}$ is unique by Corollary 3.1 then $f \circ T = f \circ T'$ which completes the proof. □

Proof of Corollary 3.2. First we show that the map ζ is well defined. Let $X = (x_0H, \dots, x_{k-1}H)$ and pick another representative $X = (x_0h_0H, \dots, x_{k-1}h_{k-1}H)$ Then consider

$$\begin{aligned}\zeta\left(\left((x_0h_0H, \dots, x_{k-1}h_{k-1}H\right)\right) &= \left[\left(h_0^{-1}x_0^{-1}x_1h_1H, \dots, h_0^{-1}x_0^{-1}x_{k-1}h_{k-1}H\right)\right]_{\delta_H^{k-1}}\\ &= \left[\left(x_0^{-1}x_1H, \dots, x_0^{-1}x_{k-1}H\right)\right]_{\delta_H^{k-1}}\\ &= \zeta\left(\left((x_0H, \dots, x_{k-1}H\right)\right)\end{aligned}$$

so ζ is well defined.

Now we show that ζ is a complete invariant. Indeed consider $\zeta\left(\delta_G^k(a, X)\right)$,

$$\zeta\left(\left(ax_0H, \dots, ax_{k-1}H\right)\right) = \left[\left(x_0^{-1}a^{-1}ax_1H, \dots, x_0^{-1}a^{-1}ax_{k-1}H\right)\right]_{\delta_H^{k-1}} = \zeta\left(\left(x_0H, \dots, x_{k-1}H\right)\right)$$

so ζ is an invariant. Now suppose that $X, Y \in (G/H)^k$ satisfy $\zeta(X) = \zeta(Y)$. Then there exists an $h \in H$ such that

$$\left(hx_0^{-1}x_1H, \dots, hx_0^{-1}x_{k-1}H\right) = \left(y_0^{-1}y_1H, \dots, y_0^{-1}y_{k-1}H\right).$$

So now consider $\delta_G^k\left(y_0 h x_0^{-1}, X\right)$,

$$\left(y_0 H, y_0 h x_0^{-1} x_1 H, \ldots, y_0 h x_0^{-1} x_{k-1} H\right) = \left(y_0 H, y_1 H, \ldots, y_{k-1} H\right)$$

and $[X]_{\delta_G^k} = [Y]_{\delta_G^k}$ and ζ is a complete invariant. □

3.3 Symmetric Pair Example

Let L be a group. Then L naturally acts on itself by left multiplication and right multiplication, which motivates the following definition.

Definition 3.1. Let L be a group, and $G = L \times L$. Let the map $\mu_{\mathrm{sym}} : G \times L \to L$ be given by

$$\mu_{\mathrm{sym}}\left(\left(a, b\right), x\right) = a x b^{-1} \tag{3.19}$$

for $(a, b) \in G$ and $x \in L$. This is the *symmetric action* of G on L.

Lemma 3.5. *The symmetric action μ_{sym} is transitive on L and the subgroup $G_{\mathrm{diag}} \subset G$,*

$$G_{\mathrm{diag}} = \{(a, a) : a \in L\},$$

is the stabilizer subgroup of the identity, $e \in L$.

Proof. Fix $x \in L$, and let $\rho : L \to G$ be given by

$$\rho(x) = \left(x^{-1}, e\right) \tag{3.20}$$

so that

$$\mu_{\mathrm{sym}}\left(\rho(x), x\right) = \mu_{\mathrm{sym}}\left(\left(x^{-1}, e\right), x\right) = x^{-1} x e = e,$$

and the action is transitive since $\mu_{\mathrm{sym}}\left(\left(\rho(x)\right)^{-1}, e\right) = x$ for all $x \in L$.

The isotropy of the identity e is all $(a,b) \in G$ such that $ab^{-1} = e$ so it is equal to G_{diag} as claimed. □

Theorem 2.7 and Lemma 3.5 imply that G/G_{diag} is canonically equivariantly bijective with L where G acts on G/G_{diag} by the standard action of left multiplication, denoted $\delta_{\text{sym}} : G \times G/G_{\text{diag}} \to G/G_{\text{diag}}$, and on L by the symmetric action, μ_{sym}. Moreover, if L is a Lie group then Theorem 2.11 shows that G/G_{diag} is canonically equivariantly diffeomorphic to L.

Let G_{diag} act on L by the restriction, η_{sym}, of μ_{sym} to G_{diag} and let $\mu_{\text{conj}} : L \times L \to L$ be the conjugation action

$$\mu_{\text{conj}}(a,x) = axa^{-1}. \tag{3.21}$$

Denote the induced diagonal action on L^k as in Definition 2.5 by μ^k_{conj},

$$\mu^k_{\text{conj}}\left(a,\left(x_0,\ldots,x_{k-1}\right)\right) = \left(ax_0a^{-1},\ldots,ax_{k-1}a^{-1}\right).$$

Similarly let μ^k_{sym} and η^k_{sym} be the diagonal actions of G and G_{diag} respectively. The next lemma shows that the orbit spaces L^k/η^k_{sym} and L^k/μ^k_{conj} are canonically bijective.

Lemma 3.6. *Let* Id *be the identity map on* L^k*, and let* $\pi_{\mu^k_{\text{conj}}} : L^k \to L^k/\mu^k_{\text{conj}}$ *and* $\pi_{\eta^k_{\text{sym}}} : L^k \to L^k/\eta^k_{\text{sym}}$ *be the quotient maps of* L^k *by the diagonal actions* μ^k_{conj} *and* η^k_{sym} *respectively.*

The map $\tilde{\text{Id}} : L^k/\eta^k_{\text{sym}} \to L^k/\mu^k_{\text{conj}}$ *given by*

$$\tilde{\text{Id}}\left(\left[(x_0,\ldots,x_{k-1})\right]_{\eta^k_{\text{sym}}}\right) = \left[(x_0,\ldots,x_{k-1})\right]_{\mu^k_{\text{conj}}}$$

is a canonical bijection which satisfies $\pi_{\mu^k_{\text{conj}}} \circ \text{Id} = \pi_{\eta^k_{\text{sym}}} \circ \tilde{\text{Id}}$.

Proof. Consider the identity map $\text{Id} : L^k \to L^k$. This map satisfies

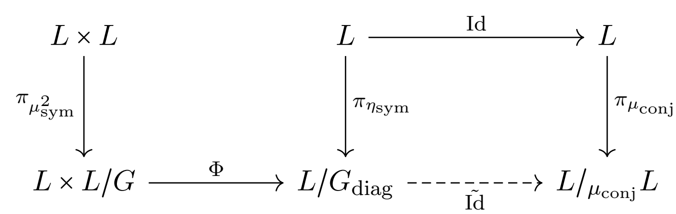
$$\text{Id}\left(\eta^k_{\text{sym}}\left((a,a),(x_0,\ldots,x_{k-1})\right)\right) = \mu^k_{\text{conj}}\left(a,(x_0,\ldots,x_{k-1})\right).$$

So by Theorem 2.4 with $\sigma : G_{\text{diag}} \to L$ given by $\sigma(a,a) = a$, the identity map induces a unique bijection of the orbit spaces $\tilde{\text{Id}} : L^k/\eta^k_{\text{sym}} \to L^k/\mu^k_{\text{conj}}$ given by

$$\tilde{\text{Id}}\left(\left[(x_0,\ldots,x_{k-1})\right]_{\eta^k_{\text{sym}}}\right) = \left[(x_0,\ldots,x_{k-1})\right]_{\mu^k_{\text{conj}}}. \tag{3.22}$$

such that $\pi_{\mu^k_{\text{conj}}} \circ \text{Id} = \pi_{\eta^k_{\text{sym}}} \circ \tilde{\text{Id}}$. □

With $k = 1$ Lemma 3.6 states that $\tilde{\text{Id}}$ is the unique bijection which makes the diagram



commute.

Now consider the case of k point joint invariants of μ_{sym}. The map ρ from Equation (3.20) is used to prove the following theorem.

Theorem 3.2. *Every Y valued invariant of μ^k_{sym} is given by*

$$f \circ T(x_0,\ldots,x_{k-1}) = f(x_0^{-1}x_1,\ldots,x_0^{-1}x_{k-1})$$

where f is a Y valued invariant of μ^{k-1}_{conj}.

The proof is a straightforward application of Lemma 3.1 and Lemma 3.6.

In the case of two point joint invariants of μ_{sym} Theorem 3.1 shows that the orbit spaces $\left(G/G_{\text{diag}} \times G/G_{\text{diag}}\right)/\delta^2_{\text{sym}}$ and $\left(G/G_{\text{diag}}\right)/\eta_{\text{sym}}$ are bijective by the map Φ. Then Lemma 3.5 identifies G/G_{diag} with L and Lemma 3.6 shows that L/η_{sym} is canonically bijective with L/μ_{conj}. That is $(L \times L)/\mu^2_{\text{sym}}$ is bijective with L/μ_{conj}.

The following corollary of Theorem 3.2 shows that class functions, functions out of L which are invariant under conjugation, determine all the invariants of μ^2_{sym}.

Corollary 3.3. *Let $T : L \times L \to L$ given by*

$$T(x_0, x_1) = x_0^{-1} x_1.$$

Every Y valued class function $f : L \to Y$ determines an invariant of μ^2_{sym} given by $f \circ T$,

$$f \circ T(x_0, x_1) = f(x_0^{-1} x_1)$$

and every invariant of μ^2_{sym} is of this form.

Finally Corollary 3.3 and Theorem 3.1 prove the following theorem.

Theorem 3.3. *If $\{f^\alpha\}$ is a set of class functions which form a complete set of invariants for μ_{conj} then the functions $\{f^\alpha \circ T\}$ are a complete set of two point invariants for μ^2_{sym}.*

In Chapter 7 we apply the above observation to compute complete invariants when L is taken to be $\mathrm{SU}(2, \mathbb{R})$ and $\mathrm{SL}(2, \mathbb{R})$.

3.4 A Remark on Free Actions

This section shows that for each subset of $(G/H)^{k-1}$ where the action of H is free guarantees a corresponding subset of $(G/H)^k$ where G acts freely.

We start by showing that the stabilizer of a point $Y \in (G/H)^{k-1}$ contains the stabilizer of any point in the preimage of a map $T : (G/H)^k \to (G/H)^{k-1}$ satisfying the conditions of Lemma 3.1.

Lemma 3.7. *Let G be a group and H a subgroup. Let μ^k_G and μ^{k-1}_H be the diagonal actions given as in Definition 2.5. Suppose that $\rho : G/H \to G$ is a map satisfying the identity in Equation* (3.3)*, and $T : (G/H)^k \to (G/H)^{k-1}$ is the map defined in Equation* (3.4)*. Then if $Y = (y_1 H, \ldots, y_{k-1} H) \in (G/H)^{k-1}$ and H_Y is its stabilizer subgroup, the stabilizer subgroup of any point $X = (x_0 H, \ldots, x_{k-1} H) \in T^{-1}(Y)$ is conjugate to a subgroup of H_Y by $\rho(x_0 H)$,*

$$\rho(x_0 H) G_X \rho(x_0 H)^{-1} \subset H_Y.$$

Proof. Let Y be as given in the theorem and consider any $X = (x_0H, \ldots, x_{k-1}H) \in T^{-1}(Y)$. Suppose that $h \in G_X$ the stabilizer for X. Then let $\rho(x_0H)h\rho(x_0H)^{-1}$ and let $\mu_G : G \times G/H \to G/H$ be the standard action of G on G/H by left multiplication. Consider

$$\mu_G\left(\rho(x_0H)h\rho(x_0H)^{-1}, eH\right) = \rho(x_0H)h\rho(x_0H)^{-1}eH = \rho(x_0H)hx_0H = \rho(x_0H)x_0H = eH.$$

That is $\rho(x_0H)h\rho(x_0H)^{-1} \in H$. Now since $T(X) = Y$ we have from Equation (3.4)

$$T(X) = (\rho(x_0H)x_1H, \ldots, \rho(x_0H)x_{k-1}H) = (y_1H, \ldots, y_{k-1}H) \tag{3.23}$$

and therefore,

$$\mu_H^{k-1}\left(\rho(x_0H)h\rho(x_0H)^{-1}, \left(y_1, \ldots, y_{k-1}\right)\right) = \left(y_1, \ldots, y_{k-1}\right)$$

so that $\rho(x_0H)h\rho(x_0H)^{-1} \in H_Y$, completing the proof. □

Now we show that any invariant subset of $(G/H)^{k-1}$ where H acts freely by the standard diagonal action of left multiplication gives an invariant subset of $(G/H)^k$ where G acts freely by the standard diagonal action of left multiplication.

Theorem 3.4. *Suppose that $U \subset (G/H)^{k-1}$ is a H invariant subset where the action μ_H^{k-1} is free. Let $\rho : G/H \to G$ be a map satisfying the identity of Equation 3.3 and $T : (G/H)^k \to (G/H)^{k-1}$ defined as in Equation* (3.4)*. Then the action μ_G^k is free on the G invariant set $T^{-1}(U) \subset (G/H)^k$.*

Proof. First we verify that $T^{-1}(U)$ is a G invariant subset of $(G/H)^k$. Pick some point $X = (x_1H, \ldots, x_{k-1}H) \in T^{-1}(U)$ and let $a \in G$. The proof of invariance follows from the next claim.

There exists some $h \in H$ such that $\rho(ax_0H)a = h\rho(x_0H)$. Using the identity in Equation (3.3) gives

$$\rho(x_0H)x_0H = \rho(ax_0H)ax_0H$$

and so there is some $h \in H$ which satisfies $\rho(x_0H)x_0 = \rho(ax_0H)ax_0h^{-1}$ for some $h^{-1} \in H$.

Now consider $T\left(\mu^k\big(a, X\big)\right)$ and use the claim above to compute,

$$
\begin{aligned}
T\left(ax_0H, \ldots, ax_{k-1}H\right) &= \left(\rho(ax_0H)ax_1H, \ldots, \rho(ax_0H)ax_{k-1}H\right) \\
&= \left(h\rho(x_0H)x_1H, \ldots, h\rho(x_0H)x_{k-1}H\right) \\
&= \mu^{k-1}\left(h, T(X)\right),
\end{aligned}
$$

and since $T(X) \in U$ and U is H invariant then $\mu^k(a, X) \in T^{-1}(U)$ verifying this subset is G invariant.

Now finally if the action μ_H^{k-1} on U is free then by Lemma 3.7 the stabilizer of any point in $T^{-1}(U)$ is conjugate to a subgroup of the trivial subgroup, that is it must be trivial and the action on this subset is free. □

When G is a Lie group acting on a product of homogeneous spaces $(G/H)^k$, Theorem 3.4 shows that for the purpose of finding an open invariant subset of points where G acts freely, one can look for an open invariant subset of $(G/H)^{k-1}$ where H acts freely instead. This along with the identification of invariants by Lemma 3.1 shows that if one can find a complete set of independent local invariants on $U \subset (G/H)^{k-1}$ where the action of H is free, then on $T^{-1}(U)$ there is a corresponding complete set of independent local invariants of G on an invariant open set where G acts freely.

CHAPTER 4

Primitive Spaces

A foliation of a homogeneous space G/H by immersed submanifolds of dimension k which satisfies that the elements of G acting by left multiplication map the immersed submanifolds of the foliation to immersed submanifolds of the foliation is called an invariant foliation, see Definition 4.4. The homogeneous spaces of a Lie group G which do not admit invariant foliations are called primitive homogeneous spaces which are formally defined in Definition 4.6. Section 4.2.2 gives an overview of the relationship between primitive homogeneous spaces of Lie groups G/H and the corresponding Lie algebra subalgebra pairs $(\mathfrak{g}, \mathfrak{h})$. We conclude this section with Theorem 4.4 and Corollary 4.4 show that the example of Section 3.3 is a primitive homogeneous space when the group L is a simple Lie group. This motivates taking $L = \mathrm{SU}(2,\mathbb{R})$ and $L = \mathrm{SL}(2,\mathbb{R})$ as minimal dimensional examples for the simple non isomorphic Lie algebras $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{su}_2(\mathbb{R})$ in Chapter 7. Theorem 4.3 due to Morosoff [13] classifies the Lie algebras of G and H when G/H is a primitive homogeneous space under the assumption that G is not simple. This classification motivates applying the reduction theory from Chapter 3 to the examples in Chapters 6 and 7.

4.1 Primitive Group Actions

First we consider a purely set based definition of what it means for an action of a group G on a set X to admit an invariant equivalence relation.

Definition 4.1. Let G be a group acting on a set X and let $\sim$ be an equivalence relation on X. Denote the equivalence class of $x \in X$ by $[x]$. If

$$\left[a \cdot x\right] = a \cdot \left[x\right]$$

for all $a \in G$ and equivalence classes $[x] \in \tilde{X}$, then $\sim$ is called a $G-$*invariant* equivalence relation, or we say that $\sim$ is invariant under the action of G.

From this definition the proof of the next lemma follows easily.

Lemma 4.1. *Let $\mu : G \times X \to X$ be a group action on X, let $\sim$ be an equivalence relation on X, and let $\tilde{X}$ be the space of equivalence classes. If $\sim$ is invariant under the action of G then*

i) there is a natural action of G on $\tilde{X}$, $\tilde{\mu} : G \times \tilde{X} \to \tilde{X}$ given by

$$\tilde{\mu}(a, \tilde{x}) = \widetilde{\mu(a, x)},$$

for $\tilde{x} \in \tilde{X}$ and $a \in G$,

ii) and the quotient map $\pi : X \to \tilde{X}$, given by $\pi(x) = \tilde{x}$, is equivariant with respect to G,

$$\pi(\mu(a, x)) = \tilde{\mu}(a, \tilde{x}).$$

Corollary 4.1. *If $\tilde{f} : \tilde{X} \to Y$ is an invariant of the G action on $\tilde{X}$ then $f = \tilde{f} \circ \pi$ is an invariant of the G action on X.*

With this definition of invariant equivalence relation we can define a primitive action on a set X for any group G.

Definition 4.2. Let G act on a set X. If there are no non-trivial equivalence relations $\sim$ on X which are invariant under the action of G, then the action of G is called *primitive.*

This definition is too general to be of use in the category of smooth manifolds, so in the next section we restrict our attention to a special class of equivalence relations defined for smooth manifolds.

4.2 Primitive Homogeneous Spaces

This section closely follows Golubitsky [10]. In order to give a suitable definition of primitive for a Lie group action on a smooth manifold M, we define equivalence relations that have equivalence classes given by a collection of k dimensional immersed submanifolds.

Definition 4.3. A k foliation, F, on M is a collection of k dimensional immersed submanifolds $\{F_m\}_{m \in M}$ such that for all $m, m' \in M$

a) the point $m \in F_m$,

b) the submanifold F_m is connected and has countable base for its topology,

c) and either $F_m = F_{m'}$ or $F_m \cap F_{m'} = \varnothing$.

The unique submanifold of the foliation containing the point m is called *the leaf through* m and F defines an equivalence relation on M by $m \sim m'$ if $F_m = F_{m'}$.

Then Definition 4.1 motivates the following class of foliations on M.

Definition 4.4. Let F be a foliation on M and let G be a group acting on M. Then F is said to be invariant under the action of G if

$$aF_m = F_{am}$$

for all $a \in G$, $m \in M$.

The trivial foliations of M into points, i.e. $F = M$, and into the connected components of M are always invariant under the action of G.

Definition 4.5. Let G be a Lie group acting on a smooth manifold M. If the only foliations of M which are invariant under the action of G are the trivial foliations into points or connected components, then the action of G is called *primitive*.

Motivated by Definition 4.2, we are interested in the possible homogeneous spaces G/H, for closed subgroups H of G, where the action of G on G/H is primitive.

Definition 4.6. Let G be a Lie group and H a closed subgroup. If G acts primitively on G/H then we call G/H a *primitive homogeneous space.*

The primitive and transitive actions of a Lie group G on a smooth manifold M were first examined in their infinitesimal form by Lie using vector field systems in $\mathbb{R}^n$, which have been classified in low dimensions see Olver [16] and Doubrov [6].

Primitive homogeneous spaces are classified by closed subgroups H which satisfy the following maximality condition.

Definition 4.7. Let G be a Lie group and H a proper Lie subgroup. If for any Lie subgroup K with $H \subset K \subset G$ then $\dim(H) = \dim(K)$ or $\dim(K) = \dim(G)$, the subgroup H is called a *maximal Lie subgroup* of G.

If H is not a maximal Lie subgroup and contained in a *closed* subgroup, then the next theorem shows how to construct a foliation on G/H that is invariant under the action of G.

Theorem 4.1. *Let G be a Lie group and H a proper closed Lie subgroup of G. Let H be nonmaximal, let K be a Lie subgroup such that $H \subset K \subset G$, and let $q : G/H \to G/K$ be a map given by*

$$q([a]_H) = [a]_K. \tag{4.1}$$

If K is closed, then there is a non-trivial foliation on G/H invariant under the action of G, $F = \{F_{[a]_H} \mid [a]_H \in G/H\}$. Where $F_{[e]_H} = q^{-1}([e]_K)_0$ is the connected component of $q^{-1}([e]_K)$ and the leaves $F_{[a]_H}$ are defined by

$$F_{[a]_H} = aF_{[e]_H}.$$

Proof. Since K is closed G/H and G/K are manifolds, and q is well defined since $H \subset K$. Let $\pi_H : G \to G/H$ and $\pi_K : G \to G/K$ be the canonical quotient maps. Then for any $a \in G$

$$q \circ \pi_H(a) = q\left([a]_H\right) = [a]_K = \pi_K(a)$$

and $q \circ \pi_H = \pi_K$. So since π_H and π_K are smooth surjective submersions then q is a smooth surjective submersion. Moreover q is an equivariant map with respect to the action of G on G/H and G/K since

$$q\left(a[e]_H\right) = q\left([a]_H\right) = [a]_K = a[e]_K = aq\left([e]_H\right).$$

Then since q is a submersion $q^{-1}([e]_K)$ is an embedded submanifold of G/H with dimension

$$\dim\left(q^{-1}([e]_K\right) = \dim(G/H) - \dim(G/K) = \dim(K) - \dim(H),$$

which is positive and strictly less than $\dim(G)$ by assumption, see Boothby [3] for the proof of this statement.

We now show that $q^{-1}([e]_K)$ is invariant under the action of $K \subset G$. Let $k \in K$ and consider the map defined by the action, $k : G/H \to G/H$ given by $k([x]_H) = [kx]_H$. We claim that k restricts to a smooth map from $q^{-1}([e]_K)$ to itself. Consider by applying k to $q^{-1}([e]_K)$, using equivariance, and that K is the stabilizer of $[e]_K$ in G gives

$$q(kq^{-1}([e]_K)) = kq(q^{-1}([e]_K)) = k[e]_K = [e]_K$$

so $kq^{-1}([e]_K) \subset q^{-1}([e]_K)$. Moreover since $q^{-1}([e]_K)$ is an *embedded* submanifold then the restriction of k is smooth, and in fact a diffeomorphism since $k^{-1} \in K$ is its inverse.

So, F, is a family of embedded submanifolds on G/H. We claim that if $[a]_H = [a']_H$ then $F_{[a]_H} = F_{[a']_H}$ so that $F_{[a]_H}$ is well defined. Since $a = a'h$ for some $h \in H$ then the claim will follow from showing $hF_{[e]_H} = F_{[e]_H}$ for all $h \in H$.

Fix some $h \in H$. Then since $h[e]_H = [e]_H$ as $H \subset G$ is the stabilizer subgroup of $[e]_H$ then $[e]_H \in hF_{[e]_H}$. As $h \in K$ from the claim above $hF_{[e]_H} \subset q^{-1}([e]_K)$ and h restricts to a diffeomorphism of $q^{-1}([e]_K)$ to itself. In particular h is a homeomorphism, so $hF_{[e]_H}$ is a connected component of $q^{-1}([e]_K)$ containing $[e]_H$ and therefore $hF_{[e]_H} = F_{[e]_H}$, so $F_{[a]_H}$ is well-defined.

Now we show that F is a non-trivial foliation of G/H which is invariant under the action of G. Clearly if F is a foliation, it will be invariant under the action of G by construction. Moreover F is non-trivial since the dimension of $F_{[e]_H}$ is $\dim(K) - \dim(H)$ which is positive and less than $\dim(G)$ by hypothesis.

We now check Definition 4.3. First if $[a]_H \in G/H$ then $[a]_H \in F_{[a]_H} = aF_{[e]_H}$ so F satisfies part a) of Definition 4.3.

Now, $F_{[e]_H}$ is the connected component of an embedded submanifold so it is connected submanifold of G/H, and since $a : G/H \to G/H$ is a diffeomorphism then $F_{[a]_H} = aF_{[e]_H}$ is connected as well, and F satisfies part b) of Definition 4.3.

Lastly we show elements of F are pairwise disjoint. Suppose that $F_{[a]_H} \cap F_{[b]_H} \neq \varnothing$. Since q is equivariant then $F_{[a]_H} \subset q^{-1}([a]_K)$ and using the definition of $F_{[a]_H}$ gives

$$F_{[a]_H} = aF_{[e]_H} = aq^{-1}\left([e]_K\right)_0 .$$

So by applying q we have $q(F_{[a]_H}) = [a]_K$. Hence, $q^{-1}\left([a]_K\right) \cap q^{-1}\left([b]_K\right) \neq \varnothing$, and by applying q to this intersection $[a]_K = [b]_K$, so $b = ak$ for some $k \in K$.

We claim this forces $F_{[a]_H} = F_{[b]_H}$, which will follow from showing $kF_{[e]_H} = F_{[e]_H}$ since the map $a : G/H \to G/H$ is a diffeomorphism. As above $kF_{[e]_H}$ is a connected component of $q^{-1}([e]_K)$. Then since

$$bF_{[e]_H} \cap aF_{[e]_H} = akF_{[e]_H} \cap aF_{[e]_H} \neq \varnothing,$$

there is a point in the intersection and $[x]_H, [y]_H \in F_{[e]_H}$ such that $ak[x]_H = a[y]_H$. Then there is a point $k[x]_H = [y]_H$ in the intersection $kF_{[e]_H} \cap F_{[e]_H}$, and as above $kF_{[e]_H} = F_{[e]_H}$. So part c) is satisfied showing that F is a foliation and completing the proof. □

The foliation on G/H is determined by the fibers of the map q from Equation (4.1), not the orbits of K acting on G/H by left multiplication.

The next lemma demonstrates that every invariant foliation of G/H comes from a subgroup of G containing H.

Lemma 4.2. *Let H be a closed subgroup of G. There exists a surjective correspondence from the set of all Lie subgroups of codimension k in G containing H to the set of all foliations of G/H of codimension k invariant under the action of G.*

For the proof see Golubitsky [10].

Corollary 4.2. *Let G be a Lie group and H a closed Lie subgroup. The action of G on G/H is primitive if and only if H is a maximal Lie subgroup.*

Corollary 4.2 shows that classifying the possible primitive homogeneous spaces of G is equivalent to classifying the closed maximal subgroups of G.

We now use Corollary 4.2 to verify the following example is a primitive homogeneous space.

Theorem 4.2. *Let L be a group containing no normal subgroups, and let $G = L \times L$ act on L by the action μ_{sym} as in Definition 3.1. The action μ_{sym} is primitive.*

Proof. First L is a homogeneous equivariantly bijective with G/G_{diag} from Lemma 3.5.

Now we show that

$$G_{\text{diag}} = \{(a,a) \,|\, a \in L\}$$

is a maximal subgroup (in the sense of groups). Let K be a subgroup such that $G_{\text{diag}} \subset K$. Then $(a,b) \in K$ if and only if $(ab^{-1}, e) \in K$. This follows from $(a,b) = (ab^{-1}, e)(b,b)$ and $(b,b) \in G_{\text{diag}} \subset K$. Now let $K_\ell = \{x \in L : (x,e) \in K\}$. We claim that K_ℓ is normal in L. Let $a \in L$ and $x \in K_\ell$, consider

$$(axa^{-1}, e) = (a,a)(x,e)(a,a)^{-1} \in K$$

since $G_{\text{diag}} \subset K$ which proves the claim. Now, since L contains no non trivial normal subgroups, there are two cases. Either $K_\ell = \{e\}$ and $K = G_{\text{diag}}$ or $(xy^{-1}, e) \in K$ for all $xy^{-1} \in L$ and $K = G$. This verifies that G_{diag} is maximal and by Corollary 4.2 the action of G on G/G_{diag} is primitive. □

Theorem 4.2 also holds under the weaker assumption that L is a simple Lie group (where it may have a discrete center), see Theorem 4.4, but requires an understanding of the relationship between a primitive homogeneous space ,G/H, and the corresponding Lie algebra-subalgebra pair $(\mathfrak{g}, \mathfrak{h})$.

The pair (G, G_{diag}) in Theorem 4.2 along with the involutive automorphism, $\sigma : G \to G$, given by $(a, b) \mapsto (b, a)$ is a symmetric pair as defined in Helgason [11]. Moreover in the case where L is compact, for example if $L = \mathrm{SU}(2, \mathbb{R})$, then the pair (G, G_{diag}) is a Riemannian symmetric pair.

4.2.1 Reduction to Effective

In this section we consider the case where the action of G on G/H is not effective. If G^* is the global isotropy, then the action of G can be reduced to an effective action of $\hat{G} = G/G^*$ which is indistinguishable from the action of G in the sense that the images $aG^*xH = axH$ for all $a \in G$ and $aG^* \in \hat{G}$. See page 33 of Bredon [5] and the book by Olver [16] for more information on this reduction.

Lemma 4.3. *Let G be a Lie group, H a closed Lie subgroup, G^* be the global isotropy subgroup, and $\hat{G} = G/G^*$.*

i) The action of G on G/H induces a transitive and effective action of $\hat{G}$ on G/H given by

$$aG^* \cdot xH = axH \tag{4.2}$$

and the stabilizer subgroup of $eH \in G/H$ is $\hat{H} = H/G^*$.

ii) The group G acts on $\hat{G}/\hat{H}$ by

$$a \cdot x\hat{H} = \pi(a)x\hat{H} \tag{4.3}$$

where $\pi : G \to \hat{G}$ is the quotient homomorphism, and there is a canonical $G-$equivariant diffeomorphism $\varphi : G/H \to \hat{G}/\hat{H}$ defined by

$$\varphi(aH) = a \cdot e\hat{H} = \pi(a)\hat{H}$$

for all $a \in G$.

Proof. We first prove part i). The action of $\hat{G}$ on G/H given in Equation (4.2) is well defined since G^* is the global isotropy, and the action is transitive because the action of G is. Let $aG^* \in G/G^*$, if

$$aG^*xH = axH = xH$$

for all $xH \in G/H$ then $a \in G^*$ and so the action is effective. The isotropy subgroup of $eH \in G/H$ is $\hat{H} = \{aG^* \mid a \in H\} = H/G^*$ completing the proof of this claim.

Now we prove part ii). The action of G on $\hat{G}/\hat{H}$ given in Equation (4.3) is well defined and transitive since $\pi : G \to \hat{G}$ is a surjective homomorphism. The stabilizer subgroup of $e\hat{H}$ using the action of G on $\hat{G}/\hat{H}$ is the subset of all $a \in G$ such that $a \cdot e\hat{H} = \pi(a)e\hat{H} = e\hat{H}$ so $a \in \pi^{-1}(e\hat{H}) = H$. By Theorem 2.11 there is a diffeomorphism $\varphi : G/H \to \hat{G}/\hat{H}$ given by

$$\varphi(aH) = a \cdot e\hat{H} = \pi(a)\hat{H},$$

which is equivariant with respect to the action of G. □

Note that if G acts smoothly and locally effectively on G/H, then the subgroup G^* is discrete and the dimensions $\dim\left(\hat{G}\right) = \dim(G)$ and $\dim\left(\hat{H}\right) = \dim(H)$.

The we will show that one of the actions Lemma 4.3 is primitive if and only if the other is in Corollary 4.5, but the proof needs the following result.

Lemma 4.4. *Let G act on smooth manifolds M and N. If $\varphi : M \to N$ is an equivariant diffeomorphism then the action of G on M is primitive if and only if the action of G on N is primitive.*

Proof. Since φ is a diffeomorphism the manifolds M and N have the same dimension, $d = \dim(M) = \dim(M)$.

Suppose that the action of G on N is primitive and let F be a non trivial foliation on M which is invariant under the action of G. The leaves of F are k dimensional with $0 < k < d$ since F is a non trivial foliation. Let $\hat{F} = \varphi(F)$, which is a foliation on N because φ is a diffeomorphism. Moreover the leaves of $\hat{F}$ satisfy

$$\hat{F}_n = \varphi\left(F_{\varphi^{-1}(n)}\right)$$

for all $n \in N$.

We now show that $\hat{F}$ is an invariant under the action of G on N. Indeed let $a \in G$. The map φ^{-1} is equivariant, so consider

$$\begin{aligned} a\hat{F}_n &= a\varphi\left(F_{\varphi^{-1}(n)}\right) \\ &= \varphi\left(aF_{\varphi^{-1}(n)}\right) \\ &= \varphi\left(F_{a\cdot\varphi^{-1}(n)}\right) \\ &= \varphi\left(F_{\varphi^{-1}(a\cdot n)}\right) \\ &= \hat{F}_{a\cdot n}. \end{aligned}$$

Then $\hat{F}$ is invariant under the action of G on N. The action of G on N is primitive, so $\hat{F}$ is trivial, and $\dim(F_n)$ is either 0 or d, contradicting that F is a non-trivial foliation.

The converse is similar. □

Lemma 4.5. *The action of G on G/H is primitive if and only if the action of $\hat{G}$ on $\hat{G}/\hat{H}$ is primitive.*

Proof. We first show that a foliation on $\hat{G}/\hat{H}$ is invariant under the action of G if and only if it is invariant under the action of $\hat{G}$. Let $\hat{F}$ be a foliation of $\hat{G}/\hat{H}$ which is invariant under the action of G on $\hat{G}/\hat{H}$. Fix an element $a \in G$. Then,

$$a \cdot \hat{F}_{\hat{H}} = \pi(a)\hat{F}_{\hat{H}} = \hat{F}_{\pi(a)\cdot\hat{H}} = \hat{F}_{a\cdot\hat{H}}$$

and $\hat{F}$ is invariant under the action of G. Conversely suppose $\hat{F}$ is a non trivial foliation invariant under the action of G on $\hat{G}/\hat{H}$. Fix some element $\hat{a} \in \hat{G}$. Pick any representative element $a \in G$ such that $\pi(a) = \hat{a}$, and consider

$$\hat{a}\hat{F}_{\hat{H}} = \pi(a)F_{\hat{H}} = a \cdot F_{\hat{H}} = F_{a\cdot\hat{H}} = F_{\hat{a}\hat{H}},$$

so $\hat{F}$ is invariant under the action of G on $\hat{G}/\hat{H}$ and proving the claim.

The claim above shows that the action of G on $\hat{G}/\hat{H}$ is primitive if and only if the action of $\hat{G}$ on $\hat{G}/\hat{H}$ is. Since φ is an equivariant bijection for the action of G on G/H and $\hat{G}/\hat{H}$ Lemma 4.4 shows that the action of G on $\hat{G}/\hat{H}$ is primitive if and only if the action of G on G/H is. The two statements together finish the proof. □

Lemma 4.3 and Corollary 4.5 justify that one can assume the action of G on G/H is effective without loss of generality. Furthermore, from Theorem 2.9, the action of G on G/H is effective if and only if H contains no normal subgroups of G. This motivates the following definition as in Golubitsky [10].

Definition 4.8. Let G be a Lie group and P be a proper closed subgroup satisfying

1) The action of G on G/P is primitive.

2) P contains no normal subgroups of G.

Then P is called a *primitive subgroup*, and (G, P) a *primitive pair*.

Note that if P is a maximal subgroup which contains no proper normal subgroups of G Corollary 4.2 implies P is a primitive subgroup and (G, P) is a primitive pair.

Classifying the primitive pairs (G, P) gives a classification of primitive homogeneous spaces. Note that the subgroup $G_{\text{diag}} \subset G = L \times L$ from Theorem 4.2 is a primitive subgroup since it contains no normal subgroups of G. However if we instead assume that L is a simple Lie group, so that its Lie algebra contains no ideals, G_{diag} may contain a discrete normal subgroup and would not satisfy part 2) of Definition 4.8.

In the next section we present a classification of primitive homogeneous spaces via classifying the possible Lie algebra subalgebra pairs that a primitive homogeneous space can have.

4.2.2 Primitive Lie algebras

In the previous section Corollary 4.2 characterized when an action of a Lie group G on G/H is primitive in terms of maximal subgroups H of G. Now we show how this notion of primitive can be characterized in terms of Lie algebras. We begin with the following definition.

Definition 4.9. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a proper Lie subalgebra. If for every subalgebra, $\mathfrak{k}$, satisfying $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$ then $\mathfrak{k} = \mathfrak{h}$ or $\mathfrak{k} = \mathfrak{g}$, then the subalgebra $\mathfrak{h}$ is *maximal.*

The following corollary to Lemma 4.2 characterizes primitive homogeneous spaces in terms of their Lie algebra-subalgebra pairs.

Corollary 4.3 (To Lemma 4.2)**.** *Let G be a Lie group and H a closed Lie subgroup with $\mathfrak{g}$ and $\mathfrak{h}$ their respective Lie algebras.*

i) If $\mathfrak{h} \subset \mathfrak{g}$ is a maximal subalgebra then the action of G on G/H is primitive.

ii) If the action of G on G/H is primitive and H is connected *then $\mathfrak{h} \subset \mathfrak{g}$ will be a maximal subalgebra.*

iii) If H is connected then the action of G on G/H is primitive if and only if the Lie algebra $\mathfrak{h}$ is a maximal subalgebra of $\mathfrak{g}$.

Proof. First we prove part 4.3. Suppose that F is an invariant k dimensional foliation on G/H. Then there is a Lie subgroup K of codimension k in G which contains H by Lemma 4.2. The connected component of H is a subgroup of K and therefore the Lie algebra $\mathfrak{k}$

of K contains $\mathfrak{h}$. Now by maximality of $\mathfrak{h}$ in $\mathfrak{g}$ then the codimension k of $\mathfrak{k}$ is either 0 or $\dim(G/H)$ and the foliation is trivial.

Now we prove part ii). Suppose that $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$ for a Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$. Then there is a connected subgroup K of G such that $H \subset K \subset G$. Since the action of G on G/H is primitive, then there are no invariant foliations of G/H and by Lemma 4.2 if $k = \operatorname{codim}_G(K)$ then $k = \operatorname{codim}_G(H)$ or $k = 0$. In the former case since $\dim(\mathfrak{h}) = \dim(\mathfrak{k})$ and $\mathfrak{h} \subset \mathfrak{k}$ then $\mathfrak{h} = \mathfrak{k}$. In the latter case $\dim(\mathfrak{k}) = \dim(\mathfrak{g})$ and $\mathfrak{k} \subset \mathfrak{g}$ so $\mathfrak{k} = \mathfrak{g}$.

The final part follows from the previous two claims. □

This motivates the following infinitesimal analog of Definition 4.8, as originally used by Morosoff [13].

Definition 4.10. Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{p}$ a subalgebra. If $\mathfrak{p}$ satisfies

i) $\mathfrak{p}$ is a proper subalgebra,

ii) $\mathfrak{p}$ is a maximal subalgebra of $\mathfrak{g}$,

iii) and $\mathfrak{p}$ contains no proper ideals of $\mathfrak{g}$

then $\mathfrak{p}$ is called a *primitive Lie subalgebra* and the pair $(\mathfrak{g}, \mathfrak{p})$ is a *primitive pair*.

Let G be a Lie group and H a connected closed subgroup and let $\mathfrak{g}$ and $\mathfrak{h}$ be their respective Lie algebras. Then if (G, H) is a primitive pair, $(\mathfrak{g}, \mathfrak{h})$ is a primitive pair. On the other hand if we suppose that $(\mathfrak{g}, \mathfrak{h})$ are a primitive pair then H may contain a discrete normal subgroup of G so that the action may only be locally effective and would not be a primitive subgroup of G since it fails part 2) of Definition 4.8.

Then given a Lie group G with Lie algebra $\mathfrak{g}$, classifying the primitive subalgebras of $\mathfrak{g}$ classifies the possible primitive homogeneous spaces G/H where H is connected. The study of these subalgebras and their connection to primitive actions was initiated by the authors Ochai [15], Morosoff [13], Komrakov [12], and Dynkin [8]. It is interesting to note that in the case where $\mathfrak{g}$ is simple, part iii) of Definition 4.10 is trivially satisfied, and classifying the primitive subalgebras reduces to classifying all maximal subalgebras of $\mathfrak{g}$.

Suppose that G is not simple so that the Lie algebra $\mathfrak{g}$ of G is not simple. The primitive subalgebras of $\mathfrak{g}$ were classified in one of the main results of Morosoff [13].

Theorem 4.3 (Morosoff)**.** *Let $\mathfrak{g}$ be a non-simple Lie algebra and let $\mathfrak{p}$ be a primitive subalgebra.*

i) If $\mathfrak{g}$ is not semi simple then there exists an Abelian ideal $\mathfrak{i}$ in $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{p} \ltimes \mathfrak{i},$$

where $\mathfrak{p}$ acts faithfully and irreducibly on $\mathfrak{i}$.

This is the affine *case.*

ii) If $\mathfrak{g}$ is semi simple then there exists a simple $\mathfrak{l}$ such that $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{l}$ and $\mathfrak{p} = \{(x,x) : x \in \mathfrak{l}\} = \mathfrak{g}_{\text{diag}}$.

This the symmetric *case.*

For the proof see Golubitsky [10] or Morosoff [13].

Now we return to the setup of Theorem 4.2 but with the weaker assumption that L is a simple Lie group, which will be of interest for the examples in Chapter 6 and 7.

Theorem 4.4. *Let L be a Lie group and let $G = L \times L$ act on L by the action μ_{sym} given in Definition 3.1. If L is a simple Lie group then the action is primitive and L is a primitive homogeneous space of G.*

Proof. From Lemma 3.5 the group L is equivariantly diffeomorphic to G/G_{diag} where $G_{\text{diag}} = \{(a,a) \,|\, a \in L\}$ and G acts on G/G_{diag} by left multiplication.

We prove that the Lie algebra, $\mathfrak{g}_{\text{diag}}$, of G_{diag}, is maximal. Let $\mathfrak{l}$ be the Lie algebra of L, and $\mathfrak{l} \oplus \mathfrak{l} = \mathfrak{g}$ be the Lie algebra of G. Since $\mathfrak{g}_{\text{diag}} = \{(x,x) \,|\, x \in \mathfrak{g}\} \simeq \mathfrak{l}$ it is simple. We show it is also maximal. Suppose $\mathfrak{k}$ is a subalgebra such that $\mathfrak{g}_{\text{diag}} \subset \mathfrak{k} \subset \mathfrak{g}$, and note that $\mathfrak{k} = \{(x,y) \,|\, x - y \in \mathfrak{k}_\ell\}$, as

$$(x,y) = (x-y,0) + (y,y)$$

and $(y,y)\in\mathfrak{k}$. Now consider the subalgebra $\mathfrak{k}_{\mathfrak{l}}=\{x\in\mathfrak{l}\,|\,(x,0)\in\mathfrak{k}\}$. We claim that $\mathfrak{k}_\ell$ is an ideal of $\mathfrak{l}$. Indeed if $z\in\mathfrak{l}$ and $x\in\mathfrak{k}_\ell$ then

$$([x,z],0)=[(x,0),(z,z)]\in\mathfrak{k}$$

since $\mathfrak{g}_{\text{diag}}\subset\mathfrak{k}$, so $[z,x]\in\mathfrak{k}_\ell$. Then either $\mathfrak{k}=\mathfrak{g}$ or $\mathfrak{k}=\{0\}$ and $\mathfrak{g}_{\text{diag}}$ is a maximal subalgebra. Finally, by Corollary 4.3 part i) the action will be primitive completing the proof. □

Note that the pair (G,G_{diag}) in Theorem 4.2 may not be a primitive pair as in Definition 4.8 as the subgroup G_{diag} may contain a discrete normal subgroup of G failing part 2) of the definition. However, the Lie algebra of $G=L\times L$ is $\mathfrak{l}\oplus\mathfrak{l}$ where $\mathfrak{l}$ is a simple Lie algebra, and the Lie algebra of G_{diag} is

$$\mathfrak{g}_{\text{diag}}=\{(x,x)\,|\,x\in\mathfrak{g}\}$$

which is isomorphic to $\mathfrak{l}$ and therefore $\mathfrak{g}_{\text{diag}}$ is simple. This observation proves the following corollary of Theorem 4.4.

Corollary 4.4. *The Lie algebra subalgebra pair* $(\mathfrak{g},\mathfrak{g}_{\text{diag}})$ *is a primitive pair of type ii) in the classification Theorem 4.3.*

Remark 4.1. If G/H is a primitive homogeneous space and G is simple and H is connected Corollary 4.3 states that the Lie subalgebra $\mathfrak{h}$ must be maximal. However, classifying the possible Lie algebra subalgebra pairs when H is not connected and G is simple is more complicated. There are primitive homogeneous spaces G/H where $\mathfrak{h}$ is not maximal, this is the main focus of Golubitsky [10]. An example is presented below in Section 5.3 in the case of $\mathrm{SL}(2,\mathbb{R})$. In the case where G is not a simple Lie group and G/H is a primitive homogeneous space $\mathfrak{h}$ will be a maximal subalgebra of $\mathfrak{g}$ and the classification of primitive subalgebras in Theorem 4.3 is the same as classifying the possible Lie algebra-subalgebra pairs for primitive homogeneous spaces.

CHAPTER 5

Two Dimensional Primitive Homogeneous Spaces of $\mathrm{PSL}(2,\mathbb{R})$.

There are three non conjugate one dimensional subalgebras, $\mathfrak{l}$, $\mathfrak{h}$, and $\mathfrak{so}_2(\mathbb{R})$ of $\mathfrak{sl}_2(\mathbb{R})$ which have bases

$$\ell = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad s = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{5.1}$$

respectively. Then

$$L(t) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}, \quad H(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}, \quad \text{and} \quad \mathrm{SO}(t) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \tag{5.2}$$

for all $t \in \mathbb{R}$ are the unique one parameter subgroups such that $\dot{L}(0) = \ell$, $\dot{H}(0) = h$, and $\dot{\mathrm{SO}}(0) = s$. Note that the images $L(\mathbb{R})$, $H(\mathbb{R})$, and $S(\mathbb{R}) = \mathrm{SO}(2,\mathbb{R})$ are one dimensional subgroups of $\mathrm{SL}(2,\mathbb{R})$. Throughout this section we will abuse notation and denote the images $L(\mathbb{R}) = L$, $H(\mathbb{R}) = H$, and $\mathrm{SO}(\mathbb{R}) = \mathrm{SO}(2,\mathbb{R})$.

Corollaries 4.2 and 4.3 allow us to identify which of the homogeneous spaces $\mathrm{SL}(2,\mathbb{R})/L$, $\mathrm{SL}(2,\mathbb{R})/H$, $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ are primitive.

Lemma 5.1. *The spaces* $\mathrm{SL}(2,\mathbb{R})/L$ *and* $\mathrm{SL}(2,\mathbb{R})/H$ *are not primitive homogeneous spaces and* $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ *is a primitive homogeneous space.*

Proof. Let B_0 be the two dimensional connected subgroup of the lower triangular matrices in $\mathrm{SL}(2,\mathbb{R})$. Then since $L, H \subset B_0$ they are not maximal, and by Corollary 4.2 the first claim follows.

For the second claim we note that $\mathrm{SO}(2,\mathbb{R})$ is connected so by Corollary 4.3 part iii) $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ is primitive if and only if $\mathfrak{so}_2(\mathbb{R})$, as defined in Equation (5.1) is a maximal subalgebra of $\mathfrak{sl}_2(\mathbb{R})$. Let $\mathfrak{k}$ be a two dimensional Lie algebra such that $\mathfrak{so}_2(\mathbb{R}) \subset \mathfrak{k} \subset \mathfrak{sl}_2(\mathbb{R})$. Let

$$x = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

be a basis for $\mathfrak{so}_2(\mathbb{R})$, and extend this to a basis for $\mathfrak{k}$, by some matrix

$$y = \begin{bmatrix} t & u \\ v & -t. \end{bmatrix}$$

Then by closure, the bracket

$$[x, y] = \begin{bmatrix} -(u+v) & 2t \\ 2t & u+v \end{bmatrix}$$

is also in $\mathfrak{k}$. However $[x, y]$ is linearly independent of x and y, so k would be three dimensional which is a contradiction. Therefore the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ is primitive. □

The subgroup $\mathrm{SO}(2,\mathbb{R})$ of $\mathrm{SL}(2,\mathbb{R})$ contains the discrete normal subgroup of $\mathrm{SL}(2,\mathbb{R})$ given by $N = \{\pm \mathrm{I}\}$, so by Theorem 2.9 the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ is not effective and $(\mathrm{SL}(2,\mathbb{R}), \mathrm{SO}(2,\mathbb{R}))$ is not a primitive pair since it does not satisfy part 2) of Definition 4.8.

Lemma 5.2. *The standard action of* $\mathrm{SL}(2,\mathbb{R})$ *on* $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ *has global isotropy subgroup given by the center of* $\mathrm{SL}(2,\mathbb{R})$*, the normal subgroup* $N = \{\pm \mathrm{I}\}$.

Proof. Let G^* be the global isotropy subgroup of $\mathrm{SL}(2,\mathbb{R})$ for the standard action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ by left multiplication. Suppose that $B \in G^*$. Since B is in the global isotropy, $B[\mathrm{I}]_{\mathrm{SO}(2,\mathbb{R})} = [\mathrm{I}]_{\mathrm{SO}(2,\mathbb{R})}$, and hence $B \in \mathrm{SO}(2,\mathbb{R})$. Let

$$B = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$$

for some $\phi \in [0, 2\pi]$ and let

$$E = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}$$

so that $E \in \mathrm{SL}(2,\mathbb{R})$. Now since B is in the global isotropy, $B[E]_{\mathrm{SO}(2,\mathbb{R})} = [E]_{\mathrm{SO}(2,\mathbb{R})}$, and there exits some $\theta \in [0, 2\pi)$ such that

$$\begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

which gives the system of equations

$$\begin{cases} \sin(\phi) = \sin(\theta) \\ \sin(\phi) + \frac{1}{2}\cos(\phi) = \frac{1}{2}\cos(\theta) + \sin(\theta) \\ \cos(\phi) - \frac{1}{2}\sin(\phi) = \cos(\theta) \end{cases} .$$

The first equation and the second equation imply that $\cos(\phi) = \cos(\theta)$ and from the third equation $\sin(\phi) = 0$. Then $\phi = 0, \pi$ and $B = \pm\mathrm{I}$. On the other hand N is a normal subgroup of $\mathrm{SL}(2,\mathbb{R})$ contained in $\mathrm{SO}(2,\mathbb{R})$ so by Theorem 2.9, N is contained in the global isotropy. □

Corollary 5.1. *The action of* $\mathrm{PSL}(2,\mathbb{R})$ *on* $\mathrm{PSL}(2,\mathbb{R})/\hat{\mathrm{SO}}(2,\mathbb{R})$ *is effective and primitive.*

Proof. From Lemma 4.3 there is a transitive effective action of $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/N$ on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ and an equivariant diffeomorphism $\varphi : \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})/\hat{\mathrm{SO}}(2,\mathbb{R})$, where $\hat{\mathrm{SO}}(2,\mathbb{R}) = \mathrm{SO}(2,\mathbb{R})/N$, is given by

$$\varphi\left([A]_{\mathrm{SO}(2,\mathbb{R})}\right) = [\hat{A}]_{\hat{\mathrm{SO}}(2,\mathbb{R})}$$

for $\hat{A} = [A]_N = \{\pm A\}$. So since $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$ is a primitive homogeneous space with respect to the standard action of $\mathrm{SL}(2,\mathbb{R})$ by left multiplication by Lemma 5.1, then by Lemma 4.5 the homogeneous space $\mathrm{PSL}(2,\mathbb{R})/\hat{\mathrm{SO}}(2,\mathbb{R})$ is primitive with respect to the standard action of $\mathrm{PSL}(2,\mathbb{R})$ by left multiplication. □

Since $\hat{\mathrm{SO}}(2,\mathbb{R})$ does not contain any normal subgroups of $\mathrm{PSL}(2,\mathbb{R})$ Corollary 5.1 implies that $(\mathrm{PSL}(2,\mathbb{R}), \hat{\mathrm{SO}}(2,\mathbb{R}))$ is a primitive pair as given in Definition 4.8.

Now we identify the one parameter subgroups of $\mathrm{PSL}(2,\mathbb{R})$ corresponding to $\mathfrak{l}$, $\mathfrak{h}$, and $\mathfrak{so}_2(\mathbb{R})$, $\hat{L}$ $\hat{H}$ and $\hat{\mathrm{SO}}(2,\mathbb{R})$.

Lemma 5.3. *Let* $\pi : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$ *be the quotient homomorphism. Then* $\pi_* : \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{psl}_2(\mathbb{R})$ *is a Lie algebra isomorphism, and the one parameter subgroups which correspond to the isomorphic images of* $\mathfrak{l}$, $\mathfrak{h}$, *and* $\mathfrak{so}_2(\mathbb{R})$ *under* π_* *are*

$$\hat{L} = \pi(L), \quad \hat{H} = \pi(H), \quad \hat{\mathrm{SO}}(2,\mathbb{R}) = \pi(\mathrm{SO}(2,\mathbb{R})), \tag{5.3}$$

where L, H, *and* $\mathrm{SO}(2,\mathbb{R})$ *are the subgroups from Equation* (5.2).

Proof. The quotient homomorphism is a smooth surjective submersion with constant rank equal to three and hence the linearization $\pi_* : \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{psl}_2(\mathbb{R})$ is an isomorphism of vector spaces. Moreover π_* is a Lie algebra homomorphism because π is a Lie group homomorphism and therefore π_* is a Lie algebra isomorphism. Under the identification from π_* the one dimensional Lie algebra corresponding to $\mathfrak{l}$ is $\pi_*(\mathfrak{l})$, and we claim that $\pi \circ L$ is the unique one parameter subgroup with $\pi_*(\ell)$ as its tangent vector at the identity $\hat{\mathrm{I}} \in \mathrm{PSL}(2,\mathbb{R})$. To prove the claim consider

$$\frac{d}{dt}\left(\pi \circ L(t)\right)\Big|_{t=0} = \pi_*\left(\dot{L}(0)\right) = \pi_*(\ell)$$

using $\dot{L}(0) = \ell$ since L is the unique one parameter subgroup with ℓ as its tangent vector at the identity, $\mathrm{I} \in \mathrm{SL}(2,\mathbb{R})$. The proof for the other cases is similar. □

The Lie algebra isomorphism π_* of $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{psl}_2(\mathbb{R})$ from Lemma 5.3 identifies the two Lie algebras, so that any of the properties we develop for $\mathfrak{sl}_2(\mathbb{R})$ in this chapter hold for $\mathfrak{psl}_2(\mathbb{R})$ as well.

Theorem 5.1. *The spaces* $\mathrm{PSL}(2,\mathbb{R})/\hat{L}$ *and* $\mathrm{PSL}(2,\mathbb{R})/\hat{H}$ *are not primitive homogeneous spaces and* $\mathrm{SL}(2,\mathbb{R})/\hat{\mathrm{SO}}(2,\mathbb{R})$ *is a primitive homogeneous space where* $\hat{L}$, $\hat{H}$, *and* $\hat{\mathrm{SO}}(2,\mathbb{R})$ *are the one parameter subgroups of* $\mathrm{PSL}(2,\mathbb{R})$ *from Lemma 5.3.*

Proof. The homogeneous space $\mathrm{PSL}(2,\mathbb{R})/\hat{\mathrm{SO}}(2,\mathbb{R})$ is primitive by Corollary 5.1. We will only show that $\mathrm{PSL}(2,\mathbb{R})/\hat{L}$ is not primitive as the argument is similar for $\mathrm{PSL}(2,\mathbb{R})/\hat{H}$. Since $\hat{L} \subset \pi(B_0)$ where B_0 is the connected two dimensional Lie subgroup of upper triangular matrices in $\mathrm{SL}(2,\mathbb{R})$, then $\pi(B_0)$ is a connected two dimensional subgroup of $\mathrm{PSL}(2,\mathbb{R})$ which contains $\hat{L}$, and it is easy to see that this containment is strict. So by Corollary 4.2 the homogeneous space $\mathrm{PSL}(2,\mathbb{R})/\hat{L}$ is not primitive. □

The next section identifies the homogeneous spaces $\mathrm{PSL}(2,\mathbb{R})/\hat{L}$, $\mathrm{PSL}(2,\mathbb{R})/\hat{H}$, and $\mathrm{PSL}(2,\mathbb{R})/\hat{\mathrm{SO}}(2,\mathbb{R})$ with orbits of the Adjoint representation of $\mathrm{PSL}(2,\mathbb{R})$.

5.1 Two Dimensional Model Spaces of $\mathrm{PSL}(2,\mathbb{R})$ Homogeneous Spaces.

The Adjoint representation of $\mathrm{SL}(2,\mathbb{R})$ on $\mathfrak{sl}_2(\mathbb{R})$ is the action $\mathrm{Ad}_{\mathrm{SL}} : \mathrm{SL}(2,\mathbb{R}) \times \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{sl}_2(\mathbb{R})$ be given by

$$\mathrm{Ad}_{\mathrm{SL}}(A,p) = ApA^{-1} \tag{5.4}$$

for $A \in \mathrm{SL}(2,\mathbb{R})$ and $p \in \mathfrak{sl}_2(\mathbb{R})$. Note that by Theorem 2.1 the action $\mathrm{Ad}_{\mathrm{SL}}$ is equivalent to the homomorphism $\Phi_{\mathrm{Ad}_{\mathrm{SL}}} : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{GL}(\mathfrak{sl}_2(\mathbb{R}))$ given by

$$\Phi_{\mathrm{Ad}_{\mathrm{SL}}}(A)[p] = (\mathrm{Ad}_{\mathrm{SL}})_A(p) = \mathrm{Ad}_{\mathrm{SL}}(A,p)$$

where we use the fact that the maps $(\mathrm{Ad}_{\mathrm{SL}})_A$ are linear to verify $\Phi_{\mathrm{Ad}_{\mathrm{SL}}} \subset \mathrm{GL}(\mathfrak{sl}_2(\mathbb{R})$.

The map $\Phi_{\mathrm{Ad}_{\mathrm{SL}}}$ is invariant under the action of $N = \{\pm \mathrm{I}\}$ by right or left multiplication on $\mathrm{SL}(2,\mathbb{R})$ and, by Theorem 2.2, there is a unique function $\Phi_{\mathrm{Ad}_{\mathrm{PSL}}} : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{GL}(\mathfrak{sl}_2(\mathbb{R}))$ such that $\Phi_{\mathrm{Ad}_{\mathrm{PSL}}} \circ \pi = \Phi_{\mathrm{Ad}_{\mathrm{SL}}}$ where π is the quotient homomorphism. It is easily checked that $\Phi_{\mathrm{Ad}_{\mathrm{PSL}}}$ is a homomorphism of groups, and so by Theorem 2.1 is equivalent to an action $\mathrm{Ad}_{\mathrm{PSL}} : \mathrm{PSL}(2,\mathbb{R}) \times \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{sl}_2(\mathbb{R})$ defined by

$$\mathrm{Ad}_{\mathrm{PSL}}\left(\hat{A}, p\right) = ApA^{-1}, \tag{5.5}$$

for $p \in \mathfrak{sl}_2(\mathbb{R})$ and any $A \in \mathrm{SL}(2,\mathbb{R})$ such that $\pi(A) = \hat{A}$. Note that by the Lie algebra isomorphism π_* this action agrees with the Adjoint representation of $\mathrm{PSL}(2,\mathbb{R})$. The two dimensional orbits of the action $\mathrm{Ad}_{\mathrm{SL}}$ are split into three types, one sheeted hyperbola, two sheeted hyperbola, and cones as shown in Figure 5.1. These orbits are identified with the two dimensional homogeneous spaces $\mathrm{PSL}(2,\mathbb{R})/\hat{L}$, $\mathrm{PSL}(2,\mathbb{R})/\hat{H}$, and $\mathrm{PSL}(2,\mathbb{R})/\hat{\mathrm{SO}}(2,\mathbb{R})$ from Theorem 5.1 in the following theorem.

Theorem 5.2. *Let* $\mathrm{Ad}_{\mathrm{PSL}}$ *be the Adjoint action given in Equation* (5.5)*, let* $\hat{L}, \hat{H}, \hat{\mathrm{SO}}(2)$ *be the connected one parameter subgroups of* $\mathrm{PSL}(2,\mathbb{R})$ *given in Theorem* (5.1)*, and let* (x,y,t) *be the coordinates determined by the basis*

$$X_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad \textit{and} \quad X_3 = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix} \tag{5.6}$$

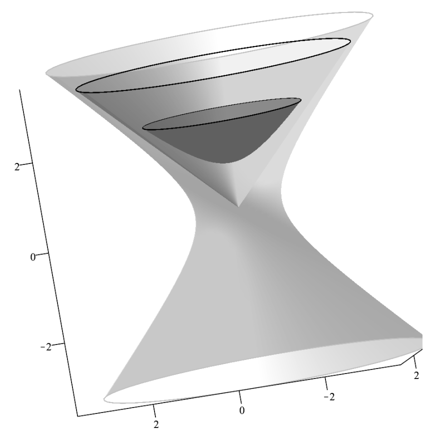
for $\mathfrak{sl}_2(\mathbb{R})$*. Then*

i) The homogeneous space $\mathrm{PSL}(2,\mathbb{R})/\hat{H}$ *is equivariantly diffeomorphic to the one sheet hyperbola* $Q_2 = \{(x,y,t)\,|\,2(x^2+y^2-t^2) = 2\}$.

ii) The homogeneous space $\mathrm{PSL}(2,\mathbb{R})/\hat{L}$ *is equivariantly diffeomorphic to the upper half cone,* $Q_0^+ = \{(x,y,t)\,|\,x^2+y^2-t^2 = 0,\ t>0\}$.

iii) $\mathrm{PSL}(2,\mathbb{R})/\hat{\mathrm{SO}}(2,\mathbb{R})$ *is equivariantly diffeomorphic to one sheet of a two sheet hyperbola* $Q_{-2}^+ = \{(x,y,t)\,|\,2(x^2+y^2-t^2) = -2,\ t>0\}$.

Fig. 5.1: Orbits of the $\mathrm{PSL}(2,\mathbb{R})$ Adjoint action.



Two Sheet Hyperbola	Cone	One Sheet Hyperbola
$\mathrm{PSL}(2,\mathbb{R})/\hat{\mathrm{SO}}(2,\mathbb{R})$	$\mathrm{PSL}(2,\mathbb{R})/\hat{L}$	$\mathrm{PSL}(2,\mathbb{R})/\hat{H}$
primitive	not primitive	not primitive

In order to prove Theorem 5.2 we begin by identifying the orbits of $\mathrm{Ad}_{\mathrm{SL}}$ and $\mathrm{Ad}_{\mathrm{PSL}}$.

Lemma 5.4. *The orbit of any point $p \in \mathfrak{sl}_2(\mathbb{R})$ with respect to the action $\mathrm{Ad}_{\mathrm{SL}}$ defined in Equation (5.4) is equal to the orbit of p with respect to $\mathrm{Ad}_{\mathrm{PSL}}$ defined in Equation (5.5).*

Proof. Let $p \in \mathfrak{sl}_2(\mathbb{R})$ and let $[p]_{\mathrm{Ad}_{\mathrm{SL}}}$ be its orbit. If $q \in [p]_{\mathrm{Ad}_{\mathrm{SL}}}$ then there exists an element $A \in \mathrm{SL}(2,\mathbb{R})$ such that $\mathrm{Ad}_{\mathrm{SL}}(A,p) = q$. Now let $\pi : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$ be the quotient homomorphism and define a map $\tilde{\pi} : \mathrm{SL}(2,\mathbb{R}) \times \mathfrak{sl}_2(\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R}) \times \mathfrak{sl}_2(\mathbb{R})$ by

$$\tilde{\pi}(A,p) = (\pi(A), q)$$

then by definition of $\mathrm{Ad}_{\mathrm{PSL}}$ from Equation (5.5) we have

$$\mathrm{Ad}_{\mathrm{PSL}} \circ \tilde{\pi} = \mathrm{Ad}_{\mathrm{SL}}. \tag{5.7}$$

Therefore, $\mathrm{Ad}_{\mathrm{PSL}}(\pi(A), p) = \mathrm{Ad}_{\mathrm{SL}}(A, p) = q$, and $[p]_{\mathrm{Ad}_{\mathrm{PSL}}} \subset [p]_{\mathrm{Ad}_{\mathrm{SL}}}$. On the other hand suppose that $q \in [p]_{\mathrm{Ad}_{\mathrm{PSL}}}$, then there exists $\hat{A} \in \mathrm{PSL}(2, \mathbb{R})$ such that $\mathrm{Ad}_{\mathrm{PSL}}(\hat{A}, p) = q$ and if $A \in \mathrm{SL}(2, \mathbb{R})$ is any element such that $\pi(A) = \hat{A}$ then from Equation (5.7)

$$q = \mathrm{Ad}_{\mathrm{PSL}} \circ \tilde{\pi}(A, p) = \mathrm{Ad}_{\mathrm{SL}}(A, p),$$

which verifies that $[p]_{\mathrm{Ad}_{\mathrm{PSL}}} = [p]_{\mathrm{Ad}_{\mathrm{SL}}}$ completing the proof. □

Now we define the Killing form on $\mathfrak{sl}_2(\mathbb{R})$. Let $\mathrm{ad}(x) : \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{sl}_2(\mathbb{R})$ be the adjoint map, $\mathrm{ad}(x) = \left[x, \cdot\right]$, where $[\cdot, \cdot]$ the commutator bracket on $\mathfrak{sl}_2(\mathbb{R})$. Then the Killing form on $\mathfrak{sl}_2(\mathbb{R})$ is

$$K_{\mathfrak{sl}_2(\mathbb{R})}(x, y) = \mathrm{tr}(\mathrm{ad}(x) \circ \mathrm{ad}(y)).$$

This is a symmetric, bilinear form which satisfies

$$K_{\mathfrak{sl}_2(\mathbb{R})}\left(\mathrm{Ad}_{\mathrm{SL}}(A, p), \mathrm{Ad}_{\mathrm{SL}}(A, q)\right) = K_{\mathfrak{sl}_2(\mathbb{R})}\left(ApA^{-1}, AqA^{-1}\right) = K_{\mathfrak{sl}_2(\mathbb{R})}(p, q)$$

for all $A \in \mathrm{SL}(2, \mathbb{R})$ and $p, q \in \mathfrak{sl}_2(\mathbb{R})$. Note that the killing form can be computed as

$$K_{\mathfrak{sl}_2}(p, q) = 4\,\mathrm{tr}(pq), \tag{5.8}$$

where tr is the usual trace function on matrices and multiplication in $\mathfrak{sl}_2(\mathbb{R})$ is matrix multiplication. The observation follows from evaluating both sides of Equation (5.8) on the basis in Equation (5.6). The Killing form defines a corresponding quadratic form,

$Q : \mathfrak{sl}_2(\mathbb{R}) \to \mathbb{R}$, by

$$Q(p) = K_{\mathfrak{sl}_2(\mathbb{R})}(p, p) = 4\,\mathrm{tr}(p^2), \tag{5.9}$$

by using Equation (5.8). The form Q is an invariant of $\mathrm{Ad}_{\mathrm{SL}}$ since the Killing form is left and right invariant. Let the level sets of Q be denoted as follows

$$Q_c = \{p \in \mathfrak{sl}_2(\mathbb{R}) \,|\, Q(p) = c\}\,. \tag{5.10}$$

The following lemma identifies the level sets of Q as two dimensional surfaces in the three dimensional Lie algebra $\mathfrak{sl}_2(\mathbb{R})$.

Lemma 5.5. *Let Q be the quadratic form defined in Equation* (5.9)*. The level sets Q_c given in Equation* (5.10) *are one of the following types.*

i) If $c > 0$ then the level sets Q_c are one sheet hyperbolas.

ii) If $c = 0$ then the level set Q_0 is a cone through the origin.

iii) If $c < 0$ then the level sets Q_c are two sheet hyperbolas.

Proof. Introduce coordinates on the vector space $\mathfrak{sl}_2(\mathbb{R})$ using the basis given in Equation (5.6). That is $p \in \mathfrak{sl}_2(\mathbb{R})$ has coordinate representation $(x, y, t) \in \mathbb{R}^3$ where x, y, t are the unique values such that $p = xX_1 + yX_2 + tX_3$.

Then from Equation (5.8) the quadratic form is $Q(p) = 4\,\mathrm{tr}(p^2)$, and its coordinate representation is,

$$Q(x, y, t) = 4\,\mathrm{tr}\left(\left(xX_1 + yX_2 + tX_3\right)^2\right) = 4\,\mathrm{tr}\left(\begin{bmatrix} \frac{1}{2}x & \frac{1}{2}(y - t) \\ \frac{1}{2}(y + t) & -\frac{1}{2}x \end{bmatrix}^2\right) = 2(x^2 + y^2 - t^2).$$

Hence the level sets of Q are described by $Q_c = \left\{(x, y, t) \,|\, 2(x^2 + y^2 - t^2) = c\right\}$, proving the claim. ∎

Lemma 5.6. *If $p \in \mathfrak{sl}_2(\mathbb{R})$, then the characteristic polynomial $P_p(\lambda)$ of p is*

$$P_p(\lambda) = \lambda^2 - \frac{1}{8}Q(p). \tag{5.11}$$

Proof. Consider the usual basis for $\mathfrak{sl}_2(\mathbb{R})$ $\{f, e, h\}$,

$$f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{5.12}$$

With respect to these coordinates the quadratic form Q in Equation (5.9) has the form

$$Q(F, H, E) = Q(Ff + Hh + Ee) = 8(EF + H^2).$$

Now consider an arbitrary matrix $p \in \mathfrak{sl}_2(\mathbb{R})$, with coordinates (F, H, E). Then the characteristic polynomial $P_p(\lambda)$ of p is

$$P_p(\lambda) = \det(Ff + Hh + Ee - \lambda \mathrm{I}) = \lambda^2 - (EF + H^2) = \lambda^2 - \frac{1}{8}Q(q),$$

which is independent of the basis for $\mathfrak{sl}_2(\mathbb{R})$ chosen since Q is invariant under conjugation.

□

Corollary 5.2. *For each $c \in \mathbb{R}$ the points $p \in Q_c$ where Q_c is the level set from Equation* (5.10) *all have the same eigenvalues.*

The sets Q_2, Q_{-2}^+, Q_{-2}^- from Theorem 5.2 are invariant subsets of $\mathrm{SL}(2,\mathbb{R})$ with respect to the adjoint action because they are connected components of level sets of Q. The next lemma proves that Q_0^+ is also an $\mathrm{SL}(2,\mathbb{R})$ invariant subset.

Lemma 5.7. *The set Q_0^+ from part ii) of Theorem 5.2 is an* $\mathrm{SL}(2,\mathbb{R})$ *invariant subspace with respect to the action* $\mathrm{Ad}_{\mathrm{SL}}$.

Proof. Let $p \in Q_0^+$. Then note that since $0 \in \mathfrak{sl}_2(\mathbb{R})$ is a fixed point of the action μ_{Ad} and $p \neq 0$, the point 0 is not an element of the orbit $[p]_{\mathrm{Ad}_{\mathrm{SL}}}$. Hence $[p]_{\mathrm{Ad}_{\mathrm{SL}}} \subset Q_0 \setminus \{0\}$. Note that the sets Q_0^+ and Q_0^-, where

$$Q_0^- = \{(x, y, t) \,|\, x^2 + y^2 - t^2 = 0,\, t < 0\},$$

are nonempty disjoint open subsets such that $Q_0 \setminus \{0\} = Q_0^+ \cup Q_0^-$ and therefore the connected subset $[p]_{\mathrm{Ad}_{\mathrm{SL}}}$ lies entirely in Q_0^+ or Q_0^-. Finally since $p \in Q_0^+$ then $[p]_{\mathrm{Ad}_{\mathrm{SL}}} \subset Q_0^+$ completing the proof. □

Lemma 5.8. *The action* $\mathrm{Ad}_{\mathrm{SL}}$ *is transitive on the sets* Q_2, Q_0^+, Q_{-2}^+ *identified in Theorem 5.2.*

Proof. There are three parts to prove.

i) Let $p \in Q_2$ from Lemma 5.5. Then from Lemma 5.6 the characteristic polynomial of p is $P_p(\lambda) = \lambda^2 - \frac{1}{4}$ and the eigenvalues of p are $\pm 1/2$. Since p has two distinct eigenvalues let $v_{1/2}, v_{-1/2}$ be eigenvectors for p and let $Z = [v_{1/2} | v_{-1/2}] \in \mathrm{GL}(n, \mathbb{R})$. Then

$$Z^{-1} p Z = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} = X_1.$$

Now $\det(Z) \neq 0$ since the eigenvectors are linearly independent, and there are two cases. If $\det(Z) > 0$ then the matrix $U = \frac{1}{\sqrt{\det(Z)}} Z \in \mathrm{SL}(2, \mathbb{R})$ satisfies $U^{-1} p U = X_1$ as well and X_1 is on the same orbit as p. If $\det(Z) < 0$ let $Y = [v_{-1/2} | v_{1/2}]$ so that $\det(Y) > 0$. Then

$$(Y)^{-1} p Y = \begin{bmatrix} -1/2 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

Let $S \in \mathrm{SL}(2,\mathbb{R})$ be the matrix,

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{5.13}$$

Then note that

$$SY^{-1}pYS^{-1} = SY^{-1}p\left(SY^{-1}\right)^{-1} = X_1$$

and p is conjugate to X_1 by an element $SY^{-1} \in \mathrm{GL}(2,\mathbb{R})$ where $\det\left(SY^{-1}\right) > 0$. By similar logic to the case where $\det(Z) > 0$ the point X_1 is in the same orbit as p verifying that $\mathrm{Ad}_{\mathrm{SL}}$ is transitive on the subset Q_2.

ii) Let $p \in Q_0^+$. Then $P_p(t) = t^2$ by Lemma 5.6 and p is nilpotent. From the Cayley–Hamilton theorem $p^2 = 0$ and $p \neq 0$ since $p \in Q_0^+$. Then there exists an element $v \in \mathbb{R}^2$ such that $pv \neq 0$. Consider the set $\{v, pv\}$. We claim this is a linearly independent set. Suppose that

$$av + bpv = 0.$$

If a or b is zero then both a and b are zero since $v, pv \neq 0$. If a and b are nonzero then $pv = \frac{a}{b}v$ so v is an eigenvector of p with eigenvalue a/b, but the eigenvalues of p are zero so $a/b = 0$ contradicting $pv \neq 0$.

Since $\{v, pv\}$ is a linearly independent set then it is a basis for $\mathbb{R}^2$. Let Z be the matrix $Z = [v|vp] \in \mathrm{GL}(2,\mathbb{R})$. If $\det(Z) > 0$ then

$$Z^{-1}pZ = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = X_2 + X_3. \tag{5.14}$$

That is q is conjugate to $X_2 + X_3$ by an element of $\mathrm{GL}(2,\mathbb{R})$ with positive determinant and therefore p is conjugate to $X_2 + X_3$ by an element of $\mathrm{SL}(2, \mathbb{R})$ and p is in the same orbit as $X_2 + X_3$. If not, $\det(Z) < 0$ then let $Y = [pv, v] \in \mathrm{GL}(2, \mathbb{R})$ so that $\det(Y) > 0$. Then

$$Y^{-1} p Y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = X_2 - X_3,$$

and p is conjugate to $X_2 - X_3$ by an element of $\mathrm{GL}(2, \mathbb{R})$ with positive determinant, and therefore p is conjugate to $X_2 - X_3$ by an element of $\mathrm{SL}(2, \mathbb{R})$. However from Lemma 5.7 the orbit of p is a subset of Q_0^+, and the coordinates of $X_2 - X_3$ from the choice of basis in Equation (5.6) are $(0, 1, -1) \in Q_0^-$. Therefore, $\det(Z) < 0$ leads to a contradiction and p is in the same orbit as $X_2 + X_3$.

iii) If $p \in Q_{-2}$ then the characteristic polynomial $P_p(t) = t^2 + \frac{1}{4}$ and p has complex conjugate eigenvalues. Then by the Cayley-Hamilton theorem, $p^2 + \frac{1}{4}\mathrm{I} = 0$, and $p \neq 0$ so there exists a non zero vector $v \in \mathbb{R}^2$ such that $p^2 v = -\frac{1}{4} v$. The set $\{v, pv\}$ is a basis for $\mathbb{R}^2$ by similar logic to part ii).

Let $Z = [v|pv] \in \mathrm{GL}(2, \mathbb{R})$ and there are two cases. If $\det(Z) > 0$ then

$$Z^{-1} p Z = \begin{bmatrix} 0 & -\frac{1}{4} \\ 1 & 0 \end{bmatrix} = \left(1 - \frac{1}{4}\right) X_2 + \left(1 + \frac{1}{4}\right) X_3 \tag{5.15}$$

where the right hand side of the equation above has coordinates $\left(0, 1 - \frac{1}{4}, 1 + \frac{1}{4}\right) \in Q_{-2}^+$ with respect to the choice of basis in Equation (5.6). Therefore since $\det(Z) > 0$ then p is in the same orbit as $\left(1 - \frac{1}{4}\right) X_2 + \left(1 + \frac{1}{4}\right) X_3$.

If $\det(Z) < 0$ then let $Y = [pv|v] \in \mathrm{GL}(2, \mathbb{R})$ with $\det(Y) > 0$. Then

$$Y^{-1} p Y = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4} & 0 \end{bmatrix} = \left(1 - \frac{1}{4}\right) X_2 - \left(1 + \frac{1}{4}\right) X_3 \tag{5.16}$$

where the right hand side has coordinates $\left(0, 1 - \frac{1}{4}, -1 - \frac{1}{4}\right)$ with respect to the choice of basis in Equation (5.6) and therefore is in Q^-_{-2}. However since $p \in Q^+_{-2}$ where Q^+_{-2} is a $\mathrm{Ad}_{\mathrm{SL}}$ invariant subset this is a contradiction and $\det(Z) > 0$.

That is every point $p \in Q^+_{-2}$ is in the same orbit as $\left(1 - \frac{1}{4}\right) X_2 + \left(1 + \frac{1}{4}\right) X_3$ which verifies that $\mathrm{Ad}_{\mathrm{SL}}$ is transitive on Q^+_{-2}.

□

For more information on the Adjoint orbits of $\mathrm{SL}(2,\mathbb{R})$ see the paper by Rubilar [18].

Corollary 5.3. *The action of* $\mathrm{Ad}_{\mathrm{PSL}}$ *is transitive on the sets* Q_2, Q^+_0, *and* Q^+_{-2}.

The proof follows from applying Lemma 5.4. The next lemma shows how to compute the isotropy subgroup for the action $\mathrm{Ad}_{\mathrm{PSL}}$ given the isotropy subgroup for the same point with respect to the action $\mathrm{Ad}_{\mathrm{SL}}$.

Lemma 5.9. *Let* $\mathrm{Ad}_{\mathrm{SL}}$ *and* $\mathrm{Ad}_{\mathrm{PSL}}$ *be the actions defined in Equations* (5.4) *and* (5.5) *respectively. For any* $p \in \mathfrak{sl}_2(\mathbb{R})$ *the isotropy subgroup* $\mathrm{PSL}(2,\mathbb{R})_p$ *of* p *with respect to the action* $\mathrm{Ad}_{\mathrm{PSL}}$ *is equal to the subgroup* $\pi(\mathrm{SL}(2,\mathbb{R})_p)$ *where* $\pi : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$ *is the quotient homomorphism and* $\mathrm{SL}(2,\mathbb{R})_p$ *is the isotropy subgroup of* p *with respect to the action* $\mathrm{Ad}_{\mathrm{SL}}$.

Proof. From Equation (5.7) we have $\mathrm{Ad}_{\mathrm{SL}} = \mathrm{Ad}_{\mathrm{PSL}} \circ \tilde{\pi}$. First let $\hat{A} \in \pi(\mathrm{SL}(2,\mathbb{R})_p)$. Then for any $A \in \mathrm{SL}(2,\mathbb{R})$ such that $\pi(A) = \hat{A}$, that is $\pm A \in \mathrm{SL}(2,\mathbb{R})_p$, Equation (5.7) implies that $\mathrm{Ad}_{\mathrm{PSL}}(\hat{A}, p) = p$ and $\hat{A} \in \mathrm{PSL}(2,\mathbb{R})_p$. On the other hand if $\hat{A} \in \mathrm{PSL}(2,\mathbb{R})_p$ then there exists an $A \in \mathrm{SL}(2,\mathbb{R})$ such that $\pi(A) = \hat{A}$ and again from Equation (5.7) $\mathrm{Ad}_{\mathrm{SL}}(A,p) = \mathrm{Ad}_{\mathrm{PSL}}(\hat{A}, p) = p$ so $A \in \mathrm{SL}(2,\mathbb{R})_p$ which completes the proof. □

Finally we prove Theorem 5.2, by identifying the homogeneous spaces arising from the subgroups $\hat{L}$, $\hat{H}$, and $\widehat{\mathrm{SO}}(2,\mathbb{R})$ with the level sets above.

Proof of Theorem 5.2. In the proof of Lemma 5.8 we showed that the orbits of X_1, $X_2 + X_3$ and $\left(1 - \frac{1}{4}\right) X_2 + \left(1 + \frac{1}{4}\right) X_3$ are Q_2, Q^+_0, and Q^+_{-2} respectively under the action $\mathrm{Ad}_{\mathrm{SL}}$. Corollary 5.3 implies that these sets are also the orbits with respect to $\mathrm{Ad}_{\mathrm{PSL}}$. Note that X_3

from Equation (5.6) has $Q(X_3) = -2$ and is given by $(0,0,1)$ in the coordinates with respect to the choice of basis in Equation (5.6). That is X_3 is in the same orbit as $\left(1-\frac{1}{4}\right)X_2 + \left(1+\frac{1}{4}\right)X_3$, and we can take X_3 as a representative for the orbit Q^+_{-2} instead of $\left(1-\frac{1}{4}\right)X_2 + \left(1+\frac{1}{4}\right)X_3$.

We now compute the isotropy of X_1, $X_2 + X_3$ and X_3, which will identify the orbits, Q_2, Q^+_0, Q^+_{-2}, with two dimensional homogeneous spaces $\mathrm{PSL}(2,\mathbb{R})$.

i) Consider the stabilizer of $X_2 + X_3$ with respect to the action $\mathrm{Ad}_{\mathrm{SL}}$, $\mathrm{SL}(2,\mathbb{R})_{X_2+X_3}$. This subgroup is defined by the solutions to

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix},$$

with $ad - bc = 1$. Then the isotropy subgroup of $X_2 + X_3$ as in Equation (5.6) is

$$\mathrm{SL}(2,\mathbb{R})_{X_2+X_3} = \left\{ \begin{bmatrix} \pm 1 & 0 \\ c & \pm 1 \end{bmatrix} \middle| \, b \in \mathbb{R} \right\}.$$

Now from Lemma 5.9 the stabilizer subgroup

$$\mathrm{PSL}(2,\mathbb{R})_{X_2+X_3} = \pi(\mathrm{SL}(2,\mathbb{R})_{X_2+X_3}) = \hat{L}.$$

Now from Theorem 2.7, $\mathrm{PSL}(2,\mathbb{R})/\hat{L}$ is equivariantly bijective with Q^+_0.

ii) Now consider the stabilizer, $\mathrm{SL}(2,\mathbb{R})_{X_1}$, of X_1 as in Equation (5.6) with respect to the action $\mathrm{Ad}_{\mathrm{SL}}$. This subgroup is given by the solutions to

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

that is the matrices in $\mathrm{SL}(2,\mathbb{R})$ that have $b = c = 0$,

$$\mathrm{SL}(2,\mathbb{R})_{X_1} = \left\{ \begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix} \middle| a \in \mathbb{R}^* \right\}.$$

From Lemma (5.9) $\mathrm{PSL}(2,\mathbb{R})_{X_1} = \pi(\mathrm{SL}(2,\mathbb{R})_{X_1}) = \hat{H}$, and $\mathrm{PSL}(2,\mathbb{R})/\hat{H}$ is equivariantly diffeomorphic to Q_2.

iii) Finally we compute $\mathrm{SL}(2,\mathbb{R})_{X_3}$. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $ad - bc = 1$, The equation $AX_3 = X_3 A$ is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\begin{bmatrix} b/2 & -a/2 \\ d/2 & -c/2 \end{bmatrix} = \begin{bmatrix} -c/2 & -d/2 \\ a/2 & b/2 \end{bmatrix}.$$

This gives the system of equations

$$\begin{cases} c + b = 0 \\ d - a = 0 \\ ad - bc = 1 \end{cases}.$$

So $a = d$ and $c = -b$ and from $ad - bc = 1$ we have $a^2 + b^2 = 1$. That is A is an orthogonal matrix so we can write the stabilizer in terms of a parameter t as

$$\mathrm{SL}(2,\mathbb{R})_{X_3} = \left\{ \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \theta \in [0, 2\pi] \right\} = \mathrm{SO}(2,\mathbb{R}).$$

As in the previous parts $\mathrm{PSL}(2,\mathbb{R})_{X_3} = \hat{\mathrm{SO}}(2,\mathbb{R})$ so that $\mathrm{PSL}(2,\mathbb{R})/\hat{\mathrm{SO}}(2,\mathbb{R})$ is equivariantly diffeomorphic to Q^+_{-2} completing the proof.

□

5.2 Invariant Foliations of $\mathrm{PSL}(2,\mathbb{R})/\hat{H}$ and $\mathrm{PSL}(2,\mathbb{R})/\hat{L}$.

In this section we will prove the following theorems

Theorem 5.3. *Let* $\mathrm{PSL}(2,\mathbb{R})/\hat{L}$ *and* $\mathrm{PSL}(2,\mathbb{R})/\hat{H}$ *be the non primitive* $\mathrm{PSL}(2,\mathbb{R})$ *homogeneous spaces from Theorem 5.1.*

i) There is an invariant foliation of $\mathrm{PSL}(2,\mathbb{R})/\hat{H}$ *with respect to the standard action of* $\mathrm{PSL}(2,\mathbb{R})$ *by left multiplication,* $F^{\hat{H}}$*, which is defined by its leaf through the identity coset,*

$$F^{\hat{H}}_{[\hat{\mathrm{I}}]_{\hat{H}}} = \left\{[\hat{E}]_{\hat{H}} \,|\, \hat{E} \in \hat{L}\right\} = \left\{ \left[\begin{pmatrix} \pm 1 & 0 \\ s & \pm 1 \end{pmatrix}\right]_{\hat{H}} \,\middle|\, s \in \mathbb{R} \right\}.$$

ii) There is a foliation of $\mathrm{PSL}(2,\mathbb{R})/\hat{L}$ *which is invariant under the standard action of* $\mathrm{PSL}(2,\mathbb{R})$ *by left multiplication,* $F^{\hat{L}}$*, with leaf through the identity*

$$F^{\hat{L}}_{[\hat{\mathrm{I}}]_{\hat{L}}} = \left\{[\hat{D}]_{\hat{L}} \,|\, \hat{D} \in \hat{H}\right\} = \left\{ \left[\begin{pmatrix} \pm s & 0 \\ 0 & \pm s^{-1} \end{pmatrix}\right]_{\hat{L}} \,\middle|\, s \in \mathbb{R}_{>0} \right\}.$$

Now consider the basis for $\mathfrak{sl}_2(\mathbb{R})$ given in Equation (5.6). We can define corresponding invariant foliations on $\mathfrak{sl}_2(\mathbb{R})$ by using the equivariant diffeomorphism between the coset spaces $\mathrm{PSL}(2,\mathbb{R})/\hat{H}$ and $\mathrm{PSL}(2,\mathbb{R})/\hat{L}$ with Q_2 and Q_0^+ respectively as given in the following theorem.

Theorem 5.4. *There are invariant foliations with respect to* $\mathrm{Ad}_{\mathrm{PSL}}$,

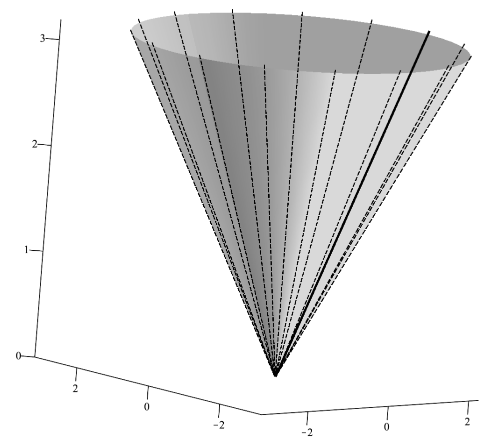
i) $\hat{F}^{\hat{H}}$ *of* Q_2 *determined by the leaf through the point* X_1

$$\hat{F}^{\hat{H}}_{X_1} = \left\{X_1 + s\,(X_2 + X_3) \,\middle|\, s \in \mathbb{R}\right\},$$

ii) and $\hat{F}^{\hat{L}}$ *of* Q_0^+ *given by the leaf through* $X_2 + X_3$

$$\hat{F}^{\hat{L}}_{X_2+X_3} = \left\{\frac{1}{s^2}\,(X_2 + X_3) \,\middle|\, s \in \mathbb{R}^*\right\}.$$

Fig. 5.2: Invariant foliation of the Upper Cone.



The foliations of Q_2 and Q_0^+ are depicted in Figure 5.3 and Figure 5.2 respectively.

Proof of Theorem 5.3. We will only prove the theorem for $F^{\hat{H}}$ since the proof for $F^{\hat{L}}$ is similar. Let $q : \mathrm{PSL}(2,\mathbb{R})/\hat{H} \to \mathrm{PSL}(2,\mathbb{R})/\hat{B}$ given by $q([\hat{A}]_{\hat{H}}) = [\hat{A}]_{\hat{B}}$ where $\hat{B} = \pi(B_0)$ and B_0 is the two dimensional subgroup of lower triangular matrices in $\mathrm{SL}(2,\mathbb{R})$. Then Theorem 4.1 implies that there is an invariant foliation of $\mathrm{PSL}(2,\mathbb{R})/\hat{H}$ which is invariant under the action of $\mathrm{PSL}(2,\mathbb{R})$ by left multiplication, $F^{\hat{H}} = \left\{F_{[\hat{A}]_{\hat{H}}} \,|\, F_{[\hat{A}]_{\hat{H}}} = \hat{A}F_{[\hat{\mathrm{I}}]_{\hat{H}}}\right\}$ where the leaf through the identity coset, $F_{[\hat{\mathrm{I}}]_{\hat{H}}} = q^{-1}([\hat{\mathrm{I}}]_{\hat{B}})^0$, the connected component of $q^{-1}([\hat{\mathrm{I}}]_B)$ which contains the identity coset $[\hat{\mathrm{I}}]_{\hat{H}}$. We will show that $F^{\hat{H}}_{[\hat{\mathrm{I}}]_{\hat{H}}}$, is the set $\hat{L}/\hat{H} = \left\{[\hat{E}]_{\hat{H}} \,|\, \hat{E} \in \hat{L}\right\}$. First note that $\hat{L} \subset \mathrm{PSL}(2,\mathbb{R})$ is a connected subgroup of $\mathrm{PSL}(2,\mathbb{R})$ and the projection $\pi_{\hat{H}} : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})/\hat{H}$ is a continuous map, the image $\hat{L}/\hat{H}$ is a connected subset of $\mathrm{PSL}(2,\mathbb{R})/\hat{H}$ containing the identity coset $[\hat{\mathrm{I}}]_{\hat{H}}$. Now note that

$$q([\hat{E}]_{\hat{H}}) = [\hat{E}]_{\hat{B}} = [\hat{\mathrm{I}}]_{\hat{B}}$$

since $\hat{L} \subset \hat{B}$, and the set $\hat{L}/\hat{H}$ is contained in the preimage $q^{-1}([\hat{\mathrm{I}}]_{\hat{B}})$. Hence, $\hat{L}/\hat{H} \subset F^{\hat{H}}_{[\hat{\mathrm{I}}]_{\hat{H}}}$. Now suppose that $[A]_{\hat{H}} \in F^{\hat{H}}_{[\hat{\mathrm{I}}]_{\hat{H}}}$. Then since $A \in \mathrm{PSL}(2,\mathbb{R})$ there exist $R \in \mathrm{SO}(2,\mathbb{R})$, $E \in L$, and $D \in H$, where $\mathrm{SO}(2,\mathbb{R})$ L and H are the connected one dimensional subgroups from Equation (5.2), such that

$$\pi(RED) = \pi(R)\pi(E)\pi(D) = A$$

and $\pi : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$ is the quotient map. Then consider $[A]_{\hat{H}} = [\pi(RED)]_{\hat{H}} = \pi(R)[\pi(E)]_{\hat{H}}$ since $\pi(D) \in \hat{H}$. Now using the equivariance of q with respect to the standard action of left multiplication of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathrm{PSL}(2,\mathbb{R})/\hat{H}$ and $\mathrm{PSL}(2,\mathbb{R})/\hat{B}$ we have

$$[\hat{\mathrm{I}}]_{\hat{B}} = q([A]_{\hat{H}}) = q(\pi(R)[\pi(E)]_{\hat{H}}) = \pi(R)[\pi(E)]_{\hat{B}} = \pi(R)[\hat{\mathrm{I}}]_{\hat{B}}$$

since $\pi(E) \in \hat{B}$. But then $\pi(R) \in \hat{B}$ and $\pi(R) = \hat{\mathrm{I}}$. verifying that $[A]_{\hat{H}} = [\pi(E)]_{\hat{H}} \in \hat{L}/\hat{H}$ and completing the proof. □

Corollary 5.4. *Let* $\pi : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$ *be the quotient homomorphism.*

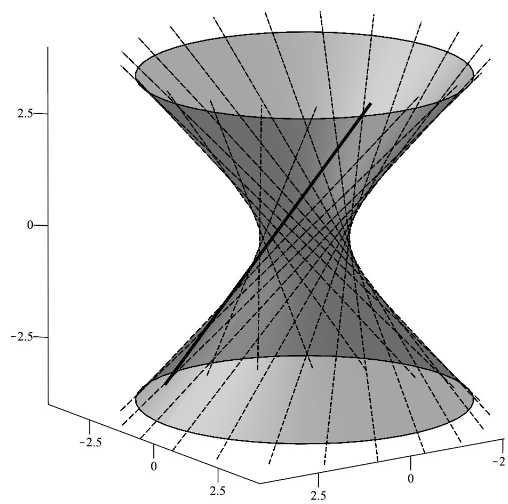
i) Every leaf of $F^{\hat{H}}$ *can be written as* $\pi(R)F^{\hat{H}}_{[\hat{\mathrm{I}}]_H}$ *for some* $R \in \mathrm{SO}(2,\mathbb{R})$. *That is there exists a unique* $\theta \in [0,\pi)$ *such that*

$$F^{\hat{H}}_{[\hat{A}]_{\hat{H}}} = \left\{ \left[\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \right]_{\hat{H}} \middle| \, s \in \mathbb{R} \right\}.$$

ii) Every leaf $F^{\hat{L}}_{[\hat{A}]_{\hat{L}}} \in F^{\hat{L}}$ *can be written as* $\pi(R)F^{\hat{L}}_{[\hat{\mathrm{I}}]_{\hat{L}}}$ *for some* $R \in \mathrm{SO}(2,\mathbb{R})$. *That is there exists some* $\theta \in [0,\pi)$ *such that*

$$F^{\hat{L}}_{[\hat{A}]_{\hat{L}}} = \left\{ \left[\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \right]_{\hat{L}} \middle| \, s \in \mathbb{R}_{>0} \right\}.$$

Fig. 5.3: The invariant foliation of one sheet hyperbola orbit.



Proof. Since every $A \in \mathrm{SL}(2,\mathbb{R})$ can be written as RDE or RED for $R \in \mathrm{SO}(2,\mathbb{R})$ $D \in H$ and $L \in D$, the claim follows from the invariance of the foliations $F^{\hat{H}}$ and $F^{\hat{L}}$ with respect to the standard action of $\mathrm{PSL}(2,\mathbb{R})$ by right multiplication, and from the fact that every $\pi(R)$ for $R \in \mathrm{SO}(2,\mathbb{R})$ can be written as

$$\pi(R) = \pi\left(\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\right)$$

for some unique $\theta \in [0,\pi)$ which follows from

$$-\mathrm{I} = \begin{bmatrix} \cos(\pi) & -\sin(\pi) \\ \sin(\pi) & \cos(\pi) \end{bmatrix}$$

and $\pi(R) = \pi(-R)$. □

The invariant foliations $F^{\hat{H}}$ of $\mathrm{PSL}(2,\mathbb{R})/\hat{H}$ and $F^{\hat{L}}$ of $\mathrm{PSL}(2,\mathbb{R})/\hat{L}$ with respect to the standard action of $\mathrm{PSL}(2,\mathbb{R})$ by left multiplication induce invariant foliations with respect to the action $\mathrm{Ad}_{\mathrm{PSL}}$ of the orbits Q_2 and Q_0^+ as described in Theorem 5.2.

Proof of Theorem 5.4. Again we will prove the claim for $\hat{F}^{\hat{H}}$ as the statement for $\hat{F}^{\hat{L}}$ is similar. By Theorem 2.7, the map $(\tilde{\mathrm{Ad}}_{\mathrm{PSL}})_{X_1} : \mathrm{PSL}(2,\mathbb{R})/\hat{H} \to Q_2$ given by

$$(\tilde{\mathrm{Ad}}_{\mathrm{PSL}})_{X_1}([\hat{A}]_{\hat{H}}) = \mathrm{Ad}_{\mathrm{PSL}}(\hat{A}, X_1) = AX_1A^{-1}$$

for any $A \in \mathrm{SL}(2,\mathbb{R})$ such that $\pi(A) = \hat{A}$. is an equivariant bijection of $\mathrm{PSL}(2,\mathbb{R})/\hat{H}$ with Q_2

Then by the proof of Lemma 4.4, $\hat{F}^{\hat{H}} = (\tilde{\mathrm{Ad}}_{\mathrm{PSL}})_{X_1}(F^{\hat{H}})$ is an invariant foliation of Q_2 with respect to the action $\mathrm{Ad}_{\mathrm{PSL}}$. We can compute the leaf through the point X_1 by the image of $F^{\hat{H}}_{[\hat{\mathrm{I}}]_{\hat{H}}}$ which is

$$(\tilde{\mathrm{Ad}}_{\mathrm{PSL}})_{X_1}\left(F^{\hat{H}}_{[\hat{\mathrm{I}}]_{\hat{H}}}\right) = \mathrm{Ad}_{\mathrm{PSL}}\left(\left\{[\hat{E}]_{\hat{H}} \,|\, \hat{E} \in \hat{L}\right\},\, X_1\right) = EX_1E^{-1}$$

for all $E \in \mathrm{SL}(2,\mathbb{R})$ such that $\hat{E} = \pi(E)$. That is

$$\begin{bmatrix} \pm 1 & 0 \\ s & \pm 1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}\begin{bmatrix} \pm 1 & 0 \\ s & \pm 1 \end{bmatrix}^{-1} = X_1 + s(X_2 + X_3)$$

for all $s \in \mathbb{R}$ which completes the proof. □

Using Corollary 5.4 each leaf of $\hat{F}^{\hat{H}}$ can be written in coordinates as the line

$$\cos(2\theta)X_1 + \sin(2\theta)X_2 + s\Big(\cos(2\theta)X_2 - \sin(2\theta)X_1 + X_3\Big)$$

for a unique $\theta \in [0, \pi)$. By using the coordinates from the choice of basis in Equation (5.6) for each θ the leaf of $\hat{F}^{\hat{H}}$ through the point $(\cos(2\theta), \sin(2\theta), 0)$ is the line

$$\Big(\cos(2\theta), \sin(2\theta), 0\Big) + s\Big(-\sin(2\theta), \cos(2\theta), 1\Big).$$

Similarly the leaf $\hat{F}^{\hat{L}}_{X_2+X_3}$ is given as

$$\frac{1}{s^2}\left(\sin(2\theta)X_1 + \cos(2\theta)X_2 + X_3\right)$$

and in coordinates

$$\frac{1}{s^2}(\sin(2\theta), \cos(2\theta), 1).$$

5.3 Primitive Homogenous Space With Submaximal Lie Algebra:

Let $N(H)$ be the normalizer of of the diagonal subgroup H from Equation (5.2),

$$N(H) = \left\{A \in \mathrm{SL}(2, \mathbb{R}) \,|\, ABA^{-1} \in H,\ \forall B \in H\right\}.$$

This section shows that the homogeneous space $\mathrm{SL}(2, \mathbb{R})/N(H)$ is primitive, but the Lie algebra for $N(H)$ is not a maximal subalgebra of $\mathfrak{sl}_2(\mathbb{R})$. The subgroup $N(H)$ is not connected, demonstrating that the condition that H be connected in Corollary 4.3 part ii) is necessary in the case where G is a simple Lie group.

We start by giving a concrete description of the normalizer $N(H)$ in the following lemmas.

Lemma 5.10. *The centralizer, $Z(H)$, of the one parameter subgroup H given in Equation (5.2) is*

$$Z(H) = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \,\middle|\, a \in \mathbb{R}^* \right\}.$$

Proof. Let $A \in \mathrm{SL}(2,\mathbb{R})$ such that $AUA^{-1} = U$ for all $U \in H$ and let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that $ad - bc = 1$. Then $AU = UA$ implies that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} ta & t^{-1}b \\ tc & t^{-1}d \end{bmatrix} = \begin{bmatrix} ta & tb \\ t^{-1}c & t^{-1}d \end{bmatrix}$$

for all $t > 0$. Then $b = t^2 b$ and $t^2 c = c$ for all $t > 0$ implies that $b, c = 0$. Using $ad - bc = 1$ gives $d = a^{-1}$ and A has the desired form. Conversely, if $a \in \mathbb{R}^*$ then

$$\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}^{-1} = \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$$

which completes the proof. □

Lemma 5.11. *Let H be the one parameter subgroup in Equation* (5.2)*, Then the normalizer subgroup, $N(H)$, is given by*

$$N(H) = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, \begin{bmatrix} 0 & a \\ -a^{-1} & 0 \end{bmatrix} \middle| a \in \mathbb{R}^* \right\}.$$

Proof. Let

$$K = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, \begin{bmatrix} 0 & a \\ -a^{-1} & 0 \end{bmatrix} \middle| a \in \mathbb{R}^* \right\}.$$

Then $K \subset N(H)$ is a straightforward computation that we will omit from the proof.

If $A \in Z(H)$ then it is an element of K by Lemma 5.10 so suppose that $A \in N(H) \setminus Z(H)$. Then there exist $U_1, U_2 \in H$ with $U_1 \neq U_2$, such that $AU_1 = U_2A$. Note that $U_1 = \mathrm{I}$ if and only if $U_2 = \mathrm{I}$ because matrix multiplication is linear and A is invertible, so $U_1, U_2 \neq \pm\mathrm{I}$ as well. Now Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad U_1 = \begin{bmatrix} u_1 & 0 \\ 0 & u_1^{-1} \end{bmatrix}, \quad \text{and} \quad U_2 = \begin{bmatrix} u_2 & 0 \\ 0 & u_2^{-1} \end{bmatrix}$$

where $u_1, u_2 > 0$, $u_1 \neq u_2$ and $ad - bc = 1$. Then $AU_1 = U_2A$ is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u_1 & 0 \\ 0 & u_1^{-1} \end{bmatrix} = \begin{bmatrix} u_2 & 0 \\ 0 & u_2^{-1} \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\begin{bmatrix} u_1a & u_1^{-1}b \\ u_1c & u_1^{-1}d \end{bmatrix} = \begin{bmatrix} u_2a & u_2b \\ u_2^{-1}c & u_2^{-1}d \end{bmatrix}.$$

The condition that $u_1 \neq u_2$ gives that $a = 0$. Then using $ad - bc = 1$ we have $bc \neq 0$ and $c = -1/b$. The off diagonal entries give $u_2^{-1} = u_1$, and this then forces $\left(u_2^{-1} - u_2\right)d = 0$ so $d = 0$ since $U_2 \neq \mathrm{I}$. Therefore

$$A = \begin{bmatrix} 0 & b \\ -b^{-1} & 0 \end{bmatrix},$$

so $A \in K$ completing the proof. □

Lemma 5.12. *The subgroup $N(H)$ is not connected.*

Proof. The subsets $Z(H)$ and $N(H) \setminus Z(H)$ are disjoint and open such that $Z(H) \cup (N(H) \setminus Z(H)) = N(H)$. □

The next lemma shows that the subgroup $N(H)$ is maximal, and therefore by Corollary 4.2 the homogeneous space $\mathrm{SL}(2,\mathbb{R})/N(H)$ is primitive.

Theorem 5.5. *The action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathrm{SL}(2,\mathbb{R})/N(H)$ is a primitive action.*

Proof. We will show that $N(H)$ is not contained in any subgroups of $\mathrm{SL}(2,\mathbb{R})$ with dimension strictly between $\dim(N(H)) = 1$ and $\dim(\mathrm{SL}(2,\mathbb{R})) = 3$. Indeed let $N(H) \subset K$ with $1 < \dim(K) < 3$. Since H is a connected Lie subgroup contained in K then the Lie algebra $\mathfrak{h}$ of H will be contained in the Lie algebra $\mathfrak{k}$ of K.

We claim that this forces $\mathfrak{k}$ to be a subalgebra of triangular matrices. Let $\{f, h, e\}$ be the standard basis for $\mathfrak{sl}_2(\mathbb{R})$ as in Equation (5.12). Then since $h \in \mathfrak{k}$ and it has dimension 2 we can extend h to a basis for $\mathfrak{k}$, $\{h, x\}$. Now let x be given by

$$x = \begin{bmatrix} t & u \\ v & -t \end{bmatrix}$$

and suppose that u and v are both non-zero. Then the bracket $[h, x] = x' \in \mathfrak{k}$ and

$$x' = \begin{bmatrix} 0 & 2u \\ -2v & 0 \end{bmatrix}.$$

Since $\mathfrak{k}$ is a subalgebra then $[h, x'] = x'' \in \mathfrak{k}$ and

$$x'' = \begin{bmatrix} 0 & 4u \\ 4v & 0 \end{bmatrix}.$$

But now consider

$$\frac{1}{4u}x' + \frac{1}{8u}x'' = e \quad \text{and} \quad \frac{1}{8v}x'' - \frac{1}{4v}x' = f$$

so $\mathfrak{k} = \mathfrak{sl}_2(\mathbb{R})$, a contradiction in assuming that both u and v are nonzero.

Then either u or v are zero, but not both since x and h are independent.

Suppose that v is zero. Since $\{x, h\}$ is a basis for $\mathfrak{k}$ then

$$\mathfrak{k} = \left\{ \begin{bmatrix} \sigma & \alpha \\ 0 & -\sigma \end{bmatrix} \middle| \sigma, \alpha \in \mathbb{R} \right\}.$$

then K contains the connected subgroup,

$$B_0 = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \middle| \forall\, a, b \in \mathbb{R},\, a > 0 \right\}$$

Let $S \in N(H)$ be the matrix

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then $S \in K$ as well and

$$S \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} S^{-1} = \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix}$$

is in K by closure under multiplication for all real $b > 0$. Hence the Lie algebra $\mathfrak{k}$ will contain the element

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

which is independent of both x and h meaning $\mathfrak{k}$ has dimension three, a contradiction. A similar argument can be used to show that $\mathfrak{k}$ is three dimensional in the case that u is zero.

Corollary 4.2 implies the action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathrm{SL}(2, \mathbb{R})/N(H)$ is primitive and the proof is complete. □

Finally we note that since $H \subset N(H)$ and $N(H)$ is a one dimensional Lie group, then the Lie algebra $\mathfrak{h}$ of H is the Lie algebra of $N(H)$. However $\mathfrak{h}$ is not a maximal subalgebra of $\mathfrak{sl}_2(\mathbb{R})$ as it is contained in the Lie algebra of B_0.

CHAPTER 6

Joint Invariants of Primitive Homogeneous Spaces with Lie Algebra-Subalgebra Pairs of Affine Type

In this chapter we consider the groups $\mathrm{A}(n)$ and $\mathrm{SA}(n)$ acting on $\mathbb{R}^n$ by the standard affine action.

Definition 6.1. Let $\mathrm{A}(n) = \{(A, a) \,|\, A \in \mathrm{GL}(n),\, a \in \mathbb{R}^n\}$, be the real affine group. Then the action, $\mu_{\mathrm{A}(n)} : \mathrm{A}(n) \times \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\mu_{\mathrm{A}(n)}\big((A, a), x\big) = Ax + a \tag{6.1}$$

for $x \in \mathbb{R}^n$ and $(A, a) \in \mathrm{A}(n)$ is called the *standard affine action* of $\mathrm{A}(n)$ on $\mathbb{R}^n$.

In Section 6.1 we prove Lemma 6.2 which shows that $\mathbb{R}^n$ is a primitive homogeneous space of $\mathrm{A}(n)$ and $\mathrm{SA}(n)$. We conclude Section 6.1 by proving Lemma 6.3 which shows that the Lie algebra subalgebra pair for $\mathbb{R}^n$ as a primitive homogeneous space of $\mathrm{A}(n)$ is of type i) in the classification Theorem 4.3. In Section 6.2 we prove Theorem 6.1 by using the reduction method to show that the joint invariants for $\mu_{\mathrm{A}(n)}$ and $\mu_{\mathrm{SA}(n)}$ is equivalent to to finding joint invariants of the standard representations of $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{SL}(n, \mathbb{R})$. The chapter concludes with Section 6.3 which determines the points of $(\mathbb{R}^n)^k$ which are in general position for $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{SL}(n, \mathbb{R})$ and constructs the joint invariants of $\mu_{\mathrm{A}(n)}$ and $\mu_{\mathrm{SA}(n)}$ in Corollary 6.3 and Corollary 6.5.

The invariants of affine type presented in this chapter have been developed in the work of Olver [17] for the special affine group $\mathrm{SA}(n)$.

The application of the reduction theory demonstrated on $\mathrm{A}(n)$ and $\mathrm{SA}(n)$ holds for the Euclidean group $\mathrm{Euc}^+(n)$ as well, showing that the joint invariants of the standard affine action restricted to $\mathrm{Euc}^+(n)$, that is the Euclidean transformations of $\mathbb{R}^n$, are determined by the joint invariants for the standard representation of $\mathrm{SO}(n, \mathbb{R})$. However, the construction of joint invariants deviates from our treatment of $\mathrm{A}(n)$ and $\mathrm{SA}(n)$ due to the standard representation of $\mathrm{SO}(n,\mathbb{R})$ not being transitive on $\mathbb{R}^n \setminus \{0\}$. For this reason the method of slices described in Section 7.1.3 is more appropriate for determining the joint invariants using the reduction method. To see a direct construction of the joint invariants for the standard representation of $\mathrm{SO}(n,\mathbb{R})$ see Olver [17].

6.1 Primitive Verification

Lemma 6.1. *The action $\mu_{\mathrm{A}(n)}$ is transitive on $\mathbb{R}^n$ and the stabilizer of the point $0 \in \mathbb{R}^n$ is the subgroup $H_{\mathrm{A}(n)} = \{(A,0) \mid A \in \mathrm{GL}(n)\} \subset \mathrm{A}(n)$.*

Proof. Define a map $\rho : \mathbb{R}^n \to \mathrm{A}(n)$ by

$$\rho(x) = (\mathrm{I}, -x) \tag{6.2}$$

which satisfies $\rho(x) \cdot x = 0$ and so the action is transitive since $\rho(x)^{-1} \cdot 0 = x$.

Now we compute the isotropy subgroup. If $(A,a) \in \mathrm{A}(n)$, then acting by (A,a) on 0 gives $(A,a) \cdot 0 = a$, and the stabilizer subgroup of 0 is $H_{\mathrm{A}(n)} = \{(A,0) \mid A \in \mathrm{GL}(n)\}$ which completes the proof. □

The action $\mu_{\mathrm{A}(n)}$ is extended to the diagonal action $\mu^k_{\mathrm{A}(n)} : \mathrm{A}(n) \times (\mathbb{R}^n)^k \to (\mathbb{R}^n)^k$ as in Definition 2.5. Let $\mu_{H_{\mathrm{A}(n)}} = \mu_{\mathrm{A}(n)}|_{H_{\mathrm{A}(n)}}$ be the restriction of $\mu_{\mathrm{A}(n)}$ to $H_{\mathrm{A}(n)}$ in the first argument. That is $\mu_{H_{\mathrm{A}(n)}}$ is the isotropy action induced by $\mu_{\mathrm{A}(n)}$, and the corresponding diagonal action of the isotropy subgroup is denoted $\mu^k_{H_{\mathrm{A}(n)}}$.

The action $\mu_{\mathrm{SA}(n)}$ are defined by restricting $\mu_{\mathrm{A}(n)}$ to the subgroup $\mathrm{SA}(n)$ which is transitive on $\mathbb{R}^n$ by the same argument as Lemma 6.1. The isotropy subgroup of $0 \in \mathbb{R}^n$ is

$$H_{\mathrm{SA}(n)} = \{(A,0) \in \mathrm{A}(n) \mid \det(A) = 1\}$$

for $\mathrm{SA}(n)$.

We now verify that $\mathrm{A}(n)/H_{\mathrm{A}(n)}$ is a primitive homogeneous space, and moreover that $(\mathrm{A}(n), H)$ is a primitive pair. The section will provide two proofs of this statement, one by showing that $H_{\mathrm{A}(n)}$ is a maximal subgroup containing no normal subgroups of $\mathrm{A}(n)$ and the other showing that the Lie algebra $\mathfrak{h}_{\mathrm{A}(n)}$ of $H_{\mathrm{A}(n)}$ is a maximal Lie subalgebra of $\mathfrak{a}_n(\mathbb{R})$, the Lie algebra of $\mathrm{A}(n)$. The proof that the subgroup $H_{\mathrm{A}(n)}$ is maximal applies similarly to the subgroup $H_{\mathrm{SA}(n)}$ of $\mathrm{SA}(n)$, but the Lie algebra proof must be modified to hold for the Lie algebra $\mathfrak{h}_{\mathrm{SA}(n)}$.

Lemma 6.2. *The action of* $\mathrm{A}(n)$ *on* $\mathrm{A}(n)/H_{\mathrm{A}(n)}$ *is primitive, and* $(\mathrm{A}(n), H_{\mathrm{A}(n)})$ *is a primitive pair.*

Proof of Lemma 6.2. We will show that $H_{\mathrm{A}(n)}$ is maximal Suppose $H_{\mathrm{A}(n)}$ is contained in a Lie subgroup K of $\mathrm{A}(n)$, $H_{\mathrm{A}(n)} \subset K \subset \mathrm{A}(n)$. Suppose that $K \neq H_{\mathrm{A}(n)}$ then there is some element $(B, b) \in K$ such that $b \neq 0$. We can factor (B, b) into the product

$$(B, b) = (B, 0)(\mathrm{I}, B^{-1}b),$$

where $B^{-1}b = b' \neq 0$. Since $(B, 0) \in K$ we can assume without loss of generality that $(B, b) = (\mathrm{I}, b)$.

We proceed by showing that $K = \mathrm{A}(n)$. Fix some $(C, c) \in \mathrm{A}(n)$. If $c = 0$ then $(C, c) \in H$ so suppose not. Then by transitivity of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n \setminus \{0\}$ there exists an $A \in \mathrm{GL}(n, \mathbb{R})$ such that $Ab = c$ as $b, c \neq 0$. Now factor

$$(C, c) = (A, 0)(\mathrm{I}, b)(A^{-1}C, 0),$$

where the right hand side is a product of elements in K and so $(C, c) \in K$. This proves that $H_{\mathrm{A}(n)}$ is a maximal subgroup and by Corollary 4.2 $\mathrm{A}(n)$ acts primitively on $\mathrm{A}(n)/H_{\mathrm{A}(n)}$ proving the first part of the claim.

Now we prove that $H_{\mathrm{A}(n)}$ contains no normal subgroups of $\mathrm{A}(n)$ to verify $(\mathrm{A}(n), H_{\mathrm{A}(n)})$ is a primitive pair. Suppose that $N \subset H_{\mathrm{A}(n)}$ is a normal subgroup of $\mathrm{A}(n)$. Then fix some

$(X, 0) \in N$ and consider conjugation by an arbitrary $(A, a) \in \mathrm{A}(n)$,

$$(A, a)(X, 0)\left(A^{-1}, -A^{-1}a\right) = \left(AXA^{-1}, (\mathrm{I} - AXA^{-1})a\right).$$

Since N is normal in $\mathrm{A}(n)$ then this must land back in N, and must land back in $H_{\mathrm{A}(n)}$, giving the equation

$$(\mathrm{I} - AXA^{-1})a = 0.$$

Since a is arbitrary then $\mathrm{I} = AXA^{-1}$ but I is fixed under conjugation so $X = \mathrm{I}$. Hence the only normal subgroup of $\mathrm{A}(n)$ that $H_{\mathrm{A}(n)}$ contains is the trivial subgroup. ☐

Remark 6.1. The key point in this proof is that $\mathrm{GL}(n, \mathbb{R})$ acts transitively on $\mathbb{R}^n \setminus \{0\}$, which is also true for $\mathrm{SL}(n, \mathbb{R})$ so the proof that $\mathbb{R}^n$ is a primitive homogeneous space of $\mathrm{SA}(n)$ is similar.

Corollary 6.1. *The action of* $\mathrm{A}(n)$ *on* $\mathbb{R}^n$ *in Equation* (6.1) *is primitive.*

Proof. Since $\mathrm{A}(n)$ acts transitively on $\mathbb{R}^n$ with isotropy $H_{\mathrm{A}(n)}$ then by Theorem 2.11 there is a canonical equivariant diffeomorphism between $\mathrm{A}(n)/H_{\mathrm{A}(n)}$ and $\mathbb{R}^n$. Then from Lemma 4.4 the action on $\mathbb{R}^n$ is primitive. ☐

Now we consider the Lie algebra approach. Since $\mathrm{A}(n)$ is a semidirect product of $\mathrm{GL}(n, \mathbb{R})$ and $\mathbb{R}^n$, the Lie algebra $\mathfrak{a}_n(\mathbb{R})$ of $\mathrm{A}(n)$ is given by the semidirect sum,

$$\mathfrak{gl}(n) \ltimes \mathbb{R}^n = \{(X, x) \mid X \in \mathfrak{gl}(n),\, x \in \mathbb{R}^n\}, \tag{6.3}$$

with the bracket,

$$\left[(X, x), (Y, y)\right] = \left((XY - YX), Xy - Yx\right) = \left([X, Y], Xy - Yx\right),$$

using that $\mathbb{R}^n$ is abelian as a Lie algebra.

$\mathfrak{a}_n(\mathbb{R})$ contains an abelian ideal $\mathfrak{i} \subset \mathfrak{a}(n)$, given by

$$\mathfrak{i} = \{(0,x) \,|\, x \in \mathbb{R}^n\}. \tag{6.4}$$

And the Lie algebra $\mathfrak{h}_{\mathrm{A}(n)}$ of $H_{\mathrm{A}(n)}$ is given by

$$\mathfrak{h}_{\mathrm{A}(n)} = \{(X,0) \,|\, X \in \mathfrak{gl}(n)\}.$$

We claim that $(\mathfrak{a}(n), \mathfrak{h}_{\mathrm{A}(n)})$ is a primitive pair of type i) as in Theorem 4.3.

Lemma 6.3. *Let* $(\mathfrak{a}(n), \mathfrak{h}_{\mathrm{A}(n)})$ *be the Lie algebras in Equation* (6.3). *Then*

i) $\mathfrak{h}_{\mathrm{A}(n)}$ *is a maximal subalgebra of* $\mathfrak{a}(n)$

ii) $\mathfrak{h}_{\mathrm{A}(n)}$ *acts faithfully and irreducibly on the abelian ideal* $\mathfrak{i}$ *given in Equation* (6.4).

iii) $\mathfrak{h}_{\mathrm{A}(n)}$ *contains no ideals of* $\mathfrak{a}(n)$.

Proof. Part i) Suppose $\mathfrak{l}$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{h}_{\mathrm{A}(n)} \subset \mathfrak{l} \subset \mathfrak{a}(n)$. Then either $\mathfrak{l} = \mathfrak{h}_{\mathrm{A}(n)}$ or there exists a point $(X,x) \in \mathfrak{l}$ with $x \neq 0$. By bracketing with $(\mathrm{I},0)$ we have

$$\big[(\mathrm{I},0),(X,x)\big] = (0,x) \in \mathfrak{l}.$$

Assume that $(X,x) = (0,x)$ without loss of generality.

Now let $(Z,z) \in \mathfrak{a}(n)$ and suppose that $z = 0$, since if not then $(Z,z) \in \mathfrak{h}_{\mathrm{A}(n)} \subset \mathfrak{l}$. Then there exists a $Y \in \mathfrak{gl}(n)$ such that $Yx = z$ and

$$\big[(Y,0),(0,x)\big] = (0,z) \in \mathfrak{l}.$$

but because $\mathfrak{l}$ is closed under addition and $(Z,0) \in \mathfrak{h}_{\mathrm{A}(n)} \subset \mathfrak{l}$ then the element $(Z,0)+(0,z) = (Z,z) \in \mathfrak{l}$, completing the proof of part i).

Part ii) Now let $(0,x) \in \mathfrak{i}$ and consider for each $(Y,0) \in \mathfrak{h}_{\mathrm{A}(n)}$ the linear map $\mathrm{ad}_{(Y,0)} : \mathfrak{i} \to \mathfrak{i}$ given by

$$\mathrm{ad}_{(Y,0)}((0,x)) = \left[(Y,0),(0,x)\right] = (0,Yx)$$

so that $\mathfrak{i}$ is a representation of $\mathfrak{h}_{\mathrm{A}(n)}$, isomorphic to the standard representation of $\mathfrak{gl}(n)$ on $\mathbb{R}^n$ which is faithful and irreducible completing the proof of part ii).

Part iii) Now suppose that $\mathfrak{k} \subset \mathfrak{h}_{\mathrm{A}(n)}$ is an ideal of $\mathfrak{a}(n)$. If $(X,0) \in \mathfrak{k}$ and $(Z,z) \in \mathfrak{a}(n)$ then

$$\left[(X,0),(Z,z)\right] = \left(\left[X,Z\right],Xz\right) \in \mathfrak{k}$$

so in particular $Xz = 0$ and $z \in \ker(X)$, but this must hold for all $z \in \mathbb{R}^n$ so $X = 0$ completing the proof. □

Note this proof is special to $\mathfrak{h}_{H_{\mathrm{A}(n)}}$ and needs to be adapted for the subgroup $\mathrm{SA}(n)$.

6.2 Reduction

The action $\mu_{\mathrm{A}(n)}$ makes $\mathbb{R}^n$ into a primitive homogeneous space of $\mathrm{A}(n)$ and $\mathrm{SA}(n)$, as discussed in Section 6.1. In this section we will identify the joint invariants of $\mathrm{A}(n)$ with those of the standard representation for $\mathrm{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$. The group $\mathrm{GL}(n,\mathbb{R})$ acts on $\mathbb{R}^n$ by the standard representation $\mu_{\mathrm{GL}(n,\mathbb{R})} : \mathrm{GL}(n,\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n$ defined by matrix multiplication,

$$\mu_{\mathrm{GL}(n,\mathbb{R})}\left(A,z\right) = Az \tag{6.5}$$

for $A \in \mathrm{GL}(n,\mathbb{R})$ and $z \in \mathbb{R}^n$. We denote the corresponding diagonal action of $\mathrm{GL}(n,\mathbb{R})$ on k copies by $\mu^k_{\mathrm{GL}(n,\mathbb{R})}; \mathrm{GL}(n,\mathbb{R}) \times (\mathbb{R}^n)^k \to (\mathbb{R}^n)^k$. The map ρ in Equation (6.2) satisfies the identity in Equation (3.3), and so by Lemma 3.1 the k point joint invariants of $\mu_{\mathrm{A}(n)}$

are all given by $f \circ T$ where f is a $k-1$ point joint invariant of $\mu_{H_{\mathrm{A}(n)}}$ and the map $T : (\mathbb{R}^n)^k \to (\mathbb{R}^n)^{k-1}$, is given by

$$T(x_0, \ldots, x_{k-1}) = (x_1 - x_0, \ldots, x_{k-1} - x_0). \tag{6.6}$$

Theorem 6.1. *Let $f : (\mathbb{R}^n)^{k-1} \to Y$ be a $k-1$ point joint invariant of $\mu_{\mathrm{GL}(n,\mathbb{R})}$. Then $f \circ T$ is a k point joint invariant of $\mu_{\mathrm{A}(n)}$ where T is given in Equation* (6.6)*, and every joint invariant of* A(n) *is of this form.*

proof of Theorem 6.1. Using the identity map $\mathrm{Id} : (\mathbb{R}^n)^{k-1} \to (\mathbb{R}^n)^{k-1}$ and the isomorphism $\sigma : H \to \mathrm{GL}(n, \mathbb{R})$ given by

$$\sigma\left((A, 0)\right) = A$$

Theorem 2.4 gives a unique bijection of the orbit spaces $\tilde{\mathrm{Id}} : \mathbb{R}^n / H_{\mathrm{A}(n)} \to \mathbb{R}^n / \mathrm{GL}(n, \mathbb{R})$

$$\tilde{\mathrm{Id}}\left(\left[(z_1, \ldots, z_{k-1})\right]_{H_{\mathrm{A}(n)}}\right) = \left[(z_1, \ldots, z_{k-1}\right]_{\mathrm{GL}(n,\mathbb{R})}$$

which satisfies $\tilde{\mathrm{Id}} \circ p_{\mu^{k-1}_{H_{\mathrm{A}(n)}}} = p_{\mu^{k-1}_{\mathrm{GL}(n,\mathbb{R})}} \circ \mathrm{Id}$. So if $f : (\mathbb{R}^n)^{k-1} \to Y$ is a Y valued $k-1$ point joint invariant of $\mathrm{GL}(n, \mathbb{R})$ then by pulling back along the identity map $f \circ \mathrm{Id} = f$ is also a joint invariant of $H_{\mathrm{A}(n)}$, and clearly every joint invariant of $H_{\mathrm{A}(n)}$ is of this form.

Hence by Lemma 3.1 every k point joint invariant of A(n) is given by

$$f \circ T(x_0, \ldots, x_{k-1}) = f(x_1 - x_0, \ldots, x_{k-1} - x_0)$$

where f is a $k-1$ point joint invariant of the standard representation of $\mathrm{GL}(n, \mathbb{R})$. □

6.3 Invariants

Now we will determine complete local joint invariants of the groups A(n) and SA(n) by considering the joint invariants of $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{SL}(n, \mathbb{R})$ acting on the subset of points in general position in $(\mathbb{R}^n)^k$.

Theorem 6.2. *The set of points in general position for the standard representation of* $\mathrm{GL}(n,\mathbb{R})$ *on* $(\mathbb{R}^n)^k$ *are the sets,*

$$U_k = \{(z_1,\ldots,z_k)\,|\, z_i \text{ are independent}\}\,. \tag{6.7}$$

Remark 6.2. Note that the preimage $T^{-1}(U_k)$ are all the points $(x_0,x_1,\ldots,x_k)$ such that $(x_1-x_0,\ldots,x_k-x_0)$ are independent. Geometrically these are the sets of vectors based at x_0 which are independent.

The proof of Theorem 6.2 uses the following lemma and its corollary.

Lemma 6.4. *For any* $1 \le k \le n$ *the action* $\mu^k_{\mathrm{GL}(n,\mathbb{R})}$ *is transitive on* $U_k \subset (\mathbb{R}^n)^k$ *and the dimension of the stabilizer of a point* $Z \in U_k$ *is* $k(n-k)$.

Proof. Fix some $1 \le k \le n$. First we note that U_k is invariant under $\mu^k_{\mathrm{GL}(n,\mathbb{R})}$ since an invertible linear transformation takes linearly independent sets to linearly independent sets. Let $Z = (z_1,\ldots,z_k) \in U_k$ and let $\{e_i\}$ be the standard basis for $\mathbb{R}^n$. Extend $\{z_i\}$ to a basis and consider the matrix, A, which has this basis as its columns. Then $A^{-1}z_i = e_i$ and

$$\mu^k_{\mathrm{GL}(n,\mathbb{R})}\left(A,\left(z_1,\ldots,z_k\right)\right) = (e_1,\ldots,e_k)$$

which verifies that the action is transitive.

The isotropy subgroup for Z is conjugate to the isotropy of $(e_1,\ldots,e_k)$ since they are in the same orbit. Let A be in the stabilizer of $(e_1,\ldots,e_k)$. then the first k columns of A are $(e_1,\ldots,e_k)$ so A is block upper triangular

$$A = \begin{bmatrix} \mathrm{I}_{k\times k} & B_{k\times(n-k)} \\ 0_{(n-k)\times k} & C_{(n-k)\times(n-k)} \end{bmatrix} \tag{6.8}$$

with B some $k\times(n-k)$ matrix and $C \in \mathrm{GL}(n-k,\mathbb{R})$. So the isotropy subgroup has dimension $k(n-k)+(n-k)^2 = n(n-k)$ as claimed. ∎

The description of the isotropy subgroup in Equation (6.8) implies the following corollary.

Corollary 6.2. *The action of* $\mathrm{GL}(n)$ *on* U_n *is free.*

proof of Theorem 6.2. Fix some k and pick a point $Z = (z_1, \ldots, z_{k-1}) \in (\mathbb{R}^n)^k$. If $Z = 0$ then it is a fixed point and has a zero dimensional orbit. So now assume $Z \neq 0$ and consider two cases on k.

If $k \leq n$ then if $Z \in U_k$ the dimension of the stabilizer for Z is $k(n-k)$ from Lemma 6.4. Now suppose $Z = (z_1, \ldots, z_k) \notin U_k$. Since Z is non zero let $\hat{Z}$ be a maximal linearly independent subset of Z with $\ell < k$ elements. Then the stabilizer of Z is contained in the stabilizer of $\hat{Z}$ by linearity of the action $\mu^k_{\mathrm{GL}(n,\mathbb{R})}$. The dimension of the isotropy subgroup for $\hat{Z}$ is $n(n-\ell)$ from Lemma 6.4, which is larger than $n(n-k)$ and therefore Z is not in general position.

Now consider the case $k > n$. For Z to be in general position its first n points must be in general position for $(\mathbb{R}^n)^n$, and hence are elements of U_n. But from Corollary 6.2 the action of $\mathrm{GL}(n,\mathbb{R})$ is free on U_n and so every point in $U_n \times (\mathbb{R}^n)^{k-n}$ has a zero dimensional stabilizer. □

The local invariants of $\mu^{n+1}_{\mathrm{GL}(n,\mathbb{R})}$ defined on $U_k \times \mathbb{R}^n$ are given in the following theorem.

Theorem 6.3. *There are* n *independent local* $n+1$ *point joint invariants for the standard representation of* $\mathrm{GL}(n,\mathbb{R})$, $\mu_{\mathrm{GL}(n,\mathbb{R})}$ *on* $(\mathbb{R}^n)^{n+1}$ *given by the functions* $\alpha^i : U_n \times \mathbb{R}^n \to \mathbb{R}$ *which are defined by the equation*

$$z_{n+1} = \alpha^i(z_1, \ldots, z_{n+1}) z_i.$$

Corollary 6.3. *The functions* $\alpha^i \circ T : \left(T^{-1}(U_n) \times \mathbb{R}^n)\right) \to \mathbb{R}$ *given by*

$$\alpha^i \circ T(x_0, \ldots, x_{n+1}) = \alpha^i(x_1 - x_0, \ldots, x_{n+1} - x_0),$$

where T is as defined in Equation (6.6), are a complete set of local $n+2$ point joint invariants for the standard affine action $\mu_{\mathrm{A}(n)}$.

The proof of Theorem 6.3 follows from the next lemma.

Lemma 6.5. *The orbit through every point, $Z = (z_1, \dots, z_{n+1}) \in U_n \times \mathbb{R}^n$, has a representative $(e_1, \dots, e_n, \alpha^i e_i)$ for a unique collection of α^i depending only on the point Z.*

Proof. Let $(z_1, \dots, z_{n+1}) \in U_n \times \mathbb{R}^n$. Then since $\{z_i\}$ are a basis for $\mathbb{R}^n$ the final point $z_{n+1} = \alpha^i z_i$ for a unique collection of components $\{\alpha^i\}$. Let A be the matrix which takes the first n points to $(e_1, \dots, e_n)$, then

$$\big(e_1, \dots, e_n, \alpha^i e_i\big). \tag{6.9}$$

□

Proof of Theorem 6.3. The functions α^i are well defined since any point in U_n is a basis for $\mathbb{R}^n$. Then suppose that $Z = (z_1, \dots, z_n, z_{n+1})$ and $A \in \mathrm{GL}(n)$. Then

$$(Az_1, \dots Az_n, Az_{n+1}) = (Az_1, \dots, Az_n, A\alpha^i z_i) = \big(Az_1, \dots, Az_n, \alpha^i (Az_i)\big)$$

so that $\alpha^i(Z) = \alpha^i(A \cdot Z)$ and these are invariant functions.

Now suppose that $W = (w_1, \dots, w_{n+1})$ and $V = (v_1, \dots, v_{n+1})$ are chosen, by Lemma 6.5 they have representatives $(e_1, \dots, \alpha^i(W)e_i)$ and $(e_1, \dots, \alpha^i(V)e_i)$ which are equal, and hence W and V are conjugate, if and only if $\alpha^i(W) = \alpha^i(V)$. □

Remark 6.3. For an explicit description of these invariants consider the function $F : U_n \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$F(z_1, \dots, z_{n+1}) = \big[z_1 | \cdots | z_n\big]^{-1} z_{n+1}$$

where $\big[z_1 | \cdots | z_n\big]$ is the matrix with columns given by $(z_1, \dots, z_n)$. The components of this function are the invariants α^i given in Theorem 6.3.

Now consider the special affine group $\mathrm{SA}(n)$.

Lemma 6.6. *The standard representation of* $\mathrm{SL}(n)$ *is transitive on* U_k *for* $k = 1, \ldots, n-1$. *and the stabilizer of a point* $Z \in U_k$ *has dimension* $n(n-k) - 1$.

Proof. The proof of transitivity is similar to Lemma 6.4. Let $Z = (z_1, \ldots, z_k) \in U_k$ and extend to an *oriented* basis of $\mathbb{R}^n$, $\hat{Z} = (z_1, \ldots, z_k, \ldots, z_n)$ such that $\det\big(\hat{Z}\big) = 1$. Then there is an element $A^{-1} \in \mathrm{SL}(n)$ such that $A^{-1}e_i = z_i$ and $\mu^k_{\mathrm{SL}(n,\mathbb{R})}(A, Z) = (e_1, \ldots, e_k)$.

Now let A be an element of the stabilizer for $(e_1, \ldots, e_k)$. As in the previous case A takes the form given in Equation (6.8), but now the block C is an element of $\mathrm{SL}(n-k, \mathbb{R})$, so the dimension of the stabilizer is $n(n-k) + (n-k)^2 - 1 = n(n-k) - 1$ as claimed. □

When $k = n$ the action $\mu^n_{\mathrm{SL}(n,\mathbb{R})}$ is not transitive, and every element of U_n has a canonical form given in the next lemma.

Lemma 6.7. *If* $(z_1, \ldots, z_n) \in U_n$ *then there is a matrix* $A \in \mathrm{SL}(n)$ *such that*

$$\mu^n_{\mathrm{SL}(n,\mathbb{R})}\big(A, (z_1, \ldots, z_n)\big) = (e_1, \ldots, e_{n-1}, \lambda e_n) \tag{6.10}$$

where $\lambda = \det(z_1, \ldots, z_n)$.

Proof. From Lemma 6.4 there exists $\tilde{A} \in \mathrm{GL}(n)$ such that

$$\mu^n_{\mathrm{GL}(n,\mathbb{R})}\big(\tilde{A}, (z_1, \ldots, z_n)\big) = (e_1, \ldots, e_n).$$

Taking the determinant of $\big(\tilde{A}z_1, \ldots, \tilde{A}z_n\big)$ gives the equation $\det\big(\tilde{A}\big)\det(z_1, \ldots, z_n) = 1$. Let

$$\lambda = \det\big(\tilde{A}\big)^{-1} = \det(z_1, \ldots, z_n),$$

and define A by scaling the last column of $\tilde{A}$ by λ so that $A \in \mathrm{SL}(n)$ and

$$\mu^n_{\mathrm{SL}(n,\mathbb{R})}\big(A, (z_1, \ldots, z_n)\big) = (e_1, \ldots, \lambda e_n).$$

□

Corollary 6.4. *The action* $\mu^n_{\mathrm{SL}(n,\mathbb{R})}$ *is free on* U_n.

The proof follows from computing the isotropy subgroup of $(e_1, \dots, \lambda e_n)$ which is the identity.

Theorem 6.4. *The function* $\det : U_n \to \mathbb{R}$ *is a complete invariant of the* $\mathrm{SL}(n)$ *action on* $(\mathbb{R}^n)^n$.

Theorem 6.4 gives the immediate corollary by using Theorem 6.1 in the special affine case.

Corollary 6.5. *The function* $\det \circ T : T^{-1}(U_n) \to \mathbb{R}$

$$\det \circ T(x_0, x_1, \dots, x_n) = \det \left(x_1 - x_0, \dots, x_n - x_0 \right) \tag{6.11}$$

is a complete local $n+1$ *point joint invariant of the action of* $\mathrm{SA}(n)$ *on* $\mathbb{R}^n$.

Proof of Theorem 6.4. Since $\det(A) = 1$ for all $A \in \mathrm{SL}(n)$ then

$$\det(Az_1, \dots, Az_n) = \det(A) \det(z_1, \dots, z_n)$$

gives that det is an invariant. Further since any two points $V, W \in U_n$ can be taken to the representative given in Lemma 6.7 then they are on the same orbit if and only if $\det(V) = \det(W)$. □

CHAPTER 7

Joint Invariants of Primitive Homogeneous Spaces with Low Dimensional Lie Algebra-Subalgebra Pairs of Symmetric Type

In this chapter we consider two examples of primitive homogeneous spaces of symmetric type, $\mathrm{SU}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{R})$. We will apply the reduction theory developed in Lemma 3.5 and Theorem 3.2 to these examples. These results show that the k point joint invariants for the symmetric action μ_{sym} from Definition 3.1 when L is taken to be $\mathrm{SU}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{R})$ are determined by the $k-1$ point joint invariants of the conjugation action μ_{conj} as defined in Equation 3.21. The chapter is split into two sections. Section 7.1 constructs the two and three point joint invariants of $\mathrm{SU}(2,\mathbb{R})$ and presents an additional technique for determining the joint invariants of the intransitive action μ_{conj} on the set of point in general position. Section 7.2 constructs a complete set of two point joint invariants of μ_{sym} given by the group $\mathrm{SL}(2,\mathbb{R})$.

7.1 $\mathrm{SU}(2,\mathbb{R})$ Two and Three Point Example

We consider the group $\mathrm{SU}(2,\mathbb{R})$,

$$\mathrm{SU}(2,\mathbb{R}) = \left\{ \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} \middle| \, z, w \in \mathbb{C} \, |z|^2 + |w|^2 = 1 \right\},$$

which is a compact simple Lie group of dimension three.

Let $L = \mathrm{SU}(2,\mathbb{R})$ and $G = \mathrm{SU}(2,\mathbb{R}) \times \mathrm{SU}(2,\mathbb{R})$ act on $\mathrm{SU}(2,\mathbb{R})$ by the symmetric action μ_{sym} given in Definition 3.1. Lemma 3.5 identifies $\mathrm{SU}(2,\mathbb{R})$ as the homogeneous space $\mathrm{SU}(2,\mathbb{R})^2/\mathrm{SU}(2,\mathbb{R})^2_{\text{diag}}$, and Theorem 4.4 shows that the action μ_{sym} of $\mathrm{SU}(2,\mathbb{R})^2$ on $\mathrm{SU}(2,\mathbb{R})$ is primitive. However, the pair $\left(\mathrm{SU}(2,\mathbb{R})^2, \mathrm{SU}(2,\mathbb{R})^2_{\text{diag}}\right)$ is not a primitive pair since $\mathrm{SU}(2,\mathbb{R})^2_{\text{diag}}$ contains the normal subgroup $(\pm \mathrm{I}, \pm \mathrm{I})$ failing part 2) of Definition 4.8. Note that the Lie group $\mathrm{SU}(2,\mathbb{R})$ is the universal covering group of $\mathrm{SO}(3,\mathbb{R})$ and that $\mathrm{SO}(3,\mathbb{R})$ is a Lie group which contains no normal subgroups, so by Theorem 4.2 the subgroup $\mathrm{SO}(3,\mathbb{R})^2_{\mathrm{diag}}$ is a primitive subgroup of $\mathrm{SO}(3,\mathbb{R})^2$ and $(\mathrm{SO}(3,\mathbb{R})^2, \mathrm{SO}(3,\mathbb{R})^2_{\mathrm{diag}})$ is a primitive pair.

Subsection 7.1.1 provides the proofs of the results for the two point case which are summarized below. By Corollary 3.3 the two point joint invariants of μ_{sym} are identified with the class functions on $\mathrm{SU}(2,\mathbb{R})$ and the trace function is a complete invariant of conjugation as stated in the next theorem which will be proved in.

Theorem 7.1. *The map* $\frac{1}{2}\operatorname{tr} : \mathrm{SU}(2,\mathbb{R}) \to [-1,1]$ *is a surjective function and a complete global invariant of* $\mathrm{SU}(2,\mathbb{R})$ *acting on itself by conjugation.*

Corollary 7.1. *The function* $\frac{1}{2}\operatorname{tr}\circ T : \mathrm{SU}(2,\mathbb{R}) \times \mathrm{SU}(2,\mathbb{R}) \to [-1,1]$ *given by*

$$\frac{1}{2}\operatorname{tr}(X_0, X_1) \circ T = \frac{1}{2}\operatorname{tr}(X_0^{-1} X_1)$$

is a a complete global two point joint invariant of μ_{sym}.

Note that this invariant is a scalar multiple of the Frobenius inner product of matrices since $X^{-1} = X^*$ for matrices in $\mathrm{SU}(2,\mathbb{R})$.

There are two fixed points, $\pm\mathrm{I}$, of the action μ_{conj}. Lemma 7.4 below shows that the stabilizer for any other point in $\mathrm{SU}(2,\mathbb{R})$ is conjugate to $\mathrm{SO}(2,\mathbb{R})$ and is proved in.

The points in general position for μ_{conj} are given below.

Lemma 7.1. *The subset*

$$\mathrm{SU}(2,\mathbb{R})^* = \mathrm{SU}(2,\mathbb{R}) \setminus \left\{ \pm\,\mathrm{I} \right\}$$

of $\mathrm{SU}(2,\mathbb{R})$ *are the points in general position for the action* μ_{conj}.

We now summarize the results of the three point case which are proved in Subsection 7.1.2. The three point joint invariants of μ_{sym} are determined by the two point joint invariants of μ_{conj} from Theorem 3.2.

The points in general position for the two point diagonal action μ^2_{conj} are given in the next lemma.

Lemma 7.2. *The set of points in general position for the action* μ^2_{conj} *is the subset of* $\mathrm{SU}(2,\mathbb{R}) \times \mathrm{SU}(2,\mathbb{R})$ *which are not simultaneously diagonalizable.*

Moreover the action μ^2_{conj} is infinitesimally free on this subset, the stabilizer of every point is given by the global isotropy subgroup ±I.

We use a canonical form for the orbits to determine a complete set of invariants.

Theorem 7.2. *The functions*

$$f_1(Z_1, Z_2) = \frac{1}{2}\operatorname{tr}\big(Z_1\big), \quad f_2(Z_1, Z_2) = \frac{1}{2}\operatorname{tr}\big(Z_2\big), \quad \textit{and} \quad f_3(Z_1, Z_2) = \frac{1}{2}\operatorname{tr}\big(Z_1^* Z_2\big)$$

are a complete set of global two point joint invariants for the diagonal action of $\mathrm{SU}(2,\mathbb{R})$ *by conjugation.*

The corresponding joint invariants of μ^3_{sym} are given in the following corollary.

Corollary 7.2. *Let* $\hat{T} : \mathrm{SU}(2,\mathbb{R})^3 \to \mathrm{SU}(2,\mathbb{R})^2$ *be given by*

$$\hat{T}(X_0, X_1, X_2) = (X_0^* X_1, X_0^* X_2)$$

then the functions

$$\begin{aligned}
F_1 &= f_1 \circ \hat{T}(X_0, X_1, X_2) = \frac{1}{2}\operatorname{tr}\big(X_0^* X_1\big),\\
F_2 &= f_2 \circ \hat{T}(X_0, X_1, X_2) = \frac{1}{2}\operatorname{tr}\big(X_0^* X_2\big),\\
F_3 &= f_3 \circ \hat{T}(X_0, X_1, X_2) = \frac{1}{2}\operatorname{tr}\big(X_1^* X_2\big),
\end{aligned}$$

form a complete set of three point global invariants for the action μ_{sym}.

Section 7.1.3 shows that $\mathrm{SU}(2,\mathbb{R})^*$ is equivariantly bijective with the set $N \times \mathrm{SU}(2,\mathbb{R})/H$ where N are the diagonal matrices of $\mathrm{SU}(2,\mathbb{R})^*$ and H is the common isotropy subgroup for each $D \in N$. and uses this equivariant diffeomorphism to construct the joint invariants from a different perspective, See Theorem 7.6 and Lemma 7.8.

7.1.1 Proof of Results for Two Point Case

The orbits of μ_{conj} are the conjugacy classes of points in $\mathrm{SU}(2,\mathbb{R})$. The following theorem states that any matrix in $\mathrm{SU}(2,\mathbb{R})$ is conjugate to a diagonal matrix by an element of the unitary group $\mathrm{U}(2,\mathbb{R})$.

Theorem 7.3. *Let Z be a complex $n \times n$ matrix. Then Z is normal if and only if Z is unitarily equivalent to a diagonal matrix.*

This is a standard fact from linear algebra, see Friedberg [9] for proof.

The following lemma gives an explicit description for a representative of every conjugacy class in $\mathrm{SU}(2,\mathbb{R})$.

Lemma 7.3. *If $Z \in \mathrm{SU}(2,\mathbb{R})$ then there exists $\theta \in [0,\pi]$ such that Z is conjugate to the diagonal matrix*

$$D_\theta = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}, \tag{7.1}$$

by an element of $\mathrm{SU}(2,\mathbb{R})$.

Proof. Fix $Z \in \mathrm{SU}(2,\mathbb{R})$. Since $Z \in \mathrm{U}(2,\mathbb{R})$, the Hermitian conjugate, $Z^* = \overline{Z^t}$, is the inverse of Z, and Z is normal. So by Theorem 7.3 there exists a unitary matrix $P \in \mathrm{U}(2,\mathbb{R})$ such that $P^*ZP = D$ for some diagonal matrix $D \in \mathrm{SU}(2,\mathbb{R})$. Since $P \in \mathrm{U}(2)$ then $\det(P)$ is a unitary complex number and therefore $\sqrt{\det(P)}$ is also a unitary complex number. So let $Q = \frac{1}{\sqrt{\det(P)}}P$ where $QQ^* = Q^*Q = \mathrm{I}$ and

$$\mu_{\text{conj}}(Q^*, Z) = Q^*ZQ = P^*ZP = D,$$

proving that Z is conjugate to D by an element of $\mathrm{SU}(2,\mathbb{R})$.

Now let

$$Z = \begin{bmatrix} z & w \\ -\bar{w} & \bar{z}, \end{bmatrix}$$

where the characteristic polynomial of Z is

$$P_Z(t) = t^2 - (z + \bar{z})t + 1,$$

so since the coefficients are real the eigenvalues must be real or complex conjugates. Suppose that the eigenvalues are real and denote them by λ_1, λ_2. Then since the constant term of $P_Z(t)$ is 1 the eigenvalues satisfy $\lambda_1 \lambda_2 = 1$ and $\lambda_1, \lambda_2 \neq 0$ so $\lambda_2 = 1/\lambda_1$.

From the linear term they must sum to $z + \bar{z}$ and from $|z|^2 + |w|^2 = 1$ we have the upper bound $|z| \leq 1$ and hence $|z + \bar{z}| \leq 2$. Then we have

$$\left| \lambda_1 + \frac{1}{\lambda_1} \right| \leq 2$$

so rearranging and factoring gives $(|\lambda_1| - 1)^2 \leq 0$ which is only possible when $|\lambda_1| = 1$ so $\lambda_1 = \pm 1$ are the only possible real roots.

Conversely, if the eigenvalues are complex conjugates, their product must be 1 from the constant term and hence are unitary.

Let the eigenvalues be given in polar form, $e^{i\theta}$ and $e^{-i\theta}$, for some $\theta \in [0, \pi]$. The diagonal representative of Z, $D = Q^* Z Q$, is one of the two forms, $D_\theta = \operatorname{diag}\left(e^{i\theta}, e^{-i\theta}\right)$ or $D_{-\theta} = \operatorname{diag}\left(e^{-i\theta}, e^{i\theta}\right)$. But the matrix

$$S = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

is in $\mathrm{SU}(2, \mathbb{R})$ and satisfies the equation, $S D_\theta S^* = D_{-\theta}$, and so D_θ and $D_{-\theta}$ are conjugate in $\mathrm{SU}(2, \mathbb{R})$. Hence, we can always take D_θ to be the diagonal representative of Z which completes the proof. □

The diagonal representatives D_θ for $\mathrm{SU}(2,\mathbb{R})$ conjugacy classes allow us to prove Theorem 7.1.

Proof of Theorem 7.1. First we show that $f = \frac{1}{2}\,\mathrm{tr}$ is a surjective map. Let $r \in [-1,1]$. Then let $\theta \in [0,\pi]$ such that $\cos(\theta) = r$. The matrix $D_\theta = \mathrm{diag}\left(e^{i\theta}, e^{-i\theta}\right)$, satisfies

$$\mathrm{tr}(D_\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \cos(\theta) = r,$$

and $f(D_\theta) = r$ verifying f is onto.

The function f is an invariant since the trace function is a class function, so we now show that f is a complete invariant. Let $Z, W \in \mathrm{SU}(2,\mathbb{R})$ satisfy,

$$\frac{1}{2}\,\mathrm{tr}(Z) = \frac{1}{2}\,\mathrm{tr}(W). \tag{7.2}$$

Let $D_\theta = \mathrm{diag}\left(e^{i\theta}, e^{-i\theta}\right)$ and $D_\phi = \mathrm{diag}\left(e^{i\phi}, e^{-i\phi}\right)$ for $\theta, \phi \in [0,\pi]$ be the diagonal representatives of Z, W respectively.

Then $f(Z) = \frac{1}{2}\,\mathrm{tr}\left(D_\theta\right) = \cos(\theta)$ and $f(W) = \frac{1}{2}\,\mathrm{tr}\left(D_\phi\right) = \cos(\phi)$ where $\theta, \phi \in [0,\pi]$, and from Equation (7.2) $\theta = \phi$, as cosine is injective on the interval $[0,\pi]$. So $D_\theta = D_\phi$ and Z is conjugate to W which completes the proof. □

Remark 7.1. From Theorem 7.1 the complete invariant $\frac{1}{2}\,\mathrm{tr}$ is surjective, and so the orbit space $\mathrm{SU}(2,\mathbb{R})/\mu_{\mathrm{conj}}$ is canonically bijective with $[-1,1]$. In Appendix B we show that this bijection is a homeomorphism with respect to the quotient topology on $\mathrm{SU}(2,\mathbb{R})/\mu_{\mathrm{conj}}$ and the subspace topology of $[-1,1]$.

Now we will determine the points in general position for μ_{conj}. First we compute the isotropy subgroup for the elements of $\mathrm{SU}(2,\mathbb{R})$ which are not the fixed points $\pm\mathrm{I}$.

Lemma 7.4. *The isotropy of every point in* $\mathrm{SU}(2,\mathbb{R}) \setminus \{\pm\mathrm{I}\}$ *is conjugate to the subgroup* $\mathrm{SO}(2,\mathbb{R})$.

Proof. Let $Z \in \mathrm{SU}(2,\mathbb{R}) \setminus \{\pm\mathrm{I}\}$. Then Z is conjugate to a diagonal representative D_θ as given in Equation (7.1), where $\theta \in (0,\pi)$ since $Z \neq \pm\mathrm{I}$.

The stabilizer of D_θ is all elements of $\mathrm{SU}(2,\mathbb{R})$ which satisfy the equation,

$$\begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}.$$

That is $\beta e^{-i\theta} = e^{i\theta}\beta$ where $\theta \neq 0, \pi$ gives $\beta = 0$ and $|\alpha| = 1$. So the stabilizer of D_θ is

$$\mathrm{SU}(2,\mathbb{R})_{D_\theta} = \left\{ \mathrm{diag}\,(\alpha, \bar{\alpha}) \;\middle|\; \alpha \in \mathbb{C},\, |\alpha| = 1 \right\}.$$

This is conjugate to $\mathrm{SO}(2,\mathbb{R})$. Suppose that $\mathrm{diag}\,(\alpha,\bar{\alpha}) = \mathrm{diag}\left(e^{it}, e^{-it}\right)$ for some $t \in [-\pi, \pi]$. Consider $S \in \mathrm{SU}(2,\mathbb{R})$ given by,

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} i & -1 \\ 1 & -i \end{bmatrix}.$$

By conjugating $\mathrm{diag}\,(\alpha,\bar{\alpha})$ by S we have,

$$S\,\mathrm{diag}\left(e^{it}, e^{-it}\right) S^* = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}.$$

So $S\,\mathrm{SU}(2,\mathbb{R})_{D_\theta} S^* \subset \mathrm{SO}(2,\mathbb{R})$. On the other hand

$$S^* \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} S = \begin{bmatrix} \cos(t) + i\sin(t) & 0 \\ 0 & \cos(t) - i\sin(t) \end{bmatrix}$$

so $\mathrm{SO}(2,\mathbb{R}) \subset S\,\mathrm{SU}(2,\mathbb{R})_{D_\theta} S^*$ completing the proof. □

Proof of Lemma 7.1. The stabilizer of any point other than ±I has dimension one, while the points ±I are fixed points and so have stabilizers of dimension three. □

7.1.2 Proof of Results for Three Point Case

This section proves Theorem 7.2 and Corollary 7.2. We first identify a canonical form for the orbits of this action.

Lemma 7.5. *Any point* $(Z_1, Z_2) \in \mathrm{SU}(2,\mathbb{R}) \times \mathrm{SU}(2,\mathbb{R})$ *is conjugate to*

$$(D_\theta, R) = \left(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}, \begin{bmatrix} z & r \\ -r & \bar{z} \end{bmatrix} \right)$$

for unique $\theta \in [0, \pi]$, $z \in \mathbb{C}$, *and* $r \in [0, 1]$ *such that* $|z|^2 + r^2 = 1$.

Proof. From Lemma 7.3, Z_1 is conjugate to D_{θ_1} as in Equation (7.1) by some element $\Lambda_1 \in \mathrm{SU}(2,\mathbb{R})$. Consider $\mu^2_{\mathrm{conj}}\left(\Lambda_1^*, (Z_1, Z_2)\right) = \left(D_{\theta_1}, \Lambda_1^* Z_2 \Lambda_1\right)$.

Let $Z_2' = A^* Z_2 A$ be given by the matrix

$$Z_2' = \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} \tag{7.3}$$

for some unique $z, w \in \mathbb{C}$ with $|z|^2 + |w|^2 = 1$.

First suppose that $w = 0$ so that $|z| = 1$. That is (Z_1, Z_2) is simultaneously diagonalizable by the matrix Λ_1. This is in the form required with $r = 0$.

Now suppose that $w \neq 0$. then $w = re^{i\phi}$ for a well defined $\phi \in [-\pi, \pi]$ and $0 < r \leq 1$. Let $\zeta = -\phi/2$. From Lemma 7.4 the matrix

$$A = \mathrm{diag}\left(e^{i\zeta}, e^{-i\zeta}\right) \tag{7.4}$$

is in the stabilizer of D_θ, and

$$AZ_2'A^* = \begin{bmatrix} z & e^{2i\zeta} w \\ -\overline{(e^{2i\zeta} w)} & \bar{z} \end{bmatrix} = \begin{bmatrix} z & r \\ -r & \bar{z} \end{bmatrix} = R$$

where $|z|^2 + r^2 = 1$ since $AZ_2'A^* \in \mathrm{SU}(2,\mathbb{R})$. Then $\mu^2_{\mathrm{conj}}(A\Lambda_1^*, (Z_1, Z_2)) = (D_\theta, R)$ as desired. □

Proof of Lemma 7.2. Consider $(Z_1, Z_2) \in \mathrm{SU}(2,\mathbb{R}) \times \mathrm{SU}(2,\mathbb{R})$. The points in general position have minimal stabilizer dimension of both Z_1 with respect to μ_{conj} and (Z_1, Z_2) with respect to μ^2_{conj}. So $(Z_1, Z_2) \subset \mathrm{SU}(2,\mathbb{R})^* \times \mathrm{SU}(2,\mathbb{R})$.

Now suppose that $(Z_1, Z_2) \in \mathrm{SU}(2,\mathbb{R})^* \times \mathrm{SU}(2,\mathbb{R})$. If (Z_1, Z_2) is not simultaneously diagonalizable then from Lemma 7.5 there is a unique representative (D_θ, R) with $r > 0$ for the orbit of (Z_1, Z_2). The stabilizer of this representative, $\mathrm{SU}(2,\mathbb{R})_{(D_\theta,R)}$ must be an element of $\mathrm{SU}(2,\mathbb{R})_{D_\theta} \cap \mathrm{SU}(2,\mathbb{R})_R$, that is an element of the form $\operatorname{diag}(\alpha, \bar{\alpha})$ with $|\alpha| = 1$ which solves the equation

$$\begin{bmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{bmatrix} \begin{bmatrix} z & r \\ -r & \bar{z} \end{bmatrix} = \begin{bmatrix} z & r \\ -r & \bar{z} \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{bmatrix}.$$

Hence, $(\alpha - \bar{\alpha}) = 0$ since we're assuming $r > 0$, and α is real and from $\alpha\bar{\alpha} = 1$, so $\alpha = \pm \mathrm{I}$. That is in the case where (Z_1, Z_2) is not simultaneously diagonalizable the isotropy has dimension 0, and is given by the global isotropy subgroup for μ^2_{conj}.

If (Z_1, Z_2) is simultaneously diagonalizable there is a pair of diagonal matrices $(D_{\theta_1}, D_{\theta_2})$ in the same orbit as (Z_1, Z_2) which have the same stabilizer of dimension 1 and are not in general position. □

Proof of Theorem 7.2. We first check that f_3 is an invariant, the others are similar. Let $A \in \mathrm{SU}(2,\mathbb{R})$ and $(Z_1, Z_2) \in \mathrm{SU}(2,\mathbb{R}) \times \mathrm{SU}(2,\mathbb{R})$ and consider,

$$f_3\left(AZ_1A^*, AZ_2A^*\right) = \frac{1}{2}\operatorname{tr}(A^* Z_1^* A A^* Z_2 A) = \frac{1}{2}\operatorname{tr}(A^* Z_1^* Z_2 A) = \frac{1}{2}\operatorname{tr}(Z_1^* Z_2),$$

which verifies f_3 is an invariant.

Now we will show that if $(X_1, X_2), (Y_1, Y_2) \in \mathrm{SU}(2,\mathbb{R}) \times \mathrm{SU}(2,\mathbb{R})$ satisfy $f_i(X_1, X_2) = f_i(Y_1, Y_2)$ then they are in the same orbit.

By Lemma 7.5 the points (X_1, X_2) and (Y_1, Y_2) have canonical forms,

$$(D_{\theta_1}, Z_1) = \left(\begin{bmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{bmatrix}, \begin{bmatrix} z_1 & r_1 \\ -r_1 & \overline{z_1} \end{bmatrix} \right)$$
$$(D_{\theta_2}, Z_2) = \left(\begin{bmatrix} e^{i\theta_2} & 0 \\ 0 & e^{-i\theta_2} \end{bmatrix}, \begin{bmatrix} z_2 & r_2 \\ -r_2 & \overline{z_2} \end{bmatrix} \right)$$

For $r_1, r_2 \in [0,1]$ and $\theta_1, \theta_2 \in [0, \pi]$.

First consider the case that $\theta = 0, \pi$ then $D_{\theta_1} = D_{\theta_2} = \pm \mathrm{I}$ which is a fixed point under conjugation so $X_1 = Y_1 = \pm \mathrm{I}$. Hence $(X_1, X_2) = (\pm \mathrm{I}, X)$ and $(Y_1, Y_2) = (\pm \mathrm{I}, Y)$ and therefore since $\mathrm{tr}(X) = \mathrm{tr}(Y)$ and the trace function is a complete invariant of conjugation then (X_1, X_2) is on the same orbit as (Y_1, Y_2).

If $\theta_1, \theta_2 \in (0, \pi)$ then since f_i are invariant functions they give equations for these representatives,

$$\begin{aligned} \mathrm{tr}\left(D_{\theta_1}^* Z_1\right) &= \mathrm{tr}\left(D_{\theta_2}^* Z_2\right) \\ \mathrm{tr}\left(D_{\theta_1}\right) &= \mathrm{tr}\left(D_{\theta_2}\right) \\ \mathrm{tr}\left(Z_1\right) &= \mathrm{tr}\left(Z_2\right). \end{aligned} \tag{7.5}$$

Let $z_1 = a_1 + ib_1 = R_1 e^{i\phi_1}$ and $z_2 = a_2 + ib_2 = R_2 e^{i\phi_2}$ be the corresponding Cartesian and polar forms for the complex numbers z_1, z_2. Then the system (7.5) becomes

$$\begin{aligned} e^{i\theta_1} z_1 + e^{-i\theta_1} \overline{z_1} &= e^{i\theta_2} z_2 + e^{-i\theta_2} \overline{z_2} \\ \cos(\theta_1) &= \cos(\theta_2) \\ \mathfrak{Re}(z_1) &= \mathfrak{Re}(z_2). \end{aligned}$$

As in the two point case we have $\theta_1 = \theta_2 = \theta$. and now also we have $a_1 = a_2 = a$, so substituting this in gives

$$e^{i\theta}(a + ib_1) + e^{-i\theta}(a - ib_1) = e^{i\theta}(a + ib_2) + e^{-i\theta}(a - ib_2)$$
$$i(e^{i\theta} - e^{-i\theta})b_1 = i(e^{i\theta} - e^{-i\theta})b_2$$
$$\sin(\theta)b_1 = \sin(\theta)b_2.$$

So since this is the case where $\theta \neq 0, \pi$, then $b_1 = b_2$ and $z_1 = z_2$. Then since $r_1 = 1 - |z_1|^2$ and $r_2 = 1 - |z_2|^2$ then $r_1 = r_2$ as well and these representatives are equal, verifying that (X_1, X_2) is in the same orbit as (Y_1, Y_2) and completing the proof. □

7.1.3 Slice Method

This section of the chapter provides an additional technique for determining the k point joint invariants of μ_{conj} when $L = \text{SU}(2, \mathbb{R})$.

To determine the joint invariants for the intransitive action μ_{conj} we introduce the idea of a slice.

Definition 7.1. Let $\mu : G \times X \to X$ be a group action. A strong slice is a subset $N \subset X$ which satisfies the following conditions.

i) If $x \in X$ then there exists a unique point $n \in N$ such that $N \cap [x] = \{n\}$.

ii) There is a subgroup H of G such that for all $n \in N$ the stabilizer $G_n = H$.

In the example of μ_{conj} where $G = \text{SU}(2, \mathbb{R})$ and $X = \text{SU}(2, \mathbb{R})^* = \text{SU}(2, \mathbb{R}) \setminus \{\pm \text{I}\}$ the set of diagonal matrices in $\text{SU}(2, \mathbb{R})^*$ are a strong slice.

Under the assumption that a strong slice exists for the action μ on a set X we can define a map $\Psi : N \times G/H \to X$ by

$$\Psi(n, [a]_H) = an$$

which is an equivariant bijection. The action on $N \times G/H$ is $\mu_{\text{slice}} : G \times (N \times G/H) \to (N \times G/H)$

$$\mu_{\text{slice}}(a, (n, [x]_H)) = (n, [ax]_H).$$

The equivariant bijection Ψ then identifies the set of μ invariants with the set of μ_{slice} invariants. Hence, the k point μ invariants are identified with the k point μ_{slice} invariants.

The k point μ_{slice} invariants fall into two types.

i) The invariants which arise from projecting onto the slice N, which are the maps $\pi_1^k : (N \times G/H)^k \to N^k$ given by

$$\pi_1^k\left((n_1, [a_0]_H, \ldots, (n_k, [a_k]_H))\right) = (n_1, \ldots, n_k).$$

ii) Invariants of the action of G on G/H by left multiplication from the equivariant map $\pi_2^k : (N \times G/H)^k \to (G/H)^k$

$$\pi_2^k\left((n_1, [x_1]_H), \ldots, (n_k, [x_k]_H)\right) = ([x_1]_H), \ldots, [x_k]_H).$$

The k point joint invariants of μ_{slice} which arise from π_2^k are of primary interest since they are exactly the k point joint invariants of G on G/H by left multiplication, and the reduction theory from Chapter 3 shows that these are equivalent to the $k-1$ point joint invariants of the subgroup H.

The method is motivated by the theory of slices, for more information about the general case see Bredon [5]

Let

$$H = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{bmatrix} \middle| \; \alpha \in \mathbb{C}, \, |\alpha| = 1 \right\}$$

be the isotropy subgroup of any diagonal matrix D_θ from Lemma 7.3.

Consider the set $N = \{D_\theta \,|\, \theta \in (0,\pi)\}$. The map $\mu_{\text{slice}} : \mathrm{SU}(2,\mathbb{R}) \times (N \times \mathrm{SU}(2,\mathbb{R})/H) \to (N \times \mathrm{SU}(2,\mathbb{R})/H)$ given by

$$\mu_{\text{slice}}\left(A, \left(D_\theta, [\Lambda]_H\right)\right) = \left(D_\theta, [A\Lambda]_H\right) \tag{7.6}$$

is an action of $\mathrm{SU}(2,\mathbb{R})$ on $N \times \mathrm{SU}(2,\mathbb{R})/H$.

We will equivariantly identify the space $\mathrm{SU}(2,\mathbb{R})^*$ of points in general position for μ_{conj} with $N \times \mathrm{SU}(2,\mathbb{R})/H$.

Theorem 7.4. *Consider the set $N \times \mathrm{SU}(2,\mathbb{R})/H$. Let μ_{slice} be the action defined in Equation (7.6). The map $\Psi : (N \times \mathrm{SU}(2,\mathbb{R})/H) \to \mathrm{SU}(2,\mathbb{R})^*$ defined by*

$$\Psi\left(D_\theta, [\Lambda]_H\right) = \Lambda D_\theta \Lambda^*$$

is an equivariant bijection with respect to the actions μ_{conj} and μ_{slice}.

Proof. Consider the map $\psi : N \times \mathrm{SU}(2,\mathbb{R}) \to \mathrm{SU}(2,\mathbb{R})^*$ defined by

$$\psi\left(D_\theta, \Lambda\right) = \Lambda D_\theta \Lambda^*.$$

Let $\mu_H : H \times (N \times \mathrm{SU}(2,\mathbb{R})) \to (N \times \mathrm{SU}(2,\mathbb{R}))$ be given by

$$\mu_H\left(A, (D_\theta, \Lambda)\right) = (D_\theta, \Lambda A)$$

where $(D_\theta, \Lambda) \in N \times \mathrm{SU}(2,\mathbb{R})$. The orbit space of μ_H is the set $N \times \mathrm{SU}(2,\mathbb{R})/H$, and we will show that ψ is a complete surjective invariant for μ_H, so that by Corollary 2.1 ψ factors through the quotient by a unique bijection Ψ.

The map ψ is surjective since every element in $\mathrm{SU}(2,\mathbb{R})^*$ is conjugate to a representative D_θ via μ_{conj}. Now suppose $(D_\theta, \Lambda), (D_\phi, \Omega) \in N \times \mathrm{SU}(2,\mathbb{R})$ such that $\psi(D_\theta, \Lambda) = \psi(D_\phi, \Omega)$.

Then

$$\Lambda D_\theta \Lambda^* = \Omega D_\phi \Omega^* \tag{7.7}$$

and $\mathrm{tr}(D_\theta) = \mathrm{tr}(D_\phi)$ which implies that $\theta = \phi$. Then conjugating both sides of Equation (7.7) by Ω^{-1} we have the equation

$$(\Omega^* \Lambda) D_\theta (\Omega^* \Lambda)^* = D_\theta.$$

Hence, $\Omega^* \Lambda \in H$ and there exists some $A \in H$ with $\Lambda = \Omega A$. Act on (D_ϕ, Ω) by A via μ_H to find

$$\mu_H \left(A\big(D_\phi, \Omega\big)\right) = (D_\theta, \Lambda)$$

and (D_θ, A) and (D_ϕ, B) are in the same orbit of μ_H. Now Corollary 2.1 verifies Ψ is a bijection of $N \times \mathrm{SU}(2,\mathbb{R})/H$ and $\mathrm{SU}(2,\mathbb{R})^*$.

Finally we verify equivariance of Ψ. Consider

$$\Psi(D_\theta, [A\Lambda]_H) = (A\Lambda)\, D_\theta (A\Lambda)^* = \mu_{\text{conj}}(A, \Lambda D_\theta \Lambda^*) = \mu_{\text{conj}} \left(A, \Psi\big(D_\theta, \Lambda\big)\right)$$

completing the proof. □

Note that the inverse of Ψ from Lemma 7.4, $\Psi^{-1} : \mathrm{SU}(2,\mathbb{R})^* \to (N \times \mathrm{SU}(2,\mathbb{R})/H)$ is given by

$$\Psi^{-1}(Z) = (D_\theta, [\Lambda]_H) \tag{7.8}$$

for the unique $\theta \in (0, \pi)$ and $[\Lambda]_H \in \mathrm{SU}(2,\mathbb{R})/H$ such that $Z = \Lambda D_\theta \Lambda^*$ for any representative Λ of $[\Lambda]_H$.

The equivariant bijection Ψ^{-1} Induces a unique bijection between the orbit spaces $\mathrm{SU}(2,\mathbb{R})^*/\mu_{\text{conj}}$ and $(N \times \mathrm{SU}(2,\mathbb{R})/H)/\mu_{\text{slice}}$ by Theorem 2.4. So by finding a complete set of invariants for μ_{slice} the map Ψ^{-1} induces a complete set of invariants for μ_{conj}.

Theorem 7.5. *Let* $\pi_1 : N \times \mathrm{SU}(2,\mathbb{R})/H \to N$ *be the projection onto the first factor* $\pi_1(D_\theta, [A]_H) = D_\theta$. *The map* π_1 *is a complete invariant of* μ_{slice} *on* $N \times \mathrm{SU}(2,\mathbb{R})/H$ *and* $\pi_1 \circ \Psi^{-1}$ *is a complete invariant of* μ_{conj} *on* $\mathrm{SU}(2,\mathbb{R})^*$.

Proof. The map π_1 is an invariant of the action μ_{slice} since μ_{slice} acts trivially on the first factor. Now we show that π_1 is complete. Suppose that $(D_\theta, [\Lambda]_H)$ and $(D_\phi, [\Omega]_H)$ are two elements of $N \times \mathrm{SU}(2,\mathbb{R})/H$ such that $\pi_1(D_\theta, [\Lambda]_H) = \pi_1(D_\phi, [\Omega]_H)$, that is, $\theta = \phi$. Then the diagonal matrices $D_\theta = D_\phi$ and the orbits $D_\theta \times \mathrm{SU}(2,\mathbb{R})/H = D_\phi \times \mathrm{SU}(2,\mathbb{R})/H$. Hence the invariant π_1 is complete.

Finally since Ψ^{-1} is an equivariant bijection the composition $\pi_1 \circ \Psi^{-1}$ is a complete invariant of μ_{conj}. □

Now consider μ^2_{conj} restricted to the invariant subset $\mathrm{SU}(2,\mathbb{R})^* \times \mathrm{SU}(2,\mathbb{R})^* \subset \mathrm{SU}(2,\mathbb{R}) \times \mathrm{SU}(2,\mathbb{R})$. Let $\Psi_2 : (N \times \mathrm{SU}(2,\mathbb{R})/H)^2 \to (\mathrm{SU}(2,\mathbb{R})^*)^2$ be

$$\Psi_2\Big(\big(D_{\theta_1}, [A_1]_H\big), \big(D_{\theta_2}, [A_2]_H\big)\Big) = \Big(\Psi\big(D_{\theta_1}, [A_1]_H\big), \Psi\big(D_{\theta_2}, [A_2]_H\big)\Big).$$

The map Ψ_2^{-1} is an equivariant bijection with respect to μ^2_{conj} and μ^2_{slice} respectively. The inverse Ψ_2^{-1} is given by

$$\Psi^{-1}(Z_1, Z_2) = \big(\Psi^{-1}(Z_1), \Psi^{-1}(Z_2)\big).$$

So the equivariant bijection Ψ_2^{-1} induces a unique bijection between the orbit spaces $(\mathrm{SU}(2,\mathbb{R})^*)^2/\mu^2_{\text{conj}}$ and $(N \times \mathrm{SU}(2,\mathbb{R})/H)^2/\mu^2_{\text{slice}}$ and a complete set of invariants for μ^2_{slice} determines a complete set of invariants for μ^2_{conj}.

As above the map $\pi_1^2 : (N \times \mathrm{SU}(2,\mathbb{R})/H)^2 \to N^2$ is an invariant function. However in this case it is not complete, so we consider the map $\pi_2^2 : (N \times \mathrm{SU}(2,\mathbb{R})/H)^2 \to$

$(\mathrm{SU}(2,\mathbb{R})/H)^2$ defined by

$$\pi_2^2\big((D_{\theta_1},[\Lambda_1]_H)(D_{\theta_2},[\Lambda_2]_H)\big) = \big([\Lambda_1]_H,[\Lambda_2]_H\big). \tag{7.9}$$

Which is equivariant with respect to the two point diagonal action of $\mathrm{SU}(2,\mathbb{R})$ on $\mathrm{SU}(2,\mathbb{R})/H$ by left multiplication and μ^2_{slice}. We will determine the two point joint invariants of μ_{slice} by constructing two point joint invariants for the action of $\mathrm{SU}(2,\mathbb{R})$ on $\mathrm{SU}(2,\mathbb{R})/H$.

Lemma 7.6. *Let $f : (\mathrm{SU}(2,\mathbb{R})/H)^2 \to Y$ be a Y valued two point joint invariant of* $\mathrm{SU}(2,\mathbb{R})$ *on* $\mathrm{SU}(2,\mathbb{R})/H$*. Then the function $f \circ \pi_2^2$ is an invariant of the action μ^2_{slice}.*

The proof is immediate from the equivariance of π_2^2 with respect to the two point diagonal actions.

Now we consider the invariants of $\mathrm{SU}(2,\mathbb{R})$ acting on $\mathrm{SU}(2,\mathbb{R})/H \times \mathrm{SU}(2,\mathbb{R})/H$. The invariants of this action can be determined by using Lemma 3.1 once a suitable map ρ satisfying the identity in Equation (3.3) is determined. In this case we let $\rho : \mathrm{SU}(2,\mathbb{R})/H \to H$ be given by

$$\rho([\Lambda]_H) = \Lambda^{-1}$$

where we chose a particular representative for each $[\Lambda]_H$ by the Axiom of Choice in order to assure ρ is well defined. Then let $T : \mathrm{SU}(2,\mathbb{R})/H \times \mathrm{SU}(2,\mathbb{R})/H \to \mathrm{SU}(2,\mathbb{R})/H$ given by

$$T([\Lambda_1]_H,[\Lambda_2]_H) = \Lambda_1^*\Lambda_2.$$

By Lemma 3.1 The invariants of $\mathrm{SU}(2,\mathbb{R})$ on $\mathrm{SU}(2,\mathbb{R})/H \times \mathrm{SU}(2,\mathbb{R})/H$ are in one to one correspondence with the invariants of H on $\mathrm{SU}(2,\mathbb{R})/H$ by precomposition with T.

Lemma 7.7. *Let $q : \mathrm{SU}(2,\mathbb{R})/H \to [-1,1]$ be given by*

$$q\left(\left[\begin{bmatrix} z & w \\ -\bar{w} & \bar{z}\end{bmatrix}\right]_H\right) = |z|^2 - |w|^2.$$

The map q is a complete invariant of the action of H on $\mathrm{SU}(2,\mathbb{R})/H$ by left multiplication.

Proof. First we show that q is a well defined invariant function. Consider $\hat{q} : \mathrm{SU}(2,\mathbb{R}) \to [-1,1]$ defined by

$$q\left(\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}\right) = |z|^2 - |w|^2.$$

We claim that $\hat{q}$ is an invariant of the action of H by right multiplication on $\mathrm{SU}(2,\mathbb{R})$ and so factors through the quotient by this action via the map q. Let $\mathrm{diag}(\alpha,\bar{\alpha}) \in H$ where $|\alpha| = 1$ and consider

$$q\left(\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{bmatrix}\right) = |z|^2 - |w|^2.$$

Now we show that p is a complete invariant function of the action by H via left multiplication on $\mathrm{SU}(2,\mathbb{R})/H$. Suppose that $[\Lambda_1]_H, [\Lambda_2]_H \in \mathrm{SU}(2,\mathbb{R})/H$ such that $p\left([\Lambda_1]_H\right) = p\left([\Lambda_2]_H\right)$. This equation is independent of the representatives chosen for the H cosets, so let

$$\Lambda_1 = \begin{bmatrix} z_1 & r_1 \\ -\bar{r}_1 & \bar{z}_1 \end{bmatrix}, \quad \text{and} \quad \Lambda_2 = \begin{bmatrix} z_2 & r_2 \\ -\bar{r}_2 & \bar{z}_2 \end{bmatrix}$$

where r_1, r_2 are non-negative real numbers, and

$$|z_1|^2 - r_1^2 = |z_2|^2 - r_2^2$$
$$|z_1|^2 - r_1^2 = |z_2|^2 - r_2^2,$$

which implies that $|z_1| = |z_2|$ and $r_1 = r_2$.

To show that $[\Lambda_1]_H$ and $[\Lambda_2]_H$ are in the same orbit of H acting on $\mathrm{SU}(2,\mathbb{R})/H$ by left multiplication. There are two cases, if $z_1 = 0$ then $z_2 = 0$ using $|z_1| = |z_2|$ so $\Lambda_1 = \Lambda_2$ using $r_1 = r_2$ so that $[\Lambda_1]_H = [\Lambda_2]_H$.

On the other hand if $z_1 \neq 0$ then $z_2 \neq 0$ as well since $|z_1| = |z_2|$. Now consider $\beta = z_2/z_1$ which satisfies $|\beta| = 1$ from $|z_1| = |z_2|$. Pick any $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$ and $\alpha^2 = z_1/z_2$ which exists since $|z_1| = |z_2|$.

Consider the elements $B = \operatorname{diag}\left(\frac{z_2}{z_1}, \frac{\bar{z}_2}{\bar{z}_1}\right) \in H$ and $A = \operatorname{diag}(\alpha, \bar{\alpha}) \in H$. Then

$$(BA)\Lambda_1\left(A^{-1}\right) = \begin{bmatrix} z_2/z_1 & 0 \\ 0 & \bar{z}_2/\bar{z}_1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{bmatrix}\begin{bmatrix} z_1 & r_1 \\ -r_1 & \bar{z}_1 \end{bmatrix}\begin{bmatrix} \bar{\alpha} & 0 \\ 0 & \alpha \end{bmatrix} = \begin{bmatrix} z_1 & r_2 \\ -r_2 & \bar{z}_2 \end{bmatrix} = \Lambda_2$$

which verifies that $BA[\Lambda_1]_H = [\Lambda_2]_H$ since $A^{-1} \in H$.

So if $q([\Lambda_1]_H) = q([\Lambda_2]_H)$ then $[\Lambda_1]_H$ and $[\Lambda_2]_H$ are in the same orbit of H acting by left multiplication and p is a complete invariant. □

Then by Lemma 7.6 The map $q \circ T \circ \pi_2^2$ is an invariant of the action μ^2_{slice} and we complete the analysis of this section by proving the following theorem

Theorem 7.6. *The maps π_1^2 and $q \circ T \circ \pi_2^2$ are a complete set of invariants for μ^2_{slice}.*

Proof. Suppose that

$$\pi_1^2\left(\left(\theta_1, [\Lambda_1]_H\right), \left(\theta_2, [\Lambda_2]_H\right), \right) = \pi_1^2\left(\left(\phi_1, [\Omega_1]_H\right), \left(\phi_2, [\Omega_2]_H\right)\right)$$
$$\left(\theta_1, \theta_2\right) = \left(\phi_1, \phi_2\right)$$

then the points $\left(\left(\theta_1, [\Lambda_1]_H\right), \left(\theta_2, [\Lambda_2]_H\right)\right)$ and $\left(\left(\phi_1, [\Omega_1]_H\right), \left(\phi_2, [\Omega_2]_H\right)\right)$. If

$$q \circ T \circ \pi_2^2\left(\left(\theta_1, [\Lambda_1]_H\right), \left(\theta_2, [\Lambda_2]_H\right)\right) = q \circ T \circ \pi_2^2\left(\left(\phi_1, [\Omega_1]_H\right), \left(\phi_2, [\Omega_2]_H\right)\right)$$

then there exists an $A \in H$ such that

$$\left(\left([A\Lambda_1]_H\right), \left([A\Lambda_2]_H\right)\right) = \left(\left([\Omega_1]_H\right), \left([\Omega_2]_H\right)\right)$$

and therefore

$$\mu^2_{\text{slice}}\left(A,\left((\theta_1,[\Lambda_1]_H),(\theta_2,[\Lambda_2]_H)\right)\right)=\left((\theta_1,[\Omega_1]_H),(\theta_2,[\Omega_2]_H)\right)=\left((\phi_1,[\Omega_1]_H),(\phi_2,[\Omega_2]_H)\right)$$

completing the proof. □

The two point joint invariants of μ_{conj} from Theorem 7.2 are functionally related to the μ_{conj} invariants determined by Theorem 7.6 by precomposition with the equivariant diffeomorphism Ψ^{-1}. We will show the relationship for the two point invariant $q\circ T\circ\pi_2^2\circ\Psi^{-1}$.

Lemma 7.8. *The function* $q\circ T\circ\pi_2^2\circ\Psi^{-1}:\mathrm{SU}(2,\mathbb{R})^*\times\mathrm{SU}(2,\mathbb{R})^*\to\mathbb{R}$ *is given by*

$$q(Z_1,Z_2)=\frac{\frac{1}{4}\left(2\operatorname{tr}(Z_1^*Z_2)-\operatorname{tr}(Z_1)\operatorname{tr}(Z_2)\right)}{\sqrt{1-\frac{1}{4}\operatorname{tr}(Z_1)^2}\sqrt{1-\frac{1}{4}\operatorname{tr}(Z_2)^2}}.$$

Proof. The proof is broken into a sequence of claims. The first claim is that for any $Z\in\mathrm{SU}(2,\mathbb{R})^*$ where

$$Z=\begin{bmatrix} z & w\\ -\bar{w} & \bar{z}\end{bmatrix}=\begin{bmatrix} x+iy & u+iv\\ -u+iv & x-iy\end{bmatrix}$$

there is a representative for the coset $[\Lambda]_H$ in the image $\Psi(Z)=(\theta,[\Lambda]_H)$, given by

$$\Lambda=\begin{bmatrix} \frac{-iu+v}{\sqrt{2u^2+2v^2+2y^2-2\sqrt{u^2+v^2+y^2}\,y}} & -\frac{\sqrt{u^2+v^2+y^2}-y}{\sqrt{2u^2+2v^2+2y^2-2\sqrt{u^2+v^2+y^2}\,y}}\\ \frac{\sqrt{u^2+v^2+y^2}-y}{\sqrt{2u^2+2v^2+2y^2-2\sqrt{u^2+v^2+y^2}\,y}} & \frac{iu+v}{\sqrt{2u^2+2v^2+2y^2-2\sqrt{u^2+v^2+y^2}\,y}}\end{bmatrix}.$$

This follows from computing

$$\Lambda^{-1}Z\Lambda=\begin{bmatrix} \frac{(iu^2+iv^2+iy^2-xy)\sqrt{u^2+v^2+y^2}-(u^2+v^2+y^2)(iy-x)}{u^2+v^2+y^2-\sqrt{u^2+v^2+y^2}\,y} & 0\\ 0 & \frac{(-iu^2-iv^2-iy^2-xy)\sqrt{u^2+v^2+y^2}+(x+iy)(u^2+v^2+y^2)}{u^2+v^2+y^2-\sqrt{u^2+v^2+y^2}\,y}\end{bmatrix}$$

where

$$\Lambda^{-1} = \begin{bmatrix} \frac{iu+v}{\sqrt{2u^2+2v^2+2y^2-2\sqrt{u^2+v^2+y^2}\,y}} & \frac{\sqrt{u^2+v^2+y^2}-y}{\sqrt{2u^2+2v^2+2y^2-2\sqrt{u^2+v^2+y^2}\,y}} \\ -\frac{\sqrt{u^2+v^2+y^2}-y}{\sqrt{2u^2+2v^2+2y^2-2\sqrt{u^2+v^2+y^2}\,y}} & \frac{-iu+v}{\sqrt{2u^2+2v^2+2y^2-2\sqrt{u^2+v^2+y^2}\,y}} \end{bmatrix} = \Lambda^*$$

so that $\Lambda \in \mathrm{SU}(2,\mathbb{R})$ and $\Lambda^{-1}X\Lambda$ is a diagonal matrix. This proves the first claim.

Now note that if we define a function $\eta : \mathrm{SU}(2,\mathbb{R}) \to \mathbb{C}$ by $\eta(Z) = 2zw$ and consider $q : \mathrm{SU}(2,\mathbb{R}) \to \mathbb{R}$ given by $q(Z) = |z|^2 - |w|^2$, then

$$\eta(\Lambda) = \frac{-i\,(-u+iv)}{\sqrt{u^2+v^2+y^2}}$$
$$r(\Lambda) = \frac{y}{\sqrt{u^2+v^2+y^2}}.$$

The second claim is that for any pair $\Lambda_1, \Lambda_2 \in \mathrm{SU}(2,\mathbb{R})$ given by

$$\Lambda_1 = \begin{bmatrix} z_1 & w_1 \\ -\bar{w}_1 & \bar{z}_1 \end{bmatrix} = \begin{bmatrix} x_1 + iy_1 & u_1 + iv_1 \\ -u_1 + iv_1 & x_1 - iy_1 \end{bmatrix}$$
$$\Lambda_2 = \begin{bmatrix} z_2 & w_2 \\ -\bar{w}_2 & \bar{z}_2 \end{bmatrix} = \begin{bmatrix} x_2 + iy_2 & u_2 + iv_2 \\ -u_2 + iv_2 & x_2 - iy_2 \end{bmatrix}$$

Then if we let

$$\eta_1 = 2z_1w_1 \quad r_1 = |z_1|^2 - |w_1|^2$$
$$\eta_2 = 2z_2w_2 \quad r_2 = |z_2|^2 - |w_2|^2$$

then the value of

$$q(\Lambda_1^*\Lambda_2) = r_1r_2 + \frac{1}{2}\left(\eta_1\bar{\eta}_2 + \bar{\eta}_1\eta_2\right).$$

This is a straightforward computation that we will leave out of the proof.

Now let $Z_1, Z_2 \in \mathrm{SU}(2,\mathbb{R})$ as denoted above. Using the first claim on Z_1, Z_2 gives representatives Λ_1 and Λ_2 such that $\Lambda_1^{-1} Z_1 \Lambda_1 = D_{\theta_1}$ and $\Lambda_2^{-1} Z_2 \Lambda_2 = D_{\theta_2}$. Then we can compute directly that

$$q(\Lambda_1^{-1}\Lambda_2) = \frac{u_1 u_2 + v_1 v_2 + y_1 y_2}{\sqrt{{u_1}^2 + {v_1}^2 + {y_1}^2}\sqrt{{u_2}^2 + {v_2}^2 + {y_2}^2}}.$$

Now we use that $\operatorname{tr}(Z_i) = 2x_i$, $x_i^2 + y_i^2 + u_i^2 + v_i^2 = 1$, and the identity,

$$\frac{1}{4}\left(\operatorname{tr}(Z_1^* Z_2) - \operatorname{tr}(Z_1)\operatorname{tr}(Z_2)\right) = u_1 u_2 + v_1 v_2 + y_1 y_2,$$

to conclude the proof. □

7.2 SL(2, ℝ) Two Point Example

This section constructs a complete set of two point joint invariants for $\mathrm{SL}(2,\mathbb{R})$. In contrast to the $\mathrm{SU}(2,\mathbb{R})$ example where all points excluding the fixed points $\pm I$ had conjugate isotropy subgroups, $\mathrm{SU}(2,\mathbb{R})$ is partitioned into three invariant subsets which have non conjugate isotropy subgroups.

Theorem 7.7. *Let* $\operatorname{tr} : \mathrm{SL}(2,\mathbb{R}) \to \mathbb{R}$ *be the trace function. Then* $\mathrm{SL}(2,\mathbb{R})$ *is partitioned into the invariant subsets*

$$\begin{aligned} Q_{>2} &= \left\{ Z \in \mathrm{SL}(2,\mathbb{R}) \,\middle|\, |\operatorname{tr}(Z)| > 2 \right\} \\ Q_{\pm 2} &= \left\{ Z \in \mathrm{SL}(2,\mathbb{R}) \,\middle|\, \operatorname{tr}(Z) = \pm 2 \right\} \\ Q_{<2} &= \left\{ Z \in \mathrm{SL}(2,\mathbb{R}) \,\middle|\, |\operatorname{tr}(Z)| < 2 \right\}. \end{aligned} \tag{7.10}$$

The joint invariants for each of these invariant subsets are given below and the proofs delegated to Section 7.2.1. Subsection 7.2.2 identifies each of the orbits as $\mathrm{SL}(2,\mathbb{R})$ homogeneous spaces from Theorem 5.2.

Theorem 7.8. *Let* $Q_{>2}$ *be as defined in Equation* (7.10)*. Then the function* $\operatorname{tr} : \mathrm{SL}(2,\mathbb{R}) \to \mathbb{R}$ *is a well defined complete local invariant of conjugation.*

Theorem 7.9. *Let Q_2 and Q_{-2} be the invariant subsets of* $\mathrm{SL}(2,\mathbb{R})$ *defined in Equation* (7.10)*. The set Q_2 has three invariant subsets*

$$\begin{aligned} Q_{2,0} &= \left\{ Z \in Q_2 \,\middle|\, \dim(\ker(Z - \mathrm{I})) = 0 \right\} \\ Q_{2,1} &= \left\{ Z \in Q_2 \,\middle|\, \dim(\ker(Z - \mathrm{I})) = 1 \right\} \\ Q_{2,2} &= \left\{ Z \in Q_2 \,\middle|\, \dim(\ker(Z - \mathrm{I})) = 2 \right\} \end{aligned} \tag{7.11}$$

and the $\mathbb{Z}_3$ valued invariant function $f_{\dim} : \mathrm{SL}(2,\mathbb{R}) \to \mathbb{Z}_3$ given by

$$f_{\dim}(Z) = \begin{cases} 0 & Z \in Q_{2,0} \\ 1 & Z \in Q_{2,1} \\ 2 & Z \in Q_{2,2} \end{cases} \tag{7.12}$$

determines which subset Z belongs to. Each of these subsets has a complete local invariant.

a) The function $f_{2,0} : Q_{2,0} \to \mathbb{Z}_2$ defined by

$$f_{2,0} = \begin{cases} 0 & Z \sim_{\mathrm{SL}(2,\mathbb{R})} \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \\ 1 & Z \sim_{\mathrm{SL}(2,\mathbb{R})} \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \end{cases}$$

is a complete local invariant of $Q_{2,0}$.

b) The function $f_{2,1} : Q_{2,1} \to \mathbb{Z}_2$ *given by*

$$f_{2,0} = \begin{cases} 0 & Z \sim_{\mathrm{SL}(2,\mathbb{R})} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ 1 & Z \sim_{\mathrm{SL}(2,\mathbb{R})} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{cases}$$

is a complete local invariant of $Q_{2,1}$.

c) $Q_{2,2}$ *is the fixed point* I.

Similarly for Q_{-2} *there are subsets,* $Q_{-2,0}, Q_{-2,1}, Q_{-2,2}$, *and invariants,* $f_{-2,0} : Q_{2,0} \to \mathbb{Z}_2$ *defined by*

$$f_{2,0} = \begin{cases} 0 & Z \sim_{\mathrm{SL}(2,\mathbb{R})} \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \\ 1 & Z \sim_{\mathrm{SL}(2,\mathbb{R})} \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} \end{cases} \tag{7.13}$$

and $f_{2,1} : Q_{2,1} \to \mathbb{Z}_2$ *given by*

$$f_{2,1} = \begin{cases} 0 & Z \sim_{\mathrm{SL}(2,\mathbb{R})} \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \\ 1 & Z \sim_{\mathrm{SL}(2,\mathbb{R})} \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \end{cases} \tag{7.14}$$

which form complete sets of local invariants where they are defined.

Theorem 7.10. *Let $Q_{<2}$ be defined as in Equation (7.10). Then the function $f_{rational}$: $Q_{<2} \to \mathrm{SL}(2,\mathbb{R})$ given by*

$$f_{\text{rational}}(Z) = \begin{cases} 0 & Z \sim_{\mathrm{SL}(2,\mathbb{R})} \begin{bmatrix} 0 & -1 \\ 1 & \mathrm{tr}(Z) \end{bmatrix}, \\ \\ 1 & Z \sim_{\mathrm{SL}(2,\mathbb{R})} \begin{bmatrix} \mathrm{tr}(Z) & 1 \\ -1 & 0 \end{bmatrix} \end{cases} \tag{7.15}$$

is a well defined complete local invariant of conjugation on $Q_{<2}$.

Corollary 3.3 implies the corresponding complete set of joint invariants for points $(X_0, X_1) \in \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ are given by the functions above precomposed with $T(X_0, X_1) = X_0^{-1} X_1$.

7.2.1 Proofs for Two Point Results

We split the proofs into three sections for each of the $\mathrm{SL}(2,\mathbb{R})$ invariant subsets, $Q_{>2}, Q_{\pm 2}, Q_{<2}$.

The case of $Q_{>2}$.

Let $Z \in \mathrm{SL}(2,\mathbb{R})$. If $|\mathrm{tr}(Z)| > 2$ then Z is diagonalizable and we have the following lemma.

Lemma 7.9. *Let $Z \in \mathrm{SL}(2,\mathbb{R})$. If $|\frac{1}{2}\mathrm{tr}(Z)| > 1$ then Z is diagonalizable over $\mathrm{SL}(2,\mathbb{R})$ with diagonal representative*

$$\begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix} \tag{7.16}$$

where $\lambda \in \mathbb{R}^*$.

Proof. Let $Z \in \mathrm{SL}(2,\mathbb{R})$ have $\left|\frac{1}{2}\operatorname{tr}(Z)\right| > 1$ then the characteristic polynomial of Z,

$$P_Z(t) = t^2 - 2\operatorname{tr}(Z)t + 1$$

implies that the eigenvalues of Z are nonzero real distinct values $\lambda, 1/\lambda$. Then Z is conjugate to either $\mathrm{diag}(\lambda, 1/\lambda)$ or $\mathrm{diag}(1/\lambda, \lambda)$ by an element of $P \in \mathrm{GL}(n,\mathbb{R})$. Now possibly by permuting the columns of X and rescaling the matrix by $\frac{1}{\sqrt{\det(P)}}$ we can assume $P \in \mathrm{SL}(2,\mathbb{R})$ without loss of generality, and since

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

is in $\mathrm{SL}(2,\mathbb{R})$ and $S\mathrm{diag}(\lambda, 1/\lambda)S^{-1} = \mathrm{diag}(1/\lambda, \lambda)$ then the proof is complete. □

Then the proof of Theorem 7.8 follows from applying Lemma 7.9.

proof of Theorem 7.8. Let $Z, W \in Q_{>2}$ with eigenvalues λ, ν respectively the equation $\lambda + \frac{1}{\lambda} = \nu + \frac{1}{\nu}$ implies either $\lambda = \nu$ or $\lambda = \frac{1}{\nu}$ and the trace is a complete invariant. □

The case of $Q_{\pm 2}$

Let $\operatorname{tr}(Z) = 2$ we will provide a detailed discussion of this case, as $\operatorname{tr}(Z) = -2$ is similar. The characteristic polynomial of Z is $P_Z(t) = t^2 - 2t + 1$ and we have the following lemma showing that f_{dim} from Equation (7.12) is well defined.

Lemma 7.10. *The dimension of* $\ker(\mathrm{I} - Z)$ *is invariant under conjugation of* Z.

Proof. Suppose that v is an eigenvector of Z. Then if $A \in \mathrm{SL}(2,\mathbb{R})$ we have $AZA^{-1}Av = Av$ and Av is an eigenvector of AZA^{-1}. So if $\{v_i\}$ is a basis for $\ker(\mathrm{I} - Z)$ then $\{Av_i\}$ is a basis for $\ker\left(\mathrm{I} - AZA^{-1}\right)$ verifying that they are the same dimension. □

Now consider $Q_{2,0}$ the following lemma gives representatives for the conjugacy classes of points in this set, and verifies that the function $f_{2,0}$ from Equation (7.13) is a well defined complete local invariant there.

Lemma 7.11. *If* $\dim(\mathrm{I} - Z) = 0$ *then* Z *is conjugate to one of the two forms*

$$\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \quad or \quad \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

and these forms are non conjugate.

We first prove the preliminary result

Lemma 7.12. *If* $|\alpha| \le 2$ *then the matrices,* $\begin{bmatrix} 0 & -1 \\ 1 & \alpha \end{bmatrix}$ *and* $\begin{bmatrix} \alpha & 1 \\ -1 & 0 \end{bmatrix}$, *are not conjugate in* $\mathrm{SL}(2,\mathbb{R})$.

Proof. Suppose there is $A \in \mathrm{SL}(2,\mathbb{R})$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & \alpha \end{bmatrix} = \begin{bmatrix} \alpha & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

This leads to the two equations $d = -a$ and $b = \alpha a + c$ so A must have the form,

$$A = \begin{bmatrix} a & \alpha a + c \\ c & -a \end{bmatrix}$$

and since $\det(A) = 1$ we have the necessary condition that a, c must solve the equation $a^2 + \alpha ac + c^2 + 1 = 0$. However this equation does not have any solutions until $|\alpha| > 2$ which can be verified by substituting $a = r(t)\cos(t)$ and $c(t) = r(t)\sin(t)$ for some arbitrary function $r(t)$. Then substituting into the equation and simplifying one arrives at the equation

$$r^2\left(1 + \frac{1}{2}\alpha \sin(2t)\right) + 1 = 0$$

but $\left|\frac{1}{2}\alpha\right| < 1$ so this equation cannot have any solutions proving the claim. □

Proof of Lemma 7.10. Let $Z \in Q_{2,0}$. Then Z has no real eigenvalues and the matrix $P =$

$[e_1|Ae_1]$, where e_1 is the first standard basis vector, is an element of $\mathrm{GL}(2,\mathbb{R})$. So conjugating by P^{-1} gives

$$P^{-1}ZP = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}.$$

There are two cases on $\det(P)$. If $\det(P) > 0$ then the matrix $R = \frac{1}{\sqrt{\det(P)}}$ has $R^{-1}ZR = P^{-1}ZP$ and Z is conjugate in $\mathrm{SL}(2,\mathbb{R})$ to the form claimed. Otherwise $\det(P) < 0$ and the matrix $P' = [Ze_1, e_1]$ has $\det(P') > 0$. Conjugating by $(P')^{-1}$ gives

$$(P')^{-1}Z(P') = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

and $R' = \frac{1}{\sqrt{\det(P')}}P'$ is in $\mathrm{SL}(2,\mathbb{R})$ such that $(R')^{-1}ZR' = (P')^{-1}ZP'$. Then from Lemma 7.12 the representatives are not conjugate and the proof is complete. □

Now we show that $f_{2,1}$ from Equation (7.14) is a well defined complete invariant on $Q_{2,1}$.

Lemma 7.13. *If* $\dim(\mathrm{I} - Z) = 1$ *then* Z *is conjugate to exactly one of the two forms*

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad or \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Proof. Let v be an eigenvector of Z so that $Zv = v$. Then since $\dim(\ker(Z - \mathrm{I})) = 1$ there exists $w, u \neq 0$ which is independent of v such that

$$(Z - \mathrm{I})w = u$$

or $Zw = u + w$ and w is not an eigenvector of Z.

We claim that u is an eigenvector of Z. Suppose not, then u is in the image of $(Z - \mathrm{I})$ and therefore a scalar multiple of w, a contradiction in w not being an eigenvector of Z.

So there exists w such that $(Z - \mathrm{I})w = v$ (possibly by rescaling w) and the matrix $P = [v|w]$ satisfies

$$P^{-1}ZP = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \tag{7.17}$$

If $\det(P) > 0$ then Z is conjugate to the desired form over $\mathrm{SL}(2,\mathbb{R})$. Otherwise if $\det(P) < 0$ then we need to take $P' = [w|v]$ and Z is conjugate over $\mathrm{SL}(2,\mathbb{R})$ to the other form completing the proof. □

Lemma 7.14. *If* $\dim(\mathrm{I} - Z) = 2$ *then* $Z = \mathrm{I}$.

The proof is clear so it will be omitted.

Proof of Theorem 7.8. From Lemma 7.10 the subsets $Q_{2,i}$ for $i = 0, 1, 2$ are invariant under $\mathrm{SL}(2,\mathbb{R})$ and from Lemmas 7.11 7.13 and 7.14 the functions $f_{2,i}$ for $i = 0, 1$ are well defined complete local invariants of the subsets $Q_{2,i}$ for $i = 0, 1$. □

The Case of $Q_{<2}$

When $|\operatorname{tr}(Z)| < 1$ then the matrix Z is not diagonalizable over the real numbers, the next lemma shows that the rational canonical form is a complete invariant of the conjugation action.

Lemma 7.15. *Let* $Z \in \mathrm{SL}(2,\mathbb{R})$ *with* $\left|\frac{1}{2}\operatorname{tr}(Z)\right| < 1$. *Then* Z *is conjugate over* $\mathrm{SL}(2,\mathbb{R})$ *to one of the forms*

$$\begin{bmatrix} 0 & -1 \\ 1 & \operatorname{tr}(Z) \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \operatorname{tr}(Z) & 1 \\ -1 & 0 \end{bmatrix}$$

which are not conjugate.

Proof. Let $\alpha = \operatorname{tr}(Z)$. The representatives are not conjugate by Lemma 7.12

Now consider the characteristic polynomial of Z, $P_Z(t) = t^2 - \alpha t + 1$. By the Cayley-Hamilton theorem Z satisfies its own characteristic polynomial and $Z^2 - \alpha Z + \mathrm{I} = 0$. Now let

e_1 be the first element of the standard basis for $\mathbb{R}^2$. Let $P = [e_1|Ae_1]$. This is an element of $\mathrm{GL}(2,\mathbb{R})$ since otherwise Ae_1 would be a real multiple of e_1, but this is impossible since A has complex eigenvalues when $|\alpha| < 2$.

There are two cases. If $\det(P) > 0$ then in the basis $\{e_1, Ze_1\}$ then

$$P^{-1}ZP = \begin{bmatrix} 0 & -1 \\ 1 & \mathrm{tr}(Z) \end{bmatrix}.$$

Then by letting $R = \frac{1}{\sqrt{\det(P)}}P$ $R^{-1}ZR = P^{-1}ZP$ and Z is conjugate to this form over $\mathrm{SL}(2,\mathbb{R})$.

On the other hand suppose that $\det(P) < 0$. Then we let $P' = [Ze_1|e_1]$ which again is in $\mathrm{GL}(2,\mathbb{R})$ and satisfies

$$(P')^{-1}ZP' = \begin{bmatrix} \alpha & 1 \\ -1 & 0 \end{bmatrix}$$

and by similar logic to the previous case Z is conjugate in $\mathrm{SL}(2,\mathbb{R})$ to the form

$$\begin{bmatrix} \mathrm{tr}(Z) & 1 \\ -1 & 0 \end{bmatrix}.$$

□

Then Lemma 7.15 implies that the function f_{rational} defined in Equation (7.15) is a well defined complete local invariant on $Q_{<2}$.

7.2.2 Isotropy of orbits

To conclude the analysis of μ_{conj} for the group $\mathrm{SL}(2,\mathbb{R})$ we will compute the isotropy subgroups for each of the orbit representatives given above and identify them with the homogeneous spaces of Chapter 5.

1. For $Z \in Q_{>2}$ the orbits have a diagonal representative

$$\begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix}$$

where $\lambda \in \mathbb{R}^*$. So the isotropy are $A \in \mathrm{SL}(2,\mathbb{R})$ such that $A\mathrm{diag}(\lambda, 1/\lambda) = \mathrm{diag}(\lambda, 1/\lambda)A$,

$$\begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and the isotropy of Z is conjugate to $\begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix}$ for any $a \in \mathbb{R}^*$. So these orbits are diffeomorphic to the one sheet hyperbola Q_2 as described in part i) of Theorem 5.2.

2. If $Z \in Q_2$ there are three representatives to compute the isotropy. If $Z = \mathrm{I}$ then its orbit is a fixed point, otherwise the stabilizer is given by solutions to the following equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

or

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and the isotropy of Z is conjugate to $\begin{bmatrix} \pm 1 & t \\ 0 & \pm 1 \end{bmatrix}$ or $\begin{bmatrix} \pm 1 & 0 \\ t & \pm 1 \end{bmatrix}$. These subgroups are conjugate over $\mathrm{SL}(2,\mathbb{R})$ and the corresponding orbits are diffeomorphic to the cones Q_0^+ as described in part ii) of Theorem 5.2.

3. If $Z \in Q_{<2}$ Then we have two possible representatives for its conjugacy class,

$$\begin{bmatrix} 0 & -1 \\ 1 & \alpha \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \alpha & 1 \\ -1 & 0 \end{bmatrix}.$$

Let the first representative be denoted R. We will compute its isotropy subgroup which is the same as for the other representative, and identify it with $\mathrm{SO}(2,\mathbb{R})$. The isotropy equation, $RA = AR$ gives the solution

$$A = \begin{bmatrix} a & -c \\ c & \alpha c + a \end{bmatrix} \quad \text{where} \quad a^2 + \alpha ac + c^2 = 1.$$

If we let $X \in \mathrm{SL}(2,\mathbb{R})$ be

$$X = \begin{bmatrix} 0 & \frac{-1}{\sqrt{2}}\left(4-\alpha^2\right)^{1/4} \\ \sqrt{2}\left(4-\alpha^2\right)^{-1/4} & \frac{\alpha}{\sqrt{2}}\left(4-\alpha^2\right)^{-1/4} \end{bmatrix}.$$

Then

$$Q = XAX^{-1} = \begin{bmatrix} \frac{\alpha c}{2} + a & -\frac{c}{2}\left(4-\alpha^2\right)^{1/2} \\ \frac{c}{2}\left(4-\alpha^2\right)^{1/2} & \frac{\alpha c}{2} + a \end{bmatrix},$$

where Q satisfies

$$Q^T Q = \begin{bmatrix} a^2 + \alpha ca + c^2 & 0 \\ 0 & a^2 + \alpha ca + c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So $Q \in \mathrm{SO}(2,\mathbb{R})$ and Q is the isotropy of $X^{-1}RX$, so that the isotropy of Z is conjugate to $\mathrm{SO}(2,\mathbb{R})$ and the orbit of Z is diffeomorphic to Q^{+}_{-2} as described in part iii) of Theorem 5.2.

This concludes our investigation of the $\mathrm{SL}(2,\mathbb{R})$ conjugacy classes.

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APPENDICES

APPENDIX A

Code

The code used to generate signature manifolds in the landmark method from the introduction:

```
import numpy as np
import matplotlib.pyplot as plt


def delta(x,y):
    dlist = []
    for i in range(0, len(y)):
        dist = np.sqrt(
            (x[0] - y[i][0])**2 + (x[1] - y[i][1])**2
        )
        dlist.append(dist)
    return dlist
#delta takes single point and list of points and spits out the list of distances.


def parax(t):
    f = t
    return f*np.cos(2*t)
def paray(t):
    g = t
    return g*np.sin(2*t)


# import data from .txt files
X1 = np.loadtxt('X1data')
```

```
Y1 = np.loadtxt('Y1data')

X2 = np.loadtxt('X2data')
Y2 = np.loadtxt('Y2data')


data1, data2 = [], []
for i in range(0, X1.size):
    data1.append([X1[i], Y1[i]])
    data2.append([X2[i], Y2[i]])


delta11 = delta([X1[0], Y1[0]], data1)
delta12 = delta([X1[len(X1) - 1], Y1[len(Y1) - 1]], data1)
delta21 = delta([X2[0], Y2[0]], data2)
delta22 = delta([X2[len(X2) - 1], Y2[len(Y2) - 1]], data2)


fig, [ax1, ax2] = plt.subplots(1, 2, figsize=(10,5))


ax1.scatter(x=X1, y=Y1, s=1, marker='o', color='r')
ax1.scatter(x=X2,y=Y2, s=1, marker='x', color='b')
ax1.set_title('curves')
ax1.set_xlim(-5, 15)
ax1.set_ylim(-5, 15)
```

```
ax2.scatter(x=delta11, y=delta12, s=1, marker='o', color='r')
ax2.scatter(x=delta21, y=delta22, s=1, marker='x', color='b', alpha=0.5)
ax2.set_title('signatures')
ax2.set_xlabel('$\delta_1$')
ax2.set_ylabel('$\delta_2$')


plt.show()
```

APPENDIX B

Topologically Identifying $\mathrm{SU}(2,\mathbb{R})/\mu_{\mathrm{conj}}$ with $[-1,1]$

The following appendix proves the topological claim made in Remark 7.1. First we need some general results about invariants of topological spaces, this discussion will closely follow Munkres [14].

Definition B.1. Let X and Y be topological spaces. Let $\pi : X \to Y$ be a surjective map. Then the map π is said to be a *topological quotient* map provided a subset U of Y is open in Y if and only if $\pi^{-1}(U)$ is open in X.

Definition B.2. Let X and Y be topological spaces. The map $f : X \to Y$ is called *open* if for any open set $U \subset X$ the image $f(U)$ is open.

Lemma B.1. *Let X and Y be topological spaces. If $f : X \to Y$ is a surjective open continuous map then f is a quotient map.*

Proof. Since f is surjective and continuous we need only show that if $f^{-1}(U)$ is open for some set $U \subset Y$ then U is open. But since f is open $f(f^{-1}(U)) = U$ is open completing the proof. □

If G is a Lie group and $\mu : G \times X \to X$ is an action which is also a continuous map with respect to the product topology on $G \times X$ then with respect to the quotient topology on X/G, the projection map $\pi : X \to X/G$ sending each element of X to its orbit is an open topological quotient map, see Boothby [3] for details. In this situation we have the following extension of Theorem 2.2 for continuous invariants.

Lemma B.2. *Let G be a Lie group, X a topological space, and $\mu : G \times X \to X$ a continuous group action with open quotient map $\pi : X \to X/G$. Let $f : X \to Y$ be an invariant function with respect to the action μ and let $\tilde{f} : X/G \to Y$ be the unique function such that $\tilde{f} \circ \pi = f$. The function $\tilde{f}$ is continuous if and only if f is continuous, and $\tilde{f}$ is a topological quotient map if and only if f is a topological quotient map.*

Proof. If $\tilde{f}$ is continuous then $f = \tilde{f} \circ \pi$ is a composition of continuous maps and therefore a continuous map.

Now suppose that f is continuous. If $V \subset Y$ is open then $U = f^{-1}(V)$ is open and $\pi(U) \subset X/G$ is open because the projection is an open map. We claim that $\pi(U) = \tilde{f}^{-1}(V)$. From $\tilde{f} \circ \pi = f$ it's immediate that $\tilde{f}(\pi(U)) = V$ so that $\pi(U) \subset \tilde{f}^{-1}(V)$. Fix $\tilde{x} \in \tilde{f}^{-1}(V)$. Since π is surjective there exists some $x \in X$ such that $\pi(x) = \tilde{x}$, and the image of $\pi(x)$ under $\tilde{f}$ is

$$\tilde{f}(\tilde{x}) = \tilde{f}(\pi(x)) = f(x),$$

so $f(x) \in V$ because $\tilde{x} \in \tilde{f}^{-1}(V)$. Hence, $x \in f^{-1}(V) = U$ and $\tilde{x} = \pi(x) \in \pi(U)$ which completes the proof that $\pi^{-1}(U) = \tilde{f}^{-1}(V)$ so $\tilde{f}$ is continuous.

Now if $\tilde{f}$ is a topological quotient map then $\tilde{f} \circ \pi = f$ is a composition of quotient maps and so f is a quotient map.

On the other hand suppose that f is a quotient map. Then since f is surjective $\tilde{f}$ is surjective. From the previous part since f is continuous then $\tilde{f}$ is continuous as well. So consider some $V \in Y$. We will show that V is open in Y if $\tilde{U} = \tilde{f}^{-1}(V)$ is open in X/G. $\pi^{-1}(\tilde{U})$ is open because π is continuous. We claim that $\pi^{-1}(\tilde{U}) = f^{-1}(V)$. From

$$f(\pi^{-1}(\tilde{U})) = \tilde{f} \circ \pi(\pi^{-1}(\tilde{U})) = \tilde{f}(\tilde{U}) = V$$

then $\pi^{-1}(\tilde{U}) \subset f^{-1}(V)$. Now fix $x \in f^{-1}(V)$. Then $\tilde{f}(\pi(x)) = f(x) \in V$ so $\pi(x) \in \tilde{U}$ and $x \in \pi^{-1}(\tilde{U})$. Then since $\pi^{-1}(\tilde{U}) = f^{-1}(V)$ is open and f is a topological quotient map then V is open completing the proof. □

Lemma B.3. *If $f : X \to Y$ is a topological quotient map and a complete invariant of the continuous group action $\mu : G \times X \to X$ then $\tilde{f} : X/G \to Y$ is a canonical homeomorphism.*

Proof. From Corollary 2.1 $\tilde{f}$ is a canonical bijection, and from Lemma B.2 $\tilde{f}$ is a topological quotient map. Then since $\tilde{f}$ is a topological quotient map and a bijection it is a homeomorphism. □

We now give a sufficient condition for a surjective continuous complete invariant function to be open.

Lemma B.4. *Let X and Y be topological spaces, $f : X \to Y$ a surjective continuous map. If there exists a continuous map $g : Y \to X$ such that $f \circ g = \mathrm{Id}_Y$ then f is a quotient map.*

Proof. Let $V \subset Y$ be a subset and suppose that $f^{-1}(V)$ is an open set. Then since g is continuous $g^{-1}(f^{-1}(V))$ is open. Now we claim that this set is equal to V. Indeed let $y \in g^{-1}(f^{-1}(V))$ then $g(y) \in f^{-1}(V)$ and $f(g(y)) = y \in V$ since $f \circ g = \mathrm{Id}_Y$. Hence $g^{-1}(f^{-1}(V)) \subset V$. On the other hand if $y \in V$ then $g(y)$ satisfies $f(g(y)) = y \in V$ so $g(y) \in f^{-1}(V)$ and hence $y \in g^{-1}(f^{-1}(V))$ which completes the proof of the claim and $g^{-1}(f^{-1}(V)) = V$. Then V is open and therefore f is a quotient map. □

Now consider $\mathrm{SU}(2,\mathbb{R})$. This is a closed Lie subgroup of $\mathrm{GL}(2,\mathbb{C})$ and therefore an embedded submanifold which is a topological manifold with respect to the subspace topology, see Boothby [3] or Helgason [11].

The group $\mathrm{GL}(2,\mathbb{C})$ is an open submanifold of $\mathcal{M}_n(\mathbb{C})$ the real vector space of $n \times n$ matrices with complex entries. The standard topology in $\mathcal{M}_n(\mathbb{C})$ is equivalent to the topology induced by the inner product,

$$\langle A, B \rangle = \frac{1}{2}\,\mathrm{tr}(A^* B),$$

and so the topology on $\mathrm{SU}(2,\mathbb{R})$ is the subspace topology with respect to the distance function

$$d(A, B) = \frac{1}{2}\,\mathrm{tr}((A - B)^*(A - B)).$$

Note that the map tr is continuous as it is a polynomial in the entries of elements of $\mathrm{SU}(2,\mathbb{R})$. We now prove the trace function is a topological quotient map.

Lemma B.5. *The map $\frac{1}{2}\,\mathrm{tr} : \mathrm{SU}(2,\mathbb{R}) \to [-1, 1]$ is a quotient map.*

Proof. $\frac{1}{2}\operatorname{tr}$ is a continuous map as a restriction of a continuous map to a subspace. Moreover $\frac{1}{2}\operatorname{tr}$ is surjective from Theorem 7.1.

We will show that there exists $g : [-1,1] \to \mathrm{SU}(2,\mathbb{R})$ which is continuous and serves as a right inverse of $\frac{1}{2}\operatorname{tr}$. Indeed let

$$g(r) = \begin{bmatrix} r + i\sqrt{1-r^2} & 0 \\ 0 & r - i\sqrt{1-r^2} \end{bmatrix}.$$

we note that $\det(g(r)) = r^2 + 1 - r^2 = 1$ and

$$g(r)g(r)^* = \begin{bmatrix} r + i\sqrt{1-r^2} & 0 \\ 0 & r - i\sqrt{1-r^2} \end{bmatrix} \begin{bmatrix} r - i\sqrt{1-r^2} & 0 \\ 0 & r + i\sqrt{1-r^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so g is well defined. The map g is also continuous since each component of $g(r)$ is continuous in r. Now we verify that $f \circ g(r) = r$. Indeed this follows from the construction of $g(r)$ so that

$$\frac{1}{2}\operatorname{tr}(g(r)) = \frac{1}{2}(2r) = r$$

and by Lemma B.4 $\frac{1}{2}\operatorname{tr}$ is a quotient map. □

Now $\frac{1}{2}\operatorname{tr} : \mathrm{SU}(2,\mathbb{R}) \to [-1,1]$ is continuous surjective and open, so it is a quotient map, and a complete invariant of the action μ_{conj} of $\mathrm{SU}(2,\mathbb{R})$ on itself by conjugation. Then Lemma B.3 shows there is a canonical homeomorphism between $\mathrm{SU}(2,\mathbb{R})/\mu_{\text{conj}}$ and $[-1,1]$.