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Generalised soft multi-mode real options model (fuzzy-stochastic approach)

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A B S T R A C T

Researchers and practitioners are dealing intensively with the real option valuation. One of the generalised types is reversible the multi-mode American real options. These options are solved mainly by applying the stochastic discrete binomial models. Uncertainty is a typical feature of valuation, and two basic types of representation are distinguished: risk (stochastic) and imprecision (fuzzy). The fuzzy-stochastic models indicate the generalised real options modelling containing both aspects. The objective of the paper is to develop and apply the generalised fuzzy-stochastic multi-mode real options model. This model is based on fuzzy numbers, the discrete binomial model, and the decomposition principle. Input data, particularly underlying cash-flows, are given by fuzzy-random numbers; fuzzy numbers give terminal values, risk-free rate, switching cost. Furthermore, assumptions and computation procedures are also described. The proposed optimisation problem is used for the fuzzy multi-mode real option value calculation. Results are compared with sub-problems, crisp-stochastic multi-modes real options and partial fuzzy-stochastic multi-mode real options models. A stylised illustrative operational flexibility example of comparing the fuzzy-stochastic multi-mode real options models is presented and discussed. The model can serve to valuation, decision-making, generalised sensitivity analysis and control under a fuzzy-stochastic environment.

1. Introduction

The intention of the paper is to propose and apply the generalised fuzzy-stochastic multi-mode real options model. It is a generalisation of the previous research phases. Model application is useful when input data is possible to state only imprecisely. Gradual research development phases can be described as follows. Real option methodology was evolved, and crisp-stochastic multi-mode real options represent a generalised approach (phase a). The American crisp-stochastic multi-mode binomial real option models were formulated (phase b). The stochastic input data uncertainty was proposed in the valuation of European continuous financial and real options. The fuzzy-stochastic European continuous real options models were proposed (phase c). The fuzzy-stochastic binomial American real options with various fuzzy parameters were evolved, including the generalised fuzzy-stochastic binomial real option model with all fuzzy parameters (phase d). However, the fuzzy-stochastic multi-mode real options with all fuzzy parameters have not been proposed yet.

(Phase a) The real options methodology is nowadays one of the generalised flexible valuation approaches, successively developed and applied. It is valid that the more modes and bigger volatility, the greater real option and flexibility value. It mainly serves for asset and projects valuation, dynamic decision-making and control. Trigeorgis and Tsekrekos (2017) surveyed and reviewed real option topics and models by various aspects: types, computation conceptions and methods, input data environment, sectors usage, and practical (empirical) applications.

Various real options alternatives were developed: switching, multi-phase, multi-stage, sequential, learning, compound and multi-mode options. The notion mode was coined by Kulatilaka (1988) under real option publications. The term multi-mode is used uniquely and commonly in the paper because of developing the generalised model with reversibility feasibility. The multi-modes real options models generalise the specific real options valuation models, e. g. Guthrie (2009) argues the same idea (p. 207). The various types of real options can be formulated and solved by the model: e. g. valuation of the company, investment projects valuation, investment project postpone (wait), investment portfolio project choice, acquisition valuation, research projects choice, construction projects valuation, technological alternatives choice, operational flexibility, machinery replacement, manufacturing flexibility, inventory management. The problem's generalisation consists of the general term mode encompassing different situations, e. g., expansion, contraction, temporarily shut down,

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abandonment production, technological processes, research programs, research stages, innovation steps, asset types, inventory lines, technological units, construction stages. Furthermore, more than two modes (phases, stages) are supposed, switching possibilities are assumed among all modes, reversibility is enabled, and switching costs are considered asymmetric. Another generalisation means that the generalised multi-mode real options model is of reversible type. Subproblems given by irreversibility are sequential or stage options, so returning to the previous modes is forbidden. All real options alternatives introduced can be formulated in a methodological way as the generalised multi-mode real options models.

The multi-mode real options are stochastic control optimisation models with valuation functions based on the real option value. The problem can be solved by applying Bellman's dynamic programming procedure if the problems are convex (feasible set is convex and objective function is convex); simultaneously, the objective functions are separable. Multi-mode real options fulfil these conditions usually.

The martingale principle of valuation is the generalised approach of valuation of assets (options as well) presented in Harrison and Kreps (1979) and Musiela and Rutkowski (2011). Under the principle, a value has to be equal to the expected future value. In the case of the risk-neutral approach, the no-arbitrage assumption, the risk-free asset is numeraire. The ratio of a random value and the risk-free asset is a martingale – which must be fulfilled - and the option value is the present value of the risk-neutral expected future value. The practical valuation approach coinciding with the martingale principle is the replication strategy in a risk-neutral world.

(Phase b) The multi-mode real options allow switching in time; therefore, the American options are used. Since model complexity, mainly numerical approximation, and especially binomial models are applied. The generalised multi-mode real options are introduced in Kulatilaka (1988), Kulatilaka and Trigeorgis (1994), Trigeorgis (1998), Guthrie (2009), and Zmeškal (2008). Examples of the multi-mode real options applications in discrete time are the following: energy sector (Carmona & Ludkovski, 2010; Bastian-Pinto, Brandão & Alves, 2010; Glensk & Madlener, 2018. Kulatilaka (1988); Kulatilaka, 1993; Marreco & Carpio, 2006; Ronn et al., 2002; Varympopiotis, Tolis & Rentizelas, 2014), metallurgical sector (Bastian-Pinto, Brandão & Ozório, 2016), pharmaceutical sector (Brandao, Fernandes & Dyer, 2018), IT investment (Pendharkar, 2010), high-tech investment (Song et al., 2017), equity valuation (Zmeškal, 2008).

Input data availability, validity, quality and precision is important aspect of real options modelling, which substantially influences the solutions, e. g. Carlsson and Fuller (2003) introduce it (p. 302). Another influential characteristic is model uncertainty. Traditionally, the stochastic (probability, risk) apparatus is used for uncertainty modelling. However, the term uncertainty has another aspect, an imprecision (non-preciseness, ambiguity, softness, vagueness, fuzziness, possibility), which is often neglected and can be modelled and represented, besides others, by fuzzy sets and fuzzy methodology. So, the general term uncertainty comprises both introduced aspects: risk (stochastic) and imprecision (fuzzy) representation. The term "imprecision" is used solely in the paper, including all aspects of non-stochastic uncertainty, because of modelling and representing by fuzzy numbers and terminology uniqueness. The models including both aspects simultaneously are titled hybrid models and can be solved, among others, by fuzzy-stochastic modelling. Encompassing both types of uncertainty is a generalised feature of the models.

(Phase c) Researches have dealt with fuzzy approaches to financial options and, in particular, real options valuation because the inclusion of imprecision is an inherent feature of modelling and reflects valuation condition more realistically. Two types of input model data can be stated imprecisely: underlying process (fuzzy or fuzzy-random distribution) and other input data (terminal value, risk-free rate). Two fundamental objectives exist: obtaining the decision (crisp) value or the sensitivity (fuzzy) value. Two methods can obtain the decision value. Firstly, the defuzzification of fuzzy input data with calculating a crisp problem can be used. Secondly, calculating the fuzzy problem with fuzzy input data and subsequent defuzzification of results can be applied. The sensitivity (fuzzy) value means investigating and evaluating the sensitivity of value. European and American options due to different time exercises are distinguished. For European options, a closed-form solution exists derived from continuous processes. American options are solved predominantly by a discrete approximation.

European fuzzy financial and real options approaches are solved in two ways: fuzzy numbers and decomposition using $\varepsilon -cuts$ (Carlsson & Fuller 2003; Chrysafis & Papadopoulos, 2009; Guerra et al., 2007; Simonelli 2001; Thiagarajaha, Appadoob & Thavaneswaranc, 2007; Yoshida, 2003; Wu, 2005, 2007; Zmeškal, 2001), an application of fuzzy measures (Liyan Han & Wenli Chen, 2006; Driouchi, Trigeorgis & Gao, 2015, Jinliang, Hubin & Wansheng, 2006).

(Phase d) The American real options have to be solved by a discrete binomial model. Therefore, the fuzzy-stochastic (hybrid) discrete binomial options methodology is yet to be developed and used. Authors dealt with various fuzzy input parameters; see e. g. approaches with fuzzy volatility, Muzzioli and Reynaerts (2007), Muzzioli and Reynaerts (2008), Muzzioli and Torricelli (2004), Yoshida (2002), Yoshida et al. (2005), or simultaneously fuzzy volatility and risk-free fuzzy rate, Few-Lee et al. (2005), comprehensive (all) fuzzy parameters Zmeškal (2010), survey Muzzioli and De Baets (2017). The fuzzy-stochastic multi-mode real options model with all fuzzy parameters has not been developed yet.

Therefore, the paper's objective is to propose and apply the generalised fuzzy-stochastic multi-mode real options model. Model generalisation consists of methodological generalisation because it includes many subproblems (e. g., irreversibility) and various uncertainty representations, stochastic (risk) and imprecision (fuzzy) representations. The usefulness of model application is especially in strategic investment and decision-making, particularly in situations with greater flexibility (number of modes), bigger volatility and imprecision of input data.

Model is developed on the fuzzy-stochastic discrete binomial method, backward induction, fuzzy numbers (T-numbers), $\varepsilon -cut$ and decomposition principle.

The impact of imprecision is investigated by comparison of three multi-mode real options model alternatives: the crisp-stochastic multi-mode real options (CSMMRO) model; the partial fuzzy-stochastic multi-mode real options (PFSMMRO) model with imprecisely determined switching (transaction) cost; and the generalised fuzzy-stochastic multi-mode real options (GFSMMRO) model with all input data stated imprecisely including fuzzy-stochastic (fuzzy-random) probability distribution.

GFSMMRO model includes imprecision comprehensively so that all input data are stated imprecisely: risk-free rate, terminal values, switching (transition) cost, underlying modes cash-flow development as fuzzy numbers (T-numbers). The model is based methodologically on the fuzzy-stochastic discrete binomial method, backward induction, fuzzy risk-neutral probabilities, fuzzy numbers, $\varepsilon -cut$, and the Decomposition principle (Resolution identity). It is shown that the GFSMMRO model is possible to solve by applying several constrained optimisation problems. The results are represented in a fuzzy number that can be compared by any binary fuzzy relation method or simplistically by any selected defuzzification method.

The paper is organised in the following way. The description of the fuzzy-stochastic apparatus applied is contained in the second section. The third section includes characteristics and a description of crisp-stochastic multi-mode real options models. The fourth section presents the computation procedure of a traditional crisp-stochastic multi-mode real options (CSMMRO) model. The partial fuzzy-stochastic multi-mode real option (PFSMMRO) model and generalised fuzzy-stochastic multi-mode real option (GFSMMRO) model procedures are included in the fifth section. Finally, the last section presents the stylised illustrative example and computation of operational flexibility. The multi-mode real option values of three chosen model types (CSMMRO, PFSMMRO, GFSMMRO) are calculated, including comparison and evaluation.

2. Fuzzy-stochastic methodology description under fuzzy numbers

Hybrid fuzzy-stochastic models can be constructed in various conceptions. The essential fuzzy-stochastic elements, which are very useful from an application point of view, are the following: (i) fuzzy set, (ii) fuzzy number and T-number, (iii) $\varepsilon-cut$ (iv), fuzzy-random variable, (v) extension principle, (vi) decomposition principle, (vii) fuzzy-probability function.

Definition 1. *A fuzzy set (depicted with a tilde) is commonly defined by a membership function*

(μ) as represented by E^n (Euclid n-dimensional space, n > 1) to a set of E^1, especially to the interval of [0;1], $\widetilde{s} \equiv \mu_{\widetilde{s}}(x)$, where $\widetilde{s}$ is the fuzzy set, x is the vector and $x \in X \subset E^n$, $\mu_{\widetilde{s}}(x)$ is the membership function.

It is well known that many fuzzy set types exist. The most common type of fuzzy set meeting the specified preconditions of normality, convexity and continuity with the upper semi-continuous membership function is the very well-known *fuzzy number*, see, e.g. (Dubois & Prade 1980, p. 10; Wu 2005, p. 91; Zadeh 1965, p. 339). The set of the fuzzy number is depicted $F_N(E^n)$. One of the most widely applied fuzzy number types is the *T-number*.

Definition 2. *A fuzzy set meeting preconditions of normality, convexity, continuity with the upper semi-continuous membership function and closeness and being defined as the quadruple* $\widetilde{s} = (s^L, s^U, s^\alpha, s^\beta)$, *where* $\phi(x)$ *is a non-decreasing function and* $\psi(x)$ *is a non-increasing function, is defined as follows,*

$$\widetilde{s} \equiv \mu_{\widetilde{s}}(x) = \left\{ \begin{array}{c} 0 \; for \; x \leqslant s^L - s^\alpha; \; \phi(x) \; for \; s^L - s^\alpha < x < s^L; \\ 1 \; for \; s^L \leqslant x \leqslant s^U; \; \psi(x) \; for \; s^U < x < s^U + s^\beta; \\ 0 \; for \; x \geqslant s^U + s^\beta \end{array} \right\}$$

and is thus called the *T-number*. See e. g. (Carlsson & Fuller, 2003, p. 300; Ramík & Rommelfanger, 1996, p. 78). Let us denote the set of T-numbers by $F_T(E)$.

Definition 3. *The* $\varepsilon-cut$ *of the fuzzy set* $\widetilde{s}^\varepsilon$, *depicted* $\widetilde{s}^\varepsilon$, *is defined as follows.* $\widetilde{s}^\varepsilon = \{x \in E^n; \mu_{\widetilde{s}}(x) \geqslant \varepsilon = [{}^-s^\varepsilon; {}^+s^\varepsilon]$, *where* ${}^-s^\varepsilon = \inf\{x \in E^n; \mu_{\widetilde{s}}(x) \geqslant \varepsilon\}$, ${}^+s^\varepsilon = \sup\{x \in E^n; \mu_{\widetilde{s}}(x) \geqslant \varepsilon\}$ *. See e.g.* (Dubois & Prade, 1980, p. 19, Wu 2005, p. 91; Zadeh, 1975, p. 223). *Remark, the* $\varepsilon-cut$, $\widetilde{s}^\varepsilon$, *of linear T-numbers is calculated as follows,* $\widetilde{s}^\varepsilon = [{}^-s^\varepsilon; {}^+s^\varepsilon] = [s^L - (1-\varepsilon)\cdot s^\alpha; s^U + (1-\varepsilon)\cdot s^\beta]$.

The crucial category in fuzzy-stochastic modelling is the fuzzy-random variable.

Definition 4. *It is said,* $\widetilde{\widetilde{s}}: \Omega \rightarrow F_T(E)$ *is the fuzzy-random variable (depicted with a tilde and a line) if for every* $w \in \Omega$ and $\varepsilon \in]0,1]$, $\widetilde{\widetilde{s}}^\varepsilon_w = \{x: x \in E^n, \widetilde{\widetilde{s}}_w \geqslant \varepsilon\} = [{}^-s^\varepsilon_w; {}^+s^\varepsilon_w]$, *is a random interval (random* $\varepsilon-cut$*). Here* ${}^-s^\varepsilon_w$, ${}^+s^\varepsilon_w$ *are two random variables (or finite, measurable functions). Let us denote the set of the fuzzy-random variables* $FR(\Omega, P)$ *here* $P: \Omega \rightarrow [0, 1]$ *and thus* $\widetilde{\widetilde{s}} \in FR(\Omega, P)$. Puri and Ralescu (1986) *introduced the fuzzy-random variable term, p.* 413. *See other authors, e. g.* (Kacprzyk & Fedrizzi, 1988, p. 66; Kruse & Meyer, 1987, p. 63; Luhandjula, 1996, p. 297; Van Hop, 2007, p.79; Wu, 2005, p. 93).

The definition implies that

a) $\widetilde{\widetilde{s}} = \bigcup_\varepsilon \widetilde{\widetilde{s}}^\varepsilon$, because $\forall\, w \in \Omega$, $\widetilde{\widetilde{s}}_w = \bigcup_\varepsilon \widetilde{\widetilde{s}}^\varepsilon_w$. Here $\widetilde{\widetilde{s}}_w$ is a fuzzy set and $\widetilde{\widetilde{s}}^\varepsilon_w$ is a random interval,
b) supposing $\widetilde{\widetilde{s}}^\varepsilon_w$ is a random interval, then $\widetilde{\widetilde{s}}$ is the fuzzy-random variable, it means $\widetilde{\widetilde{s}} = \bigcup_{\varepsilon,w} \widetilde{\widetilde{s}}^\varepsilon_w$ for every $w \in \Omega$ *and* $\varepsilon \in]0,1]$.

The extension principle is a beneficial and powerful instrument usable for calculating a function of fuzzy sets.

Definition 5. *The extension principle is the sup min composition between fuzzy sets* $\widetilde{r}_1 ... \widetilde{r}_n$ *and* $\widetilde{s} = f(\widetilde{r}_1 ... \widetilde{r}_n)$ *as follows. Let* $f: E^n \rightarrow E^1$, *then, the membership function of a fuzzy set function* $\widetilde{s} = f(\widetilde{r}_1 ... \widetilde{r}_n)$ *is defined by* $\mu_{\widetilde{s}}(y) \equiv \widetilde{s} = \sup_{\substack{x_1 x_n \\ y = f(x_1 ..... x_n)}} \min[\mu_{\widetilde{r}_1}(x_1) ...... \mu_{\widetilde{r}_n}(x_n)]$, $x_i, y \in E^1$ *. See, e.g.* (Dubois & Prade, 1980, p. 72; Zadeh, 1965, p. 339).

In general, an analytic solution, according to the extension principle, is not available. However, assuming a fuzzy set is of a fuzzy number type (T-number as well), then it is possible to solve the function of fuzzy numbers $\widetilde{s} = f(\widetilde{r}_1 ... \widetilde{r}_n)$ following the extension principle by the decomposition principle as the approximate procedure of $\varepsilon-cut$*.*

Definition 6. *The Decomposition principle (Resolution identity) is defined as follows,*

$\mu_{\widetilde{s}}(y) = \sup_\varepsilon\{\varepsilon \cdot I_{\widetilde{s}}; y \in \widetilde{s}^\varepsilon\}$ for any $y \in E^n$ and $\varepsilon \in [0;1]$, where $\widetilde{s}^\varepsilon = [{}^-s^\varepsilon; {}^+s^\varepsilon]$ is ε-cut, y = f(x), ${}^-s^\varepsilon(x) = \min_{x \in \widetilde{x}^\varepsilon \subset E^n} f(x)$, ${}^+s^\varepsilon(x) = \max_{x \in \widetilde{x}^\varepsilon \subset E^n} f(x)$. Here $I_{\widetilde{s}}$ is the characterisation function, $I_{\widetilde{s}} = \left\{ \begin{array}{l} 1 \; if \; y \in [{}^-s^\varepsilon; {}^+s^\varepsilon] \\ 0 \; if \; y \notin [{}^-s^\varepsilon; {}^+s^\varepsilon] \end{array} \right\}$. See e. g. (Ralescu, 1992, p. 309; Viertl, 1996, p. 10; Wu, 2005, p. 92; Zadeh (1971, p. 180).

The generalised application possibility is the advantage of the decomposition principle, especially for fuzzy numbers (T-numbers).

The function of fuzzy numbers $\widetilde{s} = f(\widetilde{r}_1 ... \widetilde{r}_n)$ whilst applying Definition 6 could be transformed and solved by several mathematical programming problems for stated ε in this way.

Problem P1. *max s* $= {}^+s^\varepsilon$, *or min s* $= {}^-s^\varepsilon$,

s.t. $x_i \in [{}^-x^\varepsilon_i; {}^+x^\varepsilon_i]$ for $i \in \{1; 2......n\}$, and $\varepsilon \in [0;1]$, where $s = f(x_1 .... x_n)$.

The following definition helps evaluate and compute fuzzy-random variables' functions and construct the fuzzy probabilities and the fuzzy-expected values.

Definition 7. *Let us suppose that* $\widetilde{\widetilde{s}} \in FR(\Omega, P)$ *and* $s^\varepsilon_w = [{}^-s^\varepsilon_w; {}^+s^\varepsilon_w]$, $\varepsilon \in [0;1]$, $x \in E^n$. *Let* ${}^-g^\varepsilon_w$ (resp. ${}^+g^\varepsilon_w$) *be a function* ${}^-s^\varepsilon_w$ (resp. ${}^+s^\varepsilon_w$). *Then we call* $\widetilde{\widetilde{g}}(y) = \sup_\varepsilon\{\varepsilon \cdot I_{\widetilde{\widetilde{g}}}; y \in \widetilde{\widetilde{g}}^\varepsilon\}$ *for any* $y \in E^n$ and $\varepsilon \in [0;1]$, *the fuzzy probability function or fuzzy-expected function* $\widetilde{s}$. *See, e.g.* (Puri & Ralescu, 1986, p. 415).

Definition 8. *Application of the Decomposition principle for a function of fuzzy numbers allows expressing selected fuzzy operations* $\widetilde{*}$ *among fuzzy numbers directly by intervals, as follows:*

$$\widetilde{w} = \widetilde{s} \widetilde{*} \widetilde{r} = \bigcup_\varepsilon \varepsilon(w^\varepsilon) = \bigcup_\varepsilon \varepsilon(s^\varepsilon * r^\varepsilon)$$

Fuzzy addition, $s^\varepsilon + r^\varepsilon = [{}^-s^\varepsilon + {}^-r^\varepsilon; {}^+s^\varepsilon + {}^+r^\varepsilon]$.

Fuzzy subtraction, $s^\varepsilon - r^\varepsilon = [{}^-s^\varepsilon - {}^+r^\varepsilon; {}^+s^\varepsilon - {}^-r^\varepsilon]$.

Fuzzy scalar product, $k \cdot s^\varepsilon = [k \cdot {}^-s^\varepsilon; k \cdot {}^+s^\varepsilon]$ for $k \geqslant 0$, $k \cdot s^\varepsilon = [k \cdot {}^+s^\varepsilon; k \cdot {}^-s^\varepsilon]$ for $k < 0$.

Fuzzy multiplication, $s^\varepsilon \cdot r^\varepsilon = [{}^-s^\varepsilon \cdot {}^-r^\varepsilon; {}^+s^\varepsilon \cdot {}^+r^\varepsilon]$ for $\widetilde{s} > 0, \widetilde{r} > 0$, $s^\varepsilon \cdot r^\varepsilon = [{}^-s^\varepsilon \cdot {}^+r^\varepsilon; {}^+s^\varepsilon \cdot {}^-r^\varepsilon]$ for $s < 0$, $\widetilde{r} > 0$, $s^\varepsilon \cdot r^\varepsilon = [{}^+s^\varepsilon \cdot {}^+r^\varepsilon; {}^-s^\varepsilon \cdot {}^-r^\varepsilon]$ for $s < 0$, $r < 0$.

Fuzzy division, $s^\varepsilon : r^\varepsilon = [{}^-s^\varepsilon : {}^+r^\varepsilon; {}^+s^\varepsilon : {}^-r^\varepsilon]$ for $s > 0$, $r > 0$, $s^\varepsilon : r^\varepsilon = [{}^+s^\varepsilon : {}^+r^\varepsilon; {}^-s^\varepsilon : {}^-r^\varepsilon]$ for $s < 0$, $\widetilde{r} > 0$, $s^\varepsilon : r^\varepsilon = [{}^+s^\varepsilon : {}^-r^\varepsilon; {}^-s^\varepsilon : {}^+r^\varepsilon]$ for $s < 0$, $r < 0$.

Fuzzy max, $max(s^\varepsilon) = [max\, {}^-s^\varepsilon; max\, {}^+s^\varepsilon] = [maxmin\, {}^-s^\varepsilon; maxmax\, {}^+s^\varepsilon] = [-max - ({}^-s^\varepsilon); max({}^+s^\varepsilon)]$.

Here $\widetilde{s} > 0$ is a positive fuzzy number, positive $\widetilde{s} = \{x;$ *for which* $\mu_{\widetilde{s}} \geqslant 0\}$ and simultaneously $x \in E^+$(set of positive numbers), negative $\widetilde{s} = \{x;\ for\ which\ \mu_{\widetilde{s}} \geqslant 0\}$ and simultaneously $x \in E^-$(set of negative numbers). See e. g. (Wang & Qiao 1993, p. 297).

We can show that a result of the fuzzy operation of addition, subtraction, and scalar product for linear T-numbers is also a linear T-number. For other operations, it is not valid, and the results are fuzzy numbers. Sometimes the approximation is applied so that the results are also linear T-numbers (see e. g. (Dubois & Prade, 1980, p. 57; Gao & Zhang, 2009, p. 334).

3. Multi-mode real options models

We assume a generalised multi-mode real options valuation model with the real option valuation function. We suppose the knowledge of the development of random cash flow and the terminal value. Furthermore, we generally suppose the possibility of transferring (switching) among more than two modes and all possible transfers are feasible (reversibility). Transferring is connected with non-symmetrical switching cost and revenues, respectively. Valuation approaches come out of the stochastic dynamic programming on Bellman's optimisation principle because the procedure fulfils the required assumptions. The multi-mode real options are of the American options type, and discrete binomial models are applied. Replication valuation in a risk-neutral world is supposed.

Examples of the generalised multi-modes real options models formulation are introduced in (Kulatilaka (1988); Kulatilaka & Trigeorgis, 1994, p. 791; Guthrie, 2009, p. 226; Zmeškal, 2008, p. 261).

The replication strategy valuation in the risk-neutral world is generalised valuation conception described e. g. in (Guthrie, 2009, p. 29; Musiela & Rutkowski, 2011, p. 15; Schwartz & & Trigeorgis, 2004, p. 47; Trigeorgis & Mason, 1987, p. 14). Various processes can calibrate this model. For example, the successful Cox, Ross and Rubinstein (1979) method can be used if the underlying asset follows the multiplicative binomial process.

$$V^m_{t,s} = \max_{q\in S}\left\{x^q_{t,s} + c_{m,q} + (1+R)^{-1}\cdot\left[p_{t,s}\cdot V^q_{t+1,s+1} + \left(1-p_{t,s}\right)\cdot V^q_{t+1,s-1}\right]\right\},\ t\in[0;T-1]. \tag{2}$$

The applied computation procedure of generalised multi-mode real options, according to notation in Zmeškal (2008), is the following. The backward induction means that the state's values in modes are calculated backwardly for every time step. Firstly, the terminal value $V^m_{T,s}$ is stated (calculated). Secondly, recurrent valuation for other steps according to (1) is computed with risk-neutral probability.

$$V^m_{t,s} = \max_{q\in S}\left\{x^q_{t,s} + c_{m,q} + (1+R)^{-1}\cdot\left[p_{t,s}\cdot V^q_{t+1,s+1} + \left(1-p_{t,s}\right)\cdot V^q_{t+1,s-1}\right]\right\} \tag{1}$$

where $p_{t,s}$ and $1-p_{t,s}$ are the risk-neutral probabilities of up/down movement respectively, $p_{t,s} = \frac{x_{t,s}\cdot(1+R)-x_{t+1,s-1}}{x_{t+1,s+1}-x_{t+1,s-1}}$. The no-arbitrage condition indicates $p_{t,s}\in[0;1]$, hence $x_{t,s}\cdot(1+R)\in\left[x_{t+1,s-1};x_{t+1,s+1}\right]$, and $\left(x_{t+1,s+1}-x_{t+1,s-1}\right).>0$.

Here, m is the initial mode, q is the subsequent mode, s is a random state, T is terminal time, t is the time step, $V^q_{T,s}$ are known terminal values, $V^m_{t,s}$, $V^q_{t,s}$ are calculated values, $x^q_{t,s}$ is cash-flow, R is a risk-free rate. The symbol $c_{m,q}$ depicts switching cost and revenues, respectively, between two modes, where $c_{m,q}\geqslant 0$ means revenue, $c_{m,q}<0$ indicates cost.

A constant risk-free rate is supposed for simplicity. However, there is no difficulty in applying a variable risk-free rate according to the given yield curve. Applying the backwards induction procedure, we get the initial real option value V^m_0 for the mode m.

4. Crisp-stochastic multi-modes real options (CSMMRO) model

Here, the dynamic binomial crisp-stochastic model is described. Input data (terminal value, risk-free rate, switching cost) are crisp numbers, and underlying cash-flows are crisp random numbers. The basic conception is given by underlying cash-flow modelling by a binomial model and using replication no-arbitrage calculation of the risk-neutral probability. So, the volatility and risk-neutral probability can change every time and state. The valuation procedure of CSMRO is performed in the following steps.

Valuation procedure of CSMMRO model

(i) Cash flow modelling so as an underlying asset by a binomial model

a) A common approach by expert estimation or calibration with non-proportional volatility.
b) An example of the specific case of calibration by a proportional multiplicative (Brown's geometrical) process is the Cox, Ross and Rubinstein (1979) calibration. Here $x^q_{t+1,s+1} = x^q_{t,s}\cdot U$; $x^q_{t+1,s-1} = x^q_{t,s}\cdot D$, where $U = e^{\sigma\cdot\sqrt{\Delta t}}$, $D = e^{-\sigma\cdot\sqrt{\Delta t}}$, and U, D respectively is the up and down coefficient and σ is volatility.

(ii) Stating (computing) crisp terminal values at a time T

The input value $V^m_{T,s}$ indicates the terminal value, where s is the state and q is the mode.

(iii) Valuation formula for other time steps under the backward induction recurrent procedure

(iv) The real option value is calculated by the recurrent procedure of the binomial tree to the initial values for states s and initial modes m and subsequent modes m of the particular period.

Here $p_{t,s}$ and $\left(1-p_{t,s}\right)$ is the risk-neutral probability of up/down movement respectively, $p_{t,s} = \frac{x_{t,s}\cdot(1+R)-x_{t+1,s-1}}{x_{t+1,s+1}-x_{t+1,s-1}}$. The no-arbitrage condition indicates $p_{t,s}\in[0;1]$, hence $x_{t,s}\cdot(1+R)\in\left[x_{t+1,s-1};x_{t+1,s+1}\right]$, and $\left(x_{t+1,s+1}-x_{t+1,s-1}\right).>0$.

The valuation formula V^m_0 is the initial value.

(v) Identification of the mode decision (optimal control mode $Q^m_{t,s}$)

For every initial mode, state, and time, optimal switching modes are calculated.

$$Q^m_{t,s} = \underset{q\in S}{argmax}\left\{x^q_{t,s} + c_{m,q} + (1+R)^{-1}\cdot\left[p_{t,s}\cdot V^q_{t+1,s+1} + \left(1-p_{t,s}\right)\cdot V^q_{t+1,s-1}\right]\right\}. \tag{3}$$

5. Fuzzy-stochastic multi-mode real options model

We suppose the dynamic binomial fuzzy-stochastic model with fuzzy numbers; see Definition 1. The Decomposition principle, according to Definition 6, is applied. Moreover,$\varepsilon-cut$ due to Definition 3 and operations between fuzzy numbers were used according to Definition 8. We investigated two types of fuzzy-stochastic models: the partial fuzzy-stochastic multi-mode real options (PFSMMRO) model with only fuzzy switching cost and the generalised fuzzy-stochastic multi-mode real options (GFSMMRO) model with all fuzzy input data and fuzzy-stochastic distribution of the underlying asset.

5.1. Partial fuzzy-stochastic multi-mode real options (PFSMMRO) model

Analogically to the CSMMRO model, underlying cash flow is modelled by a binomial model, and a replication no-arbitrage calculation of the risk-neutral probability is applied. So, the volatility and risk-neutral probability can change every time and state. In the model, switching costs $\widetilde{c}_{m,q}$ are fuzzy numbers, other input data (terminal value $V^q_{T,s}$, risk-free rate R) are crisp numbers, and underlying cash-flows $x^q_{t,s}$ are crisp random numbers. The risk-neutral probability is crisp as well because of crisp underlying cash flow. The switching costs are in additive relation; therefore, only $\varepsilon-cut$ of the fuzzy addition operation is applied due to Definition 8. Solution conception uses the Decomposition principle, and therefore, the following procedure is represented by $\varepsilon-cut$ and contains the following steps.

Valuation procedure of PFSMMRO model (for given $\varepsilon-cut$)

(i) Cash flow modelling so as an underlying asset by a binomial model.

a) A common approach by expert estimation or calibration with non-proportional volatility
b) An example of the specific case of calibration by a multiplicative proportional (Brown's geometrical) process is the Cox, Ross and Rubinstein (1979) calibration. Here, $x^q_{t+1,s+1} = x^q_{t,s}\cdot U$; $x^q_{t+1,s-1} = x^q_{t,s}\cdot D$, and $U = e^{\sigma\cdot\sqrt{\Delta t}}$, $D = e^{-\sigma\cdot\sqrt{\Delta t}}$, where U and D respectively is the up and down coefficient and σ is volatility.

(ii) Stating (computing) crisp terminal value at a time T

The input value $V^m_{T,s}$ indicates the terminal value, where s is the state and m is the mode.

(iii) Valuation formula for other time steps $t \in [0; T-1]$ under the backward induction recurrent procedure

$$ {}^{-}V^{\varepsilon,m}_{t,s} = \max_{q\in S}\left\{x^q_{t,s} + {}^{-}c^{\varepsilon}_{m,q} + (1+R)^{-1}\cdot\left[p_{t,s}\cdot{}^{-}V^{\varepsilon,q}_{t+1,s+1} + \left(1-p_{t,s}\right)\cdot{}^{-}V^{\varepsilon,q}_{t+1,s-1}\right]\right\}, \tag{4} $$

$$ {}^{+}V^{\varepsilon,m}_{t,s} = \max_{q\in S}\left\{x^q_{t,s} + {}^{+}c^{\varepsilon}_{m,q} + (1+R)^{-1}\cdot\left[p_{t,s}\cdot{}^{+}V^{\varepsilon,q}_{t+1,s+1} + \left(1-p_{t,s}\right)\cdot{}^{+}V^{\varepsilon,q}_{t+1,s-1}\right]\right\}. \tag{5} $$

Here $p_{t,s}$ and $\left(1-p_{t,s}\right)$ is generally the risk-neutral probability of up/down movement respectively $p_{t,s} = \frac{x_{t,s}\cdot(1+R)-x_{t+1,s-1}}{x_{t+1,s+1}-x_{t+1,s-1}}$. The no-arbitrage condition indicates $p_{t,s}\in[0;1]$, hence $x_{t,s}\cdot(1+R) \in \left[x_{t+1,s-1}; x_{t+1,s+1}\right]$, and $\left(x_{t+1,s+1} - x_{t+1,s-1}\right). > 0$.

For the solution, the operations of the fuzzy addition, the fuzzy scalar product and the fuzzy max are applied; see Definition 8.

The initial real option value is $V^{\varepsilon,m}_0 = \left[{}^{-}V^{\varepsilon,m}_0, {}^{+}V^{\varepsilon,m}_0\right]$.

(iv) Identification of the mode decision (optimal control mode $Q^{\varepsilon,m}_{t,s}$)

For every initial mode, state, time and $\varepsilon-cut$ optimal switching modes are calculated.

$$ {}^{-}Q^{\varepsilon,m}_{t,s} = \underset{q\in S}{argmax}\left\{x^q_{t,s} + {}^{-}c^{\varepsilon}_{m,q} + (1+R)^{-1}\cdot\left[p_{t,s}\cdot{}^{-}V^{\varepsilon,q}_{t+1,s+1} + \left(1-p_{t,s}\right)\cdot{}^{-}V^{\varepsilon,q}_{t+1,s-1}\right]\right\}, \tag{6} $$

$$ {}^{+}Q^{\varepsilon,m}_{t,s} = \underset{q\in S}{argmax}\left\{x^q_{t,s} + {}^{+}c^{\varepsilon}_{m,q} + (1+R)^{-1}\cdot\left[p_{t,s}\cdot{}^{+}V^{\varepsilon,q}_{t+1,s+1} + \left(1-p_{t,s}\right)\cdot{}^{+}V^{\varepsilon,q}_{t+1,s-1}\right]\right\}. \tag{7} $$

For the solution, the operations of the fuzzy addition, the fuzzy scalar product and the fuzzy max are applied; see Definition 8.

5.2. Generalised fuzzy-stochastic multi-mode real options (GFSMMRO) model

A binomial model models an underlying cash flow, and a replication no-arbitrage calculation of the risk-neutral probability is applied in a fuzzy-stochastic environment. So, the volatility and risk-neutral probability can change every time and state. Optimisation problem P2 is applied because the fuzzy number is different in every state and time, and there is no proportionality. No strict stochastic process is used, only binomial model representation. It means volatility and risk-neutral probability is a fuzzy number changing due to state and time. The fuzzy risk-neutral probability is impossible to calculate explicitly, but it is the implicit optimal result of the optimisation problem.

Input parameters (switching cost $\widetilde{c}_{m,q}$, terminal values $\widetilde{V}^q_{T,s}$, and constant risk-free-rate $\widetilde{R}$) are fuzzy numbers, and fuzzy-random numbers represent underlying cash-flows $\widetilde{x}^q_{t,s}$. The risk-neutral fuzzy probability is considered because underlying cash flows are fuzzy random numbers. Operations fuzzy addition, fuzzy multiplication, fuzzy division and fuzzy max are applied due to Definition 8. Moreover, solution conception uses the Decomposition principle with optimisation Problem P1.

The following procedure is represented by $\varepsilon-cut$ and contains the following steps.

Valuation procedure of GFSMMRO model (for given $\varepsilon-cut$)

(i) Fuzzy cash flow modelling so as an underlying asset by a binomial model

a) The common approach is given by expert estimation and calibration of non-proportional fuzzy cash flow development
b) The example of a specific approach is given by calibration with imprecision represented by fuzzy numbers multiplicative non-proportional process due to the Cox, Ross and Rubinstein (1979) calibration, then $x^{\varepsilon,q}_{t+1,s+1} = \left[x^{\varepsilon,q}_{t,s}\cdot{}^{-}U^{\varepsilon}; x^{\varepsilon,q}_{t,s}\cdot{}^{+}U^{\varepsilon}\right]$, $x^{\varepsilon,q}_{t+1,s-1} = \left[x^{\varepsilon,q}_{t,s}\cdot{}^{-}D^{\varepsilon}; x^{\varepsilon,q}_{t,s}\cdot{}^{+}D^{\varepsilon}\right]$, and $U^{\varepsilon} = e^{\sigma^{\varepsilon}\cdot\sqrt{\Delta t}}$, $D^{\varepsilon} = e^{-\sigma^{\varepsilon}\sqrt{\Delta t}}$, here U^{ε}, D^{ε} and σ^{ε} are $\varepsilon-cut$ of up/down movement respectively and volatility coefficients. Here $\varepsilon-cuts$ are employed due to fuzzy scalar product due to Definition 8.

(ii) Stating (computing) fuzzy terminal values at a time T

The input value $V^{\varepsilon,q}_{T,s} \in \left[{}^{-}V^{\varepsilon,q}_{T,s}; {}^{+}V^{\varepsilon,q}_{T,s}\right]$ indicates the fuzzy terminal value, where the symbol ε means a cut, s is the state, and m is the mode.

(iii) The formula for other time steps $t\in[0;T-1]$ under the backward induction recurrent procedure

$$ {}^{-}V^{\varepsilon,m}_{t,s} = \max_{q\in S}\left[{}^{-}c^{\varepsilon}_{m,q} + {}^{-}F^{\varepsilon,q}_{t,s}\right] \tag{8} $$

$$ {}^{+}V^{\varepsilon,m}_{t,s} = \max_{q\in S}\left[{}^{+}c^{\varepsilon}_{m,q} + {}^{+}F^{\varepsilon,q}_{t,s}\right] \tag{9} $$

here ${}^{-}F^{\varepsilon,q}_{t,s}$, ${}^{+}F^{\varepsilon,q}_{t,s}$ are calculated values through the optimisation Problem P2.

Problem P2. ${}^{-}F^{\varepsilon,q}_{t,s} = \min G^{\varepsilon,q}_{t,s}$ or ${}^{+}F^{\varepsilon,q}_{t,s} = \max G^{\varepsilon,q}_{t,s}$

s.t. $x^{\varepsilon,q}_{t,s}\cdot(1+R) \geqslant \left(x^{\varepsilon,q}_{t+1,s+1} - x^{\varepsilon,q}_{t+1,s-1}\right)$, $\left(x^{\varepsilon,q}_{t+1,s+1} - x^{\varepsilon,q}_{t+1,s-1}\right). > 0$,

$x^q_{t,s} \in \left[{}^{-}x^{\varepsilon,q}_{t,s}; {}^{+}x^{\varepsilon,q}_{t,s}\right] \frac{1}{n} x^q_{t+1,s+1} \in \left[{}^{-}x^{\varepsilon,q}_{t+1,s+1}; {}^{+}x^{\varepsilon,q}_{t+1,s+1}\right]$

$x^q_{t+1,s-1} \in \left[{}^{-}x^{\varepsilon,q}_{t+1,s-1}; {}^{+}x^{\varepsilon,q}_{t+1,s-1}\right] V^q_{t+1,s+1} \in \left[{}^{-}V^{\varepsilon,q}_{t+1,s+1}; {}^{+}V^{\varepsilon,q}_{t+1,s+1}\right]$

$V^q_{t+1,s-1} \in \left[{}^{-}V^{\varepsilon,q}_{t+1,s-1}; {}^{+}V^{\varepsilon,q}_{t+1,s-1}\right] R \in \left[{}^{-}R^{\varepsilon}; {}^{+}R^{\varepsilon}\right]$

where $G^{\varepsilon,q}_{t,s} = \left\{x^q_{t,s} + (1+R)^{-1}\cdot\left[p^q_{t,s}\cdot V^q_{t+1,s+1} + \left(1-p^q_{t,s}\right)\cdot V^q_{t+1,s-1}\right]\right\}$ $p^q_{t,s} = \frac{x^{\varepsilon,q}_{t,s}\cdot(1+R) - x^{\varepsilon,q}_{t+1,s-1}}{x^{\varepsilon,q}_{t+1,s+1} - x^{\varepsilon,q}_{t+1,s-1}}$.

For the solution, fuzzy addition due to Definition 8 is used. Furthermore, the convex optimisation Problem P2 due to Decomposition principle for $\varepsilon-cut$ is applied, Definition 8. Convexity stems from closed intervals of variables, and that particular functions (constraints, objective) are continuous non-decreasing or non-increasing functions,

The initial real option value at the beginning is $V_0^{\varepsilon,m} = \left[{}^{-}V_0^{\varepsilon,m},\ {}^{+}V_0^{\varepsilon,m}\right]$.

(iv) Identification of the mode decision (optimal control modes $Q_{t,s}^{\varepsilon,m}$)

For every initial mode, state, time and $\varepsilon-cut$ optimal switching modes are calculated.

$$ {}^{-}Q_{t,s}^{\varepsilon,m} = \underset{q\in S}{argmax}\left[{}^{-}c_{m,q}^{\varepsilon} + {}^{-}F_{t,s}^{\varepsilon,q}\right] \tag{10} $$

$$ {}^{+}Q_{t,s}^{\varepsilon,m} = \underset{q\in S}{argmax}\left[{}^{+}c_{m,q}^{\varepsilon} + {}^{+}F_{t,s}^{\varepsilon,q}\right] \tag{11} $$

here ${}^{-}F_{t,s}^{\varepsilon,q}$, ${}^{+}F_{t,s}^{\varepsilon,q}$ are calculated values through the optimisation Problem P2.

Detailed procedure of computation the generalised (GFSMMRO) model value is presented in Algorithm 1.

Algorithm 1 Generalised fuzzy-stochastic multi-modes real option (GFSMMRO) model

level of cut ε, time t, state s, modes m, q
Inputs: fuzzy cash-flow matrices $\widetilde{X}^m$, fuzzy switching cost matrix $\widetilde{C}$,
fuzzy terminal values vectors $\widetilde{V}_T^m$, risk-free fuzzy rate $\widetilde{R}$, number of $\varepsilon-cuts$ N
Outputs: the fuzzy value of the modes $\widetilde{V}_0^m$,
the optimal control switching modes matrices ${}^{-}Q_{t,s}^{\varepsilon,m}$, ${}^{+}Q_{t,s}^{\varepsilon,m}$
for $k \in \{0;1;2......N-1\}$, $\varepsilon \leftarrow \left(0 + k\cdot\frac{1}{N-1}\right)$
compute $\varepsilon-cuts$ of ${}^{-}c_{m,q}^{\varepsilon} \in {}^{-}C^{\varepsilon}$, ${}^{+}c_{m,q}^{\varepsilon} \in {}^{+}C^{\varepsilon}$,
${}^{-}x_{t,s}^{\varepsilon,m} \in {}^{-}X^{\varepsilon,m}$, ${}^{+}x_{t,s}^{\varepsilon,m} \in {}^{+}X^{\varepsilon,m}$, ${}^{-}V_{T,s}^{\varepsilon,m} \in {}^{-}V_T^{\varepsilon,m}$, ${}^{+}V_{T,s}^{\varepsilon,m} \in {}^{+}V_T^{\varepsilon,m}$, ${}^{-}R^{\varepsilon}$, ${}^{+}R^{\varepsilon}$
for $t = \{T, T-1.....0\}$
for $m \in S$
for $s \in W$
if $t = T$ **then** state (calculate) ${}^{-}V_{T,s}^{\varepsilon,m}$, ${}^{+}V_{T,s}^{\varepsilon,m}$
else compute ${}^{-}F_{t,s}^{\varepsilon,q}$, and ${}^{+}F_{t,s}^{\varepsilon,q}$ due to **Problem P2**, furthermore
${}^{-}V_{T,s}^{\varepsilon,m}$, and ${}^{+}V_{T,s}^{\varepsilon,m}$ according to Eq. (8) and Eq. (9)
${}^{-}Q_{t,s}^{\varepsilon,m}$, and ${}^{+}Q_{t,s}^{\varepsilon,m}$ according to Eq. (10) and Eq. (11)
End
End
End
End

6. Application of stylised illustrative example of fuzzy-stochastic multi-mode real options models

The generalised fuzzy-stochastic multi-mode real options model can be applied for various real options problems. One of the reversible ones is the operation flexibility problem with switching among operational (technological) modes. Strategic investment of optimal modes compositions is a crucial problem. Empirical studies and applications concerning power plants units switching are e. g. in Glensk and Madlener (2018), Marreco and Carpio (2006), Varympopiotis, Tolis and Rentizelas (2014), and operational, technological units in the metallurgical sector, see Bastian-Pinto, Brandão and Ozório (2016). By the real options methodology, a value of various combinations of operational modes is calculated and compared. The value of flexibility is computed, and the best initial mode and ranking of the initial operational modes are investigated. The optimal composition of operational (technological) modes considering investment expenditures is analysed.

The stylised fuzzy-stochastic example is focused on operational flexibility problems derived from the above-mentioned empirical studies in a simplified way. The strategic investment is analysed and justified. The influence of chosen fuzzy-stochastic uncertainty (crisp-stochastic, partial fuzzy -stochastic and generalised fuzzy-stochastic) is investigated. Analogically to Varympopiotis, Tolis and Rentizelas (2014), a combination of three fuel/procedure power plant units (modes) with full reversibility are considered: modes A, B, and C. It is supposed that all modes are constructed and operating. And 5 years remain to the closing time all operational modes. The terminal value of operational modes equals a scrap value. Change in prices and demand influence cash flows and is a reason to switch between modes. Underlying cash-flows connected with particular modes and non-symmetrical switching costs among operational modes are stated. The strategic decision consists of choosing the optimal initial operational mode and comparison with other modes values. The generalised fuzzy-stochastic multi-mode real options model is applied.

Table 1
Input data of the crisp and the fuzzy switching cost.

Switching cost $c_{m,q}$		Crisp numbers s			T-numbers $\widetilde{s} = \left[s^L, s^U, s^{\alpha}, s^{\beta}\right]$		
		Subsequent mode q			Subsequent mode q		
		A	B	C	A	B	C
Initialmode m	A	0	−4	−3	0	−5; −3;2;2	−4;-2;1;2
	B	−6	0	−7	−7;-5;3;2	0	−8;-6;1;1
	C	−8	−9	0	−9;-7;3;2	−10;-8;4;4	0

The problem is formulated as the American binomial real option model applying fuzzy-stochastic methodology and procedures described in Chapters 4 and 5. Three alternatives of the multi-mode real option models depending on a fuzzy-stochastic uncertainty are compared: the crisp-stochastic CSMMRO model with crisp input data (Alternative 1), the partial fuzzy-stochastic PFSMMRO model with only fuzzy switching cost (Alternative 2) and the generalised fuzzy-stochastic GFSMMRO model with all fuzzy input data (Alternative 3). Fuzzy input data are in the form of the linear T-numbers $\widetilde{s} = \left[s^L, s^U, s^{\alpha}, s^{\beta}\right]$. The $\varepsilon-cut$ of the fuzzy set $\widetilde{s}^{\varepsilon}$ is calculated for linear T-numbers as follows, $\widetilde{s}^{\varepsilon} = \left[{}^{-}s^{\varepsilon},\ {}^{+}s^{\varepsilon}\right] = \left[s^L - (1-\varepsilon)\cdot s^{\alpha}; s^U + (1-\varepsilon)\cdot s^{\beta}\right]$, see remark of Definition 3.

6.1. Input data of the models

We suppose in the example that the inputs are given imprecisely as linear T-numbers or crisply as crisp numbers.

The following variables for the stated modes (A, B, C), states and time are given imprecisely: fuzzy switching cost $\widetilde{c}_{m,q}$, fuzzy-random underlying cash-flows $\widetilde{x}_{t,s}^{q}$, fuzzy scrap values $\widetilde{V}_{T,s}^{m}$, and constant fuzzy risk-free-rate $\widetilde{R}$. The fuzzy values are indicated by the linear T-numbers. The Appendix shows input underlying cash flow elements, $x_{t,s}^{q,L}, x_{t,s}^{q,U}$, and scrap values elements $V_{T,s}^{q,L}$ $V_{T,s}^{q,U}$, which are constant for all modes and states because of closing all operating modes. Simplistically, the spreads are stated identically for all fuzzy input cash-flows, $x_{t,s}^{q,\alpha} = 0.8$ $x_{t,s}^{q,\beta} = 0.5$, and the scrap values, $V_{T,s}^{q,\alpha} = 0.8$, $V_{T,s}^{q,\beta} = 0.5$. Fuzzy switching costs (T-numbers) are introduced in Table 1. Input fuzzy risk-free rate is constant for all years $\widetilde{R} = [0.1; 0.105; 0.01; 0.015]$, initial fuzzy cash-flows per modes $\widetilde{x}_0^A = [9.8; 10.3; 0.8; 0.5]$, $\widetilde{x}_0^B = [11.3; 11.8; 0.8; 0.5]$, and $\widetilde{x}_0^C = [8.3; 8.8; 0.8; 0.5]$.

Crisp input data are for the stated modes (A, B, C), states and time the following: crisp switching costs $c_{m,q}$, crisp underlying cash-flows $x_{t,s}^{q}$, crisp scrap values $V_{T,s}^{q}$, and constant fuzzy risk-free-rate $\widetilde{R}$. Crisp switching costs presents Table 1. Underlying cash flows development $x_{t,s}^{q}$ and constant scrap values $V_{T,s}^{q}$ are in the Appendix. Values of crisp initial cash-flows are $x_0^A = 10$, $x_0^C = 11.5$, and $x_0^C = 8.5$, crisp risk-free rate $R = 0.1$.

Term switching cost reflects the expenditure (negative value) and revenue (positive value) connected with the transition between two modes. This parameter significantly influences decision-making and optimal control. In the example, all transitions are enabled (reversibility). Crisp and fuzzy switching costs are introduced in Table 1.

Table 2
Results of initial value per mode for fuzzy PFSMMRO and crisp CSMMRO models.

T-number	Mode A		Mode B		Mode C	
ε-cut	$^{-}V_0^A$	$^{+}V_0^A$	$^{-}V_0^B$	$^{+}V_0^B$	$^{-}V_0^C$	$^{+}V_0^C$
1	71.1276	73.1276	76.1276	76.1276	66.1276	68.1276
0.75	70.6276	73.6276	76.1276	76.1276	65.1276	69.1276
0.5	70.1276	74.1276	76.1276	76.1276	64.1276	70.1276
0.25	69.6276	74.6276	76.1276	76.1276	63.1276	71.1276
0	69.1276	75.1276	76.1276	76.1276	62.1276	72.1276
Crisp	72.1276		76.1276		67.1276	

6.2. Computation and results description

The models, all alternatives, were computed by the Matlab software. The function fmincon solved the optimisation procedure of Problem P2 with the interior-point algorithm. The options of fmincon function were always left default, i.e. we computed with the precision 1E-6. Three initial points are considered: the lower and upper bound values and their mean. The obtained solutions have usually differed in the fifth decimal place. Low dimensional models, especially Alternative 1 and Alternative 2, could be solved by Excel software.

The crisp-stochastic (CSMMRO) model is computed by the procedure introduced in chapter 4. The partial fuzzy-stochastic (PFSMMRO) model is solved by the procedure described in subchapter 5.1. The results in the *ε–cut* form and crisp model (Alternative 1, Alternative 2) for modes A, B and C are presented in Table 2 and Fig. 1.

The procedure, including Algorithm 1 of the generalised fuzzy-stochastic (GFSMMRO) model, is described in subchapter 5.2. Results comparison in the $\varepsilon - cut$ form of this model with the crisp-stochastic (CSMMRO) model (Alternative 1, Alternative 3) for modes A, B and C is presented in Table 3 and Fig. 2.

6.3. Results interpretation and discussion

The results can be used in several ways. Firstly, information about the value of stated initial modes is obtained, modes are ranked, and the best one is chosen. Secondly, the optimal modes switching (control) is obtained for every initial mode due to states, time and $\varepsilon - cut$ is obtained.

In case of crisp CSMMRO model values are the following: mode A = 72.1276, mode B = 76.1276, mode C = 67.1276 (Table 3). Comparing the three investigated alternative results, the best is an initial mode B, the second-best is initial mode A, and the worst is initial mode C.

For fuzzy PSMMRO, GFSMMRO models, the values of initial operational modes are in the form of fuzzy normal numbers. Fuzzy binary relations (e.g. necessity, possibility) is correct to apply. Approximately and simplistically, crisp values by defuzzification methods can be used for initial modes ranking. A defuzzification method is a technical operation to find a representative value from an interval introduced as a simplified tool for ranking fuzzy numbers. Examples of methods are the bisection method, the centre of gravity area, the first of maxima, the last of maxima, the mean of maxima, the centre of maxima (median), the centroid method. For instance, results per the bisection methods in the example are for the PFSMMRO model the following: mode A = 70.6276, mode B = 76.6276, mode C = 64.6276, and results for GFSMMRO model are: mode A = 47.2135, mode B = 53.2135, mode C = 41.2135. It is apparent that the ranking of modes is in the example identical to the crisp CSMMRO model. Again, the best is initial mode B, the second-best is initial mode A, and the worst is initial mode C.

Fuzzy values of initial modes in $\varepsilon - cut$ representation are helpful information. Each $\varepsilon - cut$ presents a so-called aspiration level. For stated aspiration level, initial modes can be ranked. The intervals have to be compared. Three approaches can be used, for instance: optimistic $^{+}V_0^m$, pessimistic $^{-}V_0^m$, and middle $(^{-}V_0^m + {}^{+}V_0^m)/2$.

The presented calculation can be applied to compare different compositions of operating modes value. Furthermore, investment outlays can be taken into account for modes composition choice. And the best composition of operational modes can be justified, assessed and proposed as the best strategic investment decision.

Table 3
Results of initial value per mode for fuzzy GFSMMRO and crisp CSMMRO models.

T-number	Mode A		Mode B		Mode C	
ε-cut	$^{-}V_0^A$	$^{+}V_0^A$	$^{-}V_0^B$	$^{+}V_0^B$	$^{-}V_0^C$	$^{+}V_0^C$
1	57.9244	74.7151	62.9244	77.7151	52.9244	69.7151
0.75	52.3391	79.8634	57.8391	82.3634	46.8391	75.3634
0.5	46.7274	84.9212	52.7274	86.9212	40.7274	80.9212
0.25	41.0888	88.6250	47.5888	90.1250	34.5888	85.1250
0	35.4732	91.3443	42.4732	92.3443	28.4732	88.3443
Crisp	72.1276		76.1276		67.1276	

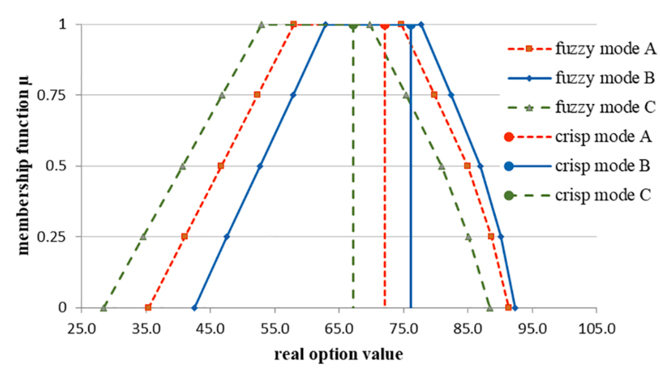

Fig. 2. Results of initial value per fuzzy GFSMMRO and crisp CSMMRO models.

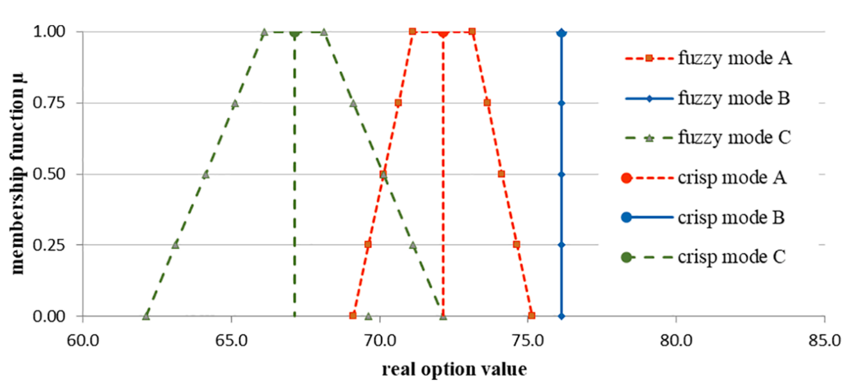

Fig. 1. Results of initial value per mode for PFSMMRO and CSMMRO models.

7. Conclusion

Application of the hybrid fuzzy-stochastic models is undoubtedly a proper conception and procedure in situations where input data is possible to determine only imprecisely. One of the crucial topics in financial modelling and decision-making is the real options methodology. Modelling conditions, decision-making terms, input data imprecision (availability, quality and precision) determine undoubtedly and definitely approaches and models applied.

The paper's objective was to develop and apply the generalised soft multi-mode real options model based on the fuzzy-stochastic methodology. Simultaneously, generality is understood especially in a methodological way. It was assumed that all input parameters are given imprecisely by fuzzy-random numbers, particularly cash-flow development, and by fuzzy numbers, including terminal values, risk-free rate, and switching cost. The proposed model was based on the Decomposition (resolution) principle and $\varepsilon-cut$. The application of the $\varepsilon-cut$ is a suitable and practical approach and procedure. Since the problem outline was convex, the optimisation problems were effectively used. The solution procedure and computation algorithm were proposed, described and applied.

Another intention of the paper was to analyse the impact of impreciseness on real option value. Therefore, a generalised fuzzy-stochastic model was compared with sub-problems, especially with the partial fuzzy-stochastic model and crisp-stochastic model. The models were described, applied and compared, including the stylised illustrative example showing characteristics of the generalised soft multi-mode real options model application possibilities. The stylised example concerns the operational flexibility problems with full reversibility.

The advantage of fuzzy numbers applied is that a crisp number is a subset of a fuzzy number. Therefore, the GFSMMRO model can solve problems with a combination of crisp and fuzzy numbers, and specific models need not be designated. Furthermore, it was shown that simplified procedures and algorithms could solve sub-problems even if it were shown that simplified procedures and algorithms could solve sub-problems. Hence, by the generalised model, the sub-problems should be effectively solved.

The analyst and manager can find the appropriate value intervals and decision-making alternatives. The results should be considered the soft multi-mode real option values respecting input data uncertainty, in both risk and imprecision. Thus, the result can be used for valuation analysis, sensitivity analysis, decision-making, optimal control and interpreted in several ways.

The fuzzy results can provide information about the value and decision-making space, depending on input data imprecision. Fuzzy preference relations or defuzzification methods can be used for decision-making and choosing the best alternative. The model can be used for optimal control under a fuzzy-stochastic environment.

The developed model can suitable reflects uncertainty types, imprecise input data and decision-making conditions. It was confirmed that the imprecision of results mirrors the imprecision of input data. The proposed fuzzy-stochastic multi-mode real options could serve as the generalised sensitivity analysis and decision-making device of the real option value determination.

CRediT authorship contribution statement

Zdeněk Zmeškal: Conceptualization, Methodology, Writing – original draft, Software. **Dana Dluhošová:** Methodology, Supervision, Funding acquisition. **Petr Gurný:** Writing – review & editing, Visualization, Methodology. **Aleš Kresta:** Software, Validation, Formal analysis.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix. Input data (cash-flows, values) of the fuzzy-stochastic (s^L, s^R) and the crisp-stochastic (s) binomial model.

		Mode A							Mode B							Mode C						
		cash-flow x_t						V_5	cash-flow x_t						V_5	cash-flow x_t						V_5
	n/t	0	1	2	3	4	5	5	0	1	2	3	4	5	5	0	1	2	3	4	5	5
crisp S	0	10.000	12.000	14.400	17.280	20.736	24.883	30.0	11.500	13.800	16.560	19.872	23.846	28.616	30.0	8.500	10.200	12.240	14.688	17.626	21.151	30.0
	-2		8.333	10.000	12.000	14.400	17.280	30.0		9.583	11.500	13.800	16.560	19.872	30.0		7.083	8.500	10.200	12.240	14.688	30.0
	-4			6.944	8.333	10.000	12.000	30.0			7.986	9.583	11.500	13.800	30.0			5.903	7.083	8.500	10.200	30.0
	-6				5.787	6.944	8.333	30.0				6.655	7.986	9.583	30.0				4.919	5.903	7.083	30.0
	-8					4.823	5.787	30.0					5.546	6.655	30.0					4.099	4.919	30.0
	-10						4.019	30.0						4.622	30.0		;				3.416	30.0
fuzzy S^L	0	9.800	11.800	14.200	17.080	20.536	24.683	25.0	11.300	13.600	16.360	19.672	23.646	28.416	25.0	8.300	10.000	12.040	14.488	17.426	20.951	25.0
	-2		8.133	9.800	11.800	14.200	17.080	25.0		9.383	11.300	13.600	16.360	19.672	25.0		6.883	8.300	10.000	12.040	14.488	25.0
	-4			6.744	8.133	9.800	11.800	25.0			7.786	9.383	11.300	13.600	25.0			5.703	6.883	8.300	10.000	25.0
	-6				5.587	6.744	8.133	25.0				6.455	7.786	9.383	25.0				4.719	5.703	6.883	25.0
	-8					4.623	5.587	25.0					5.346	6.455	25.0					3.899	4.719	25.0
	-10						3.819	25.0						4.422	25.0						3.216	25.0
fuzzy S^U	0	10.300	12.300	14.700	17.580	21.036	25.183	35.0	11.800	14.100	16.860	20.172	24.146	28.916	35.0	8.800	10.500	12.540	14.988	17.926	21.451	35.0
	-2		8.633	10.300	12.300	14.700	17.580	35.0		9.883	11.800	14.100	16.860	20.172	35.0		7.383	8.800	10.500	12.540	14.988	35.0
	-4			7.244	8.633	10.300	12.300	35.0			8.286	9.883	11.800	14.100	35.0			6.203	7.383	8.800	10.500	35.0
	-6				6.087	7.244	8.633	35.0				6.955	8.286	9.883	35.0				5.219	6.203	7.383	35.0
	-8					5.123	6.087	35.0					5.846	6.955	35.0					4.399	5.219	35.0
	-10						4.319	35.0						4.922	35.0						3.716	35.0

Legend: Value is indicated by time t and state s, where the state is implied $s = t + n$.

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