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Euler Calculus, Euler Integral Transforms, And Combinatorial Species

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Abstract

Euler integration is an integration theory with the Euler characteristic acting as the measure, and similar to classical analysis, it comes equipped with a collection of integral transforms. In this thesis, we focus on two such integral transforms: the persistent homology transform and the Fourier-Sato transform. We prove the invertibility of the former using the technique of Radon transform, and show the connection of the latter to Euler convolution and inner product. We also provide a new way to interpret the Euler integral through a generalization of combinatorial species, which also extends to magnitude homology and configuration spaces.

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ABSTRACT

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Euler integration is an integration theory with the Euler characteristic acting as the measure, and similar to classical analysis, it comes equipped with a collection of integral transforms. In this thesis, we focus on two such integral transforms: the persistent homology transform and the Fourier-Sato transform. We prove the invertibility of the former using the technique of Radon transform, and show the connection of the latter to Euler convolution and inner product. We also provide a new way to interpret the Euler integral through a generalization of combinatorial species, which also extends to magnitude homology and configuration spaces.

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Chapter 1

Introduction

1.1 Euler integral, an overview

The Euler characteristic is a topological invariant that also possesses some measure-theoretic traits. That is, under some mild conditions on the underlying spaces, the Euler characteristic satisfies the inclusion-exclusion principle as well as being multiplicative under Cartesian product. This gives rise to the development of an Euler integration theory, where the Euler characteristic acts as the measure.

The theory of Euler integral can be built from different perspectives. On one side of the coin, there are works by Kashiwara, Schapira [22], and independently by MacPherson [26], where the theory of constructible sheaves is investigated, which is the foundation that Euler integral can be built upon. On the other hand, Euler integral can also be constructed combinatorially [35] [29], and later formalized through the framework of o-minimal structures [9], which guarantees a well-defined notion of Euler characteristic for the spaces of interest.

Euler integration has received a lot of attention in recent years and has been used in a variety of applications, both in theoretical and engineering settings [3][17][5].

There are some natural questions one can raise while investigating the Euler integral and comparing it to classical integration theories. Following are a few of those questions:

1. We knew that the Euler-Radon transform is invertible, which provides us a topological summary of objects in an Euclidean space. Are there other such topological summaries?

2. In classical analysis, the Fourier transform lets us move freely between the frequency domain and the time domain through the means of convolution. That is, it converts convolution on the frequency domain to multiplication on the time domain, and vice versa (by the inverse Fourier transform). This conversion is computationally advantageous, as multiplication is an easier operation to calculate. Is this conversion also true for the Fourier-Sato transform?

3. Treating the integral as a linear functional on an appropriate space of test functions, we can start talking about distributions, i.e. "generalized functions." Is there such a notion in the world of Euler integral and constructible functions? If there is, how do they interact with the integral transforms?

4. Euler characteristic is a generalization of cardinality, or in other words, cardinality is the discrete version of Euler characteristic. There are a lot of known combinatorial constructions involving cardinalities, so can we extend these any of these constructions to Euler characteristic? We can also ask whether Euler characteristic, and Euler integral in particular, can be realized through such a combinatorial process.

The main goal of this thesis is to answer (some of) the above questions.

1.2 Outline of the thesis

Excluding the introduction, the thesis will be divided into seven chapters.

In chapter 2, we will give a quick introduction to sheaves. This will include the homotopy and the derived category of sheaves of R-modules, together with some operations on sheaves. We will also briefly discuss sheaf cohomology from several perspectives. This leads to the development of constructible sheaves, and the chapter will end with the construction of the Grothendieck group of the (derived) category of constructible sheaves on a (stratified) space.

In chapter 3, Euler integral will be formally introduced. First we will be looking at the Euler integral from the combinatorial point of view with connection to o-minimal structures, and then the relationship with sheaves. To be more specific, we will present the isomorphism between the group of constructible functions and the Grothendieck group that we constructed at the end of chapter 2. This isomorphism allows us to transfer the existing operations on sheaves to operations on constructible functions and the Euler integral. Next, we introduce the notion of Euler integral transform and review some famous results by Schapira.

In chapter 4, we prove the invertibility of the persistent homology transform, which is a special case of Euler integral transform. A quick overview of persistent homology is also given.

In chapter 5, the main topic is the Fourier-Sato transform on conic constructible functions, which is another Euler integral transform. We will prove the compatibility of the Fourier-Sato transform with the Euler convolution. We also introduce a notion of (pseudo) inner product on the space of constructible functions and discuss the relationship between this inner product and some integral transforms. To conclude the chapter, we define a notion of tempered distributions on constructible functions, which can be understood as some "generalized" functions. This allows us to extend the Fourier-Sato transform to non-conic constructible functions.

In chapter 6, we develop a new object called generalized species. This is a generalization of combinatorial species, which was first introduced by Joyal to prove the Cayley's theorem on the number of spanning trees on complete graphs. We quickly recall the classical constructions, together with the power series associated to the combinatorial species. We then define generalized species and its connection to some known mathematical objects. First, magnitude and magnitude (co)homology of finite graphs. Second, configuration spaces and the associated power series.

Third and also last, the Euler integral interpreted as an instance of this new notion of species.

The last chapter contains some final remarks, together with some potential questions for future research.

Chapter 2

Preliminaries on sheaves and constructible sheaves

2.1 Sheaves and the category of complexes

Let X be a topological space. Let $\mathrm{Open}(X)$ be the category of open subsets on X, where the morphisms are inclusions.

A presheaf $\mathscr{F}$ is a contravariant functor from $\mathrm{Open}(X)^{\mathrm{op}}$, the opposite category of $\mathrm{Open}(X)$, to another category. In this thesis, the target category is usually chosen to be the category of R-modules for some ring R. When $R = \mathbb{Z}$, we obtain the category of abelian groups **Ab**. Given a presheaf $\mathscr{F}$ and open subsets $U \subset V \subset W$

in X, we have the following commutative diagrams

$$\begin{array}{ccc} U & \hookrightarrow & W \\ & \searrow & \uparrow \\ & & V \end{array} \qquad \begin{array}{ccc} \mathscr{F}(U) & \longleftarrow & \mathscr{F}(W) \\ & \nwarrow & \downarrow \\ & & \mathscr{F}(V) \end{array}$$

For any open set U in X, the morphism induced from the inclusion $U \subset U$ is the identity map $\mathscr{F}(U) \xrightarrow{\text{id}} \mathscr{F}(U)$. The elements in $\mathscr{F}(U)$ are called the *sections* of the presheaf $\mathscr{F}$ over U, and the elements in $\mathscr{F}(X)$ are called the *global sections* of $\mathscr{F}$.

Let x be a point in X. The *stalk* of $\mathscr{F}$ at x, denoted by $\mathscr{F}_x$, is defined as the direct limit of $\mathscr{F}(U_x)$ over all open subsets in X containing x.

$$\mathscr{F}_x := \varinjlim_{\{U_x \ni x\}} \mathscr{F}(U_x)$$

This limit describes the behavior of the sheaf *infinitesimally* around the point x.

For any two presheaves $\mathscr{F}$ and $\mathscr{G}$, if we consider them as 2 contravariant functors, then a morphism between $\mathscr{F}$ and $\mathscr{G}$ is a natural transformation between the two functors.

A presheaf is a collection of data over open sets of a topological space. However, though these data are forced to obey the composition rule, in a lot of applications, one might want to have some more restrictions over the data. For example, given two open sets U and V with non-empty intersection $U \cap V$, there should be a coherent way to relate the data over the aforementioned sets, which is currently not enforced in a presheaf. This is the motivation for the definition of a **sheaf**.

Definition 2.1.1. A presheaf $\mathscr{F}$ is a sheaf if the following conditions are satisfied:

1. For any open set $U \subset X$, given any covering $U = \bigcup U_i$, if s and t are sections in $\mathscr{F}(U)$ such that $s|_{U_i} = t|_{U_i}$ for all i then $s = t$.

2. For any open set $U \subset X$, given any covering $U = \bigcup U_i$, if for each i there is a section $s_i \in \mathscr{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all pairs (i, j), then there exists a section $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$.

Condition 1 is usually called the locality condition, while condition 2 is called the gluing condition.

Given a presheaf $\mathscr{F}$, there is a *sheafification functor* that turns $\mathscr{F}$ into a sheaf. The construction is usually divided into two steps.

First, for any open set $U \subset X$, define an equivalence relation $\sim$ on elements of $\mathscr{F}(U)$ as follows: for $s, t \in \mathscr{F}(U)$, $s \sim t$ iff there exists an open cover $\bigcup U_i = U$ such that for all i, $s|_{U_i} = t|_{U_i}$. Let $\mathscr{F}^{(1)} := \mathscr{F}(U)/\sim$. $F^{(1)}$ is also a presheaf on X, and it satisfies condition 1 for sheaves.

Now, for any open set U and any open covering $\{U_i\}_{i \in I}$ of U, let

$$\mathscr{F}^{(2)}(U, \{U_i\}) := \left\{ (s_i) \in \prod \mathscr{F}^{(1)}(U_i) : s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \text{ for all } i, j \in I \right\}$$

Let

$$\mathscr{F}^{+}(U) = \varinjlim_{\{U_i\}} \mathscr{F}^{(2)}(U, \{U_i\})$$

where the direct limit is taken over the direct system of all open coverings of U with respect to refinement.

Proposition 2.1.2. *$\mathscr{F}^{+}$ is a sheaf and is called the sheafification of $\mathscr{F}$.*

Note that the sheafification process preserves sheaves, i.e. if $\mathscr{F}$ is a presheaf that is also a sheaf, then $\mathscr{F}^+ = \mathscr{F}$. Sheafification also preserves stalks, i.e. $\mathscr{F}_x^+ = F_x$ for all $x \in X$.

The sheaves of R-modules on a topological space X form a category on their own, denoted by $\mathscr{M}od(R_X)$. This category is in fact an abelian category, i.e. $\mathscr{M}od(R_X)$ is *additive* and *exact*, whose definitions will be recalled below. The sheafification functor is the left adjoint functor of the forgetful functor from the category of sheaves to the category of presheaves. For a more detailed discussion, see [11].

Definition 2.1.3. A category $\mathscr{C}$ is additive if $\mathscr{C}$ is enriched over **Ab** (the category of abelian groups), has zero objects, and for any two objects in $\mathscr{C}$, their product exists.

Definition 2.1.4. A category $\mathscr{C}$ is exact if

1. It has zero objects.

2. Any morphism in $\mathscr{C}$ has a kernel and a cokernel.

3. For any morphism $f \in \mathrm{Mor}(\mathscr{C})$, the induced morphism $\mathrm{CoIm}(f) \to \mathrm{Im}(f)$ is an isomorphism.

Given an abelian category, since the category is readily equipped with kernels and cokernels, it is natural to talk about its exact sequences and/or its complexes.

Recall that a **complex** in a category $\mathscr{C}$ is given by

$$\dots \xrightarrow{d^{n-2}} A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \xrightarrow{d^{n+1}} \dots$$

where the composition of any two consecutive differentials is equal to 0:

$$d^n \circ d^{n-1} = 0 \text{ for all } n \in \mathbb{Z}.$$

The superscript on the differentials will be omitted whenever there is no confusion. We denote such complex by $A^\bullet$. We say that a complex $A^\bullet$ is bounded below (resp. bounded above) if there exists an integer N so that $A^n = 0$ for all $n \leq N$ (resp. $n \geq N$). A complex is bounded if it is both bounded above and bounded below.

A morphism between 2 complexes $A^\bullet$ and $B^\bullet$ is a sequence of maps f_i so that the following diagram is commutative:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & A^{n+1} & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\ \dots & \longrightarrow & B^{n-1} & \longrightarrow & B^n & \longrightarrow & B^{n+1} & \longrightarrow & \dots \end{array}$$

Let $\mathrm{Hom}(A^\bullet, B^\bullet)$ be the set of all morphisms between 2 complexes $A^\bullet$ and $B^\bullet$, and $C(\mathscr{A})$ be the category of complexes in a category $\mathscr{A}$.

It can be shown that if $\mathscr{A}$ is an abelian category then so is $C(\mathscr{A})$. Since the category $\mathscr{M}od(A_X)$ of sheaves of A-modules on X is an abelian category, $C(\mathscr{M}od(A_X))$ is also abelian. If the ring A is understood, we simplify the notation by writing $C(X)$ instead. We also denote the full subcategory of bounded complexes of sheaves by $C^b(X)$. The notation $C^*(A)$ is used when we consider both categories at the same time.

In homological algebra, given a cochain complex of abelian groups or modules, the fact that $d^n \circ d^{n-1} = 0$ implies that the kernel of the map d^{n-1} contains the image of the previous map d^n. Hence, we can define the cohomology of such complex by taking the quotient $\ker(d^{n-1})/\mathrm{im}(d^n)$. Similarly, given an abelian category $\mathscr{A}$, since every map is guaranteed to have a kernel and a cokernel, the cohomology of a complex of objects in $\mathscr{A}$ can be defined the same way. For any integer n, the n^{th}-cohomology of a complex $A^\bullet$ is an object in A.

A lot of results in homological algebra also carry over. For example, given a short exact sequence of objects in $\mathscr{A}$ (i.e. a complex of the form $0 \to A^1 \to A^2 \to A^3 \to 0$), it gives rise to a long exact sequence of cohomology, as a result of the snake lemma [20]:

$$... \to H^n(A^1) \to H^n(A^2) \to H^n(A^3) \to H^{n+1}(A^1) \to ...$$

Given two complexes $A^\bullet$ and $B^\bullet$ and a morphism $f : A^\bullet \to B^\bullet$, there is an induced morphism when cohomology is taken. That is, for each integer n, there is a morphism in $\mathscr{A}$ between the cohomology objects:

$$H^n(f) : H^n(A^\bullet) \to H^n(B^\bullet)$$

If these induced morphisms are isomorphisms for all n, then the morphism f is called a **quasi-isomorphism**.

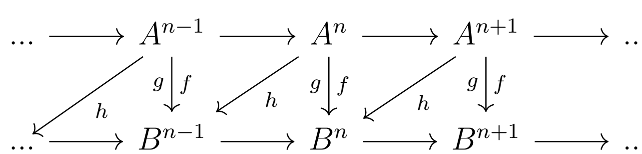
It is clear that quasi-isomorphisms are not always isomorphisms, but one might want to work in a category where that is the case, hence the introduction of the derived functor and the derived category of an abelian category.

2.2 The homotopy category and the derived category of sheaves

Before giving the definition of a derived category, we need to define another notion related to (co)chain complexes.

Let $A^\bullet$ be a complex in $\mathscr{A}$. The *shift functor*, denoted by $[n]$ for any integer n, is a functor from $C^*(\mathscr{A})$ to itself, taking a complex $A^\bullet$ to another complex $A[n]^\bullet$, where $A[n]^k = A^{n+k}$ and $d[n]^k = (-1)^n d^k$.

Definition 2.2.1. Let $f, g : A^\bullet \to B^\bullet$ be morphisms of complexes in $\mathscr{A}$. f and g are said to be **homotopic** if there exists a morphism $h : A^\bullet \to B[-1]^\bullet$ satisfying the relation: $f - g = dh + hd$.



Definition 2.2.2. For an abelian category $\mathscr{A}$, the **homotopy category** of $\mathscr{A}$ is the category $K^*(\mathscr{A})$ whose objects are the same as those of the category of complexes $C^*(\mathscr{A})$, and the morphisms between any two objects $A^\bullet$ and $B^\bullet$ are $\mathrm{Hom}(A^\bullet, B^\bullet)$ modulo homotopy.

Note that $K^*(\mathscr{A})$ is an additive category, but not an abelian category. However, it is a triangulated category. A triangulated category is an additive category with a shift operator (usually denoted by $A[1]$ for an object A) and a class of distinguished triangles, which satisfy a list of axioms. These distinguished triangles serve as an analogue of short exact sequences in abelian categories, and they are compatible with the notion of cohomology functors in triangulated categories. Interested readers can refer to [11] for a more detailed discussion on triangulated category as well as other topics in this chapter.

Definition 2.2.3. Let $\mathscr{A}$ be an abelian category. The derived category of $\mathscr{A}$ is the localization of the category $K^*(\mathscr{A})$ by quasi-isomorphisms. That is, we formally invert all the quasi-isomorphisms in $K^*(\mathscr{A})$.

Similar to the case of the homotopy category of $\mathscr{A}$, the derived category $D^*(\mathscr{A})$ is also a triangulated category but not necessarily an abelian category.

Let A be a commutative ring and X be a topological space. As the category of complexes of sheaves of A-modules on X is denoted by $C^*(A_X)$, the homotopy and derived category of sheaves of A-modules on X will be denoted by $K^*(A_X)$ and $D^*(A_X)$ respectively.

In the newly defined derived category, all of the weak equivalences in $C^*(A)$, namely the quasi-isomorphisms, become isomorphisms. Note that in the homotpy category $K^*(A)$, this is not the case. In other words, from the homotopy theory perspective, the derived category is the *true homotopy category* one might want to work with.

2.3 Operations on sheaves and the Grothendieck's six operations

A (pre)sheaf $\mathscr{F}$ on a topological space X is a collection of data on the open sets on X. If we have a continuous map from X to another space Y, it is possible to transfer this data to Y by the means of **direct image**.

Definition 2.3.1. Let $f : X \to Y$ be a continuous map. The direct image functor (or the pushforward functor) with respect to f is a functor from the category $\mathscr{Mod}(A_X)$ of sheaves on X to the category $\mathscr{Mod}(A_Y)$ by taking any sheaf $\mathscr{F}$ to the sheaf $f_*\mathscr{F}$ where $f_*\mathscr{F}(U) = \mathscr{F}(f^{-1}(U))$ for any open set $U \subset Y$.

Given a continuous map $f : X \to Y$, one might want to pullback the sheaves in the other direction. That is, given a sheaf on Y, is there a way to construct a sheaf on X from the given data on Y. This process proves to be more challenging than the pushforward defined previously, and there are two main complications here. First, the image of an open set under a continuous map is not necessarily open, hence one needs to invoke a limiting process. Second, even after applying the direct limit, the result does not guarantee the gluing and locality properties of a sheaf, hence sheafification is needed.

Definition 2.3.2. Let $f : X \to Y$ be a continuous map. The inverse image functor (or the pullback functor) with respect to f is a functor from the category $\mathscr{Mod}(A_Y)$ of sheaves on Y to the category $\mathscr{Mod}(A_X)$ by taking any sheaf $\mathscr{G}$ to the sheafification of following presheaf:

$$\varinjlim_{f(U)\subset V} \mathscr{G}(V) \text{ for any open set } U \in X$$

Since the construction of the inverse image functor already involves direct limit, the inverse image is compatible with stalks of sheaves. More precisely, for any point $x \in X$ and any sheaf $\mathscr{G}$ on Y,

$$(f^{-1}\mathscr{G})_x = \mathscr{G}_{f(x)}$$

As mentioned earlier, since the derived category of sheaves identifies all weak equivalences as isomorphisms and is the natural playground for complexes of sheaves from the homotopy point of view, we would want to expand the notion of direct and inverse images to complexes of sheaves, considered as objects in the derived category $D^*(A_X)$.

Let $\mathscr{A}$ be an abelian category with enough injectives, i.e. for any object A in $\mathscr{A}$, there exists an injective object I and a monomorphism $A \to I$. The category $\mathscr{M}od(X)$ of sheaves of abelian groups on X is an example of such a category.

For any object A in $\mathscr{A}$, we have an injective resolution

$$0 \to A \to I^1 \to I^2 \to ... \tag{2.3.1}$$

where each I^i is injective.

Let $F : \mathscr{A} \to \mathscr{B}$ be a left exact functor. Applying F to the injective resolution 2.3.1 yields a chain complex, and denote the i^{th}-cohomology of the complex by

$R^iF(A)$. R^iF is called the i^{th} **right derived functor** of F. The left derived functor to a right exact functor G can be defined similarly for an abelian category with enough projectives, and will be denoted by L^iG.

Remark 2.3.3. Note that our definition of the right derived functor relies on the choice of injective resolution. It can be shown that R^iF is independent of this choice and hence well-defined [20] [22].

Now, consider the abelian category $\mathcal{Mod}(X)$ of sheaves of abelian groups on X, together with the direct and inverse image functors defined above. Notice that while the inverse image functor is both left and right exact, the direct image functor is only left exact and not necessarily right exact. Therefore, the direct image functor has a right derived functor Rf_*, while the inverse image functor just extends naturally to the derived category.

The direct and inverse image functors, together with their derived versions, are the first two in a collection of six operations commonly known as **the Grothendieck's six operations**. The other four operations are the following:

1. The proper direct image $f_!$
2. The proper inverse image $f^!$
3. The internal tensor product $\otimes$
4. The internal Hom

The first two operations, the proper direct image and the proper inverse image functors, can be thought of as the compactly supported analogues of the direct and inverse image functors. These operations form two adjoint pairs, which is also the case for the last two functors, internal tensor product and internal Hom. In fact, the proper inverse image is often defined as the right adjoint of the (derived) proper direct image functor.

The internal tensor product of two sheaves $\mathscr{F}$ and $\mathscr{G}$ is the sheafification of the presheaf $U \mapsto \mathscr{F}(U) \otimes \mathscr{G}(U)$. The internal Hom is defined similarly, though note that we do not need sheafification here — the presheaf $U \mapsto Hom(\mathscr{F}_{|U}, \mathscr{G}_{|U})$ is already a sheaf of abelian groups).

The inner tensor product gives rise to another functor called the **external tensor product**. Given two sheaves on X, the inner tensor product returns another sheaf on X, while the external tensor product gives us a sheaf on the Cartesian product $X \times X$. The external tensor product is, in fact, more general, as it can take in sheaves on different spaces.

Let $q_X : X \times Y \to X$, $q_Y : X \times Y \to Y$ be the projections, and let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on X and Y respectively. Then the external tensor product of $\mathscr{F}$ and $\mathscr{G}$, denoted by $\mathscr{F} \boxtimes \mathscr{G}$, is the sheaf

$$\mathscr{F} \boxtimes \mathscr{G} := q_X^{-1}\mathscr{F} \otimes q_Y^{-1}\mathscr{G}$$

We denote the (left) derived external tensor product by $\boxtimes^L$.

This external tensor product is essential in order to define a notion of convolution for sheaves. First, let's recall the classical convolution for real-valued functions. Let $f, g : \mathbb{R} \to \mathbb{R}$. Then the convolution of f and g, denoted by $f * g$, is the following integral

$$f * g(x) = \int_{\mathbb{R}} f(t)g(x-t)dt$$

If we look at the Cartesian product $\mathbb{R} \times \mathbb{R}$ then this integral can be rewritten as the integral of f times g over the pair of points $(t_1, t_2) \in \mathbb{R} \times \mathbb{R}$ such that $t_1 + t_2 = x$. In other words, if we index the diagonal of $\mathbb{R} \times \mathbb{R}$ by x, then the convolution is nothing but the integral over the anti-diagonal perpendicular to x.

This exact idea gives rise to the convolution of sheaves on vector spaces. Let E be a vector space, and $s : E \times E \to E : (t_1, t_2) \mapsto t_1 + t_2$ be the addition map.

Definition 2.3.4. Let $\mathscr{F}$ and $\mathscr{G}$ be objects in the bounded derived category of sheaves on E. The **convolution** of $\mathscr{F}$ and $\mathscr{G}$, denoted by $\mathscr{F} * \mathscr{G}$, is

$$\mathscr{F} * \mathscr{G} := Rs_!(\mathscr{F} \boxtimes^L \mathscr{G})$$

Remark 2.3.5. Similar to the classical case where the convolution of two functions on $\mathbb{R}$ is also a function on $\mathbb{R}$, given two sheaves on a vector space E, the sheaf convolution produces another object in the (bounded derived) category of sheaves on E.

Before ending this section, we would like to define sheaf cohomology, which is another very important functor on sheaves. Similar to how a right derived functor is constructed through an injective resolution, the cohomology of a sheaf can be defined through the process of applying the global section functor to a resolution and then taking cohomology.

More precisely, let $\mathscr{F}$ be a sheaf of abelian groups on X. There exists a morphism from $\mathscr{F}$ into an injective sheaf I, where injective here means an injective object in the category of sheaves of abelian groups. In other words, the category $\mathscr{Mod}(X)$ of sheaves of abelian groups on X has enough injectives.

Therefore, we can form an injective resolution for $\mathscr{F}$

$$0 \to \mathscr{F} \to I^1 \to I^2 \to ...$$

Now, apply the global section functor to the above resolution. Since global section is left exact, we obtain a cochain complex

$$0 \to \Gamma(I^1, X) \to \Gamma(I^2, X) \to ...$$

The i^{th} cohomology of this complex, denoted by $H^i(\mathscr{F})$, is called the i^{th} cohomology group of the sheaf $\mathscr{F}$.

Similar to the definition of a derived functor, the cohomology of a sheaf $\mathscr{F}$ does not depend on the choice of injective resolution. Indeed, sheaf cohomology is simply the higher direct image of the global section functor [20] [22].

The cohomology of complexes of sheaves is defined in the same manner, where an injective resolution is taken in the category of complexes of sheaves. A cochain map a complex $\mathscr{F}^\bullet$ to another complex $I^\bullet$ is injective if (1) all objects in I are injective (2) the map is a quasi-isomorphism, and (3) the cochain map is a monomorphism for every index. We finish by applying the global section functor to the injective resolution and taking cohomology, and note that condition (2) on quasi-isomorphism guarantees well-definedness.

Even though this approach to sheaf cohomology has the advantage of abstraction with a clear roadmap to prove theorems, it is not easily computable. For any given sheaf, it is not clear how to get such an injective resolution — the resolution is only guaranteed to exist, but not constructive. Fortunately for us, there is a more combinatorial approach to this problem through the means of Cech cohomology. This is similar to the situation in homology where singular homology is the superior tool when it comes to proving theorems and functoriality, but when computation is needed, we have simplicial and cellular homology (and sometimes even deRham cohomology, when applicable) at our disposal.

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. For every integer k, a k-cochain of $\mathscr{F}$ is an element of $\prod \mathscr{F}(U_{i_0} \cap ... \cap U_{i_k})$, where the product is taken over all possible intersections of $(k+1)$ sets.

The coboundary operator is now defined as the alternative sum of the restriction of the sections. One can check that this is indeed a differential, and the cohomology of $\mathscr{F}$ with respect to the open cover U, denoted $H^\bullet(\mathcal{U}, \mathscr{F})$, is the cohomology of this cochain complex.

Definition 2.3.6. The Cech cohomology of the sheaf $\mathscr{F}$ is the directed limit

$$\check{H}^\bullet(X, \mathscr{F}) = \varinjlim_{\mathcal{U}} H^\bullet(\mathcal{U}, \mathscr{F})$$

where the limit is taken over the directed system of open covers with respect to refinements.

It should be noted that in practice, the formal process of taking limit might not be necessary. For some *good cover* $\tilde{\mathscr{U}}$ where pairwise intersections of the open sets are all contractible,

$$\check{H}^{\bullet}(X, \mathscr{F}) = H^{\bullet}(\tilde{\mathscr{U}}, \mathscr{F})$$

This makes Cech cohomology particularly tractable and is suitable for computational purposes.

2.4 Constructible sheaves

In this section, all sheaves are sheaves of R-modules for some ring R. Let X be a *definable* space, whose exact definition will be presented in section 3.1. The important feature of a definable space is that it possesses a special class of subsets called definable sets, i.e. subsets which are topologically tractable, and most importantly, has a well-defined notion of Euler characteristic. In this section, it suffices to replace *definable* with *semi-algebraic*.

Definition 2.4.1. A sheaf $\mathscr{F}$ on X is said to be weakly constructible if there exists a partition of X into definable subsets X_i so that $\mathscr{F}|_{X_i}$ is locally constant for all i.

Definition 2.4.2. A sheaf $\mathscr{F}$ on X is said to be constructible if it is weakly constructible and all of the stalks are finitely generated R-modules.

We say that a complex of sheaves is constructible if for each integer i, the i^{th} cohomology sheaf is constructible. The category of constructible sheaves (resp. the homotopy category of constructible sheaves, the derived category of constructible sheaves) is denoted $\mathscr{M}od_c(X)$ (resp. $K_c^*(X)$, $D_c^*(X)$).

Constructibility is a local property, i.e. if a sheaf is constructible then the restriction of the sheaf to any open set is also constructible. Moreover, similar to the case of the derived category of sheaves, the derived category of constructible sheaves is also a triangulated category. See [33] for a formal treatment of the subject.

We will discuss the importance of constructible sheaves in the next chapter, and we will end this chapter with the definition of the Grothendieck group of a triangulated category.

Definition 2.4.3. Given a triangulated category $\mathscr{C}$, the Grothendieck group of $\mathscr{C}$ is the free abelian group generated by objects in $\mathscr{C}$ quotiented by the relation $[A] - [B] + [C] = 0$ for any distinguished triangle $A \to B \to C \to A[1]$ in $\mathscr{C}$. This group is denoted by $K(\mathscr{C})$.

Note that we previously discussed another object of interest with similar notation, namely the homotopy category of complexes $K^*(\mathscr{C})$ for any abelian category $\mathscr{C}$. However, one is a category while the other is an abelian group, hence there should be no confusion if they are mentioned with context.

The choice of notation for the Grothendieck group is intentional — the Grothendieck group we define above is an instance of the K-theory of triangulated categories [27].

The Grothendieck group is very important if one wants to investigate the Euler characteristic. In particular, the Euler characteristic χ factors through a morphism $\phi : K(\mathscr{C}) \to \mathbb{Z}$ such that $\phi([A]) = \chi(A)$, and this morphism is unique.

Chapter 3

Euler Integration and Integral Transforms

Euler characteristic is a generalization of a counting function; that is, it is compatible with the inclusion-exclusion principle and is multiplicative under Cartesian products. With a suitable choice of sets, we can start building up an integration theory with Euler characteristic acting as the measure. We will be discussing these notions in the first 2 sections of this chapter.

Every integration theory comes with its own set of integral transforms, and the Euler integral is no exception. The most common transforms are the Fourier-Sato transform (an analogue of the classical Fourier transform) and the Euler-Radon transform (an analogue of the classical Radon transform). These will be the main content of the second half of this chapter and also the next two chapters.

3.1 Constructible functions

First, let us formally define the notion of constructible functions we use in this thesis. For more in depth discussions, see [9].

Definition 3.1.1. A function $f : X \to \mathbb{Z}$ is said to be *constructible* if each level set is locally finite and semi-algebraic.

In general context, the definition of constructible functions can be extended to a much bigger class, where semi-algebraic sets are replaced by *definable sets* (with respect to some pre-determined o-minimal structure). All the results regarding Euler integration in this thesis work for the general case as well.

Definition 3.1.2. An **o-minimal structure** $\mathscr{O}$ on $\mathbb{R}$ is a sequence of boolean algebras $\{\mathscr{O}_n\}$ of subsets on $\mathbb{R}^n$ satisfying the following conditions:

1. $\mathscr{O}$ is closed under cartesian products.

2. For any set $A \in \mathscr{O}_n$, $\pi_i(A) \in \mathscr{O}_{n-1}$ where π_i is the projection defined by $\pi_i : \mathbb{R}^n \to \mathbb{R}^{n-1} : (x_1, ...x_i, ..., x_n) \mapsto (x_1, ..., \overline{x_i}, ...x_n)$

3. $\mathscr{O}$ contains all algebraic sets.

4. $\mathscr{O}_1$ is the boolean algebra on **finite** unions of (open) intervals and discrete points. These open intervals are allowed to be semi-infinite.

The subsets of $\mathscr{O}$ are called **tame** or **definable** sets. Semi-algebraic sets in $\mathbb{R}^n$ form an o-minimal structure.

One can start working with o-minimal structures by saying a map between definable spaces to be definable if the graph of the map is definable. Such a map is called a definable homeomorphism if it is also a bijection.

An important property of definable sets is triangulation invariant, which is the content of the Triangulation Theorem [12].

Theorem 3.1.3. *Any definable set is definably homeomorphic to a disjoint union of open simplices in a finite simplicial complex.*

This triangulation theorem allows us to have a well-defined notion of Euler characteristic on any o-minimal structure; that is, any definable set has an Euler characteristic defined by the open simplices up to triangulation and definable homeomorphism.

3.2 Euler Integration

Let X be a definable space, and let $CF(X)$ be the group of constructible functions on X.

As discussed earlier, for each semi-algebraic set, there is a well-defined (compactly supported) Euler Characteristic associated to it. Since Euler Characteristic has the benefit of being compatible with the inclusion-exclusion principle, we can get an integration theory with Euler Characteristic acting as our "measure."

Definition 3.2.1. Let $f : X \to \mathbb{Z}$ be a constructible function. Then

$$\int_X f\, d\chi = \sum_{-\infty}^{\infty} s\, \chi(f^{-1}(s))$$

Note that since the map f is constructible, the preimage $f^{-1}(s)$ is definable for all s and hence the Euler characteristic is well-defined. This definition is similar to the measure-theoretic way of defining Lebesgue integrals.

There is a morphism from the K-theory of the derived category of bounded constructible sheaves on X to the group of constructible functions on X by taking its stalkwise Euler characteristic. In fact, this morphism is an isomorphism of groups.

Proposition 3.2.2 ([11]). *The Grothendieck group of the bounded derived category of constructible sheaves on a definable space X is isomorphic to the group of constructible functions on X. More precisely, the following map*

$$\chi : K(D^b_c(Sh(X)) \to CF(X) : \mathcal{F} \to \{x \mapsto \chi(\mathcal{F}_x)\}$$

is an isomorphism.

First and foremost, given a constructible function f, there exists a (class of) constructible sheaf $\mathscr{F}$ related to f under the above isomorphism. The Euler integral of f is now simply the Euler characteristic of the sheaf $R\pi_!\mathscr{F}$, where π is the constant map from X to a point:

$$\int_X f d\chi = \chi(R\pi_!\mathscr{F})$$

Remark 3.2.3. The fact that we are using complexes of sheaves instead of just sheaves is important. Given a sheaf of vector spaces, the Euler characteristic of the stalk is equal to the dimension of the stalk, which is always non-negative. However, this is not the case for constructible functions, as they can take negative values. Complexes of sheaves do not have this problem because they provide another grading. More precisely, if a sheaf $\mathscr{F}$ corresponds to a non-negative constructible function f, then the complex with $\mathscr{F}$ in degree 1 and the 0-sheaf everywhere else corresponds to the constructible function $(-f)$.

The isomorphism allows us to convert all the operations on the world of sheaves, e.g. convolution, Fourier-Sato transform, Verdier dual, etc., to the world of constructible functions [31]. Some theorems also carry over as direct consequences.

Definition 3.2.4. Let f and g be constructible functions on $\mathbb{R}^n$. The convolution of f and g, denoted by $f * g$, is

$$f * g(x) = \int_X f(t)g(x-t)d\chi(t)$$

Definition 3.2.5. Let f be a constructible function on $\mathbb{R}^n$. The Verdier dual of f is a constructible function $Df \in CF(\mathbb{R}^n)$ such that for all $x \in \mathbb{R}^n$,

$$Df(x) = \lim_{r\to 0} \int_{D_r(x)} f(t)d\chi(t)$$

where $D_r(x)$ is the open disk of radius r around x.

Theorem 3.2.6 (Fubini Theorem for Euler integrals [9])**.** *Let $\phi : X \to Y$ be a tame* *map. Then for all constructible functions f in X,*

$$\int_X f d\chi(x) = \int_Y \big(\int_{\phi^{-1}(y)} f d\chi(x)\big) d\chi(y)$$

3.3 Euler-Radon Transform and Schapira's inversion formula

The Euler-Radon transform in the world of Euler calculus and constructible functions is equivalent to the idea of general integral transform in classical analysis. It turns out that the more specific (classical) Radon transform would be analogous to an instance of the Euler-Radon transform with a chosen kernel. This will be discussed in more details in the last part of this section. That being said, we will be referring to the Euler-Radon transform as Radon transform if there is no confusion.

Let X and Y be two definable spaces.

Definition 3.3.1. A kernel K is a constructible function on the product space $X \times Y$ with the product definable structure.

Definition 3.3.2. Given any constructible function $f \in CF(X)$, the Euler-Radon transform of f with respect to a kernel $K \in CF(X \times Y)$ is defined by

$$R_K(f)(y) = \int_X f(x)K(x,y)d\chi(x) \tag{3.3.1}$$

for all $y \in Y$.

As K is chosen to be constructible, it can easily be seen that $R_K(f)$ (or sometimes simply written without the parentheses as $R_K f$) is a constructible function on Y.

Considered altogether, R_K can be seen as an integral transform that takes any constructible function from X to another constructible function Y:

$$R_K : CF(X) \to CF(Y)$$

X will sometimes be referred to as the **target space**, while Y will be referred to as the **sensor space**. The reason for this naming convention stems from the situation where all of the concerned constructible functions are characteristic functions, and the kernel being non-zero at $(x, y) \in X \times Y$ is equivalent to the point x in X (the target space) being "detected" by the point y in Y (the sensor space).

However, if we look back at equation 3.3.1, there is no restriction on whether we should take the integral over X or over Y; the direction of the Radon transform was determined by choice. In other words, given a kernel $K \in CF(X \times Y)$, there is also an integral transform that takes a constructible function from Y to a constructible function on X. To avoid confusion, these two situtations are differentiated as follows: if we indicate K as a kernel in $CF(X \times Y)$, the Radon transform is understood to be from $CF(X)$ to $CF(Y)$, while a kernel $K \in CF(Y \times X)$ would result in a Radon transfrom from $CF(Y)$ to $CF(X)$.

A natural question arises here: given a Radon transform $R_K : CF(X) \to CF(Y)$, is the transform by the same kernel for the other direction the inverse of

R_K? The answer is, unfortunately, negative. An easy counter-example is to take K to be the characteristic function on the point $(0, 0)$ for the Euclidean spaces X and Y. Under this kernel, the Radon transform of constructible function $f \in CF(X)$ would be a pointed constructible function at $y = 0$, which is clearly not invertible.

Of course, this begs another question: given a Radon transform $R_K : CF(X) \to CF(Y)$, is there any other Radon transform $R_{K'} : CF(Y) \to CF(X)$ so that they are *inverse* of each other? Loosely speaking, is there a way to reconstruct a constructible given its Radon transform? This question is answered by a formula given by Schapira (often referred to as the **Schapira's Inversion Formula**) in [32], which provides a positive answer for the above question if there exists an (inverse) kernel which satisfied some Euler characteristic conditions.

Theorem 3.3.3 ([32])**.** *Let K be a kernel $K \in CF(X \times Y)$. If there is a kernel $K' \in CF(Y \times X)$ and some integer constants μ and λ satisfying the following equation*

$$\int_Y K(x, y)K'(y, x')d\chi(y) = (\mu - \lambda)\delta_\Delta + \lambda \tag{3.3.2}$$

where δ_Δ is the characteristic function on the diagonal $\Delta \in X \times X$, then for any constructible function $f \in CF(X)$

$$(R_{K'} \circ R_K)(f) = (\mu - \lambda)f + \lambda\Big(\int_X f d\chi(x)\Big)\mathbb{1}_X$$

Remark 3.3.4. When a kernel K' and such constants μ and λ exists so that $\mu \neq \lambda$, applying the inverse Radon transform $R_{K'}$ to a known $R_K(f)$ recovers the original constructible function f (up to a global shift and a scaling).

If K and K' are both characteristic functions of some underlying sets S and S' in $X \times Y$ and $Y \times X$ respectively, let S_x be the set $\{y \in Y | (x, y) \in S\}$, and similarly for S'_x. Equation 3.3.2 can be rewritten as a set of conditions as follows:

1. For any point $x \in X$,

$$\int_Y K(x,y)K'(y,x)d\chi(y) = \int_Y \mathbb{1}_S(x,y)\mathbb{1}_{S'}(y,x)d\chi(y)$$

$$= \chi(S_x \cap S'_x)$$

   Since $\delta_\Delta(x,x) = 1$, the above equation is equilvalent to

$$\chi(S_x \cap S'_x) = (\mu - \lambda) + \lambda = \mu$$

2. For two distinct points $x \neq x'$ in X,

$$\int_Y K(x,y)K'(y,x')d\chi(y) = \int_Y \mathbb{1}_S(x,y)\mathbb{1}_{S'}(y,x')d\chi(y)$$

$$= \chi(S_x \cap S'_{x'})$$

   while $\delta_\Delta(x,x') = 0$. Therefore,

$$\chi(S_x \cap S'_{x'}) = (\mu - \lambda) \cdot 0 + \lambda = \lambda$$

In short, the kernel K is invertible if there are constants μ and λ so that the fibers satisfy the conditions $\chi(S_x \cap S'_x) = \mu$ for all x, $\chi(S_x \cap S'_{x'}) = \lambda$ for all $x' \neq x$, and $\mu \neq \lambda$.

Remark 3.3.5. The conditions on Euler characteristics here are strict: the numbers μ and λ have to be the same for all x and x'. This rigidity is the main obstruction in finding inverse kernels for Radon transforms.

One of the most remarkable results that Schapira proved in his first paper on this subject is the fact that the **level-set transform** is invertible [32]. This transform is the direct analogue of the classical Radon transform, hence it is sometimes called **topological tomography**.

Theorem 3.3.6. *Let* $X = \mathbb{R}^n$, $Y = S^{n-1} \times \mathbb{R}$ *for* $n \geq 2$. *Let* K *be the characteristic function on the set* $\{(x, (\xi, t)) \in \mathbb{R}^n \times S^{n-1} \times \mathbb{R} | x \cdot \xi = t\}$. *Then* K *is self-invertible.*

Proof. We check the conditions on the fibers of K.

For any $x \in X$, the set K_x is the collection of (ξ, t) so that $x \cdot \xi = t$. For each $\xi \in S^{n-1}$, one and only one t satisfies this equality. Therefore, K_x is homotopy equivalent to S^{n-1}.

For $x' \neq x$, the intersection $K_x \cap K_{x'}$ is the collection of pairs (ξ, t) so that $x \cdot \xi = t = x' \cdot \xi$. These vectors lie on the equator of S^{n-1} that is parallel to the bisector hyperplane of the line segment $\overline{xx'}$ in $\mathbb{R}^n$, and for each vector there is a unique t satisfying the equation. Therefore, for all $x' \neq x$, $K_x \cap K_{x'}$ is homotopy equivalent to S^{n-2}.

Since $\chi(S^{n-1}) \neq \chi(S^{n-2})$, K is its own Radon inverse. $\square$

Chapter 4

Persistent Homology Transform

Persistent Homology is at the core of the rise of TDA in the last few decades. It provides a summary of a given set of data, and it is a natural question to ask whether we can reconstruct the original data from persistence diagrams. The answer turns out to be positive, if we are given samplings from all directions on the unit sphere. This is the idea behind Persistent Homology Transform, and we will present a proof of this fact using the Schapira's Inversion Formula.

4.1 Persistent Homology

The basic question at the start of persistent homology is following: given a point cloud in an (Euclidean) space, possibly coming from a reasonable sampling of an object with some significant topological features, is there a way to recover the topology from this discrete collection of points? If so, can we make sure that our topological summary is robust enough so that it is not susceptible to noise?

The answer to this seemingly simple question is an assemble of tasks commonly referred to as *persistence*. Perhaps there is no better word to capture this whole process: they record the features and characteristics that persist for a long time. Of course we have to provide a quantifiable way to perceive this persistence, as time is relative.

The first step, which is also the most crucial step, is to choose a topological realization of the discrete point cloud. One might want to consider the weighted graph whose vertices are the points in the point cloud and the edges are weighted according to the Euclidean distance. This approach is not very satisfying, since a graph is topologically primitive and it fails to capture higher topological features in the original object. As a result, one might want to consider the higher-dimensional version of graphs — a simplicial complex. It is relatively simple to construct such a complex from the point cloud, and in fact there are many different ways to do so. For example, consider an ϵ thickening of the point cloud. That is, at every point, draw a disk of radius ϵ. Then, form a simplex of dimension n whenever n of these disks intersect. This creates a simplicial complex, a topological representation of the point cloud. However, it is again very naive to think that this single simplicial complex can recover the topological feature of the object. For an ϵ too small, the complex will not be any different from the discrete point cloud, while for an ϵ too big, every feature will be merged together into one big simplex of very high dimension, which is topogically uninteresting. In other words, the choice of ϵ is very important, and it is hard to decide which ϵ to use without prior domain knowledge.

Persistent homology offers a solution to this conundrum: what if we consider *all* of these radii altogether? This will give us an $\mathbb{R}$-worth collection of simplical complexes, one for each real number. This, of course, has its own difficulties. How are we going to track the changes of the complexes through time?

Here comes the second step. We need to provide a topogical invariant that is both robust but at the same time computable, and homology turns out to be the ideal candidate. For example, it has the advantage over homotopy groups for its computability, and over the Euler characteristic for being more resourceful. This is not to say that other invariants do not work; in fact, the Euler characteristic is chosen as the invariant for the persistent homology transform, which will be discussed later in the chapter.

Homological algebra also provides us an incredible toolkit to deal with this problem. For two radii $r < t$, there is a natural morphism from the homology groups at index r to the homology groups at index t, which gives us a sequence of morphisms that tracks the development of all homology classes through time. Even better, homological algebra comes equipped with a certain theorem — Gabriel's theorem — which allows us to decompose the sequence into smaller representations, called *barcodes*. For a more detailed introduction to persistent homology, see [16] [8][13] .

4.2 Persistent Homology Transform

The idea of the persistent homology transform resembles that of the classical Radon transform, but with a different kernel. Given a compact subset in $\mathbb{R}^n$, instead of considering the Euler characteristics of all intersections with hyperplanes, we choose the sublevel-sets defined by them. The formal definition is as follows.

Definition 4.2.1. The **persistent homology transform** of a constructible function $f \in CF(\mathbb{R}^n)$ is the Radon transform of f with respect to the kernel $K = \{(x, (\xi, t)) \in \mathbb{R}^n \times S^n \times \mathbb{R} | x \cdot \xi \leq t\}$

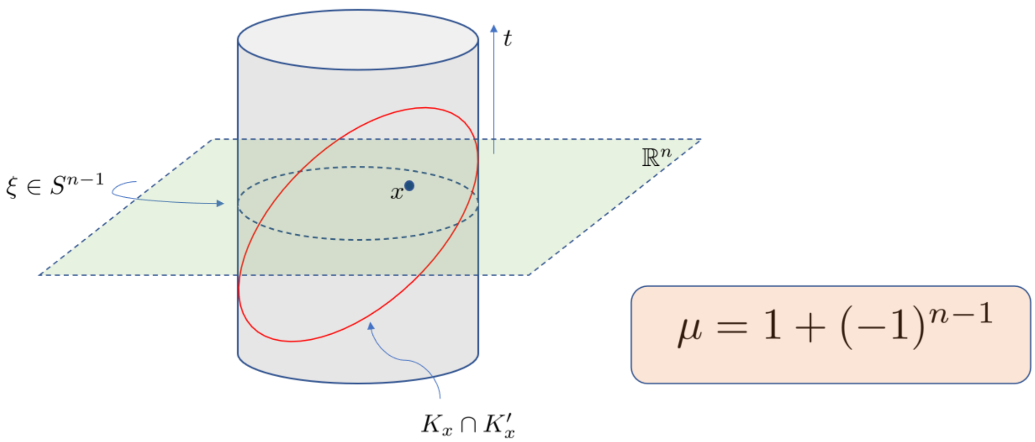
Using the technique of Radon transform, we provide an inverse kernel to show that the persistent homology transform satisfies the Schapira's inversion formula with suitable constants μ and λ.

Theorem 4.2.2. *Let f be a compactly supported constructible function on $\mathbb{R}^n$. Then the persistent homology transform of f is invertible.*

Proof. Let K' be the following kernel:

$$K' = \{((\xi, t), x) \in S^n \times \mathbb{R} \times \mathbb{R}^n | x \cdot \xi \geq t\}$$

We claim that K' acts as an inverse kernel for K.



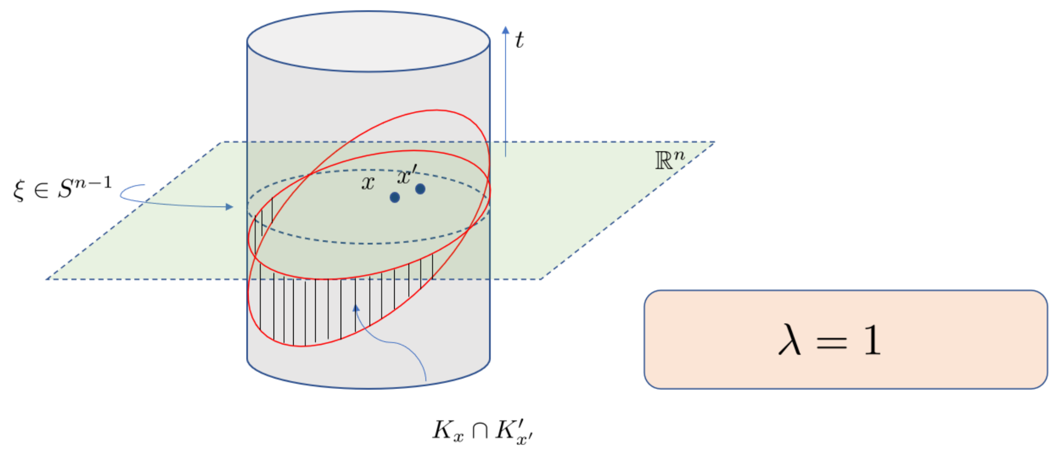
For any $x \in \mathbb{R}^n$, K_x is the collection of (ξ, t) so that $x \cdot \xi \leq t$, while K'_x satisfies the inequality $x \cdot \xi \geq t$. Therefore, their intersection is the collection of (ξ, t) so that $x \cdot \xi = t$. For each $\xi \in S^{n-1}$, there is exactly one t satisfying this condition, hence the intersection $K_x \cap K'_x$ is homeomorphic to S^{n-1}. Therefore,

$$\chi(K_x \cap K'_x) = \chi(S^{n-1}) = 1 + (-1)^{n-1}$$

Now, for two different points $x' \neq x$ in $\mathbb{R}^n$, the intersection $K_x \cap K'_{x'}$ is the collection of points $\{(\xi, t) | x \cdot \xi \leq t, x' \cdot \xi \geq t\}$. There are two cases:

1. If $x \cdot \xi < x' \cdot \xi$, there is no value of t that satisfies the two inequalities in $K_x \cap K'_{x'}$. Hence, this contributes no point to the aforementioned set.

2. If $x \cdot \xi \geq x' \cdot \xi$, t can be taken to be in the compact interval bounded by $x \cdot \xi$ and $x' \cdot \xi$. The collection of such ξ and t is therefore a compact contractible subset of $S^{n-1} \times \mathbb{R}$. The Euler characteristic of such set is equal to 1.



Therefore, $\chi(K_x \cap K'_{x'}) = 1$ for $x' \neq x$.

Applying the Schapira's Inversion formula to the kernels K' and K, and notice that $1-(-1)^{n-1} \neq 1$ for all n, we conclude that K' is an inverse kernel of K. In other words, the persistent homology transform of any compactly supported constructible function on $\mathbb{R}^n$ is invertible. $\square$

Remark 4.2.3. A proof of this theorem is also discovered independently in [10]. A constructive proof for the case $n = 3$ is given in [34]. Another approach for the case of simplicial complexes is discussed in [6].

Following is a stereographic version of the Persistent Homology Transform:

Theorem 4.2.4. *Let X be the closed unit ball $\mathbb{D}^n$ and Y be the space $\partial\mathbb{D}^n \times \mathbb{R}_{\geq 0}$.*

Let $\|\cdot\|$ be the standard norm in $\mathbb{R}^n$. Then the Radon transform with respect to the kernel

$$K = \{(x, y, r) \in \mathbb{D}^n \times \partial\mathbb{D}^n \times \mathbb{R}_{\geq 0} : \|x - y\| \leq r\}$$

is invertible.

Proof. Consider the kernel $K' = \{(x, y, r) \in \mathbb{D}^n \times \partial\mathbb{D}^n \times \mathbb{R}_{\geq 0} : \|x - y\| \geq r\}$.

For every $x \in X$, the intersection $K_x \cap K'_x$ is homeomorphic to $\partial\mathbb{D}^n = S^{n-1}$.

For $x \neq x'$ in X, the intersection $K_x \cap K'_{x'}$ is contractible, and hence has Euler characteristic 1.

Therefore, the transform is invertible. □

Chapter 5

Fourier-Sato Transform

Fourier-Sato transform was first introduced by Sato et al. [30] and then further developed by Kashiwara and Schapira as a tool to study sheaves of microfunctions. Fourier-Sato transform takes in a conic constructible sheaf on a vector bundle and returns a corresponding conic constructible sheaf on the dual. Since there is a way to turn constructible sheaves into constructible functions, we obtain a notion of Fourier-Sato transform for (conic) constructible functions. This integral transform possesses some desirable properties of the usual Fourier transform, including compatibilities with the convolution and the L^2 (pseudo) inner product, which we will provide some new proofs.

We will also discuss an analogue of tempered distributions in the context of Euler integral to extend the notion of Fourier-Sato transform to non-conic functions.

5.1 Fourier-Sato transform of conic constructible sheaves

Let X be a locally compact space with an $\mathbb{R}_+$-action, where $\mathbb{R}_+$ stands for the group of positive real numbers under multiplication. The standard example is $X = \mathbb{R}^n$ and the $\mathbb{R}_+$ action is scalar multiplication.

Let $\mathscr{Mod}(X)$ be the category of sheaves of $\mathbb{R}$-modules on X.

Definition 5.1.1. Let $\mathscr{Mod}_+(X)$ be the full subcategory of $\mathscr{Mod}(X)$ whose elements are sheaves that are locally constant when restricted to any orbit under the $\mathbb{R}_+$ action, i.e. if $F \in \mathrm{Ob}(\mathscr{Mod}_+(X))$, then for any orbit $\alpha \subset X$ of R_+, $F\Big|_\alpha$ is locally constant.

Definition 5.1.2. The category $D^+_{\mathbb{R}_+}$ of derived conic sheaves is the full subcategory of $D^+(X)$ such that if $F \in D^+_{\mathbb{R}_+}$ then all of its cohomology sheaves $H^i(F)$ are in $\mathscr{Mod}_+(X)$.

From now on, unless otherwise specified, **conic sheaves** will be understood as the objects in the derived category of conic sheaves.

As proved in [22], these conic sheaves are stable under the six Grothendieck operations, i.e. the results of applying these operations to conic sheaves are again conic sheaves.

A natural setting to talk about conic sheaves is on (real) vector bundles, where there is a canonical action of $\mathbb{R}_+$ on the space, namely scaling the vectors on the fibers by positive real numbers.

Let $\pi : E \to B$ be a vector bundle of rank n, where B is locally compact.

Definition 5.1.3. Let A be a subset of E. A is said to be **conic** if the intersection of A with any fiber of $\pi : E \to B$ is conic, i.e. invariant under the $\mathbb{R}_+$ action.

Let a be the antipodal map on E. This is the map that takes a point to its additive inverse on the fiber under the local trivialization. A subset $A \in E$ is conic if and only if $A^a := a(A)$ is conic.

The Fourier-Sato transform, as mentioned earlier, was constructed first by Sato to further study (pseudo) differential equation. As the differential operators act on the space of functions, when we work on vector bundles, it is unavoidable to start talking about dual spaces, and that the dual is a natural space for our notion of Fourier-Sato transform lives on.

Let $\pi^* : E^* \to B$ be the dual vector bundle, and let $\langle \cdot, \cdot \rangle$ be the pairing.

Definition 5.1.4. Given a set $A \in E$, the polar set of A, denoted by A°, is a subset of E^* such that $y \in A^\circ$ if and only if $\pi^*(y) \in \pi(A)$ and for every $x \in A$ in the fiber $\pi^{-1}\pi^*(y)$, $\langle x, y \rangle \geq 0$.

Consider the simplest case where the set B is a point and E is a real vector space of dimension n, then the polar set A° of a set $A \in E$ is the set of points in the dual space E^* that are positively paired with every point in A. The polar set of the positive span of a vector v is a half-space in E^*.

In order to study both of these sets at the same time, consider the product bundle $E \times_B E^*$ over B (or just $E \times E^*$ if the base space is understood) and consider the following subsets:

$$P = \Big\{(x, y) \in E \times E^* | \langle x, y\rangle \geq 0\Big\}$$

$$P' = \Big\{(x, y) \in E \times E^* | \langle x, y\rangle \leq 0\Big\}$$

We are now ready to define the Fourier-Sato transform and its inverse.

Definition 5.1.5. Let $\pi : E \to B$ be a vector bundle of finite rank on a locally compact space B, and let $F \in D^+_{\mathbb{R}_+}(E)$ and $G \in D^+_{\mathbb{R}_+}(E^*)$. Let p_1 and p_2 be the projections of $E \times E^*$ to E and E^* respectively.

Then $F^\wedge$, called **the Fourier-Sato transform of** F, is a conic sheaf in E^* defined by the formula

$$F^\wedge = Rp_{2!}(p_1^{-1}F)_{P'} \tag{5.1.1}$$

$G^\vee$, called **the inverse Fourier-Sato transform of** F, is a conic sheaf in E defined by the formula

$$F^\vee = Rp_{1!}(p_2^! G)_P \tag{5.1.2}$$

These two functors provide an equivalence of categories between the derived category of conic sheaves on E and the derived category of conic sheaves on E^*.

The Fourier-Sato transform is compatible with the six Grothendieck operations. For a list of those properties, see [22]. We only choose to mention here that the Fourier-Sato transform preserves the external tensor product, namely if F and G

are two conic sheaves in E_1 and E_2 (both are vector bundles over some B), then the Fourier-Sato transform of the external tensor product of F and G is isomorphic to the external tensor product of the corresponding Fourier-Sato transforms.

Before moving on to the next section, let us recall the notion of convolution for sheaves. This notion of convolution is used extensively in [23] to define a pseudo-distance on the (bounded) derived category of sheaves on vector spaces, which is analogous to the bottleneck distance for barcodes in persistent homology.

Let V be an $\mathbb{R}$ vector space of finite dimension, and denote by s the summation function on X, i.e. $s(x+y) = x+y$ for $x, y \in V$. Recall that $D^b(V)$ stands for the bounded derived category of sheaves on V.

Definition 5.1.6. Given $F, G \in D^b(V)$, then

$$F * G := R_{s!}(F \boxtimes G)$$

where $\boxtimes$ is the external tensor product.

The idea here is very similar to taking the convolution of real-valued functions. The convolution focuses on the affine subspaces $x+y=a$ in $V \times V$ for each $a \in V$. These subspaces are normal to the main diagonal $x=y$ and what the convolution calculates is the (compactly supported) cohomology of F and G along on these subspaces. This is analogous to calculating the integral

$$f * g(x) = \int_{\mathbb{R}} f(t)g(x-t)dt$$

along the off-diagonals corresponding to x in $\mathbb{R}^2$ with L^1 norm.

5.2 Constructible functions and their Fourier-Sato transforms

In this section, let the base space B of the vector bundle $\pi : E \to B$ be a point. In other words, E is a finite dimensional real vector space of dimension n. Under the obvious identification, we consider the vector space E and its dual E^* to be the same. This simple identification allows us to define the Fourier-Sato transform on constructible functions to be an endomorphism from the space of constructible functions to itself, instead of having to go to the dual.

First, for any $\xi \in E^*$, let

$$\begin{aligned} \phi_\xi(x) &= 1 \text{ if } \langle x, \xi \rangle \leq 0 \\ &= 0 \text{ otherwise} \end{aligned}$$

Recall that in the definition of the Fourier-Sato transform of conic sheaves 5.1.2,

$$F^\wedge = Rp_{2!}(p_1^{-1}F)_{P'}$$

we consider the (derived) sheaf of sections on P', where P' is the set of pairs $(x, y) \in E \times E^*$ such that $\langle x, y \rangle \leq 0$. This function ϕ_ξ fulfills the same role as the kernel P', sweeping over the half space opposite of ξ.

Let $CF_{\mathbb{R}_+}(E)$ be the set of conic constructible functions on E, i.e. functions that are constant under the $\mathbb{R}_+$ action on E.

Definition 5.2.1. For any $f \in CF_{\mathbb{R}_+}(E)$, **the Fourier-Sato transform of** f, denoted by Ff or $f^\wedge$, is defined by

$$Ff(\xi) = \int_E f(x)\phi_\xi(x)d\chi(x)$$

This is similar to the L^1 situation of classical Fourier transform, where we can get the Fourier transform from taking the inner product with an element in L^∞, i.e. $\hat{f}(\xi) = \langle \phi_\xi, f \rangle$ where $\phi_\xi(x) = e^{i\xi x}$. A notion of inner product for constructible functions will be introduced later in section 5.4.

For a conic constructible function f, let f^a be the image of f under the antipodal map:

$$f^a(x) = f(-x)$$

It is more convenient to define the inverse Fourier-Sato transform through the forward Fourier-Sato transform:

Definition 5.2.2. For a conic constructible function f on E, **the inverse Fourier-Sato transform of** f, denoted by $F^{-1}f$ or $f^\vee$, is defined by the formula

$$F^{-1}f = (-1)^n(Ff)^a$$

where n is the dimension of E.

Remark 5.2.3. There are different notations for Fourier and the inverse Fourier transform, which will be used interchangeably throughout this thesis.

$$Ff = f^\wedge = \hat{f}$$

$$F^{-1}f = F^{*}f = f^{\vee} = \check{f}$$

If we have conic constructible sheaves F on E and F is corresponds to f under the isormophism 3.2.2, then the Fourier-Sato transform of F also corresponds to Ff under 3.2.2.

Therefore, since the Fourier-Sato transform and its inverse give an equivalence of categories between $D^{+}_{\mathbb{R}_+}(E)$ and $D^{+}_{\mathbb{R}_+}(E*)$ by being inverses of each other, we have the following proposition.

Proposition 5.2.4. *If f is a conic constructible function on E, then*

$$F^{-1}(Ff) = f$$

and

$$F(F^{-1}f) = f$$

Example 5.2.5. Let $E = \mathbb{R}$. Consider $f = \mathbb{1}_{[0,\infty)}$, and we would like to compute the Fourier-Sato transform of f.

Recall that $\phi_{\xi}(x) = 1$ if $\langle x, \xi \rangle \leq 0$.

For $\xi \in E^{*}$, if $\xi > 0$ then $f\phi_{\xi} = \mathbb{1}_{0}$, whose Euler integral is equal to 1.

If $\xi \leq 0$, $f\phi_{\xi} = \mathbb{1}_{[0,\infty)}$, whose Euler integral is equal to 0.

Therefore, $Ff = \mathbb{1}_{(0,\infty)}$.

This example is representative: given the characteristic function f on a closed convex cone C, then the Fourier-Sato transform of f is the characteristic function of $\text{Int}(C^0)$, the interior of the polar cone of C (see for the definition of the polar cone). The proof is a case by case computation similar to the above example and hence will be omitted.

We also have a version of the Verdier dual for constructible functions, and unsurprisingly, is compatible with the Fourier-Sato transform. This is also a direct consequence from the sheaf-theoretic result.

Definition 5.2.6. The Verdier dual of a constructible function f is given by the formula

$$Df(x) := \lim_{\epsilon\to 0} \int_X \mathbb{1}_{B_x(\epsilon)} f(t) d\chi(t)$$

The existence of this limit is guaranteed from the constructibility of f and the tameness of the underlying sets.

Proposition 5.2.7. *For any $f \in CF_{\mathbb{R}^+}(X)$,*

$$D(f^{\wedge}) = (Df)^{\vee}$$

The Fourier-Sato transform is also compatible with the Fubini theorem. If we consider a conic constructible function on the product space $X\times Y$, then the Fourier-Sato transform of each slice is equal to the Fourier-Sato transform of the given function, evaluated at the coressponding component in the dual.

Proposition 5.2.8. *Let X, Y be two finite dimensional Euclidean spaces, and let g be a conic constructible function on $X \times Y$. $f(x) = \int_Y g(x,y)d\chi(y)$. Then for any $\xi \in X^*$, $Ff(\xi) = Fg(\xi, 0)$.*

Proof.

$$\begin{aligned} Ff(\xi) &= \int_X f(x)\phi_\xi(x)d\chi(x) \\ &= \int_X \Big(\int_Y g(x,y)d\chi(y)\Big)\phi_\xi(x)d\chi(x) \\ &= \int_X \int_Y g(x,y)\phi_{(\xi,0)}(x,y)d\chi(y)d\chi(x) \\ &= Fg(\xi,0) \end{aligned}$$

where the last equality is by Fubini. □

5.3 Fourier-Sato transform and convolution

Definition 5.3.1. Let $f, g \in CF(E)$. Then $f * g(x) = \int_E f(t)g(x-t)\,d\chi$

Again, this is the composition of convolution on sheaves of constructible functions and the K-theoretic functor; we get the convolution by integrating along the anti-diagonal planes in $E \times E^*$.

Similar to the classic Fourier transform, the Fourier-Sato transform of a convolution is a product of the Fourier-Sato transforms. We provide a new proof of this fact.

Theorem 5.3.2. *Let $f, g \in CF_{\mathbb{R}^+}(E)$. Then*

$$F(f * g) = F(f)F(g)$$

Proof.

$$\begin{aligned}
F(f*g)(\xi) &= \int_E (f*g)(x)\phi_\xi(x)d\chi(x) \\
&= \int_E \Big(\int_E f(t)g(x-t)d\chi(t)\Big)\phi_\xi(x)d\chi(x) \\
&= \int_E \int_E f(t)g(x-t)\phi_\xi(x)d\chi(t)d\chi(x) \\
&= \int_E \int_E f(t)g(x-t)(\phi_\xi(t)\phi_\xi(x-t)+h_\xi(x,t))d\chi(t)d\chi(x) \\
&= F(f)(\xi)F(g)(\xi) + \int_E \int_E f(t)g(x-t)h_\xi(x,t)d\chi(x)d\chi(t)
\end{aligned}$$

where $h_\xi(x,t) = 1$ if $\langle \xi, x\rangle \leq 0$ and either $\langle \xi, t\rangle > 0$ or $\langle \xi, x-t\rangle > 0$, and 0 otherwise.

We claim that $\int_E \int_E f(t)g(x-t)h_\xi(x,t)d\chi(x)d\chi(t) = 0$, in particular $\int_E f(t)g(x-t)h_\xi(x,t)d\chi(x) = 0$ for all $t \in E$.

Note that $h_\xi(x,t) = 0$ if either $\xi = 0$ or $t = 0$. Also, if the two conditions $\langle \xi, t\rangle > 0$ and $\langle \xi, x-t\rangle > 0$ happen at the same time then $\langle \xi, x\rangle > 0$, and hence $h_\xi(x,t) = 0$. Therefore, we only need to consider them separately. For non-zero ξ and t:

1. If $\langle \xi, t\rangle > 0$, the function $g(x-t)h_\xi(x,t)$ is the function g translated to t, restricted to the half plane $\langle \xi, x\rangle \leq 0$, which t does not belong. This restricted function is a formal sum of the characteristic functions on rays, which have Euler characteristic 0.

2. If $\langle \xi, x-t\rangle > 0$, the function $g(x-t)h_\xi(x,t)$ is the function g translated to t, restricted to the region bounded by the inequalities $\langle \xi, x-t\rangle > 0$ and

$\langle \xi, x \rangle \leq 0$. Note that this does not contain the point $x = t$. The restricted function is a formal sum of the characteristic functions on half-open intervals, which also have Euler characteristic 0.

Therefore, the integral $\int_E f(t)g(x-t)h_\xi(x,t)d\chi(x) = 0$ for all $t \in E$.

As a consequence, $F(f * g)(\xi) = F(f)(\xi)F(g)(\xi)$ for all ξ. □

5.4 A pseudo inner product with respect to the Euler Integration

Similar to the way L^2 inner product is usually defined for real-valued measurable functions on measure spaces, we can construct a version of L^2 (pseudo) inner product with respect to the Euler integral.

Definition 5.4.1. Let f and g be constructible functions on X. Then the inner product of f and g is

$$\langle f, g \rangle_X = \int_X fg d\chi$$

This is clearly not an honest inner product, as it is not positive definite. As a counter example, $\langle \mathbb{1}_A, \mathbb{1}_A \rangle$ can be negative for any set A of negative Euler Characteristic.

However, we can still discuss the notion of adjoint operators, which a lot of the previously mentioned operators on constructible functions turn out to be. For the rest of this thesis, we will drop the prefix *pseudo* whenever there is no confusion.

Remark 5.4.2. This inner product for conic constructible functions is equivariant under the action of $SO(E)$ on E (in other words, the action of $SO(E)$ on E is unitary), i.e. for any $\phi \in SO(E)$, $\langle f, g\rangle = \langle \phi_* f, \phi_* g\rangle$.

Remark 5.4.3. This inner product also gives rise to a basis for conic functions. For any vector x in E, let $\Phi_x = \mathbb{1}_{\mathbb{R}_+ x}$. Then for any conic function f,

$$f = \langle f, \Phi_0\rangle \Phi_0 + \sum_{x \in B(1)} \langle f, \Phi_x\rangle \Phi_x$$

Notice that $\langle \Phi_x, \Phi_y\rangle = 0$ for x, y either lying on the unit ball or being equal to 0.

5.5 Adjoint operators

Even though the inner product with respect to the Euler integration is not an honest inner product, it is still possible to talk about adjunction.

Two operators $\Phi : CF(X) \to CF(Y)$ and $\Phi' : CF(Y) \to CF(X)$ are said to be an adjoint pair if for any $f \in CF(X)$, $g \in CF(Y)$, $\langle \Phi(f), g\rangle = \langle f, \Phi'(g)\rangle$.

It turns out that a lot of traditionally considered integral transform falls into this category.

Theorem 5.5.1. *Fourier-Sato transform is a self-adjoint operator with respect to this inner product.*

Proof. Let $f \in CF_{\mathbb{R}_+}(E)$, $h \in CF_{\mathbb{R}_+}(E^*)$.

$$\begin{aligned}
\langle Ff, h\rangle &= \int_{E^*} Ff(\xi)h(\xi)d\chi(\xi) \\
&= \int_{E^*} \Big(\int_E f(x)\phi_\xi(x)d\chi(x)\Big)h(\xi)d\chi(\xi) \\
&= \int_{E^*}\int_E f(x)\phi_\xi(x)h(\xi)d\chi(x)d\chi(\xi) \\
&= \int_{E^*}\int_E f(x)\phi_x(\xi)h(\xi)d\chi(x)d\chi(\xi) \\
&= \int_E\int_{E^*} f(x)\phi_\xi(x)h(\xi)d\chi(\xi)d\chi(x) \\
&= \int_E \Big(\int_{E^*} \phi_\xi(x)h(\xi)d\chi(\xi)\Big)f(x)d\chi(x) \\
&= \int_E F^{-1}h(x)f(x)d\chi(x) \\
&= \langle f, Fh\rangle
\end{aligned}$$

□

Corollary 5.5.2. *Given* $f, g \in CF_{R_+}(E)$, $\langle Ff, F^{-1}g\rangle = \langle f, g\rangle$

Proof. From proposition 5.2.4, we have $f^{\wedge\vee} = f$. Applying this relation to the previous theorem proves the result. □

This corollary is slightly different from the situation in classical Fourier transform, where the Fourier transform is dual to the inverse Fourier transform.

The Fourier transform preserves the L^2 norm by Plancherel's theorem, and as a consequence it preserves the L^2 inner product, i.e. given L^2 functions f, g, we have $\langle Ff, Fg\rangle = \langle f, g\rangle$. Therefore, the dual operator of the Fourier transform with respect to the L^2 norm is the inverse Fourier transform, in contrast with the Fourier-Sato transform on constructible functions, where the transform is self-dual.

We have a similar result for the level-set Radon transform.

Theorem 5.5.3. *The level-set Radon transform and its inverse are adjoint.*

Proof. Let $f \in CF(\mathbb{R}^n)$, $g \in CF(S^{n-1} \times \mathbb{R})$.

Let K be the indicator function on the set $\{(x, (\xi, t)) \colon x \cdot \xi = t\}$

$$(\mathscr{R}_K f)(\xi, t) = \int_{\mathbb{R}^n} f(x)\, K(x, \xi, t)\, d\chi(x).$$

$$\begin{aligned}
\langle \mathscr{R}_K f, g \rangle &= \int_{S^{n-1}\times\mathbb{R}} \mathscr{R}_K f(\xi, t) g(\xi, t)\, d\chi(\xi, t) \\
&= \int_{S^{n-1}\times\mathbb{R}} \Big(\int_{\mathbb{R}^n} f(x) K(x, \xi, t) d\chi(x) \Big) g(\xi, t) d\chi(\xi, t) \\
&= \int_{\mathbb{R}^n} \Big(\int_{S^{n-1}\times\mathbb{R}} K(x, \xi, t) g(\xi, t) d\chi(\xi, t) \Big) f(x) d\chi(x) \\
&= \int_{\mathbb{R}^n} \mathscr{R}_K g(x) f(x)\, d\chi(x) \\
&= \langle f, \mathscr{R}_K g \rangle
\end{aligned}$$

□

It is natural to ask if the persistent homology transform is dual to its inverse or not, since the persistent homology transform can be thought of as a by product of the X-ray (Radon) transform. However, this is not the case. X-ray transform is self-dual with respect to the L^2 inner product because it is self-dual with respect to the Schapira's inversion: the inverse kernel is the original kernel itself. This does not happen often, and indeed does not work for the persistent homology transform.

5.6 Distributions and the Fourier-Sato transforms of distributions

In the classical theory of Fourier transforms, one usually works with the space of Schwartz functions. This is the space of complex-valued functions on the real line whose derivatives all decay rapidly. The Schwartz space is a good function space for the Fourier transform to work on, since the Fourier transform of a Schwartz function is again a Schwartz function. In other words, the Fourier transform is a linear isomorphism between the Schwartz space and itself, with the inverse being the inverse Fourier transform. Denote the space of Schwartz functions by $S(\mathbb{R})$.

One natural question to ask is whether one can define an appropriate notion of Fourier transform for functions that have less regularity, where the Fourier integrals do not converge, or sometimes even the derivatives do not exist. This is the motivation to construct a notion of *generalized functions*, where regularity is no longer an issue.

In functional analysis, one considers the dual of the Schwartz space, namely the space of (continuous) linear functionals on $S(\mathbb{R})$. This dual is usually denoted by $S'(\mathbb{R})$, and its elements are called *tempered distributions* (the word *tempered* is often dropped if there is no confusion). These distributions are exactly the generalized functions that we want to investigate.

The reason is rather simple: given any (locally) integrable function f, there is a natural linear functional associated to it, namely

$$T_f : S(\mathbb{R}) \to \mathbb{C}$$
$$g \mapsto \int_{\mathbb{R}} f \cdot g$$

If the functions f and g are Schwartz, then we have the identity

$$\int_{\mathbb{R}} \hat{f} \cdot g = \int_{\mathbb{R}} f \cdot \hat{g}$$

which is simply a routine application of Fubini.

One can use this identity as an inspiration to define the Fourier transform for a distributions T by

$$T^{\wedge}(\varphi) := T(\varphi^{\wedge})$$

where $T^{\wedge}$ is also a tempered distribution on the space of Schwartz functions.

Moving to the world of constructible functions, since the Fourier-Sato transform of a conic constructible function is also conic, it is reasonable to use the space of conic constructible functions $CF_{\mathbb{R}_+}(X)$ on a vector space X as the space of test functions.

Definition 5.6.1. An Euler distribution on X is a linear functional from the space of conic constructible functions $CF_{\mathbb{R}_+}(X)$ to $\mathbb{Z}$.

Similar to the case of Schwartz functions, associated to any constructible function f is a regular distribution defined by

$$T_f : CF_{\mathbb{R}_+}(X) \to \mathbb{Z}$$
$$g \mapsto \int_X (f \cdot g) d\chi$$

Since the Fourier transform of a conic function is again conic, let

$$T^\wedge(\varphi) := T(\varphi^\wedge)$$

for any Euler distribution T.

Applying the Fourier transform to the regular distribution of constructible functions extends the previous notion of Fourier-Sato transform — that is, Fourier-Sato transform is no longer restricted to the space of *conic* functions, as it can now be defined for any constructible functions treated as distributions.

One of the most important distributions in functional analysis is the Dirac delta, usually denoted δ_p for some point $p \in X$. When applied to a constructible f, δ_p simply returns the value $f(p) \in \mathbb{Z}$. In the world of constructible functions, the equivalence of the Dirac delta is exactly the regular distribution associated to the characteristic function on the point p. That is,

$$\delta_p = \int_X \mathbb{1}_p \cdot (-) d\chi$$

Then, the Fourier transform of the Dirac delta is

$$\delta_p^\wedge(f) = \delta_p(f^\wedge) = f^\wedge(p)$$

By definition, $f^\wedge(p) = \int_H f d\chi$, where H is the half space "opposite" to p (i.e. the collection of points in $x \in X$ so that $\langle p, x \rangle \leq 0$). Therefore, the Fourier transform of δ_p is equal to the regular distribution associated to the characteristic function on H. In other words, we can say that the Fourier transform of the pointed function $\mathbb{1}_p$ is the characteristic function on the half space opposite to p.

Chapter 6

Combinatorial Species

Combinatorial species is an organized way of documenting structures on finite labelled sets. For each species, the theory offers a counting method for these structures using generating functions. Combinatorial species is pioneered by Andre Joyal in the celebrated paper [21], where he proves the Cayley's formula for the number of trees on a given set of vertices using this mintly invented language. The theory has been picked up by mathematicians working in a wide range of fields [1][15], with the most recent development being in Homotopy Type Theory [36].

The theory of combinatorial species is especially interesting because associated to each species are a wide varieties of *generating functions* and other power series, which are compatible with the operations on species. This compatibility arises from a rather simple fact: the cardinality of a disjoint union of two finite sets is the sum of the cardinalities, and the cardinality of a product of two finite sets is the product of the cardinalities.

We notice that these properties is not unique to cardinalities; the Euler characteristic (of reasonable sets) also satisfies additivity and multiplicativity. In fact, there are other Euler-characteristic-like invariants that also possess these properties, and if a notion of *generalized species* is defined carefully, then the associated generating functions with respect to these Euler-characteristic-like invariants are equal to some known objects of interest. Among them are magnitudes, configuration spaces, and the Euler integral, unsurprisingly.

6.1 Combinatorial species and the associated generating functions

In this section, we quickly recall some general setups for combinatorial species, most of which can be found in [7].

Let $\mathscr{B}$ be the category of finite sets with the morphisms being isomorphisms between the sets (i.e. $\mathscr{B}$ is the groupoid of finite sets). Sometimes $\mathscr{B}$ is referred to as the *categorification* of the set of natural numbers $\mathbb{N}$, as its morphisms identify sets of the same cardinality. Sometimes we may also want to *decategorify* this category $\mathscr{B}$ by ignoring its categorical structure and considering just the underlying set of natural numbers instead, which will be discussed later.

Now, we will define a combinatorial species.

Definition 6.1.1. A combinatorial species is a functor $F : \mathscr{B} \to \mathscr{B}$.

For each finite set A in $\mathrm{Ob}(\mathscr{B})$, $F(A)$ is called a **structure** on A. The isomorphism σ between two sets in $\mathscr{B}$ is sometimes called the **transport** from U to V.

Example 6.1.2.

1. Species E of sets: $E(A) = \{A\}$.
2. Species E_n of $n - sets$: $E_n(A) = \{A\}$ if $\mathbf{card}(A) = n$, $\emptyset$ otherwise.
3. Species S of permutations: $S(A)$ is the set of all permutations on A.
4. Species L of linear orders: $L(A)$ is the set of all linear orders on A.

5. Species P or partitions: $P(A)$ is the set of all partitions on A.

Two species are said to be isomorphic if the two functors are naturally isomorphic.

Since F is a functor and the morphisms on $\mathscr{B}$ are only isomorphisms on sets of the same cardinality, the cardinality of $F(A)$ depends only on the cardinality of A, and we can write the cardinality of $F(A)$ as $\mathbf{card}(F(n))$ where $\mathbf{card}(A) = n$ without any confusion.

The cardinality, however, does not guarantee isomorphism: for example, the species of permutations S and the species of linear orders L has the same cardinality $\mathbf{card}(S(n)) = \mathbf{card}(L(n)) = n!$ for all n, but there exists no natural isomorphism between the two functors. A quick look at the case $n = 2$, where both species have 2 elements, shows this non-functoriality. If we apply the bijection that swaps the place of the two elements in the original set, then the 2 elements in the species of permutation stays the same, while the 2 elements in the species of linear orders swap their places.

That being said, the cardinality still provides an invariant for combinatorial species and its importance is of highest essence. The proof by Joyal in [21] uses the fact that the species of bi-pointed trees and the species of endofunctions have the same cardinality everywhere in order to count the number of rooted trees, even though these two species are not isomorphic, since the construction depends on a particular choice of functions.

For each species F, we define two corresponding generating functions, namely the ordinary generating function (ogf) and the exponential generating function (egf), as follows:

Definition 6.1.3. Given a combinatorial species F,

$$\mathrm{ogf}(F) := \sum_{n=0}^{\infty} \mathbf{card}(F(n))x^n$$

$$\mathrm{egf}(F) := \sum_{n=0}^{\infty} \mathbf{card}(F(n))\frac{x^n}{n!}$$

In the literature, most of the time when *the generating function of* F or $F(x)$ is mentioned, the exponential generating function egf is usually the one being referred to.

The ordinary generating function is closesly related to, but not to be confused with, the *unlabelled generating function of* F or type generating function, which is defined to be $\sum_n(\mathbf{card}\tilde{F}(n))x^n$, where $\tilde{F}(n)$ is the orbit of the combinatorial structures under the S_n-action.

Example 6.1.4. 1. Species E of sets: $\mathbf{card}(E(A)) = 1$ for all A

$$\mathrm{ogf}(E) = \sum x^n = \frac{1}{1-x}$$

$$\mathrm{egf}(E) = \sum \frac{x^n}{n!} = e^x$$

2. Species E_n of $n - sets$: $\mathbf{card}E(k) = 1$ if $k = n$ and 0 otherwise.

$$\mathrm{ogf}(E_n) = x^n$$

$$\mathrm{egf}(E_n) = \frac{x^n}{n!}$$

3. Species S of permutations and the species of linear orders L have the same generating functions: $\mathbf{card}(S(n)) = \mathbf{card}(L(n)) = n!$

$$\mathrm{egf}(S) = \mathrm{egf}(L) = \sum_{n \geq 0} n! \frac{x^n}{n!} = \frac{1}{1-x}$$

6.2 Operations on combinatorial species

What makes combintorial species special is that we can define different operations on the category of species, and that the generating functions behave nicely under these operations. We will only introduce a few basic operations here, as they will have direct analogues for generalized species: (1) addition, (2) multiplication, and (3) differentiation. Interested reader can refer to [7] for more operations on combinatorial species.

6.2.1 Addition

Let F and G be two species. Define $F + G$ to be the functor taking a set A to the disjoint union $F(A) \sqcup G(A)$, and a morphism α to its corresponding component in the disjoin union.

We have $\mathbf{card}(F + G)(A) = \mathbf{card}(F(A)) + \mathbf{card}(G(A))$, and as a consequence,

$$\mathrm{ogf}(F) + \mathrm{ogf}(G) = \mathrm{ogf}(F + G)$$

$$\mathrm{egf}(F) + \mathrm{egf}(G) = \mathrm{egf}(F + G)$$

Example 6.2.1. 1. Let E be the species of sets, $\mathbb{1}$ be the species of empty-set (taking the value $\{A\}$ if and only if A is empty), and E_+ be the species of non-empty sets (taking the value $\{A\}$ if and only if A is non-empty).

Then $E = 1 + E_+$. The corresponding ogf equation reads:

$$\text{ogf}(1) + \text{ogf}(E_+) = 1 + \frac{x}{1-x} = \frac{1}{1-x} = \text{ogf}(E)$$

2. Let E_{even} and E_{odd} be the species of sets of even and odd cardinalities, respectively.

Then $E = E_{\text{even}} + E_{\text{odd}}$. The corresponding ogf equation reads:

$$\text{ogf}(E_{\text{even}}) + \text{ogf}(E_{\text{odd}}) = \frac{1}{1-x^2} + \frac{x}{1-x^2} = \frac{1}{1-x} = \text{ogf}(E)$$

3. In a similar manner, $E = \sum_{n \geq 0} E_n$ where E_n's are species of n-sets.

6.2.2 Multiplication

There are a few types of products we can define for combinatorial species, but for the purpose of this thesis, we only choose to discuss the following two products.

Definition 6.2.2. (Cartesian Product)

$(F \times G)(A) := F(A) \times G(A)$

Definition 6.2.3. (Partitional Product, Cauchy Product)

$(F \cdot G)(A) := \coprod_{A=U_1+U_2} F(U_1) \times G(U_2)$ where the coproduct is taken over all possible partitions of A into U_1 and U_2.

For any $(F \cdot G)$-structure (f, g) on U, if σ is a transport $\sigma : U \to V$, then $(F \cdot G)(\sigma)(f, g) := (F(\sigma_{|U_1})(f), G(\sigma_{|U_2})(g))$, namely we assign the transport partition-wise.

As it turns out, the Cartesian product is not quite suitable to the theory of combinatorial species since it does not preserve the product of the generating functions; we would have to introduce a new product on the space of generating functions called the Hadamard product. It is defined by the formula

$$(\sum a_n \frac{x^n}{n!}) \times (\sum b_n \frac{x^n}{n!}) := \sum (a_n b_n) \frac{x^n}{n!},$$

and it is clear that $\text{egf}(F) \times \text{egf}(G) = \text{egf}(F \times G)$.

On the other hand, the partitional product works well with the usual product on generating functions (where we multiply generating functions as power series).

Proposition 6.2.4. *Given two combinatorial species F and G, we have*

$$\mathit{egf}(F) \cdot \mathit{egf}(G) = \mathit{egf}(F \cdot G)$$

and

$$\widetilde{F} \cdot \widetilde{G} = \widetilde{F \cdot G}$$

The proof of this proposition is a simple manipulation of generating function [7].

Note that here we again mention the type generating function $\widetilde{F}$ instead of ogf(F). These two sometimes coincide (especially in the cases of our interests) but they are not exactly the same entity.

Let 0 be the zero species, mapping every set to the empty set. Then $0 \cdot F = 0$ for every F.

Recall the empty species 1 to be the species that has 1 structure for only the empty set. Then $1 \cdot F = F$ for all F.

If we define $n := 1 + ... + 1$, then the species $n \cdot F$ is isomorphic to the species $F + ... + F$, since the partitional product can be proved to be commutative (i.e. $F \cdot G$ is isomorphic to $G \cdot F$).

Example 6.2.5. Let sE be the species of subsets, $sE(A) = \{S|S \subset A\}$. Then sE is isomorphic to the partitional product $E \cdot E$, where E is again the species of sets.

We have egf(sE) $= \sum_n 2^n \frac{x^n}{n!} = e^{2x}$, which is equal to egf$(E)^2 = (e^x)^2 = e^{2x}$.

6.2.3 Differentiation

Another operation that we can define on combinatorial species is the derivative. Since the derivative of a generating function simply moves the indices by 1 (with an appropriate scaling in the case of the exponential generating function), this suggests a similar process if we want to define such a notion for combinatorial species.

Definition 6.2.6. Given a combinatorial species F, let F' to be the species that maps $F'(A)$ to $F(A + [1])$, where [1] is the set of 1 element, and the morphisms to the extended morphisms that takes [1] to itself.

Proposition 6.2.7. *Given a species F, we have*

$$\mathit{egf}(F') = \frac{d}{dx}\mathit{egf}(F)$$

Example 6.2.8. The derivative of the species of sets E is itself, since this species always return the set of 1 element. Since egf$(E) = e^x$, this simply gives us back the equation $\frac{d}{dx}e^x = e^x$.

6.3 Generalized species

Notice that in order to define the egf and ogf, we only rely on the fact that the cardinality of the sets $F(A)$ only depends on the cardinality of A. This means that if we replace the category of finite sets with any other category with a good notion of cardinality that is invariant under isomorphisms (e.g. Euler characteristic), what we get is a generalized species that is still suitable for the generating functions.

This approach has been discussed in ([21], [1], [36]), but not with emphasis on the generating functions. Joyal ([21]) was trying to extend species to vector spaces, while [36] was mainly on the homotopy type/computer science aspects of species. In [1], they associate to each species a groupoid and then show that the egf of the species is the same as the Euler characteristic of the groupoid.

Go back to the previous discussion, since a combinatorial species is simply a functor from the category $\mathscr{B}$ of finite sets to itself, formally we can define a generalized species as a functor F from $\mathscr{B}$ to a category $\mathscr{C}$ with an Euler characteristic χ.

Note that this is an Euler characteristic **on** $\mathscr{C}$, not to be confused with an Euler characteristic **of** $\mathscr{C}$. This has not been properly defined, but for the purpose of this thesis, we define our Euler characteristic on a category $\mathscr{C}$ to be a mapping from $\mathrm{Ob}(\mathscr{C})$ to $\mathbb{Z}$ that satisfies the following conditions:

1. χ is invariant under isomorphism, i.e. if A and B are isomorphic in $\mathscr{C}$, then $\chi(A) = \chi(B)$.

2. χ is additive with respect to finite coproduct.

$$\chi(A \coprod B) = \chi(A) + \chi(B)$$

3. χ is multiplicative with respect to finite product.

$$\chi(A \prod B) = \chi(A) \cdot \chi(B)$$

In the case of combinatorial species, our category $\mathscr{C}$ is the category $\mathscr{B}$ of finite sets, and the Euler characteristic is the cardinality of the finite sets.

In this category, since the coproduct is the disjoint union of finite sets, and the product is the Cartesian product of finite sets, cardinality satisfies our conditions for the Euler characteristic.

Now, we are ready to define our generalized species, their associated generating functions, and the operations on them.

Definition 6.3.1. A generalized species is a functor from the category of finite sets $\mathscr{B}$ to a category $\mathscr{C}$ (where finite products and coproducts exist) with an Euler characteristic χ.

Definition 6.3.2. Given a generalized species F on a category $\mathscr{C}$ with an Euler characteristic χ, the associated generating functions are defined as

$$\mathrm{ogf}(F) := \sum_{n\geq 0} \chi(F[n])x^n$$
$$\mathrm{egf}(F) := \sum_{n\geq 0} \chi(F[n])\frac{x^n}{n!}$$

Note that these generating functions are well-defined because of the condition we impose on Euler characteristic, namely two objects that are isomorphic will have the same Euler characteristic.

Definition 6.3.3. Let F and G be two generalized species on a category $\mathscr{C}$. Then $F + G$ is the functor that takes $A \in \mathrm{Ob}(\mathscr{B})$ to $F(A) \coprod G(A)$, and a morphism ϕ on A to $F(A)(\phi) \coprod G(A)(\phi)$.

We can also define the Cauchy product and partitional (Hadamard) product as follows:

Definition 6.3.4. Let F and G be two generalized species on a category $\mathscr{C}$. Then the Cauchy product $F \times G$ is the functor that takes A to $F(A) \prod G(A)$, and the partitional product $F \cdot G$ is the functor that takes A to $\coprod_{U_1+U_2=A} F(U_1) \prod G(U_2)$, where the coproduct is taken over all partitions of A into finite subsets U_1 and U_2. Morphisms are taken to the coproducts and products accordingly.

Definition 6.3.5. The derivative of a generalized species F is a functor F' that takes a set $A \in \mathrm{Ob}(\mathscr{B})$ to $F(A + [1])$, where [1] is the set of 1 element, and the morphisms to the extended morphisms where [1] is mapped to itself.

With these definition, the addition, multiplication, and differentiation rules for generating functions soon follow.

Proposition 6.3.6. *Given two generalized species F and G on a category $\mathscr{C}$, we have:*

$$\mathit{ogf}(F + G) = \mathit{ogf}(F) + \mathit{ogf}(G)$$

$$\mathit{egf}(F + G) = \mathit{egf}(F) + \mathit{egf}(G)$$

$$\mathit{egf}(F \cdot G) = \mathit{egf}(F) \cdot \mathit{egf}(G)$$

$$\mathit{egf}(F') = \frac{d}{dx}\mathit{egf}(F)$$

The proof is similar to the case of combinatorial species, and hence omitted.

In the next few sections, we will explore examples of some known objects that can be described by the language of generalized species.

6.4 Magnitude as a species

Magnitude is an invariant on enriched categories. Under appropriate conditions for the underlying categories, magnitudes share some properties of cardinality and Euler characteristic, namely additive and multiplicative, so it is not too surprising that we can formalize magnitude in a framework that Euler characteristic is heavily involved.

A particular instance of enriched category that has been well investigated is finite graphs, and in this thesis, we also focus mainly on magnitudes for graphs. For a more detailed introduction to the subject, see [25].

Let G be a connected, finite graph, and let N be the number of vertices.

Let Z_G be an $N \times N$ matrix such that at each entry (i, j) corresponding to the pair of vertices (i, j), the value of the matrix is $Z_G(i, j) = q^{d(i,j)}$, where $d(i, j)$ is the distance between the two vertices on the graph, i.e. the length of the shortest path on the graph to go from vertex i to vertex j. This distance is finite since our graph is connected. In the general case where we allow the graph to be non-connected, the convention is to set those entries on the matrix to 0.

This matrix is invertible in $\mathbb{Q}(q)$, since the determinant has constant term 1 (when we evaluate the matrix at $x = 0$, we get the identity matrix).

Let $\#G(q) = \sum_{(i,j)} Z_G^{-1}(i, j)$. We call $\#G(q)$ the **magnitude of the graph** G.

This construction is general and is an instance of the **Mobius inversion**. The technique was first considered by Rota in [28] as a method to define Euler characteristic for a poset. It has since been generalized to define Euler characteristics for groupoids [1] and for categories [24].

$\#G(x)$ is not just an element in $\mathbb{Q}(x)$; it is a power series. In fact, we can write the coefficients explicitly in terms of the number of chains of specific length. The following proposition is quoted verbatim from [25].

Proposition 6.4.1 ([25], Proposition 3.9)**.** *For any graph G,*

$$\#G(q) = \sum_{k=0}^{\infty} \sum_{x_o \neq x_1 \neq ... \neq x_k} q^{d(x_0,x_1)+...+d(x_{k-1},x_k)} \in \mathbb{Z}[[q]]$$

where $x_0, ..., x_k$ denote vertices of G. That is, writing

$$\#G(q) = \sum_{n=0}^{\infty} c_n q^n \in \mathbb{Z}[[q]], \tag{6.4.1}$$

$$c_n = \sum_{k=0}^{\infty} (-1)^k |\{(x_0, ...x_k) : x_0 \neq x_1 \neq ... \neq x_k, d(x_0, x_1) + ... + d(x_{k-1}, x_k) = n\}| \tag{6.4.2}$$

The alternating sum in c_n suggests that c_n is Euler-characteristic-like in nature, in the sense that there is a hidden homology (or cohomology) theory where the cardinalities of the above sets are merely the Betti numbers.

This gives rise to the notions of magnitude (co)homology, which was first introduced by Hepworth and Willerton in [19], [18].

We define a magnitude chain complex $MC_{*,*}$ by:

$$MC_{k,l} = \mathbb{Z}\{(x_0, ...x_k) | x_i \neq x_{i+1}, d(x_0, x_1) + ... + d(x_{k-1}, x_k) = l\}$$

where k and l are non-negative integers.

The boundary map $\partial : MC_{k,l} \to MC_{k-1,l}$ is the alternative sum of the maps ∂_i, $\partial = \sum_{i=1}^{k-1} (-1)^i \partial_i$, where

$$\partial_i(x_0, ...x_k) = (x_0, ...\hat{x}_i, ...x_k)$$

if the length of the chain is preserved (and 0 otherwise).

Taking the homology of this chain complex, we obtain the **magnitude homology** of G. Note that this is a bigraded homology theory: for each fixed l, we have a collection of homology groups $H_{(k,l)}$ defined by the above boundary map ∂.

For a fixed n, consider the chain complex $M_{*,n}$ and the homology groups $H_{*,n}$. Since the alternating sum of the Betti numbers is equal to the alternating sum of the rank of the groups in the chain (these are all free group generated by chains of the form $(x_0, ..., x_k)$) for finite chain complex, we obtain formula 6.4.2 (see [19]):

$$\begin{aligned}
\chi(H_{*,n}) &= \sum_k (-1)^k \operatorname{rank}(H_{k,n}) \\
&= \sum_k (-1)^k \operatorname{rank}(M_{k,n}) \\
&= \sum_k (-1)^k |\{(x_0, ...x_k) : x_0 \neq x_1 \neq ... \neq x_k, d(x_0, x_1) + ... + d(x_{k-1}, x_k) = n\}| \\
&= c_n
\end{aligned}$$

In other words, equation 6.4.1 can be rewritten as

$$\#G(q) = \sum_{n \geq 0} \chi(H_{*,n}) q^n \tag{6.4.3}$$

This formula is our motivation to define for each finite graph G a generalized species M_G so that the associtated ogf is exactly its magnitude.

Definition 6.4.2. Given a finite graph G, the species M_G of magnitude associated to G is the functor from the category of finite sets $\mathscr{B}$ to the category **ChCplx** of chain complexes as follows:

$$M_G : \mathscr{B} \to \mathbf{ChCplx}$$

$$A \mapsto MC_{*,\mathrm{card}(A)}$$

Consider the Euler characteristic on the category of chain complexes. All of the chain complexes that are in the image of this functor are finite (i.e. only finitely many non-zero terms) since the chain has maximum length $\mathbf{card}(A)$. Therefore, equation 6.4.3 is satisfied, and hence the ogf associated to M_G is equal to the magnitude of the graph (renaming the variables appropriately):

$$\mathrm{ogf}(M_G) = \sum_n \chi(MC_{*,n})q^n = \#G(q)$$

6.5 Configuration spaces as a species

In the previous section, we were merely using finite sets as a grading (i.e. only using their cardinalities) and somewhat ignoring the labelled nature of species (combinatorial species was first introduced as a method to categorize labelled structures). In this section, we will consider a different class of objects: configuration spaces. As these spaces can be defined using labeled points in a finite set, they fit more naturally into the language of species.

First, recall the definition of the configuration spaces.

Let X be a simplicial complex. For $n > 0$, let $\mathrm{Conf}_n(X) = X^n \setminus \bigcup \Delta_{i,j}$, where $\delta_{i,j} = \{(x_1, ...x_n | x_i = x_j\}$ is the diagonal in the (i, j) plane. A configuration space can be thought of as a collection of distinguished points in the space X, i.e. a collection of configuration.

As a convention, $\text{Conf}_0(X)$ is the set of 1 element, and $\text{Conf}_1(X) = X$.

Since the points x_i are restricted to be different in the configuration space, we can define $\text{Conf}_n(X)$ as a set of injective maps. Specifically, for a finite set A, consider the space of maps $\{f : A \hookrightarrow X\}$ with the point-open topology. This space is homeomorphic to the configuration space $\text{Conf}_n(X)$, where n is the cardinality of the set A.

With this in mind, we can construct a configuration space generalized species associated to a simplicial complex.

Definition 6.5.1. Given a simplicial space X, let the generalized species $\text{Conf}(X)$ be the following functor:

$$Conf(X) : \mathscr{B} \to \mathbf{Top}$$

$$S \mapsto \{f : A \hookrightarrow X\}$$

$$(S \xrightarrow{\phi} T) \mapsto (. \circ \phi)$$

Let χ be the Euler characteristic of the configuration spaces. Then the egf of $\text{Conf}(X)$ is the object of interest in [14] where the exponential generating function is computed in terms of the Euler characteristic of its simplices:

Theorem 6.5.2 (Gal, [14]). *For any cell σ in X, let L_σ be the normal link. Then*

$$egf(Conf(X)) = \prod_{\sigma} \left(1 + (-1)^{dim\ \sigma}(1 - \chi(L_\sigma))x\right)^{(-1)^{dim\ \sigma}}$$

Remark 6.5.3. This result is extended in [2]. Baryshnikov investigates the formal power series associated to exotic configuration spaces, whose definition depends on the choice of diagonals being excluded from the set.

6.6 Euler integration as a species

In another direction, on a similar line of thoughts as [1], we can consider "decategorifying" combinatorial species by replacing the category of finite sets and groupoids by the set of natural numbers and groups respectively. The only requirement is to havea good notion of Euler characteristic on these groups. One such candidate is the group of constructible functions $CF(X)$ on a space X, with the Euler characteristic being the Euler integral.

Definition 6.6.1. A simplified species is a map from the natural number to a group G with an Euler characteristic χ.

Definition 6.6.2. The ogf and egf of a simplified species are the generating functions

$$\mathrm{ogf}(F) = \sum_n \chi(F(n))x^n$$
$$\mathrm{ogf}(F) = \sum_n \chi(F(n))\frac{x^n}{n!}$$

Definition 6.6.3. For a given constructible function $f \in CF(X)$, we associate with it a simplified species exc(f) called **the excursion species of** f by:

$$\begin{aligned}\mathrm{exc}(f) : \mathbb{N} &\to CF(X)\\ n &\mapsto \mathbb{1}_{\{f>n\}} - \mathbb{1}_{\{f<-n\}}\end{aligned}$$

It might seem unnatural to define a define a simplified species in terms of excursion sets instead of level sets, but as it turns out, using excursion proves to have some certain advantages. First, the corresponding ordinary generating function ogf of this simplified species recovers the Euler integral of the given constructible function.

Proposition 6.6.4. *Given a constructible function $f \in CF(X)$, if we evaluate the ordinary generating function of the species exc(f) associated to the function f, we get the Euler integral of the given function. In other words,*

$$ogf(exc(f))(1) = \int f d\chi$$

Proof. Given a constructible function f and its associated excursion species exc(f), we have

$$\begin{aligned}\mathrm{ogf}(\mathrm{exc}(f))(1) &= \sum_{n\geq 0}\chi(\mathrm{exc}(f)(n)) \\ &= \sum_{n\geq 0}\int(\mathbb{1}_{\{f>n\}} - \mathbb{1}_{\{f<-n\}})d\chi \\ &= \sum_{n\geq 0}(\chi(\{f>n\}) - \chi(\{f<-n\}) \\ &= \sum_{-\infty}^{\infty}(n\chi(\{f=n\}) \\ &= \int f d\chi\end{aligned}$$

□

The second advantage that excursion sets have over level sets is that in practice, excursions are much more robust and stable when it comes to computation. For example, either noise or under sampling will certainly come up, and such problem would potentially return the wrong Euler characteristics, as the underlying topology of the sets is miscalculated. However, if the constructible function is reasonably nice, the Euler integral of such functions can be recovered by just the zeroth Betti number, i.e. the number of connected components, of the excursion sets [9].

Also, for positive constructible functions on $\mathbb{R}^n$, when restricted to a line, the ogf's are in 1-to-1 correspondence to their (discrete) persistent homology transform. The n^{th} coefficient in the power series corresponds to the Euler characteristic of the n^{th} excursion set of the function. Note that requiring a constructible function to be positive is not reasonable: if the function is bounded, apply a global up-shift by adding $c\mathbb{1}_{\mathbb{R}^n}$, which can be later subtracted from the integral.

That being said, the level set species can still possess some interesting property.

Definition 6.6.5. For a constructible function $f \in CF(X)$, **the level set species of** f $\text{lvl}(f)$ is a simplified species defined by:

$$\text{lvl}(f) : \mathbb{N} \to CF(X)$$
$$n \mapsto \mathbb{1}_{\{f=n\}}$$

Proposition 6.6.6. *The Euler integral of a positive constructible function f is equal to the derivative of the ordinary generating function of lvl(f) evaluated at 1.*

Proof.

$$\begin{aligned}\frac{d}{dx}\text{ogf}(\text{lvl}(f)) &= \sum_{n\geq 0} \chi(\text{lvl}(n))x^n \\ &= \frac{d}{dx}\sum_{n\geq 0}\Big(\int \mathbb{1}_{\{f=n\}} d\chi\Big)x^n \\ &= \sum_{n\geq 0} n\chi(\{f=n\})x^{n-1}\end{aligned}$$

Setting $x = 1$ returns the Euler integral. □

We can also ask the question of whether the Fubini theorem would hold for the simplified species. If we insist on having the two generating functions to be equal (not just when evaluated at 1), then we need the pushforward to preserve the Euler characteristic of the excursion set.

Proposition 6.6.7. *Let $G : X \to Y$ be a tame map. If G satisfies the condition that*

$$\int_Y (\mathbb{1}_{G_*f>n} - \mathbb{1}_{G_*f<-n})d\chi(y) = \int_X (\mathbb{1}_{f>n} - \mathbb{1}_{f<-n})d\chi(x)$$

then $ogf(exc(f)) = ogf(exc(G_*f))$

One instance of such a map is a translation, which can be rewritten as a convolution by a pointed-function g. In this particular case, we can rewrite the conclusion above as $ogf(\text{exc}(f * g)) = ogf(\text{exc}(f)) \cdot ogf(\text{exc}(g))$. Though this is vacuous in itself (as $ogf(\text{exc}(g)) = 1$ here) and it is nothing more than just a statement on the classic Euler integral, it still begs the question of when that is true. An excursion-preserving convolution would be interesting, since in practice calculating Euler integral of excursions is much more stable than that of level sets, in general.

6.7 Magnitude as a cardinality

Even in classical combinatorial species, sometimes the structures are not only sets but they can also contain some additional underlying information. For example, the sets can have a natural graph defined on them, as in the case of the species of permutations, where we can define a graph for each set of permutations. Each permutation is connected to another by an edge if and only if there is they differ by a transposition, namely a permutation that only exchanges two elements in the set.

The graph of permutations has a magnitude, which is itself an Euler-characteristic-like invariant. Note that magnitudes do not quite satisfy our condition of an Euler characteristic: it fails to be multiplicative. Magnitudes is only multiplicative with respect to the Cartesian product $\square$ of graphs, while the product of the category of graphs is the tensor product $\times$. Even though these two products are different, they have the same number of vertices, which is the free coefficient in the magnitude power series. At the same time, the number of vertices is simply the cardinality of the set without the graph structure, which arises from the original combinatorial species. Therefore, though we cannot use magnitude (in the form of a power series), it suggests that we can use the valuation of the generating function as a valid choice of Euler characteristic. This is a motivation for us to define a notion of generalized (and simplified) 2-species.

6.8 Generalized and simplified 2-species

Definition 6.8.1. A generalized 2-species is a functor from the category $\mathscr{B}$ of finite sets to a full subcategory of the category of species (with morphisms being isomorphisms) where the evaluation of the (exponential) generating functions at fixed value a all converge, and the evaluation is the Euler characteristic of this category.

One instance of this phenomenom is the real-valued Euler Integral. This integration theory is a natural extension to the integer-valued constructible Euler integration. Consider a space X with an o-minimal structure (see [12] for a more thorough discussion on o-minimal structure). The purpose of this o-minimal structure is to specify a collection of *tame* sets, which behave nicely under topological set operations. Let Def(X) be the set of definable functions from X to $\mathbb{R}$ with the standard o-minimal structure. Now, we want to construct an integration theory on this set of functions.

Definition 6.8.2. Let f be a definable function on X. Then the upper and lower Euler integral of f are

$$\int_X f\lfloor d\chi\rfloor = \lim_{n\to\infty}\frac{1}{n}\int_X \lfloor nf\rfloor d\chi \tag{6.8.1}$$

$$\int_X f\lceil d\chi\rceil = \lim_{n\to\infty}\frac{1}{n}\int_X \lceil nf\rceil d\chi \tag{6.8.2}$$

These limits can be proved to be well-defined. There are also formulas to compute the upper and lower integrals using excursion sets, similar to the case of integer-valued constructible functions (see [4]).

Proposition 6.8.3. *Let f be a definable function, then*

$$\int_X f\lfloor d\chi\rfloor = \int_{-\infty}^{\infty} \chi(\{f\geq s\}) - \chi(\{f<-s\})ds$$

$$\int_X f\lceil d\chi\rceil = \int_{-\infty}^{\infty} \chi(\{f> s\}) - \chi(\{f\leq -s\})ds$$

We are now ready to talk about the (simplified) 2-species of definable function. For a given $f\in$ Def(X), let $\underline{f}$ be the simplified 2-species that maps each integer n to the excursion species of the constructible $\lfloor (n+1)f\rfloor$, and $\overline{f}$ be the 2-species that maps n to the excursion species of $\lceil (n+1)f \rceil$. This 1-shift is essential to get the right grading in the limit formula.

Evaluating the ordinary generating function of these two excursion species at $x = 1$ gives $\int_X \lfloor (n+1)f \rfloor d\chi$ and $\int_X \lceil (n+1)f \rceil d\chi$.

Hence, the ordinary generating functions of the upper and lower 2-species are

$$\underline{\text{ogf}} = \sum_{n \geq 0} \int_X \lfloor (n+1)f \rfloor d\chi x^n$$

$$\overline{\text{ogf}} = \sum_{n \geq 0} \int_X \lceil (n+1)f \rceil d\chi x^n$$

(the notations are shorthanded to $\underline{\text{ogf}}$ and $\overline{\text{ogf}}$ in this section since there is no confusion).

Taking the antiderivative of $\underline{\text{ogf}}$ and $\overline{\text{ogf}}$ yields $\sum_{n \geq 0} \frac{1}{n+1} \big(\int_X \lfloor (n+1)f \rfloor d\chi \big) x^{n+1}$ and $\sum_{n \geq 0} \frac{1}{n+1} \big(\int_X \lceil (n+1)f \rceil d\chi \big) x^{n+1}$ respectively.

The coefficients in these power series are exactly the terms in the definition of definable Euler integration 6.8.1 and 6.8.2.

In other words, if we denote the antiderivative of $\underline{\text{ogf}}$ and $\overline{\text{ogf}}$ by $\underline{\text{Ogf}}$ and $\overline{\text{Ogf}}$, then

$$\int_X f \lfloor d\chi \rfloor = \lim_{n \to \infty} \frac{1}{n!} \frac{d^n}{dx^n} \Big|_{x=1} \underline{\text{Ogf}}$$
$$\int_X f \lceil d\chi \rceil = \lim_{n \to \infty} \frac{1}{n!} \frac{d^n}{dx^n} \Big|_{x=1} \overline{\text{Ogf}}$$

Chapter 7

Conclusion and future questions

This thesis gives the results to the following questions:

1. Provide a proof for the invertibility of the persistent homology transform through the means of Schapira's Radon transform and inversion formula.

2. Prove the compatibility theorem between the Fourier-Sato transform and the convolution with respec to the Euler integral.

3. Introduce new notions of L^2 inner product and tempered distributions for the Euler integral.

4. Introduce a generalized version of combinatorial species, which defines magnitude cohomology and Euler integral with the appropriate choice of functors.

There are still many open questions that follow the ones being answered in this thesis. The following list is by no means exhaustive:

1. The Euler Integral is connected to the theory of sheaves through the isomorphism between the Grothendieck group of the derived category of constructible sheaves and the group of constructible functions. However, the Grothendieck group is only the K_0 in the K-theory of that category. There is already a vast literature on K-theory of triangulated categories, so one might wonder if this leads to constructions of higher Euler integration theory.

2. The Fourier-Sato transform is currently defined on vector bundles and/or on Euclidean space. Though our work extend the original notion of Fourier-Sato transform on conic constructible functions to generalized functions, it still begs the question of whether there is such a transform on cellular sheaves, similar to how there is a Fourier transform on graphs.

3. Does the combinatorial species realization of the Euler integral and magnitudes provide a computational tool to calculate some of these invariants?

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