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LORENTZIAN POLYNOMIALS, HIGHER HESSIANS, AND THE HODGE-RIEMANN PROPERTY FOR CODIMENSION TWO GRADED ARTINIAN GORENSTEIN ALGEBRAS

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Abstract. We study the Hodge-Riemann property (HRP) for graded Artinian Gorenstein (AG) algebras. We classify AG algebras in codimension two that have HRP in terms of higher Hessian matrices and positivity of Schur functions associated to certain rectangular partitions.

1. Introduction

In this paper we introduce the Hodge Riemann property (HRP) on an arbitrary graded oriented Artinian Gorenstein (AG) algebra defined over $\mathbb{R}$, and we give a criterion on the higher Hessian matrix of its Macaulay dual generator (Theorem 3.1). AG algebras can be regarded as algebraic analogues of cohomology rings (in even degrees) of complex manifolds, and the HRP is analogous to the Hodge-Riemann relations (HRR) satisfied by cohomology rings of complex Kähler manifolds. Higher Hessians were introduced by the fourth author [10] to study the strong Lefschetz property (SLP) of an AG algebra defined over an arbitrary field of characteristic zero (see also [4]); over the real numbers, HRP implies SLP (Lemma 2.3).

In a recent paper [2], Brändén and Huh introduced a remarkable class of real homogeneous polynomials, extending the class of (real) stable polynomials, which they called Lorentzian polynomials. Among other things, they showed that the first Hessian matrix of a Lorentzian polynomial has exactly one positive eigenvalue and the others negative. Murai-Nagaoka-Yazawa [5] further showed that the first Hessian of a Lorentzian polynomial is non-singular, which, in terms of Macaulay duality, implies that its associated Artinian Gorenstein (AG) algebra satisfies the HRR in degree one, an analogue of the Hodge index theorem for Kähler manifolds. In this paper, we extend the aforementioned result in two ways in the special case of two variables. First, we prove conversely that if an AG algebra satisfies HRR in degree one, then it has a Macaulay dual generator that is Lorentzian. Second, we give sufficient conditions for a polynomial to generate an AG algebra satisfying HRR in higher degrees, using a normalization operator introduced by Brändén and Huh.

Theorem A. *(Theorem 5.3) Let $F = F(X, Y)$ be any real homogeneous polynomial of degree d in $n = 2$ variables and let $A = \mathbb{R}[x, y]/\operatorname{Ann}(F)$ be the AG algebra it generates. If*

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A satisfies HRR in degree one, then there exists a linear change of coordinates $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that the polynomial $G(X,Y) = F(T(X,Y))$ is strictly Lorentzian.

Following [2], we define the normalization operator $N\colon \mathbb{R}[X,Y] \to \mathbb{R}[X,Y]$ as the $\mathbb{R}$-linear operator defined on monomials by

$$N(X^aY^b) = \frac{X^a}{a!}\frac{Y^b}{b!}.$$

Theorem B. *(Theorem 5.6) Let $F = F(X,Y) = \sum_{i=0}^{d} c_iX^{d-i}Y^i$ be a real homogeneous polynomial of degree d in $n = 2$ variables with $c_d \neq 0$, and let $A = \mathbb{R}[x,y]/\operatorname{Ann}(N(F))$ be the AG algebra generated by the normalization of F.*

1. *If the degree d univariate polynomial $f(t) = F(1,t)$ has only real roots, then $A_{N(F)}$ satisfies HRR in degree one.*
2. *If F is Lorentzian then $A_{N(F)}$ satisfies HRR in degree 2.*
3. *If F is positively stable, then $A_{N(F)}$ has the HRP.*

In $n = 2$ variables, our Hessian condition boils down to a sign condition on the detem-inants of the higher Hessians of $N(F)$ up to what we call the plateau degree of $A_{N(F)}$. By evaluating the Hessians at a special point, we identify their determinants in terms of Schur polynomials in the roots of the univariate polynomial $f(t) = F(1,t)$.

Theorem C. *(Theorem 5.4) Let $F = F(X,Y) = \sum_{i=0}^{d} c_iX^{d-i}Y^i$ be a real homogeneous polynomial of degree d in $n = 2$ variables with $c_d \neq 0$, and let $A_{N(F)}$ be the AG algebra generated by the normalization of F with plateau degree r. Let $f(t) = F(1,t)$ be the real univariate polynomial of degree d with possibly complex roots $\alpha_1,\dots,\alpha_d$, and let $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{C}^d$ be the vector with those entries. Then for each $0 \leq i \leq r$, the determinant of the i^{th} Hessian of $N(F)$, evaluated at $X = 0$ and $Y = 1$, is given by*

$$\det\left(\mathrm{Hess}_i(F)|_{(0,1)}\right) = (-1)^{\lfloor\frac{i+1}{2}\rfloor}\cdot K_i \cdot s_{\lambda(i)}(\alpha)$$

where $0 \neq K_i \in \mathbb{R}$, $\operatorname{sgn}(K_i) = \operatorname{sgn}(c_d^{i+1})$, and $s_{\lambda(i)}(\alpha)$ is the Schur polynomial for the rectan-gular partition $\lambda(i) = (i+1)^i$ evaluated at α.

Theorem C allows us to give a sort of classification of HRR Macaulay dual generators in two variables: $A_{N(F)}$ satisfies HRR in degree i if and only if the determinant of the Toeplitz matrix

$$s_{\lambda(j)} = \det\left(\left(e_{j-p+q}\right)_{0\leq p,q\leq j}\right) = \det\begin{pmatrix} e_j & \cdots & e_{2j} \\ \vdots & \ddots & \vdots \\ e_0 & \cdots & e_j \end{pmatrix}$$

is positive for each $0 \leq j \leq i \leq r$, where $(e_d,\dots,e_0)$ is the coefficient sequence of F and $e_0 > 0$ (Corollary 5.5).

This paper is organized as follows. In Section 2 we introduce the HRP and HRR in degree i for AG algebras, and compare them to the more familiar strong Lefschetz property (SLP). We also give a matrix criterion HRR. In Section 3 we give necessary and sufficient conditions for an AG algebra to have HRP in terms of the higher Hessian matrices of its Macaulay dual generator. In Section 4 we recall the definitions of Lorentzian and stable polynomials, together with some fundamental results about them from [2]. In Section 5 we prove Theorem A, Theorem C and Theorem B.

2. Hodge-Riemann Bilinear Relations

A graded *Artinain Gorenstein (AG)* $\mathbb{F}$*-algebra* B of socle degree b is a graded algebra with homogeneous graded components B_i such that $d = \max\{i : \dim_{\mathbb{R}}(B_i) > 0\}$, $\dim_{\mathbb{R}}(B_d) = 1$, and for each $0 \le i \le \lfloor \frac{b}{2} \rfloor$ multiplication defines a nondegenerate pairing $B_i \times B_{b-i} \to B_b$. A choice of an $\mathbb{R}$-linear isomorphism $\int_B : B_b \to \mathbb{R}$ is called an *orientation* on B and the pair $\left(B, \int_B\right)$ is termed a graded *oriented AG algebra* over $\mathbb{R}$ of socle degree b. The element $\sigma_B \in B_d$ that satisfies $\int_B \sigma_B = 1$ will be called the distinguished socle generator of B.

Fix a linear form $\ell \in B_1$. For each degree i, $0 \le i \le \left\lfloor \frac{b}{2} \right\rfloor$, define the i^{th} *primitive subspace* to be the kernel of the multiplication map $\times \ell^{b-2i+1} : B_i \to B_{b-i+1}$, i.e.

$$P_{i,\ell} = \left\{\beta \in B_i \mid \ell^{b-2i+1} \cdot \beta = 0\right\} \subset B_i.$$

Definition 2.1 (Strong Lefschetz Property, Hodge-Riemann Property)**.**

(1) The linear form $\ell \in B_1$ is strong Lefschetz (SL) for B if the multiplication maps

$$\times \ell^{b-2i} : B_i \to B_{b-i}$$

are isomorphisms for each $0 \le i \le \left\lfloor \frac{b}{2} \right\rfloor$. We also say that the pair (B, ℓ) has the strong Lefschetz property (SLP).

(2) The linear form $\ell \in B_1$ is Hodge-Riemann (HR) in degree i if for every $0 \le j \le i$ and every $\beta \in P_{j,\ell}$, we have

$$(-1)^j \cdot \int_B \ell^{b-2j} \cdot \beta^2 > 0.$$

We also say that the pair (B, ℓ) satisfies HRR in degree i. We say that the pair (B, ℓ) has the Hodge-Riemann property (HRP) if it satisfies HRR in degree $\left\lfloor \frac{b}{2} \right\rfloor$.

Since B is Gorenstein, multiplication defines a symmetric bilinear and non-degenerate pairing for each $0 \le i \le \left\lfloor \frac{b}{2} \right\rfloor$:

$$\begin{aligned} B_i \times B_{b-i} &\longrightarrow \mathbb{R} \\ (\beta_1, \beta_2) &=\!=\!= \int_B \beta_1 \cdot \beta_2 \end{aligned}$$

Using the linear form $\ell \in B_1$ we get the i^{th} Lefschetz pairing on B_i with respect to ℓ:

$$\begin{aligned} B_i \times B_i &\longrightarrow \mathbb{R} \\ (\beta_1, \beta_2)^i_\ell &=\!=\!= \int_B \ell^{b-2i} \cdot \beta_1 \cdot \beta_2 \end{aligned}$$

The i^{th} Lefschetz pairing is clearly symmetric and bilinear. The following result is known as primitive decomposition.

Lemma 2.2. *Assume that for some fixed degree* $0 \le i-1 \le \left\lfloor \frac{b}{2} \right\rfloor$*, the* $(i-1)^{st}$ *Lefschetz map* $\times \ell^{b-2i-1} : B_{i-1} \to B_{b-i+1}$ *is an isomorphism. Then in degree i there is vector space decomposition*

$$B_i = P_{i,\ell} \oplus \ell \cdot B_{i-1}$$

which is orthogonal with respect to the i^{th} Lefschetz pairing. In particular, if $\ell \in B_1$ is SL for B, then B admits an orthogonal decomposition with respect to the Lefschetz pairing:

$$B = \bigoplus_{i=0}^{\lfloor \frac{b}{2} \rfloor} \bigoplus_{j=0}^{i} \ell^{i-j} \cdot P_{j,\ell}$$

called the primitive decomposition with respect to ℓ.

Proof. There is a commutative diagram of linear maps

$$\begin{tikzcd} B_{i-1} \arrow[r, "\times\ell^{b-2(i-1)}"] \arrow[d, "\times\ell"'] & B_{b-i+1} \\ B_i \arrow[ur, "\times\ell^{b-2i+1}"'] & \end{tikzcd} \quad . \tag{2.1}$$

Since the top horizontal map is an isomorphism, it follows that the diagonal map must map the image of the vertical map $\ell(B_{i-1}) \subset B_i$ isomorphically onto its image, and since its kernel is $P_{i,\ell}$ the decomposition follows from linear algebra. The orthogonality of the decomposition follows directly from the definitions and its proof is left to the reader. If $\ell \in B_1$ is SL for B, its primitive decomposition follows from an easy inductive argument whose details are again left to the reader. □

Lemma 2.3. *A linear form $\ell \in B_1$ is SL for B if and only if the i^{th} Lefschetz pairing with respect to ℓ is non-degenerate on the primitive subspace $P_{i,\ell}$, for each* $0 \le i \le \left\lfloor \frac{b}{2} \right\rfloor$.

Proof. Assume that $\ell \in B_1$ is SL for B. It follows that the i^{th} Lefschetz pairing must be non-degenerate, and by orthogonality of the primitive decomposition, it follows that it must also be non-degenerate on the i^{th} primitive subspace.

Conversely, assume that $\ell \in B_1$ is not SL for B, let $0 \le i \le \left\lfloor \frac{b}{2} \right\rfloor$ be any index for which $\times\ell^{b-2i} \colon B_i \to B_{b-i}$ is not an isomorphism, and suppose that $\alpha \in B_i$ is a non-zero element of its kernel. Then $\alpha \in P_{i,\ell}$ and we must have for every $\beta \in B_i$

$$(\alpha, \beta)^i_\ell = \int_A \ell^{b-2i} \alpha\beta = 0$$

which implies that the i^{th} Lefschetz pairing is degenerate on $P_{i,\ell}$. □

Lemma 2.3 shows that $\ell \in B_1$ is HR for B implies ℓ is SL for B. The converse does not hold however; see Example 5.9.

Next we derive some matrix conditions for verifying HRR.

Recall that a real symmetric $n \times n$ matrix M has real eigenvalues $\lambda_1, \ldots, \lambda_n$. The *signature* of M is then defined to be the difference between the number of positive eigenvalues and the number of negative ones:

$$\mathrm{sgn}(M) = \#\{\lambda_i > 0\} - \#\{\lambda_i < 0\}.$$

Sylvester's Law of inertia implies that signature is invariant under the congruence relation, i.e. if Q is a non-singular $n \times n$ matrix, and $M' = Q \cdot M \cdot Q^T$ where Q^T is the transpose of Q, then $\mathrm{sgn}(M) = \mathrm{sgn}(M')$.

Since B is Gorenstein, we may choose a homogeneous $\mathbb{R}$-basis $\mathcal{E} = \sqcup_{i=0}^{b}\mathcal{E}_i$ for B such that $\mathcal{E}_0 = \{1\}$, $\mathcal{E}_b = \{\sigma_B\}$ (the distinguished socle generator for B), and for each $1 \le i \le b-1$, $\mathcal{E}_i$ and $\mathcal{E}_{b-i}$ are dual bases, that is, $\mathcal{E}_i = \{e_1, \ldots, e_s\}$ and $\mathcal{E}_{b-i} = \{f_1, \ldots, f_s\}$ with $\int_B e_i f_j = \delta_{ij}$. We shall call such a basis a symmetric basis for B. Let $M_\ell^i = M_\ell^i(\mathcal{E})$ denote the matrix for the i^{th} Lefschetz map $\times\ell^{b-2i}\colon B_i \to B_{b-i}$ with respect to a symmetric basis $\mathcal{E}$. Note that if $\mathcal{E}'$ is another symmetric basis for B, then there exists a non-singular matrix Q for which $M_\ell^i(\mathcal{E}') = Q \cdot M_\ell^i(\mathcal{E}) \cdot Q^T$. In particular, the signature of the matrix $M_\ell^i(\mathcal{E})$ is independent of our choice of $\mathcal{E}$.

Lemma 2.4. *Let B be a graded AG algebra of socle degree b. Assume that $\ell \in B_1$ is SL for B. Then ℓ is HR for B in degree i if and only if for each degree $0 \le j \le i$ and for some (any) choice of symmetric basis $\mathcal{E}$*

$$\operatorname{sgn}(M_\ell^i(\mathcal{E})) = \operatorname{sgn}(M_\ell^{i-1}(\mathcal{E})) + (-1)^i \cdot \underbrace{(\dim(B_i) - \dim(B_{i-1}))}_{\dim(P_{i,\ell})}. \tag{2.2}$$

Proof. According to Lemma 2.2 we have

$$B_i = P_{i,\ell} \oplus \ell \cdot B_{i-1} \tag{2.3}$$

and also

$$B_{b-i} = \ell^{b-2i} \cdot P_{i,\ell} \oplus \ell^{b-2i+1} \cdot B_{i-1}. \tag{2.4}$$

Denote by $-^* = \operatorname{Hom}_{\mathbb{R}}(-, \mathbb{R})$ the $\mathbb{R}$-vector space duality functor. Because B is AG we have $B_{b-i} \cong B_i^*$. Diagram (2.1) gives an isomorphism $\ell^{b-2i+1} : (\ell B_{i-1}) \to B_{b-i+1}$. Dualizing one finds an isomorphism $\ell^{b-2i+1} : (B_{b-i+1})^* \to (\ell B_{i-1})^*$. This establishes that $(\ell B_{i-1}) \cong \ell^{b-2i+1}(B_{b-i+1})^* \cong \ell^{b-2i+1}B_{i-1}$. By orthogonality of the decomposition, it follows from the above displayed equations that $(P_{i,\ell})^* \cong \ell^{b-2i} \cdot P_{i,\ell}$.

If we choose our basis $\mathcal{E}$ compatible with the decomposition (2.3) in each degree $0 \le i \le \left\lfloor \frac{b}{2} \right\rfloor$, then it follows that the dual basis to $\mathcal{E}$ will be compatible with the decomposition (2.4). Thus matrix $M_\ell^i(\mathcal{E})$ will have a block diagonal form:

$$M_\ell^i(\mathcal{E}) = \left(\begin{array}{c|c} A_\ell^i & 0 \\ \hline 0 & M_\ell^{i-1}(\mathcal{E}) \end{array}\right)$$

where A_ℓ^i is the matrix for the multiplication map $\times\ell^{b-2i}\colon P_{i,\ell} \to \ell^{b-2i} \cdot P_{i,\ell}$. Since

$$\operatorname{sgn}(M_\ell^i) = \operatorname{sgn}(A_\ell^i) + \operatorname{sgn}(M_\ell^{i-1})$$

the result follows. $\square$

In the special case where the primitive subspace is one dimensional in each degree (as in the codimension two case), the HRP can be checked by the following determinantal condition.

Lemma 2.5. *Assume that ℓ is SL for B, and assume that $\dim_{\mathbb{R}}(P_{i,\ell}) \le 1$ for each $0 \le i \le \left\lfloor \frac{b}{2} \right\rfloor$. Then ℓ is HR for B in degree i if and only there exists a symmetric basis $\mathcal{E}$ such that for each $0 \le j \le i$,*

$$(-1)^j \cdot \frac{\det\left(M_\ell^j(\mathcal{E})\right)}{\det\left(M_\ell^{j-1}(\mathcal{E})\right)} > 0.$$

In particular, if $\dim_{\mathbb{R}}(P_{i,\ell}) = 1$ *for* $0 \leq i \leq r$ *and* $P_{i,\ell} = 0$ *for* $i > r$*, then* ℓ *is HR for* B *in degree* i *if and only if for each* $0 \leq j \leq i \leq r$*,*

$$\operatorname{sgn}\left(\det\left(M_\ell^j(\mathcal{E})\right)\right) = (-1)^{\left\lfloor \frac{j+1}{2} \right\rfloor}.$$

Proof. Let $\mathcal{E}$ be any symmetric basis respecting the primitive decomposition of B with respect to ℓ. Then for each $0 \leq i \leq \left\lfloor \frac{b}{2} \right\rfloor$, the matrix for the i^{th} Lefschetz map is a block matrix of the form

$$M_\ell^i(\mathcal{E}) = \left(\begin{array}{c|c} \int_B \ell^{b-2i} \cdot \beta^2 & 0 \\ \hline 0 & M_\ell^{i-1}(\mathcal{E}) \end{array}\right), \tag{2.5}$$

where $\beta \in \mathcal{E} \cap P_{i,\ell}$, and the first assertion follows. Note that if $P_{i,\ell}$ is one dimensional for $0 \leq i \leq r$ and zero for $i > r$, and if $0 \leq i \leq r$, then the matrix in Equation 2.5 is block diagonal with 1×1 blocks, and hence the second assertion follows by induction on i. □

3. A Hessian Criterion for HRP

Let $R = \mathbb{R}[x_1, \ldots, x_n]$ and $Q = \mathbb{R}[X_1, \ldots, X_n]$ be polynomial rings where R acts on Q by differentiation, i.e.

$$x_i \circ F = \frac{\partial F}{\partial x_i}, \quad 1 \leq i \leq n, \ F \in Q.$$

A homogeneous polynomial $F \in Q$ of degree d determines an oriented graded AG algebra $A = R/\operatorname{Ann}(F)$ of socle degree d with orientation given by $\int_A \alpha = (\alpha \circ F)(0), \ \forall\, \alpha \in A$. Fix any homogeneous basis $\mathcal{E}$ for A, and for each degree $0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor$ suppose that $\mathcal{E}_i = \left\{e_1^i, \ldots, e_m^i\right\}$. Define the i^{th} Hessian matrix of F with respect to $\mathcal{E}$ as the $m \times m$ polynomial matrix

$$\operatorname{Hess}_i(F) = \operatorname{Hess}_i(F, \mathcal{E}) = \left(e_j^i e_k^i \circ F\right)_{1 \leq j,k \leq m}.$$

Note that the entries of $\operatorname{Hess}_i(F)$ are polynomials in the variables $X_1, \ldots, X_n$. Given real numbers $C_1, \ldots, C_n$, we shall write $C = (C_1, \ldots, C_n) \in \mathbb{R}^n$ in vector notation and write $\operatorname{Hess}_i(F)|_C$ to mean the numerical matrix obtained by substituting the real number C_i for the variable X_i for each $1 \leq i \leq n$.

Theorem 3.1. *Let* $\mathcal{E}$ *be any symmetric basis for* $A = R/\operatorname{Ann}(F)$ *and let* $\ell = \ell(C) = C_1 x_1 + \cdots + C_n x_n \in A_1$ *be any linear form in* A*. Then for* $C = (C_1, \ldots, C_n) \in \mathbb{R}^n$

$$\operatorname{Hess}_i(F, \mathcal{E})|_C = M_{\ell(C)}^i(\mathcal{E}).$$

In particular, the signature of $\operatorname{Hess}_i(F, \mathcal{E})|_C$ *is independent of our choice of basis* $\mathcal{E}$*.*

Proof. The key observation here is the following formula: for any homogeneous form $G \in Q$ of any degree a, and any linear form $\ell = C_1 x_1 + \cdots + C_n x_n \in R_1$ as above, we have

$$\ell^a \circ G = a! \cdot G(C_1, \ldots, C_n).$$

To see this, note first that it holds for $G = X_1^{e_1} \cdots X_n^{e_n}$ a monomial:

$$\ell^a \circ G = \frac{a!}{e_1! \cdots e_n!} C_1^{e_1} x_1^{e_1} \cdots C_n^{e_n} x_n^{e_n} \circ X_1^{e_1} \cdots X_n^{e_n} = a! G(C_1, \ldots, C_n),$$

and then it must hold for all homogeneous G by linearity of the R-action on Q. Since the orientation on A satisfies

$$\int_A \alpha = (\alpha \circ F)(0), \ \forall \alpha \in A,$$

the (j,k)-entry of $M^i_{\ell(C)}(\mathcal{E})$ is

$$m^i_{j,k} = \int_A e^i_k \ell^{d-2i} e^i_j = \left(\ell^{d-2i} e^i_k e^i_j \circ F\right)(0) = \ell^{d-2i} \circ \left(e^i_j e^i_k \circ F\right) = \left(e^i_j e^i_k \circ F\right)\Big|_C,$$

which is the (j,k)-entry of $\mathrm{Hess}_i(F)|_C$ as claimed by the first assertion. The second assertion follows from the first. □

Theorem 3.1 gives a criterion on the Macaulay dual generator for the associated AG algebra $A = R/\operatorname{Ann}(F)$ to have HRP, via Lemma 2.4. We state this as a corollary below. Incidentally Theorem 3.1 also shows that the signature (and the determinant) of the i^{th} Hessian matrix of a homogeneous form $F = F(X_1, \ldots, X_n)$ is invariant under the action of $\mathrm{GL}(n, \mathbb{R})$.

Corollary 3.2. *Given an oriented graded AG algebra $A = \mathbb{R}[x_1, \ldots, x_n]/\operatorname{Ann}(F)$, a linear form $\ell = \ell(C) = C_1x_1 + \cdots + C_nx_n \in A_1$ is HR for A in degree i if and only if* $\det(\mathrm{Hess}_j(F)|_C) \neq 0$ *and*

$$\operatorname{sgn}\left(\mathrm{Hess}_j(F)|_C\right) = \operatorname{sgn}\left(\mathrm{Hess}_{j-1}(F)|_C\right) + (-1)^j \left(\dim_{\mathbb{R}}(A_i) - \dim_{\mathbb{R}}(A_{i-1})\right)$$

for all $j \leq i$.

In particular, $\ell \in A_1$ satisfies HRR in degree 1 *if and only if* $F(C) > 0$*,* $\det(\mathrm{Hess}_1(F)|_C) \neq 0$*, and* $\operatorname{sgn}(\mathrm{Hess}_1(F)|_C) = 1 - (n-1) = 2 - n$*.*

4. Lorentzian and Stable Polynomials

Let $Q = \mathbb{R}[X_1, \ldots, X_n]$ be the standard graded polynomial ring in n-variables with real coefficients, let $Q_d \subset Q$ be the degree d graded piece, i.e. the set of homogeneous polynomials of degree d, and let $P_d \subset Q_d$ denote the subset of homogeneous polynomials whose coefficients are positive.

The following definitions are taken directly from [2, Defintion 2.1].

Definition 4.1. Define the set of strictly Lorentzian polynomials $\mathring{L}_d \subset P_d$ inductively as follows: $\mathring{L}_0 = P_0$, $\mathring{L}_1 = P_1$, and

$$\mathring{L}_2 = \{F \in P_2 \mid \operatorname{sgn}(\mathrm{Hess}_1(F)) = 0\}.$$

Then for $d > 2$ define

$$\mathring{L}_d = \left\{F \in P_d \mid x_i \circ F \in \mathring{L}_{d-1}\right\}.$$

Definition 4.2. The set of stable polynomials $S_d \subset Q_d$ consists of homogeneous polynomials F with non-negative coefficients satisfying the following condition: For some $U = (U_1, \ldots, U_n) \in \mathbb{R}^n_{\geq 0}$, $F(U) > 0$ and for every $V = (V_1, \ldots, V_n) \in \mathbb{R}^n$, the univariate polynomial $F(tU - V) \in \mathbb{R}[t]$ has only real roots. We shall use the term positively stable for the set of stable polynomials with positive coefficients, and denote this set by $S_d \cap P_d$.

Definition 4.3. A subset $J \subset \mathbb{N}^n$ is called M-convex if it satisfies the following exchange property: For each $\alpha, \beta \in J$ and each index i satisfying $\alpha_i > \beta_i$, there exists an index j satisfying $\alpha_j < \beta_j$ and $\alpha - e_i + e_j \in J$, where $e_i \in \mathbb{N}^n$ is the i^{th} standard basis vector. A homogeneous polynomial $F = \sum_{\alpha \in \mathbb{N}^n} c_\alpha X^\alpha \in Q_d$ is called M-convex if its support $\operatorname{supp}(F) = \{\alpha \mid c_\alpha \neq 0\} \subset \mathbb{N}^n$ is M-convex.

According to [2], every stable polynomial is M-convex, i.e. $S_d \subset M_d$.

Definition 4.4. Define the set of Lorentzian polynomials $L_d \subset Q_d$ inductively as follows: Set $L_1 = S_1$, $L_2 = S_2$, and for $d > 2$ define

$$L_d = \{F \in M_d \mid x_i \circ F \in L_{d-1},\ \forall\ 1 \leq i \leq n\}.$$

According to [2, Theorm 2.25], the set of Lorentzian polynomials is the closure of the set of strictly Lorentzian polynomials in the Euclidean topology on Q_d. Define the normalization operator $N\colon Q \to Q$ as the $\mathbb{R}$-linear map acting on monomials by

$$N(X_1^{a_1} \cdots X_n^{a_n}) = \frac{X_1^{a_1} \cdots X_n^{a_n}}{a_1! \cdots a_n!}.$$

Fact 4.5. *The following important facts were established in [2] and [5].*

(1) *[2, Proposition 2.2] Every stable polynomial is Lorentzian, i.e. $S_d \subset L_d$.*
(2) *[2, Corollary 3.7] If F is Lorentzian, then $N(F)$ is Lorentzian.*
(3) *[2, Proposition 2.16] If F is strictly Lorentzian, then for any $C = (C_1, \ldots, C_n) \in \mathbb{R}^n_{>0}$, $\det(\operatorname{Hess}_1(F)|_C) \neq 0$ and $\operatorname{sgn}(\operatorname{Hess}_1(F)|_a) = 2 - n$.*
(4) *[5, Theorem 3.8] If F is Lorentzian, then for any $C = (C_1, \ldots, C_n) \in \mathbb{R}^n_{>0}$, the pair $(A_F, \ell(C))$ satisfies HRR in degree one.*

In $n = 2$ variables, one can derive the following simpler, i.e. non-recursive, characterizations of Lorentzian and stable polynomials; see [2, Example 2.3, 2.26]:

Fact 4.6. *Let $F = F(X, Y) = \sum_{i=0}^{d} c_i X^{d-i} Y^i \in \mathbb{R}[X, Y]$ be a homogeneous polynomial of degree d in $n = 2$ variables. Then*

(1) *F is strictly Lorentzian if and only if its coefficient sequence is strictly positive and strictly ultra log concave, i.e. $c_i > 0$ for all i and*

$$\left(\frac{c_i}{\binom{d}{i}}\right)^2 > \left(\frac{c_{i-1}}{\binom{d}{i-1}}\right)\left(\frac{c_{i+1}}{\binom{d}{i+1}}\right),\ \forall 1 \leq i \leq d-1.$$

(2) *F is Lorentzian if and only if its coefficient sequence is non-negative, ultra log concave, and has no internal zeros, i.e. $c_i \geq 0$ for all i,*

$$\left(\frac{c_i}{\binom{d}{i}}\right)^2 \geq \left(\frac{c_{i-1}}{\binom{d}{i-1}}\right)\left(\frac{c_{i+1}}{\binom{d}{i+1}}\right),\ \forall 1 \leq i \leq d-1,$$

and

$$c_i c_k > 0 \Rightarrow c_j > 0\ \forall 0 \leq i < j < k \leq d.$$

(3) *F is (resp. positively) stable if and only if the coefficients are (resp. positive) non-negative and the univariate polynomial $F(1, t)$ has only real roots.*

Remarks 4.7. (1) It is not true that every positively stable polynomial is strictly Lorentzian. For example the quadratic polynomial

$$F = X^2 + 2XY + Y^2$$

is positively stable and Lorentzian, but not strictly Lorentzian. In [2], the term "strictly stable" is used to mean the interior of S_d, and perhaps deserves the notation $\mathring{S}_d$. In fact in their proof of Fact 4.5(1) they actually show that every strictly stable polynomial is also strictly Lorentzian, i.e. $\mathring{S}_d \subset \mathring{L}_d$.

(2) In two variables, Fact 4.5(1) taken together with Fact 4.6(1) is a classical result that goes back to Newton and Maclaurin; see [7] or [6].

(3) Evidently the containment in Fact 4.5(1) is strict, even in two variables. In fact, according to [2, Example 2.3], the polynomial

$$F_\theta(X, Y) = X^3 + 18X^2Y + 12XY^2 + \theta Y^3$$

is Lorentzian for $0 \leq \theta \leq 9$, but is only stable for $0 \leq \theta \leq 8$.

(4) The condition for ultra log concavity is equivalent to

$$c_i^2 \geq c_{i-1}c_{i+1}\left(1 + \frac{1}{d-i}\right)\left(1 + \frac{1}{i}\right), \ \forall 1 \leq i \leq d-1.$$

In particular, ultra log concavity implies the usual log concavity condition.

(5) As we shall see, the hypothesis F Lorentzian or even strictly Lorentzian, as in Fact 4.5(3) and (4), is not sufficient to conclude that A_F satisfies HRR in degrees $i > 1$; cf. Example 5.9.

(6) In two variables, the stable version of Fact 4.5(2) is a well known fact that Brenti [3, Theorem 2.4.1] attributes to Pólya and Szegö.

5. HRR in Codimension Two

Our first task is to prove Theorem A. First some notation.

Let A be a graded oriented AG algebra of codimension two and socle degree d, and let $F \in \mathbb{R}[X, Y]$ be its Macaulay dual generator, so that $A = \mathbb{R}[x, y]/\operatorname{Ann}(F)$ with orientation

$$\int_A \alpha = (\alpha \circ F)(0), \quad \alpha \in A.$$

To an element $\ell \in A_1$, we associate to the quotient algebra

$$T = T_\ell = \frac{A}{(0 :_A \ell)}$$

which is also a graded oriented AG algebra of socle degree $d - 1$; see e. g. [9, Lemma 4]. There is a natural surjection $\pi\colon A \to T$ for which the Thom class is $\tau = \ell$, meaning the orientation for T is defined by the equations

$$\int_A \ell \cdot \alpha = \int_T \pi(\alpha), \quad \forall \alpha \in A. \tag{5.1}$$

If $F_1 = F_1(X, Y)$ is the Macaulay dual generator of T, then it satisfies $F_1 = \ell \circ F$.

Lemma 5.1. *If the pair (A, ℓ) satisfies HRR in degree i, then the pair $(T, \pi(\ell))$ also satisfies HRR in degree i.*

Proof. First note that for each $0 \leq i \leq \left\lfloor \frac{d-1}{2} \right\rfloor$, we have an inclusion

$$P_{i,\pi(\ell)}(T) \subset \pi\left(P_{i,\ell}(A)\right).$$

Indeed, suppose that $\beta \in P_{i,\pi(\ell)}(T)$, and fix $\alpha \in \pi^{-1}(\beta)$. Then $\pi(\ell^{d-1-2i+1}\alpha) = 0 = \pi(\ell^{d-2i}\alpha)$, and hence for any $\alpha' \in A$, we must have

$$\int_A \ell^{d-2i+1}\alpha \cdot \alpha' = \int_T \pi(\ell^{d-2i}\alpha \cdot \alpha') = 0$$

which implies that $\ell^{d-2i+1}\alpha = 0$, and hence that $\alpha \in P_{i,\ell}$.

Next, fix $\beta \in P_{i,\pi(\ell)}$ and $\alpha \in P_{i,\ell} \cap \pi^{-1}(\beta)$. Then we evidently have

$$(-1)^i \int_A \ell^{d-2i}\alpha^2 = (-1)^i \int_T \pi(\ell^{d-1-2i}\alpha^2) = (-1)^i \int_T \pi(\ell)^{d-1-2i}\beta^2 > 0$$

and the result follows. $\square$

If $F = F(X, Y) = \sum_{i=0}^{d} c_i X^{d-i} Y^i$ is the Macaulay dual generator for A, then we have

$$\int_A x^{d-i}y^i = x^{d-i}y^i \circ F = (d-i)!i! \cdot c_i = d! \cdot \left(\frac{c_i}{\binom{d}{i}}\right). \tag{5.2}$$

The following lemma is based on an idea of Adiprisito-Huh-Katz [1, Proposition 9.8].

Lemma 5.2. *Assume $y \in A_1$ is a linear form satisfying HRR in degree one. Then there exists another linearly independent linear form $x \in A_1$ for which the sequence $(b_0, \ldots, b_d)$ is positive and strictly log concave, where*

$$b_i := \int_A x^{d-i}y^i, \ 0 \leq i \leq d.$$

Proof. First, since $y \in A_1$ is HRR in degree 1, it must satisfy $\int_A y^d > 0$. Then for any linearly independent linear form $z \in A_1$, define the form $x = x(\varepsilon, z) = y + \varepsilon z$. Then it is clear that for $\varepsilon \neq 0$, x is also linearly independent from y, and also

$$\int_A x^{d-i}y^i = \int_A y^d + \varepsilon \cdot (\text{other terms}).$$

It follows that if we choose ε sufficiently small, we get $b_i = \int_A x^{d-i}y^i > 0$. Moreover by the openness of the HR condition, we may even choose ε so small so as to guarantee that x satisfies HRR for A in degree one. In fact, we can even choose ε so that $\pi^j(x) \in T^j = A/(0 :_A y^j)$ satisfies HRR in degree one for every $0 \leq j \leq d$. For such a choice of x and y, set $b_i := \int_A x^{d-i}y^i$. Then $b_i > 0$ and it remains to see that this sequence $(b_0, \ldots, b_d)$ is strictly log concave. We will show this by induction on $d \geq 1$, the base case being trivial.

For the inductive step, assume it holds for $d-1$; in other words, assume that for any graded oriented AG algebra A' of socle degree $d-1$, and for any linear forms $x', y' \in A'_1$ where y' is HR in degree one for A' and x' is HR in degree one for $A'/(0 :_{A'} (y')^j)$ for every $0 \leq j \leq d-1$, satisfying $b'_i = \int_{A'} (x')^{d-i}(y')^i > 0$, then the sequence $(b'_0, \ldots, b'_{d-1})$ is strictly log concave. Then with A, x, y, and $b_i = \int_A x^{d-i}y^i$ as above, we will show that $(b_0,\dots,b_d)$ is strictly log concave. Let $T = A/(0 :_A y)$ with $\pi\colon A \to T$ the natural surjection. By Lemma 5.1, $(T,\pi(y))$ also satisfies HRR in degree 1. By (5.1) we deduce

$$b_i' = \int_T \pi(x)^{d-1-i}\pi(y)^i = \int_A x^{d-1-i}y^{i+1} = b_{i+1} > 0.$$

Hence our induction hypothesis implies that the sequence $(b_0',\dots,b_{d-1}') = (b_1,\dots,b_d)$ is strictly log concave. Hence to complete the induction we need only show

$$b_1^2 > b_0b_2, \text{ or } \left(\int_A x^{d-1}y\right)^2 > \left(\int_A x^d\right)\cdot\left(\int_A x^{d-2}y^2\right).$$

By Lemma 2.2, we have $A_1 = \langle x\rangle \oplus P_{1,x}$, and therefore we must have $y = ax + b\alpha$ for some $a, b \in \mathbb{R}$ and $\alpha \in P_{1,x}$; moreover since y is linearly independent from x we can deduce that $b \neq 0$. Then we have

$$\begin{aligned}
b_1 &= \int_A x^{d-1}y = \int_A x^{d-1}(ax+b\alpha) = a\int_A x^d && \left(\text{since } x^{d-1}\alpha = 0\right)\\
b_0 &= \int_A x^d\\
b_2 &= \int_A x^{d-2}y^2 = \int_A x^{d-2}(ax+b\alpha)^2 = a^2\int_A x^d + b^2\int_A x^{d-2}\alpha^2 && \left(\text{since } 2ab\int_A x^{d-1}\alpha = 0\right)
\end{aligned}$$

Since $\int_A x^{d-2}\alpha^2 < 0$ because x is HRR on A, and $b^2 > 0$, it follows immediately that $b_1^2 > b_0b_2$, and hence the sequence $(b_0, b_1,\dots,b_d)$ is strictly log concave. $\square$

We are now in a position to prove Theorem A.

Theorem 5.3 (Theorem A)**.** *Let $F = F(X,Y)$ be a real homogeneous bivariate polynomial of degree $d \geq 2$ and let $A = A = \mathbb{R}[x,y]/\operatorname{Ann}(F)$ be its associated AG algebra. If A satisfies HRR in degree one, then there exists a linear change of coordinates $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that the polynomial $G(X,Y) = F(T(X,Y))$ is strictly Lorentzian.*

Proof. Assume A satisfies HRR in degree one. Then according to Lemma 5.2 we can choose linear forms $\ell_1, \ell_2 \in A_1$ for which the sequence $b_i = \int_A \ell_1^{d-i}\ell_2^i$, $0 \leq i \leq d$ is positive and strictly log concave. Let $L_1, L_2 \in Q_1$ be their Macaulay dual linear forms, meaning that $\ell_i \circ L_j = \delta_{i,j}$ the Kronecker delta, and suppose $X = aL_1 + bL_2$ and $Y = cL_1 + bL_2$. Then define $T(X,Y) = (aX+bY, cX+dY)$, and $G(X,Y) = F(T(X,Y))$ so that $G(L_1,L_2) = F(X,Y)$. Then if G has coefficient sequence $(c_0,\dots,c_d)$, we must have $b_i = \int_A \ell_1^{d-i}\ell_2^i = d!c_i/\binom{d}{i}$ by (5.2). Since the sequence $(b_0,\dots,b_d)$ is positive and strictly log concave, it follows that the sequence $(c_0,\dots,c_d)$ is positive and strictly ultra log concave, and hence G is Lorentzian by Fact 4.6 (1). $\square$

Before proving Theorem C, we recall the notion of a Schur polynomial. A partition $\lambda = (\lambda_0,\dots,\lambda_r)$ is a weakly decreasing sequence of non-negative integers; if $\lambda_r \neq 0$ we call it the length of λ and write $\ell(\lambda) = r$. Given any partition λ with $\ell(\lambda) \leq n$, define a semi-standard Young tableau $\mathrm{SSYT}(\lambda)$ as any filling of the Young diagram of λ with the numbers $\{1,\dots,n\}$ in which the rows are weakly increasing and the columns are strictly increasing (repeats allowed). For $T \in \mathrm{SSYT}(\lambda)$, let m_i denote the number of i's in T and define the monomial $\mathbf{x}^T = \prod_{i=1}^n x_i^{m_i}$. The Schur polynomial associated to λ is the sum of all the monomials $\mathbf{x}^T$, i.e.

$$s_\lambda = s_\lambda(x_1, \dots, x_n) = \sum_{T \in \mathrm{SSYT}(\lambda)} \mathbf{x}^T.$$

Recall that the normalization is an $\mathbb{R}$-linear operator $N\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined on monomials by

$$N(X^a Y^b) = \frac{X^a}{a!}\frac{Y^b}{b!}.$$

Note that in $n = 2$ variables, every graded AG algebra $A = \mathbb{R}[x, y]/I$ has Hilbert function of the form

$$H(A) = (1, 2, 3, \dots, r-1, r^s, r-1, \dots, 3, 2, 1)$$

for some $1 \le r \le \left\lfloor \frac{d}{2} \right\rfloor$ and $1 \le s \le d$. Define $r = r(A)$ to be the plateau degree of A; it is the smallest degree of a minimal generator of I.

Theorem 5.4 (Theorem C)**.** *Let $F = F(X, Y) = \sum_{i=0}^d c_i X^{d-i} Y^i$ be a real homogeneous polynomial of degree $d \ge 2$ in $n = 2$ variables with $c_d \ne 0$, let $A_{N(F)}$ be the AG algebra generated by the normalization of F, and let r be the plateau degree of A. Let $f(t) = F(1, t)$ be the real univariate polynomial of degree d with possibly complex roots $\alpha_1, \dots, \alpha_d$, and let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{C}^d$ be the vector with those entries. Then for each $0 \le i \le r$, the determinant of the i^{th} Hessian of $N(F)$, evaluated at $X = 0$ and $Y = 1$, is given by*

$$\det\left(\mathrm{Hess}_i(N(F))|_{(0,1)}\right) = (-1)^{\lfloor \frac{i+1}{2} \rfloor} \cdot K_i \cdot s_{\lambda(i)}(\alpha) \tag{5.3}$$

where $0 \ne K_i \in \mathbb{R}$, $\mathrm{sgn}(K_i) = \mathrm{sgn}(c_d^{i+1})$, and $s_{\lambda(i)}(\alpha)$ is the Schur polynomial for the rectangular partition $\lambda(i) = (i+1)^i$ evaluated at α.

Proof. Fix $0 \le i \le r$, and choose a symmetric basis $\mathcal{E}$ for $A = A_{N(F)}$ so that in degree i, $\mathcal{E}_i$ is the natural monomial basis $\left\{e^i_j = x^{i-j} y^j\right\}_{0 \le j \le i}$. Then the (j, k) entry for the i^{th} Hessian matrix for $N(F)$ with respect to $\mathcal{E}$ evaluated at $X = 0$ and $Y = 1$ is

$$\frac{\partial^{2i}}{\partial x^{2i-j-k} \partial y^{j+k}} N(F)\Big|_{(0,1)} = \frac{(2i-j-k)! \cdot (d-2i+j+k) \cdots (d-2i+1) \cdot c_{d-2i+j+k}}{(2i-j-k)!(d-2i+j+k)!} = \frac{c_{d-2i+j+k}}{(d-2i)!}.$$

Then setting $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{C}^d$ to be the vector whose entries are the negative roots of $f(t)$, we have $c_i = c_d \cdot e_{d-i}(\alpha)$, where $e_i(x)$ is the i^{th} elementary symmetric function in d-variables $x = (x_1, \dots, x_d)$. Hence the i^{th} Hessian matrix takes the form

$$\mathrm{Hess}_i(F)|_{(0,1)} = \left(\frac{c_d}{(d-2i)!} \cdot e_{2i-j-k}(\alpha)\right)_{0 \le j,k \le i}.$$

Taking the determinant then yields

$$\det\left(\mathrm{Hess}_i(F)|_{(0,1)}\right) = \underbrace{\left(\frac{c_d}{(d-2i)!}\right)^{i+1}}_{K_i} \cdot \det\left(e_{2i-j-k}(\alpha)\right)_{0 \le j,k \le i}. \tag{5.4}$$

On the other hand, the (dual) Jacobi-Trudi identity for Schur functions, see e.g. [8, Corollary 7.16.2], states that for any partition λ of length $\ell(\lambda) \leq d$ with conjugate partition λ' that the Schur polynomial satisfies

$$s_\lambda(x) = \det\left(e_{\lambda'_j - j + k}(x)\right)_{0 \leq j,k \leq \ell(\lambda')}.$$

Taking $\lambda = \lambda(i) = \underbrace{(i+1, \dots, i+1)}_{i}$, then $\lambda(i)' = \underbrace{(i, \dots, i)}_{i+1}$, and we get

$$s_{\lambda(i)}(x) = \det\left(\left(e_{i-j+k}(x)\right)_{0 \leq j,k \leq i}\right). \tag{5.5}$$

Then it is clear that the matrix in Equation (5.4) is obtained from the matrix in Equation (5.5) by exchanging column k with column $i-k$ for all $0 \leq k \leq i$, resulting in $\left\lfloor \frac{i+1}{2} \right\rfloor$ column exchanges, and evaluating at α. Therefore we see that

$$\det\left(\mathrm{Hess}_i(F)|_{(0,1)}\right) = K_i \cdot (-1)^{\left\lfloor \frac{i+1}{2} \right\rfloor} \cdot s_{\lambda(i)}(\alpha).$$

Note that we may replace the negative roots with the actual roots because the Schur polynomial $s_{\lambda(i)}$ is a homogeneous polynomial of even degree, and hence $s_{\lambda(i)}(-\alpha) = s_{\lambda(i)}(\alpha)$, and the result follows. □

We derive the following useful corollary from Theorem 5.4.

Corollary 5.5. *Let $F = F(X,Y) = \sum_{i=0}^d c_i X^{d-i} Y^i$ be a homogeneous polynomial of degree d in $n = 2$ variables with $c_d \neq 0$, and $A_{N(F)}$ the AG algebra generated by the normalization of F, and let r be the plateau degree of A. Let $f(t) = \hat{F}(1,t)$ be the real univariate polynomial of degree d with roots $\alpha_1, \dots, \alpha_d$, and let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{C}^d$. Then $(A, \ell = y)$ satisfies HRR in degree i if and only if $F(0,1) = c_d > 0$ and $s_{\lambda(j)}(\alpha) > 0$ for each $1 \leq j \leq i$. In particular, $(A, \ell = y)$ has the HRP if and only if $F(0,1) = c_d > 0$ and $s_{\lambda(i)}(\alpha) > 0$ for all $1 \leq i \leq r$.*

Proof. This follows directly from Equation (5.3) and Lemma 2.5. □

We are now in a position to prove Theorem B.

Theorem 5.6 (Theorem B)**.** *Let $F = F(X,Y)$ be a homogeneous polynomial of degree d in $n = 2$ variables with $c_d \neq 0$ and let $A_{N(F)}$ be the AG algebra generated by its normalization.*

1. *If the univariate polynomial $f(t) = F(1,t)$ (with possibly zero or negative coefficients) has only real roots with at least one non-zero, then $(A_{N(F)}, y)$ satisfies HRR in degree one.*
2. *If F is strictly Lorentzian, then $(A_{N(F)}, y)$ satisfies HRR in degree two.*
3. *If F is stable, then $(A_{N(F)}, y)$ has the HRP.*

Proof. To prove (1), we note that for $i = 1$, $\lambda(1) = \square\square$, and

$$s_{\lambda(1)}(x) = \sum_{1 \leq i,j \leq d} x_i x_j = \frac{1}{2}(x_1^2 + \cdots + x_d^2) + \frac{1}{2}(x_1 + \cdots + x_d)^2$$

which is positive definite on $\mathbb{R}^d$. It follows that if $f(t)$ has only real roots, say $\alpha = (\alpha_1, \dots, \alpha_d)$, then $s_{\lambda(1)}(\alpha) > 0$, hence by Corollary 5.5, $A_{N(F)}$ satisfies HRR in degree one.

To prove (2), assume that F is strictly Lorentzian. Then by Fact 4.6(1) its coefficient sequence $(c_0, \ldots, c_d)$ is positive and strictly ultra log-concave, and hence so is the sequence $(e_0, \ldots, e_d)$ where $e_{d-i} = c_i/c_d$. It follows that the determinants

$$s_{\lambda(1)}(\alpha) = \det \begin{pmatrix} e_1 & e_2 \\ e_0 & e_1 \end{pmatrix} = e_1^2 - e_0e_2$$

$$s_{\lambda(2)}(\alpha) = \det \begin{pmatrix} e_2 & e_3 & e_4 \\ e_1 & e_2 & e_3 \\ e_0 & e_1 & e_2 \end{pmatrix} = e_2(e_2^2 - e_1e_3) - e_3(e_1e_2 - e_0e_3) + e_4(e_1^2 - e_0e_2)$$

are both positive and since $F(0,1) = c_d > 0$ it follows from Corollary 5.5 that $(A_{N(F)}, y)$ satisfies HRR in degree two.

To prove (3), note first that if F is positively stable then $f(t) = F(1,t)$ has only real negative roots, say $-\alpha_1, \ldots, -\alpha_d$, by Fact 4.6(1). Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d_{>0}$, then it follows that $s_{\lambda(i)}(\alpha) > 0$ for each $0 \leq i \leq r(A_{N(F)})$ because Schur functions are sums of monomials with positive coefficients. Moreover since $s_{\lambda(i)}$ is homogeneous of even degree, it follows that $s_{\lambda(i)}(-\alpha) > 0$ too. Therefore it follows from Corollary 5.5 that $(A_{N(F)}, y)$ has the HRP. If F is only stable, then some of the (negative) roots $\alpha_1, \ldots, \alpha_d$ may be zero; we can label the roots so that $\alpha_1 = \cdot = \alpha_k = 0$ and $\alpha_{k+1}, \ldots, \alpha_d$ are non-zero (positive). In this case, F (and hence also $N(F)$) has the form $F = Y^k \cdot G$ where G is not a multiple of Y. It follows that $r = r(A_{N(F)})$, the plateau degree of $A_{N(F)}$, is at most the degree of G, i.e. $r \leq d - k$. It follows that for each $1 \leq i \leq r$, there is at least one semi-standard Young tableau on $\lambda(i)$ with entries consisting only of the $d-k$ indices $\{k+1, \ldots, d\}$. It follows that $s_{\lambda(i)}(\alpha) > 0$ and hence by Corollary 5.5, $(A_{N(F)}, y)$ satisfies HRP. □

Remark 5.7. There is a connection to the theory of Polya frequency (PF) sequences, studied by Brenti [3] and others. A sequence of real numbers $(a_0, \ldots, a_d)$ is called a PF sequence if the Toeplitz matrix $M = \left(a_{j-i}\right)_{0 \leq i,j \leq d}$ has all its minors non-negative (we count $a_k = 0$ if $k < 0$). Then according to Brenti [3, Theorem 2.2.4], the sequence $(a_0, \ldots, a_d)$ is PF if and only if the polynomial $A(t) = \sum_{i=0}^d a_i t^i$ has non-negative coefficients and only real roots. This together with Corollary 5.5 provides another proof of Theorem 5.6 (3).

Extrapolating from Fact 4.5(4) and Theorem 5.6, we formulate the following conjecture:

Conjecture 5.8. *If F is Lorentzian, then $(A_{N^{i-1}(F)}, y)$ satisfies HRR in degree i.*

The following example shows that $f(t) = F(1,t)$ having real roots is not sufficient for $A_{N(F)}$ to have HRR in degree two.

Example 5.9. Let $f(t) = t^4 - t^2$ so that $F = Y^4 - X^2Y^2$ and hence $N(F) = \frac{1}{24}Y^4 - \frac{1}{4}X^2Y^2 = \frac{1}{24}(Y^4 - 6X^2Y^2)$. Then we have

$$\det\left(\mathrm{Hess}_1(N(F))|_{(0,1)}\right) = \det \frac{1}{24}\begin{pmatrix} -12 & 0 \\ 0 & 12 \end{pmatrix} = -\frac{1}{4} < 0,$$

and it follows that $\ell = y$ is HR in degree one for A. However upon computing the second Hessian, we see that A cannot have any linear forms satisfying HRR in degree two:

$$\det(\mathrm{Hess}_2(N(F))) = \det \frac{1}{24}\begin{pmatrix} 0 & 0 & -24 \\ 0 & -24 & 0 \\ -24 & 0 & 0 \end{pmatrix} = 1 > 0.$$

Note however that $\ell = y$ is SL for A. Also note that Theorem 5.3 implies that $N(F)$ is $\mathrm{GL}(2, \mathbb{R})$ equivalent to a Lorentzian polynomial G. Therefore this example also shows that G Lorentzian is insufficient to conclude that A_G satisfies HRR in degree two.

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