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Additive structure in convex sets and related topics

ADDITIVE STRUCTURE IN CONVEX SETS AND RELATED TOPICS

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Abstract

This thesis predominantly discusses a handful of problems in additive combinatorics and incidence geometry.

Our particular interest within additive combinatorics is proving sumset and energy bounds for convex sets and images of structured sets under convex functions. For both types of problems, we extend existing techniques and pioneer new ones for longer sums and functions with higher convexity.

The results and approaches yield applications in short sumset and energy estimates, few product – many sum problems, and counting lattice points on convex curves.

The secondary focus of this thesis is proving incidence results in the setting of thickened points and lines (atoms and tubes). We prove a result reminiscent of the Szemerédi–Trotter Theorem and applications, while also addressing the key modifications required to adapt results from traditional incidence geometry.

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Author's Declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

Peter Joseph Bradshaw

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Contents

Notation and Assumed Knowledge

Throughout this thesis, we adopt the following notation and conventions, and assume the following background knowledge.

Conventions and Notation

Uppercase Latin letters, most often A and B, will always be finite sets. We drop the word *finite* unless we wish to emphasise it. Unless otherwise stated, such sets are subsets of $\mathbb{R}$.

Writing $[N]$ always refers to the set $\{1, 2, \ldots, N\}$. When we wish to emphasise that a set A is indexed in increasing order, we write

$$A = \{a_1 < \cdots < a_N\}.$$

We write $|A|$ to be the number of elements in A. For disjoint union we use the symbol $\sqcup$.

Write $\mathbb{1}_S$ for the indicator variable associated with a statement S, which takes the value 1 if S is true and 0 if S is false.

Arithmetic operations on sets always refer to sumsets, product sets and their generalisations. For example

$$A + A := \{a + a' \ : \ a, a' \in A\} \qquad \text{and} \qquad AA := \{aa' \ : \ a, a' \in A\}.$$

Using $\prod$ and $\sum$ with sets always refer to many-fold sumsets and product sets:

$$\sum_{i=1}^{s} A_i := A_1 + \cdots + A_s \qquad \text{and} \qquad \prod_{i=1}^{s} A_i := A_1 \ldots A_s.$$

We use the shorthand $sA := \sum_{i=1}^{s} A$ and $A^{(s)} := \prod_{i=1}^{s} A$ when the A_i are all the same.

At times we abuse the use of the word sumset, using it to refer to any aritmetic expressions of sets using combinations of $+$ and $-$. For example, we may describe either $A - A$ or $A + A - A$ as a sumset.

The *realisation function* r_{A+A} counts the number of realisations of its argument in the sumset $A + A$. That is,

$$r_{A+A}(x) := |\{(a, a') \in A^2 \ : \ a + a' = x\}|.$$

This generalises naturally to different types of sumsets. For example, we may write r_{A-B}, r_{2A-2A} etc. Where context makes it clear what sumset x is contained in, we may write simply $r(x)$.

Throughout this thesis, X_r is the set of r-rich sums:

$$X_r = \{x \ : \ r(x) \in [r, 2r)\}.$$

The particular sumset which x is taken from (for example, $A + A$, $A - B$, or $2A - 2A$) ought to be clear from context.

Asymptotic Notation

We use Vinogradov's symbol extensively. Let $X := X(N)$ and $Y := Y(N)$ be functions depending on the growing parameter N. In all of the following notation, we assume N is sufficiently large.

We write $X \ll Y$[1] to mean that $X \leq CY$ for some absolute constant C and $X \lesssim Y$ to mean that $X \leq C_1 Y (\log Y)^{C_2}$ for some absolute constants C_1, C_2. Furthermore,

[1] In the literature associate with Chapter 5, $\lesssim$ is usually used instead of $\ll$. We will ignore this convention in the interest of keeping consistent notation throughout the thesis.

if $X \ll Y$ and $Y \ll X$, we write $X \approx Y$.

If the suppressed constants depend on some other parameters, these are indicated as subscripts. For example, $X \ll_{\alpha,\beta} Y$ means that $X \leq C(\alpha, \beta)Y$, where $C(\alpha, \beta)$ is a function depending only on α and β. We similarly define $\lesssim_{\alpha,\beta}, \approx_{\alpha,\beta}$ etc.

Writing $X = O(Y)$ will mean the same thing as $X \ll Y$. We write $X = o(Y)$ to mean that $X/Y \to 0$ as $N \to \infty$. So $X = o(1)$ if $X \to 0$ as $N \to \infty$.

Inequalities

The following inequalities are used throughout without explanation:

The *Cauchy–Schwarz inequality* states that given real-valued sequences $\{a_i\}_{i \in I}, \{b_i\}_{i \in I}$, we have

$$\left(\sum_{i \in I} a_i b_i\right)^2 \leq \sum_{i \in I} a_i^2 \cdot \sum_{i \in I} b_i^2.$$

More generally, *Hölder's inequality* states that given $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\sum_{i \in I} a_i b_i \leq \left(\sum_{i \in I} a_i^p\right)^{1/p} \cdot \left(\sum_{i \in I} b_i^q\right)^{1/q}.$$

For our purposes, often $b_i = 1$ for all $i \in I$. In this case the Cauchy–Schwarz inequality and Hölder's inequality respectively become

$$\left(\sum_{i \in I} a_i\right)^2 \leq \sum_{i \in I} a_i^2 \cdot |I| \qquad \text{and} \qquad \sum_{i \in I} a_i \leq \left(\sum_{i \in I} a_i^p\right)^{1/p} \cdot |I|^{1/q}.$$

The Cauchy–Schwarz inequality applies in more general inner product spaces, in which case

$$|\langle \mathbf{u}, \mathbf{v} \rangle|^2 \leq \langle \mathbf{u}, \mathbf{u} \rangle \langle \mathbf{v}, \mathbf{v} \rangle.$$

The Pigeonhole Principle

The *pigeonhole principle* is the obvious statement that given m pigeons placed inside n pigeonholes, if $m > n$ then there must be at least two pigeons in at least one of the pigeonholes.

A more general version says that given $m = (k-1)n + 1$ objects to be placed in n

boxes, at least one box must contain at least k objects.

Dyadic pigeonholing is a way to replace a sum with another sum which is easier to manage and is approximately the same size. Take a set S with $|S| = N$ where each element $s \in S$ has an associated $\epsilon(s)$, and assume that $\epsilon(s)$ is always positive and at most polynomial in N.

Then given any function f, we may write

$$\sum_{s\in S} f(s) = \sum_{i=0}^{M} \sum_{\substack{s\in S:\\ \epsilon(s)\in[2^i,2^{i+1})}} f(s),$$

where $M \ll \log N$, whereupon the pigeonhole principle shows that for some i_0,

$$\sum_{\substack{s\in S:\\ \epsilon(s)\in[2^{i_0},2^{i_0+1})}} f(s) \gg \frac{1}{\log N} \cdot \sum_{s\in S} f(s).$$

In this thesis log will always have base 2. Since $\log N$ may be thought of as not asymptotically significant when compared to expressions which are polynomial in N, we may replace

$$\sum_{s\in S} f(s) \qquad \text{by} \qquad \sum_{\substack{s\in S:\\ \epsilon(s)\in[2^{i_0},2^{i_0+1})}} f(s),$$

and not significantly affect the result. This has the advantage that we may henceforth assume that for all $s \in S$, $\epsilon(s) \approx 2^{i_0}$, often called the *popular* value of $\epsilon(s)$.

Introduction

This thesis predominantly discusses two distinct but related types of problems. Principally, our interest is in studying the additive properties of convex sets and functions. The secondary problem is to better understand the incidence structure of atoms and tubes, which can be seen as a thickened analogue of traditional point-line incidence theory.

A convex set A is a subset of the real numbers in which adjacent spaces between elements increase along the number line. Convex sets are widely accepted to be additively unstructured sets, and there are various natural ways to quantify such a statement. A significant portion of this thesis is devoted to proving results quantifying the statement that convex sets and functions do not exhibit additive structure. These results are a combination of sumset and energy bounds as well as their corollaries and applications.

The final chapter explores the extent to which traditional results and proofs in incidence geometry have analogues when points and lines are thickened by some small factor δ. This seemingly small modification appreciably changes the nature of the problems with some of the main results in standard incidence geometry becoming manifestly false in the thickened setting. Our work explores the extent to which these theorems may be salvaged by making extra assumptions in the main theorems and adopting new techniques from areas such as analysis.

The structure and main results of the thesis are summarised below. In the interest of transparency I also explicitly state which results are my work and in which papers they appear. The relevance of these results is discussed further as they are introduced in the text.

- Chapter 1 is an introduction to the key results in additive combinatorics, both those which are explicitly discussed in this thesis and those which are included for historical depth. A section is devoted to incidence geometry, giving an overview of some main results and discussions of relations to other problems.

- Chapters 2 and 3 give proofs of various sumset and energy bounds, which constitute the main theorems of the thesis.

  In Chapter 2, Theorem 2.1.2 is a small improvement on previous results by incorporating some new techniques. Theorem 2.1.3 is also new and has no precursor. It gives the new Theorem 2.1.4 as an application. These results appear in [12]. In this chapter, we also present the Equidistribuation Lemma 2.2.2 which I introduced in [13].

  In Chapter 3, Theorem 3.1.2, Corollary 3.1.1, Theorem 3.1.3, Theorem 3.1.4, Corollary 3.1.2, Theorem 3.1.5, and Theorem 3.4.1 are all new results. They are built up out of new iterative machinery to estimate energies as well as the Equidistribution Lemma 2.2.2 to incorporate small additive doubling into additive combinatorial problems. These results all appear in [13].

- Chapter 4 contains applications of results and ideas in Chapters 2 and 3 in two categories.

  Firstly, we prove the new Theorem 4.1.2, containing various sumset and energy estimates for higher convex sets. The proof methods use existing estimating methods with a new iterating argument. These are also contained in [13].

  The other section contains new applications for the Equidistribution Lemma 2.2.2. These results are Theorems 4.2.1, 4.2.2, and 4.2.3, all appearing in [12]. Theorem 4.2.6 is new and does not appear in any literature. I proved it after, but independently of, Misha Rudnev.

- Chapter 5 contains all discussion and results relating to continuous incidences. Theorems 5.1.2 and 5.4.1 are new. Proposition 5.2.1 is a generalisation of its precursor in [32] . The exposition is also new. These results are included in [11].

Chapter 1

Introduction to Additive Combinatorics

The study of additive combinatorics and arithmetic combinatorics concerns the arithmetic structure in sets of numbers. We start generally; let A be a subset of an abelian group G and write the group operation additively. Studying arithmetic in nonabelian groups is very different and will not be discussed further. The central question for us is: to what extent does A exhibit additive structure?

1.1 Additive Structure

Two of the most fundamental quantities of interest in additive combinatorics are *sumsets* and *energy*. The sumset of A is defined as

$$A + A := \{a_1 + a_2 : a_1, a_2 \in A\},$$

and the energy $E(A)$ is defined as the number of solutions to the equation

$$a_1 + a_2 = a_3 + a_4, \tag{1.1.1}$$

where $a_1, a_2, a_3, a_4 \in A$. While much of the theory we state in this chapter holds for any group G, our particular interest is when G is a field (and more specifically $\mathbb{R}$).

Accordingly, our examples henceforth will be taken from the reals.

Before discussing additive structure more fully, we make a cursory examination of the properties of sumsets and energies, especially relating to their size. Making no assumptions whatsoever on the set A, the following elementary bounds hold:

$$|A| \ll |A+A| \ll |A|^2 \qquad \text{and} \qquad |A|^2 \ll E(A) \ll |A|^3. \tag{1.1.2}$$

The lower bound for $|A+A|$ can be seen by observing that for some $a \in A$, we have $|A| = |\{a\} + A| \leq |A+A|$. The upper bound is clear since there are at most $|A|^2$ pairs (a_1, a_2).

Finding $|A|^2$ solutions to the energy equation (1.1.1) is realised by taking all solutions with $a_1 = a_3$ and $a_2 = a_4$, yielding the lower bound for $E(A)$. Furthermore, once a_1, a_2, a_3 are fixed in (1.1.1) in $|A|^3$ ways, there is at most one possible solution for $a_4 \in A$, establishing the upper bound for $E(A)$.

Example 1.1.1. The bounds in (1.1.2) are all attainable. Let A be an arithmetic progression, say $A = \{1, 2, \ldots, N\}$. Then

$$A + A = \{2, 3, \ldots, 2N\},$$

so $|A+A| \approx N$. Using the same set A, there are $\approx N^3$ triples $(a_1, a_2, a_3) \in A^3$ such that $a_1 \leq a_3 \leq a_2$. For each such triple, we can always find $a_4 \in A$ such that

$$a_1 + a_2 = a_3 + a_4.$$

It follows that $E(A) \approx N^3$.

Alternatively, if A is a geometric progression, say $A = \{2^0, 2^1, \ldots, 2^N\}$, then each unordered pair $(a_1, a_2) \in A^2$ gives a different $a_1 + a_2$, so $|A+A| \approx N^2$. This also means that the only solutions to the energy equation are trivial, so $E(A) \approx N^2$.

Example 1.1.1 uncovers examples on both ends of the additive structure spectrum. We want to think of arithmetic progressions as highly structured and geometric progressions as highly unstructured. This motivates our options for how we measure structure; we may say a set $A \subset G$ is *additively structured* if:

(1) $A + A$ is small,

(2) $E(A)$ is large, or

(3) A resembles a sumset; ie. there exists $B \subset G$ such that $|A \cap (B + B)| \geq \frac{|A|}{2}$.

In this thesis, we measure structure with both (1) and (2). We also use (3) briefly in Chapter 3.

We have seen that an arithmetic progression A attains both the trivial lower bound for $|A + A|$ and upper bound for $E(A)$ in Example 1.1.1 above. The famous Freiman's Theorem proves that in fact, any set with small sumset "looks like" a generalised arithmetic progression.

Definition 1.1.1. A *generalised arithmetic progression (GAP)* is a finite set of the form

$$P = \{a_0 + a_1 x_1 + \cdots + a_d x_d \ : \ x_i \in \{0, \ldots, l_i\} \text{ for all } i\},$$

where $a_0, a_1, \ldots, a_d$ are fixed reals and $l_1, \ldots, l_d$ are all fixed positive integers. We say d is the *dimension* of the generalised arithmetic progression.

A generalised arithmetic progression A with dimension d also has the property that $|A + A| \approx_d |A|$.

Theorem 1.1.1 (Freiman). *Let A be a finite subset of G with $|A+A| = K|A|$. Then $A \subset P$, where P is a GAP of size at least $f(K)|A|$ and dimension at most $d(K)$, where $f(K), d(K)$ are constants depending only on K.*

A proof can be found in [79]. In light of Freiman's Theorem, we will henceforth think of $A = [N]$ as the canonical example of an additively structured subset of $\mathbb{R}$.

1.1.1 Relating sumsets and energy

It is apparent that $|A+A|$ and $E(A)$ capture similar information. In order for $|A+A|$ to be small (that is, close to its lower bound $|A|$), we need many pairs $a_1, a_2 \in A$ to give duplicates of the same sum $a_1 + a_2$. This guarantees that many solutions to (1.1.1) are realised and $E(A)$ will be close to its upper bound $|A|^3$. The following lemma captures this heuristic.

Lemma 1.1.1. *Given a finite $A \subset G$, we have*

$$|A+A| \; E(A) \geq |A|^4. \tag{1.1.3}$$

Proof. Firstly recall that $r_{A+A}(x)$ is the number of solutions to $a_1 + a_2 = x$ where $a_1, a_2 \in A$. Equipped with this notation, it can be easily checked that

$$\sum_{x \in A+A} r_{A+A}(x) = |A|^2 \qquad \text{and} \qquad \sum_{x \in A+A} r^2_{A+A}(x) = E(A).$$

Applying the Cauchy–Schwarz inequality, one obtains the desired:

$$|A|^4 = \left(\sum_{x \in A+A} r_{A+A}(x) \right)^2 \leq \left(\sum_{x \in A+A} 1^2 \right) \left(\sum_{x \in A+A} r^2_{A+A}(x) \right) = |A+A| \; E(A). \quad \square$$

It follows that if either the sumset or energy is small, the other must be large. The reverse is not true. Both the sumset and energy may be maximised simultaneously as the following example illustrates.

Example 1.1.2. Let $A = A_1 \sqcup A_2$ where A_1 is an arithmetic progression, A_2 is a geometric progression, and $|A_1| = |A_2| = N/2$. We have

$$|A+A| \geq |A_2 + A_2| \gg N^2 \qquad \text{and} \qquad E(A) \geq E(A_1) \gg N^3.$$

Since a small sumset implies a large energy, the sumset may be thought of as a stronger notion of additive structure. However, energy is perhaps a more robust measure. In the above example, the large sumset may lead one to guess that A is a totally unstructured set. But we probably wish to think of a set containing a large arithmetic progression as being structured. This is accounted for by considering the energy.

Indeed, work of Balog and Szemeredi [2] and Gowers [27] prove that this is the only required concession to prove a converse result: if a set has large additive energy, it contains a *large subset* with a small sumset. The specific formulation of the Balog–Szemerédi–Gowers Theorem below is due to Schoen [66].

Theorem 1.1.2 (Balog–Szemerédi–Gowers). *Let A be a subset of G with $E(A) = \kappa|A|^3$. Then there exists a subset $A' \subset A$ with $|A'| \gg \kappa|A|$ and $|A'+A'| \ll \kappa^{-4}|A'|$.*

1.1.2 Longer Sums: Plünnecke's Inequalities

The definition of sumset is easily generalised to include more summands (which can be taken from different sets), and including differences as well as sums. In the language of groups, subtractions are adding inverses of elements of A.

We introduce the following notation to encapsulate these definitions. For the sumset of sets $A_1, \ldots, A_s$, we write

$$A_1 + \cdots + A_s \qquad \text{or} \qquad \sum_{i=1}^{s} A_i.$$

If some of the A_i are the same we may group them together with a coefficient, so that

$$sA := \sum_{i=1}^{s} A.$$

One must be careful not to confuse this with the set of dilates of A by s which uses the same notation at some places in the literature. In this thesis sA only ever refers to s-fold sumsets. These notations are mixed in natural ways. Some sumsets of particular importance in this thesis are: $A + A - A$ and $(s+1)A - sA$ for integer s.

Longer sumsets are similarly useful metrics of additive structure. Some simple constraints on the size of long sumsets such as kA are inherited from constraints on the size of the twofold sumset $A + A$. These are summarised in a series of identities of Plünnecke [56, 57] and were subsequently extended to sums and differences with a new proof given by Ruzsa [64]. Nowadays, results of this type are referred to as Plünnecke's inequalities or Plünnecke–Ruzsa Theorems.

Theorem 1.1.3 (Plünnecke). *Let A, B be finite subsets of G, with $|A + B| = \alpha|A|$. Then there exists a nonempty subset $A' \subset A$ such that*

$$|A' + sB| \leq \alpha^s |A'|,$$

holds for all s.

A simple proof of this result can be found in [54]. The following is a refined form which will be particularly useful in later chapters. In its original form [40, Corollary 1.5] it is stated over $\mathbb{F}_p$ but is equally valid over other abelian groups.

Theorem 1.1.4 (Plünnecke). *Let A, B be finite subsets of G, with $|A+B| = \alpha|A|$. Then there exists $A' \subset A$ with $|A'| \geq |A|/2$, such that*

$$|A' + sB| \ll \alpha^s |A|,$$

holds for all s.

For completeness in this section and because of its relevance in Chapter 3, we introduce energy equations with many terms on each side. Given a set A, the s-fold energy is defined as

$$T_s(A) := |\{(a_1, \dots, a_{2s}) \in A^{2s} : \; a_1 + \dots + a_s = a_{s+1} + \dots + a_{2s}\}|.$$

It is worth noting that an analogue of Lemma 1.1.1 exists for longer sums:

$$|sA| \; T_s(A) \geq |A|^{2s}.$$

1.1.3 Sum-product Phenomena

The results discussed thus far apply in all abelian groups G. But as previously mentioned, we will generally be working over a field $\mathbb{F}$ (which will usually be $\mathbb{R}$), in which case we simultaneously consider $\mathbb{F}$ as a group under addition and $\mathbb{F}^*$ as a group under multiplication. Accordingly, we may discuss the size of both the sumset $A+A$ and the product set AA together. For the energy, $E(A)$ will always refer to the additive energy and $E^{\times}(A)$ to its multiplicative analogue.

In any field, an arithmetic progression is the example we have in mind for strong additive structure. Similarly, a geometric progression has strong multiplicative structure (as this is an arithmetic progression in the multiplicative group). Since the examples which make $A+A$ and AA small are vastly different, it is natural to guess that both cannot be small at the same time, as Erdős and Szemerédi did in 1983 [20].

Conjecture 1.1.1 (Sum-Product Conjecture). *For any positive $\delta < 1$, there exists a constant $C(\delta) > 0$, such that for any sufficiently large $A \subset \mathbb{R}$, we have*

$$\max\{|A + A|, |AA|\} \geq C(\delta)|A|^{1+\delta}. \tag{1.1.4}$$

In fact, proving (1.1.4) for a specific $\delta > 0$ would already prove that a set cannot be both additive and multiplicative. Progress of the last forty years has been in the form of proving (1.1.4) for increasing δ values. Some landmark results were $\delta = 1/4$ due to Elekes [18] and $\delta = 1/3 - o(1)$ due to Solymosi [71]. The record to date is $\delta = \frac{1}{3} + \frac{2}{1167} - o(1)$ due to Rudnev and Stevens [62].

The state of the art in other fields is weaker. The best known in various other fields are: $\delta = 1/3 + c$ in $\mathbb{C}$ (for some small constant $c > 0$) due to Basit and Lund [4], $\delta = 1/4 - o(1)$ in prime fields $\mathbb{F}_p$ due to Mohammadi and Stevens [46], and $\delta = 1/5 - o(1)$ in function fields $\mathbb{F}_q((t^{-1}))$ due to Bloom and Jones [6]. Intuitively, we expect the quantitatively strongest results in fields with lots of structure that can be exploited.

There are also many variations on the sum-product problem. These include the few sums, many products and few products, many sums problems (these will be discussed further in Chapter 4) as well as sum-product results for longer sums and products (see [10]). All such problems enshrine the idea that additive and multiplicative structure cannot both be exhibited by the same set.

1.1.4 Balog–Wooley Decomposition

It would be natural to ask whether an analogue for the sum-product theorem exists for energy. That is, can one prove that for any set $A \subset \mathbb{F}$, we have either $E(A)$ or $E^{\times}(A)$ being small. Recalling that energy quantifies a weaker notion of structure than the sumset and product set, this does not follow from the sum-product theorem. Indeed in the strictest sense, it is false. Consider A from Example 1.1.2. This gives $E(A) \approx E^{\times}(A) \approx N^3$; that is, both additive and multiplicative energy are maximized.

However, a partial sum-product-type result for energy has been obtained over the reals by Balog and Wooley [3]. Specifically they showed that any set A can be decomposed into a set which is not additive and a set which is not multiplicative.

Theorem 1.1.5 (Balog–Wooley). *Any finite $A \subset \mathbb{R}$ can be decomposed $A = A_1 \sqcup A_2$ such that*

$$E(A_1) \lesssim |A|^{3-\delta} \qquad \text{and} \qquad E^{\times}(A_2) \lesssim |A|^{3-\delta},$$

where $\delta = 2/33$.

Further progress has been made in these so-called low energy decompositions. Recently, Xue [82] improved the best known value of δ to $3/11$, and Mudgal [47] proved a similar decomposition for longer sums and products.

1.1.5 Higher Energy

In later chapters, we will occasionally use higher energies. For example, the third order additive energy, denoted by $E_3(A)$ is the number of solutions to the equation

$$a - a' = b - b' = c - c' \qquad \text{where} \qquad a, a', b, b', c, c' \in A.$$

Equipped with the previously defined notation $r_{A-A}(x)$, we can write

$$E_3(A) = \sum_{x \in A-A} r^3_{A-A}(x).$$

The third order energy is the third moment of the realisation function r_{A-A} in the same way that $E(A)$ is the second moment. Note that higher energies are expressed as moments of r_{A-A} rather than r_{A+A}. The second order energy is unique in that

$$E(A) = \sum_{x \in A-A} r^2_{A-A}(x) = \sum_{x \in A+A} r^2_{A+A}(x).$$

Using higher energies precipitates improved bounds in many problems. We give one example of how this can manifest. By a dyadic pigeonholing argument, we may instead write

$$E_3(A) = \sum_{x \in A-A} r^3_{A-A}(x) = \sum_{r \text{ dyadic}} r^3 |X_r|,$$

where X_r is the set of r-rich differences in $A - A$. In Chapter 3, we prove energy bounds for convex sets A via upper bounds for $|X_r|$. It turns out that known bounds for $|X_r|$ yield sharp bounds for $E_3(A)$ but suboptimal bounds for $E(A)$. This is a fact that will be leveraged in Chapter 4.

1.2 Convexity

A significant portion of this thesis is focused on the additive properties of convex sets and convex functions. In this section we work exclusively over $\mathbb{R}$.

A set $A = \{a_1 < \cdots < a_N\}$ is said to be *convex* if

$$a_2 - a_1 < a_3 - a_2 < \cdots < a_N - a_{N-1}.$$

Traditionally, we would describe A as concave if these adjacent differences formed a decreasing sequence. However, in this work we make no distinction, referring to both as convex sets.

Convexity is thought to pose an obstruction to additive structure in A. The first nontrivial result in this direction is due to Hegyvári [36] who showed, in response to an earlier question by Erdős, that if A is convex,

$$|A + A| \gg \frac{\log |A|}{\log \log |A|} |A| .$$

Convex sets have been widely studied since by expressing them as $A = f([N])$ where f is an increasing function with an increasing first derivative. That this is always possible is a fact we will take for granted.

Saying a function is $C^k(U)$ means its derivatives exist and are continuous up to and including order k on an open, connected set U. We say a $C^1(U)$ function f is *convex* if both f and f' are either strictly increasing or strictly decreasing on U. Throughout this thesis, when we apply such a convex function f to a set A, we implicitly assume that $A \subset U$. This definition of convex functions is not standard but will be used throughout this thesis.

The language of convex functions suggests a related problem, that of estimating the number of lattice points on a convex curve $\Gamma = \{(x, f(x)) : x \in I\}$ where f is a convex function and I is an open set. Jarník [38] addressed this problem in 1926 with a sharp bound. However, by imposing higher smoothness conditions on f, this is improved, notably by Bombieri and Pila [7]. Their result motivates the study of functions which are "more convex"; this is discussed later in the chapter.

There is an aphorism in additive combinatorics that convex functions destroy additive structure; if f is a convex function, then either A or $f(A)$ can be additively structured, but not both. Statements about the additive structure of A and $f(A)$ are more general than sum-product results. Indeed, if we have

$$\max\{|A+A|, |f(A)+f(A)|\} \gg |A|^{1+\delta} \tag{1.2.1}$$

for some positive $\delta < 1$, then one immediately obtains a sum-product theorem by choosing $f(x) = \log(x)$. Elekes, Nathanson and Ruzsa [19] proved a more general form of (1.2.1) with $\delta = 1/4$. Specifically they proved, given sets B, C, D with $|C|, |D| \gg |B|$, that

$$|B+C||f(B)+D| \gg |B|^{3/2}|C|^{1/2}|D|^{1/2}.$$

Setting $C = B$ and $D = f(B)$ yields (1.2.1). In 2021, Stevens and Warren [74] proved the quantitative improvement that given sets A, B and convex functions f, g, we have

$$|A+B||f(A)+g(B)| \gtrsim |A|^{38/49}|B|^{38/49}.$$

The best known 2-fold sumset and energy bounds to date are due to Schoen and Shkredov [67] (difference set bound), Rudnev and Stevens [62] (sumset bound) and Shkredov [68] (energy bound) and are summarised below.

Theorem 1.2.1. *If A is convex, then*

$$\begin{aligned} |A-A| &\gtrsim |A|^{8/5 = 1.6} \\ |A+A| &\gtrsim |A|^{30/19 \approx 1.579}, \\ E(A) &\lesssim |A|^{32/13 \approx 2.4615}. \end{aligned}$$

1.2.1 Higher Convexity

In the remainder of the thesis, we will need a more sophisticated language for talking about convexity, specifically to capture that some sets are more convex than others.

Definition 1.2.1. A set $A = \{a_1, a_2, \ldots, a_N\}$ is 0-convex if it is monotone (either strictly increasing or decreasing) with respect to the indexing. Inductively we say A is k-convex if it is monotone and the set of adjacent differences

$$a_2 - a_1, a_3 - a_2, \ldots, a_N - a_{N-1}$$

forms a $(k-1)$-convex set.

Under this definition, a 1-convex set is simply a convex set as defined at the start of the chapter. Henceforth, when we say convex or k-convex we will insist that the set itself and its iterated differences are not just monotone but monotone *increasing*. This will cause no problems; all arguments herein can be easily modified to adjust for this.

Lemma 1.2.1. *If $k \geq 1$ and A is a k-convex set, then for any $1 \leq d < N$, the set*

$$\Delta_d A := \{a_{i+d} - a_i,\ 1 \leq i \leq N-d\}$$

is $(k-1)$-convex.

Proof. We use induction on k. A telescoping summation proves that $\Delta_d A$ is 0-convex. In particular, this proves the base step. For the induction, assume the statement holds for all $(k-1)$-convex functions. Since $\Delta_d A$ is monotone, in order to show that $\Delta_d A$ is $(k-1)$-convex, it suffices to show that

$$\Delta_1 \Delta_d A = \{(a_{i+1+d} - a_{i+1}) - (a_{i+d} - a_i) :\ i \in [N-d-1]\}$$

is $(k-2)$-convex. Since $\Delta_1 \Delta_d A = \Delta_d \Delta_1 A$, this follows by the $(k-1)$-convexity of $\Delta_1 A$ and the induction hypothesis. $\square$

A novel aspect of this work is demonstrating that the higher the convexity, the greater the obstruction to additive structure. In fact, by assuming higher convexity we will later prove an improvement to all the bounds in Theorem 1.2.1.

Theorem 1.2.2. *If A is a 2-convex set, then*

$$\begin{aligned} |A-A| &\gtrsim |A|^{1+151/234 \approx 1.645}\,, \\ |A+A| &\gtrsim |A|^{1+229/309 \approx 1.587}\,, \\ E(A) &\lesssim |A|^{2.4554}\,. \end{aligned}$$

Incorporating higher convexity (along with clever squeezing arguments), Hanson, Roche-Newton and Rudnev recently proved new sumset bounds.

Theorem 1.2.3 (Hanson, Roche-Newton, Rudnev)**.** *Let $k \geq 1$ and A be a k-convex set. Then*

$$|2^k A - (2^k - 1)A| \gg |A|^{k+1}.$$

Let us further justify the introduction of convex functions. A convex set A may be expressed as $A = f([N])$ for a convex function f and the corresponding convexity condition becomes

$$f(2) - f(1) < f(3) - f(2) < \cdots < f(N) - f(N-1).$$

Each term $f(i+1) - f(i)$ may be thought of a discrete version of the first derivative f' near i. The fact that these terms are increasing in i reflects that convex functions have increasing first derivative.

But more can be said if A is 2-convex. Then iterating the convexity condition shows that the terms

$$(f(i+2) - f(i+1)) - (f(i+1) - f(i)) \tag{1.2.2}$$

are increasing with i. Expression (1.2.2) is a discrete version of the second derivative f'' near i. In this way 2-convexity of a set $A = f([N])$ corresponds to f having monotone derivatives up to order 2. The following discussion generalises this idea to any order of convexity.

Definition 1.2.2. We say a $C^k(U)$ function f is *k-convex* if it has monotone derivatives $f^{(0)}, f^{(1)}, \dots, f^{(k)}$ on an open, connected subset U. Here monotone means strictly increasing or decreasing.

A 1-convex function is simply a convex function and a 0-convex function is a monotone function. To streamline exposition, we henceforth assume that a k-convex function f has monotone *increasing* derivatives $f^{(0)}, f^{(1)}, \dots, f^{(k)}$. If any of them are decreasing, the proofs herein can be easily modified to handle this. Such modifications are discussed in [33].

It is useful to have some key examples of k-convex functions in mind. The function $f(x) = x^{k+1}$ is k-convex but not $(k+1)$-convex. This function is often the barrier to improving sumset and energy results for k-convex functions. Also, $f(x) = \log(x)$ is k-convex for any k, and is important for understanding sum-product phenomena.

We mostly work with k-convex functions rather than k-convex sets due to the greater generality it enables. We also concretely define "discrete derivatives". Given a function f, let its d-derivative be

$$\Delta_d f(x) := f(x+d) - f(x).$$

Lemma 1.2.2. *Let f be a k-convex function with $k \geq 1$, and let d be nonzero. Then $\Delta_d f$ is a $(k-1)$-convex function.*

Proof. We use induction on k. Suppose f is 1-convex. We have

$$\Delta_d f(x) := f(x+d) - f(x) = \int_x^{x+d} f'(y)dy.$$

Since f' is monotone, it follows that $\Delta_d f$ is also monotone, and hence 0-convex.

Next assume the statement holds for $(k-1)$-convex functions. Let f be a k-convex function. By definition, this implies that f' is a $(k-1)$-convex function. The induction hypothesis implies that $\Delta_d(f')$ is a $(k-2)$-convex function. But since $\Delta_d A$ is monotone (which follows from the base step), and $\Delta_d(f') = (\Delta_d f)'$, it follows that $\Delta_d f$ is $(k-1)$-convex, completing the induction. $\square$

The connection between k-convex functions and k-convex sets is intuitive and is summarised in the following lemma.

Lemma 1.2.3. *If f is a k-convex function, then $f([N])$ is an k-convex set.*

Proof. We proceed by induction on k. If f is a 0-convex function, then $f([N])$ is clearly ordered as a 0-convex set.

Assume the statement holds for $(k-1)$-convex functions. Letting f be a k-convex function, it follows that $(\Delta_1 f)(x)$ is a $(k-1)$-convex function. By the induction hypothesis, $(\Delta_1 f)([N-1])$ is a $(k-1)$-convex set, which, combined with the monotonicity of $f([N])$ proves that $f([N])$ is a k-convex set, completing the induction. □

Remark. We assume a converse of Lemma 1.2.3, that for every k-convex set A, there exists an k-convex function f such that $A = f([N])$. This is a nice assumption to make at several places in the thesis. However, we only require the weaker condition that $A = f([N])$ for some f which has monotone i^{th} divided differences for $i = 0, \ldots, k$. This can be obtained by spline interpolation. For more information about divided differences, see [45].

1.3 Incidence Geometry

While not strictly a topic in additive combinatorics, a discussion of incidence geometry forms a fitting end to this chapter, as it facilitates some of the foremost techniques for solving problems in additive combinatorics. Furthermore, this section can be seen as the discrete precursor to Chapter 5 which is on continuous analogues of incidence geometry.

Incidence geometry is concerned with counting how many times various geometric objects are *incident* with each other, traditionally points and lines.

Definition 1.3.1. Given a set P of points and a set L of straight lines in $\mathbb{R}^2$, we say that a point $p \in P$ lies on a line $l \in L$ if $p \in l$. Also define the incidence-counting function:

$$\mathcal{I}(P, L) := |\{(p, l) \in P \times L : p \in l\}|.$$

Specifically we are concerned with bounds on $\mathcal{I}(P, L)$. It is easy to see that if none of the points $p \in P$ are incident with any of the lines $l \in L$, then $\mathcal{I}(P, L) = 0$. More interesting is studying upper bounds for $\mathcal{I}(P, L)$.

A superficial investigation of the properties of points and lines already allows one to make nontrivial observations about the maximum size of $\mathcal{I}(P, L)$. In $\mathbb{R}^2$, a pair of points lie together on exactly one line and two lines intersect in at most one point. If we work in the projective plane $P_2(\mathbb{R})$ rather than $\mathbb{R}^2$ then two lines intersect in exactly one point. This condition alone limits the maximum size of $\mathcal{I}(P, L)$. For example, it is impossible to have two distinct points p_1, p_2 which are both incident with two distinct line l_1, l_2.

These simple properties along with application of the Cauchy–Schwarz Inequality obtains the following so-called *trivial* incidence bounds.

Theorem 1.3.1. *Let P be a set of points and L be a set of lines in $\mathbb{R}^2$. Then*

$$\mathcal{I}(P, L) \leq |L||P|^{1/2} + |P| \qquad \text{and} \qquad \mathcal{I}(P, L) \leq |P||L|^{1/2} + |L|.$$

Proof. Applying the Cauchy–Schwarz Theorem, we get

$$\begin{aligned}
\mathcal{I}(P, L)^2 &= \left(\sum_{p \in P} \sum_{l \in L} \mathbb{1}_{p \in l} \right)^2 \\
&\leq |P| \sum_{p \in P} \left(\sum_{l \in L} \mathbb{1}_{p \in l} \right)^2 \\
&= |P| \sum_{l_1, l_2 \in L} \sum_{p \in P} \mathbb{1}_{p \in l_1 \cap l_2} \\
&= |P| \left(\sum_{l_1 = l_2} \sum_{p \in P} \mathbb{1}_{p \in l_1 \cap l_2} + \sum_{l_1 \neq l_2} \sum_{p \in P} \mathbb{1}_{p \in l_1 \cap l_2} \right).
\end{aligned}$$

Observe that the first term in the brackets is simply $\mathcal{I}(P, L)$. Next note that once the distinct lines l_1, l_2 are fixed, there is at most one point p which lies on both of them, so the second term is bounded above by $|L|^2$. Rearranging the obtained equation

$$\mathcal{I}(P, L)^2 \leq |P|(\mathcal{I}(P, L) + |L|^2),$$

and making some simple estimates yields

$$\mathcal{I}(P, L) \leq |L||P|^{1/2} + |P|.$$

The other inequality is obtained by reversing the roles of points and lines in the above calculation. □

1.3.1 The Szemerédi–Trotter Theorem

In their 1983 paper [78], Szemerédi and Trotter proved a sharp upper bound for $\mathcal{I}(P, L)$, which has come to be known as the Szemerédi–Trotter Theorem.

Theorem 1.3.2 (Szemerédi–Trotter Theorem)**.** *Let P be a set of points and L be a set of lines in $\mathbb{R}^2$. Then*

$$\mathcal{I}(P, L) \ll |P|^{2/3}|L|^{2/3} + |P| + |L|. \tag{1.3.1}$$

We present a proof of the Szemerédi–Trotter Theorem which closely follows the proof of Kaplan, Matousěk and Sharir [39], which is itself based on polynomial partitioning arguments of Guth and Katz [30]. The proof of the partitioning results use the polynomial ham sandwich theorem, also known as the Stone–Tukey Theorem [75]. Szekély gives another very nice proof of the Szemerédi–Trotter Theorem which uses lower bounds on the crossing number of a particular graph [77].

The proof method below is as follows. A polynomial partitioning lemma is used to partition the plane into pieces containing roughly the same number of points, upon which we locally apply the trivial bounds of Theorem 1.3.1. This will be particularly instructive when we wish to translate incidence problems and their proofs to the continuous setting in Chapter 5.

We require a basic definition which can be found in any introductory text in algebraic geometry; see for example [23].

Definition 1.3.2. Let $f(x, y)$ be a real polynomial. Define the *set of zeroes* of f as

$$Z(f) := \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}.$$

In the proof below, we also require the following polynomial partitioning theorem. For a proof, see [30].

Theorem 1.3.3. *If S is a finite subset of $\mathbb{R}^2$ and d any positive integer, then there exists a degree d polynomial f such that each connected component of $\mathbb{R}^2 \backslash Z(f)$ contains $\ll |S|d^{-2}$ points of S.*

Proof of the Szemerédi–Trotter Theorem. Our method will be to partition $\mathbb{R}^2$ into cells using Theorem 1.3.3, then estimate the incidences in these cells and on their boundaries using Theorem 1.3.1 and some trivial bounds.

By point-line duality, we may assume without loss of generality that $|L| \geq |P|$. Using Theorem 1.3.3 with parameter d to be chosen later, we partition $\mathbb{R}^2$ into a set of open cells $\mathcal{C}$. That is, the cells of $\mathcal{C}$ are the connected components of $\mathbb{R}^2 \backslash Z(f)$. Harnack's Curve Theorem proves that a degree d polynomial in $\mathbb{R}^2$ partitions the plane into $\ll d^2$ cells. Thus $|\mathcal{C}| \ll d^2$.

In estimating $\mathcal{I}(P, L)$, we need to count three types of incidences (p, l):

1. $p \in C$ for some $C \in \mathcal{C}$,

2. $p \in Z(P)$, and l does not lie in $Z(f)$,

3. $p \in Z(P)$, and l does lie in $Z(f)$.

The core idea is that if d is large, then the number of points in each cell is small, and the second bound from Theorem 1.3.1 is strong. If d is small, then each line cannot intersect many cells (as intersections arise from solutions to a univariate polynomial of degree d) and therefore cannot contribute many incidences. We later choose d to appropriately to balance this trade-off.

We start by counting the first class of incidences. For each cell $C \in \mathcal{C}$, let L_C be the set of lines which pass through the interior of C and P_C be the set of points on the interior of C. Note that $|P_C| \ll |P|d^{-2}$ by construction. Additionally, each line intersects $\ll d$ cells, so that $\sum_C |L_C| \ll d|L|$. By applying Theorem 1.3.1 in each cell $C \in \mathcal{C}$ and then the Cauchy–Schwarz inequality, the first class of incidences can be estimated by

$$\begin{aligned}\sum_C I(P_C, L_C) &\leq \sum_C |P_C|\sqrt{|L|_C} + \sum_C |L_C| \\ &\leq \frac{|P|}{d}\sqrt{\sum_C |L_C|} + \sum_C |L_C| \\ &\ll \frac{|P|\sqrt{|L|}}{\sqrt{d}} + d|L|.\end{aligned}$$

Next, since each line intersects with $Z(f)$ at most d times, the number of incidences in the second class is at most $d|L|$.

Finally, there are at most d lines lying in $Z(f)$ (which is realised if $Z(f)$ is a union of lines). Then the number of incidences is at most $d|P| \leq d|L|$ (using the assumption that $|P| \leq |L|$).

Putting everything together we get $I(P, L) \ll \frac{|P|\sqrt{|L|}}{\sqrt{d}} + d|L|$, which we minimise by setting $d = \min\{|P|^{2/3}|L|^{-1/3}, 1\}$. If $d = 1$, then $|P| \ll |L|^{1/2}$, whence $I(P, L) \ll |L|$. Note that since we assumed that $|L| \gg |P|$, this term subsumes the $|P|$ term in the bound (1.3.1). Otherwise if $d = |P|^{2/3}|L|^{-1/3}$ then $\mathcal{I}(P, L) \ll |P|^{2/3}|L|^{2/3}$, and the proof is complete. □

An alternative formulation of the Szemerédi–Trotter Theorem gives a bound for the number $L_k(P)$ of k-rich lines induced by a set of points P (or, by duality, the number of k-rich points $P_k(L)$ induced by a set of lines L).

Definition 1.3.3. Given a set P of points, $L_k(P)$ denotes the set of lines passing through $\geq k$ points from P. We call these *k-rich lines.*
Given a set L of lines, $P_k(L)$ denotes the set of points passing through $\geq k$ lines from L. We call these *k-rich points.*

In the above definition, we intentionally did not specify the space we are working in. The same notation is used for any $\mathbb{R}^n$ (and indeed if we work over $\mathbb{F}^n$ for some other field $\mathbb{F}$).

We sometimes use the shorthand L_k (or P_k) when the point (or line) set is obvious.

Theorem 1.3.4. *The following are both equivalent to the Szemerédi–Trotter Theorem:*

(I) Let P be a set of points in $\mathbb{R}^2$ and let $k \geq 2$. Then

$$L_k(P) \ll \frac{|P|^2}{k^3} + \frac{|P|}{k}. \tag{1.3.2}$$

(II) Let L be a set of lines in $\mathbb{R}^2$ and let $k \geq 2$. Then

$$P_k(L) \ll \frac{|L|^2}{k^3} + \frac{|L|}{k}. \tag{1.3.3}$$

Proof. It is immediate by duality that (I) and (II) are equivalent, so it remains to show that (I) is equivalent to Theorem 1.3.2.

Assume that (1.3.1) holds. Given a point set P, let L_k be the set of k-rich lines induced by P, where $k \geq 2$. Then

$$k|L_k| \ll \mathcal{I}(P, L) \ll |P|^{2/3}|L_k|^{2/3} + |P| + |L_k|.$$

We now consider which is the dominant term on the right-hand side. If the first term on the right-hand side dominates, then rearranging yields $|L_k| \ll |P|^2 k^{-3}$. If the second term dominates, then $|L_k| \ll |P|k^{-1}$. The third term cannot dominate. Combining these terms, we get

$$|L_k| \ll \frac{|P|^2}{k^3} + \frac{|P|}{k}.$$

For the other direction we start with a set P of points and a set L of lines, and we suppose that (I) holds. If $|P| \gg |L|^2$, then

$$\mathcal{I}(P, L) \ll |P|^{1/2}|L| + |P| \ll |P|,$$

and we are done. Similarly, if $|L| \gg |P|^2$, we are done. We may henceforth assume that $|P|^{1/2} \ll |L| \ll |P|^2$.

Partition L as $L = L^{(1)} \sqcup L^{(2)} \sqcup L^{(3)}$, where

$$\begin{aligned} L^{(1)} &= \text{ set of lines incident with} \geq |P|^{1/2} \text{ points,} \\ L^{(2)} &= \text{ set of lines incident with} \geq 2 \text{ and} < |P|^{1/2} \text{ points,} \\ L^{(3)} &= \text{ set of lines incident with one point.} \end{aligned}$$

Then $\mathcal{I}(P,L) = \mathcal{I}(P,L^{(1)}) + \mathcal{I}(P,L^{(2)}) + \mathcal{I}(P,L^{(3)})$. Each of these terms will be bounded individually.

Firstly it is clear that $\mathcal{I}(P,L^{(3)}) = |L^{(3)}| \ll |L|$. Next, notice that (1.3.2) yields $|L^{(1)}| = |L_{|P|^{1/2}}(P)| \ll |P|^{1/2}$. Then the the trivial incidence bound gives

$$\mathcal{I}(P,L^{(1)}) \leq |L^{(1)}||P|^{1/2} + |P| \ll |P|.$$

Finally, we split $L^{(2)}$ into "poor" and "rich" lines by a parameter k^* to be chosen. For the poor lines we use a trivial incidence bound and for the rich lines we use (1.3.2) in a dyadic sum, yielding

$$\mathcal{I}(P,L^{(2)}) \ll |L|k^* + \sum_{\substack{k>k^* \\ \text{dyadic}}} k|L_k| \ll |L|k^* + \frac{|P|^2}{{k^*}^2}.$$

Note that the second term of (1.3.2) does not appear above as the first term always dominates when $|L|^{1/2} \ll |P| \ll |L|^2$. Choosing the optimal value of $k^* = |P|^{2/3}/|L|^{1/3}$ gives

$$\mathcal{I}(P,L^{(2)}) \ll (|P||L|)^{2/3}.$$

Combining these estimates yields (1.3.1). □

A slightly different way of defining k-rich is to say that given a point set P (or analogously for a line set), a line is k-rich if it passes through k' points, where $k' \in [k, 2k)$. After a dyadic decomposition it is apparent that the bounds of Theorem 1.3.4 also hold under the new definition for k-rich but with a different constant implied by the $\ll$ notation. Henceforth, we use the two definitions of k-rich interchangeably.

Use of incidence geometry in additive combinatorics dawned as a result of a beautiful paper of Elekes [18]. By artificially imposing a points and lines structure on a finite set A, as well as its sumset $A + A$ and product set AA, he applied the Szemerédi–Trotter Theorem to obtain the $\delta = 1/4$ bound in the sum-product problem over $\mathbb{R}$. Generalisations of this approach appear in [19]. Many further applications and refinements of these techniques ensued. Indeed, combinations of incidence geometry with other techniques are common; for example, the best sum-product bound in $\mathbb{R}$ [62].

There are also applications in discrete geometry. Beck's theorem states that given n points in the plane with at most $n - k$ on any line, the number of lines connecting at least two such points is $\gg nk$. In other words, there are either $\gg n$ points on a single line, or the set of points induce $\gg n^2$ lines by connecting pairs of points. Another previously mentioned problem is to estimate the number of points of the integer lattice $[N] \times [N]$ which intersect a particular convex curve. A sharp upper bound of $N^{2/3}$ exists and will be discussed later. Both of the above problems admit solutions by simple applications of the Szemerédi–Trotter Theorem. Further applications are discussed in the survey paper of Dvir [17].

1.3.2 Other results in Incidence Geometry

Several other incidence results are also widely applicable, especially when working in higher dimensions or over fields other than $\mathbb{R}$. None of the following will be used hereafter, but we mention them to have a more complete background.

Guth–Katz

If $P \subset \mathbb{R}^2$ is a finite set of points, then the *distance set* is defined as

$$\Delta(P) := \{\|p - p'\| \ : \ p, p' \in P\}.$$

The distinct distance problem seeks to lower bound the size of $\Delta(P)$ for a general point set P. In their breakthrough paper [30], Guth and Katz resolved the distinct distance problem in $\mathbb{R}^2$, giving the following bound which is sharp up to logarithmic factors:

$$|\Delta(P)| \gg \frac{|P|}{\log |P|}.$$

Their method used the Elekes–Sharir framework, a parametrisation of rigid motions in $\mathbb{R}^2$, to convert information about distances in $\mathbb{R}^2$ into information about point-line incidences in $\mathbb{R}^3$.

The key new piece of their method is an ingenious application of the polynomial method yielding the following incidence bound.

Theorem 1.3.5 (Guth–Katz)**.** *Let L be a set of lines in $\mathbb{R}^3$ with at most $|L|^{1/2}$ lines concurrent or lying in a plane or doubly ruled surface. Suppose that $k \geq 2$. Then*

$$|P_k(L)| \ll \frac{|L|^{3/2}}{k^2}.$$

Further applications of the Guth–Katz incidence bound have been found in recent years. See for example [53, 61].

Rudnev's points-planes theorem

One great drawback of the Szemerédi–Trotter Theorem is that it doesn't hold over a general field. Indeed, all known proofs assume in some way the topology of the reals. The partitioning argument we sketched in this thesis makes no sense in a field where no notion of a "cell" exists.

As alluded to, incidences between objects other than points and lines are also studied. Rudnev proved the following point-plane incidence bound in three dimensions [60] which importantly applies in any field (with characteristic not equal to 2).

Theorem 1.3.6. *Let P and Π respectively be finite sets of points and planes in projective 3-space over some field $\mathbb{F}$. Let $|P| \geq |\Pi|$ and if $\mathbb{F}$ has positive characteristic, let $p \neq 2$ and $|\Pi| \ll p^2$. Let k be the maximum number of collinear planes. Then*

$$\mathcal{I}(P, \Pi) \ll |P||\Pi|^{1/2} + k|P|.$$

The proof idea is as follows. Lines in $P_3(\mathbb{F})$ (projective 3-space) can be parametrised as points on a 4-dimensional variety in $P_5(\mathbb{F})$ known as the Klein quadric. There is a particular sense in which an incident point-plane pair (p, π) in $P_3(\mathbb{F})$ becomes a pair of incident lines in a section of the Klein quadric. Using the same machinary as Guth and Katz, a suitable bound on the number of these line-line incidences may be obtained. For a short proof (but one which hides the underlying geometry), see [16].

One of the most important applications is a pair of point-line incidence bounds in general fields due to Stevens and de Zeeuw [73].

Theorem 1.3.7 (Stevens–de Zeeuw)**.** *Let $A, B \subset \mathbb{F}$ and L be a set of lines in $\mathbb{F}^2$ with $|A| \leq |B|$ and $|A||B|^2 \leq |L|^3$. If $\mathbb{F}$ has positive characteristic p, assume $|A||L| \ll p^2$. Then*

$$\mathcal{I}(A \times B, L) \ll |A|^{3/4}|B|^{1/2}|L|^{3/4} + |L|.$$

Theorem 1.3.8 (Stevens–de Zeeuw)**.** *Let P be a set of points and L a set of lines in $\mathbb{F}^2$, with $|P|^{7/8} < |L| < |P|^{8/7}$. If $\mathbb{F}$ has positive characteristic p, also assume $|P|^{-2}|L|^{13} \ll p^{15}$. Then*

$$\mathcal{I}(P, L) \ll |P|^{11/15}|L|^{11/15}.$$

Theorem 1.3.7 was a core result in proving the best known sum-product bound over finite fields [46]. The points-planes incidence theorem is a highly influential result in the area and has seen many applications in additive combinatorics and discrete geometry. See for example [49, 55, 58].

Chapter 2

Convexity: Estimating Long Sumset Sizes

2.1 Introduction

In the previous chapter we defined convex sets and convex functions. We also defined higher convexity, a way of quantifying that some sets and functions are more convex than others.

A central intuition in the area is summarised in the following two statements:

1. *Convex functions destroy additive structure.*

2. *The more convex a function, the more additive structure it destroys.*

The main results of this chapter are sumset theorems supporting this intuition. A simple and clever *squeezing* approach for proving sumset bounds for convex sets first appeared in [63], proving the sharp lower bound

$$|A + A - A| \gg |A|^2 \tag{2.1.1}$$

where A is a convex set. Setting $A = \{1^2, 2^2, \dots, N^2\}$ demonstrates its sharpness. This approach was significantly extended in [33], proving the following estimate for longer sums of more convex sets. The right-hand bound is again optimal.

Theorem 2.1.1. *Let A be a k-convex set. Then, we have*

$$|2^k A - (2^k - 1)A| \gg_k |A|^{k+1}. \tag{2.1.2}$$

Both (2.1.1) and (2.1.2) capture the aforementioned intuition. Take (2.1.1) for example. We can rewrite this as

$$|f([N]) + f([N]) - f([N])| \gg N^2,$$

where f is a convex function. Recall that $[N]$ is the canonical example of an additively structured set. The convex function f "destroys" this additive structure, which is indicated by $f([N])$ having a large threefold sumset $f([N]) + f([N]) - f([N])$.

Moreover, by rewriting (2.1.2) as

$$|2^k f([N]) - (2^k - 1)f([N])| \gg_k N^{k+1}$$

where f is a k-convex function, we see that higher convexity elicits greater growth in the sumset (albeit a longer sumset). Note that insisting on A being k-convex is necessary; if we allowed $f(x) := x^k$ (a $(k-1)$-convex function) then

$$2^k f([N]) - (2^k - 1)f([N]) \subset \mathbb{Z} \cap [-(2^k - 1)N^k, 2^k N^k],$$

so trivially

$$|2^k f([N]) - (2^k - 1)f([N])| \ll_k N^k,$$

a contradiction to (2.1.2).

The main results of this chapter address the following question. Can similar results be obtained if we replace $[N]$ with some other additively structured set? We will consider two types: sets B with a small sumset $B + B - B$, and sets which are sumsets in their own right, say $sB - sB$. We may call this the *near*-convex setting as $A = f(B)$ may not be k-convex for any k, but may imitate some of the structure of a convex set. The results are summarised in the forthcoming Theorem 2.1.2 and Theorem 2.1.3. In what follows, we always think of k and s as being constants and therefore much smaller than $|B|$.

Theorem 2.1.2. [12] *Let B be a finite set of real numbers and f be a k-convex function. Then if $|B+B-B| \leq K|B|$, we have*

$$|2^k f(B) - (2^k-1)f(B)| \gg_k \frac{|B|^{2^{k+1}-1}}{|B+B-B|^{2^{k+1}-k-2}} \geq |B|^{k+1}K^{-(2^{k+1}-k-2)}.$$

Theorem 2.1.2 proves that guaranteeing growth in the sumset does not require k-convexity, but near k-convexity suffices. It is an analogue of Theorem 2.1.1, where $A = f(B)$ and we can quantify the effect of our choice of B in terms of the doubling constant $|B+B-B|/|B|$. Applying Plünnecke's inequality, this result can be expressed in terms of the more traditional doubling constant $|B-B|/|B|$, slightly weakening the result.

Theorem 2.1.2 is an improvement on the main result in [33] by logarithmic factors. Specifically, we use a lemma from [13] (Lemma 2.2.2 in this document) to sidestep the need for dyadic pigeonholing and streamline the proof. The right-hand bound in its precursor result from [33] was best-possible up to logarithmic factors, while in Theorem 2.1.2, it is genuinely sharp. Set $f(x) = x^{k+1}$ and $B = [N]$ to verify. Several applications to sum-product problems are also discussed in [33].

In [34], it is proved that if B is a finite real set and f is any convex function, then

$$|2f(B \pm B) - f(B \pm B)| \gg |B|^2.$$

We prove the following generalisation.

Theorem 2.1.3. [12] *Let B be a set of reals and f be a k-convex function. If k is even with $k = 2s$ then*

$$|2^k f((s+1)B - sB) - (2^k-1)f((s+1)B - sB)| \gg_k |B|^{k+1}$$

and if k is odd with $k = 2s-1$ then

$$|2^k f(sB - sB) - (2^k-1)f(sB - sB)| \gg_k |B|^{k+1}.$$

Notice that in both cases, the number of summands inside the function f is the same as the power of $|B|$ on the right-hand side. For the purposes of discussion, assume we are in the odd k case. A generic set B is expected to have $|sB - sB| = |B|^{2s} = |B|^{k+1}$, and therefore $|f(sB - sB)| = |B|^{k+1}$ as well since f is a monotone function. However, obviously $sB - sB$ may be significantly smaller than $|B|^{k+1}$. Theorem 2.1.3 affirms that even in this case, sufficiently many sums and differences of $f(sB - sB)$ guarantee the same $|B|^{k+1}$ growth. Here too, the right-hand bound $|B|^{k+1}$ cannot be improved, evinced by setting $B = [N]$ and $f(x) = x^{k+1}$.

In both Theorem 2.1.2 and Theorem 2.1.3, it is unlikely that the number of terms in the sumset is optimal. That is, in order to get growth of order $|B|^{k+1}K^{-O(k)}$ in Theorem 2.1.2 or $|B|^{k+1}$ in Theorem 2.1.3, we expect that fewer than $2^{k+1} - 1$ summands are required in the sumset.

We conjecture that the right number of summands is quadratic in k. There is some evidence in this direction. It seems that $\mathcal{N}_{k+1} := \{i^{k+1} : i = 1, \dots N\}$ is the worst possible k-convex set for growth in sumset. There has been significant work by Wooley [80, 81] and Bourgain [9] into bounding the number of representations of an integer n as the s-fold sum of $(k+1)^{\text{th}}$ powers. Using their bounds, we get $|s\mathcal{N}_k| \gg_\epsilon N^{k-\epsilon}$ whenever $s \gg k^2$. We conjecture that for any other k-convex set or near k-convex set A, the sumset sA will at least match the growth of $s\mathcal{N}_{k+1}$. This suggests that sumsets with the order of k^2 terms ought to give rise to the same sumset growth we see above. Our techniques appear insufficient to make improvements in this direction.

In Theorem 2.1.3, it is also expected that $sB - sB$ is not optimal, and we conjecture the following.

Conjecture 2.1.1. *Let B be a set of reals and f be a k-convex function. Then*

$$|2^k f(B - B) - (2^k - 1) f(B - B)| \gg_k |B|^{k+1}.$$

Moreover, in both Theorem 2.1.2 and Theorem 2.1.3, the high convexity condition cannot be relaxed. If f were only a $(k-1)$-convex function, we could set $f(x) = x^k$ and $B = [N]$ to obtain a contradiction to both theorems.

Growth for products of generalised difference sets

It was conjectured in [1] that for any $s > 0$, there exists $m = m(s)$ such that if A is a finite real set then

$$|(A - A)^{(m)}| \gg_s |A|^s, \tag{2.1.3}$$

where $(A - A)^{(m)}$ denotes an m-fold product set $\underbrace{(A - A) \dots (A - A)}_{m \text{ times}}$. This was proved in [35] for the case when $A \subset \mathbb{Q}$. Balog, Roche-Newton and Zhelezov proved (2.1.3) for $s = 3$, and additionally that for $s = 17/8$, choosing $m = 3$ suffices – the first known results for $s > 2$. Recently, Hanson, Roche-Newton and Senger proved (implicitly) [34] that (2.1.3) holds if $m = 8, s = 33/16$, but their method was stronger in the sense that some of the $A - A$ terms could be replaced with $A - a$ for specific values of $a \in A$. They use this to improve the best known lower bound for $|\Lambda(P)|$ where $\Lambda(P)$ is the set of dot products induced by a point set P in $\mathbb{R}^2$. They proved

$$|\Lambda(P)| \gtrsim |P|^{\frac{2}{3} + \frac{1}{3057}},$$

the first result to break the threshold $|P|^{2/3}$. See [59] for more connections between similar problems and growth in $A - A$.

We prove a result approaching this conjecture from a different direction, namely allowing for products of *many*-fold sums and differences.

Theorem 2.1.4. [12] *Given any natural number $s \in \mathbb{N}$, there exists $m = m(s)$ such that if A is a finite set of reals, then*

$$|(sA - sA)^{(m)}| \gg_s |A|^s.$$

Theorem 2.1.4 is an easy corollary of Theorem 2.1.3; if Conjecture 2.1.1 holds, then (2.1.3) is the natural corollary.

2.2 Squeezing Arguments

By squeezing smaller intervals inside larger intervals, one may construct many elements in certain sumsets, in turn proving that the sumsets must be large. This section will attempt to demystify both the original argument of Ruzsa, Shakan, Solymosi and Szemerédi [63], as well as the generalisations of Hanson, Roche-Newton and Rudnev [33]. Identifying the key elements of the argument will assist in developing the argument further in later sections.

The basic idea is the following. Recall that a set $A = \{a_1 < \cdots < a_N\}$ is convex if the adjacent differences between elements is increasing

$$a_2 - a_1 < a_3 - a_2 < \cdots < a_N - a_{N-1}.$$

It follows that if i is fixed, then for any $j \leq i$, the terms

$$a_i + (a_{j+1} - a_j) \in A + A - A$$

are all unique and lie in $(a_i, a_{i+1}]$. That is, for $j \leq i$ an interval of each width $a_{j+1} - a_j$ can be *squeezed* in the larger interval $(a_i, a_{i+1}]$ to obtain a unique element of $A + A - A$ (as seen in Figure 2.1). We may apply this for any i to obtain $\sum_{i=1}^{N-1} i \approx |A|^2$ elements of $|A + A - A|$, proving (2.1.1).

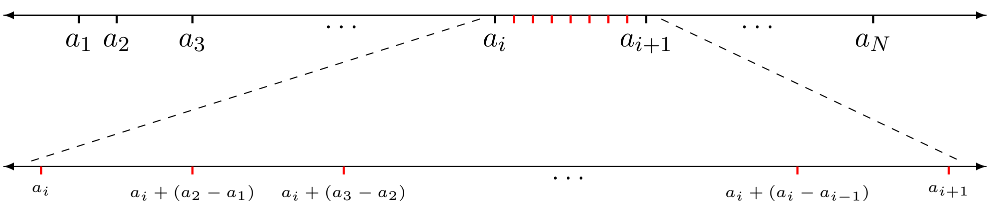

Figure 2.1: Simple squeezing

Let us generalise this proof for 2-convex sets, specifically proving Theorem 2.1.1 for $k = 2$. This ought to clarify how to prove the full theorem by induction, but complete details can be found in [33]. Let $d_i := a_{i+1} - a_i$ for $i \in [N-1]$. Then since A is convex we have

$$d_1 < d_2 < \cdots < d_{N-1},$$

but additionally, since A is 2-convex, we also have

$$d_2 - d_1 < d_3 - d_2 < \cdots < d_{N-1} - d_{N-2}. \tag{2.2.1}$$

In the previous proof, we saw that we can produce unique elements of the form

$$a_i + d_j \in A + A - A$$

by finding elements of $A - A$ which can be squeezed in $(a_i, a_{i+1}]$. But now, in light of (2.2.1), we can additionally squeeze elements of $(A-A)-(A-A)$ in $(a_i+d_j, a_i+d_{j+1}]$. Specifically, if i, j are fixed (with $j \leq i$), then for any $k \leq j \leq i$ the terms

$$a_i + d_j + (d_{k+1} - d_k) \in A + (A - A) + ((A - A) - (A - A)) = 4A - 3A$$

are all unique and lie in $(a_i + d_j, a_i + d_{j+1}]$ (as seen in Figure 2.2). We apply this for every pair i, j with $i \leq j$ to obtain $\sum_{i,j:j\leq i} j \approx N^3$ elements of $4A - 3A$, completing the proof.

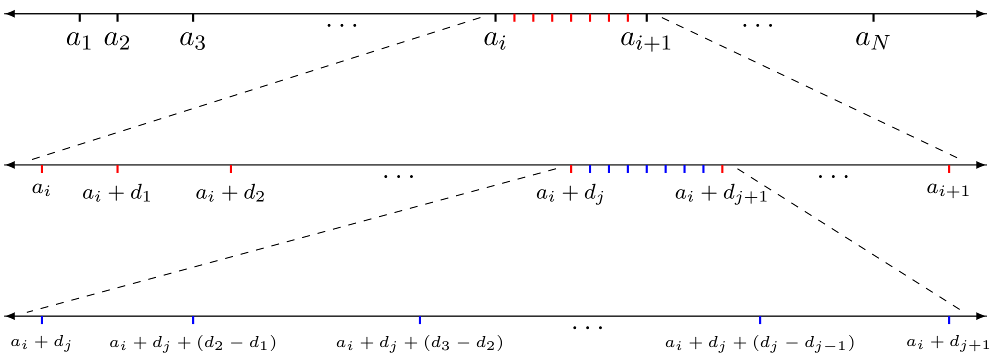

Figure 2.2: Second order squeezing

A k-fold squeezing argument involves k-iterations of the squeezing argument described above.

With the intention of expressing the above arguments in convex function notation, recall the definition of the d-derivative of a function f:

$$\Delta_d f(x) := f(x + d) - f(x).$$

If f is convex, $A = \{a_1 < \cdots < a_N\}$ and $d > 0$, then Lemma 1.2.2 implies that $\Delta_d f$ is increasing, meaning that

$$f(a_1 + d) - f(a_1) < \cdots < f(a_N + d) - f(a_N).$$

Consequently if i is fixed, then for any $j \leq i$ the terms

$$f(a_i) + f(a_j + d) - f(a_j)$$

are all different and lie in $(f(a_i), f(a_i + d)]$. We usually take d to be at most the smallest difference between adjacent elements of A to ensure that these intervals are disjoint. These observations are summarised in the following.

Lemma 2.2.1 (The squeezing lemma)**.** *Let f be a convex function and $d > 0$. Let y be a real number and Y_- a set of numbers no larger than y. Then*

$$f(y) + \Delta_d f(Y_-) \subset (f(y), f(y + d)].$$

The above discussion and squeezing lemma will provide a way of performing similar squeezing arguments with convex functions, which is the primary method in this chapter.

2.2.1 The Equidistribution Lemma

One of the most important tools in our proofs for the remainder of this thesis is an extremely simple result first appearing in [13], which we call the *Equidistribution Lemma*. It is so called because it argues that in a very particular sense, the set $A + A - A$ is uniformly spaced along the number line. Specifically we argue the following: given two elements $a, a' \in A$ with $a < a'$, if the interval $(a, a']$ is large, it contains many elements of $A + A - A$. The reason is that for a wide interval $(a, a']$ there will be a large number of smaller intervals which can be squeezed between a and a'.

We will also introduce the following notation: if $a' < a$, then

$$n_A(a', a) := |(A + A - A) \cap (a', a]|.$$

Lemma 2.2.2 (Equidistribution Lemma)**.** [13] *Let $D := \{d_1 < d_2 \cdots < d_{|D|}\}$ be the positive differences in $A - A$. Let $1 \leq Z \leq |D|$. If $a, a' \in A$ with $a' < a$ and $n_A(a', a) \leq Z$, then $a - a' \leq d_Z$.*

In other words, if there are at most Z elements of $A+A-A$ in $(a', a]$, then $a-a'$ must be among the Z smallest positive differences in $A-A$.

Proof. If not then $a - a' = d_Y$ where $Y > Z$. But then

$$a' < a' + d_i \leq a,$$

for $i = 1, \ldots, Y$. Thus there are at least $Y > Z$ elements of $A + A - A$ in $(a', a]$, contradicting that $n_A(a', a) \leq Z$. □

2.3 Proof of Theorem 2.1.3

In the language of convex functions rather than convex sets, Theorem 2.1.1 states that given a k-convex f,

$$|2^k f([N]) - (2^k - 1) f([N])| \gg_k N^{k+1}.$$

This is proved using a k-fold squeezing argument. To properly motivate Theorem 2.1.3 and its proof, one must identify the feature of the set $[N]$ which allows for k-fold squeezing.

It turns out that having many translates (about N) of the same $(k+1)$-term arithmetic progression (AP) is sufficient for k-fold squeezing to be possible. To illustrate, consider the $(k+1)$-AP $\{0, d, 2d, \ldots, kd\}$. Roughly speaking, combinations of the $k+1$ points $f(a), f(a+d), \ldots, f(a+kd)$ can produce discrete analogues of the first k derivatives of f near the point a. With many copies of the exact same $(k+1)$-AP, we can show that these discrete derivatives form monotone sequences, and therefore admit the standard squeezing argument to be applied k times. Theorem 2.1.3 follows since the aforementioned structure must exist in sets of the form $sB - sB$ and $(s+1)B - sB$.

In proving Theorem 2.1.3, we will actually prove the following stronger but more cumbersome result, because it makes the induction step more manageable. It also exactly emulates the discussed structure required for k-fold squeezing.

Proposition 2.3.1. [12] *Let $B = \{b_1 < \cdots < b_N\}$ be any set of reals (with $N \geq 2$), k be a positive integer and f be a k-convex function. Also let $d > 0$ be such that*

$$kd \leq b_i - b_j \qquad \textit{for all} \qquad j < i. \tag{2.3.1}$$

For $i = 1, \ldots, N$, set

$$P_{k,i} = \{b_i, b_i + d, \ldots, b_i + kd\}$$

and for $i = 2, \ldots, N$, set

$$S_{k,i} = \cup_{j=1}^{i-1} P_{k,j}.$$

Then the set

$$2^k f(S_{k,N}) - (2^k - 1) f(S_{k,N})$$

contains $\gg_k |B|^{k+1}$ elements in $(\min f(B), \max f(B)]$.

Proof. The proof is by induction on k. We begin with the base step. Let $d \leq b_i - b_j$ for all $j < i$. We have

$$S_{1,N} = \{b_1, b_1 + d, b_2, b_2 + d, \ldots, b_{N-1}, b_{N-1} + d\}.$$

It is worth noting that $b_i + d$ may equal b_{i+1} for some i values. This causes no problems. Since f is a convex function it follows that

$$f(b_1 + d) - f(b_1) < \cdots < f(b_{N-1} + d) - f(b_{N-1})$$

and consequently if i is fixed, then for all $j \leq i$ the terms

$$f(b_i) + f(b_j + d) - f(b_j)$$

are different and lie in the interval $I_i = (f(b_i), f(b_i + d)]$. This produces i elements of $f(S_{1,N}) + f(S_{1,N}) - f(S_{1,N})$. Since $d \leq b_i - b_j$ for all $j < i$, the intervals I_i are disjoint. Apply this argument for $i = 1, \ldots, N - 1$, producing

$$\sum_{i=1}^{N-1} i \approx |B|^2$$

elements of $f(S_{1,N}) + f(S_{1,N}) - f(S_{1,N})$ lying in $(\min f(B), \max f(B)]$, thus completing the base step.

We proceed to the induction. For $i = 2, \ldots N$, we use the squeezing lemma (Lemma 2.2.1) and (2.3.1), yielding

$$f(b_i) + \Delta_d f(S_{k-1,i}) \subset (f(b_i), f(b_i + d)] \subset (f(b_i), f(b_{i+1})].$$

Since f is k-convex, the new functions $g_i := f(b_i) + \Delta_d f$ are all $(k-1)$-convex by Lemma 1.2.2, and also

$$g_i(S_{k-1,i}) \subset (f(b_i), f(b_{i+1})].$$

From (2.3.1), the inequality $(k-1)d \leq b_i - b_j$ trivially holds for all $j < i$, so the induction hypothesis can be applied to $B_i := \{b_1, \ldots, b_i\}$, showing that the set

$$2^{k-1} g_i(S_{k-1,i}) - (2^{k-1} - 1) g_i(S_{k-1,i}) \subset 2^k f(S_{k,N}) - (2^k - 1) f(S_{k,N})$$

contains $\gg_k |B_i|^k = i^k$ elements in $(f(b_i), f(b_{i+1})]$. Applying this for each function g_i we get

$$|2^k f(S_{k,N}) - (2^k - 1) f(S_{k,N})| \gg_k \sum_{i=2}^{N} i^k \approx |B|^{k+1},$$

and all constructed elements lie in $(\min f(B), \max f(B)]$, closing the induction. $\square$

Since each k-convex set A is $f([N])$ for some k-convex function f, Theorem 2.1.1 is a special case of Proposition 2.3.1 by setting B and d appropriately such that $S_{k,N}$ is an arithmetic progression of size $\approx_k N$.

Proof of Theorem 2.1.3. Given $B = \{b_1 < \cdots < b_N\}$, let $b, b' \in B$ be such that $d_0 = b - b'$ is the smallest positive element of $B - B$.

We start by proving the case where $k = 2s$ is even. We set

$$B' = \{b_k - sd_0, \ldots, b_{kM} - sd_0\},$$

where $M = \lfloor N/k \rfloor$. Now apply Proposition 2.3.1 to B' with $d = d_0$. Since $d_0 \in B - B$ we get $S_{k,M} \subset (s+1)B - sB$, and the result follows.

If $k = 2s - 1$ is odd, then instead using

$$B' = \{b_k - sd_0 - b', \dots, b_{kM} - sd_0 - b'\}$$

completes the proof. □

We now prove Theorem 2.1.4.

Proof of Theorem 2.1.4. Set $f(x) = \log(x)$ and $B = A$ in the case $k = 2s - 1$ of Theorem 2.1.3. Using a crude upper bound on the size of the quotient set, we get that for any natural number s,

$$|(sA - sA)^{(2^k)}|^2 \gg_s \left| \frac{(sA - sA)^{(2^k)}}{(sA - sA)^{(2^k - 1)}} \right| \gg |A|^{2s}.$$

Taking square roots and setting $m(s) = 2^k = 2^{2s-1}$ completes the proof. □

2.4 Proof of Theorem 2.1.2

We already mentioned that given a k-convex f and a set B containing many translates of a $(k+1)$-AP, k-fold squeezing of $f(B)$ works. In fact, it is enough for B to contain many translates of a *generalised arithmetic progression* of the form

$$P = \{h_1 x_1 + \cdots + h_k x_d \ : \ x_i \in \{0, 1\} \text{ for all } i\}$$

with dimension k. Notice that if all the h_i are equal, P is simply a $(k+1)$-AP.

With this in mind, the proof of Theorem 2.1.2 has the following structure (after mild simplifications). Given that $|B + B - B| = K|B|$ is small, the Equidistribution Lemma 2.2.2 proves that $b_{i+1} - b_i$ may not take many values, whereupon some of these differences must have many realisations. Consider some such difference $h_1 \in B - B$ and define $B_{h_1} := \{b_i : b_{i+1} - b_i = h_1\}$. Since $|B + B - B|$ is small, so must be $|B_{h_1} + B_{h_1} - B_{h_1}|$, and we may again apply the Equidistribution Lemma to find many repetitions of some difference h_2 in $B_{h_1} - B_{h_1}$. This shows that B contains many translates of the set $\{0, h_1, h_2, h_1 + h_2\}$, the required generalised arithmetic progressions of dimension $d = 2$. Iterating the same argument yields many translates of a generalised arithmetic progression of dimension $d = k$, on which k-fold squeezing may be used. The above discussion is for explanatory purposes; many of these details are hidden in our induction argument.

Proof of Theorem 2.1.2. The proof will be by induction on k. The actual statement we will prove is the slightly stronger statement that all sums produced lie in the interval $(\min f(B), \max f(B)]$. So the base step (which we leave for now, as it is proved in the forthcoming Theorem 4.2.1) states that $f(B) + f(B) - f(B)$ contains $\gg |B|^3|B + B - B|^{-1}$ elements in $(\min f(B), \max f(B)]$.

Let $B = \{b_1 < \cdots < b_N\}$. We now prove the statement in Theorem 2.1.2 under the inductive hypothesis that for any $(k-1)$-convex function g, the set $2^{k-1}g(B) - (2^{k-1} - 1)g(B)$ contains $\gg_k \frac{|B|^{2^k-1}}{|B+B-B|^{2^k-k-1}}$ elements in $(g(b_0), g(b_N)]$.

We say that $b_i \in B$ is *good* if

$$n_B(b_i, b_{i+1}) \leq \frac{2|B + B - B|}{|B|}.$$

Since $\sum_{i=1}^{N-1} n_B(b_i, b_{i+1}) \leq |B+B-B|$, the pigeonhole principle shows that a positive proportion of all $b_i \in B$ are good. We henceforth restrict our attention to the good b_i. Consider the differences $b_{i+1} - b_i$ and let H be the set of all such differences. Since we are only considering good values of b_i, the Equidistribution Lemma 2.2.2 implies that $|H| \ll |B + B - B||B|^{-1}$. For each $h \in H$, define

$$B_h = \{b_i : b_{i+1} - b_i = h\}.$$

We furthermore know that $\sum_{h\in H} |B_h| \approx |B|$.

If $B_h = \{b_{e_1} < \cdots < b_{e_L}\}$ then let $B_h^i = \{b_{e_1} < \cdots < b_{e_i}\}$ be the truncation taking only the smallest i elements of B_h. For any $b_{e_i} \in B_h$, the squeezing lemma (Lemma 2.2.1) implies that

$$f(b_{e_i}) + \Delta_h f(B_h^i) \subset (f(b_{e_i}), f(b_{e_i+1})].$$

Since f is a k-convex function, Lemma 1.2.2 proves that $g_i := f(b_{e_i}) + \Delta_h f$ is a

$(k-1)$-convex function, and we have

$$g_i(B_h^i) \subset (f(b_{e_i}), f(b_{e_i+1})].$$

It follows from the induction hypothesis that

$$2^{k-1}g(B_h^i) - (2^{k-1}-1)g(B_h^i) \subset 2^k f(B) - (2^k-1)f(B)$$

contains $\gg_k \frac{|B_h^i|^{2^k-1}}{|B_h^i+B_h^i-B_h^i|^{2^k-k-1}}$ elements in $(f(b_{e_i}), f(b_{e_i+1})]$. This argument can be run for every element of B_h, and then also for each $h \in H$ to obtain

$$\begin{aligned}|2^k f(B) - (2^{k-1}-1)f(B)| &\gg_k \sum_{h\in H}\sum_{i=1}^{|B_h|} \frac{|B_h^i|^{2^k-1}}{|B_h^i + B_h^i - B_h^i|^{2^k-k-1}} \\ &\gg \frac{1}{|B+B-B|^{2^k-k-1}} \cdot \sum_{h\in H} |B_h|^{2^k}. \qquad (2.4.1)\end{aligned}$$

Above we have used the trivial facts that $|B+B-B| \gg |B_h^i+B_h^i-B_h^i|$ and $|B_h^i| = i$. Now by Hölder's inequality, we have

$$\sum_{h\in H} |B_h|^{2^k} \cdot |H|^{2^k-1} \geq \left(\sum_{h\in H} |B_h|\right)^{2^k} \approx |B|^{2^k}. \qquad (2.4.2)$$

Recalling that $|H| \ll |B+B-B||B|^{-1}$, (2.4.1) and (2.4.2) yield the desired

$$|2^k f(B) - (2^{k-1}-1)f(B)| \gg_k \frac{|B|^{2^{k+1}-1}}{|B+B-B|^{2^{k+1}-k-2}},$$

with all constructed elements lying in $(\min f(B), \max f(B)]$. □

Remark. Hanson, Roche-Newton, and Rudnev use [33, Lemma 3.1] to find a consecutive difference $h \in B-B$ with many realisations, and this is done by dyadic pigeonholing.

Instead we have found a large set of consecutive pairs (b_i, b_{i+1}) which don't have many elements of $B+B-B$ in between them. By the Equidistribution Lemma 2.2.2, there are few possible values that $b_{i+1}-b_i$ can take, and therefore the pigeonhole principle proves that some of these differences must be realised many times. This approach avoids the logarithmic factors intrinsic to a dyadic pigeonholing argument.

2.5 Open Problems

- While the right-hand bounds in Theorems 2.1.2 and 2.1.3 are best-possible, it is extremely unlikely that the number of summands in the sumset is optimal. Improvements in this direction will likely need to employ new techniques.

- As previously mentioned, it is conjectured that given any $s > 0$, there exists $m := m(s)$ such that given any finite real set A, we have

$$|(A - A)^{(m)}| \gg_s |A|^s.$$

  Our Theorem 2.1.4 makes an approach at this problem, but for many interesting applications (for example, the distinct dot products problem) we genuinely need a bound for $|(A - A)^{(m)}|$ rather than $|(sA - sA)^{(m)}|$. Even if we use the full strength of Proposition 2.3.1, it is not clear how to approach improvements.

  It would also be new to prove a result for any fixed $s > 3$.

Chapter 3

Convexity: Estimating Energy

3.1 Introduction

Verifying the lack of additive structure in convex sets by proving upper bounds on additive energy will be the topic of this chapter. We have already defined the energy $E(A, B)$ associated with sets A, B to be the number of solutions to the equation

$$a_1 + b_1 = a_2 + b_2$$

where $a_1, a_2 \in A$ and $b_1, b_2 \in B$. It can also be expressed as the second moment of the realisation function r_{A+B}:

$$E(A, B) = \sum_{x \in A+B} r^2_{A+B}(x).$$

More generally, "longer" convolutions are as follows.

Definition 3.1.1. Let $A_1, \dots, A_s$ be sets of reals. Define

$$T(A_1, \dots, A_s) := |\{(a_1, \dots, a_s, a'_1, \dots, a'_s) \in \prod_{i=1}^{s} A_i \times \prod_{i=1}^{s} A_i : a_1 + \cdots + a_s = a'_1 + \cdots + a'_s\}|. \tag{3.1.1}$$

We also sometimes write $T(A_1, \dots, A_s)$ as $T_s(A_1, \dots, A_s)$ in our proofs when we wish to emphasise the number of terms in the energy equation. In the case that $A_i = A$

for all i,

$$T_s(A) := T(\underbrace{A, \dots, A}_{s \text{ times}}).$$

For the special case when there are only two summands, we write $E(A) := T_2(A)$ and $E(A, B) := T(A, B)$.

As above, $T(A_1, \dots, A_s)$ is the second moment of the realisation function:

$$T(A_1, \dots, A_s) = \sum_{x \in A_1 + \dots + A_s} r^2(x),$$

where $r(x)$ counts the realisations of x as an s-fold sum in $A_1 + \dots + A_s$.

The notation T to denote energy is not as standard as E. However, our convention distinguishes between E_s which usually refers to s^{th} order energy (that is, the s^{th} moment of the realisation function r) whereas T_s is still a second moment estimate but the realisation function r considers longer sums. In general, we will write E for energies of twofold sums and T for longer energies.

The quantities $T_s(A)$ are easily interpreted via L^{2s}-norms of trigonometric polynomials with frequencies in A, see e.g. [37].

The goal of this chapter is to prove upper bounds on energies, such as $T_s(A)$ for a k-convex set A. As mentioned in Chapter 2, sumsets sA with $s > 2$ have been studied quite extensively in the context of the Erdős-Szemerédi sum-product conjecture [20], see for example [10, 14, 69]. However, although the questions about (the lack of) additive structure in convex sets may be viewed as a generalisation of pivotal questions in sum-product literature (such as the few products – many sums question, to be discussed more fully in Chapter 4), the methods developed in the sum-product literature rely heavily on algebraic ring properties of addition and multiplication that have no analogue in the convex setting.

Hence, for $s > 2$ the estimates in this chapter have no precedents, and the methodology enabling us to induct in s is new, a synthesis of a 20-year-old idea of Garaev [26], to be described shortly.

If $|A| = N$, then trivially $N^s \leq T_s(A) \leq N^{2s-1}$, and we are interested in proving non-trivial upper bounds for $T_s(A)$, under the heuristic that having enough convexity should push these upper bounds, ideally, close to the lower bound N^s.

The utility of incorporating higher convexity into the main results is as follows. For 1-convex sets, the best known estimates for $T_s(A)$ have been derived in [37] using induction and Szemerédi-Trotter bounds, namely

$$T_s(A) \ll_s N^{2s-2+2^{-(s-1)}}. \tag{3.1.2}$$

This in particular implies that $|sA| \gg_s N^{2-2^{-(s-1)}}$. This estimate cannot be improved beyond N^2, as evinced by the first N squares, which form a 1-convex (but not 2-convex) set. Consequently, the energy bound (3.1.2) is almost best-possible. Insisting on a higher degree of convexity excludes the examples where A is the set of consecutive k^{th} powers, where k is small.

One would naturally therefore hope for (and expect) better estimates for higher convex sets. Many of today's state of the art results concerning additive properties of convex sets have been obtained via a particular version of the Szemerédi–Trotter theorem, which replaces the line set L with a particular set of convex functions. This approach appears to have been first applied to convex sets by Elekes, Nathanson and Ruzsa [19]. Additionally, the special case where the point set P is a Cartesian product uses an elementary *lucky pairs* argument. The lucky pairs terminology has been adopted from J. Solymosi, in particular going back to [72].

In this chapter, we also define a notion of lucky pairs, however a different one and without proceeding towards geometric incidence arguments. Instead, we develop the idea of Garaev from [26], which underlies his elementary proof of the following energy bound. This bound was previously established by Konyagin [42] via the Szemerédi-Trotter theorem.

Theorem 3.1.1 (Konyagin–Garaev)**.** *Let A be a convex set of N elements. Then*

$$E(A) \ll N^{5/2}.$$

For completeness and contrast, both the Szemerédi–Trotter method and Garaev's elementary method will be discussed in this chapter.

Garaev's method has been brought to our attention by the work [51] by Olmezov.

In fact, the argument uses only the following weaker implication of the convexity of A, namely that for every $1 \leq d < N$, the collection of differences $\{a_{i+d} - a_i\}$ is indeed a set, rather than a multiset. In our forthcoming induction of Garaev's argument for k-convex sets, we will iterate this property, applying it to the difference sets $\{a_{i+d} - a_i\}$ as well as to A.

Our extension of Theorem 3.1.1 to k-convex sets is as follows.

Theorem 3.1.2. [13] *For $k \geq 0$, let $s \geq 2^k$ and let $A_1, \ldots, A_s$ be k-convex sets with $|A_i| \leq N$ for all $1 \leq i \leq s$. Then*

$$T(A_1, \ldots, A_s) \ll_s N^{2s-1-k+\alpha_k},$$

where $\alpha_0 = 0$ and $\alpha_k = \sum_{j=1}^{k} j 2^{-j} = 2 - 2^{-k}(k+2)$.

Loosely speaking, Theorem 3.1.2 says that provided our sets are convex enough, each time we double the number of terms s in the energy equation (3.1.1), we essentially get a "saving" of an additional factor of N off the trivial estimate N^{2s-1} for the quantity $T(A_1, \ldots, A_s)$.

We remark that in recent work, Mudgal [47] has shown, using an ingenious application of the Balog-Szemerédi-Gowers theorem on cardinality bounds such as Theorem 2.1.2, that for a k-convex set A one has the estimate $T_s(A) \leq N^{2s-1-k+\beta_k}$, with $\beta_k \to 0$ as $k \to \infty$. Mudgal's method only operates on the scale of extremely large s; it requires $s \geq 2^{C2^k \log k}$ for some fixed constant C, whereas we require $s \geq 2^k$ only. However in this range, his result is an improvement as it has $\beta_k \to 0$ whereas ours has $\alpha_k \to 2$, as $k \to \infty$.

Theorem 3.1.2 has a standard sumset implication after an application of the Cauchy–Schwarz inequality, which also illustrates the rough saving of N every time s doubles.

Corollary 3.1.1. [13] *For $k \geq 1$, let $s \geq 2^k$ and let $A_1, \ldots, A_s$ be k-convex sets with $|A_i| = N$ for all $1 \leq i \leq s$. Then*

$$|A_1 \pm A_2 \pm \cdots \pm A_s| \gg_k N^{1+k-\sum_{j=1}^{k} j 2^{-j}}.$$

We remark that for $k \geq 2$, the proof of Theorem 3.1.2 can be refined to give the improved $\alpha_k = -\frac{2}{13} + \sum_{j=1}^{k} j2^{-j}$ and slightly more for higher values of k. This follows from us bounding $\frac{2^k-1}{2^k}$ by 1 in the induction proof of Theorem 3.1.2 plus the fact that the induction can start at $s = 2$, where we have Shkredov's [68] estimate $E(A) \lesssim |A|^{32/13}$, see Theorem 4.1.1 below. However, this is not the focus of the theorem and perturbs the exposition so we only comment on the modifications needed to admit this improvement.

However, since we will use the explicit bounds for $s = 4$ in Chapter 4, we state the improved result, according to the remark above.

Theorem 3.1.3. [13] *If A_1, A_2, A_3, A_4 are 2-convex sets all of size N, then*

$$T(A_1, A_2, A_3, A_4) \lesssim N^{4+24/13} \qquad \text{and} \qquad |A_1 \pm A_2 \pm A_3 \pm A_4| \gtrsim N^{2+2/13} .$$

Moreover, for A_1, A_2, A_3, A_4 being k-convex with $k > 1$ and $1 \leq r \leq N^3$, one has

$$|\{x : r_{A_1 \pm A_2 \pm A_3 \pm A_4}(x) \geq r\}| \lesssim \frac{N^4}{r^{7/3}} E_{k-1} , \tag{3.1.3}$$

where

$$E_k = \sup_{\substack{B \ k-convex \\ |B|=N}} E(B).$$

As in Chapter 2, we again generalise k-convex sets to *near-k-convex sets* where $A = f(B)$, f is a k-convex function, and

$$|B - B + B| \leq K|B| .$$

The parameter K is again referred to as the doubling constant associated with B.

Our main result, Theorem 3.1.4, reflects the previously mentioned maxim that convex functions destroy additive structure, and that the more convex a function, the more it destroys additive structure. Its proof arises from generalising Garaev's method to longer sums and more convex sets. One can view Theorem 3.1.2 as its corollary by setting $B_1 = \cdots = B_s = [N]$.

Theorem 3.1.4. [13] *Let $B_1, \dots B_s$ be any sets with $|B_i| = N$, $|B_i + B_i - B_i| = K_i N$ for all $1 \le i \le s$. With $k \ge 0$ and $s \ge 2^k$, let $A_i = f_i(B_i)$ for some k-convex functions $f_1, \dots, f_s$. Then we have*

$$T(A_1, \dots, A_s) \ll_s \left(\prod_{i=1}^{2^k} K_i^{2-(2+2k-2\alpha_k)2^{-k}} \right) \cdot N^{2s-1-k+\alpha_k},$$

where $\alpha_0 = 0$ and $\alpha_k = \sum_{j=1}^{k} j 2^{-j}$.

By setting all the B_i and all the f_i to be the same, Theorem 3.1.4 yields the following corollary.

Corollary 3.1.2. [13] *Let B be any set with $|B| = N$ and $|B + B - B| = KN$. If $A := f(B)$ where f is a k-convex function and $s \ge 2^k$, then*

$$T_s(A) \ll K^{2^{k+1}-2-2k+2\alpha_k} \cdot N^{2s-1-k+\alpha_k},$$

where $\alpha_0 = 0$ and $\alpha_k = \sum_{j=1}^{k} j 2^{-j}$. Thus

$$|\underbrace{A \pm A \pm \cdots \pm A}_{s \text{ times}}| \gg K^{-2^{k+1}+2+2k-2\alpha_k} \cdot N^{1+k-\alpha_k}. \tag{3.1.4}$$

Corollary 3.1.1 and estimate (3.1.4) in Corollary 3.1.2 demonstrate that these energy bounds imply, at least on the qualitative level, results akin to Theorems 2.1.1 and 2.1.2. Returning to the standard k-convex examples $A = \{i^{k+1} : i \in [N]\}$ and $f(x) = x^{k+1}$, our Theorems 3.1.2 and 3.1.4 are sharp up to a factor of $N^{\alpha_k} \ll N^2$, so for large k these energy bounds do very well. Furthermore, the sumset corollaries hold for many different k-convex sets and admit any combination of plus and minus in the sumset. In this way our energy bounds are qualitatively stronger but quantitatively Theorems 2.1.2 and 2.1.1 are stronger as they are indeed sharp.

Our condition that $s \ge 2^k$ is an artifact of the induction proof we employ. However, we conjecture that the same energy bound can be obtained for $s \ge k^2$ for similar reasons to those discussed in the sumset case. Unfortunately, our methods seem insufficient to approach this.

It should be mentioned that such bounds which depend on doubling constants can be used to obtain sum-product-type results, along the lines of [33, Corollary 1.5]. Other sum-product type results in the context of convex sets can be seen in recent work of Stevens and Warren [74].

We also prove the following asymmetric two-fold energy bound:

Theorem 3.1.5. [13] *Let B, C be sets with $|B| = N$, $|B + B - B| = KN$ and $|C| = L$. If $A := f(B)$ for some convex function f, then*

$$E(A, C) \ll K^{1/2} N L^{3/2}.$$

Our result improves the following result, which is a generalisation of Theorem 3.1.1. It is proved by a straightforward extension of Konyagin's Szemerédi–Trotter proof, and appears in this form in work of Li and Roche-Newton [44]. Our improvement is in the dependence on K. Indeed, Theorem 3.1.6 follows from Theorem 3.1.5 by applying Plünnecke's inequality (Theorem 1.1.4).

Theorem 3.1.6 (Li–Roche-Newton)**.** *Let B, C be sets with $|B| = N$, $|B-B| = KN$ and $|C| = L$. If $A := f(B)$ for some convex function f, then*

$$E(A, C) \ll KNL^{3/2}.$$

The improved Theorem 3.1.5 is sharp when $|A| = |C|$. Let $A = C = f(B)$ where $B = \{x^2 : x \in [N]\}$ and $f(x) := \sqrt{x}$. Then we get $E(A, C) = K^{1/2}|A||C|^{3/2} = N^3$.

In this chapter, k and s are assumed to be small compared to other parameters. Subscripts in the asymptotic notation will be suppressed in the exposition without risk of confusion.

3.1.1 Aside: The Szemerédi–Trotter Theorem

In this section, we prove Theorem 3.1.1 using Konyagin's incidence geometric approach. Primarily this is to illustrate how it differs from our approach and its apparent limitations.

Historically, the utility in expressing a convex set as $f(B)$ for a convex f is that for many years, the state-of-the-art method for studying convex functions has been a particular version of the Szemerédi–Trotter Theorem. Technically, it is a generalisation of the Szemerédi–Trotter Theorem which instead bounds incidences between points and and a *family of pseudolines*, which is essentially a set of geometric objects whose properties mimic those of sets of lines.

Definition 3.1.1. A family of pseudolines L is a set of simple curves where each pair $l, l' \in L$ intersect at at most one point where they cross.

Theorem 3.1.7 (Szemerédi–Trotter Theorem)**.** *Let P and L respectively be a set of points and a family of pseudolines in $\mathbb{R}^2$. Then*

$$\mathcal{I}(P, L) \ll |P|^{2/3}|L|^{2/3} + |P| + |L|.$$

A proof is given in [15]. Henceforth, we do not distinguish Theorem 1.3.2 and Theorem 3.1.7, referring to both as the Szemerédi–Trotter Theorem.

Proof of Theorem 3.1.1. Let A be a convex set. Recall that

$$X_r = \{x \in A - A : r_{A-A}(x) \in [r, 2r)\},$$

that is the set of r-rich differences in $A - A$. It was mentioned in 1.1.5 that in light of the formulas

$$E(A) = \sum_{r \text{ dyadic}} r^2|X_r| \qquad \text{and} \qquad E_3(A) = \sum_{r \text{ dyadic}} r^3|X_r|,$$

a suitable upper bound on $|X_r|$ produces upper bounds on $E(A)$ and $E_3(A)$. We will prove that

$$|X_r| \ll N^3/r^3. \tag{3.1.5}$$

This will suffice to complete the proof: we write $E(A) = \sum_{r \text{ dyadic}} r^2|X_r|$ and, for some parameter r_* to be chosen, use the trivial bound $|X_r| \ll N^2/r$ for $r \leq r_*$ and estimate (3.1.5) for $r > r_*$. Choosing the optimal $r_* = N^{1/2}$ yields the desired

$$E(A) \ll N^{5/2}.$$

Identifying A with $f([N])$ for some convex function f, let $\mathcal{C} = \{(x, f(x)) : x \in I\}$ where I is an interval containing $[N]$. Let

$$P = [N] \times X_r \qquad \text{and} \qquad L = \mathcal{C} - (([-N, N] \cap \mathbb{Z}) \times A).$$

The convexity of f guarantees that any two translates of $\mathcal{C}$ can only intersect in a single point, so L is a family of pseudolines.

Take any point $(n, x) \in P$. Since $x \in X_r$, we can write

$$x = f(e_1) - f(e_1') = \cdots = f(e_r) - f(e_r').$$

Thus

$$(n, x) = (e_i, f(e_i)) - (e_i - n, f(e_i'))$$

for $i = 1, \ldots, r$, whereupon the point (n, x) lies on at least r pseudolines from L. This gives the lower bound of $r|P|$ for the number of incidences. Applying the Szemerédi–Trotter Theorem yields

$$r|P| \ll \mathcal{I}(P, L) \ll |P|^{2/3}|L|^{2/3} + |P| + |L|.$$

If $|P|$ dominates the right-hand side, then $r = 1$ and the result holds trivially. If $|L|$ dominates then $|X_r| \ll 1$ and the result again holds trivially. With $|P|^{2/3}|L|^{2/3}$ as the dominant term, rearranging gives (3.1.5). $\square$

We reprove (3.1.5) in the next section as a means to demystify Garaev's approach to the same problem, which uses only elementary combinatorial techniques and no incidence geometry. Previously, it was believed stalwartly that Szemerédi–Trotter arguments were the gold standard in studying such problems. While this may still be true, the advent of new elementary methods in recent years has resulted in several interesting discoveries and papers [12, 13, 33, 34, 63].

Konyagin's method for bounding $|X_r|$ is fairly robust in the sense that it can be generalised to sums and differences of the form $A \pm C$ where A is convex but C is any finite set. It allows for $A = f(B)$ where f is a convex function and B has small additive doubling, that is $|B+B|$ is small. However, it appears insufficient for generalising to longer sums and more convex sets, whereas the new arguments we present from [13] address both.

For proving the main results of this chapter, we dispense with any use of incidence geometry. All techniques henceforth will be purely elementary.

3.2 Methods

We begin by presenting a version of Garaev's proof of Theorem 3.1.1. This proof is essentially synthesised from its exposition by Olmezov [51], with an additional observation that convexity can be used more sparingly, which enables one to extend the estimate for $E(A)$ to $E(A,B)$, where A is a convex set and B is any set. This is based on replicating estimate (3.2.4) below, known earlier via the Szemerédi-Trotter theorem.

In the forthcoming argument (as in Konyagin's proof), we only need the following property of a *convex* set $A = \{a_1 < a_2 < \ldots < a_N\}$: for each $d < N$, the differences $a_{i+d} - a_i$, $i = 1, \ldots, N-d$ are all distinct.

Proof of Theorem 3.1.1. We are estimating the number of solutions to

$$a_{i_1} + a_{j_1} = a_{i_2} + a_{j_2} \ : (a_{i_1}, a_{j_1}, a_{i_2}, a_{j_2}) \in A^4 \, . \tag{3.2.1}$$

Unlike the previous proof of this result, let X_r be the r-rich *sums*. That is, $r \leq r_{A+A}(x) < 2r$ for each $x \in X_r$. Note that it doesn't matter whether we work with sums or differences when finding an energy bound. Write

$$x = a_{i_1} + a_{j_1} = \cdots = a_{i_r} + a_{j_r},$$

with $i_1 < i_2 < \ldots < i_r$. Since $a_{i_u} + a_{j_u} = x$ for all u, we also have $j_1 > j_2 > \ldots > j_r$. We may also assume that $j_u \geq i_u$ for all u, affecting only the multiplicative constant implied in the $\ll$ notation of the final estimate. It follows that

$$\sum_{u=1}^{r-1}(i_{u+1} - i_u) \leq N \qquad \text{and} \qquad \sum_{u=1}^{r-1}(j_u - j_{u+1}) \leq N.$$

By the pigeonhole principle, at least $3r/4$ of the summands in each sum cannot exceed $4N/r$. This implies that there is a set of indices $U \subset [r-1]$ with $|U| \geq r/2$ such that for every $u \in U$, $i_{u+1} - i_u \leq 4N/r$ and $j_u - j_{u+1} \leq 4N/r$. For $u \in U$, we say the pair $(a_{i_u}, a_{j_u}), (a_{i_{u+1}}, a_{j_{u+1}})$ is a *lucky pair*, so there are at least $r/2$ lucky pairs associated with the sum x.

Since each lucky pair gives rise to a solution to the energy equation (3.2.1), there are least $r/2$ distinct solutions of the equation

$$a_{i_1+d_1} - a_{i_1} = x = a_{i_2+d_2} - a_{i_2}\,,$$

where $i_1, i_2 \in [N]$ and $1 \leq d_1, d_2 \leq 4N/r$.

By considering all $x \in X_r$, it follows that

$$r|X_r| \ll (N/r)^2 \max_{1 \leq d_1, d_2 \ll N/r} |\{(i_1, i_2) \in [N]^2 : \, a_{i_1+d_1} - a_{i_1} = a_{i_2+d_2} - a_{i_2}\}|\,.$$

Now comes the only part of the argument where we use the convexity of A: given d_1, all differences $a_{i_1+d_1} - a_{i_1}$ are distinct, hence for any fixed d_1 and d_2, we have trivially that

$$|\{(i_1, i_2) \in [N]^2 : \, a_{i_1+d_1} - a_{i_1} = a_{i_2+d_2} - a_{i_2}\}| \leq N\,. \tag{3.2.2}$$

It follows that

$$|X_r| \ll N^3/r^3. \tag{3.2.3}$$

Substituting into a dyadic sum completes the proof. □

We remark that since the lucky pairs argument itself involves solely the pigeonhole principle and no assumptions on the set A, the above proof generalises immediately to the case of $E(A,B)$, where A is convex and B any set. Bound (3.2.3) becomes

$$|X_r| \ll |A||B|^2/r^3\,, \tag{3.2.4}$$

with X_r now being the set of r-rich sums in $A+B$. Indeed, the only necessary changes to the proof are that now d_2 pertain to the set

$$B = \{b_1 < b_2 < \ldots < b_{|B|}\},$$

so that $1 \le d_2 \ll |B|/r$, and the trivial bound (3.2.2) is replaced by $|B|$. This takes into account that given d_2, the quantities $b_{i_2+d_2} - b_{i_2}$ are not necessarily all distinct. What matters is that $a_{i_1+d_1} - a_{i_1}$ are all distinct.

Hence, Garaev's method enables one to obtain the standard corollary of estimate (3.2.4), which is usually proved using the Szemerédi-Trotter theorem.

Corollary 3.2.1. *If A is a convex set, then for any B,*

$$\begin{aligned} E_3(A,B) := \textstyle\sum_x r^3_{A\pm B}(x) &\ll |A|B|^2\log|A|\,, \\ E_{1+p}(A,B) := \textstyle\sum_x r^{1+p}_{A\pm B}(x) &\ll |A||B|^{1+p/2}\,, \quad \textit{for } 0<p<2\,. \end{aligned}$$

Earlier expositions of Garaev's method appear to overlook the fact that it generalises easily to embrace two different sets A and B, owing to an overreliance on convexity in the proof.

In order to generalise Theorem 3.1.1 to the quantity $T_s(A)$, we need to generalise the concept of lucky pairs from above. In order not to repeat ourselves, we do it in the most general setting, suitable for all the results in this chapter. In the convex set setting, $B_1,\ldots,B_s$ below are all just the interval $[N]$. In the near-convex setting, the full generality of Definition 3.2.1 and Proposition 3.2.1 will be needed.

Definition 3.2.1 (Lucky Pairs)**.** For $1 \le i \le s$, suppose B_i is a finite set of real numbers, g_i is a monotone function and $A_i = g_i(B_i)$. Given any r, where $r^{1/(s-1)} \ll |B_i + B_i - B_i|$ for all $1 \le i \le s$, let

$$X_r = \{x \in A_1 + \cdots + A_s : r \le r_{A_1+\cdots+A_s}(x) < 2r\}$$

be the r-rich sums in $A_1 + \cdots + A_s$. Suppose $P := (b_1, \ldots, b_s)$ and $P' := (b'_1, \ldots, b'_s)$ are *distinct* points, each belonging to $\prod_{i=1}^{s} B_i$. For $x \in X_r$, we say (P, P') forms a *lucky pair* associated with x if the following two conditions hold.

1. The pair (P, P') gives rise to a solution to the energy equation for the sum x. That is,
$$g_1(b_1) + \cdots + g_s(b_s) = x = g_1(b'_1) + \cdots + g_s(b'_s).$$

2. In all coordinates, there are not many elements of $B_i + B_i - B_i$ between P and P'. That is,
$$n_{B_i}(b_i, b'_i) \ll |B_i + B_i - B_i|/r^{1/(s-1)}, \tag{3.2.5}$$
for all $1 \le i \le s$ (using notation from Lemma 2.2.2).

Remark. In (3.2.5), we will always be treating the upper bound on $n_{B_i}(b_i, b'_i)$ as a positive integer. This is why we insist that $r^{1/(s-1)} \ll |B_i + B_i - B_i|$ in Definition 3.2.1. For all the results in this paper, this condition will hold trivially, so it will not be discussed further.

Proposition 3.2.1. [13] *Let $r \ge 1$, $s \ge 2$ and for $1 \le i \le s$, suppose B_i is a finite set of real numbers, g_i is a monotone function, and $A_i = g_i(B_i)$. Then for each $x \in X_r$, there are $\gg r$ lucky pairs associated with x.*

We will need the following lemma in the proof:

Lemma 3.2.1. [13] *Suppose we have an s-dimensional box in $\mathbb{R}^s$ (a Cartesian product of s orthogonal intervals) which is the union of r^s smaller (nonidentical) s-dimensional boxes (or cells) in an $r \times \cdots \times r$ grid. Then any generic hyperplane H (not parallel to any one-dimensional edge of the box) can pass through at most sr^{s-1} cells.*

Proof. By translation and scaling, we may assume that the origin is one of the corners of the box, the facets of the box are all parallel to coordinate hyperplanes and that the hyperplane H is of the form $X_1 + \cdots + X_s = C$ for some constant C.

We can index each cell by an s-tuple $(e_1, \ldots, e_s)$ which denotes its position among the cells on each axis, starting from the origin. Now for each cell in which at least one of the e_i is 1, we define its associated *diagonal* as the set of cells with indices $(e_1 + a, \dots, e_s + a)$ for $0 \le a \le r - \max_i e_i$.

Each cell lies on one of the diagonals and there are sr^{s-1} such diagonals in total. Since H intersects each diagonal in at most one cell, the proof is complete. $\square$

Proof of Proposition 3.2.1. For each i, partition $B_i + B_i - B_i$ into $r^{1/(s-1)}/4$ intervals, each containing $4|B_i + B_i - B_i|/r^{1/(s-1)}$ elements. Since $B_i \subset B_i + B_i - B_i$, this also partitions the elements of B_i. Doing this for each i partitions $\prod_i B_i$ into boxes and hence, since the g_i are all monotone functions, also partitions $\prod_i A_i$ into boxes (or cells).

Now consider some $x \in X_r$. Each solution to

$$x = g_1(b_1) + \cdots + g_s(b_s)$$

corresponds to a point $(g_1(b_1), \dots, g_s(b_s))$ on the hyperplane

$$x = X_1 + \cdots + X_s.$$

By Lemma 3.2.1 this hyperplane can pass through at most $(s/4^{s-1}) \cdot r \le r/2$ cells and the hyperplane has r points on it. By the pigeonhole principle, there must be $\gg r$ pairs of points which lie together in the same cell. By construction, these are lucky pairs, which completes the proof. $\square$

3.3 Proof of Theorem 3.1.2

Despite the fact that Theorem 3.1.2 is essentially a less general version of Theorem 3.1.4, we present its proof separately to illustrate exactly how much convexity is needed.

Proof of Theorem 3.1.2. If suffices to prove when $s = 2^k$; the full result follows by applying the trivial $T_{s+1}(A) \le N^2 \cdot T_s(A)$.

Let us denote the desired bound for $T_s(A_1, \ldots, A_s)$ as

$$\mathcal{T}_s = \mathcal{T}_s(N) := N^{2s-1-k+\alpha_k}.$$

Let $A_i := \{a_1^{(i)} < \ldots < a_N^{(i)}\}$ for each $1 \leq i \leq s$. We are counting solutions to the equation

$$a_{e_1}^{(1)} + \cdots + a_{e_s}^{(s)} = a_{e_1'}^{(1)} + \cdots + a_{e_s'}^{(s)}, \tag{3.3.1}$$

for some indices $e_1, \ldots, e_s, e_1', \ldots, e_s' \in [N]$.

The proof is by induction on k where $s = 2^k$, the base case $k = 0$ being trivial: the number of solutions of

$$a = a' : \quad a, a' \in A_1$$

is at most (in fact precisely) N.

We proceed to the induction step. Let us assume that in the equation (3.3.1) no two terms $a_{e_i}^{(i)}$ and $a_{e_i'}^{(i)}$ are the same for any $i = 1, \ldots, s$. More precisely, suppose that such non-degenerate solutions to equation (3.3.1) constitute at least half of the quantity $T_s(A_1, \ldots, A_s)$. If not then we would have

$$T_s(A_1, \ldots, A_s) \ll N \sum_{j=1}^{s} T_{s-1}(A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_s)$$

where the right-hand term is an upper bound for the number of degenerate solutions. Consider one such summand in the right-hand expression, say $T_{s-1}(A_1, \ldots, A_{s-1})$. Fix all but the first $s/2$ terms on each side of the energy equation in N^{s-2} ways. For each of these choices, we must then count the solutions to the equation

$$a_{e_1}^{(1)} + \cdots + a_{e_{s/2}}^{(s/2)} = a_{e_1'}^{(1)} + \cdots + a_{e_{s/2}'}^{(s/2)} + c,$$

for some fixed c. By a simple application of the Cauchy–Schwarz inequality, this is bounded above by $T_{s/2}(A_1, \ldots, A_{s/2})$. It follows by the induction hypothesis that

$$T_s(A_1, \ldots, A_s) \ll_s N^{s-1} \cdot \mathcal{T}_{s/2} \ll \mathcal{T}_s.$$

Thus if the degenerate solutions constituted more than half of the upper bound, the proof would be complete.

Recall that X_r is the set of r-rich sums. For each $x \in X_r$, we apply Proposition 3.2.1 with $B_1 = \ldots = B_s = [N]$ and $g_i(y) := a_y^{(i)}$, for all $1 \leq i \leq s$. [1] By considering all lucky pairs arising from any $x \in X_r$, one obtains

$$r|X_r| \ll \#\text{ solutions to (3.3.1)}\,,$$

where $|e_i - e_i'| \ll N/r^{1/(s-1)}$ for all $1 \leq i \leq s$. We now fix the combination of $d_i := e_i - e_i'$ for $1 \leq i \leq s$ which maximises the number of solutions to (3.3.1).

Notice that for each d_i, $\Delta_{d_i} A_i := \{a_{e_i'+d_i}^{(i)} - a_{e_i'}^{(i)}\}$ is a $(k-1)$-convex set (not a multiset) and has $\leq N$ elements. Here we have used that d_i is non-zero, which is a consequence of the non-degeneracy assumption. We can subtract all the elements on the right-hand side of (3.3.1), and since there are $N^s/r^{s/(s-1)}$ possible values of $(e_1 - e_1', \ldots, e_s - e_s')$, it follows that

$$r|X_r| \ll \frac{N^s}{r^{s/(s-1)}} \cdot \#\text{ of solutions to } a_1 + \ldots + a_s = 0$$

where $a_i \in \Delta_{d_i} A_i$ for $1 \leq i \leq s$. Rearranging the terms of the above equation so there are $s/2$ terms on each side of the equation and using Cauchy-Schwarz, one can then apply the induction hypothesis to obtain

$$|X_r| \ll \frac{N^s}{r^{(2s-1)/(s-1)}} \cdot \mathcal{T}_{s/2}\,. \tag{3.3.2}$$

Using $T_s(A_1, \ldots, A_s) = \sum_{r \text{ dyadic}} r^2 |X_r|$, we optimise in r by taking, for some r_* to be determined, the trivial bound $r_* N^s$ for $r \leq r_*$, and the dyadic sum with (3.3.2) over the values of $r \geq r_*$. Thus

$$T_s(A_1, \ldots, A_s) \ll r_* N^s + \frac{N^s}{r_*^{1/(s-1)}} \mathcal{T}_{s/2}\,.$$

Taking the optimal choice of

$$r_* = \mathcal{T}_{s/2}^{1-1/s}\,,$$

[1] Technically, we let g_i be an increasing function which interpolates $g_i(y) := a_y^{(i)}$.

we get

$$T_s(A_1, \ldots, A_s) \ll N^s \mathcal{T}_{s/2}^{1-1/s} = N^s \cdot N^{(s-k+\alpha_{k-1})(1-2^{-k})} \ll N^{2s-1-k+\alpha_k}.$$

This closes the induction and completes the proof. □

Remark. The step where we apply Cauchy–Schwarz is a generalisation on the well-known procedure to prove that

$$E(A,B) \leq E(A)^{1/2} E(B)^{1/2}.$$

As alluded to in the introduction, we can refine this approach to obtain a slightly better bound, specifically a smaller value of α_k. If we assume that $k \geq 2$ then $k = 2$ becomes the base case of the induction. Using the bound (3.3.2), $T(A_1, A_2, A_3, A_4)$ can be bounded in terms of $E(A)$ where A is a 1-convex set of size N. Estimating $E(A)$ using Theorem 3.1.1 produces the improvement $\alpha_k = -\frac{1}{8} + \sum_{j=1}^{k} j2^{-j}$. Using instead Shkredov's stronger bound [68]

$$E(A) \ll N^{32/13},$$

gives the further improvement $\alpha_k = -\frac{2}{13} + \sum_{j=1}^{k} j2^{-j}$. These observations constitute a proof for Theorem 3.1.3.

In the above proof, k-convexity is only used in one place. Since all the A_i are k-convex, the sets $\Delta_{d_i} A_i$ are $(k-1)$-convex. In particular, this implies that $\Delta_{d_i} A_i$ will always be a set rather than a multiset which is essential when iterating the argument.

3.4 Proofs of Theorems 3.1.5 and 3.1.4

In this section, we focus on the results pertaining to sets with small additive doubling.

Proof of Theorem 3.1.5. Let $C := \{c_1 < \cdots < c_L\}$. For each $x \in X_r$, we apply Proposition 3.2.1 with $k = 2$, with $g_1(t) := f(t), g_2(t) := c_t$ and with $B_1 = B, B_2 =$

$[L]$. [2] By considering all $x \in X_r$, this implies that the total number of lucky pairs from r-rich sums is $\gg r|X_r|$. Since lucky pairs give rise to solutions to the energy equation, it follows that

$$r|X_r| \ll \#\text{ solutions to } f(b_1) - f(b_2) = c_{e_2} - c_{e_1}, \tag{3.4.1}$$

where $n_B(b_1, b_2) \ll KN/r$ and $|e_2 - e_1| \ll L/r$. By the Equidistribution Lemma 2.2.2, there are at most KN/r possible values that $|b_2 - b_1|$ can take.

After fixing $b_1 - b_2, e_2 - e_1$ and c_{e_1} in (3.4.1), which can be done in KNL^2/r^2 ways, the energy equation admits at most one solution since $f_d(x) := f(x+d) - f(x)$ is a monotone function.

It follows that $|X_r| \ll KNL^2/r^3$. Since

$$E(A,B) \ll \sum_{r \text{ dyadic}} r^2|X_r| \ll r_* \sum_{\substack{r \text{ dyadic} \\ r \leq r_*}} r|X_r| + \sum_{\substack{r \text{ dyadic} \\ r > r_*}} \frac{KNL^2}{r} \ll r_* NL + \frac{KNL^2}{r_*}$$

we get that, upon choosing $r_* = (KL)^{1/2}$,

$$E(A,B) \ll K^{1/2} N L^{3/2}. \qquad \square$$

Proof of Theorem 3.1.4. This proof follows closely that of Theorem 3.1.2, using appropriately the Equidistribution Lemma to incorporate small doubling. As in the proof of Theorem 3.1.2, it suffices to prove the $s = 2^k$ case and again we denote the desired bound for $T_s(A_1, \ldots, A_s)$ as

$$\mathcal{T}_s(N; K_1, \ldots, K_s) := \left(\prod_{i=1}^{s} K_i^{2-(2+2k-2\alpha_k)2^{-k}} \right) \cdot N^{2^{k+1}-1-k+\alpha_k}.$$

We are counting solutions to the equation

$$f_1(b_1) + \cdots + f_s(b_s) = f_1(b_1') + \cdots + f_s(b_s'), \tag{3.4.2}$$

where $b_i, b_i' \in B_i$ for all i.

[2] Again, technically g_2 is an increasing function which interpolates $g_2(t) := c_t$.

The proof is by induction on k where again the base case $k = 0$ is trivial: the number of solutions of

$$f_1(b) = f_1(b') : \quad b, b' \in B_1$$

is at most N.

Let us assume that for each solution to (3.4.2) no two terms $f_i(b_i)$ and $f_i(b'_i)$ are equal for any $i = 1, \ldots, s$. More precisely, suppose that such non-degenerate solutions to equation (3.4.2) constitute at least half of the quantity $T_s(A_1, \ldots, A_s)$. For otherwise, as in the proof of Theorem 3.1.2, using a trivial upper bound and the induction hypothesis, we would have

$$T_s(A_1, \ldots, A_s) \ll N^{s-1} \mathcal{T}_{s/2}(N; K_{\iota_1}, \ldots, K_{\iota_{s/2}}) \ll \mathcal{T}_s(N; K_1, \ldots, K_s),$$

where $K_{\iota_1}, \ldots, K_{\iota_{s/2}}$ are the largest $s/2$ terms among all the K_i. This would complete the proof immediately.

As previously, X_r contains the sums $x \in A_1 + \cdots + A_s$ with $r \le r_{A_1+\cdots+A_s}(x) < 2r$. For each $x \in X_r$, we now apply Proposition 3.2.1 with $g_i(b) := f_i(b)$ for $1 \le i \le s$. This obtains

$$r|X_r| \ll \#\text{ solutions to (3.4.2)},$$

where $n_{B_i}(b_i, b'_i) \ll K_i N / r^{1/(s-1)}$ for all $1 \le i \le s$.

We now choose the $d_i := b_i - b'_i$ for $1 \le i \le s$ which maximise the number of solutions to (3.4.2), and then rearrange to obtain

$$(\Delta_{d_1} f_1)(b'_1) + \cdots + (\Delta_{d_{s/2}} f_{s/2})(b'_{s/2}) = (\Delta_{d_{s/2+1}} f_{s/2+1})(b_{s/2+1}) + \cdots + (\Delta_{d_s} f_s)(b_s). \tag{3.4.3}$$

By the Equidistribution Lemma 2.2.2, there are at most $\prod_{i=1}^{s}(K_i N)/r^{s/(s-1)}$ ways altogether of choosing the d_i, so we have

$$r|X_r| \ll \frac{\prod_{i=1}^{s}(K_i N)}{r^{s/(s-1)}} \cdot \#\text{ solutions to (3.4.3)}.$$

Applying Cauchy–Schwarz proves that the number of solutions to (3.4.3) is bounded above by

$$\begin{aligned} T((\Delta_{d_1} f_1)(B_1), \dots, &(\Delta_{d_{s/2}} f_{s/2})(B_{s/2}))^{1/2} \\ &T((\Delta_{d_{s/2+1}} f_{s/2+1})(B_{s/2+1}), \dots, \Delta_{d_s}(f_s)(B_s))^{1/2}. \end{aligned} \tag{3.4.4}$$

Since all the functions $\Delta_{d_i} f_i$ are $(k-1)$-convex, the induction hypothesis upper bounds (3.4.4) by

$$\mathcal{T}_{s/2}(N; K_1, \dots, K_{s/2})^{1/2} \mathcal{T}_{s/2}(N; K_{s/2+1}, \dots, K_s)^{1/2},$$

whence

$$|X_r| \ll \frac{\prod_{i=1}^{s}(K_i N)}{r^{(2s-1)/(s-1)}} \cdot \mathcal{T}_{s/2}(N; K_1, \dots, K_{s/2})^{1/2} \mathcal{T}_{s/2}(N; K_{s/2+1}, \dots, K_s)^{1/2}. \tag{3.4.5}$$

Using $T_s(A_1, \dots, A_s) = \sum_{r \text{ dyadic}} r^2 |X_r|$, we optimise in r by taking, for some r_* to be determined, the trivial bound $r_* N^s$ for $r \le r_*$, and the dyadic sum with (3.4.5) over the values of $r \ge r_*$. Thus

$$\begin{aligned} T_s(A_1, \dots, &A_s) \ll \\ &r_* N^s + \frac{\prod_{i=1}^{s}(K_i N)}{r_*^{1/(s-1)}} \mathcal{T}_{s/2}(N; K_1, \dots, K_{s/2})^{1/2} \mathcal{T}_{s/2}(N; K_{s/2+1}, \dots, K_s)^{1/2}. \end{aligned}$$

Taking the optimal choice of

$$r_* = \left(\prod_{i=1}^{s} K_i^{1-\frac{1}{s}} \right) \cdot \mathcal{T}_{s/2}(N; K_1, \dots, K_{s/2})^{(\frac{1}{2} - \frac{1}{2s})} \mathcal{T}_{s/2}(N; K_{s/2+1}, \dots, K_s)^{(\frac{1}{2} - \frac{1}{2s})},$$

it is elementary to check that

$$T_s(A_1, \dots, A_s) \ll r_* N^s \ll \left(\prod_{i=1}^{s} K_i^{2-(2+2k-2\alpha_k)2^{-k}} \right) \cdot N^{2^{k+1}-1-k+\alpha_k}.$$

This closes the induction and completes the proof. □

Similar to Theorem 3.1.2, we can refine this approach to obtain a bound with a slightly smaller (at least by 1/8) value of α_k for $k \ge 2$.

The above proofs apply to sums of length $s = 2^k$, where we start the induction with the trivial estimate for $s = 0$. One can also easily develop similar inductions that start with the quantity $T(A_1, A_2, A_3)$ and formulate analogues of Theorems 3.1.2 and 3.1.4 for $s = 3 \cdot 2^k$. We leave this to the interested reader, concluding this chapter by stating the base case $s = 3$, since it will be used once in Chapter 4.

Theorem 3.4.1. *If A_1, A_2, A_3 are 2-convex sets with N elements. Let X_r be the set of r-rich sums from $A_1 + A_2 + A_3$. Then*

$$|X_r| \ll \frac{N^{14/3}}{r^{5/2}} \,. \tag{3.4.6}$$

In particular

$$T_3(A_1, A_2, A_3) \ll N^{4+\frac{1}{9}}.$$

Proof. By the familiar lucky pairs argument

$$r|X_r| \ll N^3/r^{3/2} \cdot S_{A+B=C}\,,$$

where $S_{A+B=C}$ is the maximum number of solutions to

$$a + b = c : \quad a \in A,\, b \in B,\, c \in C$$

for some 1-convex sets A, B, C with $|A| = |B| = |C| \leq N$. It remains to show that $S_{A+B=C} \ll N^{5/3}$.

Consider the r_0-rich sums $a + b \in A + B$ and recall the corresponding bound (3.2.4). Combining with a trivial bound, we get

$$S_{A+B=C} \ll |A||B|^2/r_0^2 + |C|r_0 = N^3 r_0^{-2} + N r_0,$$

and optimising in r_0, one obtains

$$S_{A+B=C} \ll N^{5/3}\,. \qquad \square$$

3.5 Open Questions

- In Theorem 3.1.4, it takes $s = 2^k$ terms on each side of the energy equation to make a saving of $N^{k-\alpha_k}$ off the trivial bound. As in the sumset case of Chapter 2, it is highly unlikely that an energy equation with so many terms is needed to make this saving. However, again it seems that new techniques are needed to approach this.

- Additionally, Theorem 3.1.4 is sharp up to a factor N^{α_k}. As k increases, α_k approaches some small constant (depending on the formulation lying somewhere between 1 and 2). It seems plausible that this can be improved so that $\alpha_k \to 0$ as $k \to \infty$, in line with results of Mudgal [47]. However, it may involve a method other than induction, because with each iteration of the induction, the approximations compound and so are unlikely to improve.

Chapter 4

Convexity: Applications

The results of the previous chapters have led to several new and exciting applications, in particular, new methods and refinements in sumset estimation.

This chapter will be split into two significant parts. Firstly, we use the energy bounds from Chapter 3 to improve the best-known twofold sumset and energy bounds when we assume higher convexity. Secondly, we give a number of applications of the Equidistribution Lemma which allow us to prove new sumset bounds in the reals as well as the complex numbers and function fields. We also generalise a bound of Jarník which estimates the number of intersections between a regular grid and a convex curve.

4.1 Sumset bounds in 2-convex sets

The best known difference set, sumset, and energy bounds for convex sets to date are respectively due to Schoen and Shkredov [67], Rudnev and Stevens [62], and Shkredov [68], and are summarised below.

Theorem 4.1.1. *If A is convex, then*

$$\begin{aligned} |A-A| &\gtrsim |A|^{8/5\,=\,1.6} \\ |A+A| &\gtrsim |A|^{30/19\,\approx\,1.579}, \\ E(A) &\lesssim |A|^{32/13\,\approx\,2.4615}. \end{aligned}$$

We can get small improvements of all these bounds for k-convex sets, with $k \geq 2$. These estimates rely on using our new bounds for the quantity $T_4(A)$ in Theorem 3.1.3, as well as $T_3(A)$ in Theorem 3.4.1. We will incorporate them into existing methods developed by Shkredov and collaborators (see for example [50,67,68]), which rely extensively on the use of the third moment estimate

$$E_3(A) := \sum_x r^3_{A-A}(x) \ll N^3 \log N\,,$$

derived from (3.2.3). It is essential in what follows to understand how equivalence classes of triples induce third moment energy bounds, so we have attempted to make exposition in this section prerequisite-free.

One can write $E_3(A)$ (not to be confused with $T_3(A)$) in the following way:

$$E_3(A) = \sum_{x \in A-A} r^3_{A-A}(x) = |\{(a,b,c,a',b',c') \in A^6 \ : \ a-a' = b-b' = c-c'\}|.$$

In other words, if (a',b',c') is a translation of the triple (a,b,c) by a vector of the form (v,v,v), then they contribute to the third moment energy $E_3(A)$, and all contributions are of this form. Define an equivalence relation where $(a,b,c) \sim (a',b',c')$ if

$$(a,b,c) = (a',b',c') + (v,v,v)$$

for some v. Let $[a,b,c]$ denote the equivalence class containing (a,b,c) and $[A^3]$ denote the set of all such equivalence classes. If $r([a,b,c])$ is the number of triples in equivalence class $[a,b,c]$ then

$$E_3(A) = \sum_{x \in [A^3]} r^2(x)\,. \tag{4.1.1}$$

In proving a bound for $|A+A|$ a different equivalence relation will be used. We will say that $(a,b,c) \sim (a+t, b-t, c+t)$ for any $t \in \mathbb{R}$. We remark that if $r(x)$ is redefined as the number of triples in equivalence class x according to the new relation, then again (4.1.1) holds.

Let A be a k-convex set. The improvement comes from (3.1.3) which relates representations of 4-fold sums of k-convex sets to the twofold energy E_{k-1} for $(k-1)$-convex sets. When $k = 2$, this quantity is estimated by the energy bound for 1-convex sets in Theorem 4.1.1. Furthermore, for $k > 2$ this process can then be iterated to obtain incrementally better energy bounds for more convex sets, the iterations rapidly converging. We note that even the simpler energy estimate $E(A) \ll N^{5/2}$ for 1-convex sets would already improve the estimates of Theorem 4.1.1 for 2-convex sets. We present only estimates for 2-convex sets in the next theorem; the small improvements for more convex sets can be found in the forthcoming proofs.

Theorem 4.1.2. [13] *If A is a 2-convex set with $|A| = N$, then*

$$\begin{aligned} |A-A| &\gtrsim N^{1+151/234 \approx 1.645}, \\ |A+A| &\gtrsim N^{1+229/309 \approx 1.587}, \\ E(A) &\lesssim N^{2.4554}. \end{aligned}$$

The proofs for all three of the bounds in Theorem 4.1.2 are proved by lower and upper bounding the number of solutions to certain tautological equations. For the sum and difference sets $A \pm A$, these lower bounds are proved by pigeonholing arguments, whereas for the energy $E(A)$, it is proved using Shkredov's spectral (or operator) method. The method is so called because it relies on estimating the largest eigenvalue in the spectrum of a certain matrix.

The upper bounds are proved using a toolbox of standard techniques: Cauchy–Schwarz Inequality, Hölder's Inequality, dyadic pigeonholing, as well as various estimates previously discussed in this thesis.

Proof for $|A-A|$. We begin with the technically least demanding bound for the set $A - A$, where A is k-convex for $k \geq 2$. Let $|A - A| = KN$; we seek a suitable lower bound for K. Additionally, let D be the elements of $A - A$ which are realised more than the average number of times. That is,

$$D := \{x \in A - A : \ r_{A-A}(x) \geq N/4K\}.$$

Any triple $(a, b, c) \in A^3$ satisfies the tautology

$$a - c = (a - b) + (b - c).$$

What is less clear is that most triples have the additional property that $a-b$ and $b-c$ both lie in D. The definition of D means an average element of $A - A$ is expected to lie in D, so such a claim is intuitive. Specifically we show that the above tautology is valid for $\gg N^3$ triples

$$(a, b, c) \in A^3 \;:\; a - b,\; b - c \in D. \tag{4.1.2}$$

We say such a triple is *good*. Since

$$|\{(a, b, c) \in A^3 :\; a - b \notin D\}| = N \sum_{x \in (A-A)\setminus D} r_{A-A}(x) \leq N \cdot (KN) \cdot (N/4K) = N^3/4,$$

it follows that $a - b \in D$ for $\geq 3N^3/4$ of the triples (a, b, c). Similarly $b - c \in D$ for $3N^3/4$ of the triples (a, b, c), so by the pigeonhole principle, at least $N^3/2$ of the triples are good (satisfy (4.1.2)).

Next observe that all triples (a, b, c) in the same equivalence class of $[A^3]$ give rise to the same differences $a - b,\ b - c,\ a - c$, so we can think of *good* as a property of the equivalence class rather than the individual triple. Let $[S]$ be the set of good equivalence classes. Using (4.1.1) in combination with Cauchy-Schwarz, one has

$$\begin{aligned} N^6 \ll \left(\sum_{x \in [S]} r(x) \right)^2 &\ll E_3(A)\, |[S]| \qquad (4.1.3) \\ &\lesssim N^3\, |\{(d, d', d'') \in (A - A) \times D^2 :\; d = d' + d''\}| \\ &\ll N^3\, (K/N)^2 \sum_{d \in A-A} r_{A-A+A-A}(d). \end{aligned}$$

Treating $r_{A-A+A-A}(d)$ as $1 \cdot r_{A-A+A-A}(d)$ to apply Hölder's inequality and using (3.1.3) with dyadic summation yields

$$\sum_{d \in A-A} r_{A-A+A-A}(d) \lesssim (KN)^{4/7} (N^4 E_{k-1})^{3/7}\,, \tag{4.1.4}$$

with E_{k-1} as in (3.1.3). Thus

$$N^{19/7} \lesssim K^{18/7} E_{k-1}^{3/7}.$$

Using Shkredov's bound for E_{k-1} yields, for 2-convex A:

$$K \gtrsim N^{\frac{151}{234}}.$$

If A is more than 2-convex, one can asymptotically use the forthcoming bound (4.1.9) for E_{k-1}, which improves the exponent for $|A-A|$ just by slightly over 0.001. Namely, if A is sufficiently convex and N is large enough, it follows that

$$K \gg N^{.646},$$

the decimal approximation having accounted for replacing $\lesssim$ by $\ll$. $\square$

Proof for $|A+A|$. We use a slightly more involved pigeonholing technique which is exposed in full detail in Lemma 4 and the Proof of Theorem 5 in [62]. We give the skeleton below.

Suppose $|A+A| = KN$; define P to be a set of sums with $\gtrsim N/K$ realisations. Technically this proof starts by passing to a large subset with good properties. Exactly as in [62, Proof of Theorem 5], the new regularised subset of A (which we henceforth refer to simply as A) has the following key property: there exists $A' \subset A$ with $|A'| \geq 0.9|A|$ such that

- $E(A') \gg E(A)$, and
- For each $a \in A'$,
$$|(a+A) \cap P| \geq \frac{3|A|}{4}.$$

Now let D as be the set of popular differences in $A' - A'$ by energy (we will use energy to connect the difference set with the sum set). Namely D is defined as follows. By the dyadic pigeonhole principle, there exists $D \subseteq A' - A'$, and a real number $1 \leq \Delta < |A|$, such that for every $d \in D$, $\Delta \leq r_{A-A}(d) < 2\Delta$, and on top of this

$$E(A) \ll E(A') \lesssim |D|\Delta^2. \tag{4.1.5}$$

Moreover, by (3.2.3) one has $|D|\Delta^3 \ll |A|^3$ so

$$\Delta \lesssim |A|^3/E(A)\,. \tag{4.1.6}$$

Then the analogue of the tautology in the previous proof is

$$(a+b)-(b+c)=a-c \;:\quad a+b,\, b+c \in P\,, a-c\in D, \tag{4.1.7}$$

and we claim that this equation admits $\gtrsim |A||D|\Delta$ solutions. Indeed, choosing a, c such that $a-c \in D$ can be done in $|D|\Delta$ ways. For each of these, three-quarters of the b values have $a+b \in P$ and a further three-quarters of the b values have $b+c \in P$, whereupon there are $\geq N/2$ values of b which yield a solution to (4.1.7).

In this context, notice that the equivalence relation

$$(a,b,c) \sim (a+t, b-t, c+t) \;:\; t \in \mathbb{R}$$

has the property that triples from the same equivalence class give rise to the same equation underlying (4.1.7)

$$s_1 - s_2 = d,\; d \in D, s_1, s_2 \in P.$$

Thus, satisfying (4.1.7) or not, is a property which all triples (a,b,c) in the same equivalence class share. Hence, using (4.1.1) and Cauchy-Schwarz as in (4.1.3) yields

$$\begin{aligned}
(|A||D|\Delta)^2 &\ll E_3(A)\; |\{(s_1,s_2,d)\in P^2\times D \;:\; s_1-s_2=d :\}| \\
&\lesssim N^3(K/N)^2 \sum_{d\in D} r_{A-A+A-A}(d) \\
&\lesssim K^2 N^{19/7} E_{k-1}^{3/7} |D|^{4/7}\,,
\end{aligned}$$

after using Hölder's inequality and (3.1.3), as in (4.1.4).

Multiplying both sides by $\Delta^{6/7} \lesssim \left(\frac{N^3}{E(A)}\right)^{6/7}$ (see (4.1.6)) to balance the powers of $|D|$ and Δ, so one can use $|D|\Delta^2 \gtrsim E(A)$ (see (4.1.5)), yields

$$E(A)^{16/7} \lesssim K^2 E_{k-1}^{3/7} N^{23/7}\,.$$

Substituting Shkredov's bound for E_{k-1} and using the standard Cauchy-Schwarz bound

$$K \geq \frac{N^3}{E(A)}$$

yields

$$(N^3/K)^{16/7} \lesssim K^2 \cdot (N^{32/13})^{3/7} \cdot N^{23/7}.$$

Rearranging we get

$$K \gtrsim N^{\frac{229}{390}}\,.$$

Once again, if A is more than 2-convex, one can asymptotically use the forthcoming bound (4.1.9) for E_{k-1}, in which case

$$K \gtrsim N^{\frac{16}{27} \approx 0.592}\,,$$

the decimal approximation again accounting for replacing $\lesssim$ with $\ll$. $\square$

4.1.1 The Spectral Method

We now turn to the energy $E(A)$ estimate, where the analysis ends up being somewhat more involved. For the reader's convenience, we recall in the proof the key steps of Shkredov's spectral (alias operator) method [68] (for an overview of the method see [52]). The operator method is designed as a substitute for the lower bounds obtained from easy tautologies we have already seen in the $|A \pm A|$ cases.

Proof for $E(A)$. Once again, let D be the set of popular differences by energy, satisfying (4.1.5), (4.1.6).

Identifying D with its characteristic function, consider the quantity

$$\mathcal{S} := \sum_{a,b,c \in A} D(a-b)D(b-c)r_{A-A}(a-c)\,.$$

The quantity S takes triples $(a, b, c) \in A^3$ for which $a - b,\ b - c$ are in the popular set D, and counts each one the number of times the difference $a - c$ repeats itself. The spectral method enables one to get a lower bound on $\mathcal{S}$, to be compared with the upper bound we will again obtain by (4.1.1) and Cauchy-Schwarz.

After fixing an ordering $a_1, \dots, a_N$ on A, we can view D as an $N \times N$ symmetric boolean matrix with 1 at the position (i, j) if $a_i - a_j \in D$ and 0 otherwise. Similarly r_{A-A} can be seen as a square symmetric matrix R (where $R_{ij} := r_{A-A}(a_i - a_j)$), which in addition is non-negative definite (checking this is tantamount to rearrangement of the energy equation, see e.g. [68]). Thus $\mathcal{S} = \operatorname{tr} DDR$.

Let μ_1 be the unique positive real eigenvalue of D with the largest size, and $\boldsymbol{v} \geq 0$ a normalised eigenvector with all non-negative entries; this is possible by the Perron-Frobenius theorem. Since D is symmetric and real, the normalised vector which maximises $\boldsymbol{v} \cdot D\boldsymbol{v}$ is an eigenvector corresponding to the largest eigenvalue, so one can estimate

$$\mu_1 = \boldsymbol{v} \cdot D\boldsymbol{v} \geq \frac{|D|\Delta}{|A|}\,, \tag{4.1.8}$$

replacing $\boldsymbol{v}$ by the vector $\frac{1}{\sqrt{|A|}}\mathbf{1}$.

Since D is symmetric, one can write $D = Q\tilde{D}Q^\top$ so that $\tilde{D}$ is diagonal with μ_1 in the top left corner, and $\boldsymbol{v}$ is the first column of orthogonal matrix Q. The basis-invariance of trace gives

$$\mathcal{S} = \operatorname{tr}(\tilde{D}^2 Q^\top R Q).$$

Noting that R is non-negative definite, the trace can be bounded from below by the $(1,1)$-entry, whence

$$\mathcal{S} \geq \mu_1^2\ \boldsymbol{v} \cdot R\boldsymbol{v}.$$

Since $\boldsymbol{v} \geq 0$, this can be estimated from below by making the matrix R entry-wise smaller, namely replacing it with ΔD. Combining this with (4.1.8) and recalling that $\Delta^2|D| \gtrsim E(A)$ yields

$$\mathcal{S} \gtrsim \frac{(|D|\Delta)^2 E(A)}{|A|^3}\,.$$

On the other hand the quantity $\mathcal{S}$, tautologically, is the number of solutions of the equation

$$(a-b)+(b-c)=a'-c' :\ a-b,\ b-c\in D,\ a',c'\in A\,.$$

If $[S']$ is the set of equivalence classes $[(a,b,c)]$ under translation for which $a-b, b-c \in D$, then it follows from Cauchy–Schwarz that

$$\begin{aligned}\mathcal{S}^2 &= \left(\sum_{[a,b,c]\in[S']} r([a,b,c])\ r_{A-A}(a-c)\right)^2\\ &\le\ E_3(A)\cdot \sum_{d_1,d_2\in D} r^2_{A-A}(d_1+d_2),\end{aligned}$$

and since each of d_1, d_2 has at least Δ representations in $A-A$, this means

$$\mathcal{S}^2 \lesssim\ |A|^3\Delta^{-2}\sum_x r_{A-A+A-A}(x)\ r^2_{A-A}(x).$$

We partition $A-A$ into "rich and poor" sets D_1 and D_2, so that for some τ to be determined, $r_{A-A}(x)\le\tau$, for every $x\in D_1$.

We firstly consider the poor differences D_1. By Hölder's inequality

$$\sum_{x\in D_1} r_{A-A+A-A}(x)\ r^2_{A-A}(x) \le \left(\sum_{x\in D_1} r_{A-A+A-A}(x)^{7/3}\right)^{3/7}\left(\sum_{x\in D_1} r_{A-A}(x)^{7/2}\right)^{4/7}.$$

From (3.1.3) we have, once again,

$$\sum_x r_{A-A+A-A}(x)^{7/3}\lesssim N^4E_{k-1}\,,$$

and from the definition of D_1,

$$\sum_{x\in D_1} r_{A-A}(x)^{7/2}\lesssim N^3\tau^{1/2}\,.$$

As for the set D_2, we have, from (3.2.3),

$$|D_2|\le N^3/\tau^3\,.$$

Without changing the notation, we replace D_2 by its subset $\{x : \tau \leq r_{A-A}(x) < 2\tau\}$; this will not have consequences, after dyadic summation. Then the quantity to be estimated is

$$\sum_{x \in D_2} r_{A-A+A-A}(x)\, r_{A-A}^2(x) \leq \tau^2 |\{(a_1, \ldots, a_4, d) \in A^4 \times D_2 \ :\ d = a_1 + a_2 - a_3 - a_4\}|\,.$$

By Hölder's inequality, this is bounded by

$$\tau^2 \left(\sum_x r_{A+A-A}(x)^{5/2} \right)^{2/5} \left(\sum_x r_{A+D_2}(x)^{5/3} \right)^{3/5} .$$

The first bracketed term is estimated directly using (3.4.6). Moreover, by the second bound of Corollary 3.2.1,

$$\sum_x r_{A+D_2}(x)^{5/3} \ll N|D_2|^{4/3}\,.$$

Combining all these estimates, we get

$$\mathcal{S}^2 \lesssim \frac{|A|^3}{\Delta^2} \left(N^{24/7} E_{k-1}^{3/7} \tau^{2/7} + N^{73/15} \tau^{-2/5} \right) .$$

Putting together the upper and lower bounds for $\mathcal{S}^2$ gives

$$|D|^4 \Delta^6 E^2(A) \lesssim |A|^9 \left(N^{24/7} E_{k-1}^{3/7} \tau^{2/7} + N^{73/15} \tau^{-2/5} \right)$$

and optimising in τ yields

$$\tau = N^{151/72} E_{k-1}^{-5/8}\,.$$

Multiplying both sides by Δ^2, using $E(A) \lesssim |D|\Delta^2$ (see (4.1.5)) on the left and $\Delta \lesssim |A|^3/E(A)$ (see (4.1.6)) on the right yields

$$E^8(A) \lesssim N^{15+24/7+151/252} E_{k-1}^{1/4}\,.$$

It remains to substitute an estimate for E_{k-1}. If A is 2-convex we can use Shkredov's bound $E_{k-1} \lesssim N^{32/13}$, and we arrive at

$$E(A) \lesssim N^{2+1705/3744} \ll N^{2+.4554}\,.$$

One can iterate this bound for higher convexity (namely using it as E_{k-1} if $k = 3$, etc.) and it is easily seen that the iterates converge rapidly. In the limit when $E(A) = E_{k-1}$ in the above calculation one gets

$$E(A) \lesssim N^{2+127/279} \leq N^{2+.4552}\,. \tag{4.1.9}$$

□

4.2 Applications of the Equidistribution Lemma

As mentioned in Chapter 2, the Equidistribution Lemma is very useful both in generalising existing results by incorporating a small doubling component and in this section we will see its utility in proving sum-product-type bounds. We rewrite it here for ease of reference.

Lemma 4.2.1 (Equidistribution Lemma)**.** [13] *Let $D := \{d_1 < d_2 \cdots < d_{|D|}\}$ be the positive differences in $A - A$. If $a, a' \in A$ with $a' < a$ and $n_A(a', a) \leq Z$, then $a - a' \leq d_Z$.*

A new proof of a sumset bound

Recall that in [33], Hanson, Roche-Newton and Rudnev proved, given a convex function f and any finite set A, that

$$|A + A - A||f(A) + f(A) - f(A)| \gg \frac{|A|^3}{\log^3 |A|}.$$

We prove the following improvement.

Theorem 4.2.1. [12] *Let A be a finite set of reals and f be a convex function. Then*

$$|A + A - A||f(A) + f(A) - f(A)| \gg |A|^3.$$

The removal of the logarithmic factors makes the bound of Theorem 4.2.1 sharp. Indeed, if $f(x) = x^2$ and $A = [N]$ then

$$|A + A - A||f(A) + f(A) - f(A)| \approx |A|^3.$$

Theorem 4.2.1 is simply the base case for Theorem 2.1.2 in Chapter 2. It is stated separately because we present a short, new proof inspired by the very simple method by which Solymosi establishes the $\delta = 1/4$ sum-product bound in $\mathbb{C}$ [70].

Proof of Theorem 4.2.1. Let $A := \{a_1 < \cdots < a_{|A|}\}$. We say that a_i is *good* if

$$n_A(a_i, a_{i+1}) \ll \frac{|A + A - A|}{|A|} \qquad \text{and} \qquad n_{f(A)}(f(a_i), f(a_{i+1})) \ll \frac{|f(A) + f(A) - f(A)|}{|A|}.$$

Since

$$\sum_{i=1}^{|A|-1} n_A(a_i, a_{i+1}) \leq |A+A-A| \quad \text{and} \quad \sum_{i=1}^{|A|-1} n_{f(A)}(f(a_i), f(a_{i+1})) \leq |f(A)+f(A)-f(A)|,$$

by the pigeonhole principle, there is a set A' with $|A'| \gg |A|$ such that each element of A' is good.

Now consider the map

$$\Psi : a_i \mapsto (a_{i+1} - a_i, f(a_{i+1}) - f(a_i)).$$

By the mean value theorem, there exits a sequence $\{c_i\}$ where $c_i \in (a_i, a_{i+1})$ and

$$\frac{f(a_{i+1}) - f(a_i)}{a_{i+1} - a_i} = f'(c_i).$$

Once $f(a_{i+1}) - f(a_i)$ and $a_{i+1} - a_i$ are fixed, c_i is known uniquely since f' is strictly monotone. Thus a_i is also known uniquely, and Ψ is injective.

Restrict the domain of Ψ to A'. Since Ψ is injective, the size of its domain A' equals the size of its image $\Psi(A')$, proving that

$$|A| \ll |A'| = |\Psi(A')|. \tag{4.2.1}$$

A suitable upper bound for $|\Psi(A')|$ will complete the proof.

Since each $a_i \in A'$ is good it satisfies

$$n_A(a_i, a_{i+1}) \ll \frac{|A + A - A|}{|A|}.$$

The Equidistribution Lemma 4.2.1 shows that $a_{i+1}-a_i$ is among the smallest $\frac{|A+A-A|}{|A|}$ positive elements in $A-A$ and therefore, there are $\ll \frac{|A+A-A|}{|A|}$ values it can take. By an identical argument there are $\ll \frac{|f(A)+f(A)-f(A)|}{|A|}$ values $f(a_{i+1}) - f(a_i)$ can take.

This proves that $|\Psi(A')| \ll |A + A - A||f(A) + f(A) - f(A)||A|^{-2}$. It follows from (4.2.1) that

$$|A| \ll |A + A - A||f(A) + f(A) - f(A)||A|^{-2},$$

and rearranging completes the proof. □

Remark. In Chapter 2, we will claim that Theorem 4.2.1 is the base case for the induction proof of Theorem 2.1.2. In fact, we will need the slightly stronger statement that $f(A)+f(A)-f(A)$ contains $\gg |A|^3|A+A-A|^{-1}$ elements in $(\min(f(A)), \max(f(A))]$. This is immediate after modifying the definition of "good" in the above proof: say a_i is *good* if

$$n_A(a_i, a_{i+1}) \ll \frac{|A + A - A|}{|A|}, \quad \text{and}$$
$$n_{f(A)}(f(a_i), f(a_{i+1})) \ll \frac{|(f(A) + f(A) - f(A)) \cap (f(a_1), f(a_{|A|}]|}{|A|}.$$

Sum-product type results

Studying the sizes of sumsets and product sets which are not the traditional $A + A$ and AA has led to many variations of the sum-product problem. See for example [10, 50, 74]. What these and many other results *do* share with the sum-product problem is they all enshrine the philosophy that additive structure and multiplicative structure cannot coexist in the same set. We may refer to any such result as a sum-product *type* result.

The sum-product problem has been studied in other fields as well. We particularly note that for $A \subset \mathbb{C}$, (1.1.4) is known for all $\delta < 1/3 + c$ (for some small c) [4]. Previous best-known results in $\mathbb{C}$ include $\delta = 1/4$ by Solymosi [70] (using a similar method to ours in this section), $\delta = 3/11 - o(1)$ [1] a precursor to Solymosi's breakthrough approach to the sum-product problem over $\mathbb{R}$ [71], and $\delta = 1/3 - o(1)$ due to Konyagin and Rudnev [43] by generalising [71] from $\mathbb{R}$ to $\mathbb{C}$.

The function field $\mathbb{F}_q((t^{-1}))$ is defined to be the field of all Laurent series of the form

$$\sum_{i=-\infty}^{k} \alpha_i t^i, \qquad \text{where } \alpha_i \in \mathbb{F}_q \text{ for all } i.$$

For subsets A of the function field $\mathbb{F}_q((t^{-1}))$, the sum-product theorem is known for all $\delta < 1/5$ (with the implied constant C also depending on q and the closeness of δ to $1/5$) [6]. The smallness of q compared to $|A|$ is essential in their method.

It is worth mentioning that there is a correlation between the best known sum-product results and the usable structure in the field. The numerically strongest sum-product result is in $\mathbb{R}$, which has a total ordering on its elements which is preserved under addition and multiplication by positive numbers. These elementary properties form the basis for proving the best-known bounds. In contrast, the function field $\mathbb{F}_q((t^{-1}))$ admits a norm structure but this does not give a total ordering with good properties. However in finite fields $\mathbb{F}_p$, there appears to be no sensible norm structure. One would therefore expect the sum-product results to be numerically stronger in $\mathbb{F}_q((t^{-1}))$ than in $\mathbb{F}_p$. However the reverse is observed in practice: the best known is $\delta = 1/4$ in $\mathbb{F}_p$ and $\delta = 1/5 - \epsilon$ for any $\epsilon > 0$ in $\mathbb{F}_q((t^{-1}))$. This is owing to a nice incidence geometric method in finite fields [46] which has no analogue in function fields. Nevertheless, we conjecture that matching the $\delta = 1/4$ bound in $\mathbb{F}_q((t^{-1}))$ using elementary methods is a tractable problem.

In this chapter, we prove a related sum-product type result in each setting. Over the complex numbers we have the following.

Theorem 4.2.2. [12] *Let $A \subset \mathbb{C}$ be a finite set. Then the following holds:*

$$|A + A - A||AA|^2 \gg |A|^4.$$

[1] where $o(1)$ in this context means a term approaching zero as the size of the set A grows

Over function fields, we prove the following.

Theorem 4.2.3. [12] *For any finite $A \subset \mathbb{F}_q((t^{-1}))$ and any $\epsilon > 0$, we have*

$$|A + A - A|^3 |AA|^4 \gg_\epsilon q^{-2} |A|^{9-\epsilon}.$$

Theorem 4.2.3 does not extend to function fields where the base field is not finite. Furthermore, the dependence in this result on q is necessary (and cannot be improved), since $\mathbb{F}_q((t^{-1}))$ has small non-trivial subfields. Indeed, setting $A = \mathbb{F}_q$ demonstrates this sharpness. It is also worth mentioning that the same proof of Theorem 4.2.3 holds for finite subsets of any field with nonarchimedean norm and finite residue field. In particular, it holds for finite subsets of the p-adic numbers $\mathbb{Q}_p$.

A form of Plünnecke's inequality (Theorem 1.1.4) shows that given any set A in some group G, there exists $A' \subset A$ with $|A'| \geq |A|/2$ such that

$$|-A' + A + A| \ll \frac{|-A' + A|^2}{|A|}.$$

If $A' \subset A \in \mathbb{C}$, then applying Theorem 4.2.2 to A' and some simple inequalities yields

$$|A - A|^2 |AA|^2 \gg |A|^5, \tag{4.2.2}$$

which matches Solymosi's bound [70] (though a bound which has since been improved). Similarly, if $A' \subset A \in \mathbb{F}_q((t^{-1}))$, then applying Theorem 4.2.3 to A' yields

$$|A - A|^3 |AA|^2 \gg_\epsilon q^{-1} |A|^{6-\epsilon}. \tag{4.2.3}$$

This matches Bloom and Jones' bound in [6], which is the best known sum-product bound in function fields with finite residue field. Unfortunately, the bounds (4.2.2) and (4.2.3) to not follow from our theorems if $A - A$ is replaced with $A + A$.

It is in the few products, many sums framework that Theorems 4.2.2 and 4.2.3 are most relevant. The full sum-product conjecture is open and likely very difficult. A weaker version of the same problem is the *few products, many sums* problem, which also appears to be the key to understanding sum product phenomena.

Specifically, we conjecture that if $|AA| \ll M|A|$, then $|A \pm A| \gg M^{-O(1)} |A|^{2-\epsilon}$.

Over $\mathbb{R}$, the best known is that $|A-A| \gg M^{-5/3}|A|^{5/3-o(1)}$ when $|AA| \ll M|A|$ [50]. In many fields, the few products, many sums problem has not been studied explicitly and the best known bounds are realised as corollaries of sum-product results.

In various fields there are also very strong results if $|AA| \ll |A|$ and there is no small multiplicative doubling parameter M. Granville and Solymosi showed that if $A \subset \mathbb{C}$ and $|AA| \ll |A|$, then $|A+A| \gg \frac{|A|^2}{2}$ [28]. In finite fields, Shkredov and Vyugin showed for sufficiently small subgroups A of the multiplicative group $\mathbb{F}_p^*$, that $|A \pm A| \gtrsim |A|^{5/3}$.

We also mention that over the reals, the sister problem, *few sums, many products* was resolved by Solymosi [71].

Our results specifically address the few products, *many* 3*-fold sums* problem. For example, if we know that $|AA| \ll M|A|$, then for $A \subset \mathbb{C}$ we have

$$|A+A-A| \gg M^{-2}|A|^2$$

and for $A \subset \mathbb{F}_q((t^{-1}))$ we have

$$|A+A-A| \gg_\epsilon q^{-2/3}M^{-4/3}|A|^{5/3-\epsilon}.$$

Both results appear to be new. Indeed, the few products, many h-fold sums problem has been studied in [14, 41], but both appear to only apply to sums rather than sums and differences, and give quantitatively weaker bounds than Theorems 4.2.2 and 4.2.3.

The proofs of Theorems 4.2.2 and 4.2.3 are both inspired by the combination of techniques used to prove Theorem 4.2.1.

Proof of Theorems 4.2.2 and 4.2.3

The Equidistribution Lemma works equally well in fields other than $\mathbb{R}$, provided a suitable norm can be found. In particular we need a version of Lemma 4.2.1 which is applicable in $\mathbb{C}$ and in $\mathbb{F}_q((t^{-1}))$. Let $(\mathbb{F}, \|\cdot\|)$ be a field with a norm. In this section, we will use extensively the notation

$$B(a,r) = \{x \in \mathbb{F} : \|x-a\| \leq r\}.$$

That is, $B(a,r)$ is the ball around a with radius r.

Lemma 4.2.2. [12] *Let $(\mathbb{F}, \|\cdot\|)$ be a field with a norm, and let A be a finite subset of $\mathbb{F}$. Write $D := A - A = \{0\} \cup \{d_1, \ldots, d_{|D|}\}$ such that the norms of the d_i are non-decreasing. If $a, a' \in A$ and*

$$|(A + A - A) \cap B(a, \|a - a'\|)| \leq Z,$$

then $\|a - a'\| \leq \|d_Z\|$.

The proof is almost identical to the proof of Lemma 2.2.2.

Proof. If not then $a - a' = d_Y$ where $Y > Z$, whence

$$a + d_i \in B(a, \|a - a'\|),$$

for $i = 1, \ldots, Y$. This produces $Y > Z$ elements of $A + A - A$ in $B(a, \|a - a'\|)$, a contradiction. □

Importantly Lemma 4.2.2 can be applied to finite subsets of $\mathbb{C}$ with the usual complex modulus as the norm, and $\mathbb{F}_q((t^{-1}))$ with $\|x\| = q^{\deg x}$ as the norm.

Sum-product type result in $\mathbb{C}$

We prove Theorem 4.2.2 following a method of Solymosi [70] and incorporating Lemma 4.2.2 to obtain an improvement.

Proof of Theorem 4.2.2. For notation, let $B_A(a) := B(a, |a-b|)$ where b is a nearest neighbour (in A) of a (according to the standard modulus function in $\mathbb{C}$). We will say that $(a, b, c) \in A^3$ is *good* if:

$$\begin{aligned}
&b \in B_A(a)\backslash\{a\}, \\
&|(A + A - A) \cap B_A(a)| \ll \frac{|A + A - A|}{|A|}, \text{ and} \qquad (4.2.4)\\
&|(AA) \cap c \cdot B_A(a)| \ll \frac{|AA|}{|A|}. \qquad (4.2.5)
\end{aligned}$$

Note that balls of the form $B_A(a)$ have no elements of $A\backslash\{a\}$ in their interior. We argue that every complex number x is contained in at most six such balls. By translation invariance we may assume $x = 0$. Assume we have seven such balls, whose centres form the set A'. Thus for $a, a' \in A'$, $|a|$ and $|a'|$ are no larger than $|a - a'|$. This is only true if the angle subtended by a, a' at $x = 0$ is at least $60°$ for each pair $a, a' \in A'$. With seven points this is impossible.

Therefore, we have

$$\sum_{a \in A} |(A + A - A) \cap B_A(a)| = \sum_{v \in A+A-A} \sum_{a \in A} \mathbb{1}_{v \in B_A(a)} \leq 6|A + A - A|, \qquad (4.2.6)$$

and for any $c \in A$

$$\sum_{a \in A} |(AA) \cap (B_A(a))| = \sum_{v \in AA/c} \sum_{a \in A} \mathbb{1}_{v \in B_A(a)} \leq 6|AA|. \qquad (4.2.7)$$

Since each $a \in A$ has at least one nearest neighbour, applying the pigeonhole principle to (4.2.6) and (4.2.7), there exists a subset $T \in A^3$ with $|T| \gg |A|^2$ such that each triple $(a, b, c) \in T$ is good. Now consider the set T under the map

$$\Psi : (a, b, c) \mapsto (a - b, ca, cb).$$

As long as $a \neq b$ (which certainly holds for all triples in T), Ψ is injective, whereupon

$$|A|^2 \ll |T| = |\Psi(T)|.$$

We now search for an upper bound on $|\Psi(T)|$ and will do this crudely by counting, given that (a, b, c) is a good triple, how many values $a - b$ may take, and then separately, how many values the pair (ac, bc) can take.

Because (a, b, c) is good, a satisfies (4.2.4). Also, since $b \in B_A(a)$, we know that $B_A(a) = B(a, |a - b|)$, so Lemma 4.2.2 proves that there are $\ll |A + A - A||A|^{-1}$ possible values that $a - b$ may take.

The number of values of that ac may take is trivially upper-bounded by $|AA|$. Once ac is fixed, since bc lies in $c \cdot B_A(a)$ and (4.2.5) holds, there are $\ll |AA||A|^{-1}$ values that bc may take.

Putting this together we obtain

$$|A|^2 \ll |A + A - A||AA|^2|A|^{-2},$$

which we rearrange to arrive at the desired result. □

Sum-product type result in $\mathbb{F}_q((t^{-1}))$

Next we prove Theorem 4.2.3 with the method of Bloom and Jones [6] and necessary modifications to incorporate the $|A + A - A|$ term.

We will firstly introduce some notation that will be used throughout the proof. We can put a norm structure on $\mathbb{F}_q((t^{-1}))$ by saying that $\|x\| = q^{\deg x}$, where deg is the standard degree for Laurent series. Observe that the norm of a difference $\|a-a'\|$ is a measure of how similar a and a' are; specifically it determines the highest degree term on which the Laurent expansions of a and a' disagree. Next let

$$R_A(a) = \min_{a' \in A \setminus \{a\}} \|a - a'\|,$$

and

$$B_A(a) = B(a, R_A(a)).$$

We will argue that any intersecting balls in $\mathbb{F}_q((t^{-1}))$ are nested; that is, if $y \in B(x_1, r_1) \cap B(x_2, r_2)$, then either $B(x_1, r_1) \subset B(x_2, r_2)$ or $B(x_2, r_2) \subset B(x_1, r_1)$. Indeed, if $r_1 \leq r_2$ the former occurs, if $r_2 \leq r_1$ the latter occurs. It follows that if $r_1 = r_2$ then $B(x_1, r_1) = B(x_2, r_2)$.

For a proof suppose $r_1 \leq r_2$ and that $y \in B(x_1, r_1) \cap B(x_2, r_2)$. This means that the Laurent series for x_1, y agree on degrees $\geq r_1$ and x_2, y agree on degrees $\geq r_2$. Since $r_1 \leq r_2$, this implies that x_1, x_2 agree on degrees $\geq r_2$. Now if $w \in B(x_1, r_1)$, then w agrees with x_1 on degree $\geq r_1$. It follows that w agrees with x_2 on degree $\geq r_2$. This demonstrates that $w \in B(x_2, r_2)$, completing the proof.

We will use the same definition and notation for separable sets and A-chains as in [6].

Definition 4.2.1. A finite set $A \in \mathbb{F}_q((t^{-1}))$ is *separable* if its elements can be indexed as

$$A = \{a_1, \ldots, a_{|A|}\}$$

such that for each $1 \leq j \leq |A|$ there is a ball B_j with

$$A \cap B_j = \{a_1, \ldots, a_j\}.$$

We also say that $\mathcal{C} = (c_1, \ldots, c_n) \in A^n$ is an A-*chain* of length n if all the c_i are different and

$$B_A(c_1) \subset \cdots \subset B_A(c_n).$$

The following two lemmata are proved in [6]. We list them here without proof.

Lemma 4.2.3. *If $A \subset \mathbb{F}_q((t^{-1}))$ is a separable set, then for any natural numbers k, n, m such that $n + m = k$,*

$$|nA - mA| \gg_k |A|^k.$$

Remark. In [6], this Lemma is stated only for sumsets, not difference sets. However, since the result is proved by showing that the corresponding energy is minimum possible, it also proves the corresponding bound for difference sets.

Lemma 4.2.4. *If the elements of $\mathcal{C}$ form an A-chain, then $\mathcal{C}$ contains a separable set of size at least $|\mathcal{C}|/q$.*

The strategy of our proof is as follows: if we can find a suitably large A-chain, then Lemma 4.2.4 shows that it contains a large separable set U. Then Lemma 4.2.3 shows that U has a large k-fold sumset, and therefore so does A. Applying

Plünnecke's Inequality will then complete the proof. Thus the key result is the following:

Proposition 4.2.1. [12] *Let $A \subset \mathbb{F}_q((t^{-1}))$ be finite. Then A contains an A-chain $\mathcal{C}$ with*

$$|\mathcal{C}| \gg \frac{|A|^4}{|A+A-A||AA|^2(\log|A|)^3}.$$

Proof. For each $a \in A$ write $N(a)$ to be the length of the longest A-chain $(c_1, \ldots, c_k)$ where $c_k = a$. We begin by dyadic pigeonholing: for each $0 \le j \le \log|A|$, let A_j be the set of $a \in A$ such that $2^j \le N(a) < 2^{j+1}$. There exists some j_0 such that $|A_{j_0}| \ge |A|/\log|A|$.

To complete the proof, it suffices to show that

$$2^{j_0} \gg \frac{|A|^4}{|A+A-A||AA|^2(\log|A|)^3}.$$

To this end, we say that a triple $(a, b, c) \in A^3$ is *good* if

$$a \in A_{j_0}, \tag{4.2.8}$$

$$b \in B_A(a)\backslash\{a\}, \tag{4.2.9}$$

$$|(A+A-A) \cap B_A(a)| \ll \frac{2^{j_0}|A+A-A|}{|A_{j_0}|}, \text{ and} \tag{4.2.10}$$

$$|(AA) \cap c \cdot B_A(a)| \ll \frac{2^{j_0}|AA|}{|A_{j_0}|}. \tag{4.2.11}$$

Let T be the set of all good triples. We complete the proof by showing the following upper and lower bounds on $|T|$:

$$|T| \gg 2^{j_0}|A_{j_0}||A| \tag{4.2.12}$$

$$|T| \ll \frac{2^{2j_0}|A+A-A||AA|^2}{|A_{j_0}|^2}. \tag{4.2.13}$$

We begin by proving (4.2.12). Observe that

$$\sum_{a \in A_{j_0}} |(A+A-A) \cap B_A(a)| = \sum_{v \in A+A-A} \sum_{a \in A_{j_0}} \mathbb{1}_{v \in B_A(a)} = \sum_{v \in A+A-A} |C_{j_0}(v)|, \tag{4.2.14}$$

where $C_{j_0}(v)$ is the set of $a \in A_{j_0}$ such that $v \in B_A(a)$. Similarly, for any $c \in A$

$$\sum_{a\in A_{j_0}} |(AA)\cap c\cdot B_A(a)| = \sum_{u\in AA} \sum_{a\in A_{j_0}} \mathbb{1}_{u\in c\cdot B_A(a)} = \sum_{v\in AA/c} \sum_{a\in A_{j_0}} \mathbb{1}_{v\in B_A(a)} = \sum_{v\in AA/c} |C_{j_0}(v)|. \tag{4.2.15}$$

It is worth noting that unlike when we are working in $\mathbb{C}$, $|C_{j_0}(v)|$ is not bounded by any constant. However, for all $a \in C_{j_0}(v)$, the corresponding balls $B_A(a)$ all share the point v and are therefore nested. In other words, the elements of $C_{j_0}(v)$ can be ordered to form an A-chain. It follows that for some $a_0 \in C_{j_0}(v) \subset A_{j_0}$,

$$|C_{j_0}(v)| \leq N(a_0) \leq 2^{j_0+1}.$$

Applying the pigeonhole principle to (4.2.14) and (4.2.15), there is a subset $A' \subset A_{j_0}$ with $|A'| \gg |A_{j_0}|$ such that for each $a \in A'$ and $c \in A$, (4.2.12) and (4.2.13) both hold.

Given an A-chain $\mathcal{C} = (c_1, \ldots, c_{N(a)})$ with $c_{N(a)} = a \in A_{j_0}$, the definition of an A-chain shows that $c_i \in B_A(a)$ for $i = 1, \ldots, N(a)$. It follows that

$$2^{j_0} \leq N(a) \leq |B_A(a) \cap A|. \tag{4.2.16}$$

Now for any $c \in A$ and $a \in A'$, it follows that (4.2.8),(4.2.10),(4.2.11) all hold. Once a is fixed (4.2.16) shows that at least 2^{j_0} values of b will satisfy (4.2.9), completing the proof that

$$|T| \gg 2^{j_0}|A_{j_0}||A|.$$

We now prove (4.2.13). The map

$$\Psi : (a, b, c) \mapsto (a - b, ac, bc)$$

is manifestly injective when restricted to T. Similar to the proof of Theorem 4.2.2, we upper bound $|\Psi(T)|$ by

$$\frac{2^{2j_0}|A + A - A||AA|^2}{|A_{j_0}|^2}.$$

Since a satisfies (4.2.10) and $b \in B_A(a)$, Lemma 4.2.2 proves that there are $\ll \frac{2^{j_0}|A+A-A|}{|A_{j_0}|}$ possible values that $a-b$ may take. The number of values that ac may take is trivially upper-bounded by $|AA|$. Once ac is fixed, since bc lies in $c \cdot B_A(a)$ and (4.2.11) holds, there are $\ll \frac{2^{j_0}|AA|}{|A_{j_0}|}$ values that bc may take.

Putting this all together, we get

$$|T| \ll \frac{2^{2j_0}|A+A-A||AA|^2}{|A_{j_0}|^2}$$

whereupon using $|A_{j_0}| \gg \frac{|A|}{\log|A|}$, and rearranging, completes the proof. $\square$

Proof of Theorem 4.2.3. By Lemma 4.2.4 and Proposition 4.2.1, there exists a separable subset $U \subset A$ of size at least

$$H := \frac{|A|^4}{q|A+A-A||AA|^2(\log|A|)^3}.$$

Then using Lemma 4.2.3 and Plünnecke's inequality (see [54] for a short proof), we have for all positive integers k,

$$\frac{|A+A-A|^k}{|A|^{k-1}} \gg |kA-kA| \gg |kU-kU| \gg_k H^{2k}.$$

Taking k^{th} roots we get

$$|A+A-A| \gg_k H^2|A|^{1-1/k} \gtrsim \frac{|A|^{9-1/k}}{q^2|A+A-A|^2|AA|^4}.$$

Rearranging yields the desired result for sufficiently large k. $\square$

Generalisation of a theorem of Jarník

In 1926, Jarník published a paper [38] giving sharp bounds for the number of points of the integer lattice $\Lambda_N := [N] \times [N]$ which can lie on a convex curve.

Theorem 4.2.4 (Jarník). *If Γ is a convex curve, then*

$$|\Gamma \cap \Lambda_N| \ll N^{2/3}. \tag{4.2.17}$$

Furthermore, this is sharp as evinced by the so called Jarník curve Γ_J. However, if we write $\Gamma_J := \{(x, f_J(x)) : x \in U\}$ for some open set U, then f_J is a $C^1(U)$ convex function, but is not $C^2(U)$. It was conjectured that (4.2.17) could be improved if Γ is known to be differentiable to some order. Indeed, Swinnerton-Dyer proved a result in this direction [76], which was refined and improved by Schmidt [65].

Theorem 4.2.5 (Schmidt). *If f is a twice differentiable convex function with f'' weakly monotone, then*

$$|\Gamma \cap \Lambda_N| \ll N^{(3/5)+o(1)}.$$

Schmidt further conjectured that the correct power is $1/2$, which would be best possible in light of the curve $\{(x, x^{1/2})\}$. This bears resemblance to our results stating that higher convex functions permit less additive structure.

The number of lattice points lying on algebraic curves was also addressed by Bombieri and Pila [7]. They proved that given an irreducible, algebraic curve Γ of degree d in $\Lambda_N := N \times N$ we have

$$|\Gamma \cap \Lambda_N| \ll N^{(1/d)+o(1)}.$$

Since many convex functions of interest are also algebraic curves, this result is of great interest.

A different approach to generalising Jarník's work is to replace the integer grid Λ_N with $\Lambda_A := A \times A$, where $|A + A - A| = K|A|$ and K is assumed to be small. The Equidistribution Lemma allows us to modify a proof of Theorem 4.2.4 to obtain a result in this direction. It is worth mentioning that the approach of Bombieri and Pila only applies for the strict grid Λ_N, so this generalisation is genuinely novel.

Theorem 4.2.6. *Let Γ be a convex curve and $\Lambda_A = A \times A$ where $|A+A-A| = K|A|$. Then we have*

$$|\Gamma \cap \Lambda_A| \ll K^{2/3}|A|^{2/3}.$$

Proof. Without loss of generality, assume Γ is increasing at an increasing rate, and let $P_1, \dots P_r$ be all the points in $\Gamma \cap \Lambda_A$ moving from left to right.

Let $P_i = (a_i, b_i)$ for all i and notice that all the a_i and b_i lie in A. We will say that P_i is *good* if

$$|(A+A-A)\cap(a_i,a_{i+1}]| \leq \frac{4|A+A-A|}{r} \quad \text{and} \quad |(A+A-A)\cap(b_i,b_{i+1}]| \leq \frac{4|A+A-A|}{r}.$$

Let $\mathcal{P}$ be the set of all good points P_i. By the pigeonhole principle, $|\mathcal{P}| \geq r/2$. Notice that by convexity, none of the $P_{i+1} - P_i$ are the same. It follows that

$$r \ll |\mathcal{P}| = |\{(a_{i+1} - a_i, b_{i+1} - b_i) : P_i \in \mathcal{P}\}|.$$

When $P_i \in \mathcal{P}$, by the Equidistribution Lemma 4.2.1, the differences $a_{i+1} - a_i$ and $b_{i+1} - b_i$ are among the smallest $4|A+A-A|/r$ positive elements of $A - A$, and hence

$$r \ll \left(\frac{|A+A-A|}{r}\right)^2.$$

Rearranging yields the desired result. □

4.3 Open Problems

- In the preparation of this chapter, the following question arose: is it true given a finite real set A, that a positive proportion of $A+A-A$ lies in $[\min A, \max A]$. It is certainly true in mass; that is, a positive proportion of triples $(a, b, c) \in A^3$ will have $a + b - c \in [\min A, \max A]$, but is it also true in number of elements?

- Function fields appear to have more arithmetic structure than finite fields. It stands to reason that improving the sum-product bound in $\mathbb{F}_p((t^{-1}))$ to at least numerically match the bound for finite fields is tractable.

- As mentioned, Bombieri and Pila provided strong bounds for the number of intersections between an algebraic curve Γ and an integer grid Λ_N. It would be interesting to generalise to intersecting algebraic curves with $\Lambda_A := A \times A$, where A has small additive doubling.

Chapter 5

Continuous Incidence Geometry

5.1 Introduction

Incidence geometry is concerned with counting incidences between various geometric objects. For points and lines, $\mathcal{I}(P, L)$ counts the incidences between the point set P and the line set L. Related are $L_k(P)$, the set of k-rich lines induced by point set P, and $P_k(L)$, the set of k-rich points induced by line set L. The classical Szemerédi–Trotter Theorem [78] bounds sharply all three quantities.

In their 2019 paper [32], Guth, Solomon and Wang proved an analogue of the Szemerédi–Trotter Theorem akin to (1.3.3), for suitably well-spaced sets of tubes of thickness δ in $[0, 1]^2$. Furthermore, they proved a similar result in $[0, 1]^3$ which is an analogue of the seminal Guth–Katz bound [30]. Both bounds are essentially sharp. Their work expands our understanding of the Kakeya problem which studies intersections of thin tubes.

This chapter follows closely work of the author in [11]. As in [32], our objects of interest will be small δ-atoms and thin δ-tubes.

Definition 5.1.1. A δ-*atom* is a closed ball in $[0, 1]^d$ of diameter δ. A δ-*tube* is the set of all points in $[0, 1]^d$ which are within a distance $\delta/2$ of some fixed line.[1]

Unlike the discrete setting of points and lines, we need to carefully define what it means for two atoms or two tubes to be distinct. Two δ-atoms are distinct if they

[1] Throughout this chapter, distance will always refer to Euclidean distance.

do not intersect each other. Two δ-tubes are distinct if either:

- They do not intersect each other, or;
- The angle between them is greater than δ.

In this thesis, we will say that these criteria describe a set A of atoms or a set T of tubes which is *δ-separated*[2] This is a nonstandard definition; in the literature, the second criterion alone is required for a set of tubes to be δ-separated. More generally,

Definition 5.1.2. Given $\gamma \geq \delta$, we say a set of δ-atoms A (resp. δ-tubes T) are γ-separated if there is at most one atom of A (resp. tube of T) in each γ-atom (resp. γ-tube).

We say that an atom and a tube are *incident* with each other if they have a non-empty intersection. If the number of atoms from A incident with a δ-tube lies in $[k, 2k)$, then we say it is a *k-rich tube* (induced by A). Let $T_k(A)$ be the size of a maximal set of unique k-rich δ-tubes induced by A. For brevity we will often simply say the number of k-rich tubes induced by A. Owing to the above setup, a set of atoms A must always be finite.

The problem we address is upper-bounding $|T_k(A)|$. In two dimensions, this problem is dual to one of the aforementioned problems addressed in [32]. Bounding the number of incidences between a set A of atoms and a set T of tubes is an equivalent way of studying this problem.

An alternative setup involves counting approximate incidences between δ-separated points and δ-separated lines. In this setting, we would say that a point p and a line l are incident if p lies in a δ-neighbourhood of the line l. This is the setup used in [24].

Defining *distinct* atoms and *distinct* tubes in the above way is natural because it avoids degenerate configurations of atoms and tubes. To give an example, let us allow atoms to intersect. Let A be a large set of atoms which are all small perturbations of a single atom, and let T be a large set of tubes arranged in a star shape and all passing through the atoms of A. In this configuration, all atoms are incident with all tubes, which ought not to be allowed in the setup of a continuous incidence problem.

[2] Technically 2δ-separated under the forthcoming definition.

Indeed, the correct interpretation for two atoms which intersect each other is that they are two copies of the same atom. We would say that this atom appears with multiplicity two. In Section 5.2, we give an incidence result in which the atoms in A may appear with multiplicity.

However, even with these assumptions, atoms and tubes are still not a perfect model for points and lines. In discrete geometry, two points can lie on at most one line and two lines can intersect in at most one point. However, this does not hold for atoms and tubes, and constitutes one of the most important differences between the two settings. In fact, if two δ-atoms in $[0,1]^d$ are separated by a distance of x where $\delta \ll x < 1$, then there exist $\approx x^{1-d}$ distinct δ-tubes which are incident to both of them.

The following example illustrates that without further assumptions on the distribution of atoms, bounds that match the Szemerédi–Trotter Theorem (1.3.2) are unobtainable.

Example 5.1.1. Suppose A is the grid of k^2 δ-atoms that fit inside some $k\delta \times k\delta$ square in $[0,1]^2$. It is clear that $|T_k(A)| = k\delta^{-1} = \delta^{-1} \cdot \frac{|A|^2}{k^3}$. Since δ^{-1} can be arbitrarily large, any upper bounds on $|T_k(A)|$ will be very weak. Similar problematic examples can also be constructed in higher dimensions.

The following incidence bound is the dual to a result established in [24].

Theorem 5.1.1 (Fässler, Orponen, Pinamonti)**.** *Let A be a set of W^{-1}-separated atoms in $[0,1]^2$ where $1 < W \leq \delta^{-1}$. Let $T_k(A)$ be the set of k-rich tubes induced by A. Then*

$$|T_k(A)| \ll W \cdot \frac{|A|^2}{k^3}.$$

In light of Example 5.1.1, Theorem 5.1.1 is sharp when $W = \delta^{-1}$. However, whenever $k \leq \delta|A|$, this bound is worse than the trivial $|T_k(A)| \ll \delta^{-2}$ and indeed there exist sets A of atoms which attain this trivial bound.

For a set A of δ-atoms in $[0,1]^d$ which is well-distributed in some sense, we will prove a bound for $|T_k(A)|$ which essentially depends only on $|A|$ and k. In this context, well-distributed means that the atoms almost form a well-spaced grid. It is an analogue of the similar condition in [32] for tubes to be well-distributed.

Theorem 5.1.2. [11] *Let $d \geq 2$ be an integer. Given $1 < W < \delta^{-1}$, let A be a W^{-1}-separated family of δ-atoms in $[0,1]^d$ where $|A| \approx W^d$. Let $k \geq 2$. Then for every $\epsilon > 0$, there exist $C_1(\epsilon, d)$ and $C_2(\epsilon, d)$ such that if*

$$k \geq C_1(\epsilon, d)\delta^{-\epsilon} \cdot \delta^{d-1}|A|, \tag{5.1.1}$$

then

$$|T_k(A)| \leq C_2(\epsilon, d)\delta^{-\epsilon} \cdot \frac{|A|^2}{k^3}. \tag{5.1.2}$$

The condition (5.1.1) on k is necessary and has a specific meaning. If the atoms in A were randomly placed, then a simple calculation verifies that the expected richness of any δ-tube is $\delta^{d-1}|A|$. If $k \leq \delta^{d-1}|A|$, then probabilistic arguments prove that there exist configurations of atoms A such that a positive proportion of all possible δ-tubes are at least k-rich. Thus we need (5.1.1) in order to obtain nontrivial bounds for $|T_k(A)|$.

Compared to Theorem 5.1.1, our Theorem 5.1.2 has the key added assumption that $|A| \approx W^d$, which means that A is as big as a W^{-1}-separated set could be. In other words, A nearly forms a grid of atoms. In contrast, Theorem 5.1.1 applies to much sparser sets of atoms which are still W^{-1}-separated.

Given a general set T of tubes, $\mathcal{I}(A,T)$ will be the number of incidences between atoms from A and tubes from T. Concretely,

$$\mathcal{I}(A,T) := |\{(a,t) \in A \times T : a \cap t \neq \emptyset\}|.$$

We can obtain an equivalent formulation of Theorem 5.1.2 in terms of incidences by a standard argument (see the proof of Theorem 1.3.4).

Corollary 5.1.1. [11] *Let $d \geq 2$ be an integer. Given $1 < W < \delta^{-1}$, let A be a W^{-1}-separated family of δ-atoms in $[0,1]^d$ where $|A| \approx W^d$. Let T be an arbitrary set of distinct δ-tubes. Then for every $\epsilon > 0$, there exists $C_3(\epsilon, d)$, such that*

$$\mathcal{I}(A,T) \leq C_3(\epsilon, d)\delta^{-\epsilon}(|A|^{2/3}|T|^{2/3} + k_0(A,\delta)|T|), \tag{5.1.3}$$

where $k_0(A,\delta) := \max\{1, \delta^{d-1}|A|\}$.

The term $k_0(A, \delta)|T|$ in (5.1.3) plays the same role as the $|L|$ term in the Szemerédi–Trotter bound (1.3.1), namely counting the incidences from lines incident to only one point.

Let us now briefly clarify how our contribution fits in with the results from [32] upon which our methods are inspired. The main result proved in [32] bounds the number $|A_k(T)|$ of k-rich atoms induced by a set T of well-distributed tubes:

Theorem 5.1.3 (Guth–Solomon–Wang). *Let $d = 2$ or 3. Given $1 < W < \delta^{-1}$, let T be a W^{-1}-separated family of δ-tubes in $[0,1]^d$ where $|T| \approx W^{2(d-1)}$. Let $k \geq 2$. Then for every $\epsilon > 0$ there exist $C_1(\epsilon, d)$ and $C_2(\epsilon, d)$ such that if*

$$k \geq C_1(\epsilon, d)\delta^{-\epsilon} \cdot \delta^{d-1}|T|,$$

then

$$|A_k(T)| \leq C_2(\epsilon, d)\delta^{-\epsilon} \cdot \frac{|T|^{\frac{d}{d-1}}}{k^{\frac{d+1}{d-1}}}.$$

When $d = 2$, Theorem 5.1.3 is a δ-thickened version of the Szemerédi–Trotter Theorem and for $d = 3$, it is a δ-thickened version of the Guth–Katz incidence bound (Theorem 1.3.5).

We say a set A of atoms is *well-distributed* if it satisfies the conditions in Theorem 5.1.2. Similarly, we say a set T of tubes is *well-distributed* if it satisfies the conditions in Theorem 5.1.3. In dimension $d = 2$, well-distributed atoms and tubes are dual to each other. Lines in two dimensions are parametrised by two variables, so a set of well-distributed δ-tubes becomes a set of well-distributed δ-atoms when viewed in the parameter space. Thus, the $d = 2$ case of Theorem 5.1.2 follows immediately by duality from Theorem 5.1.3. This result is essentially optimal.

However for all $d \geq 3$, Theorem 5.1.2 is a new result. It cannot be obtained by reparametrising Theorem 5.1.3 or any other existing result. When $d \geq 3$, we conjecture that the bound should have k^3 replaced with k^{d+1} in the denominator of (5.1.2), but any improvement towards this appears not to be amenable to the method we use. The obstacles to obtaining stronger results for $d \geq 3$ appear to be related to the reasons that the method in [32] fails if $d > 3$. In both cases, proving a suitable modification of Proposition 5.2.1 (and its analogue from [32]) would yield improvements.

A discrete version of Theorem 5.1.2 can be obtained in any dimension $d \geq 3$ by projecting generically onto a plane and applying the Szemerédi–Trotter Theorem. However, this is not possible in the thickened setting because projecting into a plane will not preserve the well-distributed property of the set of atoms in general.

Remark. It does not matter if we define our atoms to be δ-balls with respect to the $\|\cdot\|_2$ norm or the $\|\cdot\|_\infty$ norm, that is, whether our atoms are d-dimensional balls or cubes. It only matters that there exist constants C, c such that a δ-cube is always contained in a $C\delta$-ball, and also contains a $c\delta$-ball. The choice of shape then only affects multiplicative constants in our bounds, which are of no consequence since we are primarily interested in growth rate. During our proof, we will partition the space $[0,1]^d$ into "cells", which is most natural if we view our cells as smaller d-dimensional cubes. One important upshot is that all equalities in this chapter are implicitly up to absolute constants. None of these constants are problematically large or small.

Several other recent atom–tube incidence bounds with different spacing conditions are worth noting. Fu, Gan, and Ren [21] proved a generalisation of Theorem 5.1.3 with the following more permissive spacing assumption: for $1 \leq W \leq X \leq \delta^{-1}$, T is a set of X^{-1}-separated tubes with at most X/W tubes in each $1 \times W^{-1}$ rectangle. Indeed, their result recovers the $d = 2$ case of Theorem 5.1.3 when $X = W$. Fu and Ren [22] also addressed incidence estimations where the set of atoms (resp. tubes) behave locally like α-dimensional (resp. β-dimensional) sets for fixed α, β.

Our focus in this chapter is combinatorial, but it ought to be mentioned that studying the intersections of atoms and tubes is relevant to other problems in real analysis. A discretized version of the Erdős distinct distance problem is treated in [32] and the related Falconer distance conjecture has admitted recent improvements in the plane in [29].

Incidence geometry has been used for addressing sum-product type problems since the groundbreaking paper of Elekes [18]. A δ-discretized sum-product theorem was proved by Bourgain [8], and then reproved with explicit bounds by Guth, Katz and Zahl [31]. In recent work by Gan and Harbuzova [25], Elekes' incidence geometry method was adapted to a version of the δ-discretized sum-product theorem with more restrictive assumptions on the set A. The more general discretized sum-product theorem appears impervious to being improved using continuous incidence geometry.

The structure of this chapter is as follows: in Section 5.2, we prove a general incidence result which will be needed in the proof of Theorem 5.1.2 in Section 5.3. In Section 5.4, we give an application of Theorem 5.1.2, an analogue of Beck's theorem for atoms and tubes.

5.2 A general incidence result

To assist in our proofs, we will mostly be working with incidence counts rather than directly with k-rich tubes. Furthermore, we allow for sets of *weighted* atoms and tubes. Let A be a set of atoms where each $a \in A$ has a positive integer weight $w(a)$ associated with it. This essentially means that when counting incidences, the atom a appears $w(a)$ times. Similarly define weighting for sets of tubes.

For a set of weighted atoms A with weight function w, and a set of weighted tubes T with weight function ω, we define a more general incidence counting function

$$\mathcal{I}(A, T) := \sum_{a \in A} \sum_{t \in T} w(a)\omega(t)\mathbb{1}_{\{a \cap t \neq \emptyset\}}.$$

The following incidence bound is a generalisation of [32, Proposition 2.1] and our proof is a modification of theirs. Its uses for us are twofold. Firstly, our proof of Theorem 5.1.2 requires an incidence result for a general set of atoms which may not be well-distributed. Secondly, since it applies to weighted atoms, we can sidestep some technical steps in proving the main result.

The substance of Proposition 5.2.1 below is how the incidence count $\mathcal{I}(A, T)$ behaves when the atoms in A and the tubes in T are thickened by a factor of S (in all directions except its axis). That is, each δ-atom becomes an $S\delta$-atom centred at the same point, and each δ-tube becomes an $S\delta$-tube about the same line. Importantly, after thickening, some atoms may intersect, in which case we consider them a single atom with an associated weight. If the original atoms were already weighted, then their weights sum if they intersect after thickening. Similarly, tubes may also become weighted after thickening.

Proposition 5.2.1. [11] *Fix $0 < \alpha < 1$. Let $k \geq 1$ and A be a set of distinct weighted δ-atoms in $[0,1]^d$ with weight function w. Let T be a set of distinct (not weighted) δ-tubes. Let S be such that $\delta^{-\alpha} \ll S \ll \delta^{-1}$. Then*

$$\mathcal{I}(A,T) \ll_\alpha \left(S\delta^{-(d-1)}|T| \sum_{a \in A} w(a)^2 \right)^{1/2} + \delta^{-\alpha} S^{1-d} \mathcal{I}(A^S, T^S), \tag{5.2.1}$$

where A^S and T^S are respectively the weighted sets of atoms and tubes formed by thickening A and T by a factor of S.

The proof of Proposition 5.2.1 uses some elementary Fourier analysis. We include the following definition for completeness.

Definition 5.2.1. For an integrable function $f : \mathbb{R}^d \to \mathbb{C}$, its *Fourier transform* is defined

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx.$$

For two integrable functions $f : \mathbb{R}^d \to \mathbb{C}$ and $g : \mathbb{R}^d \to \mathbb{C}$, their *convolution* is given by

$$(f * g)(x) := \int_{\mathbb{R}^d} f(y) g(x-y) dy.$$

Proof of Proposition 5.2.1. We scale the problem by δ^{-1}, so that the δ-atoms are now 1-atoms and the δ-tubes are now 1-tubes in $[0,\delta^{-1}]^d$. This will be more convenient to work with.

For any $a \in A$ and any $t \in T$, let $\chi_a(x)$ and $\chi_t(x)$ respectively be smooth bump functions approximating the indicator functions for the atom a and the tube t. Now let $f(x) := \sum_{a \in A} w(a)\chi_a(x)$ and $g(x) := \sum_{t \in T} \chi_t(x)$. In this notation,

$$\mathcal{I}(A,T) \approx \int_{[0,\delta^{-1}]^d} f(x) g(x) dx.$$

Since f and g are Lebesgue integrable functions, their Fourier transforms are well-defined. Furthermore, since they are all bounded and supported on compact sets, they are L_p functions for all p. It follows that Plancherel's Theorem holds: $\int f(x)g(x)dx = \int \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi$. Decompose this expression into high and low frequency parts:

$$\mathcal{I}(A,T) \approx \int \hat{f}\,\overline{\hat{g}}\,\eta + \int \hat{f}\,\overline{\hat{g}}\,(1-\eta),$$

where η is a smooth function taking value 1 on the ball of radius $\rho := \delta^{-\alpha/d}S^{-1}$ and supported on a ball of radius 2ρ.

Low frequency case: Assume $\mathcal{I}(A,T) \ll \int \hat{f}\,\overline{\hat{g}}\,\eta$. By Plancherel (and using that η^2 and η are essentially equal), we get

$$\mathcal{I}(A,T) \ll \int (f*h)(g*h),$$

where $\overline{\hat{h}} = \eta$. Roughly speaking, convolution with h thickens atoms and tubes by a factor of ρ^{-1}. For each atom $a \in A$, the function $\chi_a * h$ is approximately $w(a)\rho^d$ on the thickened ρ^{-1}-atom around a. Similarly, for each tube $t \in T$, the function $\chi_t * h$ is approximately ρ^{d-1} on the thickened ρ^{-1}-tube around t.

Outside of these ρ^{-1}-atoms and ρ^{-1}-tubes, the functions $f*h$ and $g*h$ are rapidly decaying. Since $S = \delta^{-\alpha/d}\rho^{-1}$, the tails of both functions are negligible outside the S-atoms A^S and S-tubes T^S. It follows that

$$\mathcal{I}(A,T) \ll \int (f*h)(g*h) \ll \rho^{d-1}\mathcal{I}(A^S,T^S) \ll_\alpha \delta^{-\alpha}S^{1-d}\mathcal{I}(A^S,T^S).$$

High frequency case: Assume that $\mathcal{I}(A,T) \ll \int \hat{f}\,\overline{\hat{g}}\,(1-\eta)$. Using the Cauchy–Schwarz inequality, we get

$$\int \hat{f}(\xi)\overline{\hat{g}(\xi)}(1-\eta(\xi))d\xi \leq \left(\int |\hat{f}(\xi)|^2 d\xi\right)^{1/2} \left(\int |\hat{g}(\xi)|^2(1-\eta(\xi))^2 d\xi\right)^{1/2}. \quad (5.2.2)$$

By Parseval's identity, the first term on the right-hand side can be evaluated as

$$\left(\int |\hat{f}(\xi)|^2 d\xi\right)^{1/2} = \left(\int |f(x)|^2 dx\right)^{1/2} \approx \left(\sum_{a\in A} w(a)^2\right)^{1/2}.$$

We now estimate the second term on the right hand side of (5.2.2). Cover the d-dimensional unit sphere with small $(d-1)$-dimensional δ-balls. These will be called *δ-caps* and are used to sort tubes in T by direction. Let T_θ be the set of all tubes from T in the direction of δ-cap θ, and let $g_\theta = \sum_{t\in T_\theta} \chi_t$. Then we need to estimate

$$\int |\hat{g}(\xi)|^2(1-\eta(\xi))^2 d\xi = \int |\sum_\theta \hat{g}_\theta(\xi)|^2(1-\eta(\xi))^2 d\xi.$$

We will apply Cauchy–Schwarz to the sum over θ. The advantage is that for fixed ξ, there are not many values of θ for which $\hat{g}_\theta(\xi)$ is non-zero.

For each $t \in T_\theta$, we have that $\hat{\chi_t}$ is mostly supported on a $1 \times \cdots \times 1 \times \delta$ slab through 0 orthogonal to θ and decays quickly outside. We call this slab $\theta^\perp$. Due to the rapidly decaying tails, the contribution of $\hat{g}_\theta$ outside the dilated slab $\delta^{-\alpha/d}\theta^\perp$ is of the order δ^B for some positive B, and is therefore negligible.

Thus, the term $\hat{g}_\theta(\xi)$ is only nonzero if ξ belongs to $\delta^{-\alpha/d}\theta^\perp$. We may assume that $|\xi| \geq \rho$, as $1-\eta(\xi)$ is otherwise zero. For such a large ξ, simple geometric arguments show that ξ belongs to $\delta^{-\alpha/d}\theta^\perp$ for at most $\ll \delta^{-\alpha/d}\rho^{-1}\delta^{-(d-2)} = S\delta^{-(d-2)}$ different θ values. Then Cauchy–Schwarz yields

$$(1-\eta(\xi))^2|\sum_\theta \hat{g}_\theta(\xi)|^2 \ll_\alpha S\delta^{-(d-2)} \sum_\theta |\hat{g}_\theta(\xi)|^2.$$

Again using Parseval's identity, it follows that

$$\begin{aligned}\int |\hat{g}(\xi)|^2(1-\eta(\xi))^2 d\xi &\ll_\alpha S\delta^{-(d-2)} \sum_\theta \int |\hat{g}_\theta(\xi)|^2 d\xi \\ &= S\delta^{-(d-2)} \sum_\theta \int |g_\theta(x)|^2 dx \\ &= S\delta^{-(d-1)}|T|.\end{aligned}$$

Substituting into (5.2.2) yields

$$\mathcal{I}(A,T) \ll \left(S\delta^{-(d-1)}|T|\sum_{a\in A} w(a)^2\right)^{1/2}. \qquad \square$$

The dominant term in (5.2.1) is determined based on whether the incidence count increases disproportionately after thickening by S. The following two examples give configurations of atoms and tubes which attain both bounds in Proposition 5.2.1, demonstrating that it is sharp up to a small factors involving S and $\delta^{-\alpha}$. For the purpose of these examples, A will not be a weighted set of atoms.

Example 5.2.1. If $A \subset [0,1]^d$ consists of all the δ-atoms in a d-dimensional box with side length $k\delta$, then $|A| = k^d$. If T is the set of induced k-rich δ-tubes, it can be shown that $|T| = \delta^{-(d-1)}k^{d-1}$. Further calculations show that

$$(S\delta^{-(d-1)}|A||T|)^{1/2} = S^{1/2}\delta^{-(d-1)}k^{d-\frac{1}{2}} \qquad \text{and} \qquad \delta^{-\alpha}S^{1-d}\mathcal{I}(A^S, T^S) = \delta^{-(d-1+\alpha)}k^d.$$

Also since all tubes in T are k-rich, we have

$$\mathcal{I}(A,T) = \delta^{-(d-1)}k^d,$$

so the second term in (5.2.1) is attained up to a $\delta^{-\alpha}$ factor (for sufficiently small S).

Example 5.2.2. If $A \subset [0,1]^d$ consists of a $(d-1)$-dimensional slice of the above configuration of δ-atoms, then we have $|A| = k^{d-1}$. Again let T be the set of induced k-rich δ-tubes, so $|T| = \delta^{-(d-1)}k^{d-3}$. In this case

$$(S\delta^{-(d-1)}|A||T|)^{1/2} = S^{1/2}\delta^{-(d-1)}k^{d-2} \quad \text{and} \quad \delta^{-\alpha}S^{1-d}\mathcal{I}(A^S, T^S) = S^{-1}\delta^{-(d-1+\alpha)}k^{d-2}.$$

Again, since all tubes in T are k-rich, we have

$$\mathcal{I}(A,T) = \delta^{-(d-1)}k^{d-2},$$

so the first term in (5.2.1) is the attained bound up to an $S^{1/2}$ factor.

5.3 The Main Result

The proof of Theorem 5.1.2 combines induction with a cell partitioning argument. Often in proofs of incidence results it is useful to partition the space into smaller cells and estimate the contribution of incidences in each cell individually. We have already seen such a proof of the Szemerédi–Trotter Theorem. Before commencing our proof, we present a higher-dimensional generalisation of the Szemerédi–Trotter Theorem in the special case that the point set is a cartesian product. The prototype for this method can be found in [72] and uses a "lucky pairs" argument.

Theorem 5.3.1. *Let $P \in \mathbb{R}^d$ be a d-fold cartesian product $P := P^{(1)} \times \cdots \times P^{(d)}$ with $|P^{(i)}| = n$ for all i, and let $k \geq 2$. Then*

$$L_k(P) \ll_d \frac{|P|^2}{k^{d+1}}.$$

Our set of well-distributed atoms looks like a perturbation of a Cartesian product. In fact, the proof of Theorem 5.3.1 works equally well for a perturbation of a Cartesian product as it does for a true Cartesian product. This equips us with a good heuristic that the true best possible bound for a set of well-distributed atoms A in $[0,1]^d$ is in fact

$$|T_k(A)| \ll_{\epsilon,\delta} \delta^{-\epsilon} \frac{|A|^2}{k^{d+1}}.$$

Furthermore, the cell partitioning is reminiscent of what we use in the proof of Theorem 5.1.2, where the cells are essentially d-dimensional rectangles in a lattice.

Proof of Theorem 5.3.1. Partition each $P^{(i)}$ into $\frac{k}{2d}$ intervals each containing the same number of elements. This in turn partitions P into $(\frac{k}{2d})^d$ cells. Let L_k be the set of k-rich lines induced by P and say that a pair of points p, p' is *lucky* if they lie together on a line in L_k and also together in the same cell. We now proceed to upper and lower-bound the number of lucky pairs.

Notice that any line $l \in L_k$ may pass through at most $k/2$ cells. Since l must pass through at least k points from P, it must induce at least $k/2$ lucky pairs by the pigeonhole principle. Furthermore, each lucky pair arises exactly once in this way. So there are at least $|L_k| \cdot \frac{k}{2}$ lucky pairs in total. We upper-bound the number of lucky pairs trivially by the number of pairs which lie together in the same cell (ignoring whether they also lie together on a line from L_k), which is

$$\left(\frac{2dn}{k}\right)^{2d} \cdot \left(\frac{k}{2d}\right)^{d} = \frac{(2d)^d |P|^2}{k^d}.$$

Combining the upper and lower bounds and rearranging yields the desired

$$L_k(P) \ll_d \frac{|P|^2}{k^{d+1}}. \qquad \square$$

In $\mathbb{R}^2$, a polynomial of degree k can induce the above partitioning into $\approx k^2$ cells. In other words, for a cartesian product, the cell partitioning described in Theorem 1.3.3 can give rise to a grid-like cell decomposition.

Our strategy for proving Theorem 5.1.2 is the following: We partition $[0,1]^d$ into cells of side length D^{-1} for some parameter D to be chosen. Proposition 5.2.1 with some thickening parameter S allows us to relate the number of k-rich tubes to an incidence count, specifically the L_2-norm of the weights of shortened tubes in all cells. This is bounded by applying the induction hypothesis in each cell.

For the proof to work, we need S to be much smaller than D, and D to be much smaller than $\delta^{-\epsilon}$. We also want S and D to be much bigger than constants. This is the motivation for the uniform choices of these parameters given in the proof.

The method is inspired by [32], but is different in several key ways. Firstly, we work with incidences to give a new exposition of this kind of proof. Secondly, by using Proposition 5.2.1 which works for weighted atoms, we sidestep several dyadic pigeonholing steps which are needed in [32]. Finally, our problem admits an additional simplification. There is a simple expression for the number of 2-rich tubes induced by A which applies in any dimension $d \geq 2$. This allows us to resolve both the "narrow" and "broad" case for small k which appear in [32], in a single simplified case.

Proof of Theorem 5.1.2. We treat ϵ and d as constants, so in what follows $\ll$ is written to mean $\ll_{\epsilon,d}$. We fix W and proceed by induction on δ. Namely we have to prove the statement for all $\delta \in (0, W^{-1})$. There are two base cases because when we apply the induction hypothesis it will be for a smaller value of both W and δ^{-1}.

The first base case will be when δ is very close to W^{-1}, namely when $\delta^{-(1-c_1\epsilon)} \leq W$ for some small fixed c_1 (we choose $c_1 < 1/(d-1)$ which assists in the following calculation). Assuming $\delta^{-(1-c_1\epsilon)} \leq W$, (5.1.1) gives

$$k \geq C_1(\epsilon, d) W^{d - \frac{d-1-\epsilon}{1-c_1\epsilon}} > C_1(\epsilon, d) W.$$

Since the distribution of atoms permits $|T_k(A)|$ to be non-zero only if $k \ll W$, we can choose $C_1(\epsilon, d)$ large enough so that $|T_k(A)| = 0$, and (5.1.2) holds trivially.

The other base case is when W is very small, say smaller than some constant c_2. In this case, $|T_k(A)| \leq c_2^{2d}$ trivially, so (5.1.2) holds for a suitable choice of $C_2(\epsilon, d)$.

We move on to the induction step. Assume the result holds for all $\delta' > K\delta$ and $W'^{-1} > KW^{-1}$, where K is sufficiently small ($K = 2$ will work). Assume (5.1.1) holds.

Firstly, we split up $[0,1]^d$ into D^d identical sub-cubes or *cells* in a d-dimensional grid, where $D = \delta^{-c_1^2\epsilon^2}$. Let a *tubelet* be the intersection of a k-rich δ-tube with one of these cells. (A tubelet looks like a section of a δ-tube of length D^{-1}.) To each tubelet t we associate a weight $w(t)$ which is the number of atoms from A intersecting t, and a multiplicity $m(t)$ which is the number of k-rich tubes containing the tubelet t. (A tubelet is "contained" in a tube if all the atoms on the tubelet also intersect the tube. A tubelet may lie on up to D^{d-1} of the k-rich tubes.) From here on, we will often abbreviate $T_k = T_k(A)$. With this notation, it is evident that

$$k|T_k| = \mathcal{I}(A, T_k) = \sum_{\text{tubelets } t} w(t)m(t).$$

It is also clear that

$$\sum_{\text{tubelets } t} m(t) = D|T_k|,$$

so by the pigeonhole principle, a positive proportion of the incidences come from tubelets t with $w(t) \gg k/D$ (with an appropriate subsumed constant). We will henceforth assume that the weights of all tubelets are at least k/D.

Now we treat separately two cases: $k \ll D$ and $k \gg D$. The reason is that we will later apply the induction hypothesis to estimate the number of k/D-rich tubelets, and the induction hypothesis only holds if k/D is greater than some constant.

Case 1: $k \ll D$. Since $D = \delta^{-c_1^2\epsilon^2}$, it follows that $\delta^{-\epsilon}k^{-3} \gg 1$. Now let's count the tubes which are at least 2-rich. For each pair of atoms $a, a' \in A$, there are $\mathrm{dist}(a,a')^{-(d-1)}$ tubes passing through both (where $\mathrm{dist}(a,a')$ is the distance between the centres of atoms a and a'). It follows that

$$|T_k(A)| \leq |T_{\geq 2}(A)| \leq \sum_{\substack{a,a' \in A \\ a \neq a'}} \mathrm{dist}(a,a')^{-(d-1)}.$$

By the grid-like configuration, we may assume that the atoms are exactly arranged in a d-dimensional integer grid (scaled down by a factor of W). At worst, this affects the above sum by a small multiplicative constant. We also lose no generality by considering only the case where a' is the atom with centre at the origin. Thus, it follows that

$$\sum_{\substack{a,a'\in A\\ a\neq a'}} \mathrm{dist}(a,a')^{-(d-1)} \approx |A|W^{d-1} \cdot \sum_{\substack{\mathbf{n}\in(\mathbb{Z}\cap[0,W])^d\\ \mathbf{n}\neq 0}} \|\mathbf{n}\|^{-(d-1)}.$$

Given $x \in [0,W]$, we have $\|\mathbf{n}\| \in [x, 2x)$ for $\approx_d x^d$ values of $\mathbf{n}$. Incorporating this into a dyadic sum, one obtains the desired

$$|T_k(A)| \ll_d |A|W^{d-1} \cdot \sum_{x \text{ dyadic}}^{W} x \ll \delta^{-\epsilon}\frac{|A|^2}{k^3}.$$

Case 2: $k \gg D$**.** We want to apply Proposition 5.2.1, but to do so globally is wasteful of the strong spacing assumptions on A, so the bound will be prohibitively weak.

If we consider any maximal set of distinct $D\delta$-tubes in $[0,1]^d$, then each δ-tube is contained in one of these $D\delta$-tubes. Indeed, this partitions the set of all δ-tubes into the $D\delta$-tubes which contain them. The rationale for this partitioning is that in each $D\delta$-tube, the tubelets behave like weighted atoms. There are $(D\delta)^{-2(d-1)}$ such $D\delta$-tubes.

Given one of these $D\delta$-tubes τ, we "stretch" it in all non-axis directions by a factor of $D^{-1}\delta^{-1}$, so it becomes $[0,1]^d$. Each tubelet in τ which runs parallel to τ becomes a D^{-1}-atom, and each k-rich δ-tube in τ becomes a D^{-1}-tube. Furthermore, each new atom a has a weight $\omega_\tau(a)$, which is the same as the weight of the corresponding tubelet. Call this set of new weighted D^{-1}-atoms $\mathbb{A}_\tau$ and the set of new D^{-1}-tubes $\mathbb{T}_\tau$. For the case $d = 2$, this procedure is indicated in Figure 5.1.

For each $D\delta$-tube τ, we count the incidences arising from tubelets which lie on, and are parallel to, τ. By the stretching procedure above (see Figure 5.1), this is

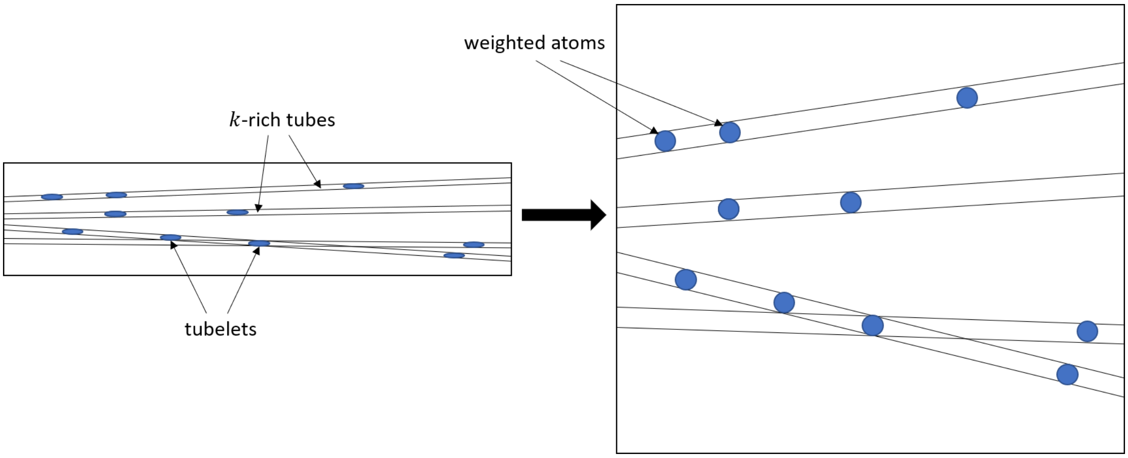

Figure 5.1: After "stretching", the incidences between tubelets and δ-tubes inside a $D\delta$-tube become incidences between weighted D^{-1}-atoms and D^{-1}-tubes.

equal to $\mathcal{I}(\mathbb{A}_\tau, \mathbb{T}_\tau)$. It follows that

$$\mathcal{I}(A, T_k) = \sum_\tau \mathcal{I}(\mathbb{A}_\tau, \mathbb{T}_\tau).$$

Applying Proposition 5.2.1 in each $D\delta$-tube τ using a thickening factor $S = \delta^{-c_1^3\epsilon^3}$ and $\delta^{-\alpha} = S^\epsilon$, and then applying Cauchy–Schwarz, we get

$$\begin{aligned}
k|T_k| \le \mathcal{I}(A, T_k) &= \sum_\tau \mathcal{I}(\mathbb{A}_\tau, \mathbb{T}_\tau) \\
&\ll \sum_\tau \left(SD^{d-1}|\mathbb{T}_\tau| \sum_{a\in\mathbb{A}_\tau} \omega_\tau(a)^2 \right)^{1/2} + S^{1-d+\epsilon} \sum_\tau \mathcal{I}(\mathbb{A}_\tau^S, \mathbb{T}_\tau^S) \\
&\le (SD^{d-1})^{1/2} \left(\sum_\tau |\mathbb{T}_\tau| \right)^{1/2} \left(\sum_\tau \sum_{a\in\mathbb{A}_\tau} \omega_\tau(a)^2 \right)^{1/2} + S^{1-d+\epsilon}\mathcal{I}(A^S, T_k^S) \\
&= (SD^{d-1})^{1/2}|T_k|^{1/2} \left(\sum_\tau \sum_{a\in\mathbb{A}_\tau} \omega_\tau(a)^2 \right)^{1/2} + S^{1-d+\epsilon}\mathcal{I}(A^S, T_k^S).
\end{aligned} \tag{5.3.1}$$

We will now have two cases based on which term in (5.3.1) dominates. [3]

Firstly suppose the second term dominates. Since there is at most one atom in

[3]Note that T_k^S is always the set of k-rich δ-tubes which are then thickened by S, not the set of k-rich $S\delta$-tubes.

each W^{-1}-cell, all thickened $S\delta$-atoms in A^S have weight one, and hence $|A^S| = |A|$. Also, the weights of tubes in T_k^S are trivially bounded above by $S^{2(d-1)}$, the maximum number of δ-tubes contained in an $S\delta$-tube. If $\tilde{T}_k^S$ is the underlying set of *unweighted* tubes, then

$$\mathcal{I}(A^S, T_k^S) \ll S^{2(d-1)} \mathcal{I}(A^S, \tilde{T}_k^S). \tag{5.3.2}$$

In order to have $k|T_k| \ll S^{1-d}\mathcal{I}(A^S, T_k^S)$, a positive proportion of these incidences must be supported on $S\delta$-tubes which are at least $S^{d-1}k$-rich in atoms from A^S. Furthermore, (5.1.1) implies that

$$S^{d-1}k \geq S^{d-1}C_1(\epsilon, d)\delta^{d-1-\epsilon}|A| \geq C_1(\epsilon, d)(S\delta)^{d-1-\epsilon}|A^S| \cdot S^{\epsilon},$$

so we can apply the induction hypothesis for any richness $k' \geq S^{d-1}k$. Standard dyadic summing of the induction hypothesis implies that

$$\mathcal{I}(A^S, \tilde{T}_k^S) \ll C_2(\epsilon, d)(S\delta)^{-\epsilon} \frac{|A|^2}{(S^{d-1}k)^2}. \tag{5.3.3}$$

Then combining (5.3.1), (5.3.2) and (5.3.3) yields

$$k|T_k| \ll S^{1-d}C_2(\epsilon, d)\delta^{-\epsilon}\frac{|A|^2}{k^2},$$

and rearranging closes the induction, as the S^{1-d} term subsumes the multiplicative constants.

Now assume the first term in (5.3.1) dominates. After rearranging, this implies that

$$|T_k| \ll \frac{SD^{d-1}\left(\sum_{\tau}\sum_{a\in\mathbb{A}_\tau}\omega_\tau(a)^2\right)}{k^2}. \tag{5.3.4}$$

Notice that the bracketed term in the numerator is a sum over all tubelets. A suitable bound on this quantity will complete the proof.

Having already partitioned $[0,1]^d$ into D^d cells, we now estimate the contribution from tubelets in each cell. For any of these cells $\mathcal{C}$, let $A_{\mathcal{C}}$ be the set of atoms from A which lie in $\mathcal{C}$ and let $T_{\mathcal{C}}$ be the set of tubelets in $\mathcal{C}$.

If we enlarge each cell $\mathcal{C}$ to $[0,1]^d$, then the δ-atoms become $D\delta$-atoms which satisfy the spacing conditions for applying the induction hypothesis.[4] Each tubelet t is now a $D\delta$-tube, and the richness of this tube, denoted by $r(t)$, is the weight of the correponding tubelet. Recall that these weights all exceed k/D. For any $m > k/D$, using (5.1.1) we get

$$\begin{aligned} m &> C_1(\epsilon,d)\delta^{d-1-\epsilon}|A|D^{-1} \\ &> C_1(\epsilon,d)(\delta D)^{d-1-\epsilon}(D^{-d}|A|) \\ &= C_1(\epsilon,d)(\delta D)^{d-1-\epsilon}|A_{\mathcal{C}}|, \end{aligned}$$

so the induction hypothesis (5.1.2) can be used in any cell $\mathcal{C}$ to bound $|T_m(A_{\mathcal{C}})|$. We get

$$\begin{aligned} \sum_{\tau}\sum_{a\in\mathbb{A}_\tau}\omega_\tau(a)^2 &= \sum_{\mathcal{C}}\sum_{t\in T_{\mathcal{C}}} r(t)^2 \\ &= \sum_{\mathcal{C}}\sum_{\substack{m \text{ dyadic} \\ m=k/D}}^{k} m^2|T_m(A_{\mathcal{C}})| \\ &\leq \sum_{\mathcal{C}}\sum_{\substack{m \text{ dyadic} \\ m=k/D}}^{k} C_2(\epsilon,d)(D\delta)^{-\epsilon}(|A|D^{-d})^2m^{-1} \\ &\ll D^d\cdot C_2(\epsilon,d)(D\delta)^{-\epsilon}(|A|D^{-d})^2\cdot(D/k). \end{aligned}$$

Substituting this into (5.3.4), and recalling that $S=\delta^{-c_1^3\epsilon^3}$ and $D=\delta^{-c_1^2\epsilon^2}$, we get

$$|T_k(A)|\leq C_2(\epsilon,d)\delta^{-\epsilon}|A|^2k^{-3},$$

closing the induction and completing the proof. □

Remark. There is a small omission in the above proof: In Case 2, it is essential that each cell contains approximately the same number of atoms from A and that they are well-distributed. This follows immediately if $D < W$. But if $D \geq W$, then δ is so ridiculously small that the $\delta^{-\epsilon}$ factor in (5.1.2) is enormous, and trivial bounds give the desired result. Concretely, if $W \leq D = \delta^{-c_1^2\epsilon^2}$, then since no pair of atoms

[4] This technically assumes that $D < W$; the reverse case is simple and discussed at the end of the proof.

can lie on more than W^{d-1} tubes,

$$|T_k(A)| \leq W^{d-1}|A|^2 \leq W^{d+2} \cdot \frac{|A|^2}{k^3} \leq \delta^{-(d+2)c_1^2\epsilon^2} \cdot \frac{|A|^2}{k^3} \leq \delta^{-\epsilon} \cdot \frac{|A|^2}{k^3}.$$

5.4 An Application

Beck's Theorem [5] is an important result in discrete geometry, and is a standard corollary of the Szemerédi–Trotter Theorem. It states that given n points in the plane with at most $n-k$ on any line, the number of lines containing at least two of the points is $\gg nk$. Theorem 5.1.2 allows us to prove a version of Beck's Theorem for a set of well-spaced atoms in any dimension $d \geq 2$. We use the same method that can be used to derive Beck's Theorem from the Szemerédi–Trotter Theorem.

Theorem 5.4.1. [11] *Let A be a set of δ-atoms in $[0,1]^d$ satisfying the spacing conditions of Theorem 5.1.2, and such that $|A| \leq \delta^{1-d}$. Then for every $\epsilon > 0$,*

$$|T_{\geq 2}(A)| \gg_{\epsilon,d} \delta^{\epsilon}|A|^2.$$

Proof. Since $|A| \leq \delta^{1-d}$, Theorem 5.1.2 can be applied and

$$|T_k(A)| \leq C_2(\epsilon/2, d)\delta^{-\epsilon/2} \cdot \frac{|A|^2}{k^3}$$

holds for all k. The number of pairs of atoms both of which lie on a tube that is at least k_0-rich is given by

$$\sum_{\substack{k \text{ dyadic} \\ k \geq k_0}} k^2|T_k(A)| \leq 2C_2(\epsilon/2, d)\delta^{-\epsilon/2}\frac{|A|^2}{k_0}.$$

Choosing $k_0 = 10C_2(\epsilon/2, d)\delta^{-\epsilon/2}$, it follows that $\gg_{\epsilon,d} |A|^2$ pairs of atoms lie together only on tubes that are less than k_0-rich. A δ-tube can have at most k_0^2 of these pairs lying on it so there are $\gg_{\epsilon,d} |A|^2/k_0^2 \approx \delta^{\epsilon}|A|^2$ tubes which are at least 2-rich, completing the proof. $\square$

5.5 Open Problems

- Our heuristics indicate that the number of k-rich tubes induced by well-spaced atoms A in $[0,1]^d$ is expected to be $\ll \delta^{-\epsilon}\frac{|A|^2}{k^{d+1}}$. Improving the denominator in (5.1.2) to $k^{3+\epsilon}$ or better still, $k^{3+\epsilon(d)}$ where $\epsilon(d)$ increases with d, would be a wonderful result.

  Proving this using the current method would involve a corresponding improvement to Proposition 5.2.1. There are heuristics for why such an improvement may be possible when $d = 3$.

- With access to an atoms and tubes version of Szemerédi–Trotter as well as Guth–Katz (in [32]), we may be able to emulate discrete arguments (see [48]) which address the δ-discretised pinned distance problem.

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