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Title:
Models of curves and Newton polytopes

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A dissertation submitted to the University of Bristol in accordance with the requirements of the degree of Doctor of Philosophy in the Faculty of Science.

July 2022

Word count: 79,720

Abstract

The purpose of this thesis is to construct explicit regular models of curves, both over fields and over discrete valuation rings. Given a perfect field k and a smooth plane curve C_0/k, we know there exists a unique non-singular projective curve $C \supseteq C_0$. The problem is to find C explicitly. Under certain conditions, a method called toric resolution describes such a curve from a certain elementary combinatorial object attached to C_0. Unfortunately, this approach does not always work. We extend this classical construction to any curve, preserving its computational and combinatorial nature.

Let K be the field of fraction of some discrete valuation ring O and C/K a hyperelliptic curve of genus g. A regular model of C over O is a regular proper flat 2-dimensional scheme $\mathcal{C} \to \operatorname{Spec} O$ with generic fibre isomorphic to C. A classical question in arithmetic geometry is how to construct such a model. An answer is known when $g \leq 2$, thanks to algorithms developed by Tate and Liu (in residue characteristic not 2). However, there was no general algorithm for an unbounded g. In this thesis, we explicitly construct a regular model of C over O with normal crossings for hyperelliptic curves of arbitrary genus, when the residue characteristic of K is not 2 (and some cases when it is 2). The description relies on a new notion we introduce: the MacLane cluster picture.

Dedication and Acknowledgements

This thesis would have not seen the light of day without the love and the support that I have received from many people over the past four years. Although only my name appears on the front cover, I truly hope that all of them feel part of this result.

I want to start thanking my supervisor, Prof Tim Dokchitser, for the countless meetings, all the time he devoted, and his general advice; he is not only a great mathematician but also a good man. For the time they will spend reading this dissertation, I am truly indebted with my VIVA examiners Prof Andrew Obus and Dr Céline Maistret. For his untiring effort to convince me not to give up, I express my gratitude to Prof Stefano Vigni, who has never missed the opportunity to let me know he believes in my capacities.

I would like to thank all my friends who have always been by my side. Unfortunately, I do not have enough space to mention them all. Please accept my apologies if your name is not explicitly reported. I express my thanks towards Giorgio, Nirvana and Pip, for their friendship and the concrete help they always gave me to face the challenges of my new life in Bristol. For the laughs, the games and the pub nights, I express my gratitude to "la gente da Ollivud", my group of Italian friends in Bristol. For walking the path of faith with me, I want to thank my parish group "TIGER". I am grateful of having two friends like Damiano and Valerio, whom I have always been able to count on.

God blessed me with a wonderful family. Mum, dad, grandma and grandpa, Claudio, if I could, I would dedicate to you much more than a paragraph. Every single word of this dissertation is the outcome of your hard work.

From the bottom of my heart I express my gratitude and love to my fiancee Valeria, my rock, my strength, the little star I brought in all dark moments I have gone through. The distance has fed the fire of our love; now I am looking forward to starting our next chapter together!

I conclude with the most important acknowledgement. I thank God with all my soul. When I felt lonely, He was there; when I was sick, He was there; when I felt that I would have never written a paper, He was there, ready to help me. Nothing would have been possible without Him.

Author's declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

SIGNED: Simone Muselli DATE: 08/07/2022

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CHAPTER 1

INTRODUCTION

Curves are the main object of this dissertation. A *curve* C defined over a field K, denoted C/K, is a scheme $C \to \operatorname{Spec} K$ of finite type, pure of dimension 1. Let C be a smooth, projective curve defined over a field K. We are interested in constructing regular models of C. Let us explain what we mean by that. Let O be an integral domain of Dedekind dimension ≤ 1 with field of fractions K.

Definition A *model* of C over O is a proper flat scheme $\mathcal{C} \to \operatorname{Spec} O$ of relative dimension 1, with generic fibre isomorphic to C.

If $\dim O = 0$, then $O = K$ and a model of C over O is a curve $\mathcal{C}$ isomorphic to C. Note that in this case, every model is regular since C is smooth. If $\dim O = 1$, then a model of C is a 2-dimensional scheme and does not have to be regular. In the following sections we present our results: explicit constructions of models of curves over perfect fields and of regular models of hyperelliptic curves over discrete valuation rings. All our descriptions rely on applying toric resolution approaches over certain Newton polygons attached to the curve C.

Each subsequent chapter of this thesis consists of one of the author's papers, is self-contained and has its own introduction and notation. Chapter 2 is [Mus1], Chapter 3 is [Mus2] and Chapter 4 is [Mus3]. In particular, the reader is not required to read the full dissertation if they are interested in a specific result.

1.1 Models of curves over perfect fields

Every smooth, projective curve C/K is uniquely determined by any dense open subset. In fact, given any affine smooth curve C_0/K there exists a unique smooth, projective curve C/K whose C_0 is a dense open subscheme. This theoretical existence and uniqueness raises a question: can we find a model of C over K knowing C_0? Indeed, describing a model of C over K would lead to the understanding of its geometry, e.g. the computation of the genus.

The problem presented above has a well-known solution, that consists of embedding C_0 in a projective space, taking its closure, and applying repeated blowing-ups to resolve all singularities. However, this procedure is usually hard to handle in practice. For this reason, alternative approaches have been developed. Here we want to focus on one of them, called *toric resolution*.

First, since all smooth, projective curves have a dense open subscheme isomorphic to a smooth curve contained in the 2-dimensional torus $\mathbb{G}^2_{m,K}$, we suppose that C_0 is of this form. A simple combinatorial object, called *Newton polygon*, is associated with C_0, and a toric variety $\mathbb{T} \supset C_0$ can be defined explicitly from it. When the closure of C_0 in $\mathbb{T}$ is smooth, it is a model of C over K.

The construction above is easy and explicit but unfortunately it does not always give a model of C over K. What can we done when it fails? In Chapter 3 we present a new approach that extends the classical toric resolution if K is perfect. On one side, our method always leads to the description of a model of C over K, called generalised Baker's model. On the other side, it preserves the computational and combinatorial nature of toric resolutions, relying on an iterative construction of Newton polygons.

1.2 Models of hyperelliptic curves over discrete valuation rings

Suppose K is a complete discretely valued field of characteristic different from 2, with ring of integers O_K and residue field k. To study the arithmetic of a smooth, projective curve C/K, it is essential to understand regular models of C over O_K. However, this is a difficult problem, even when C is a hyperelliptic curve. Similarly to the case of models over fields, a repeated blowing-ups procedure is possible but often impractical. For this reason, the study of regular models has been a very active area in recent years.

Let C/K be a hyperelliptic curve. In Chapter 2, we explicitly construct the minimal regular model with normal crossings $\mathcal{C}/O_K$ of C, under certain conditions on the curve. As an application, we also determine a basis of integral differentials of C, that is an O_K-basis for the global sections of the relative dualising sheaf $\omega_{\mathcal{C}/O_K}$. Note that this is possible due to the explicit description of $\mathcal{C}$. In some cases, the result presented in this chapter is able to produce a regular model even when the characteristic of k is 2.

In Chapter 4 a regular model over O_K is constructed for any hyperelliptic curve C/K, if $\mathrm{char}(k) \neq 2$. The description of the model is given in a closed form, thanks to a new notion we introduce, the *MacLane cluster picture*. Being a bridge between some of the objects recently used in the study of regular models, the MacLane cluster picture has the potential to have an important role in understanding the local arithmetic of hyperelliptic curves.

The constructions in both chapters follow the same spirit. We first define a toric scheme $\mathbb{T} \to \mathrm{Spec}\ O_K$ in which a certain open subscheme C_0 of C naturally embeds. The closure of C_0

in $\mathbb{T}$ is a regular model $\mathcal{C}$ of C over O_K (with strict normal crossings). It is important to point out that the construction of $\mathbb{T}$, and consequently of $\mathcal{C}$, is explicit, coming from certain Newton polygons attached to the hyperelliptic curve.

CHAPTER 2

MODELS AND INTEGRAL DIFFERENTIALS OF HYPERELLIPTIC CURVES

The purpose of this chapter is to construct regular models of hyperelliptic curves and to describe a basis of integral differentials attached to them. We will do it under certain conditions on the curve, mild when the residue characteristic is not 2. The content of this chapter can be found in the author's paper *Models and Integral Differentials of Hyperelliptic Curves* [Mus1], currently submitted for publication.

2.1 Introduction

To describe the arithmetic of curves over global fields, for example in the study of the Birch & Swinnerton-Dyer conjecture, it is essential to understand regular models and integral differentials over all primes, including those with very bad reduction. Constructing regular models of curves over discrete valuation rings is not an easy problem, even in the hyperelliptic curve case. In fact, there is no practical algorithm able to determine a model, unless the genus of the curve is 1 or we have some tameness or nondegeneracy hypothesis.

One possible approach to tackle this problem is giving a full classification of possible regular models in a fixed genus, as done by the Kodaira–Néron ([Kod], [Nér]) and Namikawa–Ueno ([NU], [Liu2]) classifications for curves of genera 1 and 2, respectively. However, this strategy seems impractical in general, since the number of models grows fast with the genus. Recently, new approaches based on clusters [D^2M^2], Newton polytopes [Dok], and MacLane valuations [OW], have been developed (see §2.1.4 for more detail).

On one side, clusters define nice and clear invariants from which one can extract information on the local arithmetic of hyperelliptic curves. Such invariants turn out to be particularly useful from a Galois theoretical point of view. However, for describing regular models, restrictions on the reduction type of the curve and on the residue characteristic of its base field ([D^2M^2], [FN]) need to be imposed. On the other side, Newton polytopes and MacLane valuations have a potential to solve the problem in general, but the respective constructions are more algorithmic and so do not give the result in closed form. Furthermore, they often depend on the chosen equation rather than on the curve itself.

In this chapter, we present a new approach that preserves both positive aspects from the above and provides a link between the two sides. We describe a model from simple invariants defined from what we call *rational cluster picture* (Definition 2.1.10). This object modifies the theory in [D^2M^2] and appears to be more suitable for our purpose (see §2.1.2). In fact, the rational cluster picture also carries intrinsic connections with the other presented approaches, as it is closely related to Newton polygons and to degree 1 MacLane valuations (see [FGMN]). When these valuations are enough to describe a regular model we say that the curve has an *almost rational cluster picture* (Definition 2.1.1; see also 2.3.29, 2.3.31). It turns out that the approach even works in residue characteristic 2, under an extra assumption that the curve is *y-regular* (Definition 2.1.4). Our main result is:

Let K be a complete[1] *discretely valued field with* $\mathrm{char}(K) \neq 2$*, and let K^{nr} be its maximal unramified extension. Let C/K be a hyperelliptic curve, having an almost rational cluster picture over K^{nr}. If the residue characteristic of K is 2, assume that $C_{K^{nr}}$ is y-regular. Then via the rational cluster picture we determine:*

(i) the minimal regular model with normal crossings $\mathcal{C}^{\min}$,

(ii) a basis of integral differentials of C.

This result applies to a wide class of curves, covering wild cases and base fields with even residue characteristic. For example, if $g = 2$, then 107 out of 120 Namikawa-Ueno types ([NU]) arise from hyperelliptic curves satisfying the conditions of our theorem.

In residue characteristic not 2, Chapter 4 constructs a regular model with string normal crossings of any hyperelliptic curve C. The strategy used there generalises the one of this chapter.

2.1.1 Main results

We will now present (a simplified version of) the main results of this chapter. We will then illustrate them with an explicit example in §2.1.3.

Let K be a complete discretely valued field of residue characteristic p, with normalised discrete valuation v and ring of integers O_K. We require $\mathrm{char}(K)$ to be not 2, but we allow $p = 2$ and $p = 0$. In this subsection we will assume for simplicity that $K = K^{nr}$. Extend the valuation v to an algebraic closure $\bar{K}$ of K. Let C/K be a hyperelliptic curve, i.e. a geometrically connected

[1]The assumption on the completeness of K is not restrictive since regular models do not change under completion of the base field.

smooth projective curve, double cover of $\mathbb{P}^1_K$. Let g be the genus of C. Assume $g \geq 1$. Fix a Weierstrass equation

$$C : y^2 = f(x).$$

Let $\mathfrak{R}$ be the set of roots of f in $\bar{K}$. Thus

$$f(x) = c_f \prod_{r \in \mathfrak{R}} (x - r).$$

For any $r, r' \in \mathfrak{R}$, with $r \neq r'$, denote by $\mathcal{D}_{r,r'}$ the smallest v-adic disc containing r and r'.

Definition 2.1.1 (Definition 2.3.26) We say that C has an *almost rational cluster picture* if for any roots $r, r' \in \mathfrak{R}$ with $r \neq r'$, either

(a) $\mathcal{D}_{r,r'} \cap K \neq \varnothing$, or

(b) $p > 0$ and $|\mathcal{D}_{r,r'} \cap \mathfrak{R}| \leq |v(r - w)|_p$ for some $w \in K$,

where $|\cdot|_p$ denotes the canonical p-adic absolute value on $\mathbb{Q}$.

The intuition behind the definition above relies on certain objects, called *MacLane clusters*, which we introduce in Chapter 4 (Definitions 4.1.2, 4.1.3). Precisely, C has an almost rational cluster picture if and only if all proper MacLane clusters have degree 1.

Definition 2.1.2 A *rational cluster* is a non-empty subset $\mathfrak{s} \subset \mathfrak{R}$ of the form $\mathcal{D} \cap \mathfrak{R}$, where $\mathcal{D}$ is a v-adic disc $\mathcal{D} = \{x \in \bar{K} \mid v(x - w) \geq \rho\}$ for some $w \in K$ and $\rho \in \mathbb{Q}$. We denote by Σ_K the set of rational clusters.

In the following definition we introduce most of the notation and quantities, associated with rational clusters, needed in order to state our main theorems.

Definition 2.1.3 For any $\mathfrak{s} \in \Sigma_K$ we say:

$\mathfrak{s}$ proper,	if $\lvert\mathfrak{s}\rvert > 1$
$\mathfrak{s}'$ is a child of $\mathfrak{s}$,	if $\mathfrak{s}' \in \Sigma_K$ and $\mathfrak{s}' \subsetneq \mathfrak{s}$ is a maximal subcluster
$\mathfrak{s}$ minimal,	if $\mathfrak{s}$ has no proper children
$\mathfrak{s}$ übereven,	if $\mathfrak{s} = \bigcup_{\mathfrak{s}' \text{ child of } \mathfrak{s}} \mathfrak{s}'$ and $\lvert\mathfrak{s}'\rvert$ even for all children $\mathfrak{s}'$ of $\mathfrak{s}$

Moreover, we write $\mathfrak{s}' < \mathfrak{s}$, or $\mathfrak{s} = P(\mathfrak{s}')$, for a child $\mathfrak{s}'$ of $\mathfrak{s}$, and $r \wedge \mathfrak{s}$ for the smallest rational cluster containing the root $r \in \mathfrak{R}$ and $\mathfrak{s}$.

Let $\mathring{\Sigma}_K$ be the set of proper rational clusters. For any $\mathfrak{s} \in \mathring{\Sigma}_K$, define its *radius*

$$\rho_{\mathfrak{s}} = \max_{w \in K} \min_{r \in \mathfrak{s}} v(r - w)$$

and the following quantities:

$$\boxed{\begin{array}{ll}
b_{\mathfrak{s}} & = \text{denominator of } \rho_{\mathfrak{s}} \\
\epsilon_{\mathfrak{s}} & = v(c_f) + \sum_{r\in\mathfrak{R}} \rho_{r\wedge\mathfrak{s}} \\
D_{\mathfrak{s}} & = 1 \text{ if } b_{\mathfrak{s}}\epsilon_{\mathfrak{s}} \text{ odd, } 2 \text{ if } b_{\mathfrak{s}}\epsilon_{\mathfrak{s}} \text{ even} \\
m_{\mathfrak{s}} & = (3-D_{\mathfrak{s}})b_{\mathfrak{s}} \\
p_{\mathfrak{s}} & = 1 \text{ if } |\mathfrak{s}| \text{ is odd, } 2 \text{ if } |\mathfrak{s}| \text{ is even} \\
s_{\mathfrak{s}} & = \frac{1}{2}(|\mathfrak{s}|\rho_{\mathfrak{s}} + p_{\mathfrak{s}}\rho_{\mathfrak{s}} - \epsilon_{\mathfrak{s}}) \\
\gamma_{\mathfrak{s}} & = 2 \text{ if } |\mathfrak{s}| \text{ is even and } \epsilon_{\mathfrak{s}} - |\mathfrak{s}|\rho_{\mathfrak{s}} \text{ is odd, } 1 \text{ otherwise} \\
p^0_{\mathfrak{s}} & = 1 \text{ if } \mathfrak{s} \text{ is minimal and } \mathfrak{s}\cap K \neq \varnothing, 2 \text{ otherwise} \\
s^0_{\mathfrak{s}} & = -\epsilon_{\mathfrak{s}}/2 + \rho_{\mathfrak{s}} \\
\gamma^0_{\mathfrak{s}} & = 2 \text{ if } p^0_{\mathfrak{s}} = 2 \text{ and } \epsilon_{\mathfrak{s}} \text{ is odd, } 1 \text{ otherwise}
\end{array}}$$

Definition 2.1.4 (Definition 2.4.10) We say that the hyperelliptic curve C is *y-regular* if either $p \neq 2$ or $D_{\mathfrak{s}} = 1$ for any $\mathfrak{s} \in \mathring{\Sigma}_K$.

Definition 2.1.5 Let $\mathfrak{s} \in \mathring{\Sigma}_K$ and let $c \in \{0,\dots,b_{\mathfrak{s}}-1\}$ such that $c\rho_{\mathfrak{s}} - \frac{1}{b_{\mathfrak{s}}} \in \mathbb{Z}$. Define

$$\tilde{\mathfrak{s}} = \{\mathfrak{s}' \in \Sigma_K \cup \{\varnothing\} \mid \mathfrak{s}' < \mathfrak{s} \text{ and } \tfrac{|\mathfrak{s}'|}{b_{\mathfrak{s}}} - c\epsilon_{\mathfrak{s}} \notin 2\mathbb{Z}\},$$

where $\varnothing < \mathfrak{s}$ if $\mathfrak{s}$ is minimal and $p^0_{\mathfrak{s}} = 2$.

The *genus* $g(\mathfrak{s})$ of a rational cluster $\mathfrak{s} \in \mathring{\Sigma}_K$ is defined as follows:

- If $D_{\mathfrak{s}} = 1$, then $g(\mathfrak{s}) = 0$.
- If $D_{\mathfrak{s}} = 2$, then $2g(\mathfrak{s})+1$ or $2g(\mathfrak{s})+2$ equals $\dfrac{|\mathfrak{s}| - \sum_{\mathfrak{s}'<\mathfrak{s}}|\mathfrak{s}'|}{b_{\mathfrak{s}}} + |\tilde{\mathfrak{s}}|$.

Notation 2.1.6 (2.4.17) Let $\alpha \in \mathbb{Z}_+$, $a, b \in \mathbb{Q}$, with $a > b$, and fix $\frac{n_i}{d_i} \in \mathbb{Q}$ so that

$$\alpha a = \frac{n_0}{d_0} > \frac{n_1}{d_1} > \ldots > \frac{n_r}{d_r} > \frac{n_{r+1}}{d_{r+1}} = \alpha b, \quad \text{with} \quad \begin{vmatrix} n_i & n_{i+1} \\ d_i & d_{i+1} \end{vmatrix} = 1,$$

and r minimal. We write $\mathbb{P}^1(\alpha, a, b)$ for a chain of $\mathbb{P}^1$s (Notation 2.4.16) of length r and multiplicities $\alpha d_1, \dots, \alpha d_r$. Denote by $\mathbb{P}^1(\alpha, a)$ the chain $\mathbb{P}^1(\alpha, a, \lfloor \alpha a - 1 \rfloor/\alpha)$.

The following theorem describes the special fibre of a regular model of C with strict normal crossings.[2] It follows from a more general result constructing a proper flat model of C unconditionally (Theorem 2.4.18). For the special fibre $\mathcal{C}_s^{\min}$ of the minimal regular model with normal crossings, the reader can refer to Theorem 2.4.22, where we also describe a defining equation for all components of $\mathcal{C}_s^{\min}$ and discuss the Galois action (for general K). Finally, note that all these models are constructed in §2.5 by giving an explicit open affine cover (see §2.5.1-2.5.3).

Theorem 2.1.7 (Regular SNC model) *Suppose C is y-regular and has almost rational cluster picture. Then we can explicitly construct a regular model with strict normal crossings $\mathcal{C}/O_K$ of C (§2.5.1-2.5.3). Its special fibre $\mathcal{C}_s/k$ is given as follows.*

[2] In this thesis a 'normal crossings' divisor is not a 'strict normal crossings' divisor in general (see e.g. [Liu4, Remark 9.1.7]).

(1) *Every $\mathfrak{s} \in \mathring{\Sigma}_K$ gives a 1-dimensional closed subscheme $\Gamma_\mathfrak{s}$ of multiplicity $m_\mathfrak{s}$. If $\mathfrak{s}$ is übereven and $\epsilon_\mathfrak{s}$ is even, then $\Gamma_\mathfrak{s}$ is the disjoint union of $\Gamma_\mathfrak{s}^- \simeq \mathbb{P}^1$ and $\Gamma_\mathfrak{s}^+ \simeq \mathbb{P}^1$, otherwise $\Gamma_\mathfrak{s}$ is a smooth geometrically integral curve of genus $g(\mathfrak{s})$ (write $\Gamma_\mathfrak{s}^- = \Gamma_\mathfrak{s}^+ = \Gamma_\mathfrak{s}$ in this case).*

(2) *Every $\mathfrak{s} \in \mathring{\Sigma}_K$ with $D_\mathfrak{s} = 1$ gives $(|\mathfrak{s}| - \sum_{\mathfrak{s}' \in \mathring{\Sigma}_K, \mathfrak{s}' < \mathfrak{s}} |\mathfrak{s}'| + p_\mathfrak{s}^0 - 2)/b_\mathfrak{s}$ open-ended $\mathbb{P}^1$s of multiplicity $b_\mathfrak{s}$ from $\Gamma_\mathfrak{s}$.*

(3) *Finally, for any $\mathfrak{s} \in \mathring{\Sigma}_K$ draw the following chains of $\mathbb{P}^1$s:*

Conditions	*Chain*	*From*	*To*
$\mathfrak{s}$ *minimal*	$\mathbb{P}^1(\gamma_\mathfrak{s}^0, -s_\mathfrak{s}^0)$	$\Gamma_\mathfrak{s}^-$	*open-ended*
$\mathfrak{s}$ *minimal*, $p_\mathfrak{s}^0/\gamma_\mathfrak{s}^0 = 2$	$\mathbb{P}^1(\gamma_\mathfrak{s}^0, -s_\mathfrak{s}^0)$	$\Gamma_\mathfrak{s}^+$	*open-ended*
$\mathfrak{s} \neq \mathfrak{R}$	$\mathbb{P}^1(\gamma_\mathfrak{s}, s_\mathfrak{s}, s_\mathfrak{s} - p_\mathfrak{s} \cdot \frac{\rho_\mathfrak{s} - \rho_{P(\mathfrak{s})}}{2})$	$\Gamma_\mathfrak{s}^-$	$\Gamma_{P(\mathfrak{s})}^-$
$\mathfrak{s} \neq \mathfrak{R}$, $p_\mathfrak{s}/\gamma_\mathfrak{s} = 2$	$\mathbb{P}^1(\gamma_\mathfrak{s}, s_\mathfrak{s}, s_\mathfrak{s} - p_\mathfrak{s} \cdot \frac{\rho_\mathfrak{s} - \rho_{P(\mathfrak{s})}}{2})$	$\Gamma_\mathfrak{s}^+$	$\Gamma_{P(\mathfrak{s})}^+$
$\mathfrak{s} = \mathfrak{R}$	$\mathbb{P}^1(\gamma_\mathfrak{s}, s_\mathfrak{s})$	$\Gamma_\mathfrak{s}^-$	*open-ended*
$\mathfrak{s} = \mathfrak{R}$, $p_\mathfrak{s}/\gamma_\mathfrak{s} = 2$	$\mathbb{P}^1(\gamma_\mathfrak{s}, s_\mathfrak{s})$	$\Gamma_\mathfrak{s}^+$	*open-ended*

When $p \neq 2$, Theorem 2.1.7 is generalised by Theorem 4.1.7, constructing a regular model with strict normal crossings for any hyperelliptic curve.

Definition 2.1.8 For any $\mathfrak{s} \in \mathring{\Sigma}_K$, an element $w_\mathfrak{s} \in K$ is called *rational centre* of $\mathfrak{s}$ if $\min_{r \in \mathfrak{s}} v(r - w_\mathfrak{s}) = \rho_\mathfrak{s}$.

If $\mathfrak{s}' < \mathfrak{s}$ and $w_{\mathfrak{s}'}$ is a rational centre of $\mathfrak{s}'$, then $w_{\mathfrak{s}'}$ is also a rational centre of $\mathfrak{s}$. For any minimal rational cluster $\mathfrak{s}'$ fix a rational centre $w_{\mathfrak{s}'}$. For any $\mathfrak{s} \in \mathring{\Sigma}_K$ fix $w_\mathfrak{s} = w_{\mathfrak{s}'}$ for some minimal rational cluster $\mathfrak{s}' \subseteq \mathfrak{s}$.

The following result gives a basis of integral differentials when $K = K^{nr}$. In Theorem 2.6.4 we extend it to the case $K \neq K^{nr}$.

Theorem 2.1.9 (Theorem 2.6.3) *Suppose C is y-regular and has almost rational cluster picture. For $i = 0, \dots, g-1$, inductively*

(i) *define* $e_i := \max_{\mathfrak{t} \in \mathring{\Sigma}_K} \left\{ \frac{\epsilon_\mathfrak{t}}{2} - \rho_\mathfrak{t} - \sum_{j=0}^{i-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{t}} \right\}$;

(ii) *choose clusters $\mathfrak{s}_i \in \mathring{\Sigma}_K$ so that $e_i = \frac{\epsilon_{\mathfrak{s}_i}}{2} - \sum_{j=0}^{i} \rho_{\mathfrak{s}_j \wedge \mathfrak{s}_i}$. If $\mathfrak{s}$ and $\mathfrak{s}'$ are two possible choices for $\mathfrak{s}_i$ satisfying $\mathfrak{s}' \subsetneq \mathfrak{s}$, then choose $\mathfrak{s}_i = \mathfrak{s}$.*

Then a basis of integral differentials is given by

$$\mu_i = \pi^{\lfloor e_i \rfloor} \prod_{j=0}^{i-1} (x - w_{\mathfrak{s}_j}) \frac{dx}{2y}, \qquad i = 0, \dots, g-1.$$

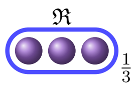
Note that given e_i as in the previous theorem, the sum $\sum_{i=0}^{g-1} \lfloor e_i \rfloor$ is the quantity, often denoted by $v(\omega^{\circ}/\omega)$, appearing in the period in the Birch and Swinnerton-Dyer conjecture (for more details see [FLS^{3}W], [vB, §1.3]).

2.1.2 Rational cluster picture

In this subsection we define the rational cluster picture and compare it with the *classical* cluster picture defined in [D^{2}M^2]. We will show, via a simple example, in which sense the new object we introduce appears to be more suitable for the study of regular models.

Definition 2.1.10 (Definition 2.3.9) Let K and C as before. The *rational cluster picture* of C is the collection of its rational clusters Σ_K together with their radii.

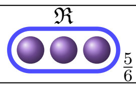
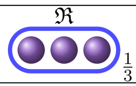
Example 2.1.11 Let p be any prime number and set $K = \mathbb{Q}_p^{nr}$. Let $E_p/\mathbb{Q}_p^{nr}$ given by $y^2 = x^3 - p$. Then E_p is an elliptic curve with Kodaira-Néron reduction type II. Therefore the minimal regular model (with normal crossings) of E_p does not depend on p. This is in accordance with the fact that the rational cluster picture of E_p is the same for all p. Indeed, the set of roots of the polynomial $x^3 - p$ is $\mathfrak{R} = \{\sqrt[3]{p}, \zeta_3\sqrt[3]{p}, \zeta_3^2\sqrt[3]{p}\}$, where ζ_3 is a primitive 3-rd of unity. Hence the rational cluster picture of E_p is



for any p,

where we denoted with bullet points the roots in $\mathfrak{R}$, with a surrounding oval the only rational cluster $\mathfrak{R}$, and with the subscript the radius $\rho_{\mathfrak{R}}$ of $\mathfrak{R}$.

A different behaviour is observed when we consider the cluster picture [D^{2}M^2, Definition 1.26] of E_p, collection of its clusters together with their depths. The cluster picture of E_p is

$p = 2$	$p = 3$	$p > 3$
cluster picture not defined		

where the subscripts represent the depth of the cluster $\mathfrak{R}$. It does depend on p and differs from the rational cluster picture when $p = 3$ (if we do not consider non-proper clusters). Thus, although the cluster picture is particularly useful for Galois theoretical problems, the rational cluster picture appears to be a more suitable object for the study of regular models of the curve.

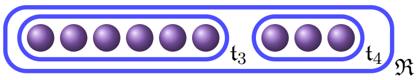
Finally, note that E_p has an almost rational cluster picture. For any two distinct roots $r, r' \in \mathfrak{R}$, the smallest v-adic disc $D_{r,r'}$ containing them also contains the whole $\mathfrak{R}$. The element $0 \in \mathbb{Q}_p^{nr}$ belongs to $D_{r,r'}$ when $p \neq 3$, while $|D_{r,r'} \cap \mathfrak{R}| = 3 = |v(r)|_p$, if $p = 3$.

The advantages of the rational cluster picture discussed in this subsection can also be observed in the following example where we study a more complex family of hyperelliptic curves having almost rational cluster picture.

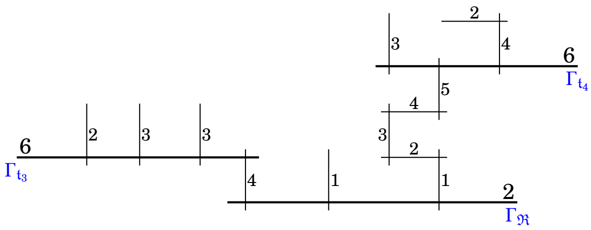
2.1.3 Example

In this subsection we are going to present an example of a family of hyperelliptic curves C_p satisfying the hypothesis of Theorems 2.1.7 and 2.1.9. Via those results we will then describe the special fibre of the minimal regular model and a basis of integral differentials of C_p. All the computations involved are explained in detail in Examples 2.3.32, 2.4.24 and 2.6.5.

For any prime number p, let $a \in \mathbb{Z}_p$, $b \in \mathbb{Z}_p^\times$ such that the polynomial $x^2 + ax + b$ is not a square modulo p. Let $C_p/\mathbb{Q}_p$ be the hyperelliptic curve of genus 4 given by $y^2 = f(x)$, where $f(x) = (x^6 + ap^4x^3 + bp^8)((x-p)^3 - p^{11})$. The curve $C_p/\mathbb{Q}_p^{nr}$ has an almost rational cluster picture and is y-regular when $p = 2$. Its rational cluster picture is



where $\rho_{\mathfrak{t}_3} = \frac{4}{3}$, $\rho_{\mathfrak{t}_4} = \frac{11}{3}$, and $\rho_{\mathfrak{R}} = 1$. From Theorem 2.1.7 we can construct a regular model with strict normal crossings of C_p with special fibre



over $\bar{\mathbb{F}}_p$. Computing the self-intersection of each irreducible component we easily see that this model coincides with the minimal regular model $\mathcal{C}^{\min}$. Theorem 2.4.22 also describes the action of the Galois group $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ on the special fibre $\mathcal{C}_s^{\min}$ of $\mathcal{C}^{\min}$. If the roots of $x^2 + ax + b \mod p$ are in $\mathbb{F}_p$ then the absolute Galois group acts trivially on each component, otherwise it swaps the 2 irreducible components of multiplicity 3 intersecting $\Gamma_{\mathfrak{t}_3}$.

From Theorem 2.1.9 it follows that, for any p, a basis of integral differentials of $C_p/\mathbb{Q}_p^{nr}$ is given by

$$\mu_0 = p^4 \cdot \frac{dx}{2y}, \quad \mu_1 = p^3(x-p) \cdot \frac{dx}{2y}, \quad \mu_2 = p(x-p)x \cdot \frac{dx}{2y}, \quad \mu_3 = (x-p)x^2 \cdot \frac{dx}{2y}.$$

In fact, this is also a basis of integral differentials of $C_p/\mathbb{Q}_p$ since they are all defined over $\mathbb{Q}_p$ (see Proposition A.2.2).

Below we will present related works of other authors concerning regular models and integral differentials of hyperelliptic curves. Note that the example presented here is not covered by [D^{2}M^2] and [Dok] since the curve C_p is not semistable and not Δ_v-regular. In fact, if $p = 3$ the curve C_p does not even have tamely potential semistable reduction. The results in [FN] assume $p > 2$ and C_p with tamely potential semistable reduction, hence they can not be used when $p = 2, 3$. Finally, there is no classification for genus 4 curves.

2.1.4 Related works of other authors

Let K be a discretely valued field with residue field k of characteristic p and let C/K be a hyperelliptic curve of genus g.

In genus 1, when k is perfect, thanks to Tate's algorithm, one can describe the minimal regular model and the space of integral differentials of an elliptic curve C (see for example [Sil2, IV.8.2], [Liu4, Theorem 9.4.35]).

If $K = \mathbb{C}(t)$ and C has genus 2, then Namikawa and Ueno [NU] and Liu [Liu5] give a full classification of the possible configurations of the special fibre of the minimal regular model of C.

If $p \neq 2$, then Liu and Lorenzini show in [LL] that regular models of C can be seen as double cover of well-chosen regular models of $\mathbb{P}^1_K$. Since the latter can be found by using the MacLane valuations ([Mac]) approach in [OW], this argument gives a way to describe any regular model of a hyperelliptic curve. At the moment there is no known closed form description of a regular model based on this approach and it has not been generalised to the $p = 2$ case.

If $p > 2$, k finite, and C is semistable, then in [$\mathrm{D}^2\mathrm{M}^2$] the authors explicitly construct a minimal regular model in terms of the cluster picture of C. Under the same assumptions, Kunzweiler [Kun] gives a basis of integral differentials rephrasing [Kau, Proposition 5.5] in terms of the cluster invariants introduced in [$\mathrm{D}^2\mathrm{M}^2$]. These results can be recovered from Theorem 2.4.22 (see Corollary 2.4.26) and Theorem 2.6.3.

If $p > 2$ and C is semistable over some tamely ramified extension L/K, then Faraggi and Nowell [FN] find the special fibre of the minimal regular model of C with strict normal crossings taking the quotient of the stable model of C_L and resolving the (tame) singularities. However, since they do not describe the charts of the model, their result does not immediately yield all arithmetic invariants, such as a basis of integral differentials.

The last work we want to recall represents an important ingredient of the strategy we will use in this chapter (described more precisely in the next subsection). T. Dokchitser in [Dok] shows that the toric resolution of C gives a regular model in case of Δ_v-regularity ([Dok, Definition 3.9]). This result, used also in [FN], holds for general curves and in any residue characteristic. In his paper, Dokchitser also describes a basis of integral differentials since his model is given as open cover of affine schemes. In Corollary 2.3.25 and Theorem 2.6.1, we will rephrase his results for hyperelliptic curves by using rational cluster picture invariants from §2.3.

2.1.5 Strategy and outline of the chapter

In [Dok], Dokchitser not only describes a regular model of C in case of Δ_v-regularity, but also constructs a proper flat model $\mathcal{C}_\Delta$ without any assumptions on C. Assume C is y-regular and has an almost rational cluster picture over K^{nr} with rational centres $w_1, \ldots, w_m \in K^{nr}$. Our approach to construct the minimal regular model with normal crossings of C is composed by the following steps:

- Consider the x-translated hyperelliptic curves $C^{w_h}/K^{nr} : y^2 = f(x+w_h)$, for $h = 1,\dots,m$. For each h, [Dok, Theorem 3.14] constructs a proper flat model $\mathcal{C}_\Delta^{w_h}$, possibly singular.
- We glue regular open subschemes of these models along common opens, and show that the result is a proper flat regular model $\mathcal{C}$ of $C_{K^{nr}}$ with strict normal crossings.
- We give a complete description of what components of the special fibre of $\mathcal{C}$ have to be blown down to obtain the minimal model with normal crossings $\mathcal{C}^{\min}$ of $C_{K^{nr}}$.
- Finally, we describe the action of the absolute Galois group G_k of k on the special fibre of $\mathcal{C}^{\min}$.

We will explicitly describe both the models $\mathcal{C}_\Delta^{w_h}$ and $\mathcal{C}$. This allows us to study the global sections of its relative dualising sheaf $\omega_{\mathcal{C}/O_K}(\mathcal{C})$.

In §2.2, we present some results on Newton polygons used in the following sections. In §2.3, we recall the basic objects and notation of [D^{2}M^2] and define the rational cluster picture. Moreover, we relate it with the notions given in §2.2. This comparison allows us to rephrase the objects in [Dok] in terms of rational clusters invariants in §2.4. In the same section we also state the theorems which describe the special fibres of a proper flat model (Theorem 2.4.18) and of the minimal regular model with normal crossings (Theorem 2.4.22) of C. The construction of these models, from which the two theorems above follow, is presented in §2.5. Finally, in §2.6, Theorems 2.6.3 and 2.6.4 describe a basis of integral differentials of C, in terms of rational clusters invariants defined in §2.3.

2.1.6 Notation

The following is main notation for fields, hyperelliptic curves and Newton polytopes.

K, v	complete field with normalised discrete valuation v
O_K, π, k, p	ring of integers, uniformiser, residue field, char(k)
$\bar{K}, \bar{k}$	fixed algebraic closure of K, residue field of $\bar{K}$
K^{s}, K^{nr}	separable closure, maximal unramified extension of K in $\bar{K}$
$O_{K^{nr}}, k^{\mathrm{s}}$	ring of integers of K^{nr}, residue field of K^{nr}
F	extension of K in $\bar{K}$, unramified in §2.4
G_K, G_k	absolute Galois groups $\mathrm{Gal}(K^{\mathrm{s}}/K), \mathrm{Gal}(k^{\mathrm{s}}/k)$
$f(x)$	$= \sum a_i x^i$, polynomial in $K[x]$, separable from §2.3
$\mathtt{NP}(f)$	Newton polygon of f, lower convex hull of $\{(i, v(a_i)) \mid a_i \neq 0\}$
$f\vert_L, \overline{f\vert_L}$	restriction and reduction of f to an edge L of $\mathtt{NP}(f)$ (2.2.5)
$g(x,y)$	$= y^2 - f(x)$, polynomial in $K[x,y]$ defining C
$f_w(x), f_h(x)$	$= f(x+w), f(x+w_h)$, for a given rational centre w_h
$g_w(x,y), g_h(x,y)$	$= y^2 - f_w(x), y^2 - f_h(x)$
C, C^w	hyperelliptic curve given by $g(x,y)=0$, $g_w(x,y)=0$
Δ^w, Δ_v^w	Newton polytopes attached to C^w as in [Dok, §1.1]
$F_{\mathrm{t}}^w, L_{\mathrm{t}}^w, V_{\mathrm{t}}^w, V_0^w$	v-faces and v-edges of Δ^w (2.4.4)

For a separable polynomial $f \in k[x]$ or a hyperelliptic curve $C/K : y^2 = f(x)$ as above, the following is the main notation for clusters.

$c_f, \mathfrak{R}$	leading coefficient and set of roots of f
Σ_f, Σ_C	cluster picture, the set of clusters of f,C (2.3.2)
$\mathfrak{s} \in \Sigma_C$	cluster, $\mathfrak{s} = \mathcal{D} \cap \mathfrak{R}$, for a v-adic disc $\mathcal{D}$ (2.3.1)
$G_\mathfrak{s}, K_\mathfrak{s}, k_\mathfrak{s}$	$G_\mathfrak{s} = \mathrm{Stab}_{G_K}(\mathfrak{s})$; $K_\mathfrak{s} = (K^{\mathrm{s}})^{G_\mathfrak{s}}$; $k_\mathfrak{s}$ residue field of $K_\mathfrak{s}$
$d_\mathfrak{s}$	$= \min_{r,r'\in\mathfrak{s}} v(r-r')$ is the depth of a cluster $\mathfrak{s}$ (2.3.1)
$\mathfrak{s}' < \mathfrak{s} = P(\mathfrak{s}')$	$\mathfrak{s}'$ is a child of $\mathfrak{s}$ and $\mathfrak{s}$ is the parent of $\mathfrak{s}'$ (2.3.3)
$\mathfrak{s} \wedge \mathfrak{t}$	smallest cluster containing $\mathfrak{s}$ and $\mathfrak{t}$ (2.3.3)
$\rho_\mathfrak{s}$	$= \max_{w\in F} \min_{r\in\mathfrak{s}} v(r-w)$, radius of $\mathfrak{s} \in \Sigma_{C_F}$ (2.3.8, 2.4.6)
$b_\mathfrak{s}$	denominator of $\rho_\mathfrak{s}$ (2.4.6)
$w_\mathfrak{s}$	rational centre of $\mathfrak{s}$ (2.3.8)
$\epsilon_\mathfrak{s}$	$= v(c_f) + \sum_{r\in\mathfrak{R}} \rho_{r\wedge\mathfrak{s}}$ (2.3.19, 2.4.6)
$\Sigma_f^{\mathrm{rat}}, \Sigma_C^{\mathrm{rat}}$	rational cluster picture (2.3.9)
$\mathfrak{s} \in \Sigma_C^{\mathrm{rat}}$	rational cluster (2.3.9)
Σ_F	$= \Sigma_{C_F}^{\mathrm{rat}}$, for some extension F/K (2.4.6)
Σ_f^z, Σ_C^z	cluster picture centred at z (2.3.34)
$\mathfrak{s} \in \Sigma_C^z$	cluster centred at z (2.3.33)
$\rho_\mathfrak{s}^z, \epsilon_\mathfrak{s}^z$	$\rho_\mathfrak{s}^z = \min_{r\in\mathfrak{s}} v(r-z)$, $\epsilon_\mathfrak{s}^z = v(c_f) + \sum_{r\in\mathfrak{R}} \rho_{r\wedge\mathfrak{s}}^z$ (2.3.35)
Σ^W, $\mathring{\Sigma}$	$\Sigma^W = \bigcup_{w\in W} \Sigma_C^w$, $\mathring{\Sigma} \subset \Sigma_{K^{nr}}$ non-removable clusters (2.4.19)
w_{hl}	$= w_h - w_l$ for fixed rational centres w_h, w_l (§2.5.1)
u_{hl}, ρ_{hl}	$u_{hl} \in O_K^\times$, $\rho_{hl} \in \mathbb{Z}$ such that $w_{hl} = u_{hl}\pi^{\rho_{hl}}$; $u_{hh} = 0$ (§2.5.1)
$D_\mathfrak{s}, m_\mathfrak{s}$	$D_\mathfrak{s} = 1$ if $b_\mathfrak{s}\epsilon_\mathfrak{s}$ odd, 2 if $b_\mathfrak{s}\epsilon_\mathfrak{s}$ even; $m_\mathfrak{s} = (3-D_\mathfrak{s})b_\mathfrak{s}$ (2.4.6)
$p_\mathfrak{s}$	$= 1$ if $\lvert\mathfrak{s}\rvert$ is odd, 2 if $\lvert\mathfrak{s}\rvert$ is even (2.4.6)
$\gamma_\mathfrak{s}$	$= 2$ if $\lvert\mathfrak{s}\rvert$ is even and $\epsilon_\mathfrak{s} - \lvert\mathfrak{s}\rvert\rho_\mathfrak{s}$ is odd, 1 otherwise (2.4.6)
$p_\mathfrak{s}^0$	$= 1$ if $\mathfrak{s}$ is minimal and $\mathfrak{s} \cap K_\mathfrak{s} \neq \varnothing$, 2 otherwise (2.4.6)
$\gamma_\mathfrak{s}^0$	$= 2$ if $p_\mathfrak{s}^0 = 2$ and $\epsilon_\mathfrak{s}$ is odd, 1 otherwise (2.4.6)
$s_\mathfrak{s}$, $s_\mathfrak{s}^0$	$s_\mathfrak{s} = \frac{1}{2}(\lvert\mathfrak{s}\rvert\rho_\mathfrak{s} + p_\mathfrak{s}\rho_\mathfrak{s} - \epsilon_\mathfrak{s})$, $s_\mathfrak{s}^0 = -\epsilon_\mathfrak{s}/2 + \rho_\mathfrak{s}$ (2.4.6)
$\overline{g_\mathfrak{s}}, \overline{g_\mathfrak{s}^0}, \overline{f_\mathfrak{s}^W}, \overline{f_\mathfrak{s}}, \tilde{f}_\mathfrak{s}$	polynomials in one variable over $k_\mathfrak{s}$ (2.4.14, 2.4.21)

2.2 Newton polygon

Let K be a complete field with a normalised valuation v, ring of integers O_K, uniformiser π, and residue field k of characteristic p. We fix $\bar{K}$, an algebraic closure of K, of residue field $\bar{k}$, and we denote by K^{s} the separable closure of K in $\bar{K}$. Denote by K^{nr} the maximal unramified extension of K in K^{s}, by $O_{K^{nr}}$ its ring of integers, and by k^{s} its residue field. Note that k^{s} is the separable closure of k in $\bar{k}$. Extend the valuation v to $\bar{K}$. Finally, write G_K, G_k for the Galois groups $\mathrm{Gal}(K^{\mathrm{s}}/K)$, $\mathrm{Gal}(k^{\mathrm{s}}/k)$, respectively.

Notation 2.2.1 Let $O_{\bar{K}} = \{a \in \bar{K} \mid v(a) \geq 0\}$. Throughout this thesis, given an element $a \in O_{\bar{K}}$, we will write $a \mod \pi$ for the reduction of a in $\bar{k}$. Similarly, given a polynomial $h \in O_{\bar{K}}[x_1, \ldots, x_n]$, namely $h = \sum a_{i_1,\ldots,i_n} \cdot x_1^{i_1} \cdots x_n^{i_n}$, we will write $h \mod \pi$ for the polynomial $\sum (a_{i_1,\ldots,i_n} \mod \pi) \cdot x_1^{i_1} \cdots x_n^{i_n} \in \bar{k}[x_1,\ldots,x_n]$.

Let $f \in K[x]$ be a non-zero polynomial of degree d, say

$$f(x) = \sum_{i=0}^{d} a_i x^i.$$

The *Newton polygon* of f, denoted $\mathtt{NP}(f)$, is

$$\mathtt{NP}(f) = \text{lower convex hull } \{(i, v(a_i)) \mid i = 0,\ldots,d,\, a_i \neq 0\} \subset \mathbb{R}^2.$$

We recall the following well-known result (see for example [Neu, II.6.3,6.4]).

Theorem 2.2.2 *Let $i_0 < \ldots < i_s = d$ be the set of indices in $\{0,\ldots,d\}$ such that the points $(i_0, v(a_{i_0})),\ldots,(i_s, v(a_{i_s}))$ are the vertices of $\mathtt{NP}(f)$. For any $j = 1,\ldots,s$, denote by ρ_j the slope of the edge of $\mathtt{NP}(f)$ which links the points $(i_{j-1}, v(a_{i_{j-1}}))$ and $(i_j, v(a_{i_j}))$. Then f has a unique factorisation over K as a product*

$$f = a_d \cdot g_0 \cdot g_1 \cdots g_s,$$

where $g_0 = x^{i_0}$ and, for all $j = 1,\ldots,s$,

- *the polynomials $g_j \in K[x]$ are monic of degree $d_j = i_j - i_{j-1}$,*
- *all the roots of g_j have valuation $-\rho_j$ in $\bar{K}$.*

In particular, $\mathtt{NP}(g_j)$ is a segment of slope $-\rho_j$.

Corollary 2.2.3 *With the notation of Theorem 2.2.2, the polynomial f has exactly d_j roots of valuation $-\rho_j$ for all $j = 1,\ldots,s$.*

Corollary 2.2.4 *If $f = \sum a_i x^i$ is irreducible of degree d and $a_0 \neq 0$, then $\mathtt{NP}(f)$ is a segment linking the points $(0, v(a_0))$ and $(d, v(a_d))$.*

Definition 2.2.5 (Restriction and reduction) Let $f = \sum_{i=0}^{d} a_i x^i \in K[x]$ and consider an edge L of its Newton polygon $\mathtt{NP}(f)$. Let $(i_1, v(a_{i_1})), (i_2, v(a_{i_2}))$, $i_1 < i_2$ be the two endpoints of L. Denote by ρ the slope of L and by n the denominator of ρ. Define the *restriction* of f to L as

$$f|_L := \sum_{i=0}^{(i_2-i_1)/n} a_{ni+i_1} x^i \in K[x].$$

Moreover we define the *reduction* of f with respect to L to be the polynomial

$$\overline{f|_L} := \pi^{-c} f|_L(\pi^{-n\rho} x) \bmod \pi \in k[x],$$

where $c = v(a_{i_1}) = v(a_{i_2}) + (i_1 - i_2)\rho$.

Remark 2.2.6. These definitions coincide with the ones given in [Dok, Definitions 3.4, 3.5] when the number of variables is 1 (for suitable choices of basis of the lattices used in the definitions).

Until the end of the section let $f \in K[x]$, consider a factorisation $f = a_d \cdot g_0 \cdot g_1 \cdots g_s$ as in Theorem 2.2.2. Denote by L_j the edge of slope ρ_j of $\mathtt{NP}(f)$, for any $j = 1 \dots s$.

Remark 2.2.7. By the lower convexity of $\mathtt{NP}(f)$, for all $j = 1, \dots, s$, note that $\overline{f|_{L_j}} = \bar{c}_j \cdot \overline{g_j|_{\mathtt{NP}(g_j)}}$ for some $\bar{c}_j \in k^\times$. In particular they define the same k-scheme in $\mathbb{G}_{m,k}$. More precisely, for any $j = 1, \dots, s$, let

$$u_j = a_d \cdot \prod_{i=j+1}^{s} g_i(0).$$

Then $\bar{c}_j = u_j/\pi^{v(u_j)} \mod \pi$.

Definition 2.2.8 We say that f is $\mathtt{NP}$-*regular* if the k-scheme

$$X_{L_j} : \{\overline{f|_{L_j}} = 0\} \subset \mathbb{G}_{m,k}$$

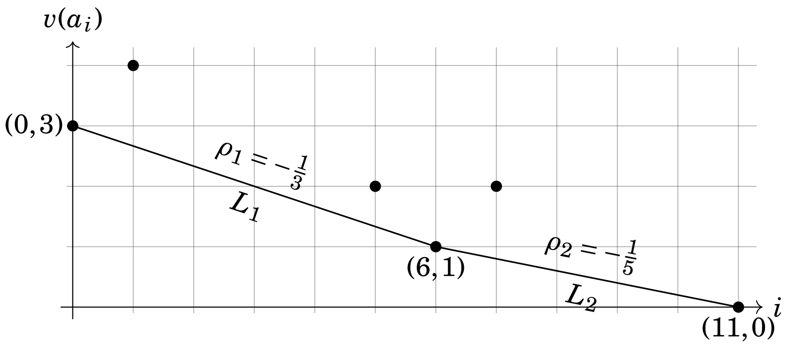
is smooth for all $j = 1, \dots, s$.

Lemma 2.2.9 *The polynomial $f = a_d \cdot g_0 \cdot g_1 \cdots g_s$ is $\mathtt{NP}$-regular if and only if g_j is $\mathtt{NP}$-regular for every $j = 1, \dots, s$.*

Proof. The lemma follows from Remark 2.2.7. □

We conclude this section with two examples.

Example 2.2.10 Let $f = x^{11} + 9x^7 - 3x^6 + 9x^5 + 81x - 27 \in \mathbb{Q}_3[x]$. Then the Newton polygon of f is



Corollary 2.2.3 implies that f has 6 roots of valuation $\frac{1}{3}$ and 5 roots of valuation $\frac{1}{5}$. Furthermore, the two polynomials g_1 and g_2 in the factorisation $f = g_1 \cdot g_2$ of Theorem 2.2.2 turn out to be

$$g_1 = x^6 + 9, \qquad g_2 = x^5 + 9x - 3.$$

Finally,

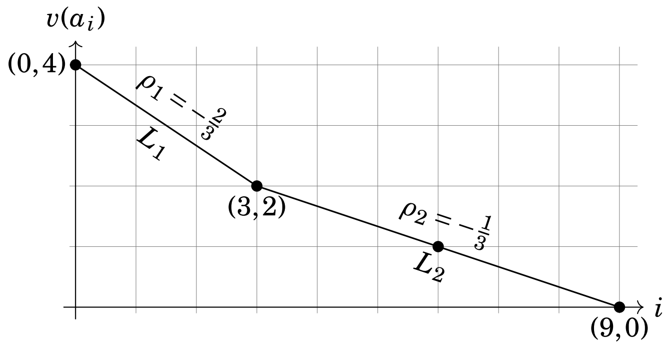
$$f|_{L_1} = -3x^2 - 27 = -3 \cdot g_1|_{\mathtt{NP}(g_1)}, \qquad f|_{L_2} = x - 3 = g_2|_{\mathtt{NP}(g_2)};$$

and

$$\overline{f|_{L_1}} = -x^2 - 1 = -(x^2+1) = -\overline{g_1|_{\mathtt{NP}(g_1)}}, \qquad \overline{f|_{L_2}} = x - 1 = \overline{g_2|_{\mathtt{NP}(g_2)}} \qquad \text{in } \mathbb{F}_3[x].$$

Thus f is $\mathtt{NP}$-regular.

Example 2.2.11 We now show an example of a polynomial that is not $\mathtt{NP}$-regular. Let $f = x^9 + 12x^6 + 36x^3 + 81 \in \mathbb{Q}_3[x]$. Then the Newton polygon of f is



Corollary 2.2.3 implies that f has 3 roots of valuation $\frac{2}{3}$ and 6 roots of valuation $\frac{1}{3}$. Furthermore, the two polynomials g_1 and g_2 in the factorisation $f = g_1 \cdot g_2$ of Theorem 2.2.2 are

$$g_1 = x^3 + 9, \qquad g_2 = x^6 + 3x^3 + 9.$$

Finally,

$$f|_{L_1} = 36x + 81 \qquad f|_{L_2} = x^2 + 12x + 36,$$
$$g_1|_{\mathtt{NP}(g_1)} = x + 9, \qquad g_2|_{\mathtt{NP}(g_2)} = x^2 + 3x + 9;$$

and

$$\overline{f|_{L_1}} = x + 1 = \overline{g_1|_{\mathtt{NP}(g_1)}}, \qquad \overline{f|_{L_2}} = (x+2)^2 = \overline{g_2|_{\mathtt{NP}(g_2)}} \qquad \text{in } \mathbb{F}_3[x].$$

Then f is not $\mathtt{NP}$-regular. In fact, in accordance with Lemma 2.2.9, g_2 is not $\mathtt{NP}$-regular.

2.3 Rational clusters

From now on, let $f \in K[x]$ be a separable polynomial and denote by $\mathfrak{R}$ the set of its roots in K^{s} and by c_f its leading coefficient. Then

$$f(x) = c_f \prod_{r \in \mathfrak{R}} (x - r).$$

Definition 2.3.1 ([D²M², Definition 1.1]) A *cluster* (for f) is a non-empty subset $\mathfrak{s} \subseteq \mathfrak{R}$ of the form $\mathcal{D} \cap \mathfrak{R}$, where $\mathcal{D}$ is a v-adic disc $\mathcal{D} = \{x \in \bar{K} \mid v(x - z) \geq d\}$ for some $z \in \bar{K}$ and $d \in \mathbb{Q}$. If $|\mathfrak{s}| > 1$ we say that $\mathfrak{s}$ is *proper* and define its *depth* $d_{\mathfrak{s}}$ to be

$$d_{\mathfrak{s}} = \min_{r, r' \in \mathfrak{s}} v(r - r').$$

Note that every proper cluster is cut out by a disc of the form

$$\mathcal{D} = \{x \in \bar{K} \mid v(x-r) \geq d_{\mathfrak{s}}\}$$

for any $r \in \mathfrak{s}$.

Definition 2.3.2 ([D²M², Definition 1.26]) The *cluster picture* of f is the collection of its clusters, together with their depths.

We denote by Σ_f the set of all clusters of f and by $\mathring{\Sigma}_f$ the subset of Σ_f of proper clusters.

Definition 2.3.3 ([D²M², Definition 1.3]) If $\mathfrak{s}' \subsetneq \mathfrak{s}$ is maximal subcluster, then we say that $\mathfrak{s}'$ is a *child* of $\mathfrak{s}$ and $\mathfrak{s}$ is the *parent* of $\mathfrak{s}'$, and we write $\mathfrak{s}' < \mathfrak{s}$. Since every cluster $\mathfrak{s} \neq \mathfrak{R}$ has one and only one parent we write $P(\mathfrak{s})$ to refer to the unique parent of $\mathfrak{s}$.

We say that a proper cluster $\mathfrak{s}$ is *minimal* if it does not have any proper child.

For two clusters (or roots) $\mathfrak{s}_1, \mathfrak{s}_2$, we write $\mathfrak{s}_1 \wedge \mathfrak{s}_2$ for the smallest cluster that contains them.

Definition 2.3.4 ([D²M², Definition 1.4]) A cluster $\mathfrak{s}$ is *odd/even* if its size is odd/even. If $|\mathfrak{s}| = 2$, then we say $\mathfrak{s}$ is a *twin*. A cluster $\mathfrak{s}$ is *übereven* if it has only even children.

Definition 2.3.5 ([D²M², Definition 1.9]) A *centre* $z_{\mathfrak{s}}$ of a proper cluster $\mathfrak{s}$ is any element $z_{\mathfrak{s}} \in K^{\mathrm{s}}$ such that $\mathfrak{s} = \mathcal{D} \cap \mathfrak{R}$, where

$$\mathcal{D} = \{x \in \bar{K} \mid v(x - z_{\mathfrak{s}}) \geq d_{\mathfrak{s}}\}.$$

Equivalently, $v(r - z_{\mathfrak{s}}) \geq d_{\mathfrak{s}}$ for all $r \in \mathfrak{s}$. The *centre* of a non-proper cluster $\mathfrak{s} = \{r\}$ is r.

Definition 2.3.6 ([D²M², Definition 1.6]) For a proper cluster $\mathfrak{s}$ set

$$\nu_{\mathfrak{s}} := v(c_f) + \sum_{r \in \mathfrak{R}} d_{r \wedge \mathfrak{s}}.$$

Definition 2.3.7 We say that Σ_f is *nested* if one of the following equivalent conditions is satisfied:

(i) there exists $z \in K^{\mathrm{s}}$ such that z is a centre for all proper clusters $\mathfrak{s} \in \Sigma_f$;

(ii) there is only one minimal cluster in Σ_f;

(iii) every non-minimal proper cluster has exactly one proper child.

Definition 2.3.8 A *rational centre* of a cluster $\mathfrak{s}$ is any element $w_{\mathfrak{s}} \in K$ such that

$$\min_{r \in \mathfrak{s}} v(r - w_{\mathfrak{s}}) = \max_{w \in K} \min_{r \in \mathfrak{s}} v(r - w).$$

If $\mathfrak{s} = \{r\}$, with $r \in K$, then $w_{\mathfrak{s}} = r$.

If $w_{\mathfrak{s}}$ is a rational centre of a proper cluster $\mathfrak{s}$, we define the *radius* of $\mathfrak{s}$ to be

$$\rho_{\mathfrak{s}} = \min_{r \in \mathfrak{s}} v(r - w_{\mathfrak{s}}).$$

Definition 2.3.9 A *rational cluster* is a cluster cut out by a v-adic disc of the form $\mathcal{D} = \{x \in \bar{K} \mid v(x-w) \geq d\}$ with $w \in K$ and $d \in \mathbb{Q}$.

The *rational cluster picture* is the collection of all rational clusters of f together with their radii.

We denote by $\Sigma_f^{\mathrm{rat}} \subseteq \Sigma_f$ the set of rational clusters and by $\mathring{\Sigma}_f^{\mathrm{rat}}$ the subset of Σ_f^{rat} of proper rational clusters.

Lemma 2.3.10 *Let $\mathfrak{s}$ be a proper cluster. Then $d_{\mathfrak{s}} \geq \rho_{\mathfrak{s}}$.*

Proof. First we want to show that

$$\min_{r,r' \in \mathfrak{s}} v(r-r') = \max_{z \in K^{\mathrm{s}}} \min_{r \in \mathfrak{s}} v(r-z).$$

Clearly $\min_{r,r' \in \mathfrak{s}} v(r-r') \leq \max_{z \in K^{\mathrm{s}}} \min_{r \in \mathfrak{s}} v(r-z)$. Let $z_{\mathfrak{s}} \in K^{\mathrm{s}}$ such that

$$\max_{z \in K^{\mathrm{s}}} \min_{r \in \mathfrak{s}} v(r-z) = \min_{r \in \mathfrak{s}} v(r-z_{\mathfrak{s}}).$$

Then, for any $r, r' \in \mathfrak{s}$, one has

$$v(r-r') \geq \min\{v(r-z_{\mathfrak{s}}), v(r'-z_{\mathfrak{s}})\} \geq \min_{r \in \mathfrak{s}} v(r-z_{\mathfrak{s}}),$$

and so

$$\min_{r,r' \in \mathfrak{s}} v(r-r') \geq \max_{z \in K^{\mathrm{s}}} \min_{r \in \mathfrak{s}} v(r-z),$$

as required. From

$$d_{\mathfrak{s}} = \min_{r,r' \in \mathfrak{s}} v(r-r') = \max_{z \in K^{\mathrm{s}}} \min_{r \in \mathfrak{s}} v(r-z) \geq \max_{w \in K} \min_{r \in \mathfrak{s}} v(r-w) = \rho_{\mathfrak{s}},$$

the lemma follows. □

Definition 2.3.11 Given a proper cluster $\mathfrak{s} \in \Sigma_f$, we define the *rationalisation* $\mathfrak{s}^{\mathrm{rat}}$ of $\mathfrak{s}$ to be the smallest rational cluster containing $\mathfrak{s}$. By definition

$$\mathfrak{s}^{\mathrm{rat}} = \mathfrak{R} \cap \{x \in \bar{K} \mid v(x-w_{\mathfrak{s}}) \geq \rho_{\mathfrak{s}}\},$$

where $w_{\mathfrak{s}}$ is a rational centre of $\mathfrak{s}$ and $\rho_{\mathfrak{s}}$ is its radius.

Lemma 2.3.12 *Let $\mathfrak{s} \in \Sigma_f^{\mathrm{rat}}$ be a proper cluster with rational centre $w_{\mathfrak{s}}$. Let $\mathfrak{s}' \in \Sigma_f^{\mathrm{rat}}$ be the child of $\mathfrak{s}$ with rational centre $w_{\mathfrak{s}}$ (let $\mathfrak{s}' = \varnothing$ if it does not exist). Then $(|\mathfrak{s}| - |\mathfrak{s}'|)\rho_{\mathfrak{s}} \in \mathbb{Z}$.*

Proof. As $\mathfrak{s} \in \Sigma_f^{\mathrm{rat}}$, one has $\mathfrak{s} = \mathfrak{s}^{\mathrm{rat}}$. Let $b_{\mathfrak{s}}$ be the denominator of $\rho_{\mathfrak{s}}$. Then $b_{\mathfrak{s}}$ divides the degree of the minimal polynomial of r, for any $r \in \mathfrak{s}$ satisfying $v(w_{\mathfrak{s}} - r) = \rho_{\mathfrak{s}}$. Then $(|\mathfrak{s}| - |\mathfrak{s}'|)\rho_{\mathfrak{s}} \in \mathbb{Z}$, where

$$\mathfrak{s}' = \mathfrak{R} \cap \{x \in \bar{K} \mid v(x-w_{\mathfrak{s}}) > \rho_{\mathfrak{s}}\},$$

as required. □

Remark 2.3.13. If a proper cluster $\mathfrak{s} \in \Sigma_f$ satisfies $d_{\mathfrak{s}} = \rho_{\mathfrak{s}}$, then a rational centre $w_{\mathfrak{s}} \in K$ of its is also a centre. Hence $\mathfrak{s}$ is a rational cluster and, in particular, is G_K-invariant. On the other hand, if a proper cluster $\mathfrak{s} \in \Sigma_f$ is G_K-invariant and $K(\mathfrak{s})/K$ is tamely ramified, then $\mathfrak{s}$ has a centre $z_{\mathfrak{s}} \in K$ by [D²M², Lemma B.1]. Thus $\rho_{\mathfrak{s}} = d_{\mathfrak{s}}$ and $\mathfrak{s} \in \Sigma_f^{\mathrm{rat}}$.

Lemma 2.3.14 *Let $\mathfrak{s}$ be a proper cluster with rational centre $w_{\mathfrak{s}}$ and let $\mathfrak{t} \in \Sigma_f$ satisfying $\mathfrak{t} \supseteq \mathfrak{s}$. Then $w_{\mathfrak{s}}$ is a rational centre of $\mathfrak{t}$ and $\rho_{\mathfrak{t}} \le \rho_{\mathfrak{s}}$. Furthermore, if $\mathfrak{s}$ is a rational cluster and $\mathfrak{t} \supsetneq \mathfrak{s}$, then $\rho_{\mathfrak{t}} < \rho_{\mathfrak{s}}$.*

Proof. It suffices to prove the lemma for $\mathfrak{t} = P(\mathfrak{s})$. Hence we first want to show that $\min_{r \in P(\mathfrak{s})} v(r - w_{\mathfrak{s}}) = \rho_{P(\mathfrak{s})}$ and $\rho_{P(\mathfrak{s})} \le \rho_{\mathfrak{s}}$. Note that

$$\min_{r \in P(\mathfrak{s})} v(r - w_{\mathfrak{s}}) \le \max_{w \in K} \min_{r \in P(\mathfrak{s})} v(r - w) = \rho_{P(\mathfrak{s})}.$$

Moreover,

$$\rho_{P(\mathfrak{s})} = \max_{w \in K} \min_{r \in P(\mathfrak{s})} v(r - w) \le \max_{w \in K} \min_{r \in \mathfrak{s}} v(r - w) = \rho_{\mathfrak{s}}.$$

If $r \in \mathfrak{s}$ then $v(w_{\mathfrak{s}} - r) \ge \rho_{\mathfrak{s}}$, by definition of $\rho_{\mathfrak{s}}$. On the other hand, if $r \in P(\mathfrak{s}) \setminus \mathfrak{s}$ then fixing $r' \in \mathfrak{s}$ we have

$$v(r - w_{\mathfrak{s}}) = v(r - r' + r' - w_{\mathfrak{s}}) \ge \min\{v(r - r'), v(r' - w_{\mathfrak{s}})\} \ge \min\{d_{P(\mathfrak{s})}, \rho_{\mathfrak{s}}\} \ge \rho_{P(\mathfrak{s})},$$

by the previous lemma. Thus $\min_{r \in P(\mathfrak{s})} v(r - w_{\mathfrak{s}}) = \rho_{P(\mathfrak{s})}$, as required.

Now suppose $\mathfrak{s} \in \Sigma_f^{\mathrm{rat}}$ with $\mathfrak{t} \supsetneq \mathfrak{s}$. From Definition 2.3.8, it follows that

$$\{x \in \bar{K} \mid v(x - w_{\mathfrak{s}}) \ge \rho_{\mathfrak{s}}\} \cap \mathfrak{R} = \mathfrak{s} \subsetneq \mathfrak{t} \subseteq \{x \in \bar{K} \mid v(x - w_{\mathfrak{s}}) \ge \rho_{\mathfrak{t}}\} \cap \mathfrak{R},$$

as $w_{\mathfrak{s}}$ is a rational centre of $\mathfrak{t}$. Thus $\rho_{\mathfrak{t}} < \rho_{\mathfrak{s}}$. ☐

Lemma 2.3.15 *Every cluster $\mathfrak{s}$ with $\rho_{\mathfrak{s}} < d_{\mathfrak{s}}$ has no rational subcluster $\mathfrak{s}' \subsetneq \mathfrak{s}$.*

Proof. Suppose by contradiction there exists $\mathfrak{s}' \in \Sigma_C^{\mathrm{rat}}$, $\mathfrak{s}' \subsetneq \mathfrak{s}$, and fix a rational centre $w_{\mathfrak{s}'}$ of $\mathfrak{s}'$. Then $w_{\mathfrak{s}'}$ is a rational centre of $\mathfrak{s}$ by the previous lemma. If $|\mathfrak{s}'| = 1$, then $w_{\mathfrak{s}'}$ is also a centre of $\mathfrak{s}$ and this contradicts $\rho_{\mathfrak{s}} < d_{\mathfrak{s}}$; so assume $\mathfrak{s}'$ proper. Let $r' \in \mathfrak{s}'$ such that $v(r' - w_{\mathfrak{s}'}) = \rho_{\mathfrak{s}'}$ and $r \in \mathfrak{s}$ such that $v(r - w_{\mathfrak{s}'}) = \rho_{\mathfrak{s}}$. But then $d_{\mathfrak{s}} \le v(r - w_{\mathfrak{s}'} + w_{\mathfrak{s}'} - r') = \rho_{\mathfrak{s}}$ again by Lemma 2.3.14. ☐

In particular, the lemma above shows that if $\mathfrak{s} \in \Sigma_f$ and $\mathfrak{s}' \in \Sigma_f^{\mathrm{rat}}$ is a maximal rational subcluster of $\mathfrak{s}$, with $\mathfrak{s}' \subsetneq \mathfrak{s}$, then $\mathfrak{s}'$ is a child of $\mathfrak{s}$. Moreover, the parent of a rational cluster is rational.

Definition 2.3.16 We say that a proper rational cluster $\mathfrak{s} \in \Sigma_f^{\mathrm{rat}}$ is *(rationally) minimal* if it does not have any proper rational child.

Lemma 2.3.17 *Let $\mathfrak{s}, \mathfrak{s}' \in \Sigma_f^{\mathrm{rat}}$ such that $\mathfrak{s}' \not\subseteq \mathfrak{s}$. If $w_{\mathfrak{s}}$ is a rational centre of $\mathfrak{s}$ then*

$$\min_{r \in \mathfrak{s}'} v(r - w_{\mathfrak{s}}) = \rho_{\mathfrak{s} \wedge \mathfrak{s}'}.$$

Proof. By Lemma 2.3.14 we have

$$\min_{r\in\mathfrak{s}\wedge\mathfrak{s}'} v(r-w_{\mathfrak{s}})=\rho_{\mathfrak{s}\wedge\mathfrak{s}'}.$$

Therefore $\min_{r\in\mathfrak{s}'} v(w_{\mathfrak{s}}-r)\geq\rho_{\mathfrak{s}\wedge\mathfrak{s}'}$. Suppose by contradiction that

$$\min_{r\in\mathfrak{s}'} v(r-w_{\mathfrak{s}})=:\rho>\rho_{\mathfrak{s}\wedge\mathfrak{s}'}.$$

It follows from Lemma 2.3.14 that

$$\min_{r\in\mathfrak{s}} v(r-w_{\mathfrak{s}})=\rho_{\mathfrak{s}}>\rho_{\mathfrak{s}\wedge\mathfrak{s}'}$$

as $\mathfrak{s}'\not\subseteq\mathfrak{s}$. But then there exists $\tilde{r}\in(\mathfrak{s}\wedge\mathfrak{s}')\smallsetminus(\mathfrak{s}\cup\mathfrak{s}')$ such that $v(\tilde{r}-w_{\mathfrak{s}})=\rho_{\mathfrak{s}\wedge\mathfrak{s}'}$. Consider the rational cluster

$$\mathfrak{t}:=\mathfrak{R}\cap\{x\in\bar{K}\mid v(x-w_{\mathfrak{s}})\geq\min\{\rho,\rho_{\mathfrak{s}}\}\}\in\Sigma_f^{\mathrm{rat}}.$$

Then $\mathfrak{s},\mathfrak{s}'\subseteq\mathfrak{t}$, but since $\tilde{r}\notin\mathfrak{t}$ we have $\mathfrak{s}\wedge\mathfrak{s}'\not\subseteq\mathfrak{t}$ that contradicts the minimality of $\mathfrak{s}\wedge\mathfrak{s}'$. □

Lemma 2.3.18 *Let $\mathfrak{t}\in\Sigma_f$ with at least two children in Σ_f^{rat}. Then $d_{\mathfrak{t}}=\rho_{\mathfrak{t}}\in\mathbb{Z}$ and $\mathfrak{t}\in\Sigma_f^{\mathrm{rat}}$. More precisely, if $\mathfrak{s},\mathfrak{s}'\in\Sigma_f^{\mathrm{rat}}$ such that $\mathfrak{s}\subsetneq\mathfrak{s}\wedge\mathfrak{s}'\supsetneq\mathfrak{s}'$, then*

$$\rho_{\mathfrak{s}\wedge\mathfrak{s}'}=v(w_{\mathfrak{s}}-w_{\mathfrak{s}'})=d_{\mathfrak{s}\wedge\mathfrak{s}'},$$

where $w_{\mathfrak{s}}$ and $w_{\mathfrak{s}'}$ are rational centres of $\mathfrak{s}$ and $\mathfrak{s}'$ respectively.

Proof. Clearly it suffices to prove the second statement as $v(w_{\mathfrak{s}}-w_{\mathfrak{s}'})\in\mathbb{Z}$. For our assumptions $\mathfrak{s}'\not\subseteq\mathfrak{s}$. Then by Lemma 2.3.17 there exists $r\in\mathfrak{s}'$ so that $v(r-w_{\mathfrak{s}})=\rho_{\mathfrak{s}\wedge\mathfrak{s}'}$. Thus,

$$v(w_{\mathfrak{s}}-w_{\mathfrak{s}'})=\min\{v(w_{\mathfrak{s}}-r),v(r-w_{\mathfrak{s}'})\}=\rho_{\mathfrak{s}\wedge\mathfrak{s}'},$$

as $v(r-w_{\mathfrak{s}'})\geq\rho_{\mathfrak{s}'}>\rho_{\mathfrak{s}\wedge\mathfrak{s}'}$ by Lemma 2.3.14. Finally, $d_{\mathfrak{s}\wedge\mathfrak{s}'}=\rho_{\mathfrak{s}\wedge\mathfrak{s}'}$ follows from Lemma 2.3.15. □

Definition 2.3.19 For a proper cluster $\mathfrak{s}$ set

$$\epsilon_{\mathfrak{s}}:=v(c_f)+\sum_{r\in\mathfrak{R}}\rho_{r\wedge\mathfrak{s}}.$$

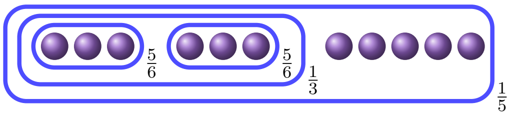
Example 2.3.20 Let $f=x^{11}-3x^6+9x^5-27\in\mathbb{Q}_3[x]$. The set of roots of f is

$$\mathfrak{R}=\{\sqrt[3]{3},\zeta_3\sqrt[3]{3},\zeta_3^2\sqrt[3]{3},-\sqrt[3]{3},-\zeta_3\sqrt[3]{3},-\zeta_3^2\sqrt[3]{3},\sqrt[5]{3},\zeta_5\sqrt[5]{3},\zeta_5^2\sqrt[5]{3},\zeta_5^3\sqrt[5]{3},\zeta_5^4\sqrt[5]{3}\},$$

where ζ_q is a primitive q-th root of unity for $q=3,5$. Then the proper clusters of f are

$$\mathfrak{s}_1=\{\sqrt[3]{3},\zeta_3\sqrt[3]{3},\zeta_3^2\sqrt[3]{3}\},\quad \mathfrak{s}_2=\{-\sqrt[3]{3},-\zeta_3\sqrt[3]{3},-\zeta_3^2\sqrt[3]{3}\},\quad \mathfrak{s}_3=\mathfrak{s}_1\cup\mathfrak{s}_2,\quad \mathfrak{R}$$

with $d_{\mathfrak{s}_1}=d_{\mathfrak{s}_2}=\frac{5}{6}$, $d_{\mathfrak{s}_3}=\frac{1}{3}$ and $d_{\mathfrak{R}}=\frac{1}{5}$. The graphic representation of the cluster picture of f is then



where the subscripts of clusters (represented as circles) are their depths.

Furthermore, note that 0 is a rational centre for all proper clusters and we have $\rho_{\mathfrak{s}_1} = \rho_{\mathfrak{s}_2} = \rho_{\mathfrak{s}_3} = \frac{1}{3}$ and $\rho_{\mathfrak{R}} = \frac{1}{5}$.

Finally, for every cluster $\mathfrak{s}$ we can also compute $\nu_{\mathfrak{s}}$ and $\epsilon_{\mathfrak{s}}$, that are

$$\nu_{\mathfrak{s}_1} = \nu_{\mathfrak{s}_2} = \frac{9}{2}, \quad \nu_{\mathfrak{s}_3} = \epsilon_{\mathfrak{s}_1} = \epsilon_{\mathfrak{s}_2} = \epsilon_{\mathfrak{s}_3} = 3, \quad \nu_{\mathfrak{R}} = \epsilon_{\mathfrak{R}} = \frac{11}{5}.$$

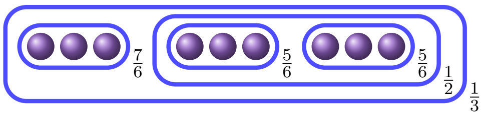
Example 2.3.21 Let $f = x^9 + 12x^6 + 36x^3 + 81 \in \mathbb{Q}_3[x]$ and fix an isomorphism $\overline{\mathbb{Q}}_3 \simeq \mathbb{C}$. Then the set of roots of f is

$$\mathfrak{R} = \{\sqrt[3]{3^2}, \zeta_3\sqrt[3]{3^2}, \zeta_3^2\sqrt[3]{3^2}, \zeta_9\sqrt[3]{3}, \zeta_9^2\sqrt[3]{3}, \zeta_9^4\sqrt[3]{3}, \zeta_9^5\sqrt[3]{3}, \zeta_9^7\sqrt[3]{3}, \zeta_9^8\sqrt[3]{3}\},$$

where $\zeta_q = e^{2\pi i/q}$ is a primitive q-th root of unity for $q = 3, 9$. Then the proper clusters of f are

$$\mathfrak{s}_1 = \{\sqrt[3]{3^2}, \zeta_3\sqrt[3]{3^2}, \zeta_3^2\sqrt[3]{3^2}\}, \quad \mathfrak{s}_2 = \{\zeta_9\sqrt[3]{3}, \zeta_9^4\sqrt[3]{3}, \zeta_9^7\sqrt[3]{3}\},$$
$$\mathfrak{s}_3 = \{\zeta_9^2\sqrt[3]{3}, \zeta_9^5\sqrt[3]{3}, \zeta_9^8\sqrt[3]{3}\}, \quad \mathfrak{s}_4 = \mathfrak{s}_2 \cup \mathfrak{s}_3, \quad \mathfrak{R}$$

with $d_{\mathfrak{s}_1} = \frac{7}{6}$, $d_{\mathfrak{s}_2} = d_{\mathfrak{s}_3} = \frac{5}{6}$, $d_{\mathfrak{s}_4} = \frac{1}{2}$, and $d_{\mathfrak{R}} = \frac{1}{3}$. The cluster picture of f is then



It is easy to see that 0 is a rational centre for all proper clusters and that $\rho_{\mathfrak{s}_1} = \frac{2}{3}$, $\rho_{\mathfrak{s}_2} = \rho_{\mathfrak{s}_3} = \rho_{\mathfrak{s}_4} = \rho_{\mathfrak{R}} = \frac{1}{3}$. Finally,

$$\nu_{\mathfrak{s}_1} = \frac{11}{2}, \quad \nu_{\mathfrak{s}_2} = \nu_{\mathfrak{s}_3} = 5, \quad \nu_{\mathfrak{s}_4} = 4, \quad \nu_{\mathfrak{R}} = 3; \qquad \epsilon_{\mathfrak{s}_1} = 4, \quad \epsilon_{\mathfrak{s}_2} = \epsilon_{\mathfrak{s}_3} = \epsilon_{\mathfrak{s}_4} = \epsilon_{\mathfrak{R}} = 3.$$

The goal of this section is to describe the $\mathtt{NP}$-regularity of $f \in K[x]$ in terms of conditions on its cluster picture.

Notation 2.3.22 If $p > 0$, we denote by $|\cdot|_p$ the standard p-adic absolute value attached to $\mathbb{Q}$, i.e. $|a|_p = p^{-v_p(a)}$ for all $a \in \mathbb{Q}$. If $p = 0$, then we write $|\cdot|_p$ for the function on $\mathbb{Q}$ identically equal to 1, i.e. $|a|_p = 1$ for all $a \in \mathbb{Q}$.

Lemma 2.3.23 *Suppose that $x \nmid f$ and that $\mathtt{NP}(f)$ is a segment L of slope $-\rho$. Let n be the denominator of ρ. Then f is $\mathtt{NP}$-regular if and only if all proper clusters $\mathfrak{s} \in \Sigma_f$ with $|\mathfrak{s}| > |\rho|_p$ satisfy $d_{\mathfrak{s}} = \rho$.*

More precisely:

(i) *If* $\mathfrak{s} \in \mathring{\Sigma}_f$ *with* $|\mathfrak{s}| > |\rho|_p$ *but* $d_{\mathfrak{s}} > \rho$*, then* $\overline{f|_L}$ *has a non-zero multiple root* $\bar{u} = \frac{r^n}{\pi^{n\rho}} \mod \pi$*, for some (any)* $r \in \mathfrak{s}$*.*

(ii) *The multiplicity of a root* $\bar{u} \in \bar{k}^\times$ *of* $\overline{f|_L}$ *equals* $|\mathfrak{s}^0|/n$*, where*

$$\mathfrak{s}^0 = \left\{ r \in \mathfrak{R} \mid \bar{u} = \frac{r^n}{\pi^{n\rho}} \mod \pi \right\}.$$

(iii) *All multiple roots of* $\overline{f|_L}$ *come from clusters* $\mathfrak{s}$ *as described in (i).*

Proof. Let q be the highest power of p dividing n (set $q = 1$ if $p = 0$). Let $m = n/q$ so that $p \nmid m$. Let $\mathfrak{R} = \{r_i \mid i = 1, \ldots, D\}$ be the (multi-)set of roots of f, where $D := \deg f$. Fix some choice of $\sqrt[n]{\pi}$ and define $\bar{u}_i \in \bar{k}^\times$ as $\bar{u}_i = r_i/\pi^\rho \mod \pi$, for all $i = 1, \ldots, D$. Firstly, note that there exists a proper cluster $\mathfrak{s}$ with $|\mathfrak{s}| > |\rho|_p$ and $d_{\mathfrak{s}} > \rho$ if and only if there exists a subset $I \subseteq \{1, \ldots, D\}$ of size $|I| > q$ such that $\bar{u}_{i_1} = \bar{u}_{i_2}$ for all $i_1, i_2 \in I$. Indeed, given $\mathfrak{s}$, then $I = \{i \in \{1, \ldots, D\} \mid r_i \in \mathfrak{s}\}$, while given I, then $\mathfrak{s} = \{r_i \mid \bar{u}_i = \bar{u}_{i_0}$, for any $i_0 \in I\}$. Secondly, recall that f is not NP-regular if and only if $\overline{f|_L}$ has a multiple root in $\bar{k}^\times$. Therefore we will prove that $\overline{f|_L}$ has a non-zero multiple root if and only if there exists a subset $I \subseteq \{1, \ldots, D\}$ with size $|I| > q$ and such that $\bar{u}_{i_1} = \bar{u}_{i_2}$ for all $i_1, i_2 \in I$.

Note that for the lower convexity of $\mathrm{NP}(f) = L$, we have

$$\overline{f|_L}(x^n) = \pi^{-(v(c_f) + D\rho)} f(\pi^\rho x) \mod \pi.$$

Hence $\{\bar{u}_i \mid i = 1, \ldots, D\}$ is the multiset of roots of $\overline{f|_L}(x^n)$. Then there exists an n-to-1 map

$$\begin{aligned} \bar{\phi} \colon \{\bar{u}_i\} &\longrightarrow \{\bar{w}_j\}\,, \\ \bar{u}_i &\longmapsto \bar{u}_i^m \end{aligned}$$

where $\{\bar{w}_j \mid j = 1, \ldots, D/n\}$ is the multiset of roots of $\overline{f|_L}$. Note that $\bar{w}_j \neq 0$ for all $j = 1, \ldots, D/n$, so all roots of $\overline{f|_L}$ are non-zero.

Now, suppose that f is not NP-regular. We want to show that there exists a subset $I \subset \{1, \ldots, D\}$ with $|I| > q$ such that $\bar{u}_{i_1} = \bar{u}_{i_2}$ for all $i_1, i_2 \in I$. Since f is not NP-regular, its reduction $\overline{f|_L}$ has a (non-zero) multiple root. Then there exist $j_1, j_2 \in \{1, \ldots, D/n\}$ so that $\bar{w}_{j_1} = \bar{w}_{j_2} =: \bar{w}$. Hence, by the definition of $\bar{\phi}$, for some (any) $\bar{u} \in \bar{\phi}^{-1}(\bar{w})$, there are at least $2q$ $\bar{u}_i$'s with $\bar{u}_i = \bar{u}$. Let I denote the set of their indices. Then $|I| \geq 2q > q$ and $\bar{u}_{i_1} = \bar{u}_{i_2}$ for all $i_1, i_2 \in I$, as required.

On the other hand, suppose that there exists a subset $I \subset \{1, \ldots, D\}$ with $|I| > q$ and such that $\bar{u}_{i_1} = \bar{u}_{i_2}$ for all $i_1, i_2 \in I$. We want to show that $\overline{f|_L}$ has a multiple root, that is there exist two indices $j_1, j_2 \in \{1, \ldots, D/n\}$ such that $\bar{w}_{j_1} = \bar{w}_{j_2}$. Suppose not and let $j \in \{1, \ldots, D/n\}$ such that $\bar{w}_j = \bar{u}_i^m = \bar{\phi}(\bar{u}_i)$ for some (all) $i \in I$. Then the polynomial $x^n - \bar{w}_j = (x^m - \bar{w}_j)^q \in \bar{k}[x]$, factor of $\overline{f|_L}(x^n)$, should have a root of order $|I| > q$. This would imply $x^m - \bar{w}_j$ is inseparable, a contradiction as $p \nmid m$.

The parts (i), (ii) and (iii) of the lemma follow from above:
(i) Given a proper cluster $\mathfrak{s} \in \Sigma_f$ with $|\mathfrak{s}| > |\rho|_p$ and $d_{\mathfrak{s}} > \rho$, we showed that $\overline{f|_L}$ has a non-zero multiple root $\bar{w}_j = \bar{u}_i^n = r_i^n/\pi^{n\rho} \mod \pi$, where r_i is any root in $\mathfrak{s}$.

(ii) By the definition of $\bar{\phi}$, given $\bar{w} \in \bar{k}$, the number of $\bar{w}_j$'s such that $\bar{w}_j = \bar{w}$ equals $|\mathfrak{s}^0|/n$, where $\mathfrak{s}^0 = \{r_i \mid \bar{u}_i^n = \bar{w}\}$.

(iii) Given a (non-zero) multiple root $\bar{w}$ of $\overline{f|_L}$ we showed that there exists $I \subseteq \{1,\dots,D\}$, with $|I| > q$ and $\bar{u}_{i_1} = \bar{u}_{i_2}$ for any $i_1, i_2 \in I$, such that $\bar{u}_i^n = \bar{w}$ for all $i \in I$. The set $\mathfrak{s} = \{r_i \mid \bar{u}_i = \bar{u}_{i_0}$, for any $i_0 \in I\}$ is a proper cluster as in (i). $\square$

Theorem 2.3.24 *Let $w \in K$ and $f_w(x) = f(x+w)$. For all clusters $\mathfrak{s} \in \Sigma_f$ define $\lambda_\mathfrak{s} = \min_{r\in\mathfrak{s}} v(r-w)$, and let b be the denominator of $\lambda_\mathfrak{s}$. Then f_w is* NP*-regular if and only if all proper clusters $\mathfrak{s} \in \Sigma_f$ with $|\mathfrak{s}| > |\lambda_\mathfrak{s}|_p$ have $d_\mathfrak{s} = \lambda_\mathfrak{s}$.*

More precisely:

(i) Let $\mathfrak{s} \in \mathring{\Sigma}_f$ with $|\mathfrak{s}| > |\lambda_\mathfrak{s}|_p$ but $d_\mathfrak{s} > \lambda_\mathfrak{s}$, and let $r \in \mathfrak{s}$ with $v(r-w) = \lambda_\mathfrak{s}$. Then $\overline{f_w|_L}$ has a non-zero multiple root $\bar{u} = \frac{(r-w)^b}{\pi^{b\lambda_\mathfrak{s}}} \mod \pi$, where L is the edge of NP(f_w) *of slope $-\lambda_\mathfrak{s}$.*

(ii) Let L be an edge of NP(f_w) *of slope $-\lambda$. Let l be the denominator of λ. The multiplicity of a root $\bar{u} \in \bar{k}^\times$ of $\overline{f_w|_L}$ equals $|\mathfrak{s}^0|/l$, where*

$$\mathfrak{s}^0 = \left\{r \in \mathfrak{R} \mid v(r-w) = \lambda \quad \textit{and} \quad \bar{u} = \tfrac{(r-w)^l}{\pi^{l\lambda}} \mod \pi\right\}.$$

(iii) For every edge L of NP(f_w)*, the multiple roots of $\overline{f_w|_L}$ come from proper clusters $\mathfrak{s}$ for f as described in (i).*

Proof. Let $\mathfrak{R}_w$ be the set of roots of f_w. Note that we have a natural bijection $\mathfrak{R} \to \mathfrak{R}_w$, $r \mapsto r-w$, which induces a bijective function $\psi : \Sigma_f \to \Sigma_{f_w}$, sending

$$\mathfrak{s} = \mathfrak{R} \cap \{x \in \bar{K} \mid v(x-z) > d\} \quad \mapsto \quad \psi(\mathfrak{s}) = \mathfrak{R}_w \cap \{x \in \bar{K} \mid v(x+w-z) > d\}.$$

In particular, if $\mathfrak{s} \in \Sigma_f$, $|\mathfrak{s}| = |\psi(\mathfrak{s})|$, $d_\mathfrak{s} = d_{\psi(\mathfrak{s})}$ and

$$\lambda_\mathfrak{s} = \min_{r\in\mathfrak{s}} v(r-w) = \min_{r\in\psi(\mathfrak{s})} v(r).$$

Hence it suffices to show the theorem for $w = 0$.

Assume $w = 0$. Let $f = c_f \cdot g_0 \cdot g_1 \dots g_t$ be a factorisation of Theorem 2.2.2. Note that if $t = 0$, then either $f \in K$ or $f \in Kx$. In both cases, f is clearly NP-regular and has no proper clusters. Then assume $t > 0$ and let $-\rho_i$ be the slope of NP(g_i) for any $i = 1,\dots,t$. Denote by $\mathfrak{R}$ the set of roots of f and by $\mathfrak{R}_i$ the set of roots of g_i for $i = 0,\dots,t$. Note that the $\mathfrak{R}_i$'s are pairwise disjoint. From Remark 2.2.7, for every edge L of NP(f) there exists i such that $\overline{f|_L} = \bar{c}_i \cdot \overline{g_i|_{\mathrm{NP}(g_i)}}$ for some $\bar{c}_i \in k^\times$. Hence, by Lemma 2.2.9 and Lemma 2.3.23, we need to prove that there exists a proper cluster $\mathfrak{s} \in \Sigma_f$ such that $|\mathfrak{s}| > |\lambda_\mathfrak{s}|_p$ and $d_\mathfrak{s} > \lambda_\mathfrak{s}$ if and only if for some $i = 1,\dots,t$ there exists a proper cluster $\mathfrak{s}_i \in \Sigma_{g_i}$ such that $|\mathfrak{s}_i| > |\lambda_{\mathfrak{s}_i}|_p = |\rho_i|_p$ and $d_{\mathfrak{s}_i} > \lambda_{\mathfrak{s}_i} = \rho_i$. We will show that one can choose $\mathfrak{s} = \mathfrak{s}_i$.

First, note that if $\mathfrak{s}$ is a proper cluster , then $\mathfrak{s} \not\subseteq \mathfrak{R}_0$, as $|\mathfrak{R}_0| \leq 1$. Furthermore, if $\mathfrak{s} \in \Sigma_f$ contains roots of different valuations, that is $\mathfrak{s} \not\subseteq \mathfrak{R}_i$ for all i, then

$$d_{\mathfrak{s}} = \min_{r,r' \in \mathfrak{s}} v(r-r') = \min_{r \in \mathfrak{s}} v(r) = \lambda_{\mathfrak{s}} = \min\{\rho_i \mid \mathfrak{R}_i \cap \mathfrak{s} \neq \varnothing\}.$$

Now suppose there exists a proper cluster $\mathfrak{s} \in \Sigma_f$ such that $|\mathfrak{s}| > |\lambda_{\mathfrak{s}}|_p$ and $d_{\mathfrak{s}} > \lambda_{\mathfrak{s}}$. For the observation above, the inequality $d_{\mathfrak{s}} > \lambda_{\mathfrak{s}}$ implies that $\mathfrak{s} \subseteq \mathfrak{R}_i$ for some $i = 1, \dots, t$. Let $\mathcal{D}$ be the v-adic disc such that $\mathfrak{s} = \mathcal{D} \cap \mathfrak{R}$. Since $\mathfrak{s} \subseteq \mathfrak{R}_i$, one has $\mathfrak{s} = \mathcal{D} \cap \mathfrak{R}_i$ which means that $\mathfrak{s} \in \Sigma_{g_i}$, as required.

Finally suppose that for some $i = 1, \dots, s$, there exists a proper cluster $\mathfrak{s}_i \in \Sigma_{g_i}$ such that $|\mathfrak{s}_i| > |\rho_i|_p$ and $d_{\mathfrak{s}_i} > \rho_i$. Let $r_i \in \mathfrak{s}_i$. Then

$$\mathfrak{s}_i = \{x \in \bar{K} \mid v(x - r_i) \geq d_{\mathfrak{s}_i}\} \cap \mathfrak{R}_i.$$

Consider the cluster $\mathfrak{s} := \{x \in \bar{K} \mid v(x - r_i) \geq d_{\mathfrak{s}_i}\} \cap \mathfrak{R}$ of f. Clearly $\mathfrak{s}_i \subseteq \mathfrak{s}$. Therefore

$$\lambda_{\mathfrak{s}_i} = \min_{r \in \mathfrak{s}_i} v(r) \geq \min_{r \in \mathfrak{s}} v(r) = \lambda_{\mathfrak{s}},$$

which implies

$$d_{\mathfrak{s}} = d_{\mathfrak{s}_i} > \rho_i = \lambda_{\mathfrak{s}_i} \geq \lambda_{\mathfrak{s}},$$

where $d_{\mathfrak{s}} = d_{\mathfrak{s}_i}$ by construction. Again from the observation above the inequality $d_{\mathfrak{s}} > \lambda_{\mathfrak{s}}$ implies that $\mathfrak{s}$ is contained in $\mathfrak{R}_j$ for some j. As $\mathfrak{s} \cap \mathfrak{R}_i \supseteq \mathfrak{s}_i \cap \mathfrak{R}_i = \mathfrak{s}_i$, we must have $\mathfrak{s} \subseteq \mathfrak{R}_i$. Thus $\mathfrak{s} = \mathfrak{s}_i$, that concludes the proof. □

Corollary 2.3.25 *Let $f \in K[x]$ be a separable polynomial. Let $w \in K$ and $f_w(x) = f(x+w)$. Then f_w is* NP*-regular if and only if all proper clusters $\mathfrak{s} \in \Sigma_f$ have rational centre w and those with $|\mathfrak{s}| > |\rho_{\mathfrak{s}}|_p$ satisfy $d_{\mathfrak{s}} = \rho_{\mathfrak{s}}$.*

Proof. If f_w is NP-regular, then, from the previous theorem, all proper clusters $\mathfrak{s} \in \Sigma_f$ with $|\mathfrak{s}| > |\lambda_{\mathfrak{s}}|_p$ have $d_{\mathfrak{s}} = \lambda_{\mathfrak{s}}$, where $\lambda_{\mathfrak{s}} = \min_{r \in \mathfrak{s}} v(r-w)$. First let $\mathfrak{s} \in \Sigma_f$ proper and assume $|\mathfrak{s}| > |\lambda_{\mathfrak{s}}|_p$. Then

$$d_{\mathfrak{s}} = \lambda_{\mathfrak{s}} = \min_{r \in \mathfrak{s}} v(r-w) \leq \max_{z \in K} \min_{r \in \mathfrak{s}} v(r-z) = \rho_{\mathfrak{s}} \leq d_{\mathfrak{s}},$$

so $d_{\mathfrak{s}} = \lambda_{\mathfrak{s}} = \rho_{\mathfrak{s}}$, and w is a rational centre of $\mathfrak{s}$. Now assume $|\mathfrak{s}| \leq |\lambda_{\mathfrak{s}}|_p$. In particular, $p > 0$ and $\lambda_{\mathfrak{s}} \notin \mathbb{Z}$, and so

$$\min_{r \in \mathfrak{s}} v(r-w) = \lambda_{\mathfrak{s}} \neq v(w - w_{\mathfrak{s}}),$$

where $w_{\mathfrak{s}}$ is a rational centre of $\mathfrak{s}$. Let $r \in \mathfrak{s}$ such that $v(r-w) = \lambda_{\mathfrak{s}}$. Then

$$\rho_{\mathfrak{s}} \leq v(r - w + w - w_{\mathfrak{s}}) = \min\{\lambda_{\mathfrak{s}}, v(w - w_{\mathfrak{s}})\} \leq \lambda_{\mathfrak{s}}.$$

Clearly

$$\rho_{\mathfrak{s}} = \max_{z \in K} \min_{r \in \mathfrak{s}} v(r-z) \geq \min_{r \in \mathfrak{s}} v(r-w) = \lambda_{\mathfrak{s}},$$

that implies $\rho_{\mathfrak{s}} = \lambda_{\mathfrak{s}} = \min_{r\in\mathfrak{s}} v(r-w)$. Hence w is a rational centre of $\mathfrak{s}$.

On the other hand, suppose that all proper clusters $\mathfrak{s}\in\Sigma_f$ have rational centre $w\in K$ and those with $|\mathfrak{s}|>|\rho_{\mathfrak{s}}|_p$ satisfy $d_{\mathfrak{s}}=\rho_{\mathfrak{s}}$. Then $\rho_{\mathfrak{s}} = \min_{r\in\mathfrak{s}} v(r-w)$ for any $\mathfrak{s}\in\Sigma_f$. Thus f_w is NP-regular again by Theorem 2.3.24. □

Definition 2.3.26 We say that f has an *almost rational cluster picture* if all proper clusters $\mathfrak{s}\in\Sigma_f$ with $|\mathfrak{s}|>|\rho_{\mathfrak{s}}|_p$ have $d_{\mathfrak{s}}=\rho_{\mathfrak{s}}$.

In the following we give different characterisations of the previous definition.

Corollary 2.3.27 *Suppose that $K(\mathfrak{R})/K$ is a tamely ramified extension. Then f has an almost rational cluster picture if and only if every proper cluster $\mathfrak{s}\in\Sigma_f$ is G_K-invariant.*

Proof. Since $K(\mathfrak{R})/K$ is tamely ramified, every cluster $\mathfrak{s}\in\Sigma_f$ has $|\rho_{\mathfrak{s}}|_p\leq 1$. Therefore the corollary follows from Remark 2.3.13. □

Corollary 2.3.28 *Suppose that $K(\mathfrak{R})/K$ is a tamely ramified extension. Then f_w is NP-regular for some $w\in K$ if and only if Σ_f is nested.*

Proof. First note that every cluster $\mathfrak{s}\in\Sigma_f$ has $|\rho_{\mathfrak{s}}|_p\leq 1$, as $K(\mathfrak{R})/K$ is tamely ramified. Therefore from Corollary 2.3.25, we need to prove that Σ_f is nested if and only if all clusters $\mathfrak{s}\in\Sigma_f$ have $d_{\mathfrak{s}}=\rho_{\mathfrak{s}}$ and rational centre w, for some $w\in K$. But this follows from Remark 2.3.13. □

Corollary 2.3.29 *The polynomial f has an almost rational cluster picture if and only if for every $r\in\mathfrak{R}\setminus K$, there exists $w\in K$ so that $r_w^b := \frac{(r-w)^b}{\pi^{b\cdot v(r-w)}} \mod \pi$ is a simple root of $f_w|_L$, where b is the denominator of $v(r-w)$, $f_w(x)=f(x+w)$ and L is the edge of $\mathrm{NP}(f_w)$ of slope $-v(r-w)$.*

Proof. Fix $\tilde{r}\in\mathfrak{R}\setminus K$ and let $\mathfrak{s}$ be the smallest proper cluster containing $\tilde{r}$. Let $w_{\mathfrak{s}}$ be a rational centre of $\mathfrak{s}$. Note that $v(\tilde{r}-w_{\mathfrak{s}})=\rho_{\mathfrak{s}}=\min_{r\in\mathfrak{s}} v(r-w_{\mathfrak{s}})$, for the choice of $\mathfrak{s}$, as $\tilde{r}\notin K$. Moreover, for any proper cluster $\mathfrak{t}$ containing $\tilde{r}$, we have $\mathfrak{s}\subseteq\mathfrak{t}$. In particular, $w_{\mathfrak{s}}$ is a rational centre of all such clusters. Let L be the edge of $\mathrm{NP}(f_{w_{\mathfrak{s}}})$ of slope $-\rho_{\mathfrak{s}}$. Theorem 2.3.24 shows that $\tilde{r}_{w_{\mathfrak{s}}}^{b_{\mathfrak{s}}}$ is a multiple root of $f_{w_{\mathfrak{s}}}|_L$ if and only if there exists $\mathfrak{t}\in\Sigma_f$ such that $\tilde{r}\in\mathfrak{t}$, $|\mathfrak{t}|>|\rho_{\mathfrak{t}}|_p$ and $d_{\mathfrak{t}}>\rho_{\mathfrak{t}}$. Therefore if f has an almost rational cluster picture, then $\tilde{r}_{w_{\mathfrak{s}}}^{b_{\mathfrak{s}}}$ is a simple root.

Suppose there exists $\mathfrak{t}\in\Sigma_f$ such that $|\mathfrak{t}|>|\rho_{\mathfrak{t}}|_p$ and $d_{\mathfrak{t}}>\rho_{\mathfrak{t}}$. Then $\mathfrak{t}\cap K=\varnothing$. By Theorem 2.3.24, it remains to show that for any $w\in K$, we have $|\mathfrak{t}|>|\lambda_{\mathfrak{t}}|_p$ and $d_{\mathfrak{t}}>\lambda_{\mathfrak{t}}$, where $\lambda_{\mathfrak{t}}=\min_{r\in\mathfrak{t}} v(r-w)$. First note $d_{\mathfrak{t}}>\rho_{\mathfrak{t}}\geq\lambda_{\mathfrak{t}}$. Moreover, in the proof of Corollary 2.3.25, we saw that $|\mathfrak{t}|\leq|\lambda_{\mathfrak{t}}|_p$ implies $\rho_{\mathfrak{t}}=\lambda_{\mathfrak{t}}$, which contradicts $|\mathfrak{t}|>|\rho_{\mathfrak{t}}|_p$. □

Lemma 2.3.30 *Suppose f has an almost rational cluster picture. Let $\mathfrak{s}\in\Sigma_f$ proper. If $d_{\mathfrak{s}}>\rho_{\mathfrak{s}}$, then $p>0$ and $|\mathfrak{s}|$ is a p-power. In particular, if $w_{\mathfrak{s}}$ is a rational centre of $\mathfrak{s}$, for any $r\in\mathfrak{s}$, the elements $r-w_{\mathfrak{s}}$ are all the roots of a monic polynomial with coefficients in K^{s}, and constant term c such that $|v(c)|_p\geq 1$.*

Proof. Let $\mathfrak{s} \in \Sigma_f$ proper, with $d_\mathfrak{s} > \rho_\mathfrak{s}$. Since f has an almost rational cluster picture, we must have $|\mathfrak{s}| \leq |\rho_\mathfrak{s}|_p$. Since $\mathfrak{s}$ is proper, $p > 0$. Let $b_\mathfrak{s}$ be the denominator of $\rho_\mathfrak{s}$. Then $v_p(b_\mathfrak{s}) > 1$. Fix a rational centre $w_\mathfrak{s}$ of $\mathfrak{s}$ and a root $r \in \mathfrak{s}$ such that $v(r - w_\mathfrak{s}) = \rho_\mathfrak{s}$. Consider $\mathfrak{s}' = \{x \in \mathfrak{R} \mid v(x - r) > \rho_\mathfrak{s}\}$. Then $\mathfrak{s} \subseteq \mathfrak{s}' \leq \mathfrak{s}^{\mathrm{rat}}$ and $|\mathfrak{s}'| \leq |\rho_\mathfrak{s}|_p$ (as $d_{\mathfrak{s}'} > \rho_\mathfrak{s} = \rho_{\mathfrak{s}'}$). Let I_w be the wild inertia subgroup of G_K. As $v(r - w_\mathfrak{s}) = \rho_\mathfrak{s}$ there exist $\sigma_1 = id, \sigma_2, \ldots, \sigma_{|\rho_\mathfrak{s}|_p} \in I_w$ such that $\sigma_i(r) \neq \sigma_j(r)$ if $i \neq j$. Moreover, $v(\sigma_i(r) - r) > \rho_\mathfrak{s}$ from the definition of I_w. Therefore $\sigma_i(r) \in \mathfrak{s}'$ for all i and so $|\rho_\mathfrak{s}|_p \leq |\mathfrak{s}'|$. Thus $|\mathfrak{s}'| = |\rho_\mathfrak{s}|_p$ and $\mathfrak{s} \subseteq \mathfrak{s}' = \{\sigma_i(r) \mid i = 1, \ldots, |\rho_\mathfrak{s}|_p\}$. Finally, as $\mathfrak{s}'$ contains only conjugates of $r \in \mathfrak{s}$, the cluster $\mathfrak{s}'$ is union of orbits of $\mathfrak{s}$. In particular, $|\mathfrak{s}| \mid |\mathfrak{s}'| = |\rho_\mathfrak{s}|_p$, and so $|\mathfrak{s}|$ is a p-power. The rest of the lemma follows. □

Proposition 2.3.31 *The polynomial f has an almost rational cluster picture if and only if for every proper cluster $\mathfrak{s} \in \Sigma_f$ one of the following is satisfied:*

(a) the smallest disc containing $\mathfrak{s}$ also contains a rational point;

(b) $p > 0$ and after a translation by an element of K, the elements in $\mathfrak{s}$ are all the roots of a polynomial with coefficients in K^{s} of p-power degree and constant term c such that $|v(c)|_p \geq 1$.

Proof. First of all note that point (a) is equivalent to requiring $d_\mathfrak{s} = \rho_\mathfrak{s}$. Therefore by Lemma 2.3.30 it only remains to show that if $d_\mathfrak{s} > \rho_\mathfrak{s}$ and (b) is satisfied, then $|\mathfrak{s}| \leq |\rho_\mathfrak{s}|_p$. Let $F \in K^{\mathrm{s}}[x]$ be the polynomial in (b) and let $w \in K$ such that $r - w$, for $r \in \mathfrak{s}$, are all the roots of F. We have $\rho_s \geq \min_{r \in \mathfrak{s}} v(r - w)$. Fix $r \in \mathfrak{s}$ such that $\rho_\mathfrak{s} \geq v(r - w) =: \rho$. Since $d_\mathfrak{s} > \rho_\mathfrak{s} \geq v(r - w)$, we have $v(r' - w) = v(r - w) = \rho$ for any $r' \in \mathfrak{s}$. Then

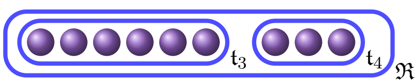
$$|\mathfrak{s}| = \deg F = |1/\deg F|_p \leq |v(c)/\deg F|_p = |\rho|_p.$$

Let $w_\mathfrak{s}$ be a rational centre of $\mathfrak{s}$. Suppose by contradiction that $\rho_\mathfrak{s} > \rho$. Let $r_\mathfrak{s} \in \mathfrak{s}$ such that $v(r_\mathfrak{s} - w_\mathfrak{s}) = \rho_\mathfrak{s}$. Hence

$$v(w - w_\mathfrak{s}) = v(w - r_\mathfrak{s} + r_\mathfrak{s} - w_\mathfrak{s}) = \min\{\rho, \rho_\mathfrak{s}\} = \rho.$$

But then $\rho \in \mathbb{Z}$, which contradicts $|\mathfrak{s}| \leq |\rho|_p$. □

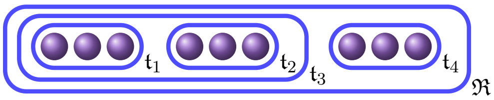
Example 2.3.32 Let p be a prime number and let $a \in \mathbb{Z}_p$, $b \in \mathbb{Z}_p^\times$ such that the polynomial $x^2 + ax + b$ is not a square modulo p. Let $f \in \mathbb{Q}_p[x]$ given by $f(x) = (x^6 + ap^4x^3 + bp^8)((x - p)^3 - p^{11})$. For any prime p the rational cluster picture of f is



where $\rho_{\mathfrak{t}_3} = \frac{4}{3}$, $\rho_{\mathfrak{t}_4} = \frac{11}{3}$, and $\rho_\mathfrak{R} = 1$.

If $p \neq 3$, then the proper clusters of Σ_f coincide with the rational clusters above and $d_\mathfrak{s} = \rho_\mathfrak{s}$ for any $\mathfrak{s} = \mathfrak{t}_3, \mathfrak{t}_4, \mathfrak{R}$. In particular, f has an almost rational cluster picture when $p \neq 3$.

Suppose $p = 3$. Then the cluster picture of f is



where $d_{\mathfrak{t}_1} = d_{\mathfrak{t}_2} = \frac{11}{6}$, $d_{\mathfrak{t}_3} = \rho_{\mathfrak{t}_1} = \rho_{\mathfrak{t}_2} = \frac{4}{3}$, $d_{\mathfrak{t}_4} = \frac{25}{6}$ and $d_{\mathfrak{R}} = 1$. Thus f has an almost rational cluster picture for all p.

We conclude this section by showing that the *cluster picture centred at* 0 completely determines the Newton polygon of f.

Definition 2.3.33 Let $z \in \bar{K}$. A *cluster centred at* z is a cluster cut out by a v-adic disc of the form $\mathcal{D} = \{x \in \bar{K} \mid v(x-z) \geq d\}$ for some $d \in \mathbb{Q}$.

Definition 2.3.34 Let $z \in \bar{K}$. Define Σ_f^z to be the set of all clusters centred at z. Write $\mathring{\Sigma}_f^z$ for the set $\Sigma_f^z \smallsetminus \{\{z\}\}$. Note that Σ_f^z is *nested*, i.e. every cluster $\mathfrak{s} \in \Sigma_f^z$ has at most one child in Σ_f^z.

Definition 2.3.35 Let $z \in \bar{K}$, and let $\mathfrak{s} \in \Sigma_f \setminus \{\{z\}\}$. The *radius of* $\mathfrak{s}$ *with respect to the centre* z is

$$\rho_{\mathfrak{s}}^z = \min_{r \in \mathfrak{s}} v(r-z).$$

The *cluster picture centred at* z of f is the collection of all clusters in Σ_f^z together with their radii with respect to z. Finally set

$$\epsilon_{\mathfrak{s}}^z := v(c_f) + \sum_{r \in \mathfrak{R}} \rho_{r \wedge \mathfrak{s}}^z.$$

Remark 2.3.36. From the definitions above, if $\mathfrak{s}$ is a cluster centred at $z \in K^{\mathrm{s}}$, then $\mathfrak{s} = \mathfrak{R} \cap \{x \in \bar{K} \mid v(x-z) \geq \rho_{\mathfrak{s}}^z\}$. But this does not mean z is a centre for $\mathfrak{s}$, that is false in general. For example, $\mathfrak{R}$ is clearly a cluster centred at any $z \in K^{\mathrm{s}}$, but there are elements of K^{s} which are not centres of $\mathfrak{R}$, e.g. any $z \in K^{\mathrm{s}}$ with valuation $v(z) < \min_{r \in \mathfrak{R}} v(r)$.

Remark 2.3.37. Let $\mathfrak{s} \in \Sigma_f$ be a proper cluster with centre z and rational centre w. Then $\mathfrak{s} \in \Sigma_f^z$, $d_{\mathfrak{s}} = \rho_{\mathfrak{s}}^z$, $\nu_{\mathfrak{s}} = \epsilon_{\mathfrak{s}}^z$, $\rho_{\mathfrak{s}} = \rho_{\mathfrak{s}}^w$, and $\epsilon_{\mathfrak{s}} = \epsilon_{\mathfrak{s}}^w$. Furthermore, $\mathfrak{s} \in \Sigma_f^{\mathrm{rat}}$ if and only if $\mathfrak{s} \in \Sigma_f^w$.

Lemma 2.3.38 *Let $w \in K$ and let $f_w(x) = f(x+w)$. Then there is a 1-to-1 correspondence between the clusters in $\mathring{\Sigma}_f^w$ and the edges of $\mathrm{NP}(f_w)$. More explicitly, let $\mathfrak{s}_1 \subset \cdots \subset \mathfrak{s}_n = \mathfrak{R}$ be the clusters in $\mathring{\Sigma}_f^w$ and let $\mathfrak{s}_0 = \{w\}$ if $\{w\} \in \Sigma_f^w$ or $\mathfrak{s}_0 = \varnothing$ otherwise. Then $\mathrm{NP}(f_w)$ has vertices Q_i, $i = 0, \dots, n$, where*

- $Q_n = (|\mathfrak{R}|, \epsilon_{\mathfrak{R}}^w - |\mathfrak{R}|\rho_{\mathfrak{R}}^w) = (\deg f, v(c_f))$,
- $Q_i = (|\mathfrak{s}_i|, \epsilon_{\mathfrak{s}_i}^w - |\mathfrak{s}_i|\rho_{\mathfrak{s}_i}^w) = (|\mathfrak{s}_i|, \epsilon_{\mathfrak{s}_{i+1}}^w - |\mathfrak{s}_i|\rho_{\mathfrak{s}_{i+1}}^w)$, *for $i = 1, \dots, n-1$,*
- $Q_0 = (|\mathfrak{s}_0|, \epsilon_{\mathfrak{s}_1}^w - |\mathfrak{s}_0|\rho_{\mathfrak{s}_1}^w)$.

and edges L_i, $i = 1, \dots, n$, of slope $-\rho_{\mathfrak{s}_i}^w$ linking Q_{i-1} and Q_i.

Furthermore, for any $i = 1, \dots, n$ we have

$$\overline{f_w|_{L_i}}(x^{b_i}) = \tfrac{u}{\pi^{v(u)}} \textstyle\prod_{r \in \mathfrak{s}_i \setminus \mathfrak{s}_{i-1}} (x + \tfrac{w-r}{\pi^{\rho_i}}) \mod \pi, \qquad u = c_f \prod_{r \in \mathfrak{R} \setminus \mathfrak{s}} (w-r),$$

where $\rho_i = \rho_{\mathfrak{s}_i}^w$, and b_i is the denominator of ρ_i.

Proof. Without loss of generality we can assume $w = 0$ so that $f_w = f$. First note that the coordinates of Q_n are trivial. Now consider a factorisation $f = c_f \cdot g_0 \cdot g_1 \cdots g_s$ of Theorem 2.2.2. Recall the polynomials g_j are monic and $g_0 \mid x$. Let $\mathfrak{R}_j$ be the set of roots of g_j. It follows from the definition of cluster centred at 0 that

$$n = s, \quad \text{and} \quad \mathfrak{s}_i = \bigcup_{j=0}^{i} \mathfrak{R}_j \quad \text{for all } i = 0,\dots,n.$$

Therefore $\mathfrak{s}_0 = \mathfrak{R}_0$ and $\mathfrak{R}_i = \mathfrak{s}_i \setminus \mathfrak{s}_{i-1}$ for any $i = 1,\dots,n$.

Let $i = 1,\dots,n-1$. Then the x-coordinate of Q_i follows as

$$|\mathfrak{s}_i| = \sum_{j=0}^{i} |\mathfrak{R}_j| = \sum_{j=0}^{i} \deg g_j = \deg \prod_{j=0}^{i} g_j.$$

The y-coordinate of Q_i equals the sum of $v(c_f)$ and the valuation of the constant term of $\prod_{j=i+1}^{n} g_j$, so

$$Q_i = \Big(|\mathfrak{s}_i|, v(c_f) + \sum_{j=i+1}^{n} |\mathfrak{R}_j| v(r_j)\Big),$$

where r_j is any root in $\mathfrak{R}_j$. But since $\mathfrak{s}_i = \bigcup_{j=0}^{i} \mathfrak{R}_j$, we have $v(r_j) = \rho^0_{\mathfrak{s}_j}$. Therefore

$$v(c_f) + \sum_{j=i+1}^{n} |\mathfrak{R}_j| v(r_j) = v(c_f) + \sum_{j=i+1}^{n} (|\mathfrak{s}_j| - |\mathfrak{s}_{j-1}|)\rho^0_{\mathfrak{s}_j} = \epsilon^0_{\mathfrak{s}_i} - |\mathfrak{s}_i|\rho^0_{\mathfrak{s}_i}.$$

Moreover,

$$\epsilon^0_{\mathfrak{s}_i} - |\mathfrak{s}_i|\rho^0_{\mathfrak{s}_i} = \epsilon^0_{\mathfrak{s}_{i+1}} - |\mathfrak{s}_i|\rho^0_{\mathfrak{s}_{i+1}}$$

from the easy computation $\epsilon^0_{\mathfrak{s}_i} - \epsilon^0_{\mathfrak{s}_{i+1}} = |\mathfrak{s}_i|\big(\rho^0_{\mathfrak{s}_i} - \rho^0_{\mathfrak{s}_{i+1}}\big)$. Finally the x-coordinate of Q_0 is trivial, while its y-coordinate equals

$$v(c_f) + \sum_{j=1}^{n} |\mathfrak{R}_j| v(r_j) = v(c_f) + \sum_{j=1}^{n} (|\mathfrak{s}_j| - |\mathfrak{s}_{j-1}|)\rho^0_{\mathfrak{s}_j} = \epsilon^0_{\mathfrak{s}_1} - |\mathfrak{s}_0|\rho^0_{\mathfrak{s}_1},$$

that concludes the first part of the proof as $|\mathfrak{s}_0| = |\mathfrak{R}_0| = \deg g_0$.

The computation of $f|_{L_i}$ follows from Remark 2.2.7. Indeed, let $i = 1,\dots,n$, and define $\bar{c}_i = u/\pi^{v(u)} \mod \pi$, where $u = c_f \prod_{j=i+1}^{n} g_j(0)$. Then $\overline{f|_{L_i}}(x^{b_i}) = \bar{c}_i \cdot \overline{g_i|_{\mathrm{NP}(g_i)}}(x^{b_i})$, where b_i is the denominator of $\rho^0_{\mathfrak{s}_i}$. But

$$\overline{g_i|_{\mathrm{NP}(g_i)}}(x^{b_i}) = g_i\big(\pi^{\rho^0_{\mathfrak{s}_i}} x\big)/\pi^{\rho^0_{\mathfrak{s}_i} \deg g_i} \mod \pi.$$

Thus the lemma follows as $\mathfrak{R}_i = \mathfrak{s}_i \setminus \mathfrak{s}_{i-1}$. □

Notation 2.3.39 Let $\mathfrak{s} \in \mathring{\Sigma}^w_f$. Following the notation of Lemma 2.3.38, let $i \in \{1,\dots,n\}$ be such that $\mathfrak{s} = \mathfrak{s}_i$. We will write $L^w_{\mathfrak{s}}$ for the edge L_i.

2.4 Description of a regular model

From now on, assume $\mathrm{char}(K) \neq 2$ and let C/K be a hyperelliptic curve, i.e. a geometrically connected, smooth, projective curve, equipped with a separable morphism $C \to \mathbb{P}^1_K$ of degree 2. Let $y^2 = f(x)$ be a Weierstrass equation of C. Suppose $\deg f > 1$. Let g be the genus of C. Accordingly with [D²M²] we define the *cluster picture* of C as the cluster picture of f. Analogously, all definitions and notations attached to f given in §2.3 (e.g. Σ_f, Σ_f^{rat}, Σ_f^z) are given for C in the same way (e.g. Σ_C, Σ_C^{rat}, Σ_C^z). In particular, we will say that C has an almost rational cluster picture if f does (Definition 2.3.26).

In this section we present the main results that follow from the construction of a model of C we develop in §2.5. In particular, Theorem 2.4.22 describes the special fibre of the minimal regular model of C with normal crossings when C has an almost rational cluster picture and is y-regular (Definition 2.4.10).

For the following sections we will use the main definitions, notations and results of [Dok, §3]. In particular, we recall (without stating) the definitions of Newton polytopes Δ and Δ_v attached to a polynomial $g \in K[x,y]$, v-vertices/edges/faces of Δ, the denominator δ_λ of a v-face/edge λ, the slopes s_1^λ, s_2^λ of a v-edge λ.

Notation 2.4.1 We denote by Δ_v^w and Δ^w respectively the polytopes Δ_v and Δ attached to the polynomial $g_w(x,y) = y^2 - f(x+w)$. The piecewise affine function $v : \Delta^w \to \mathbb{R}$ determining the bijection $\Delta^w \to \Delta_v^w$, $P \mapsto (P, v(P))$, will be denoted by v (with a little abuse of notation). For a v-face F of Δ^w, denote by $v_F : \Delta^w \to \mathbb{R}$ the linear function equal to v on F. Since the projection $\Delta_v^w \to \Delta^w$ is a bijection, given a vertex/edge/face λ of Δ_v^w we will denote by the same symbol λ the corresponding v-vertex/edge/face of Δ^w. Since they are mainly used for indexing, this will not cause confusion.

Notation 2.4.2 Given a v-edge λ of Δ^w, we will denote by r_λ the smallest non-negative integer such that we can fix $\frac{n_i}{d_i} \in \mathbb{Q}$, for $i = 0, \ldots, r_\lambda + 1$ so that

$$s_1^\lambda = \frac{n_0}{d_0} > \frac{n_1}{d_1} > \ldots > \frac{n_{r_\lambda}}{d_{r_\lambda}} > \frac{n_{r_\lambda+1}}{d_{r_\lambda+1}} = s_2^\lambda, \quad \text{with} \quad \begin{vmatrix} n_i & n_{i+1} \\ d_i & d_{i+1} \end{vmatrix} = 1.$$

Thanks to Lemma 2.3.38 we can explicitly relate the Newton polytope Δ_v^w of $g_w(x,y)$ and the cluster picture centred at w of C.

Lemma 2.4.3 *Let $w \in K$. Then there is a 1-to-1 correspondence between the clusters in $\mathring{\Sigma}_C^w$ and the faces of the Newton polytope Δ_v^w. More explicitly, let $\mathfrak{s}_1 \subset \cdots \subset \mathfrak{s}_n = \mathfrak{R}$ be the clusters in $\mathring{\Sigma}_C^w$ and let $\mathfrak{s}_0 = \{w\}$ if $\{w\} \in \Sigma_C^w$ or $\mathfrak{s}_0 = \varnothing$ otherwise. Then Δ_v^w has vertices T, Q_i, $i = 0, \ldots, n$, where*

- *$T = (0,2,0)$,*
- *$Q_n = (|\mathfrak{R}|, 0, v(c_f))$,*

- $Q_i = (|\mathfrak{s}_i|, 0, \epsilon^w_{\mathfrak{s}_{i+1}} - |\mathfrak{s}_i|\rho^w_{\mathfrak{s}_{i+1}})$ *for* $i = 0, \dots, n-1$,

and edges L_i *(*$i = 1, \dots, n$*), linking* Q_{i-1} *and* Q_i*, and* V_j *(*$j = 0, \dots, n$*), linking* Q_j *and* T*. Furthermore, (possible choices for) the slopes of the v-edges of* Δ^w *are:*

- $$s_1^{V_n} = \delta_{V_n} \frac{-v(c_f) + (|\mathfrak{R}| - 2g)\rho^w_{\mathfrak{R}}}{2} \quad \textit{and} \quad s_2^{V_n} = \lfloor s_1^{V_n} - 1 \rfloor;$$

- $$\begin{aligned} s_1^{V_i} &= \delta_{V_i}\left(-\frac{\epsilon^w_{\mathfrak{s}_i}}{2} + \left(\left\lfloor \frac{|\mathfrak{s}_i|}{2} \right\rfloor + 1\right)\rho^w_{\mathfrak{s}_i}\right), \\ s_2^{V_i} &= \delta_{V_i}\left(-\frac{\epsilon^w_{\mathfrak{s}_{i+1}}}{2} + \left(\left\lfloor \frac{|\mathfrak{s}_i|}{2} \right\rfloor + 1\right)\rho^w_{\mathfrak{s}_{i+1}}\right) \end{aligned} \qquad \textit{for all } i = 1, \dots, n-1;$$

- $$s_1^{V_0} = \delta_{V_0}\left(\frac{\epsilon^w_{\mathfrak{s}_1}}{2} - \rho^w_{\mathfrak{s}_1}\right) \quad \textit{and} \quad s_2^{V_0} = \lfloor s_1^{V_0} - 1 \rfloor;$$

- $$s_1^{L_i} = \delta_{L_i}\left(-\frac{\epsilon^w_{\mathfrak{s}_i}}{2} + \left(\left\lfloor \frac{|\mathfrak{s}_i|}{2} \right\rfloor + 1\right)\rho^w_{\mathfrak{s}_i}\right) \quad \textit{and} \quad s_2^{L_i} = \lfloor s_1^{L_i} - 1 \rfloor,$$

  for all $i = 1, \dots, n$*. In particular, as* δ_{L_i} *is the denominator of* $\rho^w_{\mathfrak{s}_i}$*,*

  $$r_{L_i} = \begin{cases} 1 & \textit{if } \delta_{L_i}\epsilon^w_{\mathfrak{s}_i} \textit{ is odd,} \\ 0 & \textit{if } \delta_{L_i}\epsilon^w_{\mathfrak{s}_i} \textit{ is even.} \end{cases}$$

Finally, for suitable choices of basis of the lattices in [Dok, 3.4, 3.5], we have

$$\overline{g_w|_{L_i}}(x^{b_i}) = -\tfrac{u}{\pi^{v(u)}} \textstyle\prod_{r \in \mathfrak{s}_i \setminus \mathfrak{s}_{i-1}} (x + \tfrac{w-r}{\pi^{\rho_i}}) \mod \pi, \qquad u = c_f \prod_{r \in \mathfrak{R} \setminus \mathfrak{s}_i} (w - r),$$

for any $i = 1, \dots, n$*, where* $\rho_i = \rho^w_{\mathfrak{s}_i}$*, and* b_i *is the denominator of* ρ_i*;*

$$\overline{g_w|_{V_j}}(y) = y^{|\bar{V}_j(\mathbb{Z})_{\mathbb{Z}}| - 1} - \tfrac{u}{\pi^{v(u)}} \mod \pi, \qquad u = c_f \textstyle\prod_{r \in \mathfrak{R} \setminus \mathfrak{s}_j} (w - r),$$

for any $j = 0, \dots, n$*, where* $|\bar{V}_j(\mathbb{Z})_{\mathbb{Z}}|$ *is the number of integer points* P *on the v-edge* V_j *with* $v(P) \in \mathbb{Z}$*, endpoints included.*

Proof. The structure of Δ^w_v follows from Lemma 2.3.38. For the computation of the slopes, we only need to individuate, for all the v-edges, the two points P_0 and P_1 of [Dok, Definition 3.12]. It is easy to see that the followings are admissible choices.

- For V_i and L_i ($i = 1, \dots, n$), choose $P_0 = (|\mathfrak{s}_i|, 0)$ and $P_1 = \left(\left\lfloor \frac{|\mathfrak{s}_i| - 1}{2} \right\rfloor, 1\right)$.
- For V_0, choose $P_0 = (0, 2)$ and $P_1 = (1, 1)$;

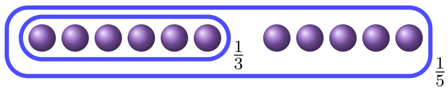
The second part of the lemma then follows from the first one. The computations of the reductions also follows from Lemma 2.3.38 by choosing the lattices $Q_{i-1}+(b_i,0)\mathbb{Z}$ for $g_w|_{L_i}$ and $Q_i+(-|\mathfrak{s}_i|/a, 2/a)\mathbb{Z}$ for $g_w|_{V_j}$, where $a=|\bar{V}_j(\mathbb{Z})_{\mathbb{Z}}|-1$. $\square$

Notation 2.4.4 Let C be as above and let $w\in K$. For every cluster $\mathfrak{s}\in\mathring{\Sigma}_C^w$ denote by $F_{\mathfrak{s}}^w$ the v-face of the Newton polytope Δ^w of $g_w(x,y)=y^2-f(x+w)$ that corresponds to $\mathfrak{s}$.

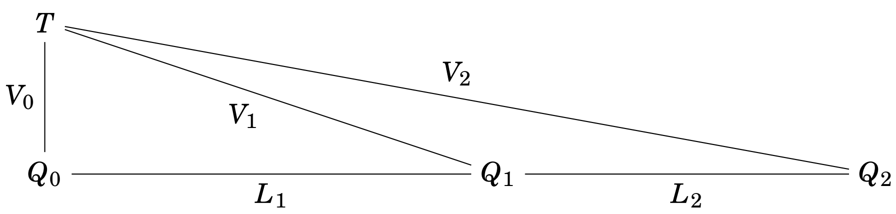
Following the notation of Lemma 2.4.3, let $i\in\{1,\dots,n\}$ be such that $\mathfrak{s}=\mathfrak{s}_i$. We will write $L_{\mathfrak{s}}^w$, $V_{\mathfrak{s}}^w$, V_0^w for the v-edges L_i, V_i, V_0, respectively.

Example 2.4.5 Let C be the hyperelliptic curve over $\mathbb{Q}_3$ given by the equation $y^2=f(x)$ where $f(x)=x^{11}-3x^6+9x^5-27$ is the polynomial of Example 2.3.20.

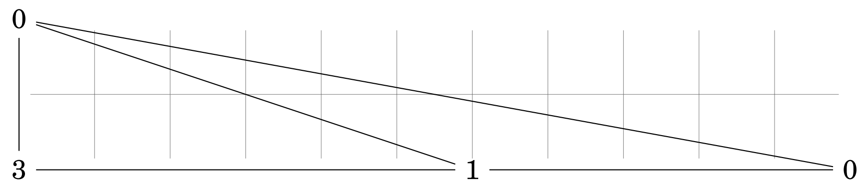
Its cluster picture centred at 0 is



where the subscripts represent the radii with respect to 0. As we can see, Σ_C^0 consists of two clusters: $\mathfrak{s}_1$ of size 6, radius $\frac{1}{3}$ and $\epsilon_{\mathfrak{s}_1}^0=3$, and $\mathfrak{s}_2=\mathfrak{R}$ of size 11, radius $\frac{1}{5}$ and $\epsilon_{\mathfrak{s}_2}^0=\frac{11}{5}$. Therefore the picture of Δ^0 broken into v-faces will be



where $T=(0,2)$, $Q_0=(0,0)$, $Q_1=(6,0)$, and $Q_2=(11,0)$. Denoting the values of v on vertices, the picture becomes



To state the theorems which describe the special fibre of the proper flat model $\mathcal{C}$ of C we will construct in §2.5, we need some definitions.

Definition 2.4.6 Let F/K be an unramified extension and let $\Sigma_F=\Sigma_{C_F}^{\mathrm{rat}}$ (i.e. set of clusters cut out by discs with centre in F). For any proper $\mathfrak{s}\in\Sigma_F$ let $G_{\mathfrak{s}}=\mathrm{Stab}_{G_K}(\mathfrak{s})$ and $K_{\mathfrak{s}}=(K^{\mathrm{s}})^{G_{\mathfrak{s}}}$. We define the following quantities:

$\mathfrak{s} \in \Sigma_F$, proper	
radius	$\rho_\mathfrak{s} = \max_{w \in F} \min_{r \in \mathfrak{s}} v(r-w)$
	$b_\mathfrak{s} =$ denominator of $\rho_\mathfrak{s}$
	$\epsilon_\mathfrak{s} = v(c_f) + \sum_{r \in \mathfrak{R}} \rho_{r \wedge \mathfrak{s}}$
	$D_\mathfrak{s} = 1$ if $b_\mathfrak{s}\epsilon_\mathfrak{s}$ odd, 2 if $b_\mathfrak{s}\epsilon_\mathfrak{s}$ even
multiplicity	$m_\mathfrak{s} = (3 - D_\mathfrak{s}) b_\mathfrak{s}$
parity	$p_\mathfrak{s} = 1$ if $\lvert\mathfrak{s}\rvert$ is odd, 2 if $\lvert\mathfrak{s}\rvert$ is even
slope	$s_\mathfrak{s} = \frac{1}{2}(\lvert\mathfrak{s}\rvert\rho_\mathfrak{s} + p_\mathfrak{s}\rho_\mathfrak{s} - \epsilon_\mathfrak{s})$
	$\gamma_\mathfrak{s} = 2$ if $\mathfrak{s}$ is even and $\epsilon_\mathfrak{s} - \lvert\mathfrak{s}\rvert\rho_\mathfrak{s}$ is odd, 1 otherwise
	$p_\mathfrak{s}^0 = 1$ if $\mathfrak{s}$ is minimal and $\mathfrak{s} \cap K_\mathfrak{s} \neq \varnothing$, 2 otherwise
	$s_\mathfrak{s}^0 = -\epsilon_\mathfrak{s}/2 + \rho_\mathfrak{s}$
	$\gamma_\mathfrak{s}^0 = 2$ if $p_\mathfrak{s}^0 = 2$ and $\epsilon_\mathfrak{s}$ is odd, 1 otherwise

Lemma 2.4.7 *Keep the notation of the previous definition and let $\mathfrak{s} \in \Sigma_K$. Then $\mathfrak{s} \in \Sigma_F$ but the quantities in Definition 2.4.6 do not depend on F.*

Proof. A cluster $\mathfrak{s} \in \Sigma_F$ belongs to Σ_K if and only if $\sigma(\mathfrak{s}) = \mathfrak{s}$ for any $\sigma \in G_K$. Then the result follows from Lemma A.1.1. □

Remark 2.4.8. Lemma 2.4.3 shed some light on the quantities we defined in Definition 2.4.6. Let $\mathfrak{s} \in \Sigma_F$. Fix a rational centre $w_\mathfrak{s} \in F$ of $\mathfrak{s}$ such that $w_\mathfrak{s} \in K_\mathfrak{s}$ if $p_\mathfrak{s}^0 = 1$. Denoting $V = V_\mathfrak{s}^{w_\mathfrak{s}}$, $L = L_\mathfrak{s}^{w_\mathfrak{s}}$, and $V_0 = V_0^{w_\mathfrak{s}}$, we have:

- $b_\mathfrak{s} = \delta_L$ and $r_L = 2 - D_\mathfrak{s}$.
- $\gamma_\mathfrak{s} = \delta_V$, $p_\mathfrak{s}/\gamma_\mathfrak{s} = \bar{V}(\mathbb{Z})_\mathbb{Z} - 1$ and $s_1^V = \gamma_\mathfrak{s} s_\mathfrak{s}$. If V is internal, that is $\mathfrak{s} \neq \mathfrak{R}$, then $s_2^V = \gamma_\mathfrak{s}(s_\mathfrak{s} - p_\mathfrak{s}\frac{\rho_\mathfrak{s} - \rho_{P(\mathfrak{s})}}{2})$.
- If $\mathfrak{s}$ is minimal and so V_0 is an edge of $F_\mathfrak{s}^{w_\mathfrak{s}}$, then $\gamma_\mathfrak{s}^0 = \delta_{V_0}$, $p_\mathfrak{s}^0/\gamma_\mathfrak{s}^0 = \bar{V}_0(\mathbb{Z})_\mathbb{Z} - 1$ and $s_1^{V_0} = -\gamma_\mathfrak{s}^0 s_\mathfrak{s}^0$.

Lemma 2.4.9 *Let $\mathfrak{s} \in \Sigma_C^{\mathrm{rat}}$ with rational centre $w \in K$. Then $D_\mathfrak{s} = 1$ if and only if $v_{F_\mathfrak{s}^w}((a,1)) \notin \mathbb{Z}$, for every $a \in \mathbb{Z}$.*

Proof. If $D_\mathfrak{s} = 1$ then $r_{L_\mathfrak{s}^w} = 1$ by Lemma 2.4.3, and so $v_{F_\mathfrak{s}^w}((a,1)) \notin \mathbb{Z}$, for every $a \in \mathbb{Z}$. Now let $c, d \in \mathbb{Z}$ such that $\rho_\mathfrak{s} \cdot c + d = 1/b_\mathfrak{s}$. If $D_\mathfrak{s} = 2$, then $b_\mathfrak{s}\epsilon_\mathfrak{s} \in 2\mathbb{Z}$, so

$$v_{F_\mathfrak{s}^w}(cb_\mathfrak{s}\epsilon_\mathfrak{s}/2, 1) = \frac{v_{F_\mathfrak{s}^w}((cb_\mathfrak{s}\epsilon_\mathfrak{s}, 0))}{2} = \frac{\epsilon_\mathfrak{s} - (cb_\mathfrak{s}\epsilon_\mathfrak{s})\rho_\mathfrak{s}}{2} = \frac{db_\mathfrak{s}\epsilon_\mathfrak{s}}{2} \in \mathbb{Z},$$

as required. □

Definition 2.4.10 We say that C is *y-regular* if $p \nmid D_\mathfrak{s}$ for every proper $\mathfrak{s} \in \Sigma_C^{\mathrm{rat}}$, i.e. if either $p \neq 2$ or $D_\mathfrak{s} = 1$ for any proper $\mathfrak{s} \in \Sigma_C^{\mathrm{rat}}$.

Remark 2.4.11. Let F/K be an unramified extension. Then from Lemma 2.4.7, if C_F is y-regular then C is y-regular.

The next lemma gives a characterisation of the Δ_v-regularity for hyperelliptic curves. In fact, C is Δ_v-regular along the horizontal edges of $\Delta = \Delta^0$ if f is NP-regular, and is Δ_v-regular along the non-horizontal edges of Δ if C is y-regular.

Lemma 2.4.12 *The hyperelliptic curve C is Δ_v-regular if and only if C is y-regular and f is* NP*-regular.*

Proof. Let $g(x,y) = y^2 - f(x)$. If C is y-regular and f is NP-regular, then C is Δ_v-regular by Lemma 2.4.3 and Lemma 2.4.9.

Conversely, if C is Δ_v-regular, then f is NP-regular, and all clusters have rational centre 0 by Corollary 2.3.25. It remains to show that if $p = 2$ then $D_{\mathfrak{s}} = 1$ for every proper $\mathfrak{s} \in \Sigma_C^{\mathrm{rat}}$. Suppose there exists $\mathfrak{s} \in \Sigma_C^{\mathrm{rat}}$ such that $D_{\mathfrak{s}} = 2$. Consider the variety $\bar{X}_{F^0_{\mathfrak{s}}}$ ([Dok, Definition 3.7]). By Lemma 2.4.9, the smoothness of $\bar{X}_{F^0_{\mathfrak{s}}}$ implies there exists $\mathfrak{s}' \in \Sigma_C^{\mathrm{rat}}$, such that $|\mathfrak{s}| - |\mathfrak{s}'| = 1$. Hence $\rho_{\mathfrak{s}} \in \mathbb{Z}$ from Lemma 2.3.12. Therefore $\bar{F}^0_{\mathfrak{s}}(\mathbb{Z}) = \bar{F}^0_{\mathfrak{s}}(\mathbb{Z})_{\mathbb{Z}}$, by Lemma 2.4.9. But this gives a contradiction as it forces either $\overline{g|_{V^0_{\mathfrak{s}'}}}$ or $\overline{g|_{V^0_{\mathfrak{s}}}}$ to be a square. ☐

Definition 2.4.13 Let $\mathfrak{s} \in \Sigma_F$ be a proper cluster and let $c \in \{0, \dots, b_{\mathfrak{s}} - 1\}$ such that $c\rho_{\mathfrak{s}} - \frac{1}{b_{\mathfrak{s}}} \in \mathbb{Z}$. Define

$$\tilde{\mathfrak{s}} = \{\mathfrak{s}' \in \Sigma_F \cup \{\varnothing\} \mid \mathfrak{s}' < \mathfrak{s} \text{ and } \tfrac{|\mathfrak{s}'|}{b_{\mathfrak{s}}} - c\epsilon_{\mathfrak{s}} \notin 2\mathbb{Z}\},$$

where $\varnothing < \mathfrak{s}$ if $\mathfrak{s}$ is minimal and $p^0_{\mathfrak{s}} = 2$.

The *genus* $g(\mathfrak{s})$ of a rational cluster $\mathfrak{s} \in \Sigma_F$ is defined as follows:

- If $D_{\mathfrak{s}} = 1$, then $g(\mathfrak{s}) = 0$.
- If $D_{\mathfrak{s}} = 2$, then $2g(\mathfrak{s}) + 1$ or $2g(\mathfrak{s}) + 2$ equals

$$\frac{|\mathfrak{s}| - \sum_{\mathfrak{s}' \in \Sigma_F, \mathfrak{s}' < \mathfrak{s}} |\mathfrak{s}'|}{b_{\mathfrak{s}}} + |\tilde{\mathfrak{s}}|.$$

Definition 2.4.14 Let $\Sigma_C^{\min}$ be the set of rationally minimal clusters of C and let $\Sigma \subseteq \Sigma_C^{\min}$. For each cluster $\mathfrak{s} \in \Sigma$, fix a rational centre $w_{\mathfrak{s}}$; if possible, choose $w_{\mathfrak{s}} \in \mathfrak{s}$. Let W be the set of these rational centres and define $\Sigma^W = \bigcup_{w \in W} \Sigma_C^w$. For any proper cluster $\mathfrak{s} \in \Sigma^W$ fix a rational centre $w_{\mathfrak{s}} \in W$. Denote $r_{\mathfrak{s}} = \frac{w_{\mathfrak{s}} - r}{\pi^{\rho_{\mathfrak{s}}}}$ for $r \in \mathfrak{R}$. Define reductions $\overline{f^W_{\mathfrak{s}}}(x) \in k[x]$, $\overline{g_{\mathfrak{s}}} \in k[y]$, and for $\mathfrak{s} \in \Sigma$ also $\overline{g^0_{\mathfrak{s}}} \in k[y]$ by

$$\begin{aligned}
\overline{f^W_{\mathfrak{s}}}(x^{b_{\mathfrak{s}}}) &= \tfrac{u}{\pi^{v(u)}} \prod_{r \in \mathfrak{s} \setminus \bigcup_{\mathfrak{s}' < \mathfrak{s}} \mathfrak{s}'} (x + r_{\mathfrak{s}}) \mod \pi, && u = c_f \textstyle\prod_{r \in \mathfrak{R} \setminus \mathfrak{s}} r_{\mathfrak{s}}, \\
\overline{g_{\mathfrak{s}}}(y) &= y^{p_{\mathfrak{s}}/\gamma_{\mathfrak{s}}} - \tfrac{u}{\pi^{v(u)}} \mod \pi, && u = c_f \textstyle\prod_{r \in \mathfrak{R} \setminus \mathfrak{s}} r_{\mathfrak{s}}, \\
\overline{g^0_{\mathfrak{s}}}(y) &= y^{p^0_{\mathfrak{s}}/\gamma^0_{\mathfrak{s}}} - \tfrac{u}{\pi^{v(u)}} \mod \pi, && u = c_f \textstyle\prod_{r \in \mathfrak{R} \setminus \{w_{\mathfrak{s}}\}} r_{\mathfrak{s}}.
\end{aligned}$$

where the union runs through all $\mathfrak{s}' \in \Sigma^W$, $\mathfrak{s}' < \mathfrak{s}$. Finally define the k-schemes

1. $X_{\mathfrak{s}}^{W} : \{\overline{f_{\mathfrak{s}}^{W}} = 0\} \subset \mathbb{G}_{m,k}$;

2. $X_{\mathfrak{s}} : \{\overline{g_{\mathfrak{s}}} = 0\} \subset \mathbb{G}_{m,k}$;

3. $X_{\mathfrak{s}}^{0} : \{\overline{g_{\mathfrak{s}}^{0}} = 0\} \subset \mathbb{G}_{m,k}$ if $\mathfrak{s} \in \Sigma$.

Notation 2.4.15 Given a scheme $\mathcal{X}/O_K$ we will denote by $\mathcal{X}_\eta$ its generic fibre $\mathcal{X} \times_{\mathrm{Spec}\ O_K} \mathrm{Spec}\ K$, and by $\mathcal{X}_s$ its special fibre $\mathcal{X} \times_{\mathrm{Spec}\ O_K} \mathrm{Spec}\ k$.

Notation 2.4.16 If $C = C_1 \cup \cdots \cup C_r$ is a chain of $\mathbb{P}^1_k$s of length r and multiplicities $m_i \in \mathbb{Z}$ (meeting transversely), then $\infty \in C_i$ is identified with $0 \in C_{i+1}$, and $0, \infty \in C$ are respectively $0 \in C_1$ and $\infty \in C_r$. Finally, if $r = 0$, then $C = \mathrm{Spec}\ k$ and $0 = \infty$.

Notation 2.4.17 Let $\alpha \in \mathbb{Z}_+$, $a, b \in \mathbb{Q}$, with $a > b$, and fix $\frac{n_i}{d_i} \in \mathbb{Q}$ so that

$$\alpha a = \frac{n_0}{d_0} > \frac{n_1}{d_1} > \ldots > \frac{n_r}{d_r} > \frac{n_{r+1}}{d_{r+1}} = \alpha b, \quad \text{with} \quad \begin{vmatrix} n_i & n_{i+1} \\ d_i & d_{i+1} \end{vmatrix} = 1,$$

and r minimal. We write $\mathbb{P}^1(\alpha, a, b)$ for a chain of $\mathbb{P}^1_k$s of length r and multiplicities $\alpha d_1, \ldots, \alpha d_r$. We denote by $\mathbb{P}^1(\alpha, a)$ the chain $\mathbb{P}^1(\alpha, a, \lfloor \alpha a - 1 \rfloor / \alpha)$. Moreover, we write $\bar{\mathbb{P}}^1(\alpha, a, b)$, $\bar{\mathbb{P}}^1(\alpha, a)$ for $\mathbb{P}^1(\alpha, a, b) \times_{\mathrm{Spec}\ k} \mathrm{Spec}\ k^{\mathrm{s}}$, $\mathbb{P}^1(\alpha, a) \times_{\mathrm{Spec}\ k} \mathrm{Spec}\ k^{\mathrm{s}}$, respectively.

Theorem 2.4.18 and Theorem 2.4.22 will follow from §2.5.

Theorem 2.4.18 *Let C/K be a hyperelliptic curve given by a Weierstrass equation $y^2 = f(x)$. Suppose* $\deg f > 1$ *and let Σ, W and Σ^W as in Definition 2.4.14. Then there exists a proper flat model $\mathcal{C}/O_K$ (constructed in §2.5) of C such that its special fibre $\mathcal{C}_s/k$ consists of* 1*-dimensional schemes given below in (1),(2),(3),(4),(5), glued along* 0*-dimensional transversal intersections:*

(1) Every proper cluster $\mathfrak{s} \in \Sigma^W$ gives a 1*-dimensional closed subscheme $\Gamma_{\mathfrak{s}}$ of multiplicity $m_{\mathfrak{s}}$. $\Gamma_{\mathfrak{s}}$ is not integral if and only if $D_{\mathfrak{s}} = 2$, $\tilde{\mathfrak{s}} \cap (\Sigma^W \cup \{\varnothing\}) = \varnothing$ and $\overline{f_{\mathfrak{s}}^W}$ is a square. When this happens, if $p = 2$ then $\Gamma_{\mathfrak{s}}$ is not reduced and $(\Gamma_{\mathfrak{s}})_{\mathrm{red}}$ is irreducible of multiplicity* 2 *in $\Gamma_{\mathfrak{s}}$, if $p \neq 2$ then $\Gamma_{\mathfrak{s}}$ is reducible, namely $\Gamma_{\mathfrak{s}} = \Gamma_{\mathfrak{s}}^+ \cup \Gamma_{\mathfrak{s}}^-$, with $\Gamma_{\mathfrak{s}}^\pm = \mathbb{P}^1_k$.*

(2) Every proper cluster $\mathfrak{s} \in \Sigma^W$ with $D_{\mathfrak{s}} = 1$ gives the closed subscheme $X_{\mathfrak{s}}^W \times \mathbb{P}^1_k$, of multiplicity $b_{\mathfrak{s}}$, where $X_{\mathfrak{s}}^W \times \{0\} \subset \Gamma_{\mathfrak{s}}$.

(3) Every proper cluster $\mathfrak{s} \in \Sigma^W$ such that $\mathfrak{s} \neq \mathfrak{R}$, gives the closed subscheme $X_{\mathfrak{s}} \times \mathbb{P}^1(\gamma_{\mathfrak{s}}, s_{\mathfrak{s}}, s_{\mathfrak{s}} - p_{\mathfrak{s}} \cdot \frac{\rho_{\mathfrak{s}} - \rho_{P(\mathfrak{s})}}{2})$ where $X_{\mathfrak{s}} \times \{0\} \subset \Gamma_{\mathfrak{s}}$ and $X_{\mathfrak{s}} \times \{\infty\} \subset \Gamma_{P(\mathfrak{s})}$.

(4) Every cluster $\mathfrak{s} \in \Sigma$ gives the closed subscheme $X_{\mathfrak{s}}^0 \times \mathbb{P}^1(\gamma_{\mathfrak{s}}^0, -s_{\mathfrak{s}}^0)$ where $X_{\mathfrak{s}}^0 \times \{0\} \subset \Gamma_{\mathfrak{s}}$ (the chains are open-ended).

(5) Finally, the cluster $\mathfrak{R}$ gives the closed subscheme $X_{\mathfrak{R}} \times \mathbb{P}^1(\gamma_{\mathfrak{R}}, s_{\mathfrak{R}})$ where $X_{\mathfrak{R}} \times \{0\} \subset \Gamma_{\mathfrak{s}}$ (the chains are open-ended).

If $\Gamma_{\mathfrak{s}}$ is reducible, the two points in $X_{\mathfrak{s}} \times \{0\}$ (and $X_{\mathfrak{s}}^0 \times \{0\}$ if $\mathfrak{s} \in \Sigma$) belong to different irreducible components of $\Gamma_{\mathfrak{s}}$. Similarly, if $\mathfrak{s} \neq \mathfrak{R}$ and $\Gamma_{P(\mathfrak{s})}$ is reducible, the two points of $X_{\mathfrak{s}} \times \{\infty\}$ belong to different irreducible components of $\Gamma_{P(\mathfrak{s})}$.

Furthermore, if C has an almost rational cluster picture and is y-regular, then, by choosing $\Sigma = \Sigma_C^{\min}$, the model $\mathcal{C}$ is regular with strict normal crossings. In that case, if $\mathfrak{s}$ is übereven and $\epsilon_{\mathfrak{s}}$ is even, then $\Gamma_{\mathfrak{s}} \simeq X_{\mathfrak{s}} \times \mathbb{P}^1_k$, otherwise $\Gamma_{\mathfrak{s}}$ is irreducible of genus $g(\mathfrak{s})$.

Theorem 2.4.18 can be compared with Theorem 4.6.3 that describes a regular (proper flat) model of C when $p \neq 2$.

Definition 2.4.19 Let $\mathfrak{s} \in \Sigma_{K^{nr}}$. We say that

- $\mathfrak{s}$ is *removable* if either $|\mathfrak{s}| = 1$, or $\mathfrak{s}$ has a child $\mathfrak{s}' \in \Sigma_{K^{nr}}$ of size $2g+1$ ($\mathfrak{s} = \mathfrak{R}$ in this case).
- $\mathfrak{s}$ is *contractible* if one of the following conditions holds:
  1. $|\mathfrak{s}| = 2$ and $\rho_{\mathfrak{s}} \notin \mathbb{Z}$, $\epsilon_{\mathfrak{s}}$ odd, $\rho_{P(\mathfrak{s})} \leq \rho_{\mathfrak{s}} - \frac{1}{2}$;
  2. $\mathfrak{s} = \mathfrak{R}$ of size $2g+2$, with a child $\mathfrak{s}' \in \Sigma_{K^{nr}}$ of size $2g$, and $\rho_{\mathfrak{s}} \notin \mathbb{Z}$, $v(c_f)$ odd, $\rho_{\mathfrak{s}'} \geq \rho_{\mathfrak{s}} + \frac{1}{2}$;
  3. $\mathfrak{s} = \mathfrak{R}$ of size $2g+2$, union of its 2 odd proper children $\mathfrak{s}_1, \mathfrak{s}_2 \in \Sigma_{K^{nr}}$, with $v(c_f)$ odd, $\rho_{\mathfrak{s}_i} \geq \rho_{\mathfrak{s}} + 1$ for $i = 1, 2$.

Notation 2.4.20 Write $\mathring{\Sigma} \subseteq \Sigma_{K^{nr}}$ for the subset of non-removable clusters.

Definition 2.4.21 Choose rational centres $w_{\mathfrak{s}}$ for every $\mathfrak{s} \in \mathring{\Sigma}$, in such a way that $w_{\mathfrak{s}} \in \mathfrak{s}$ when $p_{\mathfrak{s}}^0 = 1$, and $\sigma(w_{\mathfrak{s}}) = w_{\sigma(\mathfrak{s})}$ for all $\sigma \in \mathrm{Gal}(K^{nr}/K)$. Denote $r_{\mathfrak{s}} = \frac{w_{\mathfrak{s}} - r}{\pi^{\rho_{\mathfrak{s}}}}$ for $r \in \mathfrak{R}$ and define $\overline{g_{\mathfrak{s}}}, \overline{g_{\mathfrak{s}}^0} \in k^{\mathrm{s}}[y]$ as in Definition 2.4.14, and $\overline{f_{\mathfrak{s}}}(x) \in k^{\mathrm{s}}[x]$, by

$$x^{2-p_{\mathfrak{s}}^0}\overline{f_{\mathfrak{s}}}(x^{b_{\mathfrak{s}}}) = \frac{u}{\pi^{v(u)}} \prod_{r \in \mathfrak{s} \setminus \bigcup_{\mathfrak{s}' < \mathfrak{s}} \mathfrak{s}'} (x + r_{\mathfrak{s}}) \mod \pi, \quad u = c_f \textstyle\prod_{r \in \mathfrak{R} \setminus \mathfrak{s}} r_{\mathfrak{s}},$$

where the union runs through all $\mathfrak{s}' \in \mathring{\Sigma}$, $\mathfrak{s}' < \mathfrak{s}$. Let $G_{\mathfrak{s}} = \mathrm{Stab}_{G_K}(\mathfrak{s})$, $K_{\mathfrak{s}} = (K^{\mathrm{s}})^{G_{\mathfrak{s}}}$, and let $k_{\mathfrak{s}}$ be the residue field of $K_{\mathfrak{s}}$. Then $\overline{f_{\mathfrak{s}}} \in k_{\mathfrak{s}}[x]$, $\overline{g_{\mathfrak{s}}} \in k_{\mathfrak{s}}[y]$, and for $\mathfrak{s}$ minimal $\overline{g_{\mathfrak{s}}^0} \in k_{\mathfrak{s}}[y]$.

Let $\mathfrak{s}_0 \in \mathring{\Sigma}$ be minimal and contained in $\mathfrak{s}$. Denote $\mathring{\mathfrak{s}} = \tilde{\mathfrak{s}} \setminus \{\{r\} < \mathfrak{s} \mid r \neq w_{\mathfrak{s}_0}\}$. Note that $\mathring{\mathfrak{s}}$ does not depend on the choice of $\mathfrak{s}_0$. Define $\tilde{f}_{\mathfrak{s}} \in k_{\mathfrak{s}}[x]$ by

$$\tilde{f}_{\mathfrak{s}}(x) = \prod_{\mathfrak{s}' \in \mathring{\mathfrak{s}}} \left(x - \overline{u_{\mathfrak{s}',\mathfrak{s}}}\right) \cdot \overline{f_{\mathfrak{s}}}(x),$$

where $\overline{u_{\mathfrak{s}',\mathfrak{s}}} = \frac{w_{\mathfrak{s}'} - w_{\mathfrak{s}}}{\pi^{\rho_{\mathfrak{s}}}} \mod \pi$ if $\mathfrak{s}' \neq \varnothing$ and $\overline{u_{\mathfrak{s}',\mathfrak{s}}} = 0$ otherwise.

In the next theorem we describe the special fibre of the minimal regular model of C with normal crossings. We use Definitions/Notations 2.3.1, 2.3.3, 2.3.4, 2.3.2, 2.3.8, 2.3.9, 2.3.26, 2.4.6, 2.4.10, 2.4.13, 2.4.17, 2.4.19, 2.4.20, 2.4.21 in the statement. Note that a full description of the model is developed in §2.5.

Theorem 2.4.22 (Minimal regular NC model) *Let $C/K : y^2 = f(x)$ be a hyperelliptic curve of genus ≥ 1. Suppose $C_{K^{nr}}$ has an almost rational cluster picture and is y-regular. Then the minimal regular model with normal crossings $\mathcal{C}^{\min}/O_{K^{nr}}$ of C has special fibre $\mathcal{C}_s^{\min}/k^{\mathrm{s}}$ described as follows:*

(1) Every $\mathfrak{s} \in \mathring{\Sigma}$ gives a 1-dimensional subscheme $\Gamma_{\mathfrak{s}}$ of multiplicity $m_{\mathfrak{s}}$. If $\mathfrak{s}$ is übereven and $\epsilon_{\mathfrak{s}}$ is even, then $\Gamma_{\mathfrak{s}}$ is the disjoint union of $\Gamma_{\mathfrak{s}}^{r_{\mathfrak{s},-}} \simeq \mathbb{P}^1_{k^{\mathrm{s}}}$ and $\Gamma_{\mathfrak{s}}^{r_{\mathfrak{s},+}} \simeq \mathbb{P}^1_{k^{\mathrm{s}}}$, otherwise $\Gamma_{\mathfrak{s}}$ is irreducible of genus $g(\mathfrak{s})$ (write $\Gamma_{\mathfrak{s}}^{r_{\mathfrak{s},-}} = \Gamma_{\mathfrak{s}}^{r_{\mathfrak{s},+}} = \Gamma_{\mathfrak{s}}$ in this case). The indices $r_{\mathfrak{s},-}$ and $r_{\mathfrak{s},+}$ are the roots of $\overline{g_{\mathfrak{s}}}$ (where $r_{\mathfrak{s},-} = r_{\mathfrak{s},+}$ if $\deg \overline{g_{\mathfrak{s}}} = 1$).

(2) Every $\mathfrak{s} \in \mathring{\Sigma}$ with $D_{\mathfrak{s}} = 1$ gives open-ended $\mathbb{P}^1_{k^{\mathrm{s}}}$s of multiplicity $b_{\mathfrak{s}}$ from $\Gamma_{\mathfrak{s}}$ indexed by roots of $\overline{f_{\mathfrak{s}}}$.

(3) Every non-maximal element $\mathfrak{s} \in \mathring{\Sigma}$ gives chains $\bar{\mathbb{P}}^1(\gamma_{\mathfrak{s}}, s_{\mathfrak{s}}, s_{\mathfrak{s}} - p_{\mathfrak{s}} \cdot \frac{\rho_{\mathfrak{s}} - \rho_{P(\mathfrak{s})}}{2})$ from $\Gamma_{\mathfrak{s}}$ to $\Gamma_{P(\mathfrak{s})}$ indexed by roots of $\overline{g_{\mathfrak{s}}}$.

(4) Every minimal element $\mathfrak{s} \in \mathring{\Sigma}$ gives open-ended chains $\bar{\mathbb{P}}^1(\gamma_{\mathfrak{s}}^0, -s_{\mathfrak{s}}^0)$ from $\Gamma_{\mathfrak{s}}$ indexed by roots of $\overline{g_{\mathfrak{s}}^0}$.

(5) The maximal element $\mathfrak{s} \in \mathring{\Sigma}$ gives open-ended chains $\bar{\mathbb{P}}^1(\gamma_{\mathfrak{s}}, s_{\mathfrak{s}})$ from $\Gamma_{\mathfrak{s}}$ indexed by roots of $\overline{g_{\mathfrak{s}}}$.

(6) Finally, blow down all $\Gamma_{\mathfrak{s}}$ where $\mathfrak{s}$ is a contractible cluster.

In (3) and (5), a chain indexed by r goes from $\Gamma_{\mathfrak{s}}^r$. In (3) the chain indexed by $r_{\mathfrak{s},-}$ goes to $\Gamma_{P(\mathfrak{s})}^{r_{P(\mathfrak{s}),-}}$ and the chain indexed by $r_{\mathfrak{s},+}$ goes to $\Gamma_{P(\mathfrak{s})}^{r_{P(\mathfrak{s}),+}}$.

Before blowing down in (6), the components given in (1)–(5) describe the special fibre of a regular model of $C_{K^{nr}}$ with strict normal crossings.

The Galois group G_k acts naturally, i.e. for every $\sigma \in G_k$, $\sigma(\Gamma_{\mathfrak{s}}^r) = \Gamma_{\sigma(\mathfrak{s})}^{\sigma(r)}$, and similarly on the chains.

If $\Gamma_{\mathfrak{s}}$ is irreducible, then its function field is isomorphic to $k^{\mathrm{s}}(x)[y]$ with the relation $y^{D_{\mathfrak{s}}} = \tilde{f}_{\mathfrak{s}}(x)$.

Remark 2.4.23. Note that if $\Gamma_{\mathfrak{s}}$ or $\Gamma_{P(\mathfrak{s})}$ is reducible then $p_{\mathfrak{s}}/\gamma_{\mathfrak{s}} = 2$.

Example 2.4.24 Let p be a prime number and let $a \in \mathbb{Z}_p$, $b \in \mathbb{Z}_p^\times$ such that the polynomial $x^2 + ax + b$ is not a square modulo p. Let C be the hyperelliptic curve over $\mathbb{Q}_p$ of genus 4 given by the equation $y^2 = f(x)$, where $f(x) = (x^6 + ap^4x^3 + bp^8)((x-p)^3 - p^{11})$. In Example 2.3.32, we described the rational cluster picture of C and proved that C has an almost rational cluster picture. Recall that Σ_C^{rat} consists of 3 clusters $\mathfrak{t}_3, \mathfrak{t}_4, \mathfrak{R}$ of size $6, 3, 9$ respectively such that $\mathfrak{t}_3 < \mathfrak{R}$ and $\mathfrak{t}_4 < \mathfrak{R}$. In particular, note that $\Sigma_{\mathbb{Q}_p^{nr}} = \Sigma_C^{\mathrm{rat}}$, and no cluster of $\Sigma_{\mathbb{Q}_p^{nr}}$ is removable, so $\mathring{\Sigma} = \Sigma_C^{\mathrm{rat}}$. The minimal elements of $\mathring{\Sigma}$ are then $\mathfrak{t}_3$ and $\mathfrak{t}_4$.

We want to describe the special fibre of the minimal regular model with normal crossings $\mathcal{C}^{\min}$ of C. Compute the quantities in Definitions 2.4.6 and 2.4.13, and the polynomials $\overline{f_{\mathfrak{s}}}, \overline{g_{\mathfrak{s}}}, \overline{g_{\mathfrak{s}}^0}$ of Definition 2.4.21, for any cluster in $\mathring{\Sigma}$:

	$\rho_{\mathfrak{s}}$	$b_{\mathfrak{s}}$	$\epsilon_{\mathfrak{s}}$	$D_{\mathfrak{s}}$	$m_{\mathfrak{s}}$	$p_{\mathfrak{s}}$	$s_{\mathfrak{s}}$	$\gamma_{\mathfrak{s}}$	$p^0_{\mathfrak{s}}$	$s^0_{\mathfrak{s}}$	$\gamma^0_{\mathfrak{s}}$	$g(\mathfrak{s})$	$\overline{f_{\mathfrak{s}}}(x)$	$\overline{g_{\mathfrak{s}}}(y)$	$\overline{g^0_{\mathfrak{s}}}(y)$
$\mathfrak{t}_3$	$\frac{4}{3}$	3	11	1	6	2	$-\frac{1}{6}$	2	2	$-\frac{25}{6}$	2	0	$x^2+\bar{a}x+\bar{b}$	$y+1$	$y-1$
$\mathfrak{t}_4$	$\frac{11}{3}$	3	17	1	6	1	$-\frac{7}{6}$	1	2	$-\frac{29}{6}$	2	0	$x-1$	$y-1$	$y+1$
$\mathfrak{R}$	1	1	9	1	2	1	$\frac{1}{2}$	1	2			0	1	$y-1$	

where $\bar{a},\bar{b}$ are the reductions of a,b modulo p. Then C is also y-regular for any p. Following the steps of Theorem 2.4.22 the special fibre of $\mathcal{C}^{\min}$ over $\bar{\mathbb{F}}_p$ can be described as follows:

(1) The clusters $\mathfrak{t}_3,\mathfrak{t}_4,\mathfrak{R}$ give 3 irreducible components $\Gamma_{\mathfrak{t}_3},\Gamma_{\mathfrak{t}_4},\Gamma_{\mathfrak{R}}$ of genus 0 of multiplicities $6,6,2$ respectively;

(2) The cluster $\mathfrak{t}_3$ gives 2 open-ended $\mathbb{P}^1$s of multiplicity 3 from $\Gamma_{\mathfrak{t}_3}$, while $\mathfrak{t}_4$ gives 1 open-ended $\mathbb{P}^1$ of multiplicity 3 from $\Gamma_{\mathfrak{t}_4}$.

(3) From $\gamma_{\mathfrak{t}_3}s_{\mathfrak{t}_3}=-\frac{1}{3}>-\frac{1}{2}>-1=\gamma_{\mathfrak{t}_3}\big(s_{\mathfrak{t}_3}-p_{\mathfrak{t}_3}\cdot\frac{\rho_{\mathfrak{t}_3}-\rho_{\mathfrak{R}}}{2}\big)$, the cluster $\mathfrak{t}_3$ gives 1 $\mathbb{P}^1$ of multiplicity 4 from $\Gamma_{\mathfrak{t}_3}$ to $\Gamma_{\mathfrak{R}}$. From

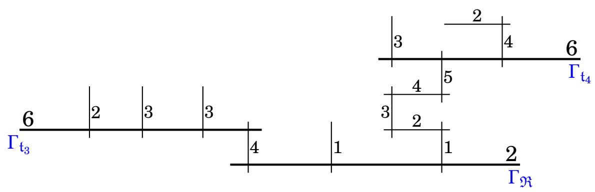
$$\gamma_{\mathfrak{t}_4}s_{\mathfrak{t}_4}=-\tfrac{7}{6}>-\tfrac{6}{5}>-\tfrac{5}{4}>-\tfrac{4}{3}>-\tfrac{3}{2}>-2>-\tfrac{5}{2}=\gamma_{\mathfrak{t}_3}\big(s_{\mathfrak{t}_4}-p_{\mathfrak{t}_4}\cdot\tfrac{\rho_{\mathfrak{t}_4}-\rho_{\mathfrak{R}}}{2}\big)$$

the cluster $\mathfrak{t}_4$ gives 1 chain of $\mathbb{P}^1$s of multiplicities $5,4,3,2,1$ from $\Gamma_{\mathfrak{t}_4}$ to $\Gamma_{\mathfrak{R}}$.

(4) From $-\gamma^0_{\mathfrak{t}_3}s^0_{\mathfrak{t}_3}=\frac{25}{3}>8>7$ the cluster $\mathfrak{t}_3$ gives 1 open-ended $\mathbb{P}^1$ of multiplicity 2 from $\Gamma_{\mathfrak{t}_3}$. From $-\gamma^0_{\mathfrak{t}_4}s^0_{\mathfrak{t}_4}=\frac{29}{3}>\frac{19}{2}>9>8$, the cluster $\mathfrak{t}_4$ gives 1 open-ended chain of $\mathbb{P}^1$s of multiplicities $4,2$ from $\Gamma_{\mathfrak{t}_4}$.

(5) From $\gamma_{\mathfrak{R}}s_{\mathfrak{R}}=\frac{1}{2}>0>-1$, the cluster $\mathfrak{R}$ gives 1 open-ended $\mathbb{P}^1$ of multiplicity 1 from $\Gamma_{\mathfrak{R}}$.

(6) There is no contractible cluster, so the components we considered in the steps above describe the special fibre of $\mathcal{C}^{\min}$ over $\bar{\mathbb{F}}_p$:



Finally, from the Galois action on the roots of the polynomials $\overline{f_{\mathfrak{s}}},\overline{g_{\mathfrak{s}}},\overline{g^0_{\mathfrak{s}}}$, for $\mathfrak{s}\in\mathring{\Sigma}$, we get that G_k acts trivially if $x^2+\bar{a}x+\bar{b}$ is reducible in $\mathbb{F}_p$, while it swaps the two components of multiplicity 3 intersecting $\Gamma_{\mathfrak{t}_3}$ (coming from (2)) otherwise.

As application of Theorem 2.4.22 we suppose k is finite of characteristic $p > 2$ and C is semistable of genus $g \geq 2$. In this setting [D^{2}M^2, Theorem 8.5] describes the minimal regular model of C in terms of its cluster picture Σ_C. We compare that result with the one obtained from Theorem 2.4.22 (Corollary 2.4.26).

First note that $C_{K^{nr}}$ is y-regular as $p \neq 2$. From [D^{2}M^2, Definition 1.7], if C is semistable then

1. the extension $K(\mathfrak{R})/K$ has ramification degree at most 2;
2. every proper cluster is $\mathrm{Gal}(K^{s}/K^{nr})$-invariant;
3. every principal cluster has $d_{\mathfrak{s}} \in \mathbb{Z}$ and $\nu_{\mathfrak{s}} \in 2\mathbb{Z}$.

It follows from Corollary 2.3.27 that $C_{K^{nr}}$ has an almost rational cluster picture.

In fact, (1) and (2) imply $\rho_{\mathfrak{s}} = d_{\mathfrak{s}}$ and $\epsilon_{\mathfrak{s}} = \nu_{\mathfrak{s}}$ for any proper cluster $\mathfrak{s}$ (Remark 2.3.13). In particular, $\mathring{\Sigma}^{\mathrm{rat}}_{C_{K^{nr}}} = \mathring{\Sigma}_C$. We will then say that $\mathfrak{s} \in \Sigma_C$ is non-removable if $\mathfrak{s}$ is proper and non-removable as rational cluster in $\Sigma_{K^{nr}}$.

Lemma 2.4.25 *Suppose k finite and $p > 2$. Assume C is semistable and let $\mathfrak{s} \in \Sigma_C$ be a non-removable cluster. Then $d_{\mathfrak{s}} \in \frac{1}{2}\mathbb{Z}$ and $\nu_{\mathfrak{s}} \in \mathbb{Z}$. Moreover, $\mathfrak{s}$ is contractible if and only if $d_{\mathfrak{s}} \notin \mathbb{Z}$ or $\nu_{\mathfrak{s}} \notin 2\mathbb{Z}$.*

Proof. Let $\mathfrak{s} \in \Sigma_C$ be a non-removable cluster. Since $K(\mathfrak{R})/K$ has ramification degree at most 2, then $d_{\mathfrak{s}} \in \frac{1}{2}\mathbb{Z}$.

By Theorem 2.4.22 the multiplicity of the 1-dimensional scheme $\Gamma_{\mathfrak{s}}$ is $m_{\mathfrak{s}}$. Furthermore, $\Gamma_{\mathfrak{s}}$ is an irreducible component of the special fibre of the minimal regular model of C if and only if $\mathfrak{s}$ is not contractible. Therefore if $\mathfrak{s}$ is not contractible, then $m_{\mathfrak{s}} = 1$, i.e. $D_{\mathfrak{s}} = 2$ and $b_{\mathfrak{s}} = 1$. It follows that $\nu_{\mathfrak{s}} \in 2\mathbb{Z}$ and $d_{\mathfrak{s}} \in \mathbb{Z}$. Suppose $\mathfrak{s}$ contractible. Then either $d_{\mathfrak{s}} \notin \mathbb{Z}$ (and $\nu_{\mathfrak{s}} \in \mathbb{Z}$) or $\mathfrak{s} = \mathfrak{R}$ of size $2g+2$, with 2 odd rational children and $v(c_f)$ odd. We want to show that in the latter case, $\nu_{\mathfrak{s}}$ is odd. By Lemma 2.3.18, $d_{\mathfrak{R}} \in \mathbb{Z}$. Then $\nu_{\mathfrak{R}} = v(c_f) + (2g+2)d_{\mathfrak{R}}$ is odd. □

Let $\mathfrak{s} \in \Sigma_C$ be a non-removable cluster. By Lemma 2.4.25, if $\mathfrak{s}$ is not contractible, then $2g(\mathfrak{s})+1$ or $2g(\mathfrak{s})+2$ equals the number of odd children of $\mathfrak{s}$. In fact, this also holds when $\mathfrak{s}$ is contractible since in that case $g(\mathfrak{s}) = 0$ and $\mathfrak{s}$ has at most 2 odd children.

Corollary 2.4.26 (Minimal regular model (semistable reduction)) *Suppose that k is finite and $p > 2$. Let C/K be a semistable hyperelliptic curve of genus $g \geq 2$. The minimal regular model $\mathcal{C}^{\min}/O_{K^{nr}}$ of C has special fibre $\mathcal{C}^{\min}_s/k^{s}$ described as follows:*

(1) Every non-removable cluster $\mathfrak{s} \in \Sigma_C$ gives a 1-dimensional subscheme $\Gamma_{\mathfrak{s}}$. If $\mathfrak{s}$ is übereven, then $\Gamma_{\mathfrak{s}}$ is the disjoint union of $\Gamma_{\mathfrak{s}}^{r_{\mathfrak{s},-}} \simeq \mathbb{P}^1$ and $\Gamma_{\mathfrak{s}}^{r_{\mathfrak{s},+}} \simeq \mathbb{P}^1$, otherwise $\Gamma_{\mathfrak{s}}$ is irreducible of genus $g(\mathfrak{s})$ (write $\Gamma_{\mathfrak{s}}^{r_{\mathfrak{s},-}} = \Gamma_{\mathfrak{s}}^{r_{\mathfrak{s},+}} = \Gamma_{\mathfrak{s}}$ in this case). The indices $r_{\mathfrak{s},-}$ and $r_{\mathfrak{s},+}$ are the roots of $\overline{g_{\mathfrak{s}}}$.

(2) Every odd proper cluster $\mathfrak{s} \in \Sigma_C$ of size $|\mathfrak{s}| \leq 2g$ gives a chain of $\mathbb{P}^1$s of length $\lfloor \frac{d_{\mathfrak{s}} - d_{P(\mathfrak{s})} - 1}{2} \rfloor$ from $\Gamma_{\mathfrak{s}}$ to $\Gamma_{P(\mathfrak{s})}$ indexed by the root of $\overline{g_{\mathfrak{s}}}$.

(3) *Every even proper cluster $\mathfrak{s} \in \Sigma_C$ of size $|\mathfrak{s}| \leq 2g$ gives a chain of $\mathbb{P}^1$s of length $\lfloor d_\mathfrak{s} - d_{P(\mathfrak{s})} - \frac{1}{2} \rfloor$ from $\Gamma_\mathfrak{s}^{r_{\mathfrak{s},-}}$ to $\Gamma_{P(\mathfrak{s})}^{r_{P(\mathfrak{s}),-}}$ indexed by $r_{\mathfrak{s},-}$ and a chain of $\mathbb{P}^1$s of same length from $\Gamma_\mathfrak{s}^{r_{\mathfrak{s},+}}$ to $\Gamma_{P(\mathfrak{s})}^{r_{P(\mathfrak{s}),+}}$ indexed by $r_{\mathfrak{s},+}$.*

(4) *Finally, blow down all $\Gamma_\mathfrak{s}$ where $\mathfrak{s}$ is a contractible cluster.*

All components have multiplicity 1, and the absolute Galois group G_k acts naturally, as in Theorem 2.4.22.

Proof. Let $\mathfrak{s} \in \Sigma_C$ be a non-removable cluster. From Lemma 2.4.25, if $\mathfrak{s}$ is not contractible, then $D_\mathfrak{s} = 2$, $\gamma_\mathfrak{s} s_\mathfrak{s} \in \mathbb{Z}$ and $\gamma_\mathfrak{s}^0 s_\mathfrak{s}^0 \in \mathbb{Z}$. Suppose $\mathfrak{s}$ contractible. If $|\mathfrak{s}| = 2$ with $d_\mathfrak{s} \notin \mathbb{Z}$ (case (1) of Definition 2.4.19), then $\gamma_\mathfrak{s}^0 s_\mathfrak{s}^0 \in \mathbb{Z}$ and $\gamma_\mathfrak{s} = 1$, $s_\mathfrak{s} \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ and so $s_\mathfrak{s} - d_\mathfrak{s} + d_{P(\mathfrak{s})} \in \mathbb{Z}$, as $P(\mathfrak{s})$ can not be contractible. If $\mathfrak{s} = \mathfrak{R}$ (cases (2), (3) of Definition 2.4.19), then $v(c_f)$ is odd, and so $\gamma_\mathfrak{s} = 2$ and $\gamma_\mathfrak{s} s_\mathfrak{s} \in \mathbb{Z}$. Therefore (2), (4) and (5) of Theorem 2.4.22 do not give any components.

Finally, as $\gamma_\mathfrak{s} = 1$ and $p_\mathfrak{s} \frac{d_\mathfrak{s} - d_{P(\mathfrak{s})}}{2} \in \frac{1}{2}\mathbb{Z}$ for any proper $\mathfrak{s}$ with size $|\mathfrak{s}| \leq 2g$ (i.e. non-maximal), the length of $\mathbb{P}^1(\gamma_\mathfrak{s}, s_\mathfrak{s}, s_\mathfrak{s} - p_\mathfrak{s} \cdot \frac{d_\mathfrak{s} - d_{P(\mathfrak{s})}}{2})$ is

$$\left\lfloor \gamma_\mathfrak{s} s_\mathfrak{s} - \gamma_\mathfrak{s} \left(s_\mathfrak{s} - p_\mathfrak{s} \cdot \frac{d_\mathfrak{s} - d_{P(\mathfrak{s})}}{2} \right) - \frac{1}{2} \right\rfloor = \left\lfloor p_\mathfrak{s} \cdot \frac{d_\mathfrak{s} - d_{P(\mathfrak{s})}}{2} - \frac{1}{2} \right\rfloor.$$

The corollary then follows from Theorem 2.4.22. □

2.5 Construction of the model

We are going to construct a proper flat model $\mathcal{C}/O_K$ of C by glueing models defined in [Dok, §4]. For this reason we will assume the reader has familiarity with the definitions and the results presented in that paper. Let us start this section by describing the strategy we will follow.

Let $\Sigma_C^{\min}$ be the set of rationally minimal clusters of C and let $\Sigma \subseteq \Sigma_C^{\min}$. For any cluster $\mathfrak{s} \in \Sigma$ fix a rational centre $w_\mathfrak{s}$ in such a way that $\mathring{\Sigma}_C^{w_\mathfrak{s}}$ consists of the proper clusters in $\Sigma_C^{w_\mathfrak{s}}$. This requirement can be satisfied by choosing $w_\mathfrak{s} \in \mathfrak{s}$ whenever possible.[3] Let W be the set of all such rational centres and define $\Sigma^W := \bigcup_{w \in W} \Sigma_C^w$. For every proper cluster $\mathfrak{t} \in \Sigma^W$ fix a rational centre $w_\mathfrak{t} \in W$ (Lemma 2.3.14). For every $w \in W$, consider the curve $C^w : y^2 = f(x + w)$, isomorphic to C, and construct the (proper flat) model $\mathcal{C}_\Delta^w/O_K$ by [Dok, §4, Theorem 3.14]. We will define an open subscheme $\mathring{\mathcal{C}}_\Delta^w$ of $\mathcal{C}_\Delta^w$ and we will show that glueing the schemes $\mathring{\mathcal{C}}_\Delta^w$, to varying of $w \in W$, along common opens, gives a proper flat model $\mathcal{C}/O_K$ of C. Furthermore, if $\Sigma = \Sigma_C^{\min}$, and C is y-regular and has an almost rational cluster picture, then $\mathring{\mathcal{C}}_\Delta^w$ is an open regular subscheme of $\mathcal{C}_\Delta^w$ and therefore $\mathcal{C}$ is also regular.

[3] This is the assumption used in Theorem 2.4.18.

2.5.1 Charts

In this subsection we explicitly describe the matrices defining the charts of the schemes C_Δ^w, $w \in W$, as presented in [Dok, §4].

Let $\Sigma = \{\mathfrak{s}_1 \dots, \mathfrak{s}_m\} \subseteq \Sigma_C^{\min}$ be a set of rationally minimal clusters and let $W = \{w_1, \dots, w_m\}$ be a set of corresponding rational centres, such that $\mathring{\Sigma}_C^{w_h}$ consists of the proper clusters of $\Sigma_C^{w_h}$, for any $h = 1, \dots, m$. Define $\Sigma^W := \bigcup_{h=1}^m \Sigma_C^{w_h}$. For any $h, l = 1, \dots, m$, $h \neq l$, define $w_{hl} := w_h - w_l$, and write $w_{hl} = u_{hl}\pi^{\rho_{hl}}$, where $u_{hl} \in O_K^\times$ and $\rho_{hl} \in \mathbb{Z}$. Note that $\rho_{hl} = \rho_{\mathfrak{s}_h \wedge \mathfrak{s}_l} = \rho_{lh}$, by Lemma 2.3.18. Set $u_{hh} := 0$. Finally, for any $h, l = 1, \dots, m$, denote by $\overline{u_{hl}} \in k$ the reduction of u_{hl} modulo π.

Definition 2.5.1 Let $h = 1, \dots, m$ and let $\mathfrak{t} \in \Sigma_C^{w_h}$ be a proper cluster. Recall the matrices and cones defined in [Dok, §4]. We say that a matrix M *is associated to* $\mathfrak{t}$ if $M = M_{L_\mathfrak{t}^{w_h}, i}$ or $M = M_{V_\mathfrak{t}^{w_h}, j}$ (or $M = M_{V_0^{w_h}, j}$ if $\mathfrak{t} = \mathfrak{s}_h$). For a matrix M associated to $\mathfrak{t}$ we denote by δ_M and σ_M respectively

- the denominator $\delta_{L_\mathfrak{t}^{w_h}}$ and the cone $\sigma_{L_\mathfrak{t}^{w_h}, i, i+1}$ if $M = M_{L_\mathfrak{t}^{w_h}, i}$,
- the denominator $\delta_{V_\mathfrak{t}^{w_h}}$ and the cone $\sigma_{V_\mathfrak{t}^{w_h}, j, j+1}$ if $M = M_{V_\mathfrak{t}^{w_h}, j}$,
- the denominator $\delta_{V_0^{w_h}}$ and the cone $\sigma_{V_0^{w_h}, j, j+1}$ if $M = M_{V_0^{w_h}, j}$.

Finally, define $X_M = \operatorname{Spec} O_K[\sigma_M^\vee \cap \mathbb{Z}^3]$ and write

$$X_\Delta^h = \bigcup_M X_M,$$

for the toric scheme defined in [Dok, §4.2].

The following lemma describes all possible matrices associated to $\mathfrak{t}$.

Lemma 2.5.2 *Let $\mathfrak{t} \in \Sigma_C^{w_h}$ be a proper cluster. Consider the v-face $F_\mathfrak{t}^{w_h}$. Let $P_0, P_1 \in \mathbb{Z}^2$ and $n_i, d_i, k_i \in \mathbb{Z}$ be as in [Dok, §4] and define*

$$\delta := \delta_M, \quad \gamma_i := \frac{n_0 d_i - n_i d_0}{\delta d_0} \quad \textit{and} \quad T_i := \begin{pmatrix} \frac{1}{\delta} & -k_i & k_{i+1} \\ 0 & \delta & 0 \\ 0 & 0 & \delta \end{pmatrix},$$

for each matrix M associated to $\mathfrak{t}$.

- *Let c be the unique element of $\{0, \dots, b_\mathfrak{t} - 1\}$ such that $\frac{1}{b_\mathfrak{t}} - \rho_\mathfrak{t} \cdot c = d \in \mathbb{Z}$. For all $i = 0, \dots, r_{L_\mathfrak{t}^{w_h}}$, choose $k_i = cn_i + d\delta d_i(\lfloor \mathfrak{t}/2 \rfloor + 1)$; then*

$$M_{L_\mathfrak{t}^{w_h}, i} = \begin{pmatrix} \delta & -c\delta d_i\left(\frac{\epsilon_\mathfrak{t}}{2} + \gamma_i\right) & c\delta d_{i+1}\left(\frac{\epsilon_\mathfrak{t}}{2} + \gamma_{i+1}\right) \\ 0 & d_i & -d_{i+1} \\ -\delta\rho_\mathfrak{t} & -d\delta d_i\left(\frac{\epsilon_\mathfrak{t}}{2} + \gamma_i\right) & d\delta d_{i+1}\left(\frac{\epsilon_\mathfrak{t}}{2} + \gamma_{i+1}\right) \end{pmatrix}, \quad M_{L_\mathfrak{t}^{w_h}, i}^{-1} = T_i \cdot \begin{pmatrix} 1 & \lfloor \frac{|\mathfrak{t}|}{2} \rfloor + 1 & 0 \\ d_{i+1}\rho_\mathfrak{t} & \frac{d_{i+1}\epsilon_\mathfrak{t}}{2} + \gamma_{i+1} & d_{i+1} \\ d_i\rho_\mathfrak{t} & \frac{d_i\epsilon_\mathfrak{t}}{2} + \gamma_i & d_i \end{pmatrix},$$

  where $P_0 = (|\mathfrak{t}|, 0)$, $P_1 = (\lfloor |\mathfrak{t}| - 1/2 \rfloor, 1)$ and $\delta = \delta_{L_\mathfrak{t}^{w_h}} = b_\mathfrak{t}$.

- *If $\mathfrak{t}$ is odd, then for all $j=0,\dots,r_{V_{\mathfrak{t}}^{w_h}}$, we have*

$$M_{V_{\mathfrak{t}}^{w_h},j}=\begin{pmatrix} -|\mathfrak{t}| & -\frac{|\mathfrak{t}|+1}{2}d_j & \frac{|\mathfrak{t}|+1}{2}d_{j+1} \\ 2 & d_j & -d_{j+1} \\ -\epsilon_{\mathfrak{t}}+|\mathfrak{t}|\rho_{\mathfrak{t}} & n_j & -n_{j+1} \end{pmatrix}, \quad M^{-1}_{V_{\mathfrak{t}}^{w_h},j}=T_j\cdot\begin{pmatrix} 1 & \frac{|\mathfrak{t}|+1}{2} & 0 \\ d_{j+1}\rho_{\mathfrak{t}}-2\cdot\gamma_{j+1} & \frac{d_{j+1}\epsilon_{\mathfrak{t}}}{2}-|\mathfrak{t}|\cdot\gamma_{j+1} & d_{j+1} \\ d_j\rho_{\mathfrak{t}}-2\cdot\gamma_j & \frac{d_j\epsilon_{\mathfrak{t}}}{2}-|\mathfrak{t}|\cdot\gamma_j & d_j \end{pmatrix},$$

  where $P_0=(|\mathfrak{t}|,0)$, $P_1=(\lfloor |\mathfrak{t}|-1/2\rfloor,1)$, $\delta=\delta_{V_{\mathfrak{t}}^{w_h}}=1$ and $k_j=k_{j+1}=0$.

- *If $\mathfrak{t}$ is even, then for all $j=0,\dots,r_{V_{\mathfrak{t}}^{w_h}}$, we have*

$$M_{V_{\mathfrak{t}}^{w_h},j}=\begin{pmatrix} -\delta\frac{|\mathfrak{t}|}{2} & -(\frac{|\mathfrak{t}|}{2}+1)d_j-k_j\frac{|\mathfrak{t}|}{2} & (\frac{|\mathfrak{t}|}{2}+1)d_{j+1}+k_{j+1}\frac{|\mathfrak{t}|}{2} \\ \delta & d_j+k_j & -d_{j+1}-k_{j+1} \\ -\delta\frac{\epsilon_{\mathfrak{t}}-|\mathfrak{t}|\rho_{\mathfrak{t}}}{2} & \frac{n_j}{\delta}-k_j\frac{\epsilon_{\mathfrak{t}}-|\mathfrak{t}|\rho_{\mathfrak{t}}}{2} & -\frac{n_{j+1}}{\delta}+k_{j+1}\frac{\epsilon_{\mathfrak{t}}-|\mathfrak{t}|\rho_{\mathfrak{t}}}{2} \end{pmatrix}, \quad M^{-1}_{V_{\mathfrak{t}}^{w_h},j}=T_j\cdot\begin{pmatrix} 1 & \frac{|\mathfrak{t}|}{2}+1 & 0 \\ d_{j+1}\rho_{\mathfrak{t}}-\gamma_{j+1} & \frac{d_{j+1}\epsilon_{\mathfrak{t}}}{2}-\frac{|\mathfrak{t}|}{2}\gamma_{j+1} & d_{j+1} \\ d_j\rho_{\mathfrak{t}}-\gamma_j & \frac{d_j\epsilon_{\mathfrak{t}}}{2}-\frac{|\mathfrak{t}|}{2}\gamma_j & d_j \end{pmatrix},$$

  where $P_0=(|\mathfrak{t}|,0)$, $P_1=(\lfloor |\mathfrak{t}|-1/2\rfloor,1)$ and $\delta=\delta_{V_{\mathfrak{t}}^{w_h}}$.

- *If $f(w_h)=0$, then for all $j=0,\dots,r_{V_0^{w_h}}$, we have*

$$M_{V_0^{w_h},j}=\begin{pmatrix} 1 & d_j & -d_{j+1} \\ -2 & -d_j & d_{j+1} \\ \epsilon_{\mathfrak{s}_h}-\rho_{\mathfrak{s}_h} & n_j & -n_{j+1} \end{pmatrix}, \quad M^{-1}_{V_0^{w_h},j}=T_j\cdot\begin{pmatrix} -1 & -1 & 0 \\ d_{j+1}\rho_{\mathfrak{s}_h}+2\cdot\gamma_{j+1} & \frac{d_{j+1}\epsilon_{\mathfrak{s}_h}}{2}+\gamma_{j+1} & d_{j+1} \\ d_j\rho_{\mathfrak{s}_h}+2\cdot\gamma_j & \frac{d_j\epsilon_{\mathfrak{s}_h}}{2}+\gamma_j & d_j \end{pmatrix},$$

  where $P_0=(0,2)$, $P_1=(1,1)$, $\delta=\delta_{V_0^{w_h}}=1$ and $k_j=k_{j+1}=0$.

- *If $f(w_h)\neq 0$, then for all $j=0,\dots,r_{V_0^{w_h}}$, we have*

$$M_{V_0^{w},j}=\begin{pmatrix} 0 & d_j & -d_{j+1} \\ -\delta & -d_j-k_j & d_{j+1}+k_{j+1} \\ \delta\frac{\epsilon_{\mathfrak{s}_h}}{2} & \frac{n_j}{\delta}+k_j\frac{\epsilon_{\mathfrak{s}_h}}{2} & -\frac{n_{j+1}}{\delta}-k_{j+1}\frac{\epsilon_{\mathfrak{s}_h}}{2} \end{pmatrix}, \quad M^{-1}_{V_0^{w_h},j}=T_j\cdot\begin{pmatrix} -1 & -1 & 0 \\ d_{j+1}\rho_{\mathfrak{s}_h}+\gamma_{j+1} & \frac{d_{j+1}\epsilon_{\mathfrak{s}_h}}{2} & d_{j+1} \\ d_j\rho_{\mathfrak{s}_h}+\gamma_j & \frac{d_j\epsilon_{\mathfrak{s}_h}}{2} & d_j \end{pmatrix},$$

  where $P_0=(0,2)$, $P_1=(1,1)$ and $\delta=\delta_{V_0^{w_h}}$.

Proof. We follow the notation of [Dok, §4]. Choose $P_0,P_1\in\mathbb{Z}^2$ as in the proof of Lemma 2.4.3.

First consider the edge $L_{\mathfrak{t}}^{w_h}$ of $F_{\mathfrak{t}}^{w_h}$. From Lemma 2.4.3 we have

$$\nu=(1,0,-\rho_t) \quad \text{and} \quad (w_x,w_y)=(-\lfloor|\mathfrak{t}|/2\rfloor-1,1).$$

Then $M_{L_{\mathfrak{t}}^{w_h},i}$ and $M^{-1}_{L_{\mathfrak{t}}^{w_h},i}$ follow from [Dok, §4.3] as $k_i\equiv n_i(\delta\rho_{\mathfrak{t}})^{-1} \mod \delta$ and

$$\frac{n_0}{\delta d_0}=\frac{1}{\delta}s_1^{L_{\mathfrak{t}}^{w_h}}=v_{F_{\mathfrak{t}}^{w_h}}(P_1)-v_{F_{\mathfrak{t}}^{w_h}}(P_0)=-\frac{\epsilon_{\mathfrak{t}}}{2}+(\lfloor|\mathfrak{t}|/2\rfloor+1)\rho_{\mathfrak{t}}$$

Now assume $\mathfrak{t}$ even and consider the edge $V_{\mathfrak{t}}^{w_h}$ of $F_{\mathfrak{t}}^{w_h}$. Since $\mathfrak{t}$ is even,

$$V_{\mathfrak{t}}^{w_h}(\mathbb{Z})=\left\{(|\mathfrak{t}|,0),\left(\frac{|\mathfrak{t}|}{2},1\right),(0,2)\right\}, \quad \nu=\left(-\frac{|\mathfrak{t}|}{2},1,-\frac{\epsilon_t}{2}+\frac{|\mathfrak{t}|}{2}\rho_{\mathfrak{t}}\right)$$

and $(w_x,w_y)=\left(-\frac{|\mathfrak{t}|}{2}-1,1\right)$ as above. Then $M_{V_{\mathfrak{t}}^{w_h},j}$ and $M^{-1}_{V_{\mathfrak{t}}^{w_h},j}$ follow again from [Dok, (4.3)] as

$$\frac{n_0}{\delta d_0}=\frac{1}{\delta}s_1^{V_{\mathfrak{t}}^{w_h}}=v_{F_{\mathfrak{t}}^{w_h}}(P_1)-v_{F_{\mathfrak{t}}^{w_h}}(P_0)=-\frac{\epsilon_{\mathfrak{t}}}{2}+\left(\frac{|\mathfrak{t}|}{2}+1\right)\rho_{\mathfrak{t}}.$$

Similar arguments and computations yield the remaining matrices. ☐

Remark 2.5.3. From the lemma above one can explicitly construct the charts of the model $\mathcal{C}_{\Delta}^{w_h}$. The description of its special fibre $\mathcal{C}_{\Delta,s}^{w_h}$ which follows from [Dok, Theorem 3.14], matches the one given in Theorem 2.4.18 in the case $W=\{w_h\}$.

2.5.2 Open subschemes

In this subsection we explicitly describe the open subschemes $\mathring{C}^w_\Delta \subseteq C^w_\Delta$, for $w \in W$.

Let $h = 1, \dots, m$ and let $\mathfrak{t} \in \Sigma_C^{w_h}$ be a proper cluster. Let M be a matrix associated to $\mathfrak{t}$. Write

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \qquad \text{and} \qquad M^{-1} = \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12} & \tilde{m}_{13} \\ \tilde{m}_{21} & \tilde{m}_{22} & \tilde{m}_{23} \\ \tilde{m}_{31} & \tilde{m}_{32} & \tilde{m}_{33} \end{pmatrix}$$

Recall that $X_M = \operatorname{Spec} R$, where

$$R = \frac{O_K[X^{\pm 1}, Y, Z]}{\left(\pi - X^{\tilde{m}_{13}} Y^{\tilde{m}_{23}} Z^{\tilde{m}_{33}}\right)} \hookrightarrow \frac{O_K[X^{\pm 1}, Y^{\pm 1}, Z^{\pm 1}]}{\left(\pi - X^{\tilde{m}_{13}} Y^{\tilde{m}_{23}} Z^{\tilde{m}_{33}}\right)} \overset{M}{\simeq} K\left[x^{\pm 1}, y^{\pm 1}\right],$$

via the change of variable

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} x^{m_{11}} y^{m_{21}} \pi^{m_{31}} \\ x^{m_{12}} y^{m_{22}} \pi^{m_{32}} \\ x^{m_{13}} y^{m_{23}} \pi^{m_{33}} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \bullet M, \quad \begin{pmatrix} x \\ y \\ \pi \end{pmatrix} = \begin{pmatrix} X^{\tilde{m}_{11}} Y^{\tilde{m}_{21}} Z^{\tilde{m}_{31}} \\ X^{\tilde{m}_{12}} Y^{\tilde{m}_{22}} Z^{\tilde{m}_{32}} \\ X^{\tilde{m}_{13}} Y^{\tilde{m}_{23}} Z^{\tilde{m}_{33}} \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \bullet M^{-1}.$$

Let $l \neq h$. Set

$$T_M^{hl}(X,Y,Z) := \begin{cases} 1 + u_{hl} X^{\rho_{hl}\tilde{m}_{13} - \tilde{m}_{11}} Y^{\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21}} Z^{\rho_{hl}\tilde{m}_{33} - \tilde{m}_{31}} & \text{if } \mathfrak{t} \supseteq \mathfrak{s}_h \wedge \mathfrak{s}_l, \\ u_{hl}^{-1} X^{\tilde{m}_{11} - \rho_{hl}\tilde{m}_{13}} Y^{\tilde{m}_{21} - \rho_{hl}\tilde{m}_{23}} Z^{\tilde{m}_{31} - \rho_{hl}\tilde{m}_{33}} + 1 & \text{if } \mathfrak{t} \not\supseteq \mathfrak{s}_h \wedge \mathfrak{s}_l, \end{cases}$$

element of $R[Y^{-1}, Z^{-1}]$. Note that

$$\begin{aligned} &\text{if} \quad \mathfrak{t} \supseteq \mathfrak{s}_h \wedge \mathfrak{s}_l \quad \text{then} \quad T_M^{hl}(X,Y,Z) \overset{M}{\longmapsto} \frac{x + w_{hl}}{x}, \\ &\text{if} \quad \mathfrak{t} \not\supseteq \mathfrak{s}_h \wedge \mathfrak{s}_l \quad \text{then} \quad T_M^{hl}(X,Y,Z) \overset{M}{\longmapsto} \frac{x + w_{hl}}{w_{hl}}. \end{aligned}$$

The following two lemmas prove that $T_M^{hl}(X,Y,Z) \in R$. Therefore, up to units, $T_M^{hl}(X,Y,Z)$ can be seen as the polynomial in $O_K[X^{\pm 1}, Y, Z]$ satisfying

$$x - w_{hl} \overset{M}{=} X^{n_X} Y^{n_Y} Z^{n_Z} T_M^{hl}(X,Y,Z),$$

with $n_X, n_Y, n_Z \in \mathbb{Z}$, such that $\operatorname{ord}_Y(T_M^{hl}) = \operatorname{ord}_Z(T_M^{hl}) = 0$.

Lemma 2.5.4 *Let $h, l = 1, \dots, m$, with $h \neq l$, let $t \in \Sigma_C^{w_h}$ be such that $\mathfrak{t} \supseteq \mathfrak{s}_h \wedge \mathfrak{s}_l$ and let M be a matrix associated to $\mathfrak{t}$. Then*

$$\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21} \geq \rho_{\mathfrak{t}}\tilde{m}_{23} - \tilde{m}_{21} \geq 0 \quad \textit{and} \quad \rho_{hl}\tilde{m}_{33} - \tilde{m}_{31} \geq \rho_{\mathfrak{t}}\tilde{m}_{33} - \tilde{m}_{31} \geq 0.$$

Furthermore if $M = M_{L_{\mathfrak{t}}^{w_h}, i}$ then

- *$\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21} = 0$ if and only if $i = r_{L_{\mathfrak{t}}^{w_h}}$ or $\mathfrak{t} = \mathfrak{s}_h \wedge \mathfrak{s}_l$,*
- *$\rho_{hl}\tilde{m}_{33} - \tilde{m}_{31} = 0$ if and only if $\mathfrak{t} = \mathfrak{s}_h \wedge \mathfrak{s}_l$;*

if $M = M_{V_{\mathfrak{t}}^{w_h},j}$ *then*

- $\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21} > 0$,
- $\rho_{hl}\tilde{m}_{33} - \tilde{m}_{31} = 0$ *if and only if* $\mathfrak{t} = \mathfrak{s}_h \wedge \mathfrak{s}_l$ *and* $j = 0$.

Proof. This result follows from Lemma 2.5.2, which gives a complete description of M and M^{-1}. We show it when $\mathfrak{t}$ is even and $M = M_{V_{\mathfrak{t}}^{w_h},j}$, and leave the other cases for the reader. First of all recall that $\rho_{hl} = \rho_{\mathfrak{s}_h\wedge\mathfrak{s}_l}$ by Lemma 2.3.18. Then

$$\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21} = \delta\left(d_{j+1}\left(\rho_{hl} - \rho_t\right) + \gamma_{j+1}\right) > \delta d_{j+1}\left(\rho_{\mathfrak{s}_h\wedge\mathfrak{s}_l} - \rho_{\mathfrak{t}}\right) \geq 0,$$

where $\gamma_j = \frac{n_0 d_j - n_j d_0}{\delta d_0}$ and $\delta = \delta_M$. Similarly,

$$\rho_{hl}\tilde{m}_{33} - \tilde{m}_{31} = \delta\left(d_j\left(\rho_{hl} - \rho_t\right) + \gamma_j\right) \geq \delta d_j\left(\rho_{\mathfrak{s}_h\wedge\mathfrak{s}_l} - \rho_{\mathfrak{t}}\right) \geq 0.$$

In particular $\rho_{hl}\tilde{m}_{33} - \tilde{m}_{31} = 0$ if and only if $\mathfrak{t} = \mathfrak{s}_h \wedge \mathfrak{s}_l$ and $j = 0$. □

Lemma 2.5.5 *Let* $\mathfrak{t} \in \Sigma_C^{w_h}$ *be a proper cluster such that* $\mathfrak{t} \not\supseteq \mathfrak{s}_h \wedge \mathfrak{s}_l$, *and let* M *be a matrix associated to* $\mathfrak{t}$. *Then*

$$\tilde{m}_{21} - \rho_{hl}\tilde{m}_{23} \geq 0 \quad \text{and} \quad \tilde{m}_{31} - \rho_{hl}\tilde{m}_{33} > 0.$$

Furthermore, $\tilde{m}_{21} - \rho_{hl}\tilde{m}_{23} = 0$ *if and only if*

- $M = M_{L_{\mathfrak{t}}^{w_h},i}$ *and* $i = r_{L_{\mathfrak{t}}^{w_h}}$, *or*
- $\mathfrak{t} < \mathfrak{s}_h \wedge \mathfrak{s}_l$, $M = M_{V_{\mathfrak{t}}^{w_h},j}$, *and* $j = r_{V_{\mathfrak{t}}^{w_h}}$.

Proof. This result follows again from Lemma 2.5.2. As in the previous lemma, we show it when $\mathfrak{t}$ is even and $M = M_{V_{\mathfrak{t}}^{w_h},j}$, and leave the other cases for the reader.

Let $r = r_{V_{\mathfrak{t}}^{w_h}}$. Note that $t \neq \mathfrak{R}$. Set $\delta = \delta_M$ and $\gamma_j = \frac{n_0 d_j - n_j d_0}{\delta d_0}$. Then

$$\tilde{m}_{31} - \rho_{hl}\tilde{m}_{33} = \delta\left(d_j\left(\rho_t - \rho_{hl}\right) - \gamma_j\right) > \delta d_j\left(\rho_{P(\mathfrak{t})} - \rho_{\mathfrak{s}_h\wedge\mathfrak{s}_l}\right) \geq 0.$$

since $d_j > 0$ and $\gamma_j/d_j < \gamma_{r+1}/d_{r+1} = \rho_{\mathfrak{t}} - \rho_{P(\mathfrak{t})}$. Similarly,

$$\tilde{m}_{21} - \rho_{hl}\tilde{m}_{23} = \delta\left(d_{j+1}\left(\rho_t - \rho_{hl}\right) - \gamma_{j+1}\right) \geq \delta d_{j+1}\left(\rho_{P(\mathfrak{t})} - \rho_{\mathfrak{s}_h\wedge\mathfrak{s}_l}\right) \geq 0,$$

In particular $\tilde{m}_{21} - \rho_{hl}\tilde{m}_{23} = 0$ if and only if $\mathfrak{t} < \mathfrak{s}_h \wedge \mathfrak{s}_l$ and $j = r$. □

Let

$$T_M^h(X,Y,Z) := \prod_{l\neq h} T_M^{hl}(X,Y,Z),$$

and define

$$V_M^h := \operatorname{Spec} R[T_M^h(X,Y,Z)^{-1}] \subset X_M, \quad \text{and} \quad \mathring{X}_\Delta^h := \bigcup_{\mathfrak{t},M} V_M^h \subseteq X_\Delta^h,$$

where $\mathfrak{t}$ runs through all proper clusters in $\Sigma_C^{w_h}$ and M runs through all matrices associated to $\mathfrak{t}$. We can then define the subscheme

$$\mathring{\mathcal{C}}_\Delta^{w_h} := \mathcal{C}_\Delta^{w_h} \cap \mathring{X}_\Delta^h \subset X_\Delta^h,$$

where $\mathcal{C}_\Delta^{w_h}/O_K$ is the model of the hyperelliptic curve $C^{w_h} : y^2 = f(x+w_h)$ described in [Dok, Theorem 3.14] (see [Dok, §4] for the construction). Explicitly, let $g_h(x,y) := y^2 - f(x+w_h)$ and define $\mathcal{F}_M^h \in O_K[X^{\pm 1}, Y, Z]$ such that $\mathrm{ord}_Y(\mathcal{F}_M^h) = \mathrm{ord}_Z(\mathcal{F}_M^h) = 0$, with all non-zero coefficients in $O_K^\times$, satisfying

$$y^2 - f(x+w_h) \overset{M}{=} Y^{n_{Y,h}} Z^{n_{Z,h}} \mathcal{F}_M^h(X,Y,Z),$$

for unique $n_{Y,h}, n_{Z,h} \in \mathbb{Z}$. Consider the subscheme

$$U_M^h := \mathrm{Spec}\, \frac{R\left[T_M^h(X,Y,Z)^{-1}\right]}{\left(\mathcal{F}_M^h(X,Y,Z)\right)} \subset V_M^h.$$

Then

$$\mathring{\mathcal{C}}_\Delta^{w_h} = \bigcup_{\mathfrak{t},M} U_M^h \subset \mathring{X}_\Delta^h,$$

where $\mathfrak{t}$ runs through all proper clusters in $\Sigma_C^{w_h}$ and M runs through all matrices associated to $\mathfrak{t}$, as before.

2.5.3 Glueing

In this subsection we show how to glue the schemes $\mathring{\mathcal{C}}_\Delta^w$, for $w \in W$, to obtain a proper flat model $\mathcal{C}$ of C (properness will be proved in §2.5.8-2.5.9).

Let $h, l = 1, \dots, m$, with $h \neq l$. Consider the isomorphism

$$\phi : K\left[x^{\pm 1}, y^{\pm 1}, \prod_{o \neq l}(x+w_{lo})^{-1}\right] \xrightarrow{\simeq} K\left[x^{\pm 1}, y^{\pm 1}, \prod_{o \neq h}(x+w_{ho})^{-1}\right] \tag{2.1}$$

sending $x \mapsto x + w_{hl}$, $y \mapsto y$. If $\mathfrak{t} \supseteq \mathfrak{s}_h \wedge \mathfrak{s}_l$ and M is a matrix associated to $\mathfrak{t}$, then ϕ gives a map

$$R[Y^{-1}, Z^{-1}, T_M^l(X,Y,Z)^{-1}] \xrightarrow{M^{-1}\circ\phi\circ M} R[Y^{-1}, Z^{-1}, T_M^h(X,Y,Z)^{-1}],$$

which sends

$$F(X,Y,Z) \mapsto F(X \cdot T_M^{hl}(X,Y,Z)^{m_{11}}, Y \cdot T_M^{hl}(X,Y,Z)^{m_{12}}, Z \cdot T_M^{hl}(X,Y,Z)^{m_{13}}).$$

Hence it induces the isomorphisms

$$R[T_M^l(X,Y,Z)^{-1}] \xrightarrow{\simeq} R[T_M^h(X,Y,Z)^{-1}], \qquad V_M^h \xrightarrow{\simeq} V_M^l. \tag{2.2}$$

Via these maps we see that $g_h(x,y) = Y^{n_{Y,h}} Z^{n_{Z,h}} \mathcal{F}_M^h(X,Y,Z)$ also equals

$$Y^{n_{Y,l}} \cdot Z^{n_{Z,l}} \cdot (T_M^{hl})^{n_{Y,l}m_{12} + n_{Z,l}m_{13}} \mathcal{F}_M^l\left(X \cdot (T_M^{hl})^{m_{11}}, Y \cdot (T_M^{hl})^{m_{12}}, Z \cdot (T_M^{hl})^{m_{13}}\right),$$

where $T_M^{hl} = T_M^{hl}(X,Y,Z)$. Since neither Y nor Z divide $T_M^{hl}(X,Y,Z)$, we have $n_{Y,h} = n_{Y,l}$, $n_{Z,h} = n_{Z,l}$ and

$$\mathcal{F}_M^h(X,Y,Z) = (T_M^{hl})^{n_{Y,l}m_{12}+n_{Z,l}m_{13}} \mathcal{F}_M^l\left(X(T_M^{hl})^{m_{11}}, Y(T_M^{hl})^{m_{12}}, Z(T_M^{hl})^{m_{13}}\right).$$

Hence (2.2) induces the isomorphisms

$$\frac{R\left[T_M^l(X,Y,Z)^{-1}\right]}{\left(\mathcal{F}_M^l(X,Y,Z)\right)} \xrightarrow{\simeq} \frac{R\left[T_M^h(X,Y,Z)^{-1}\right]}{\left(\mathcal{F}_M^h(X,Y,Z)\right)}, \qquad U_M^h \xrightarrow{\simeq} U_M^l. \tag{2.3}$$

Define the subschemes

$$V^{hl} := \bigcup_{\mathfrak{t}_l, M_l} V_{M_l}^h \subseteq \mathring{X}_\Delta^h, \qquad U^{hl} := V^{hl} \cap \mathcal{C}_\Delta^{w_h} \subseteq \mathring{\mathcal{C}}_\Delta^{w_h},$$

where $\mathfrak{t}_l$ runs through all proper clusters in $\Sigma_C^{w_h} \cap \Sigma_C^{w_l}$ (i.e. $\mathfrak{t}_l \in \Sigma^W$, $\mathfrak{s}_h \wedge \mathfrak{s}_l \subseteq \mathfrak{t}_l$) and M_l runs through all matrices associated to $\mathfrak{t}_l$. From (2.1), (2.2) and (2.3) we have isomorphisms of schemes

$$V^{hl} \xrightarrow{\simeq} V^{lh}, \qquad U^{hl} \xrightarrow{\simeq} U^{lh}. \tag{2.4}$$

Now, $U^{hl} \subset V^{hl}$ are open subschemes respectively of $\mathring{\mathcal{C}}_\Delta^{w_h} \subset \mathring{X}_\Delta^h$ for any $l \neq h$. Glueing the schemes $\mathring{\mathcal{C}}_\Delta^{w_h} \subset \mathring{X}_\Delta^h$, to varying of $h = 1,\dots,m$, respectively along the opens $U^{hl} \subset V^{hl}$ via (2.4) gives the schemes $\mathcal{C} \subset \mathcal{X}$. We will show that $\mathcal{C}/O_K$ is a proper flat[4] model of C.

2.5.4 Generic fibre

We start studying the generic fibre $\mathcal{C}_\eta$ of $\mathcal{C}$. Since it is the glueing of all $\mathring{\mathcal{C}}_{\Delta,\eta}^{w_h}$ through the glueing maps

$$U_\eta^{hl} \longrightarrow U_\eta^{lh}$$

induced by (2.4), we start focusing on $\mathring{\mathcal{C}}_{\Delta,\eta}^{w_h}$ for $h = 1,\dots,m$. In particular, as $\mathring{\mathcal{C}}_\Delta^{w_h}$ is an open subscheme of $\mathcal{C}_\Delta^{w_h}$, we study $\mathcal{C}_{\Delta,\eta}^{w_h} \smallsetminus \mathring{\mathcal{C}}_{\Delta,\eta}^{w_h} = C^{w_h} \smallsetminus \mathring{\mathcal{C}}_{\Delta,\eta}^{w_h}$.

Lemma 2.5.6 *For any $h = 1,\dots,m$,*

$$C^{w_h} \smallsetminus \mathring{\mathcal{C}}_{\Delta,\eta}^{w_h} = \operatorname{Spec} \frac{K[x,y]}{\left(g_h(x,y), \prod_{o\neq h}(x + w_{ho})\right)}.$$

Proof. For every choice of a proper cluster $\mathfrak{t} \in \Sigma_C^{w_h}$, and M associated to $\mathfrak{t}$, let

$$P_M := \left(\mathcal{C}_{\Delta,\eta}^{w_h} \smallsetminus \mathring{\mathcal{C}}_{\Delta,\eta}^{w_h}\right) \cap X_M = \operatorname{Spec} \frac{R \otimes_{O_K} K}{\left(\mathcal{F}_M^h(X,Y,Z), T_M^h(X,Y,Z)\right)}.$$

To study P_M we are going to use Lemma 2.5.2 and the definition of $T_M^h(X,Y,Z)$.

[4] Note that the flatness of $\mathcal{C}$ is trivial since it is a local property.

Suppose first $\mathfrak{t} \neq \mathfrak{R}$ and $M = M_{V_{\mathfrak{t}}^{w_h},j}$. Then $\tilde{m}_{23}, \tilde{m}_{33} > 0$, so

$$P_M = \operatorname{Spec} \frac{R[Y^{-1}, Z^{-1}]}{\left(\mathcal{F}_M^h(X,Y,Z), T_M^h(X,Y,Z)\right)} \overset{M}{\simeq} \operatorname{Spec} \frac{K[x^{\pm 1}, y^{\pm 1}]}{(g_h(x,y), \prod_o (x + w_{ho}))}, \tag{2.5}$$

where the product runs over all $o \neq h$. Now let $\mathfrak{t} = \mathfrak{R}$ and $M = M_{V_{\mathfrak{t}}^{w_h},j}$. If $j \neq r_{V_{\mathfrak{R}}^{w_h}}$, then P_M is as in the previous case (since $\tilde{m}_{23}, \tilde{m}_{33} > 0$). If $j = r_{V_{\mathfrak{R}}^{w_h}}$, then $\tilde{m}_{33} > 0$, $\tilde{m}_{23} = 0$, but $\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21} > 0$ by Lemma 2.5.4. So from the definition of $T_M^{hl}(X,Y,Z)$ we have once more the equality (2.5). Similarly, if $\mathfrak{t} = \mathfrak{s}_h$ and $M = M_{V_0^{w_h},j}$, then $\tilde{m}_{33} > 0$, and $\tilde{m}_{21} - \rho_{hl}\tilde{m}_{23} > 0$ by Lemma 2.5.5. Hence we have (2.5) again.

It remains to study P_M when $M = M_{L_{\mathfrak{t}}^{w_h},i}$. If $i \neq r_{L_{\mathfrak{t}}^{w_h}}$, then $\tilde{m}_{23}, \tilde{m}_{33} > 0$ and so P_M is as in (2.5). Let $i = r_{L_{\mathfrak{t}}^{w_h}}$. Then $\tilde{m}_{33} > 0$ but both $\tilde{m}_{23}$ and $\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21}$ equal 0. Hence $\tilde{m}_{23} = \tilde{m}_{21} = 0$, which also implies $m_{21} = m_{23} = 0$. Therefore M defines an isomorphism $R[Z^{-1}] \simeq K[x^{\pm 1}, y]$, which induces

$$P_M = \operatorname{Spec} \frac{R[Z^{-1}]}{\left(\mathcal{F}_M^h(X,Y,Z), T_M^h(X,Y,Z)\right)} \overset{M}{\simeq} \operatorname{Spec} \frac{K[x^{\pm 1}, y]}{\left(g_h(x,y), \prod_{o \neq h} (x + w_{ho})\right)}.$$

This concludes the proof. □

Regarding $\mathcal{C}_{\Delta}^{w_h}$ as a model of C via the natural isomorphism $C \xrightarrow{\sim} C^{w_h}$, we get

$$C \smallsetminus \mathring{\mathcal{C}}_{\Delta,\eta}^{w_h} = \operatorname{Spec} \frac{K[x,y]}{\left(y^2 - f(x), \prod_{o \neq h} (x - w_o)\right)}.$$

Thus the generic fibre of $\mathcal{C}$ is isomorphic to C.

2.5.5 Special fibre

We now study the structure of the special fibre $\mathcal{C}_s$ of $\mathcal{C}$. As for the generic fibre, we consider

$$\mathcal{C}_{\Delta,s}^{w_h} \smallsetminus \mathring{\mathcal{C}}_{\Delta,s}^{w_h},$$

for any $h = 1, \dots, m$. For every choice of a proper cluster $\mathfrak{t} \in \Sigma_C^{w_h}$, and M associated to $\mathfrak{t}$, let

$$S_M := \left(\mathcal{C}_{\Delta,s}^{w_h} \smallsetminus \mathring{\mathcal{C}}_{\Delta,s}^{w_h}\right) \cap X_M = \operatorname{Spec} \frac{O_K[X^{\pm 1}, Y, Z]}{\left(\mathcal{F}_M^h(X,Y,Z), T_M^h(X,Y,Z), Y^{\tilde{m}_{23}} Z^{\tilde{m}_{33}}, \pi\right)}.$$

Lemma 2.5.7 *Let $M = M_{L,i}$ for $L = L_{\mathfrak{t}}^{w_h}$. Let $l \neq h$. If $\mathfrak{t} = \mathfrak{s}_l \wedge \mathfrak{s}_h$, then $T_M^{hl}(X,Y,Z) = X^{-1}(X + u_{hl})$, otherwise*

(i) $T_M^{hl}(X,Y,0) = 1$ for $i = 0, \dots, r_L$;

(ii) $T_M^{hl}(X,0,Z) = 1$ for $i = 0, \dots, r_L - 1$.

Proof. Fix $l \neq h$. If $\mathfrak{t} \not\supseteq \mathfrak{s}_l \wedge \mathfrak{s}_h$, then by Lemma 2.5.5, we have $\tilde{m}_{21} - \rho_{hl}\tilde{m}_{23} \geq 0$ and $\tilde{m}_{31} - \rho_{hl}\tilde{m}_{33} > 0$. Moreover, if $\tilde{m}_{21} - \rho_{hl}\tilde{m}_{23} = 0$, then $i = r_L$. Therefore the equalities in (i) and (ii) follow directly from the definition of T_M^{hl}.

On the other hand, if $\mathfrak{t} \supsetneq \mathfrak{s}_l \wedge \mathfrak{s}_h$, then by Lemma 2.5.4, we have $\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21} \geq 0$ and $\rho_{hl}\tilde{m}_{33} - \tilde{m}_{31} > 0$. Moreover, if $\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21} = 0$, then $i = r_L$. Therefore we have (i) and (ii) again.

Finally, assume $\mathfrak{t} = \mathfrak{s}_l \wedge \mathfrak{s}_h$. Since $\rho_{\mathfrak{t}} = \rho_{hl} \in \mathbb{Z}$, then $\rho_{hl}\tilde{m}_{13} - \tilde{m}_{11} = -1$. Hence

$$T_M^{hl}(X,Y,Z) = 1 + u_{hl}X^{-1} = X^{-1}(X + u_{hl}),$$

by Lemma 2.5.4. □

Lemma 2.5.8 *Suppose $M = M_{L_{\mathfrak{t}}^{w_h},i}$. Then*

$$S_M = \operatorname{Spec} \frac{O_K[X^{\pm 1}, Y, Z]}{(\mathcal{F}_M^h(X,Y,Z), \prod_l (X + u_{hl}), Y^{\tilde{m}_{23}} Z^{\tilde{m}_{33}}, \pi)} \subset \mathcal{C}_\Delta^{w_h},$$

where the product runs over all $l \neq h$ such that $\mathfrak{t} = \mathfrak{s}_l \wedge \mathfrak{s}_h$.

Proof. Lemma 2.5.2 shows that $\tilde{m}_{33}$ is always different from 0, while $\tilde{m}_{23} = 0$ if and only if $i = r_{L_{\mathfrak{t}}^{w_h}}$. Thus the result follows from Lemma 2.5.7. □

Lemma 2.5.9 *Let $f_h(x) = f(x + w_h)$ and $l \neq h$. Then $\overline{u_{lh}}$ is a multiple root of $\overline{f_h|_L}$ of order $|\mathfrak{t}_l|$, where $L = L_{\mathfrak{s}_h \wedge \mathfrak{s}_l}^{w_h}$ and $\mathfrak{t}_l \in \Sigma_C^{w_l}$, $\mathfrak{t}_l < \mathfrak{s}_h \wedge \mathfrak{s}_l$.*

Furthermore, if $\Sigma = \{\mathfrak{s}_1, \dots, \mathfrak{s}_m\} = \Sigma_C^{\min}$, C has an almost rational cluster picture and $\bar{\alpha} \in \bar{k}$ is a multiple root of $\overline{f_h|_L}$ for some edge L of $\mathtt{NP}(f_h)$, then $\bar{\alpha} = \overline{u_{lh}}$ and $L = L_{\mathfrak{s}_h \wedge \mathfrak{s}_l}^{w_h}$ for some $l \neq h$.

Proof. For any proper cluster $\mathfrak{s} \in \Sigma_f$, let $\lambda_{\mathfrak{s}} = \min_{r \in \mathfrak{s}} v(r - w_h)$. Let $\mathfrak{s} \in \Sigma_C^{w_l}$, with $\mathfrak{s}_l \subseteq \mathfrak{s} \subsetneq \mathfrak{s}_h \wedge \mathfrak{s}_l$. Then w_h is not rational centre of $\mathfrak{s}$, and for every root $r \in \mathfrak{s}$, one has

$$v(r - w_h) = v(r - w_l + w_l - w_h) = \min\{v(r - w_l), \rho_{hl}\} = \rho_{hl},$$

as $v(r - w_l) \geq \rho_{\mathfrak{s}} > \rho_{hl}$. Therefore $\lambda_{\mathfrak{s}} = \rho_{hl} \in \mathbb{Z}$. In particular, $|\lambda_{\mathfrak{s}}|_p \leq 1$. Furthermore,

$$d_{\mathfrak{s}} \geq \rho_{\mathfrak{s}} > \lambda_{\mathfrak{s}} = \rho_{hl} \qquad \text{and} \qquad \frac{r - w_h}{\pi^{\rho_{hl}}} \equiv \frac{w_{lh}}{\pi^{\rho_{hl}}} \mod \pi,$$

and so Theorem 2.3.24(i) implies that $\overline{u_{lh}} = \frac{w_{lh}}{\pi^{\rho_{hl}}} \mod \pi$ is a multiple root of $\overline{f_h|_L}$, where $L = L_{\mathfrak{s}_h \wedge \mathfrak{s}_l}^{w_h}$.

Let $\mathfrak{t}_l \in \Sigma_C^{w_l}$, $\mathfrak{t}_l < \mathfrak{s}_h \wedge \mathfrak{s}_l$. Since $\mathfrak{s}_l \subseteq \mathfrak{t}_l < \mathfrak{s}_h \wedge \mathfrak{s}_l$ we have

$$\mathfrak{t}_l = \left\{ r \in \mathfrak{R} \mid \overline{u_{lh}} = \tfrac{r - w_h}{\pi^{\rho_{hl}}} \mod \pi \right\},$$

as $v(r - w_l) > \rho_{hl}$ if and only if $\overline{u_{lh}} = \frac{r - w_h}{\pi^{\rho_{hl}}} \mod \pi$. Thus the multiplicity of $\overline{u_{lh}}$ is $|\mathfrak{t}_l|$ by Theorem 2.3.24(ii).

Now let $\bar{\alpha}$ be a multiple root of $\overline{f_h|_L}$ for some edge L of $\mathtt{NP}(f_h)$ and let $\mathfrak{s} \in \Sigma_f$ associated to $\bar{\alpha}$ by Theorem 2.3.24(iii). We want to prove that if C has an almost rational cluster picture and

$\Sigma = \Sigma_C^{\min}$, then there exists $l \neq h$ so that $\bar{\alpha} = \overline{u_{lh}}$. Note first w_h is not a rational centre of $\mathfrak{s}$. Indeed, if w_h is a rational centre of $\mathfrak{s}$, then

$$|\mathfrak{s}| > |\lambda_{\mathfrak{s}}|_p = |\rho_{\mathfrak{s}}|_p, \qquad d_{\mathfrak{s}} > \lambda_{\mathfrak{s}} = \rho_{\mathfrak{s}},$$

which contradicts the fact that C has an almost rational cluster picture. As $\{\mathfrak{s}_1, \dots, \mathfrak{s}_m\} = \Sigma_C^{\min}$, we must have that w_l is a rational centre of $\mathfrak{s}$, for some $l \neq h$. Then $\mathfrak{s}_l \subseteq \mathfrak{s} \subsetneq \mathfrak{s}_h \wedge \mathfrak{s}_l$. Since $\bar{\alpha} = \frac{r - w_h}{\pi^{\lambda_{\mathfrak{s}}}}$ mod π for any $r \in \mathfrak{s}$, from above we have $\bar{\alpha} = \overline{u_{lh}}$. Finally, L is the edge of $\mathtt{NP}(f_h)$ of slope $-\lambda_{\mathfrak{s}} = -\rho_{hl}$. Thus $L = L^{w_h}_{\mathfrak{s}_h \wedge \mathfrak{s}_l}$. □

It remains to compute S_M when $M = M_{V,j}$, where $V = V_{\mathfrak{t}}^{w_h}$ or $V = V_0^{w_h}$.

Lemma 2.5.10 *Let $M = M_{V,j}$ for $V = V_{\mathfrak{t}}^{w_h}$, or $V = V_0^{w_h}$ if $\mathfrak{t} = \mathfrak{s}_h$. For any $l \neq h$ we have*

(i) $T_M^{hl}(X, Y, 0) = 1$ *except when* $\mathfrak{t} = \mathfrak{s}_l \wedge \mathfrak{s}_h$ *and* $j = 0$*;*

(ii) $T_M^{hl}(X, 0, Z) = 1$ *except when* $\mathfrak{t} < \mathfrak{s}_l \wedge \mathfrak{s}_h$ *and* $j = r_V$.

Proof. The lemma immediately follows from Lemmas 2.5.4 and 2.5.5. □

Lemma 2.5.11 *Let $M = M_{V,j}$ with $V = V_{\mathfrak{t}}^{w_h}$, or $V = V_0^{w_h}$ if $\mathfrak{t} = \mathfrak{s}_h$. Then $S_M = \varnothing$.*

Proof. For any $l \neq h$, we want to prove that

$$S_M^{hl} := \{T_M^{hl}(X, Y, Z) = Y^{\tilde{m}_{23}} Z^{\tilde{m}_{33}} = 0\} = \varnothing. \tag{2.6}$$

Lemma 2.5.2 shows that $\tilde{m}_{33}$ is always different from 0 and that $\tilde{m}_{23} = 0$ if and only if $j = r_V$, and $V = V_{\mathfrak{R}}^{w_h}$ or $V = V_0^{w_h}$. Assume that if $\mathfrak{t} = \mathfrak{s}_l \wedge \mathfrak{s}_h$ then $j \neq 0$ and that if $\mathfrak{t} < \mathfrak{s}_l \wedge \mathfrak{s}_h$ then $j \neq r_V$. Lemma 2.5.10 implies (2.6).

If $\mathfrak{t} = \mathfrak{s}_l \wedge \mathfrak{s}_h$ and $j = 0$, then $\rho_{hl}\tilde{m}_{33} - \tilde{m}_{31} = 0$ but $\rho_{hl}\tilde{m}_{23} - \tilde{m}_{21} > 0$. So

$$S_M^{hl} = \{T_M^{hl}(X, Y, Z) = Z^{\tilde{m}_{33}} = 0\} \subset \operatorname{Spec} R[Y^{-1}].$$

Similarly, if $\mathfrak{t} < \mathfrak{s}_l \wedge \mathfrak{s}_h$ and $j = r_V$, then $\tilde{m}_{21} - \rho_{hl}\tilde{m}_{23} = 0$, however $\tilde{m}_{31} - \rho_{hl}\tilde{m}_{33} > 0$. Then

$$S_M^{hl} = \{T_M^{hl}(X, Y, Z) = Y^{\tilde{m}_{23}} = 0\} \subset \operatorname{Spec} R[Z^{-1}].$$

In both cases, $S_M^{hl} \subseteq X_F$ as sets, where $F = F^{w_h}_{\mathfrak{s}_l \wedge \mathfrak{s}_h}$ ([Dok, Definition 3.7]). Let $L = L^{w_h}_{\mathfrak{s}_l \wedge \mathfrak{s}_h}$, and let $f_h(x) = f(x + w_h)$ and $g_h(x, y) = y^2 - f_h(x)$. By Lemmas 2.5.8 and 2.5.9, one has

$$S_M^{hl} \subseteq X_F \cap S_{M_{L,0}} = \varnothing,$$

as $\mathcal{F}^h_{M_{L,0}}(X, Y, 0) \mod \pi$ equals $Y^b - X^a \overline{f_h|_L}(X)$, for some $a \in \mathbb{Z}$, $b = 1, 2$ (see Lemma 2.5.17 for more details, whose proof is independent of this result). Thus if $V = V_{\mathfrak{t}}^{w_h}$ and $M = M_{V,j}$, then $S_M = \varnothing$. □

2.5.6 Components

Now that we have compared the special fibre of $\mathcal{C}$ with those of the models $\mathcal{C}_{\Delta}^{w_h}$, let us describe closed subschemes that form it. We will first study closed subschemes forming $\mathcal{C}_{\Delta,s}^{w_h}$ and then how they glue in $\mathcal{C}_s$.

Let $f_h(x) = f(x+w_h)$ and $g_h(x,y) = y^2 - f_h(x)$. According to [Dok, Theorem 3.14] the special fibre of $\mathcal{C}_{\Delta}^{w_h}$ is formed by:

- Chains of $\mathbb{P}^1_k$s coming from v-edges of Δ^{w_h}.
- 1-dimensional subschemes coming from v-faces of Δ^{w_h}.

More precisely, each v-edge E gives a scheme $X_E \times \mathbb{P}_E$, where $\mathbb{P}_E$ is a chain of $\mathbb{P}^1_k$s and $X_E \subset \mathbb{G}_{m,k}$ is given by $\overline{g_h|_E} = 0$. The multiplicities and and the length of $\mathbb{P}_E$ can be completely described by the slopes of E. On the other hand, each v-face F gives a proper scheme $\bar{X}_F$ containing an open subscheme $X_F \subseteq \mathbb{G}^2_{m,k}$ given by $\overline{g_h|_F} = 0$. How the previous schemes intersect to form $\mathcal{C}_{\Delta,s}^{w_h}$ is described by [Dok, Theorem 3.14]. The reader is pointed to [Dok] for more details.

Definition 2.5.12 Let $\mathfrak{t} \in \Sigma^W$ be a proper cluster. For any rational centre w of $\mathfrak{t}$, let $r_{\mathfrak{t},w} = \frac{w-r}{\pi^{\rho_\mathfrak{t}}}$, $u_{\mathfrak{t},w} = c_f \prod_{r\in\mathfrak{R}\setminus\mathfrak{t}} r_{\mathfrak{t},w}$ and $u^0_{\mathfrak{s}_h,w_h} = c_f \prod_{r\in\mathfrak{R}\setminus\{w_h\}} r_{\mathfrak{s}_h,w_h}$. Define $\overline{f^W_{\mathfrak{t},w}}, \overline{g_{\mathfrak{t},w}} \in k[X]$, and $\overline{g^0_{\mathfrak{s}_h,w_h}} \in k[X]$ for any $h = 1,\dots,m$, as follows:

(1) Let $u = u_{\mathfrak{t},w}$. Define $\overline{f^W_{\mathfrak{t},w}}$ by

$$\overline{f^W_{\mathfrak{t},w}}(X^{b_\mathfrak{t}}) = \frac{u}{\pi^{v(u)}} \prod_{r\in\mathfrak{t}\setminus\bigcup_{\mathfrak{s}<\mathfrak{t}}\mathfrak{s}} (X + r_{\mathfrak{t},w}) \mod \pi,$$

where the union runs through all children $\mathfrak{s}$ of $\mathfrak{t}$ in Σ^W. If $\Sigma = \Sigma_C^{\min}$ denote $\overline{f^W_{\mathfrak{t},w}}$ by $\overline{f_{\mathfrak{t},w}}$.

(2) Let $u = u_{\mathfrak{t},w}$. Define $\overline{g_{\mathfrak{t},w}}(X) := X^{p_\mathfrak{t}/\gamma_\mathfrak{t}} - \frac{u}{\pi^{v(u)}} \mod \pi$.

(3) Let $u = u^0_{\mathfrak{s}_h,w_h}$. Define $\overline{g^0_{\mathfrak{s}_h,w_h}}(X) := X^{p^0_{\mathfrak{s}_h}/\gamma^0_{\mathfrak{s}_h}} - \frac{u}{\pi^{v(u)}} \mod \pi$.

Note that the polynomials defined in Definition 2.5.12 agree with the ones in Definition 2.4.14 when $w = w_\mathfrak{t}$.

Lemma 2.5.13 *Let $\mathfrak{s},\mathfrak{t} \in \Sigma_C^{\mathrm{rat}}$, with $\mathfrak{s} \subsetneq \mathfrak{t}$. Let w', w be rational centres of $\mathfrak{s}$ and $\mathfrak{t}$ respectively, and define $\overline{u_{w'w}} = \frac{w'-w}{\pi^{\rho_\mathfrak{t}}} \mod \pi$. Then $\overline{u_{w'w}}$ does not depend on the choice of a rational centre w' of $\mathfrak{s}$.*

Proof. Suppose that w_1, w_2 are two rational centres of $\mathfrak{s}$. Then $v(w_1 - w_2) \geq \rho_\mathfrak{s} > \rho_\mathfrak{t}$, and so the lemma follows. □

Remark 2.5.14. Let $\mathfrak{t} \in \Sigma_C^{w_h}$. Let $l = 1,\dots,m$, $l \neq h$. Then $\mathfrak{t} = \mathfrak{s}_h \wedge \mathfrak{s}_l$ if and only if it has a child $\mathfrak{s} \in \Sigma_C^{w_l} \setminus \Sigma_C^{w_h}$. In particular, if this happens, Lemma 2.5.13 shows that $\overline{u_{lh}} = \frac{w-w_h}{\pi^{\rho_\mathfrak{t}}} \mod \pi$ for any rational centre w of $\mathfrak{s}$.

Definition 2.5.15 Let $\mathfrak{t}\in\Sigma_C^{w_h}$ be a proper cluster. Define $\hat{\mathfrak{t}}^W := \{\mathfrak{s}\in\Sigma^W\cup\{\varnothing\}\mid \mathfrak{s}<\mathfrak{t}\}$, where $\varnothing<\mathfrak{t}$ only if $\mathfrak{t}$ has no child in Σ^W. If $\varnothing<\mathfrak{t}$ then we will say that w_h is the rational centre of $\varnothing$.

Define $\mathbb{G}_{\mathfrak{t},w_h} := \mathbb{G}_{m,k}\setminus\bigcup_l\{\overline{u_{lh}}\}$, where the union runs through all $l\neq h$ such that $\mathfrak{s}_l\wedge\mathfrak{s}_h=\mathfrak{t}$. Note that Remark 2.5.14 shows that $\mathbb{G}_{\mathfrak{t},w_h}=\mathbb{A}^1_k\setminus\bigcup_{\mathfrak{s}\in\hat{\mathfrak{t}}^W}\{\overline{u_{w_\mathfrak{s}w_h}}\}$, where $\overline{u_{w_\mathfrak{s}w_h}}=\frac{w_\mathfrak{s}-w_h}{\pi^{\rho_\mathfrak{t}}} \mod \pi$, and $w_\mathfrak{s}$ is any rational centre of $\mathfrak{s}$.

Let $\mathfrak{t}\in\Sigma_C^{w_h}$ be a proper cluster. Let $V=V_\mathfrak{t}^{w_h}$ and $M=M_{V,j}$. In §2.5.5 we showed the special fibre of U_M^h equals $X_M\cap\mathcal{C}_{\Delta,s}^{w_h}$. Therefore the components of $\mathring{\mathcal{C}}_{\Delta,s}^{w_h}$ coming from V are the same of those of $\mathcal{C}_{\Delta,s}^{w_h}$ given by the same v-edge. Therefore V gives a closed subscheme $X_V\times\mathbb{P}_V$ of $\mathring{\mathcal{C}}_{\Delta,s}^{w_h}$, where $\mathbb{P}_V$ is a chain of $\mathbb{P}^1_k$s and $X_V:\{\overline{g_h|_V}=0\}$ over $\mathbb{G}_{m,k}$. Lemma 2.4.3 implies that $\overline{g_h|_V}=\overline{g_{\mathfrak{t},w_h}}$.

Let $V_0=V_0^{w_h}$ and $M=M_{V_0,j}$. Similarly to above, $X_M\cap\mathring{\mathcal{C}}_{\Delta,s}^{w_h}=X_M\cap\mathcal{C}_{\Delta,s}^{w_h}$ and so V_0 gives rise to a closed subscheme $X_{V_0}\times\mathbb{P}_{V_0}$ of $\mathring{\mathcal{C}}_{\Delta,s}^{w_h}$, where $\mathbb{P}_{V_0}$ is a chain of $\mathbb{P}^1_k$s and $X_{V_0}:\{\overline{g_h|_{V_0}}=0\}$ over $\mathbb{G}_{m,k}$. Note that $\overline{g_h|_{V_0}}=\overline{g^0_{\mathfrak{s}_h,w_h}}$.

Let $\mathfrak{t}\in\Sigma_C^{w_h}$ be a proper cluster. Let $L=L_\mathfrak{t}^{w_h}$ and $M=M_{L,i}$. By Lemma 2.5.8, the v-edge L gives a subscheme $X_L^W\times\mathbb{P}_L$ of $\mathring{\mathcal{C}}_{\Delta,s}^{w_h}$, where $\mathbb{P}_L$ is a chain of $\mathbb{P}^1_k$s of length r_L and $X_L^W:\{\overline{g_h|_L}=0\}$ in $\mathbb{G}_{\mathfrak{t},w_h}$. Note that $r_L=0$ or 1 by Lemma 2.4.3 and $r_L=1$ if and only if $D_\mathfrak{t}=1$. Let $\mathfrak{t}_h\in\Sigma_C^{w_h}$ be the unique child of $\mathfrak{t}$ with rational centre w_h or set $\mathfrak{t}_h=\varnothing$ if $\mathfrak{t}$ has no such child. We will show that

$$\overline{g_h|_L}(X)=-\prod_{\mathfrak{s}\in\hat{\mathfrak{t}}^W,\mathfrak{s}\neq\mathfrak{t}_h}(X+\overline{u_{w_\mathfrak{s}w_h}})^{|\mathfrak{s}|}\cdot\overline{f^W_{\mathfrak{t},w_h}}(X). \tag{2.7}$$

where $\overline{u_{w_\mathfrak{s}w_h}}=\frac{w_\mathfrak{s}-w_h}{\pi^{\rho_\mathfrak{t}}} \mod \pi$, and $w_\mathfrak{s}$ is any rational centre of $\mathfrak{s}$.

Suppose $\mathfrak{t}\neq\mathfrak{s}_h\wedge\mathfrak{s}_l$ for any $l\neq h$. Equivalently, all children of $\mathfrak{t}$ in Σ^W (at most one) belong to $\Sigma_C^{w_h}$. Then Lemma 2.4.3 shows that $\overline{g_h|_L}=-\overline{f^W_{\mathfrak{t},w_h}}$. Suppose now that $\mathfrak{t}=\mathfrak{s}_h\wedge\mathfrak{s}_l$ for some $l\neq h$. In this case $b_\mathfrak{t}=1$. We have

$$\frac{\overline{g_h|_L}(X)}{\prod_{\mathfrak{s}\in\hat{\mathfrak{t}}^W,\mathfrak{s}\neq\mathfrak{t}_h}(X+\overline{u_{w_\mathfrak{s}w_h}})^{|\mathfrak{s}|}}=\left(\frac{-\frac{u}{\pi^{v(u)}}\prod_{r\in\mathfrak{t}\setminus\mathfrak{t}_h}(X+r_{\mathfrak{t},w_h})}{\prod_{\mathfrak{s}\in\hat{\mathfrak{t}}^W,\mathfrak{s}\neq\mathfrak{t}_h}\prod_{r\in\mathfrak{s}}(X+r_{\mathfrak{t},w_h})} \mod \pi\right)=-\overline{f^W_{\mathfrak{t},w_h}}(X),$$

where $r_{\mathfrak{t},w_h}$ and $u=u_{\mathfrak{t},w_h}$ are as in Definition 2.5.12. Indeed, $\overline{u_{w_\mathfrak{s}w_h}}=r_{\mathfrak{t},w_h} \mod \pi$ for every $r\in\mathfrak{s}$ as $v(w_\mathfrak{s}-r)\geq\rho_\mathfrak{s}>\rho_\mathfrak{t}$, and since $b_\mathfrak{t}=1$, Lemma 2.4.3 implies that

$$\overline{g_h|_L}(x)=-\tfrac{u}{\pi^{v(u)}}\textstyle\prod_{r\in\mathfrak{t}\setminus\mathfrak{t}_h}(x+r_{\mathfrak{t},w_h}) \mod \pi.$$

In particular, Remark 2.5.13 and Lemma 2.5.9 shows that $(X+\overline{u_{hl}})\nmid\overline{f^W_{\mathfrak{t},w_h}}(X)$, for any $l\neq h$ such that $\mathfrak{s}_l\wedge\mathfrak{s}_h=\mathfrak{t}$. Moreover, $X\nmid\overline{f^W_{\mathfrak{t},w_h}}(X)$ by definition. Therefore the scheme X_L^W is equal to the closed subscheme $X^W_{\mathfrak{t},w_h}\subset\mathbb{A}^1_k$ given by $\overline{f^W_{\mathfrak{t},w_h}}=0$.

Let $\mathfrak{t}\in\Sigma^W$ be a proper cluster. For any $h=1,\dots,m$ such that $\mathfrak{s}_h\subseteq\mathfrak{t}$, let $\bar{X}_{F_\mathfrak{t}^{w_h}}$ be the 1-dimensional closed subscheme of $\mathcal{C}_{\Delta,s}^{w_h}$ given by $F_\mathfrak{t}^{w_h}$. Define

$$\mathring{X}_{F_\mathfrak{t}^{w_h}} := \bar{X}_{F_\mathfrak{t}^{w_h}}\cap\mathring{\mathcal{C}}_\Delta^{w_h}.$$

Denote by $\Gamma_\mathfrak{t}$ the 1-dimensional closed subscheme of $\mathcal{C}_s$, result of the glueing of the subschemes $\mathring{X}_{F_\mathfrak{t}^{w_h}}$ of $\mathring{\mathcal{C}}_{\Delta,s}^{w_h}$ to varying of h such that $\mathfrak{t}\in\Sigma_C^{w_h}$.

Lemma 2.5.16 *Let $\mathfrak{t} \in \Sigma_C^{w_h}$ be a proper cluster. The multiplicity of $\Gamma_\mathfrak{t}$ in $\mathcal{C}_s$ is $m_\mathfrak{t}$.*

Proof. Let $L = L_\mathfrak{t}^{w_h}$, $M = M_{L,0}$, and let $F = F_\mathfrak{t}^{w_h}$. The multiplicity of $\bar{X}_{F_\mathfrak{t}^{w_h}}$, and so of $\mathring{X}_{F_\mathfrak{t}^{w_h}}$ and $\Gamma_\mathfrak{t}$, is δ_F. Hence we only need to show that $m_\mathfrak{t} = \delta_F$. Let $d_0 \in \mathbb{Z}$ as in Lemma 2.5.2. Then $\delta_F = \delta_L d_0$. The result follows as $\delta_L = b_\mathfrak{t}$ and d_0, denominator of s_1^L, equals $3 - D_\mathfrak{t}$ by Lemma 2.4.3. □

Lemma 2.5.17 *Let $L = L_\mathfrak{t}^{w_h}$, $F = F_\mathfrak{t}^{w_h}$ and $M = M_{L,0}$. Let $c \in \{0, \dots, b_\mathfrak{t} - 1\}$ such that $1/b_\mathfrak{t} - \rho_\mathfrak{t} \cdot c \in \mathbb{Z}$. Then $\mathcal{F}_M^h(X,Y,0) \mod \pi$ equals the polynomial*

$$\overline{g_h|_F}(X,Y) = Y^{D_\mathfrak{t}} - \prod_{\mathfrak{s} \in \hat{\mathfrak{t}}^W} (X - \overline{u_{w_\mathfrak{s} w_h}})^{\frac{|\mathfrak{s}|}{b_\mathfrak{t}} - c\epsilon_\mathfrak{t}} \overline{f_{\mathfrak{t},w_h}^W}(X),$$

where $\overline{u_{w_\mathfrak{s} w_h}} = \frac{w_\mathfrak{s} - w_h}{\pi^{\rho_\mathfrak{t}}} \mod \pi$, and $w_\mathfrak{s}$ is any rational centre of $\mathfrak{s}$.

In particular, $\Gamma_\mathfrak{t}^h \subset \mathbb{G}_{\mathfrak{t},w_h} \times \mathbb{A}_k^1$ given by $\overline{g|_F} = 0$ is the open subscheme $U_M^h \cap \{Z = 0\}$ of $\mathring{X}_F$, and the points in S_M belong to all irreducible components of $\bar{X}_F$.

Proof. From [Dok, §3.5] and the equation of C^{w_h}, the polynomial $\mathcal{F}_M^h(X,Y,0)$ reduces modulo π to $X^{a_1} Y^b + X^{a_2} \overline{g_h|_L}(X)$, for some $b = 1,2$ and $a \in \mathbb{Z}$. Lemma 2.4.9 shows that $b = D_\mathfrak{t}$. By Lemma 2.4.3, $a_1 = 2\tilde{m}_{12}$, $a_2 = |\mathfrak{t}_h| \tilde{m}_{11} + (\epsilon_\mathfrak{t} - |\mathfrak{t}_h| \rho_\mathfrak{t}) \tilde{m}_{13}$, where $\mathfrak{t}_h \in \Sigma_C^{w_h} \cup \{\varnothing\}$, $\mathfrak{t}_h < \mathfrak{t}$. Then $a_1 = 0$ and $a_2 = \frac{|\mathfrak{t}_h|}{b_\mathfrak{t}} - c\epsilon_\mathfrak{t}$ by Lemma 2.5.2.

If $\mathfrak{t}$ has one or no child, or $D_\mathfrak{t} = 1$, then $\overline{g_h|_L} = -\overline{f_{\mathfrak{t},w_h}^W}$ by (2.7). On the other hand, if $D_\mathfrak{t} = 2$ and $\mathfrak{t}$ has two or more children in Σ_C^{rat}, then $b_\mathfrak{t} = 1$, and so $c = 0$. Therefore the equality (2.7) concludes the proof of the first part of the statement also in this case. Finally, the last part of the lemma follows from Lemma 2.5.8. □

Let c as in the previous lemma and define $\tilde{\mathfrak{t}}^W := \{\mathfrak{s} \in \hat{\mathfrak{t}}^W \mid \frac{|\mathfrak{s}|}{b_\mathfrak{t}} - c\epsilon_\mathfrak{t} \notin 2\mathbb{Z}\}$.

Proposition 2.5.18 *Let $L = L_\mathfrak{t}^{w_h}$ and $M = M_{L,0}$. The dense open subscheme $\Gamma_\mathfrak{t} \cap U_M^h$ of $\Gamma_\mathfrak{t}$ is isomorphic to the closed subscheme of $\mathbb{G}_{\mathfrak{t},w_h} \times \mathbb{A}_k^1$ given by*

$$Y^{D_\mathfrak{t}} = \prod_{\mathfrak{s} \in \tilde{\mathfrak{t}}^W} (X - \overline{u_{w_\mathfrak{s} w_h}}) \cdot \overline{f_{\mathfrak{t},w_h}^W}(X),$$

where $\overline{u_{w_\mathfrak{s} w_h}} = \frac{w_\mathfrak{s} - w_h}{\pi^{\rho_\mathfrak{t}}} \mod \pi$, and $w_\mathfrak{s}$ is any rational centre of $\mathfrak{s}$.

Proof. The proposition follows from Lemma 2.5.17 and the definition of $\mathbb{G}_{\mathfrak{t},w_h}$. □

We conclude this subsection describing how the glueing morphism (2.4) restricts to the special fibre. Suppose $\mathfrak{t} \supseteq \mathfrak{s}_l \wedge \mathfrak{s}_h$ for $l \neq h$ and let M be a matrix associated to $\mathfrak{t}$. Consider the glueing map $U_M^h \to U_M^l$ explicitly defined in §2.5.3.

Suppose first $M = M_{V,j}$ with $V = V_\mathfrak{t}^{w_l}$. By Lemma 2.5.10 the glueing morphism restricts to the identity on $X_V \times \mathbb{P}_V$.

Suppose $M = M_{L,i}$ with $L = L_\mathfrak{t}^{w_l}$. Note that $\tilde{m}_{12} = 0$ from Lemma 2.5.2. Recall the open subscheme $\Gamma_\mathfrak{t}^h$ of $\mathring{X}_{F_\mathfrak{t}^{w_h}}$ defined in Lemma 2.5.17. Then, Lemma 2.5.7 implies that the glueing map restricts to an isomorphism $\Gamma_\mathfrak{t}^h \mapsto \Gamma_\mathfrak{t}^l$ induced by the ring homomorphism sending $X \mapsto X + \overline{u_{w_h w_l}}$, where $\overline{u_{w_h w_l}} = \frac{w_h - w_l}{\pi^{\rho_t}} \mod \pi$. Similarly, it restricts to an isomorphism $X^W_{L_t^{w_h}} \times \mathbb{P}_{L_t^{w_h}} \to X^W_{L_t^{w_l}} \times \mathbb{P}_{L_t^{w_l}}$, where $\mathbb{P}_{L_t^{w_h}} \to \mathbb{P}_{L_t^{w_l}}$ is the identity and $X^W_{L_t^{w_h}} \to X^W_{L_t^{w_l}}$ is induced by the ring homomorphism sending $X \mapsto X + \overline{u_{w_h w_l}}$.

2.5.7 Regularity

In this subsection we prove that if C has an almost rational cluster picture and is y-regular, then the scheme $\mathcal{C}$ is regular.

Let $w_h \in W$. We want to show that if $\Sigma = \Sigma_C^{\min}$, and C has an almost rational cluster picture and is y-regular, then $\mathring{\mathcal{C}}_\Delta^{w_h}$ is a regular scheme.

Lemma 2.5.19 *Consider the model $\mathcal{C}_\Delta^{w_h}/O_K$ and let $f_h(x) = f(x + w_h)$. Suppose $\Sigma = \{\mathfrak{s}_1, \dots, \mathfrak{s}_m\} = \Sigma_C^{\min}$, and C has an almost rational cluster picture and is y-regular. If P is a singular point of $\mathcal{C}_\Delta^{w_h}$ then*

$$P \in \operatorname{Spec} \frac{O_K[X^{\pm 1}, Y, Z]}{(\mathcal{F}_M^h(X,Y,Z), X + u_{hl}, Y^{\tilde{m}_{23}} Z^{\tilde{m}_{33}}, \pi)} \subset \mathcal{C}_\Delta^{w_h} \cap X_M,$$

for some $l \neq h$, where $M = M_{L^{w_h}_{\mathfrak{s}_h \wedge \mathfrak{s}_l}, i}$ for $i = 0, \dots, r_{L^{w_h}_{\mathfrak{s}_h \wedge \mathfrak{s}_l}}$.

Proof. Denote by $m_\alpha(X) \in O_K[X]$ a lift of the minimal polynomial in $k[X]$ of $\bar{\alpha} \in \bar{k}$. By Lemma 2.5.9, we only need to show that if $P \in \mathcal{C}_\Delta^{w_h}$ is a singular point then

$$P \in \operatorname{Spec} \frac{O_K[X^{\pm 1}, Y, Z]}{(\mathcal{F}_{M_{L,i}}^h(X,Y,Z), m_\alpha(X), Y^{\tilde{m}_{23}} Z^{\tilde{m}_{33}}, \pi)}, \tag{2.8}$$

for some v-edge $L = L_t^{w_h}$ of Δ^{w_h}, and some multiple root $\bar{\alpha}$ of $\overline{f_h|_L}$. For any v-edge E of Δ^{w_h} and any $i = 0, \dots, r_E$, we study the polynomial $\mathcal{F}_M^h$ where $M = M_{E,i}$, using [Dok, §4.5]. Let $g_h(x,y) = y^2 - f_h(x)$. Let $L = L_t^{w_h}$ and $M = M_{L,i}$. Note that $\overline{g_h|_L} = -\overline{f_h|_L}$. We have $\mathcal{F}_M^h(X,0,Z) = \overline{g_h|_L}(X)$ for any i. On the other hand, $\mathcal{F}_M^h(X,Y,0) = \overline{g_h|_L}(X)$ if $i > 0$ and $\mathcal{F}_M^h(X,Y,0) = \overline{g_h|_F}(X,Y)$ if $i = 0$. From the description given in Lemma 2.5.17, we conclude that for these matrices M the points in (2.8) are the only possibly singular points of $\mathcal{C}_\Delta^{w_h} \cap X_M$. In particular, this proves that for any v-face F of Δ^{w_h}, the points in X_F are non-singular in $\mathcal{C}_\Delta^{w_h}$.

Let $V = V_t^{w_h}$ or $V = V_0^{w_h}$ and $M = M_{V,j}$. Since C is y-regular, $p \nmid \deg(\overline{g_h|_V})$ by Lemma 2.4.9. By [Dok, §4.5] and the fact that the points in X_F are non-singular for all v-faces F, we conclude that $\mathcal{C}_\Delta^{w_h}$ has no singular point on X_M for these matrices M, as required. □

Proposition 2.5.20 *Suppose $\Sigma = \Sigma_C^{\min}$, and C has an almost rational cluster picture and is y-regular, then $\mathcal{C}$ is a regular scheme.*

Proof. Lemmas 2.5.19 and 2.5.8 show that $\mathring{\mathcal{C}}_\Delta^{w_h}$ is regular for every h. Thus their glueing $\mathcal{C}$ is regular as well. □

2.5.8 Separatedness

It remains to prove that $\mathcal{C}$ is a proper scheme. In this subsection we show it is separated. Clearly it suffices to prove that $\mathcal{X}/O_K$ is separated. Since the schemes X_Δ^h are separated, then the open subschemes $\mathring{X}_\Delta^h$ are separated as well by [Liu4, Proposition 3.3.9]. Consider the open cover $\{V_M^h\}_{h,M}$ of $\mathcal{X}$. Let $h,l=1,\dots,m$ and let M_h and M_l be matrices associated to proper clusters $\mathfrak{t}_h \in \Sigma_C^{w_h}$ and $\mathfrak{t}_l \in \Sigma_C^{w_l}$ respectively. By [Liu4, Proposition 3.3.6] we want to show

(i) $V_{M_h}^h \cap V_{M_l}^l$ is affine,

(ii) The canonical homomorphism

$$O_{\mathcal{X}}(V_{M_h}^h)\otimes_{\mathbb{Z}} O_{\mathcal{X}}(V_{M_l}^l)\longrightarrow O_{\mathcal{X}}(V_{M_h}^h\cap V_{M_l}^l)$$

is surjective.

The definition of the glueing map (2.4) implies (i). If $h=l$, or $\mathfrak{s}_l\subseteq\mathfrak{t}_h$, or $\mathfrak{s}_h\subseteq\mathfrak{t}_l$, then (ii) follows from the separatedness of $\mathring{X}_\Delta^h$ and $\mathring{X}_\Delta^l$. So assume $l\neq h$, and $\mathfrak{t}_h,\mathfrak{t}_l\subsetneq\mathfrak{s}_h\wedge\mathfrak{s}_l$. Consider the Moebius transformation

$$\psi_l:\quad x\longmapsto\frac{x}{xw_{hl}^{-1}+1},\quad y\longmapsto\frac{y}{(xw_{hl}^{-1}+1)^{g+1}}.$$

It sends the curve C^{w_l} to the isomorphic hyperelliptic curve

$$C_l^h: y^2=(xw_{hl}^{-1}+1)^{2g+2}f\left(x(xw_{hl}^{-1}+1)^{-1}+w_l\right).$$

As

$$\begin{aligned}f_l^h(x):&=(xw_{hl}^{-1}+1)^{2g+2}f\left(x(xw_{hl}^{-1}+1)^{-1}+w_l\right)\\&=c_f w_{hl}^{|\mathfrak{R}|}(xw_{hl}^{-1}+1)^{2g+2-|\mathfrak{R}|}\prod_{r\in\mathfrak{R}\smallsetminus\{w_h\}}\frac{r-w_h}{w_{lh}}\left(xw_{hl}^{-1}+\frac{r-w_l}{r-w_h}\right),\end{aligned}$$

every cluster $\mathfrak{s}\in\Sigma_C^{w_l}$ such that $\mathfrak{s}\subsetneq\mathfrak{s}_h\wedge\mathfrak{s}_l$, corresponds to a unique cluster $\mathfrak{s}^h\in\Sigma_{C_l^h}^0$ of same size, same radius and rational centre 0. Moreover,

$$\epsilon_{\mathfrak{s}^h}=v(c_{f_l^h})+\sum_{r'\in\mathfrak{s}^h}\rho_{\mathfrak{s}^h}+\sum_{r'\notin\mathfrak{s}^h}v(r')=\epsilon_{\mathfrak{s}}.$$

Call $\mathfrak{t}_l^h$ the cluster in $\Sigma_{C_l^h}^0$ corresponding to $\mathfrak{t}_l$. Let Δ^{lh} and Δ_v^{lh} be the Newton polytopes attached to $y^2-f_l^h(x)$ and let X_Δ^{lh} be the associated toric scheme (defined in [Dok, §4.2]). Since $\mathfrak{t}_l\subsetneq\mathfrak{s}_h\wedge\mathfrak{s}_l$, the v-faces $F_{\mathfrak{t}_l}$ of Δ^{w_l} and $F_{\mathfrak{t}_l^h}$ of Δ^{lh} are identical by Lemma 2.4.3. Furthermore, note that if $\mathfrak{t}_l<\mathfrak{s}_h\wedge\mathfrak{s}_l$, then $\rho_{P(\mathfrak{t}_l^h)}\leq\rho_{hl}=\rho_{P(\mathfrak{t}_l)}$ and so $s_2^{V^0}\leq s_2^V$, where $V^0=V_{\mathfrak{t}_l^h}^0$ and $V=V_{\mathfrak{t}_l}^{w_l}$. Therefore the matrix $M:=M_l$ is also associated to $\mathfrak{t}_l^h$.

For every $o=1,\dots,m$, with $o\neq l$, define

$$w_{hlo}=\begin{cases}\frac{w_{hl}w_{lo}}{w_{ho}} & \text{if } o\neq h,\\ w_{hl} & \text{if } o=h,\end{cases}$$

and write $w_{hlo} = u_{hlo}\pi^{\rho_{hlo}}$, where $u_{hlo} \in O_K^\times$ and $\rho_{hlo} \in \mathbb{Z}$, i.e.

$$u_{hlo} = \begin{cases} \frac{u_{hl}u_{lo}}{u_{ho}} & \text{if } o \neq h, \\ u_{hl} & \text{if } o = h, \end{cases} \quad \text{and} \quad \rho_{hlo} = \begin{cases} \rho_{hl} + \rho_{lo} - \rho_{ho} & \text{if } o \neq h, \\ \rho_{hl} & \text{if } o = h. \end{cases}$$

Define

$$\tilde{T}_M^{hlo}(X,Y,Z) := \begin{cases} 1 + u_{hlo} X^{\rho_{hlo}\tilde{m}_{13}-\tilde{m}_{11}} Y^{\rho_{hlo}\tilde{m}_{23}-\tilde{m}_{21}} Z^{\rho_{hlo}\tilde{m}_{33}-\tilde{m}_{31}} & \text{if } \mathfrak{t}_l \supseteq \mathfrak{s}_o, \\ u_{hlo}^{-1} X^{\tilde{m}_{11}-\rho_{hlo}\tilde{m}_{13}} Y^{\tilde{m}_{21}-\rho_{hlo}\tilde{m}_{23}} Z^{\tilde{m}_{31}-\rho_{hlo}\tilde{m}_{33}} + 1 & \text{if } \mathfrak{t}_l \not\supseteq \mathfrak{s}_o. \end{cases}$$

We want to show $\tilde{T}_M^{hlo}(X,Y,Z) \in R$. If $o = h$ then

$$\tilde{T}_M^{hlo}(X,Y,Z) = T_M^{hl}(X,Y,Z) \in R.$$

So assume $o \neq h$. If $\mathfrak{s}_o \subseteq \mathfrak{t}_l$, then it follows from Lemma 2.5.4 as $\mathfrak{s}_l \wedge \mathfrak{s}_o \subsetneq \mathfrak{s}_l \wedge \mathfrak{s}_h$ and so $\rho_{hlo} = \rho_{lo}$. On the other hand, if $\mathfrak{s}_o \not\subseteq \mathfrak{t}_l$, then it follows from Lemma 2.5.5 as $\tilde{m}_{23}, \tilde{m}_{33} > 0$ and $\rho_{hlo} \leq \max\{\rho_{hl}, \rho_{lo}\}$. Let

$$\tilde{T}_M^{hl}(X,Y,Z) := \prod_{o \neq l} \tilde{T}_M^{hlo}(X,Y,Z).$$

The Moebius transformation

$$K[x^{\pm 1}, y^{\pm 1}, \textstyle\prod_{o\neq l}(x + w_{lo})^{-1}] \xrightarrow{\psi_l} K[x^{\pm 1}, y^{\pm 1}, \textstyle\prod_{o \neq l}(x + w_{hlo})^{-1}]$$

considered above induces an isomorphism

$$R[T_M^l(X,Y,Z)^{-1}] \xrightarrow{M^{-1}\circ\psi_l\circ M} R[\tilde{T}_M^{hl}(X,Y,Z)^{-1}],$$

sending

$$\begin{aligned} X &\mapsto X \cdot T_M^{hl}(X,Y,Z)^{-m_{11}-(g+1)m_{21}}, \\ Y &\mapsto Y \cdot T_M^{hl}(X,Y,Z)^{-m_{12}-(g+1)m_{22}}, \\ Z &\mapsto Z \cdot T_M^{hl}(X,Y,Z)^{-m_{13}-(g+1)m_{23}}. \end{aligned}$$

Then

$$\tilde{V}_M^{lh} := \operatorname{Spec} R[\tilde{T}_M^{hl}(X,Y,Z)^{-1}]$$

is an open subscheme of X_Δ^{lh}, isomorphic to V_M^l. We can clearly carry out similar constructions for t_h, M_h.

By comparing the Newton polytopes Δ_v^{lh} and Δ_v^{hl}, we see that the Moebius transformation $x \mapsto w_{hl}/(w_{lh}^{-1}x)$, $y \mapsto y/(w_{lh}^{-1}x)^{g+1}$ gives an isomorphism

$$\psi : K[x^{\pm 1}, y^{\pm 1}, \prod_{o \neq l}(x + w_{hlo})^{-1}] \longrightarrow K[x^{\pm 1}, y^{\pm 1}, \prod_{o \neq h}(x + w_{lho})^{-1}]$$

which induces a birational map $X_\Delta^{hl} \dashrightarrow X_\Delta^{lh}$, defined on the open set $\tilde{V}_{M_h}^{hl}$ of X_Δ^{hl}. In particular, there exists an open set $\tilde{V}_{M_h}^{lh}$ of X_Δ^{lh}, isomorphic to $V_{M_h}^h$ via the map induced by $\psi_h^{-1} \circ \psi$.

Recall the definition of ϕ in (2.1), which induces the glueing map between $V^l_{M_l}$ and $V^h_{M_h}$. Since the following diagram

$$\begin{array}{ccc} K[x^{\pm1},y^{\pm1},\prod_{o\neq l}(x+w_{lo})^{-1}] & \xrightarrow{\phi} & K[x^{\pm1},y^{\pm1},\prod_{o\neq h}(x+w_{ho})^{-1}] \\ \downarrow{\scriptstyle\psi_l} & & \downarrow{\scriptstyle\psi_h} \\ K[x^{\pm1},y^{\pm1},\prod_{o\neq l}(x+w_{hlo})^{-1}] & \xrightarrow{\psi} & K[x^{\pm1},y^{\pm1},\prod_{o\neq h}(x+w_{lho})^{-1}] \end{array}$$

is commutative, then the surjectivity of

$$\mathcal{O}_{\mathcal{X}}(V^h_{M_h})\otimes_{\mathbb{Z}}\mathcal{O}_{\mathcal{X}}(V^l_{M_l})\longrightarrow \mathcal{O}_{\mathcal{X}}(V^h_{M_h}\cap V^l_{M_l})$$

follows from the separatedness of X^{lh}_Δ.

2.5.9 Properness

In this subsection we prove that $\mathcal{C}$ is proper. By [EGA, IV.15.7.10], it remains to show that $\mathcal{C}_s$ is proper. From [Liu4, Exercise 3.3.11], we only need to prove that the 1-dimensional subscheme $\Gamma_{\mathfrak{t}}$ is proper for every $\mathfrak{t}=\mathfrak{s}_h\wedge\mathfrak{s}_l$. Indeed every other component is entirely contained in a model $\mathcal{C}^{w_h}_\Delta$, which is proper (see §2.5.5). Let $\mathfrak{t}=\mathfrak{s}_h\wedge\mathfrak{s}_l$ for some $h,l=1,\dots,m$, with $h\neq l$. For any $o=1,\dots,m$ such that $\mathfrak{s}_o\subset\mathfrak{t}$, let $\mathfrak{t}_o$ be the unique child of $\mathfrak{t}$ with $\mathfrak{s}_o\subseteq\mathfrak{t}_o<\mathfrak{t}$. Then $\Gamma_{\mathfrak{t}}$ is equal to the glueing of the schemes

$$\operatorname{Spec}\frac{R[T^o_M(X,Y,Z)^{-1}]}{\left(\mathcal{F}^o_M(X,Y,Z),Z,\pi\right)},\qquad M=M_{L^{w_o}_{\mathfrak{t}},0},M_{V^{w_o}_{\mathfrak{t}},0},$$

and

$$\operatorname{Spec}\frac{R[T^o_M(X,Y,Z)^{-1}]}{\left(\mathcal{F}^o_M(X,Y,Z),Y,\pi\right)},\qquad M=M_{V^{w_o}_{\mathfrak{t}_o},r_{V^{w_o}_{\mathfrak{t}_o}}},$$

for all o such that $\mathfrak{s}_o\subset\mathfrak{t}$, through the isomorphism (2.4) and the glueing maps in the definition of $\mathcal{C}^{w_o}_\Delta$. In particular, for any o as above there exists a natural birational map $s_o:\Gamma_{\mathfrak{t}}\dashrightarrow\bar{X}_{F^{w_o}_{\mathfrak{t}}}$ which is defined as the identity morphism on the dense open $\mathring{X}_{F^{w_o}_{\mathfrak{t}}}=\Gamma_{\mathfrak{t}}\cap\mathring{\mathcal{C}}^{w_o}_\Delta$.

Let D/k be a normal curve, let $P\in D$ and let $D\smallsetminus\{P\}\xrightarrow{g}\Gamma_{\mathfrak{t}}$ be a non-constant morphism of curves. We want to show that g extends to D. For every o as above, $\bar{X}_{F^{w_o}_{\mathfrak{t}}}$ is proper, so the birational map

$$g_o:=s_o\circ g:D\smallsetminus\{P\}\dashrightarrow\bar{X}_{F^{w_o}_{\mathfrak{t}}}$$

extends to a morphism $\bar{g}_o:D\longrightarrow\bar{X}_{F^{w_o}_{\mathfrak{t}}}$. If

$$P_o:=\bar{g}_o(P)\in\left(\bar{X}_{F^{w_o}_{\mathfrak{t}}}\cap\mathring{\mathcal{C}}^{w_o}_\Delta\right)=s_o\left(\Gamma_{\mathfrak{t}}\cap\mathring{\mathcal{C}}^{w_o}_\Delta\right)$$

for some o such that $\mathfrak{s}_o\subset\mathfrak{t}$ (we will later show this is always the case), then there exists an open neighbourhood U of P_o such that $U\subseteq\left(\bar{X}_{F^{w_o}_{\mathfrak{t}}}\cap\mathring{\mathcal{C}}^{w_o}_\Delta\right)$ and so $s_o|^U_{s_o^{-1}(U)}$ is an isomorphism. Since $P\in\bar{g}_o^{-1}(U)$, the map

$$\bar{g}_o^{-1}(U)\xrightarrow{\bar{g}_o|^U_{\bar{g}_o^{-1}(U)}}U\xrightarrow{\left(s_o|^U_{s_o^{-1}(U)}\right)^{-1}}s_o^{-1}(U)\hookrightarrow\Gamma_{\mathfrak{t}},$$

induces an extension $D \longrightarrow \Gamma_{\mathfrak{t}}$ of g.

Suppose that $P_o \notin \bar{X}_{F_{\mathfrak{t}}^{w_o}} \cap \mathring{\mathcal{C}}_{\Delta}^{w_o}$ for any o such that $\mathfrak{s}_o \subset \mathfrak{t}$. From §2.5.5 we have

$$P_o \in S_M = \operatorname{Spec} \frac{R}{(\mathcal{F}_M^o(X,Y,Z), \prod_l (X + u_{ol}), Z, \pi)}, \tag{2.9}$$

where $M = M_{L_{\mathfrak{t}}^{w_o},0}$, and the product runs over all $l \neq o$ such that $\mathfrak{t} = \mathfrak{s}_o \wedge \mathfrak{s}_l$. In particular P_o is a point of each irreducible component of $\bar{X}_{F_{\mathfrak{t}}^{w_o}}$ by Lemma 2.5.17. Let $h \neq o$ such that $X + u_{oh}$ vanishes at P_o. Let ξ be the generic point of D and let $\xi_o = g_o(\xi)$, $\xi_h = g_h(\xi)$ be generic points of $\bar{X}_{F_{\mathfrak{t}}^{w_o}}$ and $\bar{X}_{F_{\mathfrak{t}}^{w_h}}$ respectively. Then the birational maps s_o and s_h give

$$\begin{tikzcd}
 & & \bar{X}_{F_{\mathfrak{t}}^{w_o}} & & & k\left(\overline{\xi_o}\right) \arrow[dll, "\phi_{g_o}"'] \arrow[dd, "\simeq"] \\
D \smallsetminus \{P\} \arrow[r, "g"] & \Gamma_{\mathfrak{t}} \arrow[ur, dashed, "s_o"] \arrow[dr, dashed, "s_h"'] & \Longrightarrow & k(D) & & \\
 & & \bar{X}_{F_{\mathfrak{t}}^{w_h}} & & & k\left(\overline{\xi_h}\right) \arrow[ull, "\phi_{g_h}"]
\end{tikzcd}$$

where we denote by ϕ_{g_o} and ϕ_{g_h} the homomorphisms between function fields induced by g_o and g_h. The vertical isomorphism is induced by the map

$$\frac{R[T_M^o(X,Y,Z)^{-1}]}{\left(\mathcal{F}_M^o(X,Y,Z), Z\right)} \longrightarrow \frac{R[T_M^h(X,Y,Z)^{-1}]}{\left(\mathcal{F}_M^h(X,Y,Z), Z\right)}$$

which sends (see §2.5.3 and Lemma 2.5.7)

$$X + u_{oh} \mapsto X \cdot T_M^{ho}(X,Y,Z)^{m_{11}} + u_{oh} = X\left(1 + u_{ho}X^{-1}\right) + u_{oh} = X.$$

But the rational function $X + u_{oh}$ vanishes at P_o, while X does not vanish at P_h by (2.9). This gives a contradiction, as $\bar{g}_o(P) = P_o$ and $\bar{g}_h(P) = P_h$.

2.5.10 Genus

Suppose $\Sigma = \{\mathfrak{s}_1, \ldots, \mathfrak{s}_m\} = \Sigma_C^{\min}$, and C has an almost rational cluster picture and is y-regular. In the previous subsections we proved that $\mathcal{C}/O_K$ is a proper regular model of C. Let $\mathfrak{t} \in \Sigma_C^{w_h}$ be a proper cluster. In this subsection we want to describe the genus of the components $\Gamma_{\mathfrak{t}}$ of $\mathcal{C}_s$ introduced in §2.5.6.

Proposition 2.5.21 *Let $\mathfrak{t} \in \Sigma_C^{w_h}$. Then $\Gamma_{\mathfrak{t}}$ is isomorphic to the smooth projective 1-dimensional scheme given by*

$$Y^{D_{\mathfrak{t}}} = \prod_{\mathfrak{s} \in \tilde{\mathfrak{t}}^W} (X - \overline{u_{w_{\mathfrak{s}} w_h}}) \overline{f_{\mathfrak{t}, w_h}}(X)$$

where $\overline{u_{w_{\mathfrak{s}} w_h}} = \frac{w_{\mathfrak{s}} - w_h}{\pi^{\rho_{\mathfrak{t}}}} \mod \pi$, and $w_{\mathfrak{s}}$ is any rational centre of $\mathfrak{s}$.

In particular,

1. *if* $D_\mathfrak{t} = 1$, *then* $\Gamma_\mathfrak{t} \simeq \mathbb{P}^1_k$;
2. *if* $D_\mathfrak{t} = 2$ *and* $\mathfrak{t}$ *is übereven, then* $\Gamma_\mathfrak{t}$ *is the disjoint union of two* $\mathbb{P}^1$*s over some quadratic extension of* k;
3. *in all other cases,* $\Gamma_\mathfrak{t}$ *is a hyperelliptic curve of genus* $g(\mathfrak{t})$.

Proof. The first part of the proposition follows from Proposition 2.5.18.

For the second part of the statement note that if $D_\mathfrak{t} = 1$ then the result follows. Suppose $D_\mathfrak{t} = 2$. Then $p \neq 2$ as C is y-regular. Note that since $\Sigma = \Sigma_C^{\min}$, the proper clusters in Σ^W correspond to the proper clusters in Σ_C^{rat}. Recall the definition of $\tilde{\mathfrak{t}}$ given in Definition 2.4.13. Let $h(X) = \prod_{\mathfrak{s}\in\tilde{\mathfrak{t}}^W}(X - \overline{u_{w_\mathfrak{s} w_h}})\overline{f_{\mathfrak{t},w_h}}(X)$.

Suppose $\mathfrak{t}$ is übereven. Then all its children are (proper) rational cluster by Lemma 2.3.30 since they are even and $p \neq 2$. In particular $b_\mathfrak{t} = 1$ by Lemma 2.3.18 and so $\epsilon_\mathfrak{t} \in 2\mathbb{Z}$ and $\tilde{\mathfrak{t}} = \tilde{\mathfrak{t}}^W = \varnothing$ since it equals the set of odd rational children. Moreover, $\mathfrak{t} = \bigcup_{\mathfrak{s}<\mathfrak{t},\mathfrak{s}\,\mathrm{proper}} \mathfrak{s}$, and so $\overline{f_{\mathfrak{t},w_h}} \in k$. Thus $h(X) \in k$.

Now suppose $h(X) \in k$. Then $\tilde{\mathfrak{t}}^W = \varnothing$ and $\mathfrak{t} = \bigcup_{\mathfrak{s}<\mathfrak{t}} \mathfrak{s}$, where $\mathfrak{s}$ runs through all children $\mathfrak{s} \in \Sigma^W$ of $\mathfrak{t}$. The non-proper clusters in Σ^W are of the form $\{w_l\}$ for some $l = 1, \dots, m$. If $\{w_l\} < \mathfrak{t}$, then $\mathfrak{t} = \mathfrak{s}_l$, but in that case $\mathfrak{t}$ would not equal the union of its children in Σ^W. Hence $\mathfrak{t}$ has no non-proper children. It follows that $\tilde{\mathfrak{t}} = \tilde{\mathfrak{t}}^W$ and $\mathfrak{t}$ equals the union of its proper rational children. In particular, $\mathfrak{t}$ has two or more children in Σ_C^{rat}, so $b_\mathfrak{t} = 1$, by Lemma 2.3.18. But then $\tilde{\mathfrak{t}}$ is the set of odd children of $\mathfrak{t}$ as $\epsilon_\mathfrak{t} \in 2\mathbb{Z}$, and so all rational children of $\mathfrak{t}$ are even.

It only remains to prove that if $h(x) \notin k$, then the genus of $\Gamma_\mathfrak{t}$ is $g(\mathfrak{t})$. Since $h(X)$ is a separable polynomial, we need to show that

$$\deg h = \frac{|\mathfrak{t}| - \sum_{\mathfrak{s}\in\Sigma_C^{\mathrm{rat}},\mathfrak{s}<\mathfrak{t}} |\mathfrak{s}|}{b_\mathfrak{t}} + \tilde{\mathfrak{t}}.$$

It suffices to prove that if $\mathfrak{s} \in \Sigma_C^{\mathrm{rat}}$ is a non-proper rational child of $\mathfrak{t}$ different from $\{w_h\}$, then $b_\mathfrak{t} = 1$ and $\mathfrak{s} \in \tilde{\mathfrak{t}}$. Suppose $\mathfrak{s} = \{r\}$ is such a rational cluster. Since $r \in \mathfrak{t}$, we have $v(r - w_h) \geq \rho_\mathfrak{t}$. Suppose $v(r - w_h) > \rho_\mathfrak{t}$. Then $\mathfrak{s} \in \mathring{\Sigma}_C^{w_h}$, as $\mathfrak{s} < \mathfrak{t}$ and $r \neq w_h$. But this contradicts our choice of W. Then $\rho_\mathfrak{t} = v(r - w_h) \in \mathbb{Z}$ and so $b_\mathfrak{t} = 1$. It follows that $\tilde{\mathfrak{t}}$ is the set of odd children of $\mathfrak{t}$. Thus $\mathfrak{s} \in \tilde{\mathfrak{t}}$. □

2.5.11 Minimal regular NC model

Suppose the base extended curve $C_{K^{nr}}$ is y-regular and has an almost rational cluster picture. Consider the model $\mathcal{C}/O_{K^{nr}}$ constructed before with $\Sigma = \Sigma_{C_{K^{nr}}}^{\min}$. In this subsection we study what components of $\mathcal{C}_s$ have to be blown down to obtain the minimal regular model with normal crossings.

Recall [Dok, §5]. Let $\Sigma_{K^{nr}} = \Sigma_{C_{K^{nr}}}^{\mathrm{rat}}$ and fix a proper cluster $\mathfrak{t} \in \Sigma_{C_{K^{nr}}}^{w_h}$. Suppose first $\mathfrak{t} \neq \mathfrak{s}_h \wedge \mathfrak{s}_l$ for all $l = 1, \dots, m$ with $l \neq h$. Equivalently, $\mathfrak{t}$ has at most one proper child in $\Sigma_{K^{nr}}$. Then $\Gamma_\mathfrak{t} \simeq \bar{X}_{F_\mathfrak{t}^{w_h}}$ and can be seen entirely in $\mathring{\mathcal{C}}_\Delta^{w_h}$. In particular, if $\Gamma_\mathfrak{t}$ can be blown down then $F_\mathfrak{t}^{w_h}$ is a removable or contractible v-face (see [Dok, Theorem 5.7]). By Lemma 2.4.3, we find

- $F_{\mathfrak{t}}^{w_h}$ is removable if and only if $\mathfrak{t} = \mathfrak{R}$ with a child in $\Sigma_{K^{nr}}$ of size $2g+1$.
- $F_{\mathfrak{t}}^{w_h}$ is contractible if and only if either $|\mathfrak{t}| = 2$ and $\frac{\epsilon_{\mathfrak{t}}}{2} - \rho_{\mathfrak{t}} \in \mathbb{Z}$ or $\mathfrak{t}$ has a proper rational child $\mathfrak{s} \in \Sigma_{K^{nr}}$, of size $2g$, and $\frac{\epsilon_{\mathfrak{t}}}{2} - g\rho_{\mathfrak{t}} \in \mathbb{Z}$.

Recall Definition 2.4.19. Note that $F_{\mathfrak{t}}^{w_h}$ is removable if and only if $\mathfrak{t}$ is removable. In this case, $F_{\mathfrak{t}}^{w_h}$ can be ignored for the construction of $\mathcal{C}_{\Delta}^{w_h}$ (for any h since $\mathfrak{t} = \mathfrak{R}$), and so $\mathfrak{t}$ can be ignored for the construction of $\mathcal{C}$.

Assume now $F_{\mathfrak{t}}^{w_h}$ contractible. We want to understand when $\Gamma_{\mathfrak{t}}$ can be blown down. First consider the case $|\mathfrak{t}| = 2$ and $\frac{\epsilon_{\mathfrak{t}}}{2} - \rho_{\mathfrak{t}} \in \mathbb{Z}$. Then $\Gamma_{\mathfrak{t}}$ intersects other components of $\mathcal{C}_s$ in 2 points (as $V_{\mathfrak{t}}^{w_h}$ gives two chains of $\mathbb{P}^1$s and the v-edges $V_0^{w_h}$ and $L_{\mathfrak{t}}^{w_h}$ give no component in $\mathcal{C}_{\Delta,s}^{w_h}$). To have self-intersection -1, $\Gamma_{\mathfrak{t}}$ has to have multiplicity > 1. It follows from Lemma 2.5.16 that $\rho_{\mathfrak{t}} \notin \mathbb{Z}$, as $\frac{\epsilon_{\mathfrak{t}}}{2} - \rho_{\mathfrak{t}} \in \mathbb{Z}$. Moreover, by Lemma 2.3.12, one has $\rho_{\mathfrak{t}} \in \frac{1}{2}\mathbb{Z}$. Therefore $\epsilon_{\mathfrak{t}}$ is odd and the multiplicity of $\Gamma_{\mathfrak{t}}$ is 2. Let $r := r_{V_{\mathfrak{t}}^{w_h}}$ and consider

$$\gamma_{\mathfrak{t}} s_{\mathfrak{t}} = \frac{n_0}{d_0} > \frac{n_1}{d_1} > \cdots > \frac{n_r}{d_r} > \frac{n_{r+1}}{d_{r+1}} = \gamma_{\mathfrak{t}}\left(s_{\mathfrak{t}} - \rho_{\mathfrak{t}} + \rho_{P(\mathfrak{t})}\right)$$

given by $V_{\mathfrak{t}}^{w_h}$. If $\Gamma_{\mathfrak{t}}$ can be blown down then $d_1 = 1$. Since $\gamma_{\mathfrak{t}} s_{\mathfrak{t}} = -\frac{\epsilon_{\mathfrak{t}}}{2} + 2\rho_{\mathfrak{t}}$, we have $d_0 = 2$. In particular $d_1 = 1$ if and only if $\rho_{\mathfrak{t}} - \rho_{P(\mathfrak{t})} = \frac{n_0}{d_0} - \frac{n_{r+1}}{d_{r+1}} \geq \frac{1}{2}$ (see also [Dok, Remark 3.15]). Thus if $|\mathfrak{t}| = 2$, then $\Gamma_{\mathfrak{t}}$ can be blown down if and only if $\rho_{\mathfrak{t}} \notin \mathbb{Z}$, $\epsilon_{\mathfrak{t}}$ odd, $\rho_{P(\mathfrak{t})} \leq \rho_{\mathfrak{t}} - \frac{1}{2}$. Note that this is case (1) of Definition 2.4.19.

Second consider the case $|\mathfrak{t}| = 2g+2$ with a proper rational child $\mathfrak{s}$ of size $2g$ and $\frac{\epsilon_{\mathfrak{t}}}{2} - g\rho_{\mathfrak{t}} \in \mathbb{Z}$. The argument is very similar to the previous one. If $\Gamma_{\mathfrak{t}}$ can be blown down then it must have multiplicity > 1 and this implies $\rho_{\mathfrak{t}} \notin \mathbb{Z}$ again by Lemma 2.5.16. From Lemma 2.3.12 it follows that $(|\mathfrak{t}| - |\mathfrak{s}|)\rho_{\mathfrak{t}} \in \mathbb{Z}$, so $\rho_{\mathfrak{t}} \in \frac{1}{2}\mathbb{Z}$. Then $m_{\mathfrak{t}} = 2$ and

$$\frac{v(c_f)}{2} = \frac{\epsilon_{\mathfrak{t}}}{2} - (g+1)\rho_{\mathfrak{t}} \in \tfrac{1}{2}\mathbb{Z} \setminus \mathbb{Z},$$

so $v(c_f)$ odd. Let $r := r_{V_{\mathfrak{s}}^{w_h}}$ and consider

$$\gamma_{\mathfrak{s}} s_{\mathfrak{s}} = \frac{n_0}{d_0} > \frac{n_1}{d_1} > \cdots > \frac{n_r}{d_r} > \frac{n_{r+1}}{d_{r+1}} = \gamma_{\mathfrak{s}}(s_{\mathfrak{s}} - \rho_{\mathfrak{s}} + \rho_{\mathfrak{t}})$$

given by $V_{\mathfrak{s}}^{w_h}$. If $\Gamma_{\mathfrak{t}}$ can be blown down then $d_r = 1$. Recall that $\epsilon_{\mathfrak{s}} - |\mathfrak{s}|\rho_{\mathfrak{s}} = \epsilon_{\mathfrak{t}} - |\mathfrak{s}|\rho_{\mathfrak{t}}$. Then $\gamma_{\mathfrak{s}}(s_{\mathfrak{s}} - \rho_{\mathfrak{s}} + \rho_{\mathfrak{t}}) = -\frac{\epsilon_{\mathfrak{t}}}{2} + (g+1)\rho_{\mathfrak{t}}$, so $d_{r+1} = 2$. In particular $d_r = 1$ if and only if $\rho_{\mathfrak{s}} - \rho_{\mathfrak{t}} = \frac{n_0}{d_0} - \frac{n_{r+1}}{d_{r+1}} \geq \frac{1}{2}$. Thus if $\mathfrak{t}$ has size $2g+2$ and has a unique proper rational child $\mathfrak{s} \in \Sigma_{K^{nr}}$, then $\Gamma_{\mathfrak{t}}$ can be blown down if and only if $|\mathfrak{s}| = 2g$, $\rho_{\mathfrak{t}} \notin \mathbb{Z}$, $v(c_f)$ odd, $\rho_{\mathfrak{s}} \geq \rho_{\mathfrak{t}} + \frac{1}{2}$. This is case (2) of Definition 2.4.19.

Finally, if $|\mathfrak{t}| = 2g+1$, $\mathfrak{t}$ has a proper child $\mathfrak{s} \in \Sigma_{K^{nr}}$ of size $2g$ and $\frac{\epsilon_{\mathfrak{t}}}{2} - g\rho_{\mathfrak{t}} \in \mathbb{Z}$, then $\rho_{\mathfrak{t}} \in \mathbb{Z}$, as $(|\mathfrak{t}| - |\mathfrak{s}|)\rho_{\mathfrak{t}} \in \mathbb{Z}$. It follows that $\epsilon_{\mathfrak{t}} \in \mathbb{Z}$ and so $m_{\mathfrak{t}} = 1$. This implies the self-intersection of $\Gamma_{\mathfrak{t}}$ is not -1, since it intersects the rest of $\mathcal{C}_{\mathfrak{t}}$ in at least two points as before. Hence in this case $\Gamma_{\mathfrak{t}}$ can never be blown down.

Now assume there exists $l \neq h$ such that $\mathfrak{t} = \mathfrak{s}_h \wedge \mathfrak{s}_l$. Then $\mathfrak{t}$ is not minimal. Let $\mathfrak{t}_h, \mathfrak{t}_l \in \Sigma_{K^{nr}}$ be such that $\mathfrak{s}_h \subseteq \mathfrak{t}_h < \mathfrak{t}$ and $\mathfrak{s}_l \subseteq \mathfrak{t}_l < \mathfrak{t}$. Suppose $\Gamma_{\mathfrak{t}}$ irreducible. If $|\mathfrak{t}| \leq 2g$ (or, equivalently, $\mathfrak{t}$ is not the largest non-removable cluster), then $\Gamma_{\mathfrak{t}}$ intersects at least other 3 components of $\mathcal{C}_s$ (given by $\mathfrak{t}_h, \mathfrak{t}_l$, and $P(\mathfrak{t})$). So it cannot be contracted to obtain a model with normal crossings. A similar argument holds if there exists $o \neq l$ such that $\mathfrak{s}_o \wedge \mathfrak{s}_h = \mathfrak{t}$: at least 3 components (given by $\mathfrak{t}_h$, $\mathfrak{t}_l$ and $\mathfrak{t}_o$) intersect $\Gamma_{\mathfrak{t}}$, so blowing down $\Gamma_{\mathfrak{t}}$ would make the model lose normal crossings. Assume then $|\mathfrak{t}| > 2g$ and $\mathfrak{s}_o \wedge \mathfrak{s}_h \neq \mathfrak{t}$ for all $o \neq l$. Then $\Gamma_{\mathfrak{t}}$ intersects at least other 2 components of $\mathcal{C}_s$ given by $V_{\mathfrak{t}_h}^{w_h}$ and $V_{\mathfrak{t}_l}^{w_l}$. Firstly, if $\Gamma_{\mathfrak{t}}$ can be blown down, then $m_{\mathfrak{t}} > 1$. But $\rho_{\mathfrak{t}} = \rho_{hl} \in \mathbb{Z}$. Then $m_{\mathfrak{t}}$ is at most 2. If $m_{\mathfrak{t}} = 2$ then $D_{\mathfrak{t}} = 1$, that implies $\epsilon_{\mathfrak{t}}$ odd and $\Gamma_{\mathfrak{t}} \simeq \mathbb{P}^1$ by Proposition 2.5.21. It also follows $s_{\mathfrak{t}} \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. If $\mathfrak{t}$ is odd then this implies that $V_{\mathfrak{t}}^{w_h}$ gives a $\mathbb{P}^1$ intersecting $\Gamma_{\mathfrak{t}}$. Since that would be a third component intersecting $\Gamma_{\mathfrak{t}}$, the cluster $\mathfrak{t}$ has to be even. Hence $\mathfrak{t} = \mathfrak{R}$ and $|\mathfrak{t}| = 2g+2$. Then $\epsilon_{\mathfrak{t}}$ is odd if and only if $v(c_f)$ is odd, as $\rho_{\mathfrak{t}} \in \mathbb{Z}$. Now, $L_{\mathfrak{t}}^{w_h}$ gives some $\mathbb{P}^1$s intersecting $\bar{X}_{F_{\mathfrak{t}}^{w_h}} \subset \mathcal{C}_{\Delta,s}^{w_h}$. All these $\mathbb{P}^1$s are not in $\mathring{\mathcal{C}}_{\Delta,s}^{w_h}$ (and so in $\mathcal{C}_s$) if and only if $\mathfrak{t}_h \cup \mathfrak{t}_l = \mathfrak{t}$. In particular, $\mathfrak{t}_h$ and $\mathfrak{t}_l$ are either both even or both odd. If $\mathfrak{t}_h$ is even, then $\gamma_{\mathfrak{t}_h} = 2$, and so the component given by $V_{\mathfrak{t}_h}^{w_h}$ has multiplicity at least 2. The self-intersection of $\Gamma_{\mathfrak{t}}$ could not be -1 in this case. Assume $\mathfrak{t}_h$ is odd. Let $r := r_{V_{\mathfrak{t}_h}^{w_h}}$ and consider

$$\gamma_{\mathfrak{t}_h} s_{\mathfrak{t}_h} = \frac{n_0}{d_0} > \frac{n_1}{d_1} > \cdots > \frac{n_r}{d_r} > \frac{n_{r+1}}{d_{r+1}} = \gamma_{\mathfrak{t}_h}\left(s_{\mathfrak{t}_h} - \frac{\rho_{\mathfrak{t}_h} - \rho_{\mathfrak{t}}}{2}\right)$$

given by $V_{\mathfrak{t}_h}^{w_h}$. We want $d_r = 1$. Since

$$\gamma_{\mathfrak{t}_h}\left(s_{\mathfrak{t}_h} - \frac{\rho_{\mathfrak{t}_h} - \rho_{\mathfrak{t}}}{2}\right) = -\frac{\epsilon_{\mathfrak{t}}}{2} + \frac{|\mathfrak{t}_h| - 1}{2}\rho_{\mathfrak{t}} \in \tfrac{1}{2}\mathbb{Z} \setminus \mathbb{Z},$$

we have $d_{r+1} = 2$. As before $d_r = 1$ if and only if $\frac{\rho_{\mathfrak{t}_h} - \rho_{\mathfrak{t}}}{2} = \frac{n_0}{d_0} - \frac{n_{r+1}}{d_{r+1}} \geq \frac{1}{2}$ and similarly for $\mathfrak{t}_l$. Thus if $\mathfrak{t}$ has two or more rational children and $\Gamma_{\mathfrak{t}}$ is irreducible then it can be blown down if and only if $v(c_f)$ is odd and $\mathfrak{t} = \mathfrak{R}$ is union of its 2 odd rational children $\mathfrak{t}_h$ and $\mathfrak{t}_l$, satisfying $\rho_{\mathfrak{t}_h} \geq \rho_{\mathfrak{t}} + 1$, $\rho_{\mathfrak{t}_l} \geq \rho_{\mathfrak{t}} + 1$. This is case (3) of Definition 2.4.19.

Suppose now $\Gamma_{\mathfrak{t}}$ reducible. By Proposition 2.5.21 the cluster $\mathfrak{t}$ is übereven, $\epsilon_{\mathfrak{t}}$ is even and $\Gamma_{\mathfrak{t}}$ is the disjoint union of $\Gamma_{\mathfrak{t}}^- \simeq \mathbb{P}^1$ and $\Gamma_{\mathfrak{t}}^+ \simeq \mathbb{P}^1$. As before, both $\Gamma_{\mathfrak{t}}^-$ and $\Gamma_{\mathfrak{t}}^+$ intersect at least other two components (given by the proper children of $\mathfrak{t}$). But then neither $\Gamma_{\mathfrak{t}}^-$ nor $\Gamma_{\mathfrak{t}}^+$ has self-intersection -1, as $m_{\mathfrak{t}} = 1$.

We have showed that, for a rational cluster $\mathfrak{t} \in \Sigma_{K^{nr}}$, an irreducible component of $\Gamma_{\mathfrak{t}}$ can be blown down if and only if $\mathfrak{t}$ is contractible. Moreover, in this case, $\Gamma_{\mathfrak{t}}$ is irreducible. It remains to show that after blowing down all components $\Gamma_{\mathfrak{t}}$ where $\mathfrak{t}$ is a contractible cluster, no other component can be blown down. First note that if $\mathfrak{t}$ is a contractible cluster, then $m_{\mathfrak{t}} = 2$ and $\Gamma_{\mathfrak{t}}$ intersects one or two other components of multiplicity 1 at two points in total. If it intersects only one component, then after the blowing down, the latter will have a node and will not be isomorphic to $\mathbb{P}^1$. If $\Gamma_{\mathfrak{t}}$ intersects two components and those intersect something else in $\mathcal{C}_s$, then they will not have self-intersection -1 also when $\Gamma_{\mathfrak{t}}$ is blown down. Therefore suppose that one of those two does not intersect any other component of $\mathcal{C}_s$. If we are in case (1) or case (2), it is easy to see that this never happens. Indeed, in those cases, $\Gamma_{\mathfrak{t}}$ intersects non-open-ended chains of $\mathbb{P}^1$s. Then without loss of generality assume to be in case (3) and that $\Gamma_{\mathfrak{t}_h}$ is the component that can be blown down once $\Gamma_{\mathfrak{t}}$ has been contracted. This implies $\mathfrak{s}_h = \mathfrak{t}_h$ and $\rho_{\mathfrak{s}_h} = \rho_{\mathfrak{t}} + 1$. Then $b_{\mathfrak{s}_h} = 1$ and $\epsilon_{\mathfrak{s}_h} = \epsilon_{\mathfrak{t}} + |\mathfrak{s}_h|$. Since both $\epsilon_{\mathfrak{t}}$ and $\mathfrak{s}_h$ are odd, we have $\epsilon_{\mathfrak{s}_h} \in 2\mathbb{Z}$. So $D_{\mathfrak{s}_h} = 2$ and $\tilde{\mathfrak{s}}_h$ is the set of rational children of $\mathfrak{s}_h$. Hence $g(\mathfrak{s}_h) = \left\lfloor \frac{|\mathfrak{s}_h|-1}{2} \right\rfloor \geq 1$ since $|\mathfrak{s}_h| \geq 3$. But then $\Gamma_{\mathfrak{s}_h}$ cannot be blown down.

2.5.12 Galois action

Consider the base extended hyperelliptic curve $C_{K^{nr}}/K^{nr}$. The rational clusters of $C_{K^{nr}}$ and their corresponding rational centres are then over K^{nr}. Denote $\Sigma_{K^{nr}} = \Sigma_{C_{K^{nr}}}^{\mathrm{rat}}$. For any proper cluster $\mathfrak{s} \in \Sigma_{K^{nr}}$, let $G_{\mathfrak{s}} = \mathrm{Stab}_{G_K}(\mathfrak{s})$, $K_{\mathfrak{s}} = (K^{\mathrm{s}})^{G_{\mathfrak{s}}}$ and $k_{\mathfrak{s}}$ be the residue field of $K_{\mathfrak{s}}$. Let $\Sigma_{C_{K^{nr}}}^{\min} = \{\mathfrak{s}_1, \ldots, \mathfrak{s}_m\}$ be the set of rationally minimal clusters of $C_{K^{nr}}$. Fix a set $W = \{w_1, \ldots, w_m\} \subset K^{nr}$ of corresponding rational centres. By Lemma A.1.1, we can assume this choice to be G_K-equivariant, i.e. for any $\sigma \in G_K$, one has $\sigma(w_l) = w_h$ if and only if $\sigma(\mathfrak{s}_l) = \mathfrak{s}_h$. We can also require that $w_h \in \mathfrak{s}_h$ if $\mathfrak{s}_h \cap K_{\mathfrak{s}_h} \neq \varnothing$. Similarly, for any proper cluster $\mathfrak{t} \in \Sigma_{K^{nr}} \setminus \Sigma_{C_{K^{nr}}}^{\min}$, fix a rational centre $w_{\mathfrak{t}}$ in such a way that $w_{\sigma(t)} = \sigma(w_{\mathfrak{t}})$ for any $\sigma \in G_K$. Set $w_{\mathfrak{s}_o} := w_o$ for any $o = 1, \ldots, m$.

Lemma 2.5.22 *With the choices above, for any $h = 1, \ldots, m$, the set of proper clusters in $\Sigma_{C_{K^{nr}}}^{w_h}$ coincides with $\mathring{\Sigma}_{C_{K^{nr}}}^{w_h}$.*

Proof. Suppose by contradiction that there exists a non-proper cluster $\{r\} = \mathfrak{s} \in \Sigma_{C_{K^{nr}}}^{w_h}$, with $r \neq w_h$. Note that $r \in \mathfrak{s}_h$ and so $\mathfrak{s} < \mathfrak{s}_h$. Recall that since $\mathfrak{s}$ is a cluster centred at w_h, it is cut out by the disc $\mathcal{D} = \{x \in \bar{K} \mid v(x - w_h) \geq \rho_{\mathfrak{s}}^{w_h}\}$, with $\rho_{\mathfrak{s}}^{w_h} = v(r - w_h) > \rho_{\mathfrak{s}_h}$. This implies that $w_h \notin \mathfrak{R}$, otherwise $w_h \in \mathfrak{s}$ and $|\mathfrak{s}| \geq 2$. In particular, $w_h \notin \mathfrak{s}_h$. For our choice of w_h, it follows that $\mathfrak{s}_h \cap K_{\mathfrak{s}_h} = \varnothing$. Therefore $r \notin K_{\mathfrak{s}_h}$ and so there exists $\sigma \in G_{\mathfrak{s}_h}$ such that $\sigma(r) \neq r$. Since $w_h \in K_{\mathfrak{s}_h}$ we have

$$v(\sigma(r) - w_h) = v(\sigma(r - w_h)) = v(r - w_h) = \rho_{\mathfrak{s}}^{w_h}.$$

But then $\sigma(r) \in \mathfrak{s}$, and so $|\mathfrak{s}| \geq 2$, a contradiction. □

Assume that $C_{K^{nr}}$ is y-regular and has an almost rational cluster picture. By the previous lemma, from the set of rational centres W we can construct the proper regular model $\mathcal{C}/O_{K^{nr}}$ of $C_{K^{nr}}$ as previously presented in this section. In this subsection we show how the Galois group $\mathrm{Gal}(K^{nr}/K)$ acts on the $O_{K^{nr}}$-scheme $\mathcal{C}$. Moreover, we describe the induced action of G_k on the special fibre $\mathcal{C}_s/k^{\mathrm{s}}$, and give defining equations for the principal components of $\mathcal{C}_s$ compatibly with the action.

For any $l = 1, \ldots, m$, recall the notation $f_l(x) = f(x + w_l) \in K^{nr}[x]$ and $C^{w_l}/K^{nr} : y^2 = f_l(x)$. Fix $\sigma \in G_K$. Let $l, h = 1, \ldots, m$ such that $\sigma(\mathfrak{s}_l) = \mathfrak{s}_h$. Then $\sigma(f_l) = f_h$. Now, let $\mathfrak{t} \in \Sigma_{C_{K^{nr}}}^{w_l}$ be a proper cluster. Then $\sigma(\mathfrak{t}) \in \Sigma_{C_{K^{nr}}}^{w_h}$ and $\rho_{\mathfrak{t}} = \rho_{\sigma(\mathfrak{t})}$. It follows that most of the quantities attached to $\mathfrak{t}$, such as those of Definition 2.4.6, are the same for $\sigma(\mathfrak{t})$, e.g. $\epsilon_{\mathfrak{t}} = \epsilon_{\sigma(\mathfrak{t})}$. In particular, if M is a matrix associated to $\mathfrak{t}$ then M is associated to $\sigma(\mathfrak{t})$ as well. So $\sigma(\mathcal{F}_M^l) = \mathcal{F}_M^h$. Finally, as $\sigma(\prod_{o \neq l}(x + w_{lo})^{-1}) = \prod_{o \neq h}(x + w_{ho})^{-1}$ we also have $\sigma(T_M^l) = T_M^h$.

Hence the natural K^{nr}-isomorphism $C^{w_h} \xrightarrow{\sigma} C^{w_l}$ induces $O_{K^{nr}}$-isomorphisms of schemes

$$\mathcal{C}_{\Delta}^{w_h} \xrightarrow{\sigma} \mathcal{C}_{\Delta}^{w_l}, \qquad \mathring{\mathcal{C}}_{\Delta}^{w_h} \xrightarrow{\sigma} \mathring{\mathcal{C}}_{\Delta}^{w_l}, \qquad U_M^h \xrightarrow{\sigma} U_M^l. \tag{2.10}$$

Via the glueing morphisms (2.4), these maps describe the action of G_K on $\mathcal{C}$.

We want to study the action of G_k on the special fibre of $\mathcal{C}$ more in detail. Let $\sigma \in \mathrm{Gal}(K^{nr}/K)$ and let $\bar{\sigma} \in G_k$ corresponding to σ via the canonical isomorphism $\mathrm{Gal}(K^{nr}/K) \simeq G_k$. Let l, h and $\mathfrak{t}$ as above. In §2.5.6 we described closed 1-dimensional subschemes composing $\mathring{\mathcal{C}}_{\Delta,s}^{w_l}$ and the morphisms induced by the glueing maps. Recall the polynomials introduced in Definition 2.5.12. From (2.10) we get

$$\bar{\sigma}(\overline{g^0_{\mathfrak{s}_l,w_l}}) = \overline{g^0_{\mathfrak{s}_h,w_h}}, \quad \bar{\sigma}(\overline{g_{\mathfrak{t},w_l}}) = \overline{g_{\sigma(\mathfrak{t}),w_h}}, \quad \bar{\sigma}(\overline{g_l|_{L_{\mathfrak{t}}^{w_l}}}) = \overline{g_h|_{L_{\sigma(\mathfrak{t})}^{w_h}}}.$$

From the equality (2.7) we obtain $\bar{\sigma}(\overline{f_{\mathfrak{t},w_l}}) = \overline{f_{\sigma(\mathfrak{t}),w_h}}$. Note that the previous relations can also be recovered directly from the definitions.

Lemma 2.5.23 *Let $w_{\mathfrak{t}}$ be the rational centre of $\mathfrak{t}$ fixed above. Then*

(i) $\overline{g_{\mathfrak{t},w_{\mathfrak{t}}}}, \overline{f_{\mathfrak{t},w_{\mathfrak{t}}}} \in k_{\mathfrak{t}}[X]$;

(ii) $\overline{g_{\mathfrak{t},w_{\mathfrak{t}}}} = \overline{g_{\mathfrak{t},w_l}}$ *and* $\overline{f_{\mathfrak{t},w_{\mathfrak{t}}}}(X) = \overline{f_{\mathfrak{t},w_l}}(X + \overline{u_{w_{\mathfrak{t}}w_l}})$ *where* $\overline{u_{w_{\mathfrak{t}}w_l}} = \frac{w_{\mathfrak{t}} - w_l}{\pi^{\rho_{\mathfrak{t}}}} \mod \pi$;

Proof. For any rational centre w of $\mathfrak{t}$, let $u_{\mathfrak{t},w} = c_f \prod_{r \in \mathfrak{R} \setminus \mathfrak{t}} (w - r)$ as in Definition 2.5.12. Note that $u_{\mathfrak{t},w}/\pi^{v(u_{\mathfrak{t},w})}$ is independent of w since

$$v((w_{\mathfrak{t}} - r) - (w - r)) = v(w_{\mathfrak{t}} - w) \geq \rho_{\mathfrak{t}} > v(w_{\mathfrak{t}} - r)$$

for any $r \in \mathfrak{R} \setminus \mathfrak{t}$. Then $\overline{g_{\mathfrak{t},w_{\mathfrak{t}}}} = \overline{g_{\mathfrak{t},w_l}}$. If $\bar{\sigma} \in \mathrm{Gal}(k^{\mathrm{s}}/k_{\mathfrak{t}})$, i.e. $\sigma \in \mathrm{Gal}(K^{nr}/K_{\mathfrak{t}})$, then

$$\bar{\sigma}(\overline{g_{\mathfrak{t},w_{\mathfrak{t}}}}) = \bar{\sigma}(\overline{g_{\mathfrak{t},w_l}}) = \overline{g_{\mathfrak{t},w_h}} = \overline{g_{\mathfrak{t},w_{\mathfrak{t}}}}.$$

In particular $\overline{g_{\mathfrak{t},w_{\mathfrak{t}}}} \in k_{\mathfrak{t}}[X]$.

Since $u_{\mathfrak{t},w}/\pi^{v(u_{\mathfrak{t},w})}$ is independent of w we also have

$$\overline{f_{\mathfrak{t},w_{\mathfrak{t}}}}(X^{b_{\mathfrak{t}}}) = \overline{f_{\mathfrak{t},w_l}}((X + \overline{u_{w_{\mathfrak{t}}w_l}})^{b_{\mathfrak{t}}}).$$

Suppose $\rho_{\mathfrak{t}} \in \mathbb{Z}$. Then $b_{\mathfrak{t}} = 1$ and so the equality above implies $\overline{f_{\mathfrak{t},w_{\mathfrak{t}}}}(X) = \overline{f_{\mathfrak{t},w_l}}(X + \overline{u_{w_{\mathfrak{t}}w_l}})$. Suppose $\rho \notin \mathbb{Z}$. Then $v(w - w_{\mathfrak{t}}) > \rho_{\mathfrak{t}}$ for any rational centre w of $\mathfrak{t}$ as $v(w - w_{\mathfrak{t}}) \in \mathbb{Z}$ and $v(w - w_{\mathfrak{t}}) \geq \rho_{\mathfrak{t}}$. Hence $\overline{u_{w_{\mathfrak{t}}w_l}} = 0$. Thus $\overline{f_{\mathfrak{t},w_{\mathfrak{t}}}}(X^{b_{\mathfrak{t}}}) = \overline{f_{\mathfrak{t},w_l}}(X^{b_{\mathfrak{t}}})$, which implies $\overline{f_{\mathfrak{t},w_{\mathfrak{t}}}}(X) = \overline{f_{\mathfrak{t},w_l}}(X) = \overline{f_{\mathfrak{t},w_l}}(X + \overline{u_{w_{\mathfrak{t}}w_l}})$. If $\bar{\sigma} \in \mathrm{Gal}(k^{\mathrm{s}}/k_{\mathfrak{t}})$, i.e. $\sigma \in \mathrm{Gal}(K^{nr}/K_{\mathfrak{t}})$, then

$$\bar{\sigma}(\overline{f_{\mathfrak{t},w_{\mathfrak{t}}}})(X) = \bar{\sigma}(\overline{f_{\mathfrak{t},w_l}})(X + \bar{\sigma}(\overline{u_{w_{\mathfrak{t}}w_l}})) = \overline{f_{\mathfrak{t},w_h}}(X + \overline{u_{w_{\mathfrak{t}}w_h}}) = \overline{f_{\mathfrak{t},w_{\mathfrak{t}}}}(X),$$

and so $\overline{f_{\mathfrak{t},w_{\mathfrak{t}}}} \in k_{\mathfrak{t}}[X]$. □

Remark 2.5.24. Note that the polynomials $\overline{f_{\mathfrak{t},w_\mathfrak{t}}}$, $\overline{g_{\mathfrak{t},w_\mathfrak{t}}}$ and $\overline{g^0_{\mathfrak{s}_h,w_h}}$ equal the polynomials $\overline{f_\mathfrak{t}}$, $\overline{g_\mathfrak{t}}$ and $\overline{g^0_{\mathfrak{s}_h}}$ given in Definition 2.4.21.

Let $V = V_\mathfrak{t}^{w_l}$ and consider the subscheme $X_V \times \mathbb{P}_V$ of $\mathcal{C}_s$ given by V, where $\mathbb{P}_V$ is a chain of $\mathbb{P}^1$s and $X_V : \{\overline{g_{\mathfrak{t},w_l}} = 0\}$ over $\mathbb{G}_{m,k^s}$. If $\mathfrak{s}_h \subset \mathfrak{t}$, then the glueing map $U_M^h \to U_M^l$ induces the identity $\phi_V^{hl} : X_{V_\mathfrak{t}^{w_h}} \xrightarrow{=} X_{V_\mathfrak{t}^{w_l}}$. Define $X_\mathfrak{t} \subseteq \mathbb{G}_{m,k^s}$ given by $g_{\mathfrak{t},w_\mathfrak{t}} = 0$. By Lemma 2.5.23, $\phi_V^o : X_\mathfrak{t} \xrightarrow{\simeq} X_{V_\mathfrak{t}^{w_o}}$, for $o = h,l$, and this isomorphism is compatible with the Galois action and the glueing maps, i.e. $\sigma \circ \phi_V^h = \phi_V^l \circ \sigma$ and $\phi_V^{hl} \circ \phi_V^h = \phi_V^l$ as morphisms on $X_\mathfrak{t}$.

When $V_0 = V_0^{w_l}$ we can consider the subscheme $X_{V_0} \times \mathbb{P}_{V_0}$ given by V_0, where $\mathbb{P}_{V_0}$ is a chain of $\mathbb{P}^1$s and $X_{V_0} : \{\overline{g_{\mathfrak{s}_l,w_l}} = 0\}$ over $\mathbb{G}_{m,k^s}$. Since $X_{V_0} \times \mathbb{P}_{V_0}$ is never glued to any other component there is no need to choose a different model for it.

Let $L = L_\mathfrak{t}^{w_l}$ and consider the subscheme $X_L^W \times \mathbb{P}_L$ given by L, where $\mathbb{P}_L$ is a chain of $\mathbb{P}^1$s and $X_L^W : \{\overline{f_{\mathfrak{t},w_l}} = 0\}$ over $\mathbb{A}^1_{k^s}$. If $\mathfrak{s}_h \subset \mathfrak{t}$, then the isomorphism $\phi_L^{hl} : X^W_{L_\mathfrak{t}^{w_h}} \xrightarrow{\simeq} X^W_{L_\mathfrak{t}^{w_l}}$ given by the glueing map $U_M^h \to U_M^l$ is induced by the ring isomorphism $k^s[X] \to k^s[X]$, sending $X \mapsto X + \overline{u_{w_h w_l}}$, where $\overline{u_{w_h w_l}} = \frac{w_h - w_l}{\pi^{\rho_\mathfrak{t}}} \mod \pi$. Define $X_\mathfrak{t}^W \subseteq \mathbb{A}^1_{k^s}$ given by $\overline{f_{\mathfrak{t},w_\mathfrak{t}}} = 0$. By Lemma 2.5.23, the map $X \mapsto X + \overline{u_{w_\mathfrak{t} w_l}}$ induces an isomorphism $\phi_L^o : X_\mathfrak{t}^W \xrightarrow{\simeq} X^W_{L_\mathfrak{t}^{w_o}}$, for $o = h,l$, compatible with the Galois action and the glueing morphisms, i.e. $\sigma \circ \phi_L^h = \phi_L^l \circ \sigma$ and $\phi_L^{hl} \circ \phi_L^h = \phi_L^l$ as morphisms on $X_\mathfrak{t}^W$.

Recall the definitions of $\hat{\mathfrak{t}}^W$ and $\mathbb{G}_{\mathfrak{t},w_l} \subseteq \mathbb{A}^1_{k^s}$ given in Definition 2.5.15 and the definition of $\mathring{\mathfrak{t}}$ given in Definition 2.4.21. Note that by Lemma 2.5.22,

$$\hat{\mathfrak{t}}^W = \{\mathfrak{s} \in \Sigma_{K^{nr}} \cup \{\varnothing\} \mid \mathfrak{s} < \mathfrak{t}\} \setminus \{\{r\} \in \Sigma_{K^{nr}} \mid r \notin W\}.$$

Fix $c = 0, \dots, b_\mathfrak{t} - 1$ such that $1/b_\mathfrak{t} - c\rho_\mathfrak{t} \in \mathbb{Z}$. For any rational centre $w \in K^{nr}$ of $\mathfrak{t}$ define $\hat{f}_{\mathfrak{t},w} \in k^s[X,Y]$ by

$$\hat{f}_{\mathfrak{t},w}(X) = \prod_{\mathfrak{s} \in \hat{\mathfrak{t}}^W} (X - \overline{u_{w_\mathfrak{s} w}})^{\frac{|\mathfrak{s}|}{b_\mathfrak{t}} - c\epsilon_\mathfrak{t}} \overline{f_{\mathfrak{t},w}}(X),$$

where $\overline{u_{w_\mathfrak{s} w}} = \frac{w_\mathfrak{s} - w}{\pi^{\rho_\mathfrak{t}}} \mod \pi$ ($w_\mathfrak{s} = w_l$ if $\mathfrak{s} = \varnothing$). Let $L = L_\mathfrak{t}^{w_l}$, $F = F_\mathfrak{t}^{w_l}$ and $M = M_{L,0}$. It follows from Lemma 2.5.17 that the scheme $\Gamma_\mathfrak{t}^{w_l} = \mathring{X}_F \cap U_M^l$ is given by $Y^{D_\mathfrak{t}} = \hat{f}_{\mathfrak{t},w_l}(X)$ as a subscheme of $\mathbb{G}_{\mathfrak{t},w_l} \times \mathbb{A}^1_{k^s}$. We then obtain $\bar{\sigma}(\hat{f}_{\mathfrak{t},w_l}) = \hat{f}_{\sigma(\mathfrak{t}),w_h}$ from the action (2.10) of σ on U_M^l.

Lemma 2.5.25 *With the notation above,*

(i) $\hat{f}_{\mathfrak{t},w_\mathfrak{t}} \in k_\mathfrak{t}[X]$*;*

(ii) $\hat{f}_{\mathfrak{t},w_\mathfrak{t}}(X) = \hat{f}_{\mathfrak{t},w_l}(X + \overline{u_{w_\mathfrak{t} w_l}})$ *where* $\overline{u_{w_\mathfrak{t} w_l}} = \frac{w_\mathfrak{t} - w_l}{\pi^{\rho_\mathfrak{t}}} \mod \pi$*;*

Proof. If $\mathfrak{s} \in \mathring{\mathfrak{t}}$, then $\sigma(\mathfrak{s}) \in (\sigma(\mathring{\mathfrak{t}}))$ and $\bar{\sigma}(\overline{u_{w_\mathfrak{s} w}}) = \overline{u_{w_{\sigma(\mathfrak{s})}\sigma(w)}}$ for any rational centre w of $\mathfrak{t}$. Hence $\hat{f}_{\mathfrak{t},w_\mathfrak{t}} \in k_\mathfrak{t}[X]$ and $\bar{\sigma}(\hat{f}_{\mathfrak{t},w_l}) = \hat{f}_{\sigma(\mathfrak{t}),w_h}$ by Lemma 2.5.23(i),(iii). Since $\overline{u_{w_\mathfrak{s} w_\mathfrak{t}}} = \overline{u_{w_\mathfrak{s} w_l}} - \overline{u_{w_\mathfrak{t} w_l}}$, Lemma 2.5.23(ii) implies $\hat{f}_{\mathfrak{t},w_\mathfrak{t}}(X) = \hat{f}_{\mathfrak{t},w_l}(X + \overline{u_{w_\mathfrak{t} w_l}})$. □

Define $\Gamma_\mathfrak{t}^{w_\mathfrak{t}} \subset \mathbb{G}_{\mathfrak{t},w_\mathfrak{t}} \times \mathbb{A}^1_{k^s}$ given by $Y^{D_\mathfrak{t}} = \hat{f}_{\mathfrak{t},w_\mathfrak{t}}$. Suppose $\mathfrak{s}_h \subset \mathfrak{t}$, and let $\phi_\mathfrak{t}^{hl} : \Gamma_\mathfrak{t}^{w_h} \simeq \Gamma_\mathfrak{t}^{w_l}$ be the isomorphism coming from the glueing map $U_M^h \to U_M^l$ induced by the ring homomorphism

$X \mapsto X + \overline{u_{w_h w_l}}$. By Lemma 2.5.25, the map $X \mapsto X + \overline{u_{w_\mathfrak{t} w_l}}$ induces an isomorphism $\phi_\mathfrak{t}^o : \Gamma_\mathfrak{t}^{w_\mathfrak{t}} \simeq \Gamma_\mathfrak{t}^{w_o}$, for $o = h, l$, which is compatible with the Galois action and the glueing maps, i.e. $\sigma \circ \phi_\mathfrak{t}^h = \phi_\mathfrak{t}^l \circ \sigma$ and $\phi_\mathfrak{t}^{hl} \circ \phi_\mathfrak{t}^h = \phi_\mathfrak{t}^l$ as morphisms on $\Gamma_\mathfrak{t}^{w_\mathfrak{t}}$. Therefore $\Gamma_\mathfrak{t}$ is isomorphic to the smooth completion of $\Gamma_\mathfrak{t}^{w_\mathfrak{t}}$, and so it is given by $Y^{D_\mathfrak{t}} = \tilde{f}_\mathfrak{t}(X)$, where $\tilde{f}_\mathfrak{t}(X) = \prod_{\mathfrak{s} \in \mathring{\mathfrak{t}}} (X - \overline{u_{w_\mathfrak{s} w_\mathfrak{t}}}) \overline{f_{\mathfrak{t}, w_\mathfrak{t}}}(X)$ is the polynomial given in Definition 2.4.21.

2.6 Integral differentials

Let C be a hyperelliptic curve of genus $g \geq 1$ defined over K by a Weierstrass equation $y^2 = f(x)$. It is well-known that the K-vector space of global sections of the sheaf of differentials of C, namely $H^0(C, \Omega^1_{C/K})$, is spanned by the basis

$$\underline{\omega} = \left\{ \frac{dx}{2y}, x\frac{dx}{2y}, \dots, x^{g-1}\frac{dx}{2y} \right\}.$$

Let $\mathcal{C}$ be a regular model of C over O_K and consider its canonical (or dualising) sheaf $\omega_{\mathcal{C}/O_K}$. The free O_K-module of its global sections $H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K})$ can be viewed as an O_K-lattice in $H^0(C, \Omega^1_{C/K})$ (see [Liu4, Corollary 9.2.25(a)]). The aim of this section is to present a basis of $H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K})$ as an O_K-linear combination of the elements in $\underline{\omega}$ under the assumptions of Theorem 2.4.22. Note that by [Liu4, Corollary 9.2.25(b)] the problem is independent of the choice of model $\mathcal{C}$ but it does depend on the choice of the equation $y^2 = f(x)$ since the basis $\underline{\omega}$ does. Throughout this section let C and $\mathcal{C}/O_K$ be as above.

If C is Δ_v-regular, [Dok, Theorem 8.12] gives an O_K-basis of $H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K})$, as required. We rephrase it in terms of rational cluster invariants, by using results of §2.3 and Lemma 2.4.12.

Theorem 2.6.1 *Suppose C has an almost rational cluster picture and is y-regular, and all proper clusters $\mathfrak{s} \in \Sigma_C$ have same rational centre $w \in K$. Let $\mathfrak{s}_1 \subset \cdots \subset \mathfrak{s}_n = \mathfrak{R}$ be the proper clusters in Σ_C^{rat}. For every $j = 0, \dots, g-1$, define*

$$i_j := \min\{i \in \{1, \dots, n\} \mid 2(j+1) < |\mathfrak{s}_i|\}$$

and

$$e_j := \tfrac{1}{2}\epsilon_{\mathfrak{s}_{i_j}} - (j+1)\rho_{\mathfrak{s}_{i_j}}.$$

Then the differentials

$$\mu_j = \pi^{\lfloor e_j \rfloor}(x - w)^j \frac{dx}{2y} \qquad j = 0, \dots, g-1,$$

form an O_K-basis of $H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K})$.

Proof. Let $C^w : y^2 = f(x + w)$ be the hyperelliptic curve isomorphic to C through the change of variable $y \mapsto y$, $x \mapsto x + w$. By Corollary 2.3.25 and Lemma 2.4.12, the curve C^w is Δ_v-regular. Since $\mathring{\Sigma}_C^{\mathrm{rat}}$ consists of the proper clusters in Σ_C^w, Lemma 2.4.3 and [Dok, Theorem 8.12] implies that

$$\mu_j = \pi^{\lfloor e_j \rfloor} x^j \frac{dx}{2y} \qquad j = 0, \dots, g-1,$$

form an O_K-basis of $H^0(\mathcal{C},\omega_{\mathcal{C}/O_K})$ as a lattice in $H^0(C^w,\Omega^1_{C^w/K})$ (that is if $\mathcal{C}$ is regarded as a model of C^w). Changing variables concludes the proof. $\square$

Suppose now C has an almost rational cluster picture and is y-regular. Let $\Sigma_C^{\min}$ be the set of rationally minimal clusters and let $W=\{w_\mathfrak{s}\mid \mathfrak{s}\in\Sigma_C^{\min}\}$ be a corresponding set of rational centres, such that all clusters in $\mathring{\Sigma}_C^{w_\mathfrak{s}}$ are proper. For every proper cluster $\mathfrak{t}\in\Sigma_C^{\mathrm{rat}}$, choose a minimal cluster $\mathfrak{s}\subseteq\mathfrak{t}$ and set $w_\mathfrak{t}:=w_\mathfrak{s}$. Consider the regular model $\mathcal{C}/O_K$ of C of Theorem 2.4.18, constructed in §2.5 by glueing the open subschemes $\mathring{\mathcal{C}}^w_\Delta$ of $\mathcal{C}^w_\Delta$ for $w\in W$. We want to describe the canonical morphism $C\to\mathcal{C}$. Write $W=\{w_1,\dots,w_m\}$ as in §2.5. For any $h=1,\dots,m$, let $\mathfrak{t}\in\Sigma_C^{w_h}$ be a proper cluster and let M be a matrix associated to $\mathfrak{t}$. Let $C^{w_h}: y^2=f(x+w_h)$ and

$$y^2-f(x+w_h)\overset{M}{=}Y^{n_Y}Z^{n_Z}\mathcal{F}_M^h(X,Y,Z).$$

Then, on the affine chart X_M the projection $C\to\mathcal{C}^{w_h}_\Delta$ is induced by

$$\frac{R}{\big(\mathcal{F}_M^h(X,Y,Z)\big)}\xrightarrow{M}\frac{K[(x')^{\pm1},(y')^{\pm1}]}{\big((y')^2-f(x'+w_h)\big)}\xrightarrow{\simeq}\frac{K[x^{\pm1},y^{\pm1}]}{\big(y^2-f(x)\big)},$$

where $(X,Y,Z)=(x',y',\pi)\bullet M$ and $(x',y')=(x-w_h,y)$. In particular it follows that $(X,Y,Z)=(x-w_h,y,z)\bullet M$ and so

$$\begin{pmatrix}x-w_h\\ y\\ \pi\end{pmatrix}=\begin{pmatrix}X^{\tilde m_{11}}Y^{\tilde m_{21}}Z^{\tilde m_{31}}\\ X^{\tilde m_{12}}Y^{\tilde m_{22}}Z^{\tilde m_{32}}\\ X^{\tilde m_{13}}Y^{\tilde m_{23}}Z^{\tilde m_{33}}\end{pmatrix}=\begin{pmatrix}X\\ Y\\ Z\end{pmatrix}\bullet M^{-1}.$$

For a proper cluster $\mathfrak{t}\in\Sigma_C^{\mathrm{rat}}$ recall the definitions of $\Gamma_\mathfrak{t}$ and $m_\mathfrak{t}$.

Proposition 2.6.2 *Let $\mathfrak{t}\in\Sigma_C^{\mathrm{rat}}$ be a proper cluster. Then*[5]

$$\begin{aligned}&\mathrm{ord}_{\Gamma_\mathfrak{t}}(x-w_\mathfrak{s})=m_\mathfrak{t}\rho_\mathfrak{t},\\ &\mathrm{ord}_{\Gamma_\mathfrak{t}}\frac{dx}{2y}=-m_\mathfrak{t}\big(\tfrac12\epsilon_\mathfrak{t}-\rho_\mathfrak{t}-1\big)-1.\end{aligned}$$

for every proper cluster $\mathfrak{s}\in\Sigma_C^{\mathrm{rat}}$, $\mathfrak{s}\subseteq\mathfrak{t}$.

Proof. Let $g(x,y)=y^2-f(x)$. Let $W=\{w_1,\dots,w_m\}$ as above. Let $h=1,\dots,m$ such that $w_h=w_\mathfrak{s}$. Let $F=F_\mathfrak{t}^{w_h}$, $V=V_\mathfrak{t}^{w_h}$, $M=M_{V,0}$ and let X,Y,Z be the transformed variables $(X,Y,Z)=(x-w_\mathfrak{s},y,\pi)\bullet M$. Define $\mathcal{H}(X,Y,Z)=\pi-X^{\tilde m_{13}}Y^{\tilde m_{23}}Z^{\tilde m_{33}}$, and $\mathcal{G}(X,Y,Z)=g((X,Y,Z)\bullet M^{-1})$. We have

$$\mathcal{F}_M^h(X,Y,Z)=Y^{-n_Y}Z^{-n_Z}\mathcal{G}(X,Y,Z),$$

where $n_Z=m_\mathfrak{t}\epsilon_\mathfrak{t}$, since $\mathrm{ord}_Z(y^2)=m_\mathfrak{t}\epsilon_\mathfrak{t}$ for Lemma 2.5.2. Write $\mathcal{F}=\mathcal{F}_M^h$.

[5] If $\Gamma_\mathfrak{t}$ is reducible, say $\Gamma_\mathfrak{t}=\Gamma_\mathfrak{t}^-\cup\Gamma_\mathfrak{t}^+$, with $\mathrm{ord}_{\Gamma_\mathfrak{t}}(\cdot)$ we mean $\min\{\mathrm{ord}_{\Gamma_\mathfrak{t}^-}(\cdot),\mathrm{ord}_{\Gamma_\mathfrak{t}^+}(\cdot)\}$

Note that $d(x-w_5)=dx$ and $(g_{w_5})'_x(x-w_5)=g'_x(x)$, where $g_{w_5}(x,y)=g(x+w_5,y)$. Then, by [Dok, 8.7],

$$\begin{cases}(x-w_5)g'_x = m_{11}X\mathcal{G}'_X + m_{12}Y\mathcal{G}'_Y + m_{13}Z\mathcal{G}'_Z \\ yg'_y = m_{21}X\mathcal{G}'_X + m_{22}Y\mathcal{G}'_Y + m_{23}Z\mathcal{G}'_Z\end{cases}$$

from which it follows that

$$\begin{aligned} m_{11}yg'_y - m_{21}(x-w_5)g'_x &= (m_{11}m_{22}-m_{21}m_{12})Y\mathcal{G}'_Y - (m_{21}m_{13}-m_{11}m_{23})Z\mathcal{G}'_Z \\ &= \tilde{m}_{33}Y\mathcal{G}'_Y - \tilde{m}_{23}Z\mathcal{G}'_Z. \end{aligned}$$

We will show later that this quantity is non-zero. Moreover,

$$\tilde{m}_{33}Y\mathcal{G}'_Y - \tilde{m}_{23}Z\mathcal{G}'_Z = Y^{n_Y}Z^{n_Z}\left(\tilde{m}_{33}Y\mathcal{F}'_Y - \tilde{m}_{23}Z\mathcal{F}'_Z + (n_Y+n_Z)\mathcal{F}\right).$$

Recall that $X=(x-w_5)^{m_{11}}y^{m_{21}}\pi^{m_{31}}$. Then $\frac{dX}{X} = m_{11}\frac{dx}{x-w_5} + m_{21}\frac{dy}{y}$. Furthermore, as $0=dg=g'_x dx + g'_y dy$ in $\Omega_{C/K}$, we have

$$\frac{dX}{X} = \left(\frac{m_{11}}{x-w_5} - \frac{m_{21}}{y}\frac{g'_x}{g'_y}\right)dx = \frac{dx}{(x-w_5)yg'_y}\left(m_{11}yg'_y - m_{21}(x-w_5)g'_x\right).$$

Therefore

$$\frac{dx}{2(x-w_5)y^2} = \frac{dX}{XY^{n_Y}Z^{n_Z}\left(\tilde{m}_{33}Y\mathcal{F}'_Y - \tilde{m}_{23}Z\mathcal{F}'_Z + (n_Y+n_Z)\mathcal{F}\right)}. \tag{2.11}$$

Let $S=\operatorname{Spec} O_K$. Considering X^{-1} as an independent variable, the scheme

$$U = \operatorname{Spec}\frac{O_K[Y,Z,X^{-1},X]}{(\mathcal{F},\mathcal{H},X\cdot X^{-1}-1)}$$

defines a complete intersection in $\mathbb{A}^4_S$. Furthermore, U is an open subscheme of $\mathcal{C}^{w_h}_\Delta \cap X_M$ that restricted to $\mathbb{A}^4_S\setminus\{T^h_M(X,Y,Z)=0\}$ equals U^h_M. In particular, U is integral. From §2.5.5 it follows that $U_{\mathfrak{t}} = U\cap\{Z=0\}$ is a dense open subset of $\mathring{X}_F$. Recall that $\mathring{X}_F$ is an open subscheme of $\Gamma_{\mathfrak{t}}$. Hence it suffices to prove the proposition for $U_{\mathfrak{t}}$ instead of $\Gamma_{\mathfrak{t}}$ ([Liu4, Lemma 9.2.17(a)]). Since X and Y are units and Z vanishes to order 1 on $U_{\mathfrak{t}}$, Lemma 2.5.2 implies that

$$\operatorname{ord}_{U_{\mathfrak{t}}}(x-w_5) = \tilde{m}_{31} = m_{\mathfrak{t}}\rho_{\mathfrak{t}} \quad\text{and}\quad \operatorname{ord}_{U_{\mathfrak{t}}}y = \tilde{m}_{32} = m_{\mathfrak{t}}\tfrac{\epsilon_{\mathfrak{t}}}{2}. \tag{2.12}$$

Recall that U is integral and that U_η is isomorphic to an open subscheme of C. Then U_η is smooth. Hence, by [Liu4, Corollary 6.4.14(b)], the sheaf ω_{C/O_K} is generated on U by $\mathcal{E}^{-1}dX$ where

$$\mathcal{E} := \begin{vmatrix} \mathcal{F}'_Y & \mathcal{F}'_Z & \mathcal{F}'_{X^{-1}} \\ H'_Y & H'_Z & \mathcal{F}'_{X^{-1}} \\ 0 & 0 & X \end{vmatrix} = -\pi XY^{-1}Z^{-1}\left(\tilde{m}_{33}Y\mathcal{F}'_Y - \tilde{m}_{23}Z\mathcal{F}'_Z\right),$$

if $\mathcal{E}$ is non-zero. Suppose it is; we are going to prove it later. Thus, as $\mathcal{F}=0$ on U, from (2.11) and (2.12) we have

$$\operatorname{ord}_{U_{\mathfrak{t}}}\frac{dx}{2y} = m_{\mathfrak{t}}\left(\tfrac{1}{2}\epsilon_{\mathfrak{t}}+\rho_{\mathfrak{t}}\right) + \tilde{m}_{33} - n_Z - 1 = m_{\mathfrak{t}}\left(-\tfrac{1}{2}\epsilon_{\mathfrak{t}}+\rho_{\mathfrak{t}}+1\right) - 1.$$

It remains to show that $\mathcal{E}$ does not equal 0 on U. Suppose it does. Then from the computations above, it follows that $m_{11}yg'_y - m_{21}(x-w_{\mathfrak{s}})g'_x = 0$ in $K(C)$. Since m_{21} equals either 1 or 2 by Lemma 2.5.2, it follows that there exists a non-zero $c \in K$, such that

$$m_{11}yg'_y - m_{21}(x-w_{\mathfrak{s}})g'_x + cg = 0$$

($c \in K$ from degree analysis). Then $cf(x) = m_{21}(x-w_{\mathfrak{s}})f'(x)$. Note that m_{21} is non-zero as $\mathrm{char}(K) \neq 2$. But then a contradiction follows since f is a separable polynomial of degree ≥ 3. □

In the following two theorems we describe a basis of integral differentials of C. We use Definitions/Notations 2.3.1, 2.3.3, 2.3.2, 2.3.8, 2.3.9, 2.3.26, 2.4.6, 2.4.10 in the statements.

Theorem 2.6.3 *Let C/K be a hyperelliptic curve of genus $g \geq 1$ defined by the Weierstrass equation $y^2 = f(x)$ and let $\mathcal{C}/O_K$ be a regular model of C. Suppose C has an almost rational cluster picture and is y-regular. For $i = 0,\ldots,g-1$ choose inductively proper clusters $\mathfrak{s}_i \in \Sigma_C^{\mathrm{rat}}$ so that*

$$e_i := \frac{\epsilon_{\mathfrak{s}_i}}{2} - \sum_{j=0}^{i} \rho_{\mathfrak{s}_j \wedge \mathfrak{s}_i} = \max_{\mathfrak{t}\in\Sigma_C^{\mathrm{rat}}} \left\{ \frac{\epsilon_{\mathfrak{t}}}{2} - \rho_{\mathfrak{t}} - \sum_{j=0}^{i-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{t}} \right\},$$

where if $\mathfrak{s}$ and $\mathfrak{s}'$ are two possible choices for $\mathfrak{s}_i$ satisfying $\mathfrak{s}' \subset \mathfrak{s}$, then choose $\mathfrak{s}_i = \mathfrak{s}$. Then the differentials

$$\mu_i = \pi^{\lfloor e_i \rfloor} \prod_{j=0}^{i-1} (x - w_{\mathfrak{s}_j}) \frac{dx}{2y}, \qquad i = 0,\ldots,g-1,$$

form an O_K-basis of $H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K})$.

Proof. Since $H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K})$ is independent of the choice of regular model, we consider $\mathcal{C}$ to be the model described in Theorem 2.4.18 and constructed in §2.5.

We first show that the differentials μ_i are global sections of $\omega_{\mathcal{C}/O_K}$. It suffices to prove they are regular along all components $\Gamma_{\mathfrak{t}}$, where $\mathfrak{t} \in \Sigma_C^{\mathrm{rat}}$ proper. Indeed for the construction of $\mathcal{C}$ and the definition of the e_i's, the differentials μ_i are regular along all other components of $\mathcal{C}_s$ by Theorem 2.6.1. Fix $i = 1,\ldots,g-1$ and let $j = 0,\ldots,i-1$. Let $\mathfrak{t} \in \Sigma_C^{\mathrm{rat}}$ be a proper cluster. If $\mathfrak{s}_j \subseteq \mathfrak{t}$ then

$$\mathrm{ord}_{\Gamma_{\mathfrak{t}}}(x - w_{\mathfrak{s}_j}) = m_{\mathfrak{t}}\rho_{\mathfrak{t}} = m_{\mathfrak{t}}\rho_{\mathfrak{s}_j \wedge \mathfrak{t}},$$

by Proposition 2.6.2. If $\mathfrak{t} \subsetneq \mathfrak{s}_j$ then $w_{\mathfrak{t}}$ is a rational centre of $\mathfrak{s}_j$. Hence

$$v(w_{\mathfrak{t}} - w_{\mathfrak{s}_j}) \geq \min_{r\in\mathfrak{t}} \min\{v(r-w_{\mathfrak{t}}), v(r-w_{\mathfrak{s}_j})\} \geq \min\{\rho_{\mathfrak{t}}, \rho_{\mathfrak{s}_j}\} = \rho_{\mathfrak{s}_j} = \rho_{\mathfrak{s}_j \wedge \mathfrak{t}}.$$

Therefore Proposition 2.6.2 implies

$$\begin{aligned} \mathrm{ord}_{\Gamma_{\mathfrak{t}}}(x - w_{\mathfrak{s}_j}) &\geq \min\{\mathrm{ord}_{\Gamma_{\mathfrak{t}}}(x - w_{\mathfrak{t}}), \mathrm{ord}_{\Gamma_{\mathfrak{t}}}(w_{\mathfrak{t}} - w_{\mathfrak{s}_j})\} \\ &\geq \min\{m_{\mathfrak{t}}\rho_{\mathfrak{t}}, m_{\mathfrak{t}}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}}\} = m_{\mathfrak{t}}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}}. \end{aligned}$$

If $\mathfrak{s}_j \not\subseteq \mathfrak{t}$ and $\mathfrak{t} \not\subseteq \mathfrak{s}_j$ then from Lemma 2.3.18 it follows that

$$\mathrm{ord}_{\Gamma_\mathfrak{t}}(x - w_{\mathfrak{s}_j}) = \min\{m_\mathfrak{t}\rho_\mathfrak{t}, m_\mathfrak{t}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}}\} = m_\mathfrak{t}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}}.$$

as $\rho_\mathfrak{t} > \rho_{\mathfrak{s}_j\wedge\mathfrak{t}}$. Thus we have proved that

$$\mathrm{ord}_{\Gamma_\mathfrak{t}}(x - w_{\mathfrak{s}_j}) \geq m_\mathfrak{t}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}}, \qquad \text{where the equality holds if } \mathfrak{t} \not\subset \mathfrak{s}_j. \tag{2.13}$$

Hence it follows from Proposition 2.6.2 that

$$\mathrm{ord}_{\Gamma_\mathfrak{t}}\mu_i \geq m_\mathfrak{t}\Big(\lfloor e_i \rfloor + \sum_{j=0}^{i-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}} - \frac{\epsilon_\mathfrak{t}}{2} + \rho_t + 1\Big) - 1.$$

But

$$\lfloor e_i \rfloor \geq \Big\lfloor \frac{\epsilon_\mathfrak{t}}{2} - \rho_\mathfrak{t} - \sum_{j=0}^{i-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}} \Big\rfloor > \frac{\epsilon_\mathfrak{t}}{2} - \rho_\mathfrak{t} - \sum_{j=0}^{i-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}} - 1,$$

then $\mathrm{ord}_{\Gamma_\mathfrak{t}}\mu_i > -1$, that implies $\mathrm{ord}_{\Gamma_\mathfrak{t}}\mu_i \geq 0$, as required.

Now we need to show that the differentials μ_i span $H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K})$, i.e. the lattice they span is saturated in the global sections of $\omega_{\mathcal{C}/O_K}$. Suppose not. Then there exist $I \subseteq \{0, \dots, g-1\}$ and $u_i \in O_K^\times$ for $i \in I$ such that the differential $\frac{1}{\pi}\sum_{i\in I} u_i\mu_i$ is regular along $\Gamma_\mathfrak{t}$, for every proper cluster $\mathfrak{t} \in \Sigma_C^{\mathrm{rat}}$. First we want to show that for any $i_1, i_2 = 0, \dots, g-1$ with $i_1 < i_2$, one has $\mathfrak{s}_{i_2} \not\subset \mathfrak{s}_{i_1}$. Suppose by contradiction that $\mathfrak{s}_{i_2} \subsetneq \mathfrak{s}_{i_1}$. Then

$$\begin{aligned} e_{i_2} &\geq \frac{\epsilon_{\mathfrak{s}_{i_1}}}{2} - \rho_{\mathfrak{s}_{i_1}} - \sum_{j=0}^{i_2-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_1}} = e_{i_1} - \rho_{\mathfrak{s}_{i_1}} - \sum_{j=i_1+1}^{i_2-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_1}} \geq e_{i_1} - \rho_{\mathfrak{s}_{i_1}} - \sum_{j=i_1+1}^{i_2-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_2}} \\ &\geq \frac{\epsilon_{\mathfrak{s}_{i_2}}}{2} - \rho_{\mathfrak{s}_{i_2}} - \sum_{j=0}^{i_1-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_2}} - \rho_{\mathfrak{s}_{i_1}} - \sum_{j=i_1+1}^{i_2-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_2}} = \frac{\epsilon_{\mathfrak{s}_{i_2}}}{2} - \sum_{j=0}^{i_2}\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_2}} = e_{i_2}. \end{aligned}$$

Therefore

$$\max_{\mathfrak{t}\in\Sigma_C^{\mathrm{rat}}}\Big\{\frac{\epsilon_\mathfrak{t}}{2} - \rho_\mathfrak{t} - \sum_{j=0}^{i_2-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}}\Big\} = e_{i_2} = \frac{\epsilon_{\mathfrak{s}_{i_1}}}{2} - \rho_{\mathfrak{s}_{i_1}} - \sum_{j=0}^{i_2-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_1}},$$

and this means that $\mathfrak{s}_{i_1}$ is a possible choice for the i_2-th cluster $\mathfrak{s}_{i_2}$. But $\mathfrak{s}_{i_2} \subsetneq \mathfrak{s}_{i_1}$, so the i_2-th cluster should have been $\mathfrak{s}_{i_1}$, a contradiction.

Let $I_0 \subseteq I$ be the set of indices i such that $\gamma_i := e_i - \lfloor e_i \rfloor$ is maximal. Let $i_0 = \min I_0$ and let $\Gamma_0 = \Gamma_{\mathfrak{s}_{i_0}}$. Since $\mathfrak{s}_{i_0} \not\subset \mathfrak{s}_j$, for all $j = 0, \dots, i_0 - 1$, from (2.13) it follows that

$$\begin{aligned} m := \mathrm{ord}_{\Gamma_0}\frac{1}{\pi}\mu_{i_0} &= -m_{\mathfrak{s}_{i_0}}\gamma_{i_0} + m_{\mathfrak{s}_{i_0}}\Big(e_{i_0} - \frac{\epsilon_{\mathfrak{s}_{i_0}}}{2} + \rho_{\mathfrak{s}_{i_0}} + \sum_{j=0}^{i_0-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_0}}\Big) - 1 \\ &= -m_{\mathfrak{s}_{i_0}}\gamma_{i_0} - 1 < 0. \end{aligned}$$

Furthermore,

$$\begin{aligned} \mathrm{ord}_{\Gamma_0}\frac{1}{\pi}\mu_i &\geq -m_{\mathfrak{s}_{i_0}}\gamma_i + m_{\mathfrak{s}_{i_0}}\Big(e_i - \frac{\epsilon_{\mathfrak{s}_{i_0}}}{2} + \rho_{\mathfrak{s}_{i_0}} + \sum_{j=0}^{i-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_0}}\Big) - 1 \\ &\geq -m_{\mathfrak{s}_{i_0}}\gamma_i - 1 \geq -m_{\mathfrak{s}_{i_0}}\gamma_{i_0} - 1 = m, \end{aligned}$$

for all $i \in I$. Let $J := \{i \in I \mid \mathrm{ord}_{\Gamma_0}\frac{1}{\pi}\mu_i = m\}$. Then $J \neq \varnothing$ since $i_0 \in J$. The order of the differential $\frac{1}{\pi}\sum_{i\in J} u_i\mu_i$ along Γ_0 must be $> m$. Let $i \in I$. From the computations above $i \in J$ if and only if

(i) $\operatorname{ord}_{\Gamma_0}(x-w_{\mathfrak{s}_j}) = m_{\mathfrak{s}_{i_0}}\rho_{\mathfrak{s}_{i_0}\wedge\mathfrak{s}_j}$ for all $j=0,\dots,i-1$. Equivalently, if $\mathfrak{s}_j \supsetneq \mathfrak{s}_{i_0}$ for some $j<i$, then $v(w_{\mathfrak{s}_{i_0}}-w_{\mathfrak{s}_j}) = \rho_{\mathfrak{s}_{i_0}\wedge\mathfrak{s}_j}$.

(ii) $e_i = \frac{\epsilon_{\mathfrak{s}_{i_0}}}{2} - \rho_{\mathfrak{s}_{i_0}} - \sum_{j=0}^{i-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_0}}$. In particular, if $\mathfrak{s}_i \subseteq \mathfrak{s}_{i_0}$, then $\mathfrak{s}_i = \mathfrak{s}_{i_0}$.

(iii) $\gamma_i = \gamma_{i_0}$. Equivalently, $i \in I_0$.

Therefore $J \subseteq I_0$, $i_0 = \min J$ and

$$\lfloor e_i \rfloor - \lfloor e_{i_0} \rfloor = e_i - \gamma_i - e_{i_0} + \gamma_{i_0} = e_i - e_{i_0} = -\sum_{j=i_0}^{i-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_0}},$$

for all $i \in J$. Hence

$$\frac{1}{\pi}\sum_{i\in J} u_i\mu_i = \frac{1}{\pi}\mu_{i_0}\Big(\sum_{i\in J}\frac{u_i}{\pi^{\sum_{j=i_0}^{i-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_0}}}}\prod_{j=i_0}^{i-1}(x-w_{\mathfrak{s}_j})\Big),$$

and since $\operatorname{ord}_{\Gamma_0}\frac{1}{\pi}\mu_{i_0} = m < 0$ we must have

$$\operatorname{ord}_{\Gamma_0}\Big(\sum_{i\in J}\frac{u_i}{\pi^{\sum_{j=i_0}^{i-1}\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_0}}}}\prod_{j=i_0}^{i-1}(x-w_{\mathfrak{s}_j})\Big) > 0. \tag{2.14}$$

For any $j<i\in J$, with $i_0 \le j$ we have $\mathfrak{s}_j \not\subset \mathfrak{s}_{i_0}$. Therefore either $\mathfrak{s}_j = \mathfrak{s}_{i_0}$ or $\mathfrak{s}_j \wedge \mathfrak{s}_{i_0} \supsetneq \mathfrak{s}_{i_0}$. In the latter case,

$$\operatorname{ord}_{\Gamma_0}(x-w_{\mathfrak{s}_{i_0}}) = m_{\mathfrak{s}_{i_0}}\rho_{\mathfrak{s}_{i_0}} > m_{\mathfrak{s}_{i_0}}\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_0}} = \operatorname{ord}_{\Gamma_0}(x-w_{\mathfrak{s}_j}).$$

It follows from (2.14) that

$$\operatorname{ord}_{\Gamma_0}\Big(\sum_{i\in J} v_i\frac{(x-w_{\mathfrak{s}_{i_0}})^{\beta_i}}{\pi^{\beta_i\rho_{\mathfrak{s}_{i_0}}}}\Big) > 0,$$

where $J_i = \{j\in I \mid i_0 \le j < i \text{ and } \mathfrak{s}_j \neq \mathfrak{s}_{i_0}\}$, $v_i = u_i\prod_{j\in J_i}\frac{w_{\mathfrak{s}_{i_0}}-w_{\mathfrak{s}_j}}{\pi^{\rho_{\mathfrak{s}_j\wedge\mathfrak{s}_{i_0}}}} \in O_K^\times$, and $\beta_i = |\{i_0,\dots,i-1\}\setminus J_i|$.

To find a contradiction, we will use the explicit description of a dense open affine subset of Γ_0. Let $W=\{w_1,\dots,w_m\}$ be the set of rational centres of the rationally minimal clusters for C fixed at the beginning of the section. Let $w_h \in W$ such that $w_h = w_{\mathfrak{s}_{i_0}}$, and let $L = L_{\mathfrak{s}_{i_0}}^{w_h}$, $M = M_{L,0}$, and consider

$$U_M^h \cap \{Z=0\} = \operatorname{Spec}\frac{R[T_M^h(X,Y,Z)^{-1}]}{\big(\mathcal{F}_M^h(X,Y,Z),Z\big)} \subset \Gamma_{\mathrm{t}},$$

dense open subscheme of Γ_{t}. From Lemma 2.5.2,

$$\sum_{i\in J} v_i\frac{(x-w_h)^{\beta_i}}{\pi^{\beta_i\rho_{\mathfrak{s}_{i_0}}}} = \sum_{i\in J} v_i X^{\beta_i/b_{\mathfrak{s}_{i_0}}},$$

which is a unit since the polynomial $\mathcal{F}_M^h(X,Y,Z)$ in $\{Z=0\}$ is of the form $Y^2-G(X)$ or $Y-G(X)$ for some non-constant $G(X)\in K[X]$ (for more details see Lemma 2.5.17). This gives a contradiction and concludes the proof. □

Assume now $C_{K^{nr}}$ has an almost rational cluster picture and is y-regular as in Theorem 2.4.22. Since $|\Sigma_C|$ is finite, there exists a finite unramified extension F/K such that C_F has an almost rational cluster picture and is y-regular. Denote by O_F the ring of integers of F. Let $\Sigma_F = \Sigma_{C_F}^{\mathrm{rat}}$. Fix a rational centre $w_{\mathfrak{s}} \in F$ for every rationally minimal cluster $\mathfrak{s} \in \Sigma_F$. For all non-minimal proper clusters $\mathfrak{t} \in \Sigma_F$ choose a rational centre $w_{\mathfrak{t}} = w_{\mathfrak{s}}$ for some rationally minimal cluster $\mathfrak{s} \subseteq \mathfrak{t}$. In this setting the next theorem gives a basis of integral differentials of C.

Theorem 2.6.4 *Let C/K be a hyperelliptic curve of genus $g \geq 1$ defined by the Weierstrass equation $y^2 = f(x)$ and let $\mathcal{C}/O_K$ be a regular model of C. Suppose there exists a finite unramified extension F/K such that C_F has an almost rational cluster picture and is y-regular. For $i = 0,\dots,g-1$ choose inductively proper clusters $\mathfrak{s}_i \in \Sigma_F$ so that*

$$e_i := \frac{\epsilon_{\mathfrak{s}_i}}{2} - \sum_{j=0}^{i} \rho_{\mathfrak{s}_j \wedge \mathfrak{s}_i} = \max_{\mathfrak{t}\in\Sigma_F} \left\{ \frac{\epsilon_{\mathfrak{t}}}{2} - \rho_{\mathfrak{t}} - \sum_{j=0}^{i-1} \rho_{\mathfrak{s}_j \wedge \mathfrak{t}} \right\},$$

where if $\mathfrak{s}$ and $\mathfrak{s}'$ are two possible choices for $\mathfrak{s}_i$ satisfying $\mathfrak{s}' \subset \mathfrak{s}$, then choose $\mathfrak{s}_i = \mathfrak{s}$. Let $\beta \in O_F^\times$ such that $\mathrm{Tr}_{F/K}(\beta) \in O_K^\times$. Then the differentials

$$\mu_i = \pi^{\lfloor e_i \rfloor} \cdot \mathrm{Tr}_{F/K}\Big(\beta \prod_{j=0}^{i-1} (x - w_{\mathfrak{s}_j})\Big) \frac{dx}{2y}, \qquad i = 0,\dots,g-1,$$

form an O_K-basis of $H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K})$.

Proof. First note that without loss of generality we can suppose F/K Galois. Moreover, since F/K is unramified, $\mathrm{Gal}(F/K) \simeq \mathrm{Gal}(\mathfrak{f}/k)$, where $\mathfrak{f}$ is the residue field of F, and so the existence of β is guaranteed by the surjectivity of $\mathrm{Tr}_{\mathfrak{f}/k}$. Let $\mathcal{C}$ be the minimal regular model of C over O_K. By [Liu4, Proposition 10.1.17], the base extended scheme $\mathcal{C}_{O_F}$ is the minimal regular model of C_F over O_F. Let $\mu_0^F,\dots,\mu_{g-1}^F$ be the basis of integral differentials of C_F given by Theorem 2.6.3.

Suppose $\mu_0',\dots,\mu_{g-1}'$ is a basis of integral differentials of C_F that, for any $\sigma \in \mathrm{Gal}(F/K)$ and any $j = 0,\dots,g-1$, satisfies

$$\sigma(\mu_j') = \mu_j' + \sum_{0 \leq i < j} \lambda_{ij} \mu_i', \tag{2.15}$$

for some $\lambda_{ij} \in O_F$ (depending on σ). Note that $\mu_0^F,\dots,\mu_{g-1}^F$ is in fact such a basis. We want to prove that, for any $j = 0,\dots,g-1$, the differentials

$$\mu_0',\dots,\mu_{j-1}', \mathrm{Tr}_{F/K}(\beta\mu_j'), \mu_{j+1}',\dots,\mu_{g-1}' \tag{2.16}$$

still form a basis of $H^0(\mathcal{C}_F, \omega_{\mathcal{C}_F/O_F})$ satisfying condition (2.15). From equation (2.15) it follows that

$$\mathrm{Tr}_{F/K}(\beta\mu_j') = \sum_{\sigma\in\mathrm{Gal}(F/K)} \sigma(\beta)\sigma(\mu_j') = \mathrm{Tr}_{F/K}(\beta)\mu_j' + \sum_{i<j} \lambda_{ij}' \mu_i',$$

for some $\lambda_{ij}' \in O_F$. Since $\mathrm{Tr}_{F/K}(\beta) \in O_K^\times$, the differentials in (2.16) form a basis of $H^0(\mathcal{C}_F, \omega_{\mathcal{C}_F/O_F})$ satisfying condition (2.15).

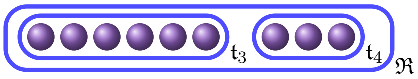
Since $\mu_0^F,\dots,\mu_{g-1}^F$ satisfies (2.15), by induction it follows that

$$\mathrm{Tr}_{F/K}(\beta\mu_0^F),\dots,\mathrm{Tr}_{F/K}(\beta\mu_{g-1}^F)$$

is a basis of $H^0(\mathcal{C}_F,\omega_{\mathcal{C}_F/O_F})$. Proposition A.2.2 concludes the proof. □

We conclude this section with an application of Theorem 2.6.3.

Example 2.6.5 Let p be a prime number and let $a\in\mathbb{Z}_p$, $b\in\mathbb{Z}_p^\times$ such that the polynomial x^2+ax+b is not a square modulo p. Let C be the hyperelliptic curve over $\mathbb{Q}_p$ of genus 4 described by the equation $y^2=f(x)$, where $f(x)=(x^6+ap^4x^3+bp^8)((x-p)^3-p^{11})$. We have already shown in Examples 2.3.32 and 2.4.24 that C satisfies the hypothesis of Theorem 2.6.3 and has rational cluster picture



We choose rational centres for the minimal clusters $\mathfrak{t}_3$ and $\mathfrak{t}_4$: $w_{\mathfrak{t}_3}=0$ and $w_{\mathfrak{t}_4}=p$. Since $\mathfrak{R}=\mathfrak{t}_3\wedge\mathfrak{t}_4$, we can set either $w_{\mathfrak{R}}=w_{\mathfrak{t}_3}$ or $w_{\mathfrak{R}}=w_{\mathfrak{t}_4}$. Let us fix $w_{\mathfrak{R}}=w_{\mathfrak{t}_3}=0$. Then to choose $\mathfrak{s}_0,\mathfrak{s}_1,\mathfrak{s}_2,\mathfrak{s}_3$ as in Theorem 2.6.3 we draw the following table:

	$\rho_{\mathfrak{t}}$	$\epsilon_{\mathfrak{t}}$	$\frac{\epsilon_{\mathfrak{t}}}{2}-\rho_{\mathfrak{t}}$	$\frac{\epsilon_{\mathfrak{t}}}{2}-\rho_{\mathfrak{t}}-\rho_{\mathfrak{s}_0\wedge\mathfrak{t}}$	$\frac{\epsilon_{\mathfrak{t}}}{2}-\rho_{\mathfrak{t}}-\sum_{j=0}^{1}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}}$	$\frac{\epsilon_{\mathfrak{t}}}{2}-\rho_{\mathfrak{t}}-\sum_{j=0}^{2}\rho_{\mathfrak{s}_j\wedge\mathfrak{t}}$
$\mathfrak{t}_3$	$\frac{4}{3}$	11	$\frac{25}{6}$	$\frac{19}{6}$	$\frac{11}{6}$	$\frac{1}{2}$
$\mathfrak{t}_4$	$\frac{11}{3}$	17	$\frac{29}{6}$	$\frac{7}{6}$	$\frac{1}{6}$	$-\frac{5}{6}$
$\mathfrak{R}$	1	9	$\frac{7}{2}$	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{1}{2}$

The numbers in red indicate that $\mathfrak{s}_0=\mathfrak{t}_4$, $\mathfrak{s}_1=\mathfrak{s}_2=\mathfrak{t}_3$ and $\mathfrak{s}_3=\mathfrak{R}$. Thus the differentials

$$\mu_0=p^4\cdot\frac{dx}{2y},\quad \mu_1=p^3\cdot(x-p)\frac{dx}{2y},\quad \mu_2=p\cdot(x-p)x\frac{dx}{2y},\quad \mu_3=(x-p)x^2\frac{dx}{2y}$$

form a $\mathbb{Z}_p$-basis of $H^0(\mathcal{C},\omega_{\mathcal{C}/\mathbb{Z}_p})$, for any regular model $\mathcal{C}/\mathbb{Z}_p$ of C.

CHAPTER 3

A GENERALISATION OF THE TORIC RESOLUTION OF CURVES

Let k be a perfect field and let C_0 be a smooth curve in the torus $\mathbb{G}_{m,k}^2$. Extending the toric resolution of C_0 with respect to its Newton polygon, we explicitly construct an explicit model over k of the smooth completion of C_0. Such a model exists for any smooth projective curve and can be described via a combinatorial algorithm using an iterative construction of Newton polygons. The content of this chapter can be found in the paper *A generalisation of the toric resolution of curves* [Mus2], submitted for publication.

3.1 Introduction

Let U be any smooth affine curve defined over a perfect field k. Up to isomorphism there exists a unique smooth projective curve C/k birational to U, called the *smooth completion* of U. In this chapter we study the problem of finding explicit *models of C over k*, i.e curves $\tilde{C}$ isomorphic to C over k. More precisely, we present an algorithm to construct a model over k of smooth projective curves which are birational to a smooth curve $C_0 \subset \mathbb{G}_{m,k}^2$. In fact, every smooth projective curve is the smooth completion of a curve C_0 as above (Corollary B.1.4). Note that a curve is not required to be connected in this work (see conventions and notations in §3.1.4).

3.1.1 Overview

When it exists, a *Baker's model* of a smooth projective curve C/k is an explicit model of C over k. It is constructed via a toric resolution of a smooth curve $C_0 \subset \mathbb{G}_{m,k}^2$, birational to C. A Baker's model helps in studying the geometry of C. For example, it gives combinatorial interpretations of the genus, the gonality, the Clifford index and the Clifford degree [CC]. Let us give a brief description of this model.

For any C and C_0 as above, let $f = \sum_{(i,j)\in\mathbb{Z}^2} c_{ij} x^i y^j \in k[x^{\pm 1}, y^{\pm 1}]$ be a Laurent polynomial defining $C_0 : f = 0$ in the torus $\mathbb{G}^2_{m,k}$. Let Δ be the Newton polygon of f. A classical construction associates a 2-dimensional toric variety $\mathbb{T}_\Delta$ to the integral polytope Δ. The Zariski closure C_1 of C_0 in $\mathbb{T}_\Delta$ is called the *completion of* C_0 *with respect to its Newton polygon*. It is an easy-to-describe projective curve, whose C_0 is a dense open. The construction of C_1 from C_0 is said *toric resolution* on $\mathbb{T}_\Delta$. If C_1 is regular, it is isomorphic to C and is said a Baker's model of C. A smooth projective curve does not always admit a Baker's model (see Appendix B.2). Its existence is closely related to another interesting property: the nondegeneracy.

For any face λ of Δ (of any dimension) let $f_\lambda = \sum_{(i,j)\in\mathbb{Z}^2\cap\lambda} c_{ij} x^i y^j$. The Laurent polynomial f is nondegenerate if for every face λ of Δ the system of equations $f_\lambda = x\frac{\partial f_\lambda}{\partial x} = y\frac{\partial f_\lambda}{\partial y} = 0$ has no solutions in $(\bar{k}^\times)^2$. The nondegeneracy of f has a geometric interpretation in terms of C_1. From the explicit description of C_1, there is a canonical way to endow the subset $C_1 \backslash C_0$ with a structure of closed subscheme. We say C_1 is *outer regular* if $C_1 \backslash C_0$ is smooth. One can prove that f is nondegenerate if and only if C_1 is outer regular. This is a sufficient condition for the regularity of C_1.

A smooth projective curve C is said nondegenerate if it admits an outer regular Baker's model. Nondegenerate curves have several applications. They have turned out to be useful in singular theory [Kou] and in the theory of sparse resultants [GKZ], as well as for studying specific classes of curves [Mik],[BP],[KWZ]. Over finite fields, nondegenerate curves have also been used in p-adic cohomology theory [AS], in the computation of zeta-functions [CDV] and in the study of the torsion subgroup of their own Jacobians [CST]. Unfortunately, nondegenerate curves are rare, especially for high genera [CV1]. In fact, recall that even a Baker's model may not exist.

Let C/k be any smooth projective curve. In this chapter we construct an explicit model C_n of C over k, called *generalised Baker's model* (Definition 3.7.1), extending the classical toric resolution without losing the connection with Newton polygons. *Every* smooth projective curve C has a generalised Baker's model and it can be constructed from *any* smooth curve $C_0 \subset \mathbb{G}^2_{m,k}$ birational to C. Similarly to the classical case, the subset $C_n \backslash C_0$ will naturally be equipped with a structure of closed subscheme. We say that C_n is *outer regular* if the subscheme $C_n \backslash C_0$ is smooth. Although not all smooth projective curves are nondegenerate, they always have an outer regular generalised Baker's model (Corollary 3.7.8). Let us describe our approach briefly.

For any smooth curve $C_0 \subset \mathbb{G}^2_{m,k}$, we construct a sequence of proper birational morphisms of curves

$$\dots \xrightarrow{s_{n+1}} C_{n+1} \xrightarrow{s_n} C_n \xrightarrow{s_{n-1}} \dots \xrightarrow{s_1} C_1, \tag{3.1}$$

where C_1 is the completion of C_0 with respect to its Newton polygon. The curves C_n are birational to C_0 and explicitly constructed over an algebraic closure $\bar{k}/k$ via an iterative construction of Newton polygons. We also describe the action of the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$ on $C_n \times_k \bar{k}$. Note that since C_1 is projective, the curves C_n will be projective as well. If C_n is regular, for some

n, then it is a model over k of C. Such C_n is what we call a generalised Baker's model of C. Thus the following theorem is a key result of the current chapter.

Theorem 3.1.1 (Theorems 3.5.10, 3.7.7) *For a sufficiently large n, the curve C_n is outer regular.*

From the explicit construction of an outer regular generalised Baker's model one can also describe the set $C(\bar{k})\backslash C_0(\bar{k})$. The result that is obtained extends the known one for nondegenerate curves. We will state them in §3.1.3, in the case of geometrically connected curves. In the next subsection we discuss one of the main motivations of this work: the study of regular models of curves over discrete valuation rings.

3.1.2 Models of curves over discrete valuation rings

Let K be a complete discretely valued field with ring of integers O_K and residue field k. Let C/K be a projective curve. A *model of C over O_K* is a proper flat scheme $\mathcal{C} \to \operatorname{Spec} O_K$ of dimension 2 such that its generic fibre $\mathcal{C}_\eta = \mathcal{C} \times_{O_K} K$ is a model of C over K. The study of regular models over O_K of geometrically connected smooth projective curves C is of great interest in Arithmetic Geometry. The understanding of such models is essential for describing the arithmetic of C and leads to the computation of important objects, such as Tamagawa numbers and integral differentials.

Let $C_0 \subset \mathbb{G}^2_{m,K}$ be an affine curve given by $f(x,y) = 0$ and let C_1 be the completion of C_0 with respect to its Newton polygon Δ. Via a toric resolution approach, [Dok] constructs a model of C_1 over O_K, denoted $\mathcal{C}_\Delta$. This is an innovative result, able to construct regular models of curves over discrete valuation rings in cases that were previously hard to tackle (such as the case of curves with wildly potential semistable reduction). However, this approach has two major limits. First, it can construct a model of a smooth projective curve C only if C admits a Baker's model. Second, although we are mainly interested in regular models, $\mathcal{C}_\Delta$ may be singular. Let us discuss more in detail this second aspect.

The scheme $\mathcal{C}_\Delta$ is given as the Zariski closure of C_0 in a toric scheme X_Σ. The ambient space X_Σ is constructed from Δ, taking into account also the valuations of the coefficients of f. The connection of $\mathcal{C}_\Delta$ with toric resolution of curves goes beyond its generic fibre. Let $\mathcal{C}^{\mathrm{red}}_{\Delta,s}$ be the reduced closed subscheme with the same underlying topological space of the special fibre $\mathcal{C}_{\Delta,s}$ of $\mathcal{C}_\Delta$. Then $\mathcal{C}^{\mathrm{red}}_{\Delta,s}$ can be decomposed in principal components $\bar{X}_F$ and chains of $\mathbb{P}^1$s. The components $\bar{X}_F$ are the completions of curves $X_F \subset \mathbb{G}^2_{m,k}$ with respect to their Newton polygons. One can see that if all $\bar{X}_F$ are outer regular, then $\mathcal{C}_\Delta$ is regular. Thus the fact that not every projective curve has an outer regular Baker's model is the main obstruction for the regularity of $\mathcal{C}_\Delta$.

Therefore the existence of outer regular generalised Baker's models, subject of this chapter, has the potential to extend Dokchitser's result to construct regular models of all smooth projective curves. Although such an extension is highly non-trivial, in [Mus1] we can already see an implicit application of generalised Baker's model towards that goal. Let us spend a few lines explaining why. In [Mus1] the author constructs a regular model $\mathcal{C}$ over O_K for a wide class of hyperelliptic curves C/K as follows. Let $C : y^2 = h(x)$ be a hyperelliptic curve in this class. One considers smooth curves $C_0^w \subset \mathbb{G}_{m,K}^2$, for $w \in W \subseteq K$, given by $y^2 = h(x+w)$ and so birational to C. For each $w \in W$, let $\mathcal{C}_{\Delta^w}$ be the model of C constructed from C_0^w by [Dok]. The regular model $\mathcal{C}$ is then obtained by glueing regular open subschemes $\mathring{\mathcal{C}}_{\Delta^w}$ of $\mathcal{C}_{\Delta^w}$, containing all points of codimension 1. In particular, for any principal component $\bar{X}_F$ of $\mathcal{C}_{\Delta^w,s}^{\mathrm{red}}$ there exists a closed subscheme Γ_{t} of $\mathcal{C}_s = \mathcal{C} \times_{O_K} k$, birational to $\bar{X}_F$. The regularity of $\mathcal{C}$ follows from the fact that Γ_{t} is an outer regular generalised Baker's model of the smooth completion of X_F (this can be checked by comparing the description of Γ_{t} in [Mus1, §5] and the construction in §3.8 of an outer regular generalised Baker's model for curves given by superelliptic equations).

3.1.3 Outer regular generalised Baker's model

Let k be a perfect field with algebraic closure $\bar{k}$. Let $f \in k[x^{\pm1}, y^{\pm1}]$ such that $C_0 : f = 0$ is a geometrically connected smooth curve over $\mathbb{G}_{m,k}^2$, and let Δ be the Newton polygon of f. If f is nondegenerate, then the completion C_1 of C_0 with respect to Δ is outer regular. In particular, C_1 is a Baker's model of the smooth completion C of C_0. From C_1 we can describe the points in $C \setminus C_0$ in an elementary way as follows.

Definition 3.1.2 For any edge ℓ of an integral 2-dimensional polytope $\mathcal{P}$, consider the unique surjective affine function $\ell^* : \mathbb{Z}^2 \to \mathbb{Z}$ given by $\ell^*|_\ell = 0$, $\ell^*|_{\mathcal{P}} \geq 0$. Write $\ell^*(i,j) = ai + bj + c$, for some $a, b, c \in \mathbb{Z}$. Then the primitive vector $(a,b) \in \mathbb{Z}^2$ will be called the *normal vector* of ℓ.

We also extend this definition to segments $\mathcal{P}$, considered as integral 2-dimensional polytopes of zero volume. In this case $\mathcal{P}$ has two edges, equal to $\mathcal{P}$ itself, with opposite normal vectors.

Notation 3.1.3 For any primitive vector $\beta = (\beta_1, \beta_2) \in \mathbb{Z}^2$ fix $\delta_\beta = (\delta_1, \delta_2) \in \mathbb{Z}^2$ such that $\delta_1\beta_2 - \delta_2\beta_1 = 1$. Note that δ_β can be freely chosen, and depends (only) on β.

For any edge ℓ of Δ:

(1) Consider its normal vector $\beta = (\beta_1, \beta_2) \in \mathbb{Z}^2$ and $\delta_\beta = (\delta_1, \delta_2) \in \mathbb{Z}^2$.

(2) Via the change of variables $x = X^{\delta_1} Y^{\beta_1}$, $y = X^{\delta_2} Y^{\beta_2}$, let $f_\ell \in k[X,Y]$ such that $X \nmid f_\ell$, $Y \nmid f_\ell$, and
$$f(x,y) = X^{n_X} Y^{n_Y} \cdot f_\ell(X,Y),$$
for some $n_X, n_Y \in \mathbb{Z}$.

Define the curve $C_\ell : f_\ell(X,Y) = 0$ in $\mathbb{G}_{m,k} \times \mathbb{A}_k^1 = \operatorname{Spec} k[X^{\pm1}, Y]$. Note that $C_\ell \cap \mathbb{G}_{m,k}^2 = C_0$. The completion of C_0 with respect to Δ is
$$C_1 = \bigcup_{\ell \subset \partial\Delta} C_\ell,$$
where the curves C_ℓ are glued along their common open subscheme C_0.

Let $P_1 = \bigsqcup_{\ell\subset\partial\Delta}\{f_\ell\}$, where ℓ runs through all edges of Δ. For any $f_\ell \in P_1$ define $f|_\ell \in k[X]$ by $f|_\ell(X) = f_\ell(X,0)$. It is easy to see that f is nondegenerate if and only if $f|_\ell$ has no multiple roots in $\bar{k}^\times$ for any edge ℓ of Δ. Then from the description of C_1 we have the following result.

Theorem 3.1.4 ([Dok, Theorem 2.2(3)]) *Suppose f nondegenerate. There is a natural bijection that preserves* $\mathrm{Gal}(\bar{k}/k)$*-action,*

$$C(\bar{k})\setminus C_0(\bar{k}) \overset{1:1}{\longleftrightarrow} \bigsqcup_{f_\ell\in P_1} \{\text{(simple) roots of } f|_\ell \text{ in } \bar{k}^\times\}.$$

If f is not nondegenerate, or, equivalently, if C_1 is not outer regular, we can construct from C_1 an outer regular generalised Baker's model C_n of C, that always exists. Then the explicit description of C_n can be used to obtain a more general version of Theorem 3.1.4 capable to describe the points in $C\setminus C_0$ unconditionally.

First we are going to define finite indexed sets P_n of polynomials in $\bar{k}[X,Y]$, for all $n \in \mathbb{Z}_+$. A polynomial in P_n will be denoted by f_ℓ for an edge ℓ of some 2-dimensional polytope. However, if $n \geq 2$ then $f_\ell \in P_n$ will be indexed not only by ℓ but also by a polynomial of P_{n-1} and a non-zero element of $\bar{k}$. For any $f_\ell \in P_n$, define $f|_\ell \in \bar{k}[X]$ by $f|_\ell(X) = f_\ell(X,0)$. Let P_1 be as above. For $n \in \mathbb{Z}_+$, we recursively construct the set P_{n+1} from P_n via the following algorithm.

Algorithm 3.1.5 *For any $f_\ell \in P_n$ and any multiple root $a \in \bar{k}^\times$ of $f|_\ell$ do:*

(1) Rename the variables of f_ℓ from X,Y to x,y.

(2) Let $f_{\ell,a} \in \bar{k}[x,y]$ given by $f_{\ell,a}(x,y) = f_\ell(x+a,y)$.

(3) Draw the Newton polygon $\Delta_{\ell,a}$ of $f_{\ell,a}$.

(4) For any edge ℓ' of $\Delta_{\ell,a}$ with normal vector $\beta = (\beta_1,\beta_2) \in \mathbb{Z}_+^2$, consider $\delta_\beta = (\delta_1,\delta_2) \in \mathbb{Z}^2$, previously fixed.

(5) Through the change of variables $x = X^{\delta_1}Y^{\beta_1}$, $y = X^{\delta_2}Y^{\beta_2}$, let $f_{\ell'} = (f_{\ell,a})_{\ell'} \in \bar{k}[X,Y]$ such that $X \nmid f_{\ell'}$, $Y \nmid f_{\ell'}$, and

$$f_{\ell,a}(x,y) = X^{n_X}Y^{n_Y} \cdot f_{\ell'}(X,Y),$$

for some $n_X, n_Y \in \mathbb{Z}$.

(6) Define $P_{\ell,a} = \bigsqcup_{\ell'\subset\partial\Delta_{\ell,a}}\{f_{\ell'}\}$, where ℓ' runs through all edges of $\Delta_{\ell,a}$ with normal vector in $\mathbb{Z}_+^2$.

Then

$$P_{n+1} := \bigsqcup_{f_\ell,a} P_{\ell,a},$$

where f_ℓ runs through all polynomials in P_n and a runs through all multiple roots of $f|_\ell$ in $\bar{k}^\times$.

For every $n \in \mathbb{Z}_+$, one can inductively define an action of $\mathrm{Gal}(\bar{k}/k)$ on P_n with the following property: for any $\sigma \in \mathrm{Gal}(\bar{k}/k)$ and $f_\ell \in P_n$ the polynomials $\sigma \cdot f_\ell$ and f_ℓ^σ are equal. Note that this property is not enough to describe the action since P_n is an indexed set.

Let $\sigma \in \mathrm{Gal}(\bar{k}/k)$. If $f_\ell \in P_1$, then define $\sigma \cdot f_\ell = f_\ell$. Let $f_{\ell'} \in P_{n+1}$ for $n \in \mathbb{Z}_+$. From Algorithm 3.1.5 it follows that $f_{\ell'} = (f_{\ell,a})_{\ell'}$ for some $f_\ell \in P_n$ and some multiple root $a \in \bar{k}^\times$ of $f|_\ell$. By inductive hypothesis $\sigma \cdot f_\ell$ is an element $f_{\sigma(\ell)}$ of P_n, and $\sigma(a)$ is a multiple root of $f|_{\sigma(\ell)}$. Moreover, $f_{\sigma(\ell),\sigma(a)} = f^\sigma_{\ell,a}$. Hence the Newton polygon $\Delta_{\sigma(\ell),\sigma(a)}$ coincides with $\Delta_{\ell,a}$. In particular, it has an edge $\sigma(\ell')$ with normal vector equal to the one of ℓ'. Then define

$$\sigma \cdot f_{\ell'} := f_{\sigma(\ell')} = (f_{\sigma(\ell),\sigma(a)})_{\sigma(\ell')} \in P_{n+1}.$$

Iterate Algorithm 3.1.5 until $P_{n+1} = \varnothing$, i.e. for all $f_\ell \in P_n$, the polynomials $f|_\ell$ have no multiple roots in $\bar{k}^\times$. The procedure terminates. Define

$$P = P_1 \sqcup \cdots \sqcup P_n.$$

Note that the Galois action on P_i for all $1 \le i \le n$ induces an action on P. For any $\sigma \in \mathrm{Gal}(\bar{k}/k)$ and $f_\ell \in P$, let $f_{\sigma(\ell)} \in P$ be the element $\sigma \cdot f_\ell$. We can now generalise Theorem 3.1.4.

Theorem 3.1.6 *There is a natural bijection*

$$C(\bar{k}) \setminus C_0(\bar{k}) \overset{1:1}{\longleftrightarrow} \bigsqcup_{f_\ell \in P} \{\textit{simple roots of } f|_\ell \textit{ in } \bar{k}^\times\},$$

that preserves $\mathrm{Gal}(\bar{k}/k)$*-action, where* $\sigma \in \mathrm{Gal}(\bar{k}/k)$ *takes a simple root* $r \in \bar{k}^\times$ *of* $f|_\ell$ *to the simple root* $\sigma(r) \in \bar{k}^\times$ *of* $f|_{\sigma(\ell)}$.

Theorem 3.1.6 is proved at the end of §3.7.

Example 3.1.7 Let $f = (x^2+1)^2 + y - y^3 \in \mathbb{F}_3[x^{\pm 1}, y^{\pm 1}]$ and let $C_0 : f = 0$ in $\mathbb{G}^2_{m,\mathbb{F}_3}$. Note that C_0 is regular. By [CV2, Proposition 3.2], the smooth completion C of C_0 is not nondegenerate. Hence Theorem 3.1.4 cannot be used. We want to describe the points in $C \setminus C_0$ via Theorem 3.1.6. First compute the set P via Algorithm 3.1.5. One has $P = P_1 \sqcup P_2$, where

- P_1 consists of 3 polynomials $f_{\ell_1}, f_{\ell_2}, f_{\ell_3}$, where $f|_{\ell_1} = (X^2+1)^2$, $f|_{\ell_2} = X^3 + X^2 - 1$, $f|_{\ell_3} = -X + 1$, up to some power of X;

- P_2 consists of 2 polynomials f_{ℓ_4}, f_{ℓ_5}, satisfying $f_{\ell_5} = f_{\sigma(\ell_4)}$, where σ is the Frobenius automorphism; furthermore, $f|_{\ell_4} = f|_{\ell_5} = -X + 1$, up to some power of X.

Thus Theorem 3.1.6 shows that $C \setminus C_0$ consists of one point coming from ℓ_4, ℓ_5 with residue field $\mathbb{F}_9$, one point coming from ℓ_2 with residue field $\mathbb{F}_{27}$ and one $\mathbb{F}_3$-rational point coming from ℓ_3.

3.1.4 Outline of the chapter and notation

For the most part of the chapter we will assume $k = \bar{k}$. In §3.2 we define toric varieties $\mathbb{T}_v$ attached to primitive integer-valued vectors v. The charts of the curves C_n in the sequence (3.1) will be the Zariski closures of dense opens of C_0 inside $\mathbb{T}_v$. In §3.4 we show how to construct the sequence (3.1) recursively and explain its connection with Newton polygons. We also prove the properties of the curves and the morphisms in (3.1) previously listed in §3.3. Section 3.5 gives the definition of generalised Baker's model and outer regularity over algebraically closed base fields. We prove some crucial results and present interesting consequences. In §3.6 we see the construction developed in previous sections from a more general point of view. This will be useful to tackle the case of non-algebraically closed base fields, treated in §3.7. Finally, §3.8 and §3.9 consist of applications of our construction. In §3.8 we discuss the case of superelliptic equations. In §3.9 we show an explicit and non-trivial example of a generalised Baker's model.

Conventions and notations

- Throughout, k will be a perfect field, algebraically closed in §3.2-3.6.
- An *algebraic variety X over k*, denoted X/k, is a scheme of finite type over Spec k. Let $\mathcal{K}_X$ be the sheaf of stalks of meromorphic functions on X ([Liu4, Definition 7.1.13]). We denote by $k(X)$ the set of global sections of $\mathcal{K}_X$, i.e. $k(X) = H^0(X, \mathcal{K}_X)$. It will be called *the ring of rational functions* or *function ring* of X. It extends the notions of field of rational functions or function field of integral varieties.
- Let X/k be an algebraic variety. Since k is perfect, X is smooth if and only if it is regular. In this context we will then use the words *smooth*, *regular*, *non-singular* interchangeably. We will denote by $\mathrm{Reg}(X)$ the open subset of regular points of X and by $\mathrm{Sing}(X)$ the closed subset of singular points of X.
- A morphism $X \to Y$ between two algebraic varieties X, Y defined over k will always be a *morphism of k-schemes*, unless otherwise specified.
- A *birational map* $f : X \dashrightarrow Y$ between algebraic varieties X, Y over k is a k-rational map ([EGA, I.7.1.2]) that comes from an isomorphism from a dense open $U \subseteq X$ onto a dense open $V \subseteq Y$. If such a map exists, we say that X is birational to Y. A *birational morphism* is a morphism which is (a representative of) a birational map ([Liu4, Definition 7.5.3]).
- A *curve* is an equidimensional algebraic variety of dimension 1. We will denote by $\mathbb{G}_m$ the affine algebraic group $\mathbb{G}_{m,k} = \mathrm{Spec}\ k[x_1^{\pm 1}, y^{\pm 1}]$ whenever k is algebraically closed.
- Given a ring A and an ideal I of A we identify the ideals of A/I with the ideals of A containing I. Furthermore, sometimes we refer to an element $a \in A$ as an element of A/I omitting the class symbol.

- Finally, the set of natural numbers will contain 0, i.e. $\mathbb{N} = \mathbb{Z}_{\geq 0}$.

3.2 Ambient toric varieties and charts

Let k be an algebraically closed field, $n \in \mathbb{Z}_+$, $A = k[x_1^{\pm 1},\dots,x_n^{\pm 1},y^{\pm 1}]$ and $\mathbb{G}_m^{n+1} = \operatorname{Spec} A$. Let $v = (v_1,\dots,v_n,v_{n+1}) \in \mathbb{Z}^{n+1}$ be a primitive vector. Define the affine function $\phi_v : \mathbb{Z}^{n+1} \to \mathbb{Z}$ given by

$$\phi_v(i_1,\dots,i_n,j) = v_1 i_1 + \dots + v_n i_n + v_{n+1} j.$$

For any $i = (i_1,\dots,i_n,j) \in \mathbb{Z}^{n+1}$, denote by $\mathbf{x}^i$ the monomial $x_1^{i_1}\cdots x_n^{i_n} y^j$ of $k[x_1^{\pm 1},\dots,x_n^{\pm 1},y^{\pm 1}]$. For any monomial $\mathbf{x}^i$ define $\operatorname{ord}_v(\mathbf{x}^i) = \phi_v(i)$. For $f \in A$, with $f \neq 0$, expand

$$f = \sum_i c_i \mathbf{x}^i, \quad c_i \in k^\times,$$

and set $\operatorname{ord}_v(f) = \min_i \operatorname{ord}_v(\mathbf{x}^i)$. We have just defined a map $\operatorname{ord}_v : A^\times \to \mathbb{Z}$, which naturally extends to a valuation $\operatorname{ord}_v : \operatorname{Frac}(A)^\times \to \mathbb{Z}$.

Definition 3.2.1 Given a primitive vector $w \in \mathbb{Z}^{n+1}$, we say that a matrix $M \in \mathrm{SL}_{n+1}(\mathbb{Z})$ is *attached to* w if its last row is w.

Fix a matrix $M = (a_{ij})$ attached to v. It gives the change of variables

$$\begin{aligned}(x_1,\dots,x_n,y) &= (X_1^{a_{11}}\cdots X_n^{a_{n1}} Y^{v_1},\dots,X_1^{a_{1(n+1)}}\cdots X_m^{a_{n(n+1)}} Y^{v_{n+1}})\\ &= (X_1,\dots,X_n,Y)\bullet M,\\ (X_1,\dots,X_n,Y) &= (x_1,\dots,x_n,y)\bullet M^{-1}.\end{aligned}$$

For any $f \in A^\times$, denoting by $\mathcal{F} \in k[X_1^{\pm 1},\dots,X_n^{\pm 1},Y^{\pm 1}]^\times$ the Laurent polynomial given by

$$\mathcal{F}(X_1,\dots,X_n,Y) = f((X_1,\dots,X_n,Y)\bullet M),$$

note that $\operatorname{ord}_v(f) = \operatorname{ord}_Y(\mathcal{F})$. We get an embedding

$$A \overset{M}{\simeq} k[X_1^{\pm 1},\dots,X_n^{\pm 1},Y^{\pm 1}] \hookleftarrow k[X_1^{\pm 1},\dots,X_n^{\pm 1},Y] =: R,$$

from which we define the affine toric variety $\mathbb{T}_v = \operatorname{Spec} R \hookleftarrow \mathbb{G}_m^{n+1}$. Since v is the last row of M, the toric variety $\mathbb{T}_v$ only depends on v up to isomorphisms that restricted to $\mathbb{G}_m^{n+1}$ equal the identity. Furthermore, up to isomorphism, the closed subvariety $\bar{\mathbb{T}}_v = \operatorname{Spec} R/(Y) \simeq \mathbb{G}_m^n$ of $\mathbb{T}_v$ only depends on v as well.

Now let I be an ideal of A defining a curve $C_{0,I} = \operatorname{Spec} A/I$ in $\mathbb{G}_m^{n+1}$. We denote by $C_{v,I}$ the Zariski closure of $C_{0,I}$ in $\mathbb{T}_v$. Then $C_{v,I}$ is determined by v and I, up to isomorphisms that preserve $C_{0,I}$. Recall that $C_{v,I} = \operatorname{Spec} R/\mathcal{I}$, where $\mathcal{I}$ is the inverse image of I under the embedding $R \hookrightarrow A$ above. Suppose $\mathcal{J} \subset R$ is an ideal such that $A/I \simeq R[Y^{-1}]/\mathcal{J}R[Y^{-1}]$ via M. Then $\mathcal{J}$ defines $C_{v,I}$

if and only if it equals its saturation with respect to Y, i.e. $\mathcal{J} = Y^\infty : \mathcal{J}$, or, equivalently, if the image of Y in $R/\mathcal{J}$ is a regular element.

Finally, let $f = f_1 \in k[x_1^{\pm1}, y^{\pm1}]$ defining a smooth curve $C_0 : f = 0$ in $\mathbb{G}_m^2 = \operatorname{Spec} k[x_1^{\pm1}, y^{\pm1}]$. For all $i = 2, \ldots, n$, let $g_i \in k[x_1^{\pm1}, y^{\pm1}]$ and denote $f_i = x_i - g_i$. Then

$$\frac{k[x_1^{\pm1}, y^{\pm1}]_{g_2\cdots g_n}}{(f)} \simeq \frac{k[x_1^{\pm1}, x_2^{\pm1}, \ldots, x_n^{\pm1}, y^{\pm1}]}{(f_1, f_2, \ldots, f_n)}.$$

Let T be the tuple $(g_2, \ldots, g_n)$ and I the ideal $(f_1, \ldots, f_n)$. Define $C_{0,T} = \operatorname{Spec} \frac{k[x_1^{\pm1}, y^{\pm1}]_{g_2\cdots g_n}}{(f)}$, an affine open of C_0. Then T gives an open immersion $C_{0,T} \hookrightarrow \mathbb{G}_m^{n+1}$ with image $C_{0,I}$. Let $v \in \mathbb{Z}^{n+1}$ be a primitive vector. Denote by $C_{v,T}$ the curve $C_{v,I}$ (closure of $C_{0,I}$ inside $\mathbb{T}_v$). We will often identify $C_{0,T}$ with the dense open image of the immersion $C_{0,T} \simeq C_{0,I} \hookrightarrow C_{v,T}$.

Let C_0 as above. For any $m \in \mathbb{Z}_+$ define

$$\Omega_m = \{(v, T) \mid v \in \mathbb{Z}^{m+1} \text{ is a primitive vector and } T \in k[x_1^{\pm1}, y^{\pm1}]^{m-1}\}.$$

If $\alpha = (v, T) \in \Omega_m$ for some $m \in \mathbb{Z}_+$, denote by $C_{0,\alpha}$, C_α, respectively the curves $C_{0,T}$, $C_{v,T}$ introduced in the previous section. Furthermore, we set $C_\alpha = C_{0,\alpha} = C_0$ when $\alpha = 0$. Define

$$\Omega = \left\{\alpha \in \textstyle\bigsqcup_{m\in\mathbb{Z}_+} \Omega_m \mid C_{0,\alpha} \text{ is dense in } C_0\right\}.$$

If $\alpha = (v, T) \in \Omega$, denote by $\bar{C}_\alpha$ the scheme-theoretic intersection of C_α and $\bar{\mathbb{T}}_v$ in $\mathbb{T}_v$. Note that, up to isomorphism, $\bar{C}_\alpha$ only depends on α.

From the open immersions with dense images $C_{0,\alpha} \hookrightarrow C_\alpha$, $C_{0,\alpha} \hookrightarrow C_0$, we have natural birational maps $s_{\alpha\alpha'} : C_\alpha \dashrightarrow C_{\alpha'}$, for all $\alpha, \alpha' \in \Omega \sqcup \{0\}$. Denote by $U_{\alpha\alpha'}$ the largest (dense) open of C_α such that $s_{\alpha\alpha'}$ comes from an open immersion $U_{\alpha\alpha'} \hookrightarrow C_{\alpha'}$. Note that $C_{0,\alpha} \cap C_{0,\alpha'}$ embeds in $U_{\alpha\alpha'}$ via the canonical open immersion $C_{0,\alpha} \hookrightarrow C_\alpha$.

Definition 3.2.2 Let $m \in \mathbb{Z}_+$ and $c \in \mathbb{Z}$. Let $v = (v_1, \ldots, v_m, v_{m+1}) \in \mathbb{Z}^{m+1}$ and $\beta = (\beta_1, \beta_2) \in \mathbb{Z}^2$ be primitive vectors. Define the primitive vector

$$\beta \circ_c v = (v_1\beta_2, v_2\beta_2, \cdots, v_m\beta_2, \beta_1 + c\beta_2, v_{m+1}\beta_2) \in \mathbb{Z}^{m+2}.$$

If $g \in k[x_1^{\pm1}, \ldots, x_m^{\pm1}, y^{\pm1}]$, define $\beta \circ_g v = \beta \circ_{\operatorname{ord}_v(g)} v$.

Definition 3.2.3 Let $m \in \mathbb{Z}_+$ and $\alpha \in \Omega_m$. Write $\alpha = (v, T)$ where $T = (g_2, \ldots, g_m)$. Fix $g \in k[x_1, \ldots, x_m, y]$ and let $g_{m+1} \in k[x_1^{\pm1}, y^{\pm1}]$ be the unique Laurent polynomial such that $g_{m+1} \equiv g \mod (f_2, \ldots, f_m)$, where $f_i = x_i - g_i$. For any primitive vector $\beta \in \mathbb{N} \times \mathbb{Z}_+$, define

$$\beta \circ_g \alpha = (\beta \circ_g v, (g_2, \ldots, g_m, g_{m+1})) \in \Omega_{m+1}.$$

Note that for any $\alpha, \alpha' \in \Omega_m$, polynomials $g, g' \in k[x_1, \ldots, x_m, y]$, and primitive vectors $\beta, \beta' \in \mathbb{N} \times \mathbb{Z}_+$, if $\beta \circ_g \alpha = \beta' \circ_{g'} \alpha'$, then $\alpha = \alpha'$.

Definition 3.2.4 Let $m \in \mathbb{Z}_+$. Given $\alpha \in \Omega_m$ and $\gamma \in \Omega_{m+1}$, we will write $\alpha < \gamma$ if there exists a polynomial $g \in k[x_1, \ldots, x_m, y]$ and a primitive vector $\beta \in \mathbb{N} \times \mathbb{Z}_+$ such that $\gamma = \beta \circ_g \alpha$. Endow Ω with a structure of partially ordered set by extending $<$ by transitivity.

3.3 Baker's resolution

Let k be an algebraically closed field and let $f \in k[x_1^{\pm 1}, y^{\pm 1}]$ be a Laurent polynomial defining a smooth curve $C_0 : f = 0$ over $\mathbb{G}_m^2$. We will construct a sequence

$$\dots \xrightarrow{s_{n+1}} C_{n+1} \xrightarrow{s_n} C_n \xrightarrow{s_{n-1}} \dots \xrightarrow{s_1} C_1$$

of proper birational morphisms of projective curves over k, birational to C_0. Such a sequence will be called a *Baker's resolution of* C_0 (Definition 3.6.2). Each curve C_n will be explicitly described and inductively constructed via Newton polygons. In particular, the curve C_1 is the completion of C_0 with respect to the Newton polygon Δ of f. In §3.5 we will show how to use Baker's resolution to desingularise C_1, by finding a regular curve C_n, model over k of the smooth completion of C_0.

For any $n \in \mathbb{Z}_+$, we aim to construct the projective curve C_n as follows:

Construction 3.3.1 We will define a finite subset $\Sigma_n \subset \Omega$. Then

$$C_n := \bigcup_{\alpha \in \Sigma_n} C_\alpha \cup C_0,$$

where the glueing morphisms are given by the birational maps $s_{\alpha\alpha'}$, for $\alpha, \alpha' \in \Sigma_n \sqcup \{0\}$. More precisely, the chart C_α is glued with $C_{\alpha'}$ along $U_{\alpha\alpha'}$ via the isomorphism $U_{\alpha\alpha'} \xrightarrow{\sim} U_{\alpha'\alpha}$ induced by $s_{\alpha\alpha'}$. In fact, for our choice of Σ_n the opens $U_{\alpha\alpha'}$ will be as small as possible, i.e. $C_\alpha \cap C_{\alpha'} = C_{0,\alpha} \cap C_{0,\alpha'}$, for any $\alpha, \alpha' \in \Sigma_n \sqcup \{0\}$, $\alpha \neq \alpha'$

Furthermore, for any $\alpha = (v, T) \in \Sigma_n$, we construct:

(a) An ideal $\mathfrak{a}_\alpha = (\mathcal{F}_2, \dots \mathcal{F}_m) \subset k[X_1^{\pm 1}, \dots, X_m^{\pm 1}, Y]$, and a matrix $M_\alpha \in \mathrm{SL}_{m+1}(\mathbb{Z})$ attached to v defining an isomorphism

$$\frac{k[x_1^{\pm 1}, \dots, x_m^{\pm 1}, y^{\pm 1}]}{(f_2, \dots, f_m)} \underset{\simeq}{\overset{M_\alpha}{}} \frac{k[X_1^{\pm 1}, \dots, X_m^{\pm 1}, Y^{\pm 1}]}{(\mathcal{F}_2, \dots, \mathcal{F}_m)},$$

where $f_i = x_i - g_i$ and $T = (g_2, \dots, g_m)$.

(b) A positive integer $j_\alpha \leq m$ such that there is an embedding

$$R_\alpha := \frac{k[X_1^{\pm 1}, \dots, X_m^{\pm 1}, Y]}{(\mathcal{F}_2, \dots, \mathcal{F}_m)} \hookrightarrow k(X_{j_\alpha}, Y),$$

taking $X_{j_\alpha} \mapsto X_{j_\alpha}$ and $Y \mapsto Y$. Moreover, Y is not invertible in R_α.

(c) A polynomial $\mathcal{F}_\alpha \in k[X_{j_\alpha}, Y]$, not divisible by Y, such that

$$C_\alpha = \operatorname{Spec} \frac{k[X_1^{\pm 1}, \dots, X_m^{\pm 1}, Y]}{(\mathcal{F}_\alpha, \mathcal{F}_2, \dots, \mathcal{F}_m)}.$$

The ideal $\mathfrak{a}_\alpha$ equals its saturation with respect to Y by (b). Therefore (a) implies that $\mathfrak{a}_\alpha$ is uniquely determined by M_α.

The homomorphism in (b) induces an injective ring homomorphism

$$\frac{R_\alpha}{(Y)} = \frac{k[X_1^{\pm 1},\dots,X_m^{\pm 1},Y]}{(\mathcal{F}_2,\dots,\mathcal{F}_m,Y)} \hookrightarrow k(X_{j_\alpha}),$$

taking $X_{j_\alpha} \mapsto X_{j_\alpha}$. Let D_α be its image. Then D_α is a localisation of $k[X_{j_\alpha}]$, isomorphic to $R_\alpha/(Y)$. More precisely, if $t_1,\dots,t_m$ are the images of $X_1,\dots,X_m$ in $k(X_{j_\alpha})$, then $D_\alpha = k[X_{j_\alpha}, t_1^{\pm 1},\dots,t_{m-1}^{\pm 1}]$.

Then, from (c), there exists a non-zero polynomial $f|_\alpha \in k[X_{j_\alpha}]$, given by $f|_\alpha(X_{j_\alpha}) = \mathcal{F}_\alpha(X_{j_\alpha},0)$, such that

$$\frac{k[X_1^{\pm 1},\dots,X_m^{\pm 1},Y]}{(\mathcal{F}_\alpha,\mathcal{F}_2,\dots,\mathcal{F}_m,Y)} \simeq \frac{D_\alpha}{(f|_\alpha)}.$$

The closed subscheme $\bar{C}_\alpha$ of C_α will be identified with Spec $D_\alpha/(f|_\alpha)$. As a set, it is finite and equals $C_\alpha \setminus C_{0,\alpha}$.

Finally, note that the injective homomorphism in (b) and the description of C_α in (c) give an open immersion $C_\alpha \hookrightarrow \operatorname{Spec} k[X_{j_\alpha},Y]/(\mathcal{F}_\alpha)$.

3.4 Construction of the sequence

Let k be an algebraically closed field and let $f \in k[x_1^{\pm 1}, y^{\pm 1}]$ be a Laurent polynomial defining a smooth curve $C_0 : f = 0$ over $\mathbb{G}_m^2$.

3.4.1 Completion with respect to Newton polygon

In this subsection we give a description of the curve C_1, completion of C_0 with respect to its Newton polygon, with the properties of 3.3.1. We will show that $C_\alpha \simeq C_0$ for all but finitely many $\alpha \in \Omega_1 \subset \Omega$. Defining $\Sigma_1 \subseteq \Omega_1$ as the subset of those exceptional elements, the curve C_1 will be the glueing of C_α, $\alpha \in \Sigma_1$, along the common open C_0.

Let $v = (a,b) \in \mathbb{Z}^2$ be any primitive vector and $\alpha = (v,()) \in \Omega_1$. Fix a matrix $M_\alpha = \left(\begin{smallmatrix} c & d \\ a & b \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ attached to v and define $\phi_v : \mathbb{Z}^2 \to \mathbb{Z}$ by $\phi_v(i,j) = ai + bj - \operatorname{ord}_v(f)$. Via the change of variables given by M_α we get

$$f((X_1,Y)\bullet M_\alpha) = X_1^* Y^{\operatorname{ord}_v(f)} \mathcal{F}_\alpha(X_1,Y), \qquad \text{where } \mathcal{F}_\alpha \in k[X_1,Y].$$

Then $\operatorname{ord}_Y(\mathcal{F}_\alpha) = 0$ and so $C_\alpha = \operatorname{Spec} k[X_1^{\pm 1},Y]/(\mathcal{F}_\alpha)$.

Note that $C_{0,\alpha} = C_0$. Let $f|_\alpha \in k[X_1]$ given by $f|_\alpha(X_1) = \mathcal{F}_\alpha(X_1,0)$. Recall that the scheme $\bar{C}_\alpha = \operatorname{Spec} k[X_1^{\pm 1}]/(f|_\alpha)$ equals $C_\alpha \setminus C_{0,\alpha}$ as a set. Therefore $C_\alpha \simeq C_0$ if and only if $f|_\alpha$ is invertible in $k[X_1^{\pm 1}]$. Expand $f = \sum_{i,j} c_{ij} x_1^i y^j$. Let Δ be the Newton polygon of f. It follows that

$$f|_\alpha = X_1^* \cdot \sum_{(i,j)\in\phi_v^{-1}(0)} c_{ij} X_1^{ci+dj}.$$

Hence $f|_\alpha$ is not invertible in $k[X_1^{\pm 1}]$ if and only if $\phi_v^{-1}(0) \cap \Delta = \text{edge}$.

Then we can explicitly construct Σ_1 as follows. For every edge ℓ of Δ consider its normal vector $v_\ell \in \mathbb{Z}^2$ (see Definition 3.1.2). Define $\Sigma_1 = \{(v_\ell, ()) \in \Omega_1 \mid \ell \text{ edge of } \Delta\}$. The next result follows from the computations above.

Proposition 3.4.1 *Let v be the normal vector of an edge ℓ of Δ and let $\alpha = (v, ()) \in \Sigma_1$. Let $(i_0, j_0), \dots, (i_l, j_l)$ be the points of $\ell \cap \mathbb{Z}^2$, ordered along ℓ counterclockwise with respect to Δ. Then*

$$f|_\alpha = X_1^d \cdot \sum_{r=0}^{l} c_{i_r j_r} X_1^r, \qquad \textit{for some } d \in \mathbb{N}.$$

Glueing C_α for any $\alpha \in \Sigma_1$ gives the curve C_1. Note that C_1 is the Zariski closure of C_0 in $\bigcup_{(v,()) \in \Sigma_1} \mathbb{T}_v$ (where the toric varieties $\mathbb{T}_v$ are glued along their common open $\mathbb{G}_m^2$).

Remark 3.4.2. Consider the toric surface $\mathbb{T}_\Delta$ of Δ. It is a complete algebraic variety. Then $\bigcup_{(v,()) \in \Sigma_1} \mathbb{T}_v$ is a (non-proper) subscheme of $\mathbb{T}_\Delta$. Nevertheless the curve C_1 is also the Zariski closure of C_0 in $\mathbb{T}_\Delta$ (see [Dok, Remark 2.6]). Thus it is projective.

Remark 3.4.3. Note that for any $\alpha \in \Sigma_1$, the points on $C_\alpha \setminus C_0$ are not visible on any other chart of C_1. Indeed for any $\alpha, \alpha' \in \Sigma_1$, where $\alpha \neq \alpha'$, consider the birational map

$$s_{\alpha\alpha'} : C_\alpha = \operatorname{Spec} k[X_1^{\pm 1}, Y]/(\mathcal{F}_\alpha) \dashrightarrow \operatorname{Spec} k[X_1^{\pm 1}, Y]/(\mathcal{F}_{\alpha'}) = C_{\alpha'}$$

given by the matrix $M_{\alpha\alpha'} = M_\alpha M_{\alpha'}^{-1}$. Since the lower left entry of $M_{\alpha\alpha'}$ is non-zero, the largest open $U_{\alpha\alpha'}$ of C_α for which $s_{\alpha\alpha'}$ comes from an open immersion $U_{\alpha\alpha'} \hookrightarrow C_{\alpha'}$ is $U_{\alpha\alpha'} = D(Y) \subset C_\alpha$, i.e. the image of C_0 in C_α. Thus $C_\alpha \cap C_{\alpha'} = C_0$ for any $\alpha, \alpha' \in \Sigma_1 \sqcup \{0\}$, $\alpha \neq \alpha'$.

3.4.2 Inductive construction of the curves

Until the end of the section let $n \in \mathbb{Z}_+$ and suppose we constructed a finite subset $\Sigma_n \subset \Omega$ and a projective curve C_n as in 3.3.1. In particular, $C_n = \bigcup_{\alpha \in \Sigma_n} C_\alpha \cup C_0$.

Remark 3.4.4. Let $\alpha \in \Sigma_n$. Recall $C_{0,\alpha}$ is smooth as so is C_0. Therefore $\operatorname{Sing}(C_\alpha) \subseteq C_\alpha \setminus C_{0,\alpha}$. Then, as an easy consequence of the Jacobian criterion, any singular point of C_α is the image of a singular point of $\bar{C}_\alpha$ under the closed immersion $\bar{C}_\alpha \hookrightarrow C_\alpha$. This fact can be observed by comparing the Jacobian matrices of C_α, defined in 3.3.1(c), and $\bar{C}_\alpha = C_\alpha \cap \{Y = 0\}$, at points of $C_\alpha \setminus C_{0,\alpha} = \bar{C}_\alpha$. In particular, if C_n is singular then $\bar{C}_\alpha$ is singular for some $\alpha \in \Sigma_n$.

Let $\alpha \in \Sigma_n$ and fix $S_n \subseteq \operatorname{Sing}(\bar{C}_\alpha)$. Via the immersion $\bar{C}_\alpha \hookrightarrow C_n$ given by the closed immersion $\bar{C}_\alpha \hookrightarrow C_\alpha$ and the inclusion $C_\alpha \subseteq C_n$, the points in S_n will be identified with their images in C_n. In this subsection we will construct a finite subset $\Sigma_{n+1} \subset \Omega$ defining a curve C_{n+1} as indicated in 3.3.1. Then, in §3.4.4 we will define a proper birational morphism $s_n : C_{n+1} \to C_n$ with exceptional locus $s_n^{-1}(S_n \cap \operatorname{Sing}(C_n))$.

Let $m \in \mathbb{Z}_+$ such that $\alpha \in \Omega_m$. Write $\alpha = (v, T)$, where $v \in \mathbb{Z}^{m+1}$ and $T \in k[x_1^{\pm 1}, y^{\pm 1}]^{m-1}$. Let $M_\alpha \in \mathrm{SL}_{m+1}(\mathbb{Z})$ be the matrix attached to v fixed by 3.3.1(a), defining a change of variables

$$(x_1, \ldots, x_m, y) = (X_1, \ldots, X_m, \tilde{Y}) \bullet M_\alpha.$$

Note that we have changed the notation for the variable Y to $\tilde{Y}$ for avoiding confusion later. Let $\mathfrak{a}_\alpha = (\tilde{\mathcal{F}}_2, \ldots, \tilde{\mathcal{F}}_m) \subset k[X_1^{\pm 1}, \ldots, X_m^{\pm 1}, \tilde{Y}]$ be the ideal in 3.3.1(a) and $\mathcal{F}_\alpha \in k[X_{j_\alpha}, \tilde{Y}]$ be the polynomial in 3.3.1(c) so that

$$C_\alpha = \mathrm{Spec}\, \frac{k[X_1^{\pm 1}, \ldots, X_m^{\pm 1}, \tilde{Y}]}{(\mathcal{F}_\alpha, \tilde{\mathcal{F}}_2 \ldots, \tilde{\mathcal{F}}_m)}.$$

Denote $A_m = k[X_1^{\pm 1}, \ldots, X_m^{\pm 1}]$.

Fix a point $p \in S_n$. Recall $\bar{C}_\alpha = \mathrm{Spec}\, D_\alpha/(f|_\alpha)$, where D_α is a (non-trivial) localisation of $k[X_{j_\alpha}]$ and $f|_\alpha \in k[X_{j_\alpha}]$ is non-zero. There exists some irreducible $\bar{\mathcal{G}}_p \in D_\alpha$ such that $(\bar{\mathcal{G}}_p)$ is the maximal ideal of $\mathcal{O}_{\bar{C}_\alpha, p}$. Then $f|_\alpha \in (\bar{\mathcal{G}}_p)^2$. We choose $\bar{\mathcal{G}}_p \in k[X_{j_\alpha}]$ monic of degree 1. Consider p as a point of C_n. Then $p \in C_\alpha \setminus C_{0,\alpha}$. In particular, $p \notin C_0$, since $C_0 \cap C_\alpha = C_{0,\alpha}$. For any $\tilde{\mathcal{G}}_p \in k[X_{j_\alpha}, \tilde{Y}]$ such that $\tilde{\mathcal{G}}_p \equiv \bar{\mathcal{G}}_p \mod \tilde{Y}$, the ideal $(\tilde{\mathcal{G}}, \tilde{Y}) + \mathfrak{a}_\alpha$ is the maximal ideal of $\mathcal{O}_{C_\alpha, p}$. We fix a choice of $\tilde{\mathcal{G}}_p$ such that $\tilde{\mathcal{G}}_p - \bar{\mathcal{G}}_p \in \tilde{Y} k[\tilde{Y}]$ and $\tilde{\mathcal{G}}_p \nmid \mathcal{F}_\alpha$.

Remark 3.4.5. Note that such a choice of $\tilde{\mathcal{G}}_p$ is always possible. Indeed, if $\deg_{\tilde{Y}}(\mathcal{F}_\alpha)$ is the degree of $\mathcal{F}_\alpha$ with respect to $\tilde{Y}$, it suffices to define

$$\tilde{\mathcal{G}}_p = \bar{\mathcal{G}}_p + \tilde{Y}^{\deg_{\tilde{Y}}(\mathcal{F}_\alpha)+1}$$

On the other hand, $\tilde{\mathcal{G}}_p = \bar{\mathcal{G}}_p$ is often admissible and better for computations. For instance, if C_0 is connected, then we can always choose $\tilde{\mathcal{G}}_p = \bar{\mathcal{G}}_p$.

Lemma 3.4.6 *Consider the principal open set $U_p = D(\tilde{\mathcal{G}}_p)$ of C_α. Then U_p is dense in C_α.*

Proof. As a consequence of 3.3.1, we saw that there is a natural open immersion

$$C_\alpha \hookrightarrow \mathrm{Spec}\, k[X_{j_\alpha}, \tilde{Y}]/(\mathcal{F}_\alpha).$$

Since $\tilde{\mathcal{G}}_p \in k[X_{j_\alpha}, \tilde{Y}]$, the image of U_p is the open subset $V_p = D(\tilde{\mathcal{G}}_p)$ of Spec $k[X_{j_\alpha}, \tilde{Y}]/(\mathcal{F}_\alpha)$. Note that if V_p is dense, then U_p is dense in C_α. In fact, V_p is dense in Spec $k[X_{j_\alpha}, \tilde{Y}]/(\mathcal{F}_\alpha)$ since $\tilde{\mathcal{G}}_p \nmid \mathcal{F}_\alpha$. □

Write $T = (g_2, \ldots, g_m)$. From 3.3.1(a) recall the isomorphism

$$\frac{k[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_m^{\pm 1}, y^{\pm 1}]}{(f_2, \ldots, f_m)} \underset{\simeq}{\overset{M_\alpha}{}} \frac{A_m[\tilde{Y}^{\pm 1}]}{\mathfrak{a}_\alpha},$$

where $f_i = x_i - g_i$ for all $i = 2, \ldots, m$. Let $g_p \in k[x_1, \ldots, x_m, y]$ such that

$$x_1^* \cdots x_m^* y^* \cdot g_p(x_1, \ldots, x_m, y) = \tilde{\mathcal{G}}_p\big((x_1, \ldots, x_m, y) \bullet M_\alpha^{-1}\big).$$

We fix a canonical choice of g_p by requiring $\mathrm{ord}_y g_p = 0$, and $\mathrm{ord}_{x_i}(g_p) = 0$ for all $i = 1, \ldots, m$.

Definition 3.4.7 We say that $g_p \in k[x_1,\dots,x_m,y]$ is *related to* $\tilde{\mathcal{G}}_p$ *by* M_α if it is defined as above. Note that it is uniquely determined by $\tilde{\mathcal{G}}_p$ and M_α.

Define $\alpha_p = (0,1)\circ_{g_p} \alpha \in \Omega_{m+1}$ (Definition 3.2.3). Fix a choice of a matrix $M_{\alpha_p} \in \mathrm{SL}_{m+2}(\mathbb{Z})$ attached to $(0,1)\circ_{g_p} v$ such that the change of variables

$$(x_1,\dots,x_m,x_{m+1},y) = (X_1,\dots,X_m,\tilde{X}_{m+1},\tilde{Y}) \bullet M_{\alpha_p}$$

restricts to the change of variables given by the matrix M_α on the subring $k[x_1^{\pm1},\dots,x_m^{\pm1},y^{\pm1}]$ of $k[x_1^{\pm1},\dots,x_{m+1}^{\pm1},y^{\pm1}]$ and gives the equality

$$x_{m+1} - g_p = X_1^{n_1}\cdots X_m^{n_m}\tilde{Y}^{\mathrm{ord}_v(g_p)}(\tilde{X}_{m+1} - \tilde{\mathcal{G}}_p), \tag{3.2}$$

for some $n_1,\dots,n_m \in \mathbb{Z}$. In particular,

$$\frac{k[x_1^{\pm1},\dots,x_{m+1}^{\pm1},y^{\pm1}]}{(f_2,\dots,f_m,x_{m+1}-g_{m+1})} \overset{M_{\alpha_p}}{\simeq} \frac{A_m[\tilde{X}_{m+1}^{\pm1},\tilde{Y}^{\pm1}]}{\mathfrak{a}_\alpha + (\tilde{X}_{m+1}-\tilde{\mathcal{G}}_p)} \hookrightarrow k(X_{j_\alpha},\tilde{Y}). \tag{3.3}$$

where $g_{m+1} \in k[x_1^{\pm1},y^{\pm1}]$ is the unique polynomial so that $g_{m+1} \equiv g_p \mod (f_2,\dots,f_m)$.

Remark 3.4.8. Such M_{α_p} is constructed as follows. Via M_α write

$$g_p = X_1^{n_1}\cdots X_m^{n_m} Y^{\mathrm{ord}_v(g_p)} \cdot \tilde{\mathcal{G}}_p$$

for some $n_1,\dots,n_m \in \mathbb{Z}$. Then

- The $(m+1)$-th row of M_{α_p} is the vector $(0,\dots,0,1,0)$;
- The $(m+1)$-th column of M_{α_p} is the vector $(n_1,\dots,n_m,1,\mathrm{ord}_v(g_p))$;
- The submatrix of M_{α_p} obtained by removing the $(m+1)$-th row and the $(m+1)$-th column equals M_α.

This construction is unique. Indeed, the $(m+1)$-th column is fixed by the equality (3.2), while all other columns are fixed by the fact that M_{α_p} defines the same change of variables of M_α on $k[x_1^{\pm1},\dots,x_m^{\pm1},y^{\pm1}]$.

Lemma 3.4.9 *With the notation above*

$$C_{\alpha_p} = \mathrm{Spec}\, \frac{k[X_1^{\pm1},\dots,X_m^{\pm1},\tilde{X}_{m+1}^{\pm1},\tilde{Y}]}{(\mathcal{F}_\alpha,\tilde{\mathcal{F}}_2,\dots,\tilde{\mathcal{F}}_m,\tilde{X}_{m+1}-\tilde{\mathcal{G}}_p)}.$$

Furthermore, C_{0,α_p} is dense in C_0, i.e. $\alpha_p \in \Omega$, and the birational map $s_{\alpha_p\alpha}$ comes from an open immersion $s_{\alpha_p\alpha}: C_{\alpha_p} \hookrightarrow C_\alpha$ with image $D(\tilde{\mathcal{G}}_p) \subset C_\alpha$. Finally, $s_{\alpha_p\alpha}$ induces $\bar{C}_{\alpha_p} \simeq \bar{C}_\alpha \setminus \{p\}$.

Proof. First note that $C_{0,\alpha_p} \subset C_{0,\alpha}$. Considering $C_{0,\alpha}$ as an open subscheme of C_α, then C_{0,α_p} equals $D(\tilde{\mathcal{G}}_p) \cap C_{0,\alpha} \subset C_\alpha$. Then C_{0,α_p} is dense in $C_{0,\alpha}$ by Lemma 3.4.6. It follows that C_{0,α_p} is dense in C_0 since so is $C_{0,\alpha}$. In other words, $\alpha_p \in \Omega$. The ring homomorphism

$$A_\alpha := \frac{A_m[\tilde{Y}]}{(\mathcal{F}_\alpha)+\mathfrak{a}_\alpha} \to \frac{A_m[\tilde{X}_{m+1}^{\pm 1}, \tilde{Y}]}{(\mathcal{F}_\alpha, \tilde{X}_{m+1} - \tilde{\mathcal{G}}_p)+\mathfrak{a}_\alpha} =: A_{\alpha_p}$$

is injective by Lemma 3.4.6 and induces the birational map $s_{\alpha_p\alpha}$ if $C_{\alpha_p} = \operatorname{Spec} A_{\alpha_p}$ from (3.3). The injectivity implies that $\tilde{Y}$ is a regular element of A_{α_p} since $\tilde{Y}$ is a regular element of A_α by definition of C_α. This concludes the proof by definition of C_{α_p}. □

Now consider the lexicographic monomial order $X_{j_\alpha} > \tilde{X}_{m+1} > \tilde{Y}$ on $k[X_{j_\alpha}, \tilde{X}_{m+1}, \tilde{Y}]$ and compute the *normal form* $\mathcal{F}_{\alpha,p}$ of $\mathcal{F}_\alpha$ by $\tilde{X}_{m+1} - \tilde{\mathcal{G}}_p$ with respect to >. In other words, the polynomial $\mathcal{F}_{\alpha,p}$ is the remainder of the complete multivariate division of $\mathcal{F}_\alpha$ by $\tilde{X}_{m+1} - \tilde{\mathcal{G}}_p$. Note that $\mathcal{F}_{\alpha,p} \in k[\tilde{X}_{m+1}, \tilde{Y}]$, as $\tilde{\mathcal{G}}_p - \bar{\mathcal{G}}_p \in \tilde{Y} k[\tilde{Y}]$ and $\bar{\mathcal{G}}_p \in k[X_{j_\alpha}]$ of degree 1.

Let $\beta \in \mathbb{Z}_+^2$ be any primitive vector. Fix a matrix $M_\beta \in \mathrm{SL}_2(\mathbb{Z})$ attached to β. Then M_β gives an isomorphism between $k[\tilde{X}_{m+1}^{\pm 1}, \tilde{Y}^{\pm 1}]$ and $k[X_{m+1}^{\pm 1}, Y^{\pm 1}]$ through the change of variables $(\tilde{X}_{m+1}, \tilde{Y}) = (X_{m+1}, Y) \bullet M_\beta$. This transformation lifts to

$$A_m[\tilde{X}_{m+1}^{\pm 1}, \tilde{Y}^{\pm 1}] \overset{I_m \oplus M_\beta}{\simeq} A_m[X_{m+1}^{\pm 1}, Y^{\pm 1}], \tag{3.4}$$

where $I_m \in \mathrm{SL}_m(\mathbb{Z})$ is the identity matrix of size m. Since $\beta \in \mathbb{Z}_+^2$, the isomorphism (3.4) restricts to a homomorphism

$$A_m[\tilde{X}_{m+1}, \tilde{Y}] \xrightarrow{I_m \oplus M_\beta} A_m[X_{m+1}^{\pm 1}, Y]$$

Let $\beta = (\beta_1, \beta_2)$ and let (δ_1, δ_2) be the first row of M_β, so $\delta_1\beta_2 - \delta_2\beta_1 = 1$. Set $A_{m+1} = A_m[X_{m+1}^{\pm 1}]$. Denote by $\mathcal{F}_2, \ldots, \mathcal{F}_m, \mathcal{G}_p \in A_{m+1}[Y]$ the images of $\tilde{\mathcal{F}}_2, \ldots, \tilde{\mathcal{F}}_m, \tilde{\mathcal{G}}_p$ under $I_m \oplus M_\beta$, respectively. Let $\mathcal{F}_{m+1} = X_{m+1}^{\delta_1} Y^{\beta_1} - \mathcal{G}_p$, image of $\tilde{X}_{m+1} - \tilde{\mathcal{G}}_p$. Then we get the homomorphism

$$\frac{A_m[\tilde{Y}]}{\mathfrak{a}_\alpha} \simeq \frac{A_m[\tilde{X}_{m+1}, \tilde{Y}]}{\mathfrak{a}_\alpha + (\tilde{X}_{m+1} - \tilde{\mathcal{G}}_p)} \xrightarrow{I_m \oplus M_\beta} \frac{A_{m+1}[Y]}{(\mathcal{F}_2, \ldots, \mathcal{F}_{m+1})}. \tag{3.5}$$

Note that since $\beta_2 > 0$ then

$$\mathcal{G}_p \equiv \bar{\mathcal{G}}_p \mod Y, \qquad \text{and} \qquad \mathcal{F}_i \equiv \bar{\mathcal{F}}_i \mod Y \quad \text{for any } i = 2, \ldots, m,$$

where $\bar{\mathcal{F}}_i$ is the unique polynomial in A_m such that $\tilde{\mathcal{F}}_i \equiv \bar{\mathcal{F}}_i \mod \tilde{Y}$.

Let $\gamma = \beta \circ_{g_p} \alpha \in \Omega_{m+1}$. By definition, $C_{0,\gamma} = C_{0,\alpha_p}$. Therefore $\gamma \in \Omega$ by Lemma 3.4.9. Let $\mathfrak{a}_\gamma$ be the ideal of $A_{m+1}[Y]$ generated by $\mathcal{F}_2, \ldots, \mathcal{F}_{m+1}$ and set $M_\gamma = (I_m \oplus M_\beta) \cdot M_{\alpha_p} \in \mathrm{SL}_{m+2}(\mathbb{Z})$. Note that the matrix M_γ is attached to $\beta \circ_{g_p} v$. Let $\mathcal{F}_\gamma \in k[X_{m+1}, Y]$, with $\operatorname{ord}_Y(\mathcal{F}_\gamma) = 0$, satisfying

$$\mathcal{F}_{\alpha,p}((X_{m+1}, Y) \bullet M_\beta) = X_{m+1}^{n_X} Y^{n_Y} \cdot \mathcal{F}_\gamma(X_{m+1}, Y),$$

for some $n_X, n_Y \in \mathbb{Z}$. Note that

$$\frac{k[x_1^{\pm 1}, \ldots, x_{m+1}^{\pm 1}, y^{\pm 1}]}{(f_2, \ldots, f_{m+1})} \overset{M_{\alpha_p}}{\simeq} \frac{k[X_1^{\pm 1}, \ldots, X_m^{\pm 1}, \tilde{X}_{m+1}^{\pm 1}, \tilde{Y}^{\pm 1}]}{(\tilde{\mathcal{F}}_2, \ldots, \tilde{\mathcal{F}}_m, \tilde{X}_{m+1} - \tilde{\mathcal{G}}_p)} \overset{I_m \oplus M_\beta}{\simeq} \frac{k[X_1^{\pm 1}, \ldots, X_{m+1}^{\pm 1}, Y^{\pm 1}]}{(\mathcal{F}_2, \ldots, \mathcal{F}_{m+1})}, \tag{3.6}$$

where $f_{m+1} = x_{m+1} - g_{m+1}$. In particular, the ideal $\mathfrak{a}_\gamma$ and the matrix M_γ satisfy 3.3.1(a) for γ. With $j_\gamma = m+1$ we are now going to show 3.3.1(b) for γ.

Recall from 3.3.1(b) there is an injective homomorphism $R_\alpha \hookrightarrow k(X_{j_\alpha}, \tilde{Y})$ taking $X_{j_\alpha} \mapsto X_{j_\alpha}$ and $\tilde{Y} \mapsto \tilde{Y}$. Since $\tilde{\mathcal{G}}_p - \bar{\mathcal{G}}_p \in \tilde{Y} k[\tilde{Y}]$ and $\bar{\mathcal{G}}_p \in k[X_{j_\alpha}]$ monic of degree 1, we have

$$\frac{A_m[\tilde{X}_{m+1}, \tilde{Y}]}{\mathfrak{a}_\alpha + (\tilde{X}_{m+1} - \tilde{\mathcal{G}}_p)} \hookrightarrow \frac{k(X_{j_\alpha}, \tilde{Y})[\tilde{X}_{m+1}]}{(\tilde{X}_{m+1} - \tilde{\mathcal{G}}_p)} \simeq k(\tilde{X}_{m+1}, \tilde{Y}).$$

Then we can construct the following commutative diagram

$$\begin{tikzcd}
\frac{A_{m+1}[Y^{\pm 1}]}{\mathfrak{a}_\gamma} = \frac{k[X_1^{\pm 1},\dots,X_{m+1}^{\pm 1},Y^{\pm 1}]}{(\mathcal{F}_2,\dots,\mathcal{F}_m,\mathcal{F}_{m+1})} \arrow[r, "\iota_\gamma"] & k(X_{m+1}, Y) \\
\frac{k[X_1^{\pm 1},\dots,X_m^{\pm 1},\tilde{X}_{m+1}^{\pm 1},\tilde{Y}^{\pm 1}]}{(\tilde{\mathcal{F}}_2,\dots,\tilde{\mathcal{F}}_m,\tilde{X}_{m+1}-\tilde{\mathcal{G}}_p)} \arrow[u, hook, two heads, "I_m \oplus M_\beta"] \arrow[r, hook] & k(\tilde{X}_{m+1}, \tilde{Y}) \arrow[u, "M_\beta"']
\end{tikzcd} \tag{3.7}$$

given by the matrix M_β. Therefore the homomorphism ι_γ is injective and takes $X_{m+1} \mapsto X_{m+1}$ and $Y \mapsto Y$.

Lemma 3.4.10 *With the notation above, there is an isomorphism*

$$\frac{k[X_1^{\pm 1}, \dots, X_{m+1}^{\pm 1}, Y]}{(\mathcal{F}_2, \dots, \mathcal{F}_{m+1}, Y)} \simeq k[X_{m+1}^{\pm 1}],$$

taking $X_{m+1} \mapsto X_{m+1}$. The images of $X_1, \dots, X_m$ in $k[X_{m+1}^{\pm 1}]$ lies in k.

Proof. Recall that for every $i = 2, \dots, m$ there exists a (unique) Laurent polynomial $\bar{\mathcal{F}}_i \in A_m$ such that $\tilde{\mathcal{F}}_i \equiv \bar{\mathcal{F}}_i \mod \tilde{Y}$. Since $\mathcal{F}_{m+1} \equiv \bar{\mathcal{G}}_p \mod Y$ and $\mathcal{F}_i \equiv \bar{\mathcal{F}}_i \mod Y$ for any $i = 2, \dots, m$, we have

$$\frac{k[X_1^{\pm 1},\dots,X_{m+1}^{\pm 1},Y]}{(\mathcal{F}_2,\dots,\mathcal{F}_{m+1},Y)} \simeq \frac{k[X_1^{\pm 1},\dots,X_{m+1}^{\pm 1},\tilde{Y}]}{(\tilde{\mathcal{F}}_2,\dots,\tilde{\mathcal{F}}_m,\bar{\mathcal{G}}_p,\tilde{Y})} \simeq \frac{D_\alpha}{(\bar{\mathcal{G}}_p)}[X_{m+1}^{\pm 1}] \simeq k[X_{m+1}^{\pm 1}],$$

and the isomorphisms take $X_{m+1} \mapsto X_{m+1}$, as required. $\square$

Proposition 3.4.11 *With the notation above, there is an injective homomorphism*

$$R_\gamma := \frac{A_{m+1}[Y]}{\mathfrak{a}_\gamma} \hookrightarrow k(X_{m+1}, Y).$$

taking $X_{m+1} \mapsto X_{m+1}$ and $Y \mapsto Y$. Furthermore, YR_γ is prime ideal.

Proof. Lemma 3.4.10 shows YR_γ is a prime ideal as $R_\gamma/(Y) \simeq k[X_{m+1}^{\pm 1}]$ is an integral domain. From (3.7) we have

$$\frac{A_{m+1}[Y]}{\mathfrak{a}_\gamma} \to \frac{A_{m+1}[Y^{\pm 1}]}{\mathfrak{a}_\gamma} \hookrightarrow k(X_{m+1}, Y)$$

taking $X_{m+1} \mapsto X_{m+1}$ and $Y \mapsto Y$. Therefore it suffices to show that the ideal $\mathfrak{a}_\gamma$ of $A_{m+1}[Y]$ equals its saturation $\mathfrak{a}_\gamma : Y^\infty$ with respect to Y. Suppose not. Consider the primary decomposition of $\mathfrak{a}_\gamma$,

$$\mathfrak{a}_\gamma = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s, \qquad \mathfrak{p}_i = \sqrt{\mathfrak{q}_i}.$$

Recall that the primary decomposition of $\mathfrak{a}_\gamma : Y^\infty$ consists of all the $\mathfrak{q}_i$'s which do not contain any power of Y. Hence there exists some $i = 1,\dots,s$ such that $\mathfrak{p}_i \supseteq (Y)+\mathfrak{a}_\gamma$. Moreover, we can choose i such that $\mathfrak{p}_i$ is a minimal prime ideal over $\mathfrak{a}_\gamma$, i.e. $\mathfrak{p}_i \in \mathrm{Min}(\mathfrak{a}_\gamma)$. Then, by Krull's height theorem, the height of $\mathfrak{p}_i$ is at most m (the number of generators of $\mathfrak{a}_\gamma$), and so $\mathrm{ht}((Y)+\mathfrak{a}_\gamma) \le m$. But

$$\dim \frac{A_{m+1}[Y]}{(Y)+\mathfrak{a}_\gamma} = 1,$$

by Lemma 3.4.10. This gives a contradiction, since

$$\mathrm{ht}((Y)+\mathfrak{a}_\gamma) + \dim \tfrac{A_{m+1}[Y]}{(Y)+\mathfrak{a}_\gamma} = \dim A_{m+1}[Y] = m+2,$$

from the regularity of $A_{m+1}[Y]$. $\square$

Proposition 3.4.12 *Let $\beta \in \mathbb{Z}_+^2$ and $\gamma = \beta \circ_{g_p} \alpha$ as above. Then*

$$C_\gamma = \mathrm{Spec}\, \frac{k[X_1^{\pm 1},\dots,X_{m+1}^{\pm 1},Y]}{(\mathcal{F}_\gamma,\mathcal{F}_2,\dots,\mathcal{F}_{m+1})}.$$

Proof. The isomorphism in (3.6) implies that $C_{0,\gamma} \simeq \mathrm{Spec}\, \frac{A_{m+1}[Y^{\pm 1}]}{(\mathcal{F}_\gamma)+\mathfrak{a}_\gamma}$ via M_γ. Then from the definition of C_γ, it suffices to show that Y is a regular element of $R_\gamma/(\mathcal{F}_\gamma)$, where $R_\gamma = A_{m+1}[Y]/\mathfrak{a}_\gamma$. From Proposition 3.4.11 there is an injective homomorphism $R_\gamma \hookrightarrow k(X_{m+1},Y)$, taking $X_{m+1} \mapsto X_{m+1}$ and $Y \mapsto Y$. Moreover, YR_γ is a prime ideal. Therefore if Y is a zero-divisor of $R_\gamma/(\mathcal{F}_\gamma)$ then $\mathcal{F}_\gamma \in YR_\gamma$, as R_γ is an integral domain. But this is not possible as Y is not invertible in R_γ and we chose $\mathcal{F}_\gamma$ such that $\mathrm{ord}_Y(\mathcal{F}_\gamma)=0$. Thus Y is a regular element of $R_\gamma/(\mathcal{F}_\gamma)$. $\square$

Notation 3.4.13 Let $\gamma = \beta \circ_{g_p} \alpha$, with $\beta \in \mathbb{Z}_+^2$ primitive. We have defined:

- $\mathfrak{a}_\gamma = (\mathcal{F}_2,\dots,\mathcal{F}_{m+1})$ and $M_\gamma = (M_\beta \oplus I_m)\cdot M_{\alpha_p}$ for some matrix M_β attached to β;
- $j_\gamma = m+1$, $R_\gamma = k[X_1^{\pm 1},\dots,X_{m+1}^{\pm 1},Y]/\mathfrak{a}_\gamma$;
- $\mathcal{F}_\gamma \in k[X_{m+1},Y]$, with $Y \nmid \mathcal{F}_\gamma$, satisfying $\mathcal{F}_{\alpha,p} \overset{M_\beta}{=} X_{m+1}^* Y^* \cdot \mathcal{F}_\gamma$, and $f|_\gamma \in k[X_{m+1}]$ given by $f|_\gamma(X_{m+1}) = \mathcal{F}_\gamma(X_{m+1},0)$.

With the notation above, γ satisfies the properties (a), (b), (c) of 3.3.1 by (3.6) and Propositions 3.4.11, 3.4.12.

Define $\bar{\mathcal{G}}_{S_n} \in k[X_{j_\alpha}]$, $\tilde{\mathcal{G}}_{S_n} \in k[X_{j_\alpha},\tilde{Y}]$ and $g_{S_n} \in k[x_1,\dots,x_m,y]$ by

$$\bar{\mathcal{G}}_{S_n} = \prod_{p \in S_n} \bar{\mathcal{G}}_p, \qquad \tilde{\mathcal{G}}_{S_n} = \prod_{p \in S_n} \tilde{\mathcal{G}}_p, \qquad g_{S_n} = \prod_{p \in S_n} g_p.$$

Then $\tilde{\mathcal{G}}_{S_n} \equiv \bar{\mathcal{G}}_{S_n} \mod \tilde{Y}$ and g_{S_n} is related to $\tilde{\mathcal{G}}_{S_n}$ by M_α, i.e. $x_1^* \cdots x_m^* y^* \cdot g_{S_n} = \tilde{\mathcal{G}}_{S_n}$ via M_α. Define $\tilde{\alpha} = (0,1) \circ_{g_{S_n}} \alpha$. Analogously to what we did for α_p in Remark 3.4.8, we can uniquely construct a matrix $M_{\tilde{\alpha}} \in \mathrm{SL}_{m+2}(\mathbb{Z})$ attached to $(0,1)\circ_{g_{S_n}} v$ in such a way that the change of variables given by $M_{\tilde{\alpha}}$ restricts to the change of variables given by M_α on $(x_1,\dots,x_m,y)$ and

$$x_{m+1} - g_{S_n} = X_1^* \cdots X_m^* \tilde{Y}^{\mathrm{ord}_v(g_{S_n})}(\tilde{X}_{m+1} - \tilde{\mathcal{G}}_{S_n}) \quad \text{via } M_{\tilde{\alpha}}.$$

In particular, denoting by $g_{m+1} \in k[x_1^{\pm 1}, y^{\pm 1}]$ the unique polynomial such that $g_{m+1} \equiv g_{S_n}$ mod $(f_2,\dots,f_m)$ one has

$$\frac{k[x_1^{\pm1},\dots,x_{m+1}^{\pm1},y^{\pm1}]}{(f_2,\dots,f_m,x_{m+1}-g_{m+1})} \overset{M_{\tilde{\alpha}}}{\simeq} \frac{A_m[X_{m+1}^{\pm1},\tilde{Y}^{\pm1}]}{\mathfrak{a}_\alpha + (X_{m+1} - \tilde{\mathcal{G}}_{S_n})} \hookrightarrow k(X_{j_\alpha},\tilde{Y}). \tag{3.8}$$

Remark 3.4.14. The construction of $M_{\tilde{\alpha}}$ is given by Remark 3.4.8 by replacing M_{α_p} with $M_{\tilde{\alpha}}$, g_p with g_{S_n}, and $\tilde{\mathcal{G}}_p$ with $\tilde{\mathcal{G}}_{S_n}$.

Lemma 3.4.15 *With the notation above*

$$C_{\tilde{\alpha}} = \operatorname{Spec} \frac{k[X_1^{\pm1},\dots,X_m^{\pm1},X_{m+1}^{\pm1},\tilde{Y}]}{(\mathcal{F}_\alpha,\tilde{\mathcal{F}}_2,\dots,\tilde{\mathcal{F}}_m,X_{m+1}-\tilde{\mathcal{G}}_{S_n})}.$$

Moreover, $C_{0,\tilde{\alpha}}$ is dense in C_0, i.e. $\tilde{\alpha} \in \Omega$, and for any $p \in S_n$ the birational maps $s_{\tilde{\alpha}\alpha}, s_{\tilde{\alpha}\alpha_p}, s_{\alpha_p\alpha}$ induce a commutative diagram of open immersions

$$\begin{tikzcd} C_{\tilde{\alpha}} \arrow[rr, hook, "s_{\tilde{\alpha}\alpha}"] \arrow[dr, hook, "s_{\tilde{\alpha}\alpha_p}"'] & & C_\alpha \\ & C_{\alpha_p} \arrow[ur, hook, "s_{\alpha_p\alpha}"'] & \end{tikzcd}$$

where $s_{\tilde{\alpha}\alpha}$ has image $D(\tilde{\mathcal{G}}_{S_n}) \subset C_\alpha$. Finally, $s_{\tilde{\alpha}\alpha}$ induces $\bar{C}_{\tilde{\alpha}} \simeq \bar{C}_\alpha \setminus S_n$.

Proof. First note that $C_{0,\tilde{\alpha}} = \bigcap_{p\in S_n} C_{0,\alpha_p}$. Then $C_{0,\tilde{\alpha}}$ is a dense open of C_0 by Lemma 3.4.9. The ring homomorphism

$$A_{\alpha_p} := \frac{A_m[\tilde{X}_{m+1}^{\pm1},\tilde{Y}]}{(\mathcal{F}_\alpha,\tilde{X}_{m+1}-\tilde{\mathcal{G}}_p)+\mathfrak{a}_\alpha} \simeq \frac{R_\alpha[\tilde{\mathcal{G}}_p^{-1}]}{(\mathcal{F}_\alpha)} \to \frac{R_\alpha[\tilde{\mathcal{G}}_{S_n}^{-1}]}{(\mathcal{F}_\alpha)} \simeq \frac{A_m[X_{m+1}^{\pm1},\tilde{Y}]}{(\mathcal{F}_\alpha,X_{m+1}-\tilde{\mathcal{G}}_{S_n})+\mathfrak{a}_\alpha} =: A_{\tilde{\alpha}}.$$

is injective by Lemma 3.4.6 and induces the birational map $s_{\tilde{\alpha}\alpha_p}$ if $\operatorname{Spec} A_{\tilde{\alpha}} = C_{\tilde{\alpha}}$ from (3.3) and (3.8). Since $\tilde{Y}$ is a regular element of A_{α_p} by Lemma 3.4.9, then $\tilde{Y}$ is a regular element of $A_{\tilde{\alpha}}$. This proves $C_{\tilde{\alpha}} = \operatorname{Spec} A_{\tilde{\alpha}}$ by definition of $C_{\tilde{\alpha}}$ and gives the required commutative diagram again by Lemma 3.4.9. □

Notation 3.4.16 Define

- $\mathfrak{a}_{\tilde{\alpha}} = \mathfrak{a}_\alpha + (\tilde{X}_{m+1} - \tilde{\mathcal{G}}_{S_n}) \subset k[X_1^{\pm1},\dots,X_m^{\pm1},\tilde{X}_{m+1}^{\pm1},\tilde{Y}]$ and $M_{\tilde{\alpha}}$ as described in Remark 3.4.14;
- $j_{\tilde{\alpha}} = j_\alpha$, $R_{\tilde{\alpha}} = R_\alpha[\tilde{X}_{m+1}^{\pm1}]/(\tilde{X}_{m+1} - \tilde{\mathcal{G}}_{S_n})$ and $D_{\tilde{\alpha}} = D_\alpha[\bar{\mathcal{G}}_{S_n}^{-1}]$;
- $\mathcal{F}_{\tilde{\alpha}} = \mathcal{F}_\alpha$ and $f|_{\tilde{\alpha}} = f|_\alpha$.

With the notation above, $\tilde{\alpha}$ satisfies the properties (a), (b), (c) of 3.3.1.

Definition 3.4.17 For any $p \in S_n$ let

$$\Sigma_p = \{\gamma = \beta \circ_{g_p} \alpha \mid \beta \in \mathbb{Z}_+^2 \text{ primitive, and } C_{0,\gamma} \subsetneq C_\gamma\} \subset \Omega,$$

and $\Sigma_{S_n} = \bigcup_{p \in S_n} \Sigma_p$. Define

$$\hat{\Sigma}_n = \Sigma_n \setminus \{\alpha\}, \qquad \tilde{\Sigma}_n = \hat{\Sigma}_n \cup \{\tilde{\alpha}\}, \qquad \Sigma_{n+1} = \Sigma_{S_n} \cup \tilde{\Sigma}_n.$$

Recall that for any $\gamma, \gamma' \in \Omega \sqcup \{0\}$ we have a canonical way to glue the curves C_γ, $C_{\gamma'}$ through the birational maps $s_{\gamma\gamma'}$. Then

$$C_{n+1} = \bigcup_{\gamma \in \Sigma_{n+1}} C_\gamma \cup C_0.$$

We also define the following curves.

Definition 3.4.18 For any $p \in S_n$ define

$$C_p := \bigcup_{\gamma \in \Sigma_p} C_\gamma, \qquad \hat{C}_n := \bigcup_{\alpha' \in \hat{\Sigma}_n} C_{\alpha'}.$$

Then $C_{n+1} = \bigcup_{p \in S_n} C_p \cup C_{\tilde{\alpha}} \cup \hat{C}_n \cup C_0$.

3.4.3 The role of Newton polygons

Let $p \in S_n$. In this subsection we show that Newton polygons can be used to obtain an explicit description of the set Σ_p. We want to find all primitive vectors $\beta \in \mathbb{Z}_+^2$ such that $C_{0,\gamma} \subsetneq C_\gamma$, where $\gamma = \beta \circ_{g_p} \alpha$.

Let $\beta = (\beta_1, \beta_2) \in \mathbb{Z}_+^2$ be a primitive vector and let $\gamma = \beta \circ_{g_p} \alpha$. Recall that $f|_\gamma(X_{m+1}) = \mathcal{F}_\gamma(X_{m+1}, 0)$. Hence $f|_\gamma \neq 0$ since $Y \nmid \mathcal{F}_\gamma$. Note that $D_\gamma = k[X_{m+1}^{\pm 1}]$ by Lemma 3.4.10. Therefore $C_\gamma = C_{0,\gamma}$ if and only if $f|_\gamma$ is invertible in $k[X_{m+1}^{\pm 1}]$. Since through the change of variables given by M_β

$$\mathcal{F}_{\alpha,p} = X_{m+1}^* Y^{\operatorname{ord}_\beta(\mathcal{F}_{\alpha,p})} \cdot \mathcal{F}_\gamma,$$

from the Newton polygon $\Delta_{\alpha,p}$ of $\mathcal{F}_{\alpha,p}$ one can see whether $f|_\gamma$ is invertible in $k[X_{m+1}^{\pm 1}]$ or not. Let $\phi : \mathbb{Z}^2 \to \mathbb{Z}$ be the affine function defined by

$$\phi(i,j) = \beta_1 i + \beta_2 j - \operatorname{ord}_\beta(\mathcal{F}_{\alpha,p}).$$

Then $f|_\gamma$ is not invertible in $k[X_{m+1}^{\pm 1}]$ if and only if $\phi^{-1}(0) \cap \Delta_{\alpha,p} = \text{edge}$. Thus Σ_p consists of all elements $\beta \circ_{g_p} \alpha$ such that $\beta \in \mathbb{Z}_+^2$ is the normal vector of some edge of $\Delta_{\alpha,p}$. All these elements are distinct as immediate consequence of Definition 3.2.3. Furthermore, note that this description shows that Σ_p is finite and non-empty as $\tilde{X}_{m+1} \mid \mathcal{F}_{\alpha,p}(\tilde{X}_{m+1}, 0)$ but $\tilde{X}_{m+1} \nmid \mathcal{F}_{\alpha,p}$.

Proposition 3.4.19 *Let $\beta \in \mathbb{Z}_+^2$ be the normal vector of an edge ℓ of the Newton polygon $\Delta_{\alpha,p}$ of $\mathcal{F}_{\alpha,p}$. Let $\gamma = \beta \circ_{g_p} \alpha$. Expand $\mathcal{F}_{\alpha,p} = \sum_{i,j} c_{ij} \tilde{X}_{m+1}^i \tilde{Y}^j$, where $c_{ij} \in k$. Let $(i_0, j_0), \ldots, (i_l, j_l)$ be the points of $\ell \cap \mathbb{Z}^2$, ordered along ℓ counterclockwise with respect to $\Delta_{\alpha,p}$. Then*

$$f|_\gamma = X_{m+1}^d \sum_{r=0}^{l} c_{i_r j_r} X_{m+1}^r$$

for some $d \in \mathbb{N}$.

Proof. Let (δ_1, δ_2) be the first row of M_β. It is easy to see that

$$f|_\gamma = \sum_{(i,j)\in\ell} c_{ij} X_{m+1}^{\delta_1 i + \delta_2 j + d'} \qquad \text{for some } d' \in \mathbb{Z},$$

with $\delta_1 i + \delta_2 j + d' \geq 0$. Note that $(i_r, j_r) = (i_0, j_0) + r(\beta_2, -\beta_1)$. Therefore, for $d = d' + (\delta_1 i_0 + \delta_2 j_0)$, the proposition follows since $\delta_1 \beta_2 - \delta_2 \beta_1 = 1$. □

3.4.4 Inductive construction of the morphisms

In this subsection we want to construct a birational morphism $s_n : C_{n+1} \to C_n$. In §3.4.5 we will prove that s_n is proper with the exceptional locus $s_n^{-1}(S_n \cap \operatorname{Sing}(C_n))$.

Remark 3.4.20. Let $p \in S_n$. Similarly to the classical case (Remark 3.4.3), for any $\gamma, \gamma' \in \Sigma_p$, $\gamma \neq \gamma'$, one has $C_\gamma \cap C_{\gamma'} = C_{0,\gamma}$. More precisely, the birational map $s_{\gamma\gamma'} : C_\gamma \dashrightarrow C_{\gamma'}$ has domain of definition $C_{0,\gamma}$ giving an isomorphism $C_{0,\gamma} \to C_{0,\gamma'}$.

Remark 3.4.21. Let $p \in S_n$. For any primitive vector $\beta \in \mathbb{Z}_+^2$, if $\gamma = \beta \circ_{g_p} \alpha$ then from (3.5) we obtain the homomorphism of rings

$$\frac{A_m[\tilde{Y}]}{(\mathcal{F}_\alpha) + \mathfrak{a}_\alpha} \simeq \frac{A_m[\tilde{X}_{m+1}, \tilde{Y}]}{(\mathcal{F}_\alpha, \tilde{X}_{m+1} - \tilde{\mathcal{G}}_p) + \mathfrak{a}_\alpha} \xrightarrow{I_m \oplus M_\beta} \frac{A_{m+1}[Y]}{(\mathcal{F}_\gamma) + \mathfrak{a}_\gamma} \tag{3.9}$$

that induces a birational morphism $C_\gamma \to C_\alpha$. In fact, from the definition of M_γ we see that it agrees with $s_{\gamma\alpha} : C_\gamma \dashrightarrow C_\alpha$ as rational map.

Lemma 3.4.22 *Let $p \in S_n$ and $\gamma = \beta \circ_{g_p} \alpha$ for some primitive $\beta \in \mathbb{Z}_+^2$. Then $s_{\gamma\alpha} : C_\gamma \to C_\alpha$ restricts to an isomorphism $C_{0,\gamma} \xrightarrow{\sim} D(\tilde{\mathcal{G}}_p) \cap C_{0,\alpha} \subset C_\alpha$and $s_{\gamma\alpha}(C_\gamma \setminus C_{0,\gamma}) \subseteq \{p\}$.*

Proof. Let $\gamma = \beta \circ_{g_p} \alpha$ for some $\beta \in \mathbb{Z}_+^2$. The first part of the lemma follows from Remark 3.4.21 and (3.4). The morphism $s_{\gamma\alpha}$ is induced by the ring homomorphism taking $\tilde{Y} \mapsto X_{m+1}^{\delta_2} Y^{\beta_2}$, with $M_\beta = \begin{pmatrix} \delta_1 & \delta_2 \\ \beta_1 & \beta_2 \end{pmatrix}$. Recall

$$C_\gamma \setminus C_{0,\gamma} = \bar{C}_\gamma = \{Y = 0\} \subset C_\gamma.$$

Since $\bar{\mathcal{G}}_p \equiv \mathcal{F}_{m+1} \mod Y$, the morphism $s_{\gamma\alpha}$ takes $\bar{C}_\gamma$ into the closed subscheme $\{\bar{\mathcal{G}}_p = 0\}$ of $\bar{C}_\alpha$. This concludes the proof as $\{\bar{\mathcal{G}}_p = 0\} = \{p\}$. □

Let $p \in S_n$. Considering p as a point of C_α, denote by $U_{\alpha,p}$ the open subscheme $D(\tilde{\mathcal{G}}_p) \cap C_{0,\alpha}$ of C_α. Recall that $C_{0,\alpha}$ is dense in C_α by definition. Hence Lemma 3.4.6 implies that $U_{\alpha,p}$ is dense. Let $C_{\alpha,p} = U_{\alpha,p} \cup \{p\}$ as subset of C_α. We want to show that $C_{\alpha,p}$ is dense and open in C_α. From the density of $U_{\alpha,p}$ it follows that $C_{\alpha,p}$ is dense and that $V_p := C_\alpha \setminus U_{\alpha,p}$ is a finite set of closed points of C_α. Thus $C_{\alpha,p}$, complement of $V_p \setminus \{p\}$, is open in C_α.

Definition 3.4.23 For any $p \in S_n$ we define $C_{\alpha,p}$ to be the dense open subset $U_{\alpha,p} \cup \{p\}$ of C_α, equipped with the canonical structure of open subscheme.

Let $p \in S_n$. By Remark 3.4.20 and Lemma 3.4.22, the maps $s_{\gamma\alpha} : C_\gamma \to C_\alpha$, for $\gamma \in \Sigma_p$, glue to a morphism $C_p \to C_{\alpha,p}$.

Definition 3.4.24 For any $p \in S_n$, define $s_p : C_p \to C_{\alpha,p}$ as the glueing of the morphisms $s_{\gamma\alpha} : C_\gamma \to C_\alpha$, for all $\gamma \in \Sigma_p$.

Lemma 3.4.25 *The morphism $s_p : C_p \to C_{\alpha,p}$ is separated.*

Proof. Consider the open immersion $\iota_p : C_{\alpha,p} \to C_\alpha$. By [Liu4, Proposition 3.3.9(e)] it suffices to prove that $\iota_p \circ s_p$ is separated. Since C_α is affine, we only have to show that C_p is separated over Spec k by [Liu4, Exercise 3.3.2]. Let $\Delta_{\alpha,p}$ be the Newton polygon of $\mathcal{F}_{\alpha,p}$. Recall from §3.4.3 that

$$C_p = \bigcup_\beta C_{\beta \circ_{g_p} \alpha} \qquad \text{with} \quad C_\gamma = \operatorname{Spec} \frac{A_m[X_{m+1}^{\pm 1}, Y]}{(\mathcal{F}_\gamma) + \mathfrak{a}_\gamma}, \quad \gamma = \beta \circ_{g_p} \alpha,$$

where β runs through all normal vectors in $\mathbb{Z}_+^2$ of edges of $\Delta_{\alpha,p}$ and the curves $C_{\beta \circ_{g_p} \alpha}$ are glued along their common open $C_{0,\alpha_p} = C_{0,\beta \circ_{g_p} \alpha}$. To avoid confusion, if $\gamma = \beta \circ_{g_p} \alpha$, rename the variables X_{m+1}, Y of $\mathcal{O}_{C_\gamma}(C_\gamma)$ to X_β, Y_β. Since closed immersions are separated and separated morphisms are stable under base changes it suffices to prove that the toric variety $\bigcup_\beta \operatorname{Spec} k[X_\beta^{\pm 1}, Y_\beta] \subset \mathbb{T}_{\Delta_{\alpha,p}}$ is separated. This follows from the classical theory on toric varieties. □

Lemma 3.4.26 *The morphism s_p induces an isomorphism $s_p^{-1}(U_{\alpha,p}) \to U_{\alpha,p}$. In particular, s_p is birational.*

Proof. This result immediately follows from Lemma 3.4.22 as $\Sigma_p \neq \emptyset$. □

Lemma 3.4.27 *The morphism $s_p : C_p \to C_{\alpha,p}$ is proper.*

Proof. By Lemma 3.4.25, the morpshism s_p is separated. We will then prove the lemma via the valuative criterion for properness. Let R be a discrete valuation ring with field of fractions K. We want to prove that any commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "t_p"] \arrow[d, hook] & C_p \arrow[d, "s_p"] \\ \operatorname{Spec} R \arrow[r, "t_\alpha"] \arrow[ur, dashed] & C_{\alpha,p} \end{tikzcd}$$

can be filled in as shown. Let π be a uniformiser of R and let $\omega = (\pi)$ be the closed point of Spec R. Since $C_{\alpha,p} = U_{\alpha,p} \cup \{p\}$ and $s_p^{-1}(U_{\alpha,p}) \to U_{\alpha,p}$ is an isomorphism by Lemma 3.4.26, we can assume $p = t_\alpha(\omega)$. Indeed, if not, then s_p^{-1} is defined on the open dense neighbourhood $U_{\alpha,p}$ of $t_\alpha(\omega)$, that therefore contains the image of t_α. Moreover, t_α can be supposed not constant, otherwise Spec $R \to C_p$ can be defined as the constant morphism of image $t_p((0))$.

Recall that the injective homomorphism $R_\alpha \hookrightarrow k(X_{j_\alpha}, \tilde{Y})$, given by 3.3.1(b), induces an open immersion $C_\alpha \hookrightarrow \operatorname{Spec}(k[X_{j_\alpha},\tilde{Y}]/(\mathcal{F}_\alpha))$. In particular, the local ring $\mathcal{O}_{C_\alpha,p}$, equal to $\mathcal{O}_{C_{\alpha,p},p}$, is naturally isomorphic to the localisation of $k[X_{j_\alpha},\tilde{Y}]/(\mathcal{F}_\alpha)$ at the prime ideal $(\tilde{\mathcal{G}}_p,\tilde{Y}) = (\bar{\mathcal{G}}_p,\tilde{Y})$. From the local homomorphism

$$\tau : \frac{k[X_{j_\alpha},\tilde{Y}]_{(\tilde{\mathcal{G}}_p,\tilde{Y})}}{(\mathcal{F}_\alpha)} \simeq \mathcal{O}_{C_{\alpha,p},p} \xrightarrow{t^\#_{\alpha,\omega}} R$$

we observe that $\operatorname{ord}_\pi(\tilde{\mathcal{G}}_p) > 0$, $\operatorname{ord}_\pi(\tilde{Y}) > 0$. We want to show that neither $\tilde{Y}$ nor $\tilde{\mathcal{G}}_p$ are taken to 0 by τ. Note that $\ker(\tau) \subsetneq (\tilde{\mathcal{G}}_p,\tilde{Y})$, since t_α is not constant. Hence it suffices to prove that $\tau(\tilde{Y}) = 0$ if and only if $\tau(\tilde{\mathcal{G}}_p) = 0$.

Suppose $\tau(\tilde{Y}) = 0$. Then $\tau(f|_\alpha) = 0$ and $\tau(\tilde{\mathcal{G}}_p) = \tau(\bar{\mathcal{G}}_p)$. Recall that $\bar{\mathcal{G}}_p$ is a factor of $f|_\alpha$. Let $h_p \in k[X_{j_\alpha}]$ with $\bar{\mathcal{G}}_p \nmid h_p$ such that $f|_\alpha = h_p \cdot (\bar{\mathcal{G}}_p)^{m_p}$, for some $m_p \in \mathbb{Z}_+$. Note that $\tau(h_p)$ is invertible as $h_p \notin (\tilde{\mathcal{G}}_p,\tilde{Y})$. Since $\tau(f|_\alpha) = 0$ and R is reduced, it follows that $\tau(\tilde{\mathcal{G}}_p) = 0$.

Suppose $\tau(\tilde{\mathcal{G}}_p) = 0$. Let $\mathcal{H}_p \in k[\tilde{Y}]$ be the normal form of $\mathcal{F}_\alpha$ by $\tilde{\mathcal{G}}_p$ with respect to the lexicographic order on $k[X_{j_\alpha},\tilde{Y}]$ given by $X_{j_\alpha} > \tilde{Y}$. Note that $\tau(\mathcal{H}_p) = 0$ as $\tau(\mathcal{F}_\alpha) = 0$, but $\mathcal{H}_p \neq 0$ as $\tilde{\mathcal{G}}_p \nmid \mathcal{F}_\alpha$. Recall that $\tilde{\mathcal{G}}_p - \bar{\mathcal{G}}_p \in \tilde{Y}k[\tilde{Y}]$. Since $\bar{\mathcal{G}}_p$ is a degree 1 factor of $f|_\alpha$ and $\mathcal{F}_\alpha - f|_\alpha \in (\tilde{Y})$, one has $\mathcal{H}_p \in \tilde{Y}k[\tilde{Y}]$. Write $\mathcal{H}_p = \tilde{Y}^t \cdot \mathcal{H}$, for $t \in \mathbb{Z}_+$ and $\mathcal{H} \notin \tilde{Y}k[\tilde{Y}]$. Note that $\tau(\mathcal{H})$ is invertible as $\mathcal{H} \notin (\tilde{\mathcal{G}}_p,\tilde{Y})$. It follows that $\tau(\tilde{Y}) = 0$ since R is reduced and $\tau(\mathcal{H}_p) = 0$.

Hence $\operatorname{ord}_\pi(\tilde{\mathcal{G}}_p), \operatorname{ord}_\pi(\tilde{Y}) \in \mathbb{Z}_+$ and so the affine function

$$\phi : \mathbb{Z}^2 \to \mathbb{Z}, \quad (i,j) \mapsto \operatorname{ord}_\pi \tilde{\mathcal{G}}_p^i \tilde{Y}^j$$

is a non-trivial linear map with a rank 1 kernel spanned by some primitive vector $(\beta_2, -\beta_1) \in \mathbb{Z}_+ \times \mathbb{Z}_-$. Set $\beta = (\beta_1, \beta_2)$ and $\gamma = \beta \circ_{g_p} \alpha$. Then

$$C_\gamma = \operatorname{Spec} \frac{A_{m+1}[Y]}{(\mathcal{F}_\gamma) + \mathfrak{a}_\gamma}$$

and $C_\gamma \subset C_p$ from the definition of C_p (also when $\gamma \notin \Sigma_p$). Hence

(3.10)
$$\begin{tikzcd} K & \frac{A_m[X_{m+1}^{\pm 1},Y]}{(\mathcal{F}_\gamma)+\mathfrak{a}_\gamma} \arrow[l] \arrow[dl, dashed] \\ R \arrow[u] & \frac{A_m[\tilde{Y}]}{(\mathcal{F}_\alpha)+\mathfrak{a}_\alpha}, \arrow[l] \arrow[u] \end{tikzcd}$$

where the ring homomorphism on the right, inducing the map

$$s_{\gamma\alpha} : C_\gamma \xrightarrow{s_p} C_{\alpha,p} \hookrightarrow C_\alpha,$$

is given by $\tilde{Y} \mapsto X_{m+1}^{\delta_2} Y^{\beta_2}$ for $M_\beta = \begin{pmatrix} \delta_1 & \delta_2 \\ \beta_1 & \beta_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. To conclude the proof it suffices to show that the commutative diagram (3.10) can be filled in as shown. Recall

$$\mathcal{F}_{m+1} = X_{m+1}^{\delta_1} Y^{\beta_1} - \mathcal{G}_p \in \mathfrak{a}_\gamma.$$

Then

$$\mathrm{ord}_\pi(X_{m+1}) = \mathrm{ord}_\pi(\tilde{\mathcal{G}}_p^{\beta_2} \tilde{Y}^{-\beta_1}) = \phi((\beta_2, -\beta_1)) = 0$$

and so $\mathrm{ord}_\pi(Y) > 0$ as $\beta \in \mathbb{Z}_+^2$. Thus (3.10) can be filled in as shown. □

Lemma 3.4.28 *If $p \in S_n$ is a regular point of C_n, then s_p is an isomorphism.*

Proof. As p is a regular point of codimension 1, the ring $\mathcal{O}_{C_{\alpha,p},p}$ is normal. Therefore there exists a normal integral open subscheme $U \subseteq C_{\alpha,p}$ containing p. Since s_p is proper birational by Lemma 3.4.27, then so is $s_U : s_p^{-1}(U) \to U$. In particular, $s_p^{-1}(U)$ is integral. It follows from [Liu4, Corollary 4.4.3]) that s_U is an isomorphism. Thus s_p is an isomorphism, since $s_p^{-1}(U_{\alpha,p}) \to U_{\alpha,p}$ is an isomorphism by Lemma 3.4.26. □

Proposition 3.4.29 *For any $\gamma \in \Sigma_{n+1}$, the curve $C_{0,\gamma}$ is dense in C_{n+1}.*

Proof. For any $\gamma \in \Sigma_{n+1}$ recall that $C_{0,\gamma}$ is dense in its closure C_γ. Therefore $C_0 = \bigcup_{\gamma \in \Sigma_{n+1}} C_{0,\gamma} \cup C_0$ is dense in $C_{n+1} = \bigcup_{\gamma \in \Sigma_{n+1}} C_\gamma \cup C_0$. Fix $\gamma \in \Sigma_{n+1}$. It suffices to show that $C_{0,\gamma}$ is dense C_0. But this holds as $\gamma \in \Omega$. □

Definition 3.4.30 Define a surjective function $\psi_n : \Sigma_{n+1} \sqcup \{0\} \to \Sigma_n \sqcup \{0\}$ by $\psi_n(0) = 0, \psi_n|_{\hat{\Sigma}_n} = id_{\hat{\Sigma}_n}, \psi_n(\Sigma_{n+1} \setminus \hat{\Sigma}_n) = \{\alpha\}$.

Let $\gamma \in \Sigma_{n+1} \sqcup \{0\}$ and denote $\alpha_\gamma = \psi_n(\gamma)$. Then the birational map $s_{\gamma\alpha_\gamma}$ has domain of definition C_γ. Indeed, it is trivial when $\gamma = 0$ or $\gamma \in \hat{\Sigma}_n$ while it follows from Remark 3.4.21 if $\gamma \in \Sigma_{S_n}$ and from Lemma 3.4.15 if $\gamma = \tilde{\alpha}$.

Theorem 3.4.31 *There exists a unique morphism $s_n : C_{n+1} \to C_n$ extending the birational maps $s_{\gamma'\alpha'} : C_{\gamma'} \dashrightarrow C_{\alpha'}$ for $\gamma' \in \Sigma_{n+1} \sqcup \{0\}$, $\alpha' \in \Sigma_n \sqcup \{0\}$. In particular,*

$$s_n|_{C_0} : C_0 \xrightarrow{id} C_0 \subseteq C_n, \quad s_n|_{\hat{C}_n} : \hat{C}_n \xrightarrow{id} \hat{C}_n \subseteq C_n,$$

and $s_n|_{C_p} : C_p \xrightarrow{s_p} C_{\alpha,p} \subseteq C_n$, for any $p \in S_n$.

Proof. For any $\gamma \in \Sigma_{n+1} \sqcup \{0\}$ let $\alpha_\gamma = \psi_n(\gamma)$. We observed that the birational maps $s_{\gamma\alpha_\gamma}$ have domain of definition C_γ, and so define morphisms

$$s_\gamma : C_\gamma \xrightarrow{s_{\gamma\alpha_\gamma}} C_{\alpha_\gamma} \subseteq C_n.$$

Note that $s_\gamma|_{C_{0,\gamma}}$ is an open immersion. This fact is trivial when $\gamma \in \hat{\Sigma}_n \sqcup \{0\}$ and follows from Lemmas 3.4.15 and 3.4.22 otherwise.

Recall the definition of the dense open $U_{\gamma\gamma'}$ of C_γ for any $\gamma, \gamma' \in \Omega \sqcup \{0\}$. We want to show that for any $\gamma, \gamma' \in \Sigma_{n+1} \sqcup \{0\}$ and any $\alpha' \in \Sigma_n \sqcup \{0\}$ the maps s_γ and $s_{\gamma'\alpha'} : C_{\gamma'} \dashrightarrow C_{\alpha'} \subseteq C_n$ agree on the intersection of their domains of definition. Let D be the domain of definition of $s_{\gamma'\alpha'}$. Then $D \supseteq U_{\gamma'\alpha'}$. Let $U = D \cap C_\gamma \subseteq C_{n+1}$ and $U_0 = C_{0,\gamma} \cap C_{0,\gamma'} \cap C_{0,\alpha'}$. Since $C_{0,\gamma'} \cap C_{0,\alpha'} \subseteq U_{\gamma'\alpha'}$, one has $U_0 \subseteq D$. Hence U_0 is an open of U, dense by Proposition 3.4.29. Now, U is reduced, C_n is separated and $s_\gamma|_{U_0} = s_{\gamma'\alpha'}|_{U_0}$ by definition. Therefore [Liu4, Proposition 3.3.11] implies the two maps coincide on U, as required.

Thus the morphisms s_γ glue to a morphism $s_n : C_{n+1} \to C_n$ and s_n extends the birational maps $s_{\gamma'\alpha'} : C_\gamma \dashrightarrow C_{\alpha'}$ for $\gamma' \in \Sigma_{n+1} \sqcup \{0\}$, $\alpha' \in \Sigma_n \sqcup \{0\}$. Then the uniqueness follows. □

Definition 3.4.32 Define $s_n : C_{n+1} \to C_n$ to be the birational morphism of k-schemes of Theorem 3.4.31. We call s_n the morphism *resolving* S_n (although $s_n^{-1}(S_n)$ is not necessarily non-singular).

Remark 3.4.33. Let $\gamma, \gamma' \in \Sigma_{n+1} \sqcup \{0\}$ and $\alpha' \in \Sigma_n \sqcup \{0\}$. Suppose there exist open subschemes $V_{\alpha'} \subseteq C_{\alpha'}$, $U_\gamma \subseteq C_\gamma$, $U_{\gamma'} \subseteq C_{\gamma'}$ such that s_n restricts to isomorphisms $U_\gamma \to V_{\alpha'}$, $U_{\gamma'} \to V_{\alpha'}$. Since s_n extends the rational maps $s_{\gamma\alpha'}$, $s_{\gamma'\alpha'}$, the map $s_{\gamma\gamma'}$ is defined on U_γ and induces an isomorphism $U_\gamma \to U_{\gamma'}$. This implies that the opens U_γ, $U_{\gamma'}$ are glued, and so are equal in C_{n+1}.

It follows that if U_1, U_2 are opens of C_{n+1} such that $s_n|_{U_1}$ and $s_n|_{U_2}$ are open immersions, then $s_n|_{U_1 \cup U_2}$ is an open immersion.

3.4.5 Geometric properties

In this subsection we will show that Σ_{n+1} and C_{n+1} satisfy all remaining properties of 3.3.1, i.e. C_n is a projective curve and $C_\gamma \cap C_{\gamma'} = C_{0,\gamma} \cap C_{0,\gamma'}$ for any $\gamma, \gamma \in \Sigma_{n+1} \sqcup \{0\}$, $\gamma \neq \gamma'$. Furthermore, we will prove that the morphism s_n defined in Theorem 3.4.31, is a proper birational morphism with exceptional locus $s_n^{-1}(S_n \cap \mathrm{Sing}(C_n))$.

Consider the principal open $D(\tilde{\mathcal{G}}_{S_n}) \subset C_\alpha$. Note that

$$\mathcal{U} = \{C_{\alpha'} \mid \alpha' \in \hat{\Sigma}_n\} \cup \{C_0\} \cup \{C_{\alpha,p} \mid p \in S_n\} \cup \{D(\tilde{\mathcal{G}}_{S_n})\} \tag{3.11}$$

is an open cover of C_n.

Lemma 3.4.34 *The morphism $s_n : C_{n+1} \to C_n$ is surjective.*

Proof. We want to show that every open in the cover (3.11) is contained in the image of s_n. Recall $s_{\tilde{\alpha}\alpha}(C_{\tilde{\alpha}}) = D(\tilde{\mathcal{G}}_{S_n})$ by Lemma 3.4.15. Moreover, the morphism $s_p : C_p \to C_{\alpha,p}$ is surjective by Lemma 3.4.22 as $\Sigma_p \neq \emptyset$. Then the lemma follows from Theorem 3.4.31. □

Lemma 3.4.35 *For any $p \in S_n$, we have*

$$s_n^{-1}(p) = C_p \setminus C_{0,\alpha_p}, \qquad \textit{and} \qquad s_n^{-1}(C_{\alpha,p}) = C_p.$$

Furthermore, the morphism $s_n^{-1}(C_n \setminus S_n) \to C_n \setminus S_n$ induced by s_n is an isomorphism.

Proof. Let $p \in S_n$. Lemma 3.4.22 shows that

$$s_p^{-1}(p) = \bigcup_{\gamma \in \Sigma_p} (C_\gamma \setminus C_{0,\gamma}) = C_p \setminus C_{0,\alpha_p},$$

where the last equality holds as $C_{0,\gamma} = C_{0,\alpha_p}$ for all $\gamma \in \Sigma_p$. Moreover, $p \notin s_n(C_q)$ for any $q \in S_n$, $q \neq p$, and also $p \notin s_n(C_{\tilde{\alpha}})$ by Lemma 3.4.15. Recall $p \notin C_0$. In particular, $p \notin C_{\alpha'}$ for any $\alpha' \in \hat{\Sigma}_n$, since $C_\alpha \cap C_{\alpha'} \subseteq C_0$ (from our assumptions on C_n). Then $s_n^{-1}(p) = s_p^{-1}(p)$ by Theorem 3.4.31.

Let $U_{S_n} = C_n \setminus S_n$. We want to show that $s_n^{-1}(U_{S_n}) \to U_{S_n}$ is an isomorphism. From above

$$s_n^{-1}(U_{S_n}) = \hat{C}_n \cup C_0 \cup C_{\tilde{\alpha}},$$

as $C_{0,\gamma} \subseteq C_0$ for any $\gamma \in \Sigma_{S_n}$. Note that $s_n|_{\hat{C}_n}$, $s_n|_{C_0}$ and $s_n|_{C_{\tilde{\alpha}}}$ are open immersions by Theorem 3.4.31. Thus $s_n^{-1}(U_{S_n}) \to U_{S_n}$ is an isomorphism from Remark 3.4.33 and Lemma 3.4.34.

Recall that $C_{\alpha,p} \setminus \{p\} = U_{\alpha,p} \subseteq U_{S_n}$ and $s_p^{-1}(U_{\alpha,p}) = C_{0,\alpha_p}$ by Lemma 3.4.22. Moreover, s_p induces an isomorphism $C_{0,\alpha_p} \to U_{\alpha,p}$ by Lemma 3.4.26. Since $s_n^{-1}(U_{S_n}) \to U_{S_n}$ is an isomorphism, $s_n^{-1}(C_{\alpha,p}) = s_p^{-1}(C_{\alpha,p}) = C_p$. □

Theorem 3.4.36 *The morphism $s_n : C_{n+1} \to C_n$ resolving $S_n \subseteq \mathrm{Sing}(\bar{C}_\alpha)$ is a surjective proper birational morphism with exceptional locus contained in $s_n^{-1}(S_n)$. In particular, the curve C_{n+1} is projective.*

Proof. First recall s_n is surjective by Lemma 3.4.34. Consider the open cover $\mathcal{U}$ of C_n introduced in (3.11). As properness is a local property on the codomain, if $s_n^{-1}(U) \to U$ is proper for any $U \in \mathcal{U}$, then s_n is proper. Lemma 3.4.35 implies that $s_n^{-1}(U) \to U$ is an isomorphism except when $U = C_{\alpha,p}$ for some $p \in S_n$. But $s_n^{-1}(C_{\alpha,p}) = C_p$ for any $p \in S_n$ again by Lemma 3.4.35. Hence Lemma 3.4.27 implies that s_n is proper. It follows that the curve C_{n+1} is complete, and then projective, since so is C_n.

Proposition 3.4.29 implies that C_0 is dense in C_{n+1}. Let $U_{S_n} = C_n \setminus S_n$, dense in C_n. Since $C_0 \subseteq s_n^{-1}(U_{S_n})$, the isomorphism $s_n^{-1}(U_{S_n}) \to U_{S_n}$ implies that s_n is birational with exceptional locus contained in $s_n^{-1}(S_n)$. □

Lemma 3.4.37 *Let $s_n : C_{n+1} \to C_n$ be the morphism resolving $S_n \subseteq \mathrm{Sing}(\bar{C}_\alpha)$. Let $p \in S_n$. Then $p \in \mathrm{Reg}(C_n)$ if and only if the exceptional locus of s_n is contained in $s_n^{-1}(S_n \setminus \{p\})$. In that case, $\bar{C}_\gamma$ is regular for all $\gamma \in \Sigma_p$.*

Proof. If $p \in \mathrm{Reg}(C_n)$ then $s_p : C_p \to C_{\alpha,p}$ is an isomorphism by Lemma 3.4.28. Then the exceptional locus of s_n is contained in $s_n^{-1}(S_n \setminus \{p\})$ by Lemma 3.4.35 and Theorem 3.4.31.

Suppose the exceptional locus of s_n is contained in $s_n^{-1}(S_n \setminus \{p\})$. In particular, there exists an open neighbourhood U of p such that $s_n^{-1}(U) \to U$ is an isomorphism. This implies that $s_p : C_p \to C_{\alpha,p}$ is an isomorphism by Theorem 3.4.31 and Lemma 3.4.35. Recall $\Sigma_p \neq \varnothing$. Let $\gamma \in \Sigma_p$ so that $\gamma = \beta \circ_{g_p} \alpha$ with $\beta \in \mathbb{Z}_+^2$. As in §3.4.2, write

$$C_\alpha = \mathrm{Spec}\, \frac{A_m[\tilde{Y}]}{(\mathcal{F}_\alpha) + \mathfrak{a}_\alpha}, \qquad C_\gamma = \mathrm{Spec}\, \frac{A_m[X_{m+1}^{\pm 1}, Y]}{(\mathcal{F}_\gamma) + \mathfrak{a}_\gamma}.$$

Consider the morphism $s_{\gamma\alpha} : C_\gamma \to C_\alpha$ induced by the ring homomorphism taking $\tilde{Y} \mapsto X_{m+1}^{\delta_2} Y^{\beta_2}$, where $\beta = (\beta_1, \beta_2)$ and $\delta_1, \delta_2 \in \mathbb{Z}$ such that $\delta_1\beta_2 - \delta_2\beta_1 = 1$. Recall that $s_{\gamma\alpha}(C_\gamma \setminus C_{0,\gamma}) = \{p\}$ by Lemma 3.4.22. As s_p is an isomorphism, $s_{\gamma\alpha}$ is an open immersion. In particular,

$$s^{\#}_{\gamma\alpha}(U_{\alpha\gamma}) : \mathcal{O}_{C_\alpha}(U_{\alpha\gamma}) \to \mathcal{O}_{C_\gamma}(C_\gamma) \tag{3.12}$$

is an isomorphism, where $U_{\alpha\gamma} = s_{\gamma\alpha}(C_\gamma)$. In fact, $U_{\alpha\gamma} = C_{\alpha,p}$. Then $p \in U_{\alpha\gamma}$ and $\mathfrak{m}_p = (\tilde{\mathcal{G}}_p, \tilde{Y}) + \mathfrak{a}_\alpha$ is the maximal ideal of $\mathcal{O}_{C_\alpha}(U_{\alpha\gamma})$ corresponding to p. Recall $\mathcal{F}_{m+1} = X_{m+1}^{\delta_1} Y^{\beta_1} - \mathcal{G}_p \in \mathfrak{a}_\gamma$. Then $\mathfrak{m}_p \mathcal{O}_{C_\gamma}(C_\gamma) \subseteq (\mathcal{F}_\gamma, Y) + \mathfrak{a}_\gamma$, which implies the equality, since $\mathfrak{m}_p \mathcal{O}_{C_\gamma}(C_\gamma)$ has to be maximal. It follows that the ring isomorphism (3.12) induces

$$k \simeq \frac{D_\alpha}{(\bar{\mathcal{G}}_p)} \simeq \frac{A_m[\tilde{Y}]_{(\tilde{\mathcal{G}}_p, \tilde{Y}) + \mathfrak{a}_\alpha}}{(\tilde{\mathcal{G}}_p, \tilde{Y}) + \mathfrak{a}_\alpha} \xrightarrow{\sim} \frac{A_m[X_{m+1}^{\pm 1}, Y]_{(\mathcal{F}_\gamma, Y) + \mathfrak{a}_\gamma}}{(\mathcal{F}_\gamma, Y) + \mathfrak{a}_\gamma} \simeq \frac{D_\gamma}{(f|_\gamma)}.$$

Therefore $\bar{C}_\gamma = \operatorname{Spec} D_\gamma/(f|_\gamma) \simeq \operatorname{Spec} k$, and so is regular. In particular, the point $w = s_n^{-1}(p)$ is a non-singular point of C_{n+1} (Remark 3.4.4). Thus C_n is regular at p, as w is not in the exceptional locus of s_n. □

Proposition 3.4.38 *Let $s_n : C_{n+1} \to C_n$ be the morphism resolving S_n. Then $S_n \subset \operatorname{Reg}(C_n)$ if and only if s_n is an isomorphism. In that case, $\bar{C}_\gamma$ is regular for all $\gamma \in \Sigma_{S_n}$.*

Proof. The proposition follows from Lemma 3.4.37. □

Recall from 3.3.1 that $C_\gamma \cap C_{\gamma'} = C_{0,\gamma} \cap C_{0,\gamma'}$ for any $\gamma, \gamma' \in \Sigma_n \sqcup \{0\}$, $\gamma \neq \gamma'$. We now want to show this fact is true for Σ_{n+1} as well.

Proposition 3.4.39 *For any $\gamma, \gamma' \in \Sigma_{n+1} \sqcup \{0\}$, if $\gamma \neq \gamma'$, then*

$$C_\gamma \cap C_{\gamma'} = C_{0,\gamma} \cap C_{0,\gamma'}.$$

Proof. Let $\gamma, \gamma' \in \Sigma_{n+1} \sqcup \{0\}$, $\gamma \neq \gamma'$. Recall $s_n^{-1}(C_0) = C_0$ and that s_n restricts to the identity $C_0 \to C_0$. Hence it suffices to show that

$$s_n(C_\gamma) \cap s_n(C_{\gamma'}) = s_n(C_{0,\gamma}) \cap s_n(C_{0,\gamma'}).$$

Consider the open $D(\tilde{\mathcal{G}}_{S_n}) \subseteq C_\alpha$ and let $U_{\alpha,S_n} = D(\tilde{\mathcal{G}}_{S_n}) \cap C_{0,\alpha}$. If both γ and γ' belong to $\tilde{\Sigma}_n \sqcup \{0\}$, we can conclude by the hypothesis on C_n (see 3.3.1), since $s_n(C_{\tilde{\alpha}}) = D(\tilde{\mathcal{G}}_{S_n})$ and $s_n(C_{0,\tilde{\alpha}}) = U_{\alpha,S_n}$ by Lemma 3.4.15. Then assume $\gamma \in \Sigma_p$ for some $p \in S_n$. Lemma 3.4.22 shows that $s_n(C_\gamma) = C_{\alpha,p}$ and $s_n(C_{0,\gamma}) = U_{\alpha,p}$. If $\gamma' \in \Sigma_p$ as well, then $C_{0,\gamma} = C_\gamma \cap C_{\gamma'} = C_{0,\gamma'}$ from Remark 3.4.20. If $\gamma' \in \Sigma_q$ for some $q \in S_n$, $q \neq p$, then

$$s_n(C_\gamma) \cap s_n(C_{\gamma'}) = C_{\alpha,p} \cap C_{\alpha,q} = U_{\alpha,p} \cap U_{\alpha,q} = s_n(C_{0,\gamma}) \cap s_n(C_{0,\gamma'}).$$

If $\gamma' = \tilde{\alpha}$, then

$$s_n(C_\gamma) \cap s_n(C_{\tilde{\alpha}}) = C_{\alpha,p} \cap D(\tilde{\mathcal{G}}_{S_n}) = U_{\alpha,p} \cap U_{\alpha,S_n} = s_n(C_{0,\gamma}) \cap s_n(C_{0,\tilde{\alpha}}).$$

Finally, suppose $\gamma' \in \hat{\Sigma}_n \sqcup \{0\}$. Note that $U_{\alpha,p} = C_{\alpha,p} \cap C_{0,\alpha}$. Then

$$s_n(C_\gamma) \cap s_n(C_{\gamma'}) = C_{\alpha,p} \cap C_{\gamma'} = U_{\alpha,p} \cap C_{0,\gamma'} = s_n(C_{0,\gamma}) \cap s_n(C_{0,\gamma'}),$$

as $C_\alpha \cap C_{\gamma'} = C_{0,\alpha} \cap C_{0,\gamma'}$ from the inductive hypothesis on C_n. □

3.5 A generalised Baker's model

Let k be an algebraically closed field. Let $f \in k[x_1^{\pm 1}, y^{\pm 1}]$ be a Laurent polynomial defining a smooth curve $C_0 : f = 0$ over $\mathbb{G}_m^2$. Let Δ be the Newton polygon of f and let C_1 be the completion of C_0 with respect to Δ.

Definition 3.5.1 Let C_0 and C_1 as above. A *simple Baker's resolution of* C_0 is a sequence of proper birational morphisms of k-schemes

$$\dots \xrightarrow{s_{n+1}} C_{n+1} \xrightarrow{s_n} C_n \xrightarrow{s_{n-1}} \dots \xrightarrow{s_1} C_1 \tag{3.13}$$

where the curves C_n/k are constructed from subsets $\Sigma_n \subset \Omega$ as described in 3.3.1 and the maps s_n are the morphisms resolving sets $S_n \subseteq \mathrm{Sing}(\bar{C}_\alpha)$, for some $\alpha \in \Sigma_n$.

We have showed how to construct simple Baker's resolutions of C_0 recursively for any choice of sets $S_n \subseteq \mathrm{Sing}(\bar{C}_\alpha)$, $\alpha \in \Sigma_n$. We want to prove that for any simple Baker's resolution of C_0, the sets S_n are eventually empty. Thus simple Baker's resolutions can be used to desingularise C_1.

Definition 3.5.2 Recall k is supposed algebraically closed. Let C/k be a smooth projective curve. A smooth curve $\tilde{C}/k$ is a *generalised Baker's model* of C if there exist a smooth curve $\tilde{C}_0 \subset \mathbb{G}_m^2$, birational to C, and a simple Baker's resolution

$$\dots \xrightarrow{s_{n+1}} C_{n+1} \xrightarrow{s_n} C_n \xrightarrow{s_{n-1}} \dots \xrightarrow{s_1} C_1$$

of $\tilde{C}_0$ so that $\tilde{C} = C_n$ for some $n \in \mathbb{Z}_+$. In this case we say that $\tilde{C}$ is a generalised Baker's model of C *with respect to* $\tilde{C}_0$. Note that $\tilde{C}$ is a *model of C over k*, i.e. $\tilde{C} \simeq C$, by Lemma B.1.3.

For the remainder of the section we fix a simple Baker's resolution of C_0

$$\dots \xrightarrow{s_{n+1}} C_{n+1} \xrightarrow{s_n} C_n \xrightarrow{s_{n-1}} \dots \xrightarrow{s_1} C_1$$

where the maps s_n are the morphisms resolving $S_n \subseteq \mathrm{Sing}(\bar{C}_\alpha)$, $\alpha \in \Sigma_n$.

Theorem 3.5.3 *There exists $h \in \mathbb{Z}_+$ such that $S_n \subset \mathrm{Reg}(C_n)$ for all $n \geq h$.*

Proof. Let $n \in \mathbb{Z}_+$ and consider $s_n : C_{n+1} \twoheadrightarrow C_n$ resolving S_n. As birational morphism between projective curves, s_n is finite ([Liu4, Lemma 7.3.10]). By Theorem 3.4.36 we have an exact sequence of sheaves

$$0 \to \mathcal{O}_{C_n} \to s_n^* \mathcal{O}_{C_{n+1}} \to \mathcal{S}_n \to 0,$$

where $\mathcal{S}_n$ is a skyscraper sheaf with support contained in S_n. Denote the arithmetic genus of a curve X/k by $p_a(X)$. Then we get

$$p_a(C_{n+1}) = p_a(C_n) - \dim_k H^0(C_n, \mathcal{S}_n). \tag{3.14}$$

Let r be the number of irreducible components of C_0. For any n, recall there is a natural open immersion $C_0 \hookrightarrow C_n$ with dense image. Therefore the curve C_n is reduced and has r irreducible components $X_1, \ldots, X_r$. Let $i = 1, \ldots, r$ and let X_i' be the normalisation of X_i. Then $H^0(X_i', \mathcal{O}_{X_i'}) = k$ as k is algebraically closed ([Liu4, Corollary 3.3.21]). Hence $p_a(X_i') \geq 0$. Therefore $p_a(C_n) \geq 1 - r$ by [Liu4, Proposition 7.5.4]. It follows from (3.14) that $(p_a(C_n))_{n \in \mathbb{Z}_+}$ is a decreasing sequence in $\mathbb{Z}$ bounded below by $1 - r$. Hence it is eventually constant, i.e. there exists $h \in \mathbb{Z}_+$ such that $p_a(C_{n+1}) = p_a(C_n)$ for all $n \geq h$. From (3.14), we have $H^0(C_n, \mathcal{S}_n) = 0$, that implies $\mathcal{S}_n = 0$, as it is a skyscraper sheaf. Hence $\mathcal{O}_{C_n} \simeq s_n^* \mathcal{O}_{C_{n+1}}$. It follows that s_n is an isomorphism since it is affine. Thus Lemma 3.4.38 shows that $S_n \subset \mathrm{Reg}(C_n)$ for any $n \geq h$. □

Remark 3.5.4. Let $n \in \mathbb{Z}_+$. In Remark 3.4.4 we noticed that any singular point of C_n is the image of a point in $\mathrm{Sing}(\bar{C}_\alpha)$ under the immersion $\bar{C}_\alpha \hookrightarrow C_n$, for some $\alpha \in \Sigma_n$. Therefore if C_n is singular, we can choose $S_n \subseteq \mathrm{Sing}(\bar{C}_\alpha)$, $\alpha \in \Sigma_n$, such that $S_n \cap \mathrm{Sing}(C_n) \neq \varnothing$.

Theorem 3.5.5 *Let $N = \{n \in \mathbb{Z}_+ \mid C_n$ is singular$\}$. Suppose $S_n \cap \mathrm{Sing}(C_n) \neq \varnothing$ for all $n \in N$. Then N is finite. In other words, there exists $h \in \mathbb{Z}_+$ so that C_n is regular for all $n \geq h$. In particular, for any $n \geq h$, the curve C_n is a generalised Baker's model of the smooth completion of C_0.*

Proof. The result follows from Theorem 3.5.3. □

Remark 3.5.6. The arithmetic genus of the curve C_1 is $p_a(C_1) = |\Delta(\mathbb{Z})|$, where $|\Delta(\mathbb{Z})|$ is the number of internal integer points of the Newton polygon of C_0 ([Dok, Remark 2.6(d)]). Therefore it can be explicitly computed. Equation (3.14) gives a recursive way to calculate the arithmetic genus of the following curves C_n.

By choosing the sets S_n as in Theorem 3.5.5, we would eventually compute the genus g of the smooth completion of C_0. Furthermore, if $h \in \mathbb{Z}_+$ is as in Theorem 3.5.5 then $g \leq |\Delta(\mathbb{Z})| - h$. Hence the number of steps needed to desingularise C_1 via a simple Baker's resolution is $\leq |\Delta(\mathbb{Z})|$.

Lemma 3.5.7 *For any $n \in \mathbb{Z}_+$,*

$$C_n \setminus C_0 = \bigsqcup_{\gamma \in \Sigma_n} C_\gamma \setminus C_{0,\gamma} = \bigsqcup_{\gamma \in \Sigma_n} \bar{C}_\gamma.$$

Proof. From 3.3.1, for any $\gamma, \gamma' \in \Sigma_n \sqcup \{0\}$, one has $C_\gamma \cap C_{\gamma'} = C_{0,\gamma} \cap C_{0,\gamma'}$. This implies that if $\gamma \in \Sigma_n$ then $C_\gamma \cap C_0 = C_{0,\gamma}$ and $C_\gamma \cap C_{\gamma'} \subseteq C_0$ for every $\gamma' \in \Sigma_n$, $\gamma' \neq \gamma$. The lemma follows. □

Theorem 3.5.8 *There exists $h \in \mathbb{Z}_+$ such that $S_n = \varnothing$ for all $n \geq h$.*

Proof. By Theorem 3.5.3 there exists $h' \in \mathbb{Z}_+$ such that $S_n \subset \mathrm{Reg}(C_n)$ for all $n \geq h'$. Let $n \geq h'$. For any $\Gamma \subseteq \Sigma_n$ let $N(\Gamma)$ be the number of points of $C_n \setminus C_0$ which are singular on $\bar{C}_\gamma$ for some $\gamma \in \Gamma$. Note that by Lemma 3.5.7, one has $N(\Gamma) = \sum_{\gamma\in\Gamma} N(\{\gamma\})$.

Let $\alpha \in \Sigma_n$ such that $S_n \subseteq \mathrm{Sing}(\bar{C}_\alpha)$. Since $C_{\tilde{\alpha}}$ embeds in C_α via s_n and $S_n = \bar{C}_\alpha \setminus s_n(\bar{C}_{\tilde{\alpha}})$ by Lemma 3.4.15, we have $N(\tilde{\Sigma}_n) = N(\Sigma_n) - |S_n|$. On the other hand, $N(\Sigma_{S_n}) = 0$ by Proposition 3.4.38, as $S_n \subset \mathrm{Reg}(C_n)$. Hence

$$N(\Sigma_{n+1}) = N(\Sigma_{S_n}) + N(\tilde{\Sigma}_n) = N(\Sigma_n) - |S_n| \leq N(\Sigma_n).$$

Then $N(\Sigma_n)_{n\geq h'}$ forms a decreasing sequence bounded below by 0. Thus it is eventually constant, i.e. there exists $h \in \mathbb{Z}_+$ such that $N(\Sigma_{n+1}) = N(\Sigma_n)$ for all $n \geq h$. But we saw above that this happens only if $S_n = \varnothing$. □

Definition 3.5.9 For any $n \in \mathbb{Z}_+$, the curve C_n is said *outer regular* if $\bar{C}_\gamma$ is regular for any $\gamma \in \Sigma_n$. In other words, C_n is outer regular if the closed subset $C_n \setminus C_0$ of C_n, equipped with the structure of closed subscheme coming from Lemma 3.5.7, is regular.

Note that from Remark 3.4.4, if C_n is outer regular, then it is regular.

Theorem 3.5.10 *Suppose $S_n \neq \varnothing$ for all $n \in \mathbb{Z}_+$ such that C_n is not outer regular. Then there exists $h \in \mathbb{Z}_+$ so that for all $n \geq h$ the closed subschemes $\bar{C}_\gamma$ are regular for all $\gamma \in \Sigma_n$. In particular, the curve C_h is an outer regular generalised Baker's model of the smooth completion of C_0.*

Proof. The result follows from Theorem 3.5.8. □

Corollary 3.5.11 *Every smooth projective curve defined over an algebraically closed field k admits an outer regular generalised Baker's model.*

Proof. By Corollary B.1.4, for any smooth projective curve C there exists a smooth curve $C_0 \subset \mathbb{G}_m^2$ birational to C. Construct a simple Baker's resolution (3.13) of C_0 recursively by choosing $S_n \neq \varnothing$ whenever C_n is not outer regular. Theorem 3.5.10 concludes the proof. □

Lemma 3.5.12 *Let $n \in \mathbb{Z}_+$. For any $\gamma \in \Sigma_n$ we have a natural bijection*

$$\mathrm{Reg}(\bar{C}_\gamma) \xleftrightarrow{1:1} \{\textit{simple roots of } f|_\gamma \textit{ in } k^\times\}.$$

Proof. For any $\gamma \in \Sigma_n$, we have

$$\mathrm{Reg}\Big(\mathrm{Spec}\,\frac{k[X_{j_\gamma}^{\pm1}]}{(f|_\gamma)}\Big) \xleftrightarrow{1:1} \{\text{simple roots of } f|_\gamma \text{ in } k^\times\}.$$

We will prove by induction on n that $\mathrm{Reg}(\bar{C}_\gamma) = \mathrm{Reg}\big(\mathrm{Spec}\, k[X_{j_\gamma}^{\pm1}]/(f|_\gamma)\big)$. If $n = 1$, the statement follows since $D_\gamma = k[X_{j_\gamma}^{\pm1}]$ for all $\gamma \in \Sigma_1$. Suppose $n > 0$ and $\gamma \in \Sigma_{n+1}$. Let $s_n : C_{n+1} \to C_n$ be the morphism resolving $S_n \subseteq \mathrm{Sing}(\bar{C}_\alpha)$, for $\alpha \in \Sigma_n$. By Definition 3.4.17 the result follows from the inductive hypothesis except when either $\gamma = \tilde{\alpha}$ or $\gamma \in \Sigma_{S_n}$. If $\gamma \in \Sigma_{S_n}$, then $D_\gamma = k[X_{j_\gamma}^{\pm1}]$ by Lemma 3.4.10, so $\bar{C}_\gamma = \mathrm{Spec}\, k[X_{j_\gamma}^{\pm1}]/(f|_\gamma)$. If $\gamma = \tilde{\alpha}$, then $\bar{C}_\gamma = \bar{C}_\alpha \setminus S_n$ by Lemma 3.4.15. Then $\mathrm{Reg}(\bar{C}_\gamma) = \mathrm{Reg}(\bar{C}_\alpha)$. Thus $\mathrm{Reg}(\bar{C}_\gamma) = \mathrm{Reg}\big(\mathrm{Spec}\, k[X_{j_\gamma}^{\pm1}]/(f|_\gamma)\big)$ since $j_{\tilde{\alpha}} = j_\alpha$ and $f|_{\tilde{\alpha}} = f|_\alpha$. □

Theorem 3.5.13 *Let $n \in \mathbb{Z}_+$. Suppose C_n is outer regular. Then we have a natural bijection*

$$C_n(k)\setminus C_0(k) \xleftrightarrow{1:1} \bigsqcup_{\gamma \in \Sigma_n} \{\textit{simple roots of } f|_\gamma \textit{ in } k^\times\}.$$

Proof. Lemma 3.5.7 shows that $C_n \setminus C_0 = \bigsqcup_{\gamma \in \Sigma_n} \bar{C}_\gamma$. Thus Lemma 3.5.12 concludes the proof. □

We conclude the section with the following two lemmas, proving that for any $n \in \mathbb{Z}_+$ the unions in Definition 3.4.17 are all disjoint. This fact is particularly useful in applications: together with Proposition 3.4.39 it implies the points in $C_\gamma \setminus C_{0,\gamma}$ for $\gamma \in \Sigma_{S_n} \cup \{\tilde{\alpha}\}$ are not visible on $\hat{C}_n$.

Recall the partial order $<$ on Ω given in Definition 3.2.4.

Lemma 3.5.14 *Let $n \in \mathbb{Z}_+$. For any $\gamma, \gamma' \in \Sigma_n$, neither $\gamma < \gamma'$ nor $\gamma' > \gamma$.*

Proof. We are going to prove the lemma by induction on n. If $n = 1$ the result is trivial. Suppose $n > 0$ and let $\alpha \in \Sigma_n$ such that $S_n \subseteq \mathrm{Sing}(\bar{C}_\alpha)$. Suppose by contradiction there exist $\gamma, \gamma' \in \Sigma_{n+1}$ such that $\gamma < \gamma'$. By definition $\Sigma_{n+1} = \tilde{\Sigma}_{S_n} \cup \hat{\Sigma}_n$, where $\tilde{\Sigma}_{S_n} = \Sigma_{S_n} \cup \{\tilde{\alpha}\}$. Let $m \in \mathbb{Z}_+$ such that $\alpha \in \Omega_m$. Then $\alpha' \in \Omega_{m+1}$ for any $\alpha' \in \tilde{\Sigma}_{S_n}$. In particular, γ and γ' cannot be both in $\tilde{\Sigma}_{S_n}$. In fact, by inductive hypothesis, either $\gamma \in \tilde{\Sigma}_{S_n}$ and $\gamma' \in \hat{\Sigma}_n$ or viceversa. Suppose $\gamma \in \tilde{\Sigma}_{S_n}$. Then $\alpha < \gamma < \gamma'$. But this gives a contradiction since $\alpha, \gamma' \in \Sigma_n$. Suppose $\gamma' \in \tilde{\Sigma}_{S_n}$. Then α is the unique element of Ω_m such that $\alpha < \gamma'$. In particular, $\gamma \leq \alpha$. But $\gamma \neq \alpha$ since $\gamma \in \hat{\Sigma}_n$. Thus $\gamma < \alpha$, contradicting the inductive hypothesis on Σ_n. □

Lemma 3.5.15 *Let $n \in \mathbb{Z}_+$ and let $\alpha \in \Sigma_n$ such that $S_n \subseteq \mathrm{Sing}(\bar{C}_\alpha)$. Then the sets $\hat{\Sigma}_n$, $\{\tilde{\alpha}\}$, and Σ_p, for $p \in S_n$, are pairwise disjoint.*

Proof. Let $p \in S_n$. First note that for every $\gamma \in \Sigma_p$ and $\gamma' \in \bigcup_{q \in S_n \setminus \{p\}} \Sigma_q \cup \{\tilde{\alpha}\}$, the images of C_γ and $C_{\gamma'}$ under s_n are different. Then $\gamma \neq \gamma'$. It remains to show that if $\gamma \in \Sigma_p \cup \{\tilde{\alpha}\}$ and $\alpha' \in \hat{\Sigma}_n$, then $\gamma \neq \alpha'$. Note that $\gamma > \alpha$. Therefore if $\gamma = \alpha'$ then $\alpha' > \alpha$, where both α and α' are elements of Σ_n. But this is not possible by Lemma 3.5.14. □

3.6 Simultaneous resolution of different charts

Let k be an algebraically closed field and let $f \in k[x_1^{\pm 1}, y^{\pm 1}]$ be a Laurent polynomial defining a smooth curve $C_0 : f = 0$ over $\mathbb{G}_m^2$. Let C_1 be the completion of C_0 with respect to its Newton polygon. In the previous sections we showed that we can construct a sequence of proper birational morphisms

$$\ldots \overset{s_{n+1}}{\twoheadrightarrow} C_{n+1} \overset{s_n}{\twoheadrightarrow} C_n \overset{s_{n-1}}{\twoheadrightarrow} \ldots \overset{s_1}{\twoheadrightarrow} C_1,$$

where the curves C_n/k are constructed from sets $\Sigma_n \subseteq \Omega$ as described in 3.3.1 and the maps s_n are the morphisms resolving $S_n \subset \mathrm{Sing}(\bar{C}_{\alpha_n})$ for $\alpha_n \in \Sigma_n$.

Let $n \in \mathbb{Z}_+$. Note that, once we have chosen the polynomials $\tilde{\mathcal{G}}_p$ for any $p \in S_n$, the construction of $\Sigma_{n+1} \setminus \hat{\Sigma}_n$ only depends on α_n and S_n by Lemma 3.5.15. Suppose $\alpha_{n+1} \in \hat{\Sigma}_n$. Then

$$\Sigma_{n+2} = \Sigma_{S_n} \cup \Sigma_{S_{n+1}} \cup \{\tilde{\alpha}_n, \tilde{\alpha}_{n+1}\} \cup (\Sigma_n \setminus \{\alpha_n, \alpha_{n+1}\}).$$

Thus Σ_{n+2} would have been defined in the same way if, instead of resolving S_n first and then S_{n+1}, we had resolved S_{n+1} first and then S_n. In other words, the construction of Σ_{n+2}, and so of C_{n+2}, from Σ_n does not depend on the order of resolution of S_n and S_{n+1}.

In this section we will show that from our construction we can *resolve* points coming from different charts simultaneously. More precisely, we will explain how to construct a sequence as in §3.3 where the morphisms s_n resolve finite sets of points $S_n \subseteq \bigsqcup_{\alpha \in \Sigma_n} \operatorname{Sing}(\bar{C}_\alpha)$. Note that by Lemma 3.5.7 we can identify the points in S_n with points of C_n via the immersions $\bar{C}_\alpha \hookrightarrow C_n$.

Suppose that, for some $n \in \mathbb{Z}_+$, we have constructed $\Sigma_n \subset \Omega$ and C_n as in 3.3.1. Let $S_n \subseteq \bigsqcup_{\alpha \in \Sigma_n} \operatorname{Sing}(\bar{C}_\alpha)$. Denote $S_{n,\alpha} = S_n \cap \bar{C}_\alpha$ for any $\alpha \in \Sigma_n$. Consider the subset $\Sigma_{n,S_n} := \{\alpha \in \Sigma_n \mid S_{n,\alpha} \neq \varnothing\}$ of Σ_n and order its elements $\alpha_0, \alpha_1, \ldots, \alpha_h$. For each $i = 0, \ldots, h$ we can recursively construct the morphism $s_{n+i} : C_{n+i+1} \to C_{n+i}$ resolving $S_{n,\alpha_i} \subseteq \operatorname{Sing}(\bar{C}_{\alpha_i})$ as described in §3.4. Indeed $\alpha_0 \in \Sigma_n$ and $\alpha_i \in \Sigma_{n+i}$ since

$$\alpha_i \in \Sigma_n \setminus \{\alpha_0, \ldots, \alpha_{i-1}\} \subseteq \hat{\Sigma}_{n+i-1} \qquad \text{for any } i \geq 1.$$

Therefore from the observation made at the beginning of the section

$$\begin{aligned}\Sigma_{n+h+1} &= \bigcup_{i=0}^{h} \Sigma_{S_{n,\alpha_i}} \cup \{\tilde{\alpha}_0, \ldots, \tilde{\alpha}_h\} \cup (\Sigma_n \setminus \{\alpha_0, \ldots, \alpha_h\}) \\ &= \bigcup_{\alpha \in \Sigma_{n,S_n}} (\Sigma_{S_{n,\alpha}} \cup \{\tilde{\alpha}\}) \cup (\Sigma_n \setminus \Sigma_{n,S_n}).\end{aligned}$$

In particular, Σ_{n+h+1} is independent of the order chosen for the elements in Σ_{n,S_n}. This approach eventually constructs a complete curve C_{n+h+1} and a surjective birational morphism

$$C_{n+h+1} \xrightarrow{s_{n+h} \circ s_{n+h-1} \circ \cdots \circ s_n} C_n,$$

with exceptional locus equal to the inverse image of $S_n \cap \operatorname{Sing}(C_n)$. This morphism does not depend on the order chosen for the elements α_i of Σ_{n,S_n}. Indeed by Theorem 3.4.31 it is the unique morphism extending the birational maps $s_{\gamma\alpha} : C_\gamma \dashrightarrow C_\alpha$ for $\gamma \in \Sigma_{n+h+1} \sqcup \{0\}$ and $\alpha \in \Sigma_n \sqcup \{0\}$.

Definition 3.6.1 We will say that $s_{n+h} \circ \cdots \circ s_n$ is the morphism *resolving* the finite set $S_n \subseteq \bigsqcup_{\alpha \in \Sigma_n} \operatorname{Sing}(\bar{C}_\alpha)$.

We can then redefine $\Sigma_{n+1} := \Sigma_{n+h+1}$ and $C_{n+1} := C_{n+h+1}$ to see that we can construct finite subsets $\Sigma_n \subset \Omega$ and projective curves C_n/k as described in 3.3.1 and a sequence of proper birational morphisms

$$\ldots \overset{s_{n+1}}{\twoheadrightarrow} C_{n+1} \overset{s_n}{\twoheadrightarrow} C_n \overset{s_{n-1}}{\twoheadrightarrow} \ldots \overset{s_1}{\twoheadrightarrow} C_1,$$

where the maps $s_n : C_{n+1} \to C_n$ are the morphisms resolving freely chosen $S_n \subseteq \bigsqcup_{\alpha \in \Sigma_n} \operatorname{Sing}(\bar{C}_\alpha)$.

Definition 3.6.2 Let C_0 and C_1 as above. A *Baker's resolution of* C_0 is a sequence of proper birational morphisms of k-schemes

$$\dots \xrightarrow{s_{n+1}} C_{n+1} \xrightarrow{s_n} C_n \xrightarrow{s_{n-1}} \dots \xrightarrow{s_1} C_1$$

where the curves C_n/k are constructed from subsets $\Sigma_n \subset \Omega$ as indicated in 3.3.1 and the maps s_n are the morphisms resolving sets $S_n \subseteq \bigsqcup_{\alpha \in \Sigma_n} \mathrm{Sing}(\bar{C}_\alpha)$.

Simple Baker's resolutions are Baker's resolution. In fact, from what discussed in this section, Baker's resolutions of C_0 are just contraptions of simple Baker's resolutions. Hence the results in §3.5 extends to Baker's resolutions. Let us explicitly restate Theorem 3.5.8 in light of the terminology introduced in the current section as an example.

Theorem 3.6.3 *For any Baker's resolution of* C_0 *given as in Definition 3.6.2, there exists* $h \in \mathbb{Z}_+$ *such that* $S_n = \varnothing$ *for any* $n \geq h$.

Baker's resolutions are not really a new concept, but rather a more general point of view which will be useful in the next section, where we tackle the case of a non-algebraically closed base field.

3.7 The case of non-algebraically closed base field

In this section let k be a perfect field with algebraic closure $\bar{k}$. Denote by G_k the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$. Let $f \in k[x_1^{\pm 1}, y^{\pm 1}]$ such that $C_{0,k} : f = 0$ is a smooth curve defined over $\mathbb{G}^2_{m,k}$. Set $C_0 = C_{0,k} \times_k \bar{k}$. In the previous section we showed how to construct a sequence of proper birational morphisms of $\bar{k}$-schemes

$$\dots \xrightarrow{s_{n+1}} C_{n+1} \xrightarrow{s_n} C_n \xrightarrow{s_{n-1}} \dots \xrightarrow{s_1} C_1,$$

called Baker's resolution of C_0, where the curves $C_n/\bar{k}$ are equipped with canonical open immersions $\iota_n : C_0 \hookrightarrow C_n$ such that $s_n \circ \iota_{n+1} = \iota_n$. Suppose that for any $n \in \mathbb{Z}_+$ one has $G_k \subseteq \mathrm{Aut}(C_n)$ and $s_n \circ \sigma = \sigma \circ s_n$ for all $\sigma \in G_k$. Then, from the universal property of quotient schemes, one has an induced sequence of proper birational morphisms of k-schemes

$$\dots \xrightarrow{s_{n+1,k}} C_{n+1,k} \xrightarrow{s_{n,k}} C_{n,k} \xrightarrow{s_{n-1,k}} \dots \xrightarrow{s_{1,k}} C_{1,k}.$$

where the curves $C_{n,k} := C_n/G_k$ are defined over k. Furthermore, the morphisms ι_n induce open immersions $\iota_{n,k} : C_{0,k} \hookrightarrow C_{n,k}$ such that $s_{n,k} \circ \iota_{n+1,k} = \iota_{n,k}$. In fact, $C_n \simeq C_{n,k} \times_k \bar{k}$ and the quotient morphism $C_n \to C_{n,k}$ is the canonical projection. Then C_n is smooth if and only if so is $C_{n,k}$.

The argument above motivates the subject of this section, which is constructing a Baker's resolution of C_0 such that $G_k \subseteq \mathrm{Aut}(C_n)$ and s_n is Galois-invariant for any $n \in \mathbb{Z}_+$. The following definition extends Definitions 3.5.2, 3.5.9 to the case of general perfect fields.

Definition 3.7.1 Let C/k be a smooth projective curve. A curve $\tilde{C}/k$ is a *generalised Baker's model* of C if $\tilde{C} \simeq C$ and there exists a smooth curve $\tilde{C}_{0,k}/k$ such that the base extended curve $\tilde{C} \times_k \bar{k}$ is a generalised Baker's model of $C \times_k \bar{k}$ with respect to $\tilde{C}_{0,k} \times_k \bar{k}$. Furthermore, a generalised Baker's model $\tilde{C}$ of C is *outer regular* if $\tilde{C} \times_k \bar{k}$ is outer regular.

Let us first describe a group action of G_k on Ω. Let $\alpha = (v, T) \in \Omega$ and $\sigma \in G_k$. Let $m \in \mathbb{Z}_+$ such that $\alpha \in \Omega_m$ and write $T = (g_2, \dots, g_m)$ for Laurent polynomials $g_i \in \bar{k}[x_1^{\pm 1}, y^{\pm 1}]$. Set $T^\sigma = (g_2^\sigma, \dots, g_m^\sigma)$ and define $(v, T)^\sigma = (v, T^\sigma)$. Recall

$$C_{0,\alpha} = \operatorname{Spec} \frac{\bar{k}[x_1^{\pm 1}, \dots, x_m^{\pm 1}, y^{\pm 1}]}{(f_1, f_2, \dots, f_m)}$$

with $f_1 = f \in k[x_1^{\pm 1}, y^{\pm 1}]$ and $f_i = x_i - g_i \in \bar{k}[x_1^{\pm 1}, y^{\pm 1}]$ for $i \geq 2$. Hence $C_{0,\alpha^\sigma} = C_{0,\alpha}^\sigma$. Then C_{0,α^σ} is dense in C_0 and so $\alpha^\sigma \in \Omega$. Thus the element α^σ is set as the image of α under the action of σ. The next lemma follows.

Lemma 3.7.2 *Let $\sigma \in G_k$ and $\alpha \in \Omega$. Let $m \in \mathbb{Z}_+$ such that $\alpha \in \Omega_m$. If $\gamma = \beta \circ_g \alpha$, for some primitive vector $\beta \in \mathbb{N} \times \mathbb{Z}_+$ and $g \in \bar{k}[x_1, \dots, x_m, y]$ then $\gamma^\sigma = \beta \circ_{g^\sigma} \alpha^\sigma$.*

We will show that if the morphisms s_n resolve Galois-invariant sets of points for any $n \in \mathbb{Z}_+$, the curves C_n can be constructed from subsets $\Sigma_n \subset \Omega$ with the properties of 3.3.1 and the following additional one:

(d) The action of G_k on Ω restricts to Σ_n. Furthermore, for any $\sigma \in G_k$ and any $\alpha \in \Sigma_n$, we have $M_{\alpha^\sigma} = M_\alpha$, $j_{\alpha^\sigma} = j_\alpha$, $\mathcal{F}_{\alpha^\sigma} = \mathcal{F}_\alpha^\sigma$.

In particular, if (d) holds for $n \in \mathbb{Z}_+$, then $G_k \subseteq \operatorname{Aut}(C_n)$.

Suppose the set Σ_n defining C_n satisfies the additional property (d). Let $\alpha \in \Sigma_n$ and let $m \in \mathbb{Z}_+$ such that $\alpha \in \Omega_m$. Let $\sigma \in G_k$. From (d) it follows that $\alpha^\sigma \in \Sigma_n$ and $\mathfrak{a}_\alpha^\sigma = \mathfrak{a}_{\alpha^\sigma}$, $R_{\alpha^\sigma} = R_\alpha^\sigma$, $C_{\alpha^\sigma} = C_\alpha^\sigma$, $f|_{\alpha^\sigma} = f|_\alpha^\sigma$. Hence $D_{\alpha^\sigma} = D_\alpha^\sigma$ and so $\bar{C}_{\alpha^\sigma} = \bar{C}_\alpha^\sigma$.

Let $p \in \operatorname{Sing}(\bar{C}_\alpha)$. Recall $\bar{\mathcal{G}}_p \in \bar{k}[X_{j_\alpha}]$ is monic of degree 1 generating the maximal ideal of $\mathcal{O}_{\bar{C}_\alpha, p}$. Since $\bar{C}_\alpha^\sigma = \bar{C}_{\alpha^\sigma}$, the ideal $(\bar{\mathcal{G}}_p^\sigma)$ is the maximal ideal of $\mathcal{O}_{\bar{C}_\alpha^\sigma, p^\sigma}$. Therefore $\bar{\mathcal{G}}_{p^\sigma} = \bar{\mathcal{G}}_p^\sigma$ as $\bar{\mathcal{G}}_p^\sigma \in \bar{k}[X_{j_{\alpha^\sigma}}]$ is linear and monic. Finally, the equality $\mathcal{F}_{\alpha^\sigma} = \mathcal{F}_\alpha^\sigma$ implies that we can choose $\tilde{\mathcal{G}}_{p^\sigma} = \tilde{\mathcal{G}}_p^\sigma$. Let $g_p \in \bar{k}[x_1, \dots, x_m, y]$ related to $\tilde{\mathcal{G}}_p$ by M_α. If $\tilde{\mathcal{G}}_{p^\sigma} = \tilde{\mathcal{G}}_p^\sigma$, then g_p^σ is the polynomial related to $\tilde{\mathcal{G}}_{p^\sigma}$ by $M_{\alpha^\sigma} = M_\alpha$; hence $g_{p^\sigma} = g_p^\sigma$.

Now let $S_n \subseteq \bigsqcup_{\alpha \in \Sigma_n} \operatorname{Sing}(\bar{C}_\alpha)$ be a G_k-invariant set. Consider the morphism $s_n : C_{n+1} \to C_n$ resolving S_n. We want to show that we can construct the collection Σ_{n+1} defining C_{n+1} in such a way that it satisfies (d). Define $S_{n,\alpha} = S_n \cap \bar{C}_\alpha$ for any $\alpha \in \Sigma_n$ and $\Sigma_{n,S_n} = \{\alpha \in \Sigma_n \mid S_{n,\alpha} \neq \varnothing\}$. Note that since S_n is G_k-invariant, so is Σ_{n,S_n}. Moreover,

$$S_{n,\alpha}^\sigma = \{p^\sigma \mid p \in S_{n,\alpha}\} = S_{n,\alpha^\sigma}$$

for any $\alpha \in \Sigma_n$ and $\sigma \in G_k$.

Let $\gamma \in \Sigma_{n+1}$ and $\sigma \in G_k$. Assume $\tilde{\mathcal{G}}_{p^\sigma} = \tilde{\mathcal{G}}_p^\sigma$ for any $p \in S_n$. If $\gamma \notin \Sigma_n$, then for some $\alpha \in \Sigma_{n,S_n}$ either $\gamma = \tilde{\alpha}$ or $\gamma = \beta \circ_{g_p} \alpha$, for some $p \in S_{n,\alpha}$, and $\beta \in \mathbb{Z}_+^2$ primitive. It follows from Lemma 3.7.2 that γ^σ equals either $\widetilde{\alpha^\sigma}$ or $\beta \circ_{g_{p^\sigma}} \alpha^\sigma$. In particular, the matrix M_{γ^σ}, the positive integer j_{γ^σ} and the polynomial $\mathcal{F}_{\gamma^\sigma}$ have been defined in §3.4.2 even when $\gamma^\sigma \notin \Sigma_{n+1}$ (see Notation 3.4.13, 3.4.16). This allows us to state the following result.

Theorem 3.7.3 *Consider the morphism $s_n : C_{n+1} \to C_n$ resolving the G_k-invariant set $S_n \subseteq \bigsqcup_{\alpha\in\Sigma_n} \mathrm{Sing}(\bar{C}_\alpha)$. Suppose Σ_n satisfies the additional property (d). Then Σ_{n+1} satisfies (d) if for all $\sigma \in G_k$, one has*

(1) $\tilde{\mathcal{G}}_{p^\sigma} = \tilde{\mathcal{G}}_p^\sigma$ for all $p \in S_n$;

(2) $M_\gamma = M_{\gamma^\sigma}$ for any $\gamma \in \Sigma_{n+1}$;

(3) $\mathrm{ord}_{X_{j_\gamma}}(\mathcal{F}_\gamma) = \mathrm{ord}_{X_{j_{\gamma^\sigma}}}(\mathcal{F}_{\gamma^\sigma})$ for any $\gamma \in \Sigma_{n+1}$.

Furthermore, if $\alpha_1, \dots, \alpha_h \in \Sigma_{n,S_n}$ so that $\Sigma_{n,S_n} = \bigsqcup_{i=1}^h G_k\, \alpha_i$, then

$$\Sigma_{n+1} = G_k \cdot \bigcup_{i=1}^{h} (\Sigma_{S_{n,\alpha_i}} \cup \{\tilde{\alpha}_i\}) \cup (\Sigma_n \setminus \Sigma_{n,S_n}).$$

Proof. Assume (1), (2) and (3) and let $\sigma \in G_k$. Let $\gamma \in \Sigma_{n+1} \setminus \Sigma_n$. Then there exists $\alpha \in \Sigma_{n,S_n}$ such that $\gamma = \tilde{\alpha}$ or $\gamma \in \Sigma_{S_{n,\alpha}}$.

Suppose $\gamma = \tilde{\alpha}$ for some $\alpha \in \Sigma_{n,S_n}$. Then $\gamma^\sigma = \widetilde{\alpha^\sigma}$ by Lemma 3.7.2 and so $\gamma^\sigma \in \Sigma_{n+1}$. Note that $j_\gamma = j_{\gamma^\sigma}$ and $\mathcal{F}_\gamma = \mathcal{F}_{\gamma^\sigma}$. Indeed, $j_{\tilde{\alpha}} = j_\alpha$, $\mathcal{F}_{\tilde{\alpha}} = \mathcal{F}_\alpha$ by construction, and $j_{\gamma^\sigma} = j_{\alpha^\sigma} = j_\alpha$ and $\mathcal{F}_{\gamma^\sigma} = \mathcal{F}_{\alpha^\sigma} = \mathcal{F}_\alpha^\sigma$, where the last equalities follow from the fact that Σ_n satisfies (d).

Suppose now that $\gamma \in \Sigma_{S_{n,\alpha}}$. Then $\gamma = \beta \circ_{g_p} \alpha$ for some $p \in S_{n,\alpha}$ and some primitive vector $\beta \in \mathbb{Z}_+^2$. Lemma 3.7.2 implies that $\gamma^\sigma = \beta \circ_{g_{p^\sigma}} \alpha^\sigma$. Let $m \in \mathbb{Z}_+$ such that $\alpha \in \Omega_m$. Note that $j_\gamma = j_{\gamma^\sigma}$. Indeed $j_\gamma = m+1$ by construction, and similarly $j_{\gamma^\sigma} = m+1$ since $\alpha^\sigma \in \Omega_m$.

Now we want to show that $\mathcal{F}_\gamma^\sigma = \mathcal{F}_{\gamma^\sigma}$. Let $\alpha_p = (0,1) \circ_{g_p} \alpha$, as in §3.4.2. Then $(\alpha_p)^\sigma = (0,1) \circ_{g_{p^\sigma}} \alpha^\sigma = (\alpha^\sigma)_{p^\sigma}$ by Lemma 3.7.2. Since $g_p^\sigma = g_{p^\sigma}$ and $M_\alpha = M_{\alpha^\sigma}$, Remark 3.4.8 shows that $M_{\alpha_p} = M_{(\alpha^\sigma)_{p^\sigma}}$. Recall that the matrix M_γ is obtained as the product $(I_m \oplus M_\beta) \cdot M_{\alpha_p}$, for some matrix M_β attached to β. Similarly, $M_{\gamma^\sigma} = (I_m \oplus M'_\beta) \cdot M_{(\alpha^\sigma)_{p^\sigma}}$ for some matrix M'_β attached to β. It follows that $M_\beta = M'_\beta$ as we are assuming $M_\gamma = M_{\gamma^\sigma}$.

We recall $\mathcal{F}_\gamma$ and $\mathcal{F}_{\gamma^\sigma}$ are constructed from $\mathcal{F}_{\alpha,p}$ and $\mathcal{F}_{\alpha^\sigma,p^\sigma}$ respectively, via the change of variables given by M_β. Explicitly,

$$\mathcal{F}_{\alpha,p} \overset{M_\beta}{=} X_{m+1}^{n_1} Y^{n_2} \cdot \mathcal{F}_\gamma, \qquad \mathcal{F}_{\alpha^\sigma,p^\sigma} \overset{M_\beta}{=} X_{m+1}^{n_3} Y^{n_4} \cdot \mathcal{F}_{\gamma^\sigma},$$

for some $n_1, n_2, n_3, n_4 \in \mathbb{Z}$. Note that $\mathcal{F}_{\alpha,p}^\sigma = \mathcal{F}_{\alpha^\sigma,p^\sigma}$ since $\tilde{\mathcal{G}}_p^\sigma = \tilde{\mathcal{G}}_{p^\sigma}$ and $\mathcal{F}_\alpha^\sigma = \mathcal{F}_{\alpha^\sigma}$. Therefore $\mathcal{F}_\gamma^\sigma = \mathcal{F}_{\gamma^\sigma}$ as $\mathrm{ord}_{X_{m+1}}(\mathcal{F}_\gamma) = \mathrm{ord}_{X_{m+1}}(\mathcal{F}_{\gamma^\sigma})$ by assumption and $\mathrm{ord}_Y(\mathcal{F}_\gamma) = \mathrm{ord}_Y(\mathcal{F}_{\gamma^\sigma}) = 0$ by construction.

To conclude the proof it only remains to show that $\bar{C}_{\gamma^\sigma} \neq \varnothing$ since this would imply $\gamma^\sigma \in \Sigma_{S_{n,\alpha^\sigma}}$. We showed $j_{\gamma^\sigma} = j_\gamma$, $M_{\gamma^\sigma} = M_\gamma$ and $\mathcal{F}_{\gamma^\sigma} = \mathcal{F}_\gamma^\sigma$, and so $\bar{C}_{\gamma^\sigma} = \bar{C}_\gamma^\sigma$. But $\bar{C}_\gamma \neq \varnothing$ since $\gamma \in \Sigma_{S_{n,\alpha}}$. Thus $\bar{C}_{\gamma^\sigma} \neq \varnothing$. ∎

Remark 3.7.4. Suppose Σ_n satisfies (d). In this remark we show that conditions (1),(2),(3) of Theorem 3.7.3 can always be obtained.

Let $\sigma \in G_k$. We have already observed that we can choose the polynomials $\tilde{\mathcal{G}}_p$, for $p \in S_n$, satisfying (1). Let $\gamma \in \Sigma_{n+1}$. If $\gamma \in \Sigma_n$, then the equalities $M_\gamma = M_{\gamma^\sigma}$ and $\mathrm{ord}_{X_{j_\gamma}}(\mathcal{F}_\gamma) = \mathrm{ord}_{X_{j_{\gamma^\sigma}}}(\mathcal{F}_{\gamma^\sigma})$ follow from the fact that Σ_n satisfies (d). Suppose $\gamma = \tilde{\alpha}$ for some $\alpha \in \Sigma_{n,S_n}$. Assuming (1), the equality $M_\gamma = M_{\gamma^\sigma}$ follows from Lemma 3.7.2 and Remark 3.4.14. Furthermore, $j_\gamma = j_{\gamma^\sigma}$ and $\mathcal{F}_\gamma^\sigma = \mathcal{F}_{\gamma^\sigma}$ as $j_\alpha = j_{\gamma^\alpha}$ and $\mathcal{F}_\alpha^\sigma = \mathcal{F}_{\alpha^\sigma}$. Suppose $\gamma \in \Sigma_{S_{n,\alpha}}$ for some $\alpha \in \Sigma_{n,S_n}$. Then $\gamma = \beta \circ_{g_p} \alpha$ for some primitive $\beta \in \mathbb{Z}_+^2$ and some $p \in S_{n,\alpha}$. Let $m \in \mathbb{Z}_+$ such that $\alpha \in \Omega_m$. In the proof of Theorem 3.7.3 we showed that $M_\gamma = (I_m \oplus M_\beta) \cdot M_{\alpha_p}$ and $M_{\gamma^\sigma} = (I_m \oplus M'_\beta) \cdot M_{\alpha_p}$, for some matrices M_β, M'_β attached to β that can be freely chosen. Therefore it suffices to choose $M_\beta = M'_\beta$ to have $M_\gamma = M_{\gamma^\sigma}$. Finally, the polynomial $\mathcal{F}_\gamma$ is fixed up to a power of X_{j_γ}, so we can easily require $\mathrm{ord}_{X_{j_\gamma}}(\mathcal{F}_\gamma) = \mathrm{ord}_{X_{j_{\gamma^\sigma}}}(\mathcal{F}_{\gamma^\sigma})$.

Remark 3.7.5. The conditions of Theorem 3.7.3 are satisfied if

(1) $\tilde{\mathcal{G}}_p = \bar{\mathcal{G}}_p$ for any $p \in S_n$;

(2) for any primitive $\beta \in \mathbb{Z}_+^2$, a fixed matrix $M_\beta \in \mathrm{SL}_2(\mathbb{Z})$ attached to β is chosen whenever choosing a matrix attached to β is required;

(3) there exists $a \in \mathbb{N}$ such that $\mathrm{ord}_{X_{j_\gamma}}(\mathcal{F}_\gamma) = a$ for any $\gamma \in \Sigma_{n+1}$.

Note that point (2) implies that if $\gamma = \beta \circ_{g_p} \alpha$, for some $\alpha \in \Sigma_n$, $p \in S_{n,\alpha}$, and some primitive vector $\beta \in \mathbb{Z}_+^2$, then we use the fixed matrix M_β to construct $M_\gamma = (I_m \oplus M_\beta) \cdot M_{\alpha_p}$.

Let C_1 be the completion of C_0 with respect to its Newton polygon. From §3.4.1 we easily see that $\gamma^\sigma = \gamma$ and $\mathcal{F}_\gamma^\sigma = \mathcal{F}_\gamma$ for any $\gamma \in \Sigma_1$ and any $\sigma \in G_k$. Hence the set $\Sigma_1 \subset \Omega$, defining C_1, satisfies (d). Theorem 3.7.3 and Remark 3.7.4 show that we can construct Baker's resolutions of C_0

$$\dots \xrightarrow{s_{n+1}} C_{n+1} \xrightarrow{s_n} C_n \xrightarrow{s_{n-1}} \dots \xrightarrow{s_1} C_1,$$

such that for all $n \in \mathbb{Z}_+$ the sets Σ_n satisfy the additional property (d). In particular, $G_k \subseteq \mathrm{Aut}(C_n)$ and Σ_n is G_k-invariant. The Galois-invariance of Σ_n makes the action on C_n easy to describe. Fix such a Baker's resolution.

Lemma 3.7.6 *Let $n \in \mathbb{Z}_+$. Then $\sigma \circ s_n = s_n \circ \sigma$, for any $\sigma \in G_k$.*

Proof. Recall that s_n restricts to the identity on C_0. Then the two morphisms of k-schemes $\sigma \circ s_n$ and $s_n \circ \sigma$ agree on C_0. But C_0 is a dense open of C_{n+1}, thus $\sigma \circ s_n = s_n \circ \sigma$ by [Liu4, Proposition 3.3.11]. □

Let $n \in \mathbb{Z}_+$. Recall that for any $\sigma \in G_k$ and $\gamma \in \Sigma_n$, we have $f|_{\gamma^\sigma} = f|_\gamma^\sigma$, as Σ_n satisfies (d). Therefore there is a natural action of G_k on the set

$$\bigsqcup_{\gamma \in \Sigma_n} \{\text{simple roots of } f|_\gamma \text{ in } \bar{k}^\times\},$$

where the simple root $r \in \bar{k}^\times$ of $f|_\gamma$ is taken to the simple root $\sigma(r)$ of $f|_{\gamma^\sigma}$.

Theorem 3.7.7 *Let $f \in k[x_1^{\pm 1}, y^{\pm 1}]$ be a Laurent polynomial defining a smooth curve $C_{0,k} : f = 0$ over $\mathbb{G}^2_{m,k}$. Denote $C_0 = C_{0,k} \times_k \bar{k}$. We can recursively construct a Baker's resolution of C_0*

$$\ldots \overset{s_{n+1}}{\twoheadrightarrow} C_{n+1} \overset{s_n}{\twoheadrightarrow} C_n \overset{s_{n-1}}{\twoheadrightarrow} \ldots \overset{s_1}{\twoheadrightarrow} C_1$$

where the maps s_n are the birational morphisms resolving G_k-invariant sets $S_n \subseteq \bigsqcup_{\alpha \in \Sigma_n} \mathrm{Sing}(\bar{C}_\alpha)$ (chosen arbitrarily) and the sets Σ_n, defining the curves $C_n/\bar{k}$, satisfy the additional property (d). For any such sequence:

(i) There exists $h \in \mathbb{Z}_+$ such that $S_n = \varnothing$ for any $n \geq h$.

(ii) If $\mathrm{Sing}(\bar{C}_\alpha) \subseteq \mathrm{Reg}(C_n)$ for all $\alpha \in \Sigma_n$, then the scheme-theoretical quotient C_n/G_k is a generalised Baker's model of the smooth completion C of $C_{0,k}$.

(iii) If C_n is outer regular, then there is a natural bijection

$$C(\bar{k}) \setminus C_{0,k}(\bar{k}) \overset{1:1}{\longleftrightarrow} \bigsqcup_{\gamma \in \Sigma_n} \{\textit{simple roots of } f|_\gamma \textit{ in } \bar{k}^\times\},$$

preserving the action of the Galois group G_k.

Proof. Theorem 3.7.3 and Remark 3.7.4 show that the sequence can be constructed recursively, for any choice of Galois-invariant $S_n \subseteq \bigsqcup_{\alpha \in \Sigma_n} \mathrm{Sing}(\bar{C}_\alpha)$. Part (i) follows from Theorem 3.6.3. Part (ii) is implied by Remark 3.4.4, Lemma 3.7.6 and the argument presented at the beginning of the current section. Now assume C_n is outer regular, i.e. $\bar{C}_\alpha$ is regular for all $\alpha \in \Sigma_n$. Therefore Lemma 3.5.12 shows that, for every $\gamma \in \Sigma_n$, from the definition $\bar{C}_\gamma = \mathrm{Spec}\, \frac{D_\gamma}{(f|_\gamma)}$ we obtain a natural bijective map

$$\bar{C}_\gamma \overset{1:1}{\longleftrightarrow} \{\text{simple roots of } f|_\gamma \text{ in } \bar{k}^\times\}.$$

By part (ii), the smooth completion C of $C_{0,k}$ is isomorphic to the quotient C_n/G_k. Therefore $C \times_k \bar{k} \simeq C_n$ and so $C(\bar{k}) \simeq C_n(\bar{k})$. Since $C_{0,k}(\bar{k}) \simeq C_0(\bar{k})$ by definition, Lemma 3.5.7 implies part (iii). □

Corollary 3.7.8 *Any smooth projective curve C defined over a perfect field k has an outer regular generalised Baker's model.*

Proof. By Corollary B.1.4, for any projective smooth curve C/k there exists a curve $C_{0,k}/k$ as in Theorem 3.7.7, birational to C. By Theorem 3.7.7 we can construct a Baker's resolution of $C_{0,k} \times_k \bar{k}$

$$\ldots \overset{s_{n+1}}{\twoheadrightarrow} C_{n+1} \overset{s_n}{\twoheadrightarrow} C_n \overset{s_{n-1}}{\twoheadrightarrow} \ldots \overset{s_1}{\twoheadrightarrow} C_1$$

where s_n are the birational morphisms resolving the Galois-invariant sets $S_n = \bigsqcup_{\alpha \in \Sigma_n} \mathrm{Sing}(\bar{C}_\alpha)$ and the sets Σ_n satisfy the additional property (d). Furthermore, by Theorem 3.7.7(i) there exists $n \in \mathbb{Z}_+$ such that $S_n = \varnothing$. It follows that $\bar{C}_\gamma$ is regular for all $\gamma \in \Sigma_n$, i.e. C_n is outer regular. Let $\tilde{C} = C_n/G_k$. Thus $\tilde{C}$ is an outer regular generalised Baker's model of C. □

In the next proof we will show how Algorithm 3.1.5 and Theorem 3.1.6 follow from previous results.

Proof of Theorem 3.1.6. Suppose $C_{0,k}$ is geometrically connected. We recursively construct a Baker's resolution of $C_0 = C_{0,k} \times_k \bar{k}$

$$\dots \overset{s_{n+1}}{\twoheadrightarrow} C_{n+1} \overset{s_n}{\twoheadrightarrow} C_n \overset{s_{n-1}}{\twoheadrightarrow} \dots \overset{s_1}{\twoheadrightarrow} C_1$$

where the morphisms s_n resolve the sets $S_n = \bigsqcup_{\alpha\in\Sigma_n} \mathrm{Sing}(\bar{C}_\alpha)$. In the construction, for any $n \in \mathbb{Z}_+$, we make the following choices:

(1) For any point $p \in S_n$ choose $\tilde{\mathcal{G}}_p = \bar{\mathcal{G}}_p$. This is always possible, since C_0 is connected (see Remark 3.4.5).

(2) Every time we need to choose a matrix $M_\beta \in \mathrm{SL}_2(\mathbb{Z})$ attached to some primitive vector $\beta = (\beta_1, \beta_2) \in \mathbb{Z}_+^2$, choose $M_\beta = \left(\begin{smallmatrix}\delta_1 & \delta_2\\ \beta_1 & \beta_2\end{smallmatrix}\right)$, where $(\delta_1, \delta_2) = \delta_\beta$ (Notation 3.1.3).

(3) For any $\gamma \in \Sigma_{n+1} \setminus \Sigma_n$, choose $\mathcal{F}_\gamma$ with $\mathrm{ord}_{X_{j_\gamma}}(\mathcal{F}_\gamma) = 0$.

With the choices above, by Theorem 3.7.3 and Remark 3.7.5, the sets Σ_n satisfy the additional property (d) and the sets S_n are Galois-invariant. Theorem 3.7.7(i) implies that there exists $n \in \mathbb{Z}_+$ such that $\bar{C}_\alpha$ is regular for all $\alpha \in \Sigma_n$. In other words, C_n is outer regular. Let n be as small as possible, i.e. such that C_h is not outer regular for every $h < n$. By Theorem 3.7.7(iii) there is a natural bijection preserving the action of the Galois group G_k,

$$C(\bar{k}) \setminus C_{0,k}(\bar{k}) \overset{1:1}{\longleftrightarrow} \bigsqcup_{\gamma\in\Sigma_n} \{\text{simple roots of } f|_\gamma \text{ in } \bar{k}^\times\},$$

where C is the smooth completion of $C_{0,k}$.

For any $h < n$ recall $S_{h,\alpha} = S_h \cap \bar{C}_\alpha$ for any $\alpha \in \Sigma_h$, and note that

$$\Sigma_{h,S_h} = \{\alpha \in \Sigma_h \mid S_{h,\alpha} \neq \varnothing\} = \{\alpha \in \Sigma_h \mid \bar{C}_\alpha \text{ is singular}\},$$

as $S_{h,\alpha} = \mathrm{Sing}(\bar{C}_\alpha)$. Define

$$\tilde{\Sigma}_h = \{\tilde{\alpha} \mid \alpha \in \Sigma_{h,S_h}\} \cup (\Sigma_h \setminus \Sigma_{h,S_h}), \quad \text{and} \quad \Sigma_{h+1}^+ = \textstyle\bigcup_{\alpha\in\Sigma_{h,S_h}} \Sigma_{S_{h,\alpha}},$$

so that $\Sigma_{h+1} = \Sigma_{h+1}^+ \cup \tilde{\Sigma}_h$. We are going to show that $\bar{C}_\gamma$ is regular for any $\gamma \in \tilde{\Sigma}_h$. From the choice of S_h, we have $\bar{C}_\gamma$ regular for any $\gamma \in \Sigma_h \setminus \Sigma_{h,S_h}$. Now let $\alpha \in \Sigma_{h,S_h}$. Lemma 3.4.15 shows that $\bar{C}_{\tilde{\alpha}}$ is isomorphic to $\bar{C}_\alpha \setminus S_{h,\alpha}$. This is a regular scheme since $S_{h,\alpha} = \mathrm{Sing}(\bar{C}_\alpha)$.

Now we want to describe the set S_h for any $h < n$. Define $\Sigma_1^+ = \Sigma_1$. If $h > 1$, then $\bar{C}_\gamma$ is regular when $\gamma \in \tilde{\Sigma}_{h-1}$. Therefore $\Sigma_{h,S_h} \subseteq \Sigma_h^+$ for all $h < n$. Now $D_\gamma = k[X_{j_\gamma}^{\pm 1}]$ for all $\gamma \in \Sigma_h^+$ from §3.4.1 (case $h = 1$) and Lemma 3.4.10 (case $h > 1$). Hence the points in S_h bijectively corresponds to non-zero multiple roots of $f|_\gamma$, $\gamma \in \Sigma_h^+$. Furthermore, given $\gamma \in \Sigma_h^+$, for any multiple root $r \in \bar{k}^\times$ of $f|_\gamma$ we have $\bar{\mathcal{G}}_{p_r} = X_{j_\gamma} - r$, where p_r is the point of S_h corresponding to r.

Let P_h, P be the indexed sets of polynomials in $\bar{k}[X,Y]$ constructed in §3.1.3 via Algorithm 3.1.5. We are going to prove the following facts:

(i) $P_h = \bigsqcup_{\gamma \in \Sigma_h^+} \{\mathcal{F}_\gamma(X,Y)\}$ for any $h \leq n$;

(ii) $\bigsqcup_{i=1}^{h} P_i = \bigsqcup_{\gamma \in \Sigma_h} \{\mathcal{F}_\gamma(X,Y)\}$ for any $h \leq n$;

(iii) $P = \bigsqcup_{\gamma \in \Sigma_n} \{\mathcal{F}_\gamma(X,Y)\}$;

where $\mathcal{F}_\gamma(X,Y)$ denotes the image of $\mathcal{F}_\gamma$ under the isomorphism

$$\bar{k}[X_{j_\gamma}, Y] \to \bar{k}[X,Y], \quad X_{j_\gamma} \mapsto X,\ Y \mapsto Y.$$

Note that (iii) concludes the proof of Theorem 3.1.6.

We prove (i) by induction on h. If $h = 1$, then $\Sigma_1^+ = \Sigma_1$, and so the equality follows from §3.4.1. Suppose $h \geq 1$. We want to show that

$$P_{h+1} = \bigsqcup_{\gamma \in \Sigma_{h+1}^+} \{\mathcal{F}_\gamma(X,Y)\}.$$

Let us recall the steps that have to be done to construct the polynomials $\mathcal{F}_\gamma$, for $\gamma \in \Sigma_{h+1}^+$. We observed that the points in S_h correspond to non-zero multiple roots of $f|_\alpha$ for $\alpha \in \Sigma_h^+$. For any $\alpha \in \Sigma_h^+$ and any multiple root $a \in \bar{k}^\times$ of $f|_\alpha$ do:

(1) Replace Y with $\tilde{Y}$ in $\mathcal{F}_\alpha$ so that $\mathcal{F}_\alpha \in \bar{k}[X_{j_\alpha}, \tilde{Y}]$.

(2) Denote by p_a the point of S_h corresponding to a. We noted that $\bar{\mathcal{G}}_{p_a} = X_{j_\alpha} - a$. Since we chose $\tilde{\mathcal{G}}_{p_a} = \bar{\mathcal{G}}_{p_a}$, the normal form $\mathcal{F}_{\alpha,p_a}$ of $\mathcal{F}_\alpha$ by $\tilde{X}_{m+1} - \tilde{\mathcal{G}}_{p_a}$ with respect to the lexicographic order given by $X_{j_\alpha} > \tilde{X}_{m+1} > \tilde{Y}$ is

$$\mathcal{F}_{\alpha,p_a}(\tilde{X}_{m+1}, \tilde{Y}) = \mathcal{F}_\alpha(\tilde{X}_{m+1} + a, \tilde{Y})$$

(here $m \in \mathbb{Z}_+$ such that $\alpha \in \Omega_m$).

(3) Draw the Newton polygon Δ_{α,p_a} of $\mathcal{F}_{\alpha,p_a}$.

(4) Let $\gamma = \beta \circ_{g_{p_a}} \alpha$ for the normal vector $\beta \in \mathbb{Z}_+^2$ of some edge of Δ_{α,p_a}. From §3.4.3, we have $\gamma \in \Sigma_{p_a}$.

(5) The fixed matrix $M_\beta = \left(\begin{smallmatrix} \delta_1 & \delta_2 \\ \beta_1 & \beta_2 \end{smallmatrix}\right)$ gives the change of variables

$$(\tilde{X}_{m+1}, \tilde{Y}) = (X_{m+1}, Y) \bullet M_\beta = (X_{m+1}^{\delta_1} Y^{\beta_1}, X_{m+1}^{\delta_2} Y^{\beta_2}).$$

Via this transformation we define $\mathcal{F}_\gamma$ to be the unique polynomial in $\bar{k}[X_{m+1}, Y]$ such that $\operatorname{ord}_{X_{m+1}} \mathcal{F}_\gamma = \operatorname{ord}_Y \mathcal{F}_\gamma = 0$, satisfying

$$\mathcal{F}_{\alpha,p_a}(\tilde{X}_{m+1}, \tilde{Y}) \overset{M_\beta}{=} X_{m+1}^{n_X} Y^{n_Y} \cdot \mathcal{F}_\gamma(X_{m+1}, Y),$$

for some $n_X, n_Y \in \mathbb{Z}$.

(6) In fact, all elements $\gamma \in \Sigma_{p_a}$ equals $\beta \circ_{g_{p_a}} \alpha$ with $\beta \in \mathbb{Z}_+^2$ normal vector of some edge of Δ_{α,p_a}.

The procedure presented here describes how to construct the polynomials $\mathcal{F}_\gamma$ for all $\gamma \in \Sigma_{h+1}^+$ knowing the polynomials $\mathcal{F}_\alpha$, for all $\alpha \in \Sigma_{h,S_n} \subseteq \Sigma_h^+$. Comparing it with Algorithm 3.1.5 we see that $P_{h+1} = \bigsqcup_{\gamma \in \Sigma_{h+1}^+} \{\mathcal{F}_\gamma(X,Y)\}$ since $\bigsqcup_{\alpha \in \Sigma_h^+} \{\mathcal{F}_\alpha(X,Y)\} = P_h$ by inductive hypothesis.

We now prove (ii) by induction on h. If $h = 1$, then (ii) follows from (i) as $\Sigma_1 = \Sigma_1^+$ by definition. Suppose then $h \geq 1$. We want to show that $\bigsqcup_{i=1}^{h+1} P_i = \bigsqcup_{\gamma \in \Sigma_{h+1}} \{\mathcal{F}_\gamma(X,Y)\}$. But $\Sigma_{h+1} = \Sigma_{h+1}^+ \sqcup \tilde{\Sigma}_h$, so, by (i) and inductive hypothesis, it suffices to show that

$$\bigsqcup_{\gamma \in \tilde{\Sigma}_h} \{\mathcal{F}_\gamma(X,Y)\} = \bigsqcup_{\gamma \in \Sigma_h} \{\mathcal{F}_\gamma(X,Y)\}.$$

But this easily follows from the definition of $\tilde{\Sigma}_h$ since $\mathcal{F}_{\tilde{\alpha}} = \mathcal{F}_\alpha$ for any $\alpha \in \Sigma_{S_h,h}$ (Notation 3.4.16).

To prove (iii), first note that from (i), for any $h \leq n$ the indexed set P_h is non-empty. Then (iii) is implied by (ii) if for any $f_\ell \in P_n$, the polynomial $f|_\ell(X) = f_\ell(X,0)$ has no non-zero multiple roots. But this follows from (i) since $\bar{C}_\alpha$ is regular for any $\alpha \in \Sigma_n$, and so $f|_\gamma$ has no multiple roots in $\bar{k}^\times$ for any $\gamma \in \Sigma_h^+$ as $D_\gamma = k[X_{j_\gamma}^{\pm 1}]$ in this case (Lemma 3.4.10). As already observed, this concludes the proof of Theorem 3.1.6. □

3.8 Superelliptic equations

Let k be a perfect field and let $\bar{k}$ be an algebraic closure of k. Denote by G_k the absolute Galois group of k. As application of the construction presented in the previous sections, we consider a curve $C_{0,k}$ in $\mathbb{G}_{m,k}^2$ defined by an equation

$$y^s = h(x),$$

for some polynomial $h \in k[x]$ and some $s \in \mathbb{Z}_+$ not divisible by $\mathrm{char}(k)$. By convention the polynomial $f(x,y)$ defining $C_{0,k}$ will be $y^s - h(x)$. Denote by C_0 the curve $C_{0,k} \times_k \bar{k}$. Note that C_0 is smooth, but may be not connected, e.g. when $h(x)$ is an s-th power. Expand

$$h(x) = \sum_{i=m_0}^{d} c_i x^i, \qquad c_i \in k,$$

where c_{m_0} and c_d are non-zero. We want to study a Baker's resolution of C_0

$$\ldots \overset{s_{n+1}}{\twoheadrightarrow} C_{n+1} \overset{s_n}{\twoheadrightarrow} C_n \overset{s_{n-1}}{\twoheadrightarrow} \ldots \overset{s_1}{\twoheadrightarrow} C_1$$

as in Theorem 3.7.7, where the Galois-invariant sets S_n which the birational morphisms s_n resolve are as large as possible, i.e. $S_n = \bigsqcup_{\alpha \in \Sigma_n} \mathrm{Sing}(\bar{C}_\alpha)$. For the purpose of the construction of the Baker's resolution $x_1 = x$.

The Newton polygon Δ of f always has at least two edges: ℓ_1 with endpoints $(m_0,0)$, $(0,s)$ and normal vector $\gcd(m_0,s)^{-1}(s,m_0)$, and ℓ_2 with endpoints $(d,0)$, $(0,s)$ and normal vector

$\gcd(d,s)^{-1}(-s,-d)$. If h is a monomial then Δ is a segment, otherwise Δ is a triangle. In the latter case, the third edge ℓ has endpoints $(m_0,0),(d,0)$ and normal vector $(0,1)$. Construct the completion C_1 of C_0 with respect to Δ, as described in §3.4.1. For any $i=1,2$ let $v_i \neq (0,1)$ be the normal vector of ℓ_i and set $\alpha_i = (v_i,()) \in \Sigma_1$. From Proposition 3.4.1 it follows that

$$f|_{\alpha_i} = X_1^* \cdot (a_l X_1^l + a_0), \qquad l \in \mathbb{Z}_+,\ a_0, a_l \in k^\times,$$

where $\mathrm{char}(k) \nmid l$. In fact, if $i=1$ then $l=\gcd(m_0,s)$, $a_l=-c_{m_0}$, $a_0=1$, while if $i=2$ then $l=\gcd(d,s)$, $a_l=1$, $a_0=-c_d$. In particular, $f|_{\alpha_i}$ has no multiple roots in $\bar{k}^\times$.

Suppose now that h is not a monomial. Let $v=(0,1)$ be the normal vector of ℓ and let $\alpha=(v,())$ be the corresponding element of Σ_1. Consider $\mathcal{F}_\alpha \in \bar{k}[X_1,Y]$. Note that since $v=(0,1)$, we can choose $M_\alpha = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and so $\mathcal{F}_\alpha = f(X_1,Y)$. In particular, $f|_\alpha = f(X_1,0) = -h(X_1)$. Since $D_\alpha = \bar{k}[X_1^{\pm1}]$, the singular points of $\bar{C}_\alpha$ correspond to the non-zero multiple roots of $f|_\alpha$, or, equivalently, to the non-zero multiple roots of h. Hence S_1 is the set of those points. If $S_1=\varnothing$, then C_1 is (outer) regular. We deduce the following lemma.

Lemma 3.8.1 *If h has no multiple root in $\bar{k}^\times$, then C_1 is an outer regular (generalised) Baker's model of the smooth completion of C_0.*

Suppose $S_1 \neq \varnothing$. Construct the morphism $s_1 : C_2 \to C_1$ resolving S_1. Let v and α as above. Rename the variable Y of $\mathcal{F}_\alpha$ to $\tilde{Y}$, so that $\mathcal{F}_\alpha \in \bar{k}[X_1,\tilde{Y}]$. Let $p \in S_1$ and let $r \in \bar{k}^\times$ be the multiple root of h corresponding to p. One has $\bar{\mathcal{G}}_p = X_1 - r$. Note that $\bar{\mathcal{G}}_p$ does not divide $\mathcal{F}_\alpha$, so choose $\tilde{\mathcal{G}}_p = \bar{\mathcal{G}}_p$. Then

$$\mathcal{F}_{\alpha,p}(\tilde{X}_2,\tilde{Y}) = \mathcal{F}_\alpha(\tilde{X}_2+r,\tilde{Y}) = f(\tilde{X}_2+r,\tilde{Y}) = \tilde{Y}^s - h(\tilde{X}_2+r).$$

It follows that the Newton polygon $\Delta_{\alpha,p}$ of $\mathcal{F}_{\alpha,p}$ has a unique edge ℓ_r with normal vector in $\mathbb{Z}_+^2$. Denoting by m_r the multiplicity of the root r of h, the endpoints of ℓ_r are $(m_r,0)$, $(0,s)$ and $\beta_r = \gcd(m_r,s)^{-1}(s,m_r)$ is its normal vector. Let $\gamma_r = \beta_r \circ_{g_p} \alpha$, where g_p is the polynomial related to $\tilde{\mathcal{G}}_p$ by M_α. Define $h_r(x) = h(x)/(x-r)^{m_r} \in \bar{k}[x]$. Then Proposition 3.4.19 implies

$$f|_{\gamma_r}(X_2) = X_2^* \cdot (-a_r X_2^{\gcd(m_r,s)} + 1),$$

where $a_r = h_r(r)$. In particular, since $\mathrm{char}(k) \nmid s$, the polynomial $f|_{\gamma_r}$ has no multiple root in $\bar{k}^\times$. Therefore $\bar{C}_{\gamma_r}$ is regular for any non-zero multiple root r of h. Moreover, $\bar{C}_{\tilde{\alpha}}$ is also regular as $\bar{C}_{\tilde{\alpha}} \simeq \bar{C}_\alpha \setminus S_1$. Recall the notation $\tilde{\Sigma}_1 = \hat{\Sigma}_1 \cup \{\tilde{\alpha}\}$, where $\hat{\Sigma}_1 = \Sigma_1 \setminus \{\alpha\}$. Since

$$\Sigma_2 = \{\gamma_r \mid r \text{ multiple root of } h\} \cup \tilde{\Sigma}_1$$

the schemes $\bar{C}_\gamma$ are regular for all $\gamma \in \Sigma_2$. We obtain the following result.

Lemma 3.8.2 *If h has multiple roots in $\bar{k}^\times$, then C_1 is singular, but C_2 is an outer regular generalised Baker's model of the smooth completion of C_0.*

Remark 3.8.3. Note that $C_2 = \bigcup_{\gamma \in \Sigma_2} C_\gamma$ since $C_0 \subseteq C_\gamma$ for any $\gamma \in \Sigma_2$.

We want to give an explicit description of the curve $C_{2,k} = C_2/G_k$, when h has multiple roots in $\bar{k}^\times$. First note that for any $\gamma \in \tilde{\Sigma}_1$ the polynomials defining the curves C_γ have coefficients in k. Therefore $G_k \subseteq \mathrm{Aut}(C_\gamma)$ for all $\gamma \in \tilde{\Sigma}_1$ and the charts C_γ/G_k of $C_{2,k}$ easily follows. It remains to describe the curve $\left(\bigcup_{\sigma\in G_k} C_{\gamma_{\sigma(r)}}\right)/G_k$ for any non-zero multiple root r of h.

Let $g \in k[x]$ be the minimal polynomial of a multiple root $r \in \bar{k}^\times$ of h. Let m_r, h_r, β_r, γ_r as above. Set $s_r = \gcd(m_r, s)$. Note that $\mathrm{ord}_g(h) = m_r$. If $\left(\begin{smallmatrix}\delta_1 & \delta_2\\ \beta_1 & \beta_2\end{smallmatrix}\right)$ is the matrix attached to β_r used for the construction of C_{γ_r} then

$$\mathcal{O}_{C_{\gamma_r}}(C_{\gamma_r}) = \frac{\bar{k}[X_1^{\pm1}, X_2^{\pm1}, Y]}{(1 - X_2^{s_r}\cdot h_r(X_1), X_2^{\delta_1}Y^{\beta_1} - X_1 + r)}.$$

Define $g_r, h_g \in \bar{k}[x]$ by $g_r(x) = g(x)/(x-r)$, $h_g(x) = h(x)/g(x)^{m_r}$. Note that $g_r(X_1)$ is invertible in $\mathcal{O}_{C_{\gamma_r}}(C_{\gamma_r})$. Consider the homomorphism

$$\phi_r : \frac{\bar{k}[X_1^{\pm1}, X_2^{\pm1}, Y]}{(1 - X_2^{s_r}\cdot h_g(X_1), X_2^{\delta_1}Y^{\beta_1} - g(X_1))} \longrightarrow \frac{\bar{k}[X_1^{\pm1}, X_2^{\pm1}, Y]}{(1 - X_2^{s_r}\cdot h_r(X_1), X_2^{\delta_1}Y^{\beta_1} - X_1 + r)},$$

taking $X_1 \mapsto X_1$, $X_2 \mapsto X_2 \cdot g_r(X_1)^{\beta_2}$, $Y \mapsto Y \cdot g_r(X_1)^{-\delta_2}$. Let $A_g := \mathrm{Dom}(\phi_r)$. Note that $\mathrm{Spec}\, A_g = C_{\gamma_g}$, where $\gamma_g = \beta_r \circ_g \alpha \in \Omega$. Then ϕ_r induces an open immersion $\iota_r : C_{\gamma_r} \hookrightarrow C_{\gamma_g}$. The glueing of the open immersions $\iota_{\sigma(r)}$, for $\sigma \in G_k$, gives an isomorphism

$$\left(\bigcup\nolimits_{\sigma\in G_k} C_{\gamma_{\sigma(r)}}\right) \simeq C_{\gamma_g},$$

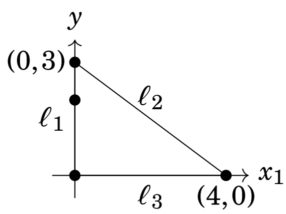
commuting with the Galois action. Since C_{γ_g} is defined by polynomials with coefficients in k, the quotient C_{γ_g}/G_k is easy to describe, as required.

3.9 Example

Let $C_{0.\mathbb{F}_2} : f = 0 \subset \mathbb{G}^2_{m,\mathbb{F}_2}$ with $f = x_1^4 + 1 + y^2 + y^3$. Note that $C_{0,\mathbb{F}_2}$ is smooth. Write $C_0 = C_{0,\mathbb{F}_2} \times_{\mathbb{F}_2} \bar{\mathbb{F}}_2$, where $\bar{\mathbb{F}}_2$ is an algebraic closure of $\mathbb{F}_2$.

3.9.1 Construction of C_1

The Newton polygon Δ of f is



We want to construct the completion C_1 of C_0 with respect to Δ as explained in §3.4.1. For any edge ℓ_i of Δ let β_i be the normal vector of ℓ_i. Then $\beta_1 = (1,0)$, $\beta_2 = (-3,-4)$, $\beta_3 = (0,1)$. Let $\alpha_i = (\beta_i, ()) \in \Sigma_1$ for $i = 1,2,3$. Then $\Sigma_1 = \{\alpha_1, \alpha_2, \alpha_3\}$ and

$$C_1 = C_{\alpha_1} \cup C_{\alpha_2} \cup C_{\alpha_3},$$

where we omitted C_0 as $C_0 \subset C_\alpha$ for every $\alpha \in \Sigma_1$. From Proposition 3.4.1 the polynomials $f|_{\alpha_1}$ and $f|_{\alpha_2}$ are separable (up to a power of X_1) and so the corresponding curves C_{α_1} and C_{α_2} are regular. On the other hand, $1 \in \mathbb{F}_2$ is a non-zero multiple root of $f|_{\alpha_3}$, so C_{α_3} may be singular. Let us compute the defining polynomial $\mathcal{F}_{\alpha_3}$. The identity matrix $I \in \mathrm{SL}_2(\mathbb{Z})$ is attached to β_3, so we fix $M_{\alpha_3} = I$. Via I we get

$$\mathcal{F}_{\alpha_3} = X_1^4 + 1 + Y^2 + Y^3.$$

Then $C_{\alpha_3} = \operatorname{Spec} \bar{\mathbb{F}}_2[X_1^{\pm 1}, Y]/(\mathcal{F}_{\alpha_3})$ is singular. Thus C_1 is not smooth, having 1 singular point, visible on C_{α_3}.

3.9.2 Construction of C_2

Rename the variable Y of C_{α_3} to $\tilde{Y}$. Let p be the singular point of C_{α_3}. Then $\bar{\mathcal{G}}_p = X_1 + 1$. Choose $\tilde{\mathcal{G}}_p = \bar{\mathcal{G}}_p$. We will construct the morphism $s_1 : C_2 \to C_1$ resolving the set $S_1 = \{p\}$. Note that $S_1 = \bigsqcup_{\alpha \in \Sigma_1} \operatorname{Sing}(\bar{C}_\alpha)$. Let $\alpha = \alpha_3$ and $\beta = \beta_3$. Then

$$\tilde{\mathcal{G}}_p\left((x_1, y) \bullet M_\alpha^{-1}\right) = x_1 + 1,$$

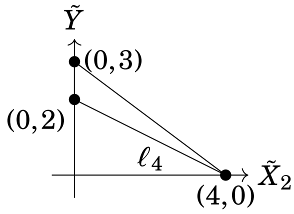
so $g_p = x_1 + 1 \in \mathbb{F}_2[x_1, y]$ is the polynomial related to $\tilde{\mathcal{G}}_p$ by M_α. Define $g_2 = g_p$ and $f_2 = x_2 - g_2$. Note that since S_1 consists of a single point, we have $\tilde{\mathcal{G}}_{S_1} = \tilde{\mathcal{G}}_p$ and $g_{S_1} = g_p$. Then $\alpha_p = \tilde{\alpha}$. Compute $\operatorname{ord}_\beta(g_p) = 0$ and $\tilde{\alpha} = \alpha_p = (0,1) \circ_{g_{S_1}} \alpha = ((0,0,1),(g_2))$. Then

$$C_{\tilde{\alpha}} = C_{\alpha_p} = \operatorname{Spec} \frac{\bar{\mathbb{F}}_2[X_1^{\pm 1}, \tilde{X}_2^{\pm 1}, \tilde{Y}]}{(\mathcal{F}_{\alpha_3}, \tilde{X}_2 + X_1 + 1)}.$$

The normal form of $\mathcal{F}_{\alpha_3}$ by $\tilde{X}_2 - \mathcal{G}$ with respect to the lexicographic order given by $X_1 > \tilde{X}_2 > \tilde{Y}$ is

$$\mathcal{F}_{\alpha,p} = \mathcal{F}_\alpha\big(\tilde{X}_2 + 1, \tilde{Y}\big) = \tilde{X}_2^4 + \tilde{Y}^2 + \tilde{Y}^3.$$

The Newton polygon of $\mathcal{F}_{\alpha,p}$ is



There is only 1 edge, denoted ℓ_4, with normal vector in $\mathbb{Z}_+^2$. The normal vector of ℓ_4 is $\beta_4 = (1,2)$. It follows that $v_4 = \beta_4 \circ_{g_p} \beta = (0,1,2)$. Hence $\gamma_4 = \beta_4 \circ_{g_p} \alpha = (v_4, (g_2))$ is the corresponding element of Σ_p. Then $\Sigma_2 = \{\alpha_1, \alpha_2, \tilde{\alpha}_3, \gamma_4\}$.

To check whether C_{γ_4} is regular, compute $\mathcal{F}_{\gamma_4}$. The matrix $M_{\beta_4} = \left(\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right)$, attached to β_4, defines the change of variables $\tilde{X}_2 = X_2 Y$, $\tilde{Y} = X_2 Y^2$, from which we get

$$\begin{aligned} \mathcal{F}_{\alpha,p} &= X_2^2 Y^4 \mathcal{F}_{\gamma_4}, & \mathcal{F}_{\gamma_4} &= X_2^2 + 1 + X_2 Y^2, \\ \tilde{X}_2 - \tilde{\mathcal{G}}_p &= \mathcal{F}_2, & \mathcal{F}_2 &= X_2 Y + X_1 + 1, \end{aligned}$$

where $\mathcal{F}_2$ is the generator of the ideal $\mathfrak{a}_{\gamma_4}$. Therefore the curve

$$C_{\gamma_4} = \operatorname{Spec} \frac{\bar{\mathbb{F}}_2[X_1^{\pm 1}, X_2^{\pm 1}, Y]}{(\mathcal{F}_{\gamma_4}) + \mathfrak{a}_{\gamma_4}}$$

is singular, and so is the projective curve $C_2 = C_{\alpha_1} \cup C_{\alpha_2} \cup C_{\tilde{\alpha}_3} \cup C_{\gamma_4}$. In the union we omitted C_0, as $C_0 \subset C_{\alpha_1}$.

3.9.3 Construction of C_3

Let q be the singular point of C_{γ_4}. We now construct the morphism $s_2 : C_3 \to C_2$ resolving $S_2 = \{q\}$. Let $\gamma = \gamma_4$. Rename the variable Y of C_γ to $\tilde{Y}$. Choose $\tilde{\mathcal{G}}_q = \bar{\mathcal{G}}_q = X_2 + 1$. By definition

$$M_\gamma = ((1) \oplus M_{\beta_4}) \cdot M_{\alpha_p} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad M_\gamma^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Then $g_q = x_2^2 + y \in \mathbb{F}_2[x_1, x_2, y]$ is the polynomial related to $\tilde{\mathcal{G}}_q$ by M_γ, as

$$\tilde{\mathcal{G}}_q\left((x_1, x_2, y) \bullet M_\gamma^{-1}\right) = x_2^2 y^{-1} + 1.$$

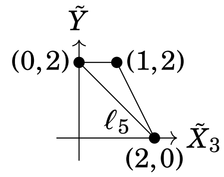
Let $g_3 = (x_1 + 1)^2 + y$ be the Laurent polynomial in $k[x_1^{\pm 1}, y^{\pm 1}]$ congruent to g_q modulo f_2. Compute $\operatorname{ord}_{v_4}(g_q) = 2$. Then

$$\tilde{\gamma} = \gamma_q = (0,1) \circ_{g_q} \gamma = ((0,1,2,2),(g_2,g_3)).$$

The normal form of $\mathcal{F}_\gamma$ by $\tilde{X}_3 - \tilde{\mathcal{G}}_q$ with respect to the lexicographic order given by $X_2 > \tilde{X}_3 > \tilde{Y}$ is

$$\mathcal{F}_{\gamma,q} = \tilde{X}_3^2 + (\tilde{X}_3 + 1)\tilde{Y}^2.$$

The Newton polygon of $\mathcal{F}_{\gamma,q}$ is



There is only 1 edge, denoted ℓ_5, with normal vector in $\mathbb{Z}_+^2$. The normal vector of ℓ_5 is $\beta_5 = (1,1)$ and so the corresponding element of Σ_q is

$$\gamma_5 = \beta_5 \circ_{g_q} \gamma = ((0,1,3,2),(g_2,g_3)).$$

Hence $\Sigma_3 = \{\alpha_1, \alpha_2, \tilde{\alpha}_3, \tilde{\gamma}_4, \gamma_5\}$.

The matrix $M_{\beta_5} = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, attached to β_5, defines the change of variables $\tilde{X}_3 = X_3 Y$, $\tilde{Y} = Y$ from which we get

$$\mathcal{F}_{\gamma,q} = Y^2 \mathcal{F}_{\gamma_5} \qquad \mathcal{F}_{\gamma_5} = X_3^2 + X_3 Y + 1,$$
$$\tilde{X}_3 - \tilde{\mathcal{G}}_q = \mathcal{F}_3 \qquad \mathcal{F}_3 = X_3 Y + X_2 + 1,$$

and $\mathcal{F}_2 = X_2Y + X_1 + 1$ is the image of the generator of $\mathfrak{a}_\gamma$ under M_{β_5}. Then $\mathfrak{a}_{\gamma_5} = (\mathcal{F}_2, \mathcal{F}_3)$ and

$$C_{\gamma_5} = \mathrm{Spec}\,\frac{\bar{\mathbb{F}}_2[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}, Y]}{(\mathcal{F}_{\gamma_5}) + \mathfrak{a}_{\gamma_5}}$$

is regular (even if $f|_{\gamma_5}$ is not separable). Therefore the curve

$$C_3 = C_{\alpha_1} \cup C_{\alpha_2} \cup C_{\tilde{\alpha}_3} \cup C_{\tilde{\gamma}_4} \cup C_{\gamma_5}$$

is regular as well, and is a generalised Baker's model of the smooth completion of C_0. It is not outer regular, since $\bar{C}_{\gamma_5}$ has a singular point. One more step is therefore necessary (and sufficient by Proposition 3.4.38) to construct an outer regular generalised Baker's model. Note that in the description of C_3 we omitted C_0, as $C_0 \subset C_{\alpha_1}$. Finally, the polynomials defining the charts C_γ, $\gamma \in \Sigma_3$ have coefficients in $\mathbb{F}_2$, so the construction of the generalised Baker's model $C_3/G_{\mathbb{F}_2}$ of the smooth completion of $C_{0,\mathbb{F}_2}$ easily follows.

CHAPTER 4

Regular models of hyperelliptic curves

This chapter is based on the paper *Regular models of hyperelliptic curves* [Mus3], submitted for publication. Let K be a complete discretely valued field of odd residue characteristic and O_K its ring of integers. We explicitly construct a regular model $\mathcal{C}$ over O_K with strict normal crossings of any hyperelliptic curve $C/K : y^2 = f(x)$. For this purpose, we introduce the new notion of *MacLane cluster picture*, that aims to be a link between clusters and MacLane valuations.

The description of the special fibre of $\mathcal{C}$, presented in Theorem 4.1.7, is being implemented in MAGMA by T. Dokchitser.

4.1 Introduction

In this paper we construct regular models of hyperelliptic curves over discrete valuation rings with residue characteristic different from 2. The understanding of regular models is essential to describe the arithmetic of curves and for example finds application in the study of the Birch & Swinnerton-Dyer conjecture over global fields.

4.1.1 Overview

Let K be a complete discretely valued field, with ring of integers O_K. Given a connected smooth projective curve C/K, a *regular model* of C over O_K is an integral regular proper flat scheme $\mathcal{C} \to O_K$ of dimension 2 with generic fibre isomorphic to C. The main result of this work can be presented as follows:

Suppose that the residue characteristic of K is not 2. Let $C/K : y^2 = f(x)$ be a hyperelliptic curve. From the MacLane clusters for f we determine a regular model of C over O_K with strict *normal crossings.*

The *MacLane clusters* for a separable polynomial $f \in K[x]$ are a new notion we introduce in this paper (see §4.1.2 for more details). It has connections with other objects used for the study of regular models: clusters [D²M²], rational clusters [Mus1], Newton polytopes [Dok], and MacLane valuations [OW]. Like (rational) clusters, MacLane clusters define nice and clear invariants from which one can give a result in a closed form. In fact, one can see that rational clusters are MacLane clusters of degree 1. On the other side, the construction of our model can be implemented from the algorithmic nature of the approaches based on Newton polytopes and MacLane valuations.

The construction of the model presented in §4.5 generalises the one showed in Chapter 2. For this reason, the author believes the approach developed in this chapter could be used to tackle some even residue characteristic cases, as we did in Chapter 2.

4.1.2 Main result

Let K be a complete discretely valued field, with normalised discrete valuation v_K, ring of integers O_K, and residue field k. Let $\bar{K}$ be an algebraic closure of K, extend v_K to $\bar{K}$. Assume $\mathrm{char}(k) \neq 2$. Let C/K be a hyperelliptic curve, i.e. a geometrically connected smooth projective curve of genus ≥ 1, double cover of $\mathbb{P}^1_K$. We can fix a Weierstrass equation $C : y^2 = f(x)$ where

$$f(x) = c_f \prod_{r \in \mathfrak{R}} (x - r) \in K[x], \quad c_f \in K,$$

such that $v_K(r) > 0$ for all $r \in \mathfrak{R}$.

Definition 4.1.1 Let $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. Given a monic irreducible polynomial $g \in K[x]$ and an element $\lambda \in \hat{\mathbb{Q}}$, the *discoid* $D(g, \lambda)$ is the set

$$D = D(g, \lambda) = \{\alpha \in \bar{K} \mid v_K(g(\alpha)) \geq \lambda\} \subset \bar{K}.$$

For any $r \in \mathfrak{R}$, denote by $D \wedge r$ the smallest discoid containing D and r.

Define $\deg D = \min\{d \in \mathbb{Z}_+ \mid D = D(g, \lambda), \deg g = d\}$.

To each discoid we can associate a pseudo-valuation (Appendix C.1) $v_D : K[x] \to \hat{\mathbb{Q}}$ defined by

$$v_D(f) = \inf_{\alpha \in D} v_K(f(\alpha)).$$

The map $D \mapsto v_D$ is injective. Therefore if $v = v_D$ denote $D_v = D$ and $d_v = \deg D$.

Definition 4.1.2 A *MacLane cluster* is a pair $(\mathfrak{s}, v)$ where $\mathfrak{s} \subseteq \mathfrak{R}$, and $v = v_D$ for some discoid D, such that

1. $\mathfrak{s} = D \cap \mathfrak{R} \neq \varnothing$;

2. if $\mathfrak{s} = D' \cap \mathfrak{R}$ for a discoid $D' \subsetneq D$ then $\deg D' > \deg D$.

The *degree* of $(\mathfrak{s},v)$ is the quantity d_v.

Definition 4.1.3 For any MacLane clusters $(\mathfrak{s},v),(\mathfrak{t},w)$ we say:

$(\mathfrak{s},v)$ proper,	if $\lvert\mathfrak{s}\rvert > d_v$
$(\mathfrak{t},w)\subseteq(\mathfrak{s},v)$,	if $D_w\subseteq D_v$
$(\mathfrak{t},w)$ is a child of $(\mathfrak{s},v)$,	if $(\mathfrak{t},w)\subsetneq(\mathfrak{s},v)$ is a maximal subcluster
$(\mathfrak{s},v)$ degree-minimal,	if $(\mathfrak{s},v)$ has no proper children of degree d_v

We write $(\mathfrak{t},w)<(\mathfrak{s},v)$ for a child $(\mathfrak{t},w)$ of $(\mathfrak{s},v)$.

For the remainder of this subsection we also assume k algebraically closed. This additional condition is not necessary for the construction of the model but it simplifies the statement of Theorem 4.1.7.

Let Σ be the set of proper MacLane clusters.

Notation 4.1.4 Let $\mathcal{P}\subset K[x]$ be the subset of monic irreducible polynomials. For any $d\in\mathbb{Z}_+$, denote $\mathcal{P}_{\le d}=\{g\in\mathcal{P}\mid \deg g\le d\}$.

Definition 4.1.5 (4.6.1) Let $(\mathfrak{s},v)\in\Sigma$. Define the following quantities:

λ_v	$=\max_{g\in\mathcal{P}_{\le d_v}}\min_{r\in\mathfrak{s}} v_K(g(r))$, called *radius*
b_v	$=$ denominator of $\lambda_v d_v$
e_v	$=b_v d_v$
ν_v	$=v_K(c_f)+\sum_{r\in\mathfrak{R}}\left(\lambda_{v_{D_v\wedge r}}/d_{v_{D_v\wedge r}}\right)$
n_v	$=1$ if $e_v\nu_v$ odd, 2 if $e_v\nu_v$ even
m_v	$=2e_v/n_v$
t_v	$=\lvert\mathfrak{s}\rvert/d_v$
p_v	$=1$ if t_v is odd, 2 if t_v is even
s_v	$=\frac{1}{2}(t_v\lambda_v+p_v\lambda_v-\nu_v)$
γ_v	$=2$ if t_v is even and $\nu_v d_v-\lvert\mathfrak{s}\rvert\lambda_v$ is odd, 1 otherwise
δ_v	$=1$ if $(\mathfrak{s},v)$ is degree-minimal, 0 otherwise
p_v^0	$=1$ if $\delta_v=1$ and $d_v=\min_{r\in\mathfrak{s}}[K(r):K]$, 2 otherwise
s_v^0	$=-\nu_v/2+\lambda_v$
γ_v^0	$=2$ if $p_v^0=2$ and $\nu_v d_v$ is an odd integer, 1 otherwise

Let $\ell_v\in\mathbb{Z}$, $0\le\ell_v<b_v$ such that $\ell_v\lambda_v d_v-\frac{1}{b_v}\in\mathbb{Z}$. Define

$$\tilde{v}=\left\{(\mathfrak{t},w)\in\Sigma\mid(\mathfrak{t},w)<(\mathfrak{s},v)\text{ and }\tfrac{\lvert\mathfrak{t}\rvert}{e_v}-\ell_v\nu_v d_w\notin 2\mathbb{Z}\right\}.$$

Let $c_v^0=1$ if $\frac{2-p_v^0}{b_v}-\ell_v\nu_v d_v\notin 2\mathbb{Z}$, and $c_v^0=0$ otherwise. Define

$$u_v=\frac{\lvert\mathfrak{s}\rvert-\sum_{(\mathfrak{t},w)<(\mathfrak{s},v)}\lvert\mathfrak{t}\rvert-d_v(2-p_v^0)}{e_v}+\lvert\tilde{v}\rvert+\delta_v c_v^0,$$

where the sum runs through the proper children $(\mathfrak{t},w)$ of $(\mathfrak{s},v)$. The *genus* $g(v)$ of a MacLane cluster $(\mathfrak{s},v)\in\Sigma$ is defined as follows:

- If $n_v = 1$, then $g(v) = 0$.
- If $n_v = 2$, then $g(v) = \max\{\lfloor (u_v - 1)/2 \rfloor, 0\}$.

We recall the following notation from Chapter 2.

Notation 4.1.6 (2.4.17) Let $\alpha \in \mathbb{Z}_+$, $a, b \in \mathbb{Q}$, with $a > b$, and fix $\frac{n_i}{d_i} \in \mathbb{Q}$ so that

$$\alpha a = \frac{n_0}{d_0} > \frac{n_1}{d_1} > \ldots > \frac{n_r}{d_r} > \frac{n_{r+1}}{d_{r+1}} = \alpha b, \quad \text{with} \quad \begin{vmatrix} n_i & n_{i+1} \\ d_i & d_{i+1} \end{vmatrix} = 1,$$

and r minimal. We write $\mathbb{P}^1(\alpha, a, b)$ for a chain of $\mathbb{P}^1_k$s of length r and multiplicities $\alpha d_1, \ldots, \alpha d_r$. Denote by $\mathbb{P}^1(\alpha, a)$ the chain $\mathbb{P}^1(\alpha, a, \lfloor \alpha a - 1 \rfloor / \alpha)$.

The following theorem describes the special fibre of the regular model of a hyperelliptic curve C/K with strict normal crossings we construct in §4.5, when k algebraically closed and $\mathrm{char}(k) \neq 2$. See Definition 4.6.1 and Theorem 4.6.3 for a more general statement which does not require k algebraically closed.

Theorem 4.1.7 (Regular SNC model) *Assume* $\mathrm{char}(k) \neq 2$. *Suppose k algebraically closed. Let C/K be a hyperelliptic curve. Then we can explicitly construct a regular model with strict normal crossings $\mathcal{C}/O_K$ of C (§4.5). Its special fibre $\mathcal{C}_s/k$ is given as follows.*[1]

(1) Every $(\mathfrak{s}, v) \in \Sigma$ gives a 1-dimensional closed subscheme Γ_v of multiplicity m_v. If $n_v = 2$ and $u_v = 0$, then Γ_v is the disjoint union of $\Gamma_v^- \simeq \mathbb{P}^1_k$ and $\Gamma_v^+ \simeq \mathbb{P}^1_k$, otherwise Γ_v is a smooth integral curve of genus $g(v)$ (write $\Gamma_v^- = \Gamma_v^+ = \Gamma_v$ in this case).

(2) Every $(\mathfrak{s}, v) \in \Sigma$ with $n_v = 1$ gives

$$\frac{1}{e_v}\Big(|\mathfrak{s}| - \sum_{\substack{(\mathfrak{t},w)\in\Sigma \\ (\mathfrak{t},w)<(\mathfrak{s},v)}} |\mathfrak{t}| + d_v(p_v^0 - 2)\Big)$$

open-ended $\mathbb{P}^1_k$s of multiplicity e_v from Γ_v.

(3) Finally, for any $(\mathfrak{s}, v) \in \Sigma$ draw the following chains of $\mathbb{P}^1_k$s:

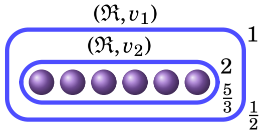
Conditions	*Chain*	*From*	*To*
$\delta_v = 1$	$\mathbb{P}^1(d_v\gamma_v^0, -s_v^0)$	Γ_v^-	*open-ended*
$\delta_v = 1,\ p_v^0/\gamma_v^0 = 2$	$\mathbb{P}^1(d_v\gamma_v^0, -s_v^0)$	Γ_v^+	*open-ended*
$(\mathfrak{s}, v) < (\mathfrak{t}, w)$	$\mathbb{P}^1(d_v\gamma_v, s_v, s_v - \frac{p_v}{2}(\lambda_v - \frac{d_v}{d_w}\lambda_w))$	Γ_v^-	Γ_w^-
$(\mathfrak{s}, v) < (\mathfrak{t}, w),\ p_v/\gamma_v = 2$	$\mathbb{P}^1(d_v\gamma_v, s_v, s_v - \frac{p_v}{2}(\lambda_v - \frac{d_v}{d_w}\lambda_w))$	Γ_v^+	Γ_w^+
$(\mathfrak{s}, v)$ *maximal*	$\mathbb{P}^1(d_v\gamma_v, s_v)$	Γ_v^-	*open-ended*
$(\mathfrak{s}, v)$ *maximal*, $p_v/\gamma_v = 2$	$\mathbb{P}^1(d_v\gamma_v, s_v)$	Γ_v^+	*open-ended*

As we pointed out in §2.1, Theorem 4.1.7 is a generalisation of Theorem 2.1.7.

[1] This theorem is being implemented by T. Dokchitser in MAGMA.

4.1.3 Example

Let $p \neq 2$ be a prime number and let $\mathbb{Q}_p^{nr}$ be the maximal unramified extension of $\mathbb{Q}_p$ in $\bar{\mathbb{Q}}_p$. Let $f = (x^2-p)^3 - p^5 \in \mathbb{Q}_p[x]$ and $C/\mathbb{Q}_p^{nr} : y^2 = f(x)$ a genus 2 hyperelliptic curve. We can represent the set of MacLane clusters as



where the bullet points denote the roots of f, the circles are the proper MacLane clusters and the superscripts and the subscripts are respectively the degree and the radius of the corresponding cluster. In fact, there are two proper MacLane clusters:

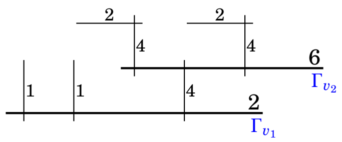
(i) $(\mathfrak{R}, v_1)$, where $D_{v_1} = D(x, 1/2)$;

(ii) $(\mathfrak{R}, v_2)$, where $D_{v_2} = D(x^2-p, 5/3)$.

Note that $\min_{r\in\mathfrak{R}}[K(r):K] = 6$ since f is irreducible. We have

	b_v	e_v	ν_v	n_v	m_v	t_v	p_v	s_v	γ_v	δ_v	p_v^0	s_v^0	γ_v^0	$g(v)$
v_1	2	2	3	2	2	6	2	$1/2$	1	1	2	-1	2	0
v_2	3	6	10	2	6	3	1	$-5/3$	1	1	2	$-10/3$	1	0

By Theorem 4.1.7, the special fibre of the regular model $\mathcal{C}$ we construct is



where all irreducible components have genus 0. In fact, by computing the self-intersections of all irreducible components, we see that $\mathcal{C}$ is the minimal regular model of C ([Liu4, Theorem 9.3.8]).

4.1.4 Related works of other authors

Let K be a discretely valued field of odd residue characteristic and let C/K be a hyperelliptic curve. In this subsection we want to present previous works studying regular models of C, possibly under some extra conditions. Note that some of the results cited below may apply to more general curves and fields.

In genus 1 there is a complete characterisation of (minimal) regular models of C (see for example [Sil2, IV.8.2] when the residue field of K is perfect). A description of all special fibre configurations is also given by Namikawa and Ueno [NU] and Liu [Liu5] for genus 2 curves, when $K = \mathbb{C}(t)$.

If C is semistable over some tamely ramified extension L/K, then [FN] describes the special fibre of the minimal regular model of C with strict normal crossings. If, in addition, $L=K$ is a local field, in [D^2M^2] we can also see an explicit construction of the model itself.

T. Dokchitser in [Dok] shows that a certain toric resolution of C gives a regular model in case of Δ_v-regularity ([Dok, Definition 3.9]). This condition is rephrased in terms of clusters in [Mus1, Corollary 3.25].

Finally, [Mus1] constructs the minimal regular model with normal crossings if C has almost rational cluster picture. One can see that the latter condition is equivalent of requiring that all MacLane clusters have degree 1.

4.2 MacLane valuations

In this section we summarise definitions and results on MacLane valuations. Our main references are [KW], [Mac], [OS1] and [Rüt].

Let K be a complete discretely valued field, with normalised discrete valuation v_K, ring of integers O_K and residue field k. Let $\bar{K}$ be an algebraic closure of K and let K^s be the separable closure of K in $\bar{K}$. Let $G_K = \mathrm{Gal}(K^s/K)$ be the absolute Galois group of K.

Let $\hat{\mathbb{V}}$ denote the set of the discrete pseudo-valuation[2] $v : K[x] \to \hat{\mathbb{Q}}$ extending v_K and satisfying $v(x) \geq 0$. Let $\mathbb{V}$ be the set of valuations in $\hat{\mathbb{V}}$. In other words, $\mathbb{V}$ consists of those pseudo-valuations $v \in \hat{\mathbb{V}}$ satisfying $v^{-1}(\infty) = 0$. We can equip $\hat{\mathbb{V}}$ with a natural partial order:

$$v \geq w \quad \text{if and only if} \quad v(g) \geq w(g) \text{ for all } g \in K[x].$$

The partially ordered set $\hat{\mathbb{V}}$ has a least element v_0, called *Gauss valuation*, defined by

$$v_0(a_m x^m + \cdots + a_1 x + a_0) = \min_i v_K(a_i) \qquad (a_i \in K).$$

Note that v_0 is a valuation, i.e. $v_0 \in \mathbb{V}$.

Every $v \in \mathbb{V}$ can be extended to a valuation $K(x) \to \hat{\mathbb{Q}}$, that will also be denoted by v.

Definition 4.2.1 For every $v \in \mathbb{V}$ define

Γ_v	the valuation group of v
e_v	the index $[\Gamma_v : \mathbb{Z}]$
A_v	the residue ring of v
$\mathbb{F}_v$	the residue field of v

Definition 4.2.2 Let $v \in \mathbb{V}$. For any $g, h \in K(x)$ we say that

- g is *v-equivalent* to h, denoted $g \sim_v h$, if $v(g-h) > v(g)$.
- g is *v-divisible* by h, denoted $h \mid_v g$, if there exists $q \in K[x]$ such that $g \sim_v qh$.

[2] See Appendix C.1 for more details.

Let $v \in \mathbb{V}$. For any $\alpha \in \Gamma_v$, define

$$O_v(\alpha) = \{g \in K[x] \mid v(g) \geq \alpha\}, \qquad O_v^+(\alpha) = \{g \in K[x] \mid v(g) > \alpha\}.$$

The *graded algebra of* v is the integral domain

$$Gr(v) := \bigoplus_{\alpha \in \Gamma_v} A_v(\alpha), \quad \text{where } A_v(\alpha) = O_v(\alpha)/O_v^+(\alpha).$$

The canonical homomoprhism $k \to A_v$ equips A_v and $Gr(v)$ with a k-algebra structure. There is a natural map $H_v : K[x] \to Gr(v)$ given by $H_v(0) = 0$ and

$$H_v(g) = g + O_v^+(v(g)) \in A_v(v(g)),$$

when $g \neq 0$. The map H_v satisfies the following properties

1. $f \sim_v g$ if and only if $H_v(f) = H_v(g)$,

2. $H_v(fg) = H_v(f)H_v(g)$,

for $f, g \in K[x]$. Let $U_v \subseteq K[x]^*$ be the multiplicative set

$$U_v = \{g \in K[x] \mid H_v(g) \text{ is a unit in } Gr(v)\}$$

and let $P_v \subseteq K(x)$ be the localisation of $K[x]$ by U_v. We extend H_v to a map $P_v \to Gr(v)$ by taking $g/u \mapsto H_v(g)H_v(u)^{-1} \in A_v(v(g/u))$, for any $g \in K[x], u \in U_v$. With a little abuse of notation we denote the extended map again by H_v. The properties (1), (2) of H_v hold for all $f, g \in P_v$.

Definition 4.2.3 We call $H_v : P_v \to Gr(v)$ the *residue map of* v.

For any $\alpha \in \Gamma_v$, let

$$P_v(\alpha) = \{g \in P_v \mid v(g) \geq \alpha\}, \qquad P_v^+(\alpha) = \{g \in P_v \mid v(g) > \alpha\}.$$

Note that H_v induces a birational map $P_v(\alpha)/P_v^+(\alpha) \to A_v(\alpha)$.

Definition 4.2.4 Let $v \in \mathbb{V}$. A monic polynomial $\phi \in K[x]$ is a *key polynomial over* v if

(1) ϕ is *v-irreducible*, i.e. if $\phi \mid_v ab$ then $\phi \mid_v a$ or $\phi \mid_v b$, for all $a, b \in K[x]$;

(2) ϕ is *v-minimal*, i.e. if $\phi \mid_v a$ then $\deg a \geq \deg \phi$, for all $a \in K[x]$.

Denote by $\mathrm{KP}(v)$ the set of key polynomials over v.

Remark 4.2.5. Let $v \in \mathbb{V}$. Then $\mathrm{KP}(v) \subseteq O_K[x]$ ([FGMN, Corollary 1.10]).

Definition 4.2.6 ([Mac, Theorem 4.2]) Let $v \in \mathbb{V}$. Let $\phi \in \mathrm{KP}(v)$ and $\lambda \in \hat{\mathbb{Q}}$, $\lambda > v(\phi)$. Define a pseudo-valuation $w \in \hat{\mathbb{V}}$, denoted $w = [v, v(\phi) = \lambda]$, by

$$w(a_m\phi^m + \cdots + a_1\phi + a_0) = \min_i (v(a_i) + i\lambda) \qquad a_i \in K[x], \deg a_i < \deg\phi.$$

We call w the *augmentation* of v with respect to (ϕ, λ).

Remark 4.2.7. Let $w = [v, v(\phi) = \lambda]$ be an augmentation of v. Then

(i) $w > v$ by [FGMN, Propositions 1.7, 1.9].

(ii) λ and $\deg\phi$ are uniquely determined by w, but not the key polynomial ϕ itself in general (see [KW, Remark 2.7]).

Definition 4.2.8 A pseudo-valuation $v \in \hat{\mathbb{V}}$ is *MacLane* if it can be attained after a finite number of augmentations starting with v_0. Write

$$v = [v_0, v_1(\phi_1) = \lambda_1, \ldots, v_m(\phi_m) = \lambda_m], \quad m \in \mathbb{N},$$

where $v_i = [v_{i-1}, v_i(\phi_i) = \lambda_i]$ is an augmentation of v_{i-1} for any $i = 1, \ldots, m$, and $v_m = v$. We will call ϕ_m a *centre* of v and λ_m the *radius* of v.[3]

Let $\hat{\mathbb{V}}_M \subset \hat{\mathbb{V}}$ denote the set of MacLane pseudo-valuations and let $\mathbb{V}_M \subset \mathbb{V}$ denote the set of MacLane valuations.

Remark 4.2.9. There are different equivalent characterisations for the sets $\mathbb{V}_M$ and $\hat{\mathbb{V}}_M$ (see [KW, §2]). In fact,

(i) $\mathbb{V}_M$ consists of those valuations $v \in \mathbb{V}$ with residue field $\mathbb{F}_v$ of transcendence degree 1 over k;

(ii) all infinite pseudo-valuations $v \in \hat{\mathbb{V}}$ are Maclane.

Notation 4.2.10 Let $v \in \hat{\mathbb{V}}_M$. Remark 4.2.7(ii) implies that the radius of v is uniquely determined by v. We will denote it by λ_v.

Definition 4.2.11 Let $v \in \hat{\mathbb{V}}_M$. An *augmentation chain (of length m)* for v is a tuple

$$((\phi_1, \lambda_1), \ldots, (\phi_m, \lambda_m)), \tag{4.1}$$

where $v = [v_0, v_1(\phi_1) = \lambda_1, \ldots, v_m(\phi_m) = \lambda_m]$. We say that (4.1) is

1. a *MacLane chain* if $\phi_{i+1} \not\sim_{v_i} \phi_i$ for any $i = 1, \ldots, m-1$.

2. *minimal* if $\deg\phi_{i+1} > \deg\phi_i$ for any $i = 1, \ldots, m-1$.

[3] By convention, if $v = v_0$, then any monic integral polynomial of degree 1 is a centre of v and 0 is the radius of v.

For any augmentation chain (4.1) we have

$$\deg\phi_1 \mid \deg\phi_2 \mid \cdots \mid \deg\phi_m,$$

by [FGMN, Lemma 2.10]. If it is a MacLane chain, then $v(\phi_i) = \lambda_i$ for any $i = 1, \ldots, m$. In particular, $\Gamma_v = \lambda_1\mathbb{Z} + \cdots + \lambda_m\mathbb{Z}$.

Remark 4.2.12. Let $v \in \hat{\mathbb{V}}_M$.

1. A minimal augmentation chain is a Maclane chain.

2. From any MacLane chain $((\phi_1, \lambda_1), \ldots, (\phi_m, \lambda_m))$ for v, we can find a minimal augmentation chain for v by removing the pairs (ϕ_i, λ_i) with $\deg\phi_i = \deg\phi_{i+1}$, for $i = 1, \ldots, m-1$ ([Mac, Lemma 15.1], [FGMN, Lemma 3.4]).

Notation 4.2.13 We will denote an augmentation chain (4.1) by

$$[v_0, v_1(\phi_1) = \lambda_1, \ldots, v_m(\phi_m) = \lambda_m],$$

where $v_i = [v_{i-1}, v_i(\phi_i) = \lambda_i]$ for all $i = 1, \ldots, m$.

Definition 4.2.14 Let $v \in \hat{\mathbb{V}}_M$ given by a MacLane chain

$$[v_0, v_1(\phi_1) = \lambda_1, \ldots, v_m(\phi_m) = \lambda_m]. \tag{4.2}$$

(a) The *degree of* v, denoted $\deg v$, is the positive integer $\deg\phi_m$.

(b) If (4.2) is minimal, then m is said the *depth of* v.

The degree and the depth of v are independent of the chosen MacLane chain (4.2) by [FGMN, Proposition 3.6].

Note that if $v \in \mathbb{V}_M$ then $\deg v \mid \deg\phi$ for any $\phi \in \mathrm{KP}(v)$.

Definition 4.2.15 Let $v \in \mathbb{V}_M$. A key polynomial $\phi \in \mathrm{KP}(v)$ is said

1. *proper* if v has a centre $\phi_v \not\sim_v \phi$.

2. *strong* if $v = v_0$ or $\deg\phi > \deg v$.

Lemma 4.2.16 *Let $w \in \hat{\mathbb{V}}_M$. A polynomial $\phi \in K[x]$ is a centre of w if and only if $\phi \in \mathrm{KP}(w)$ and* $\deg w = \deg\phi$*. Furthermore, if $w = [v, w(\phi) = \lambda]$, then any two centres of w are v-equivalent.*

Proof. Let $v \in \mathbb{V}_M$ such that $w = [v, w(\phi_w) = \lambda_w]$. If $\phi \in K[x]$ is a centre of w then $\phi \in \mathrm{KP}(w)$ by [FGMN, Proposition 1.7(4)] and $\deg w = \deg\phi$ from Remark 4.2.7(ii). Conversely, suppose $\phi \in \mathrm{KP}(w)$ and $\deg\phi = \deg w$. From the w-minimality of ϕ and ϕ_w, one has $w(\phi) = \lambda_w$. Hence

$$v(\phi - \phi_w) = w(\phi - \phi_w) \geq \lambda_w > v(\phi_w).$$

Therefore $\phi \sim_v \phi_w$. In particular, $\phi \in \mathrm{KP}(v)$ as $\deg\phi = \deg\phi_w$, and so $w = [v, w(\phi) = \lambda_w]$. Thus ϕ is a centre of w. □

Definition 4.2.17 Given a monic irreducible polynomial $\phi \in K[x]$ and an element $\lambda \in \hat{\mathbb{Q}}$, the *discoid* of centre ϕ and radius λ is the set

$$D = D(\phi, \lambda) = \{\alpha \in \bar{K} \mid v_K(\phi(\alpha)) \geq \lambda\} \subset \bar{K}.$$

Let $\mathcal{D}_K$ denote the set of discoids.

Remark 4.2.18. Let $D = D(\phi, \lambda)$ be a discoid.

1. D is finite if $\lambda = \infty$, while equals the union of the Galois orbits of a disc centred at a root of ϕ if $\lambda < \infty$ ([Rüt, Lemma 4.43]).

2. For any $D' \in \mathcal{D}_K$ such that $D \cap D' \neq \varnothing$ either $D \subseteq D'$ or $D \subseteq D'$ ([Rüt, Lemma 4.44]).

Definition 4.2.19 Given a MacLane pseudo-valuation v, define

$$D_v = \{\alpha \in \bar{K} \mid v_K(g(\alpha)) \geq v(g) \quad \text{for all } g \in K[x]\}.$$

It is a discoid by the following lemma.

Lemma 4.2.20 *If $v = [v_0, v_1(\phi_1), \ldots, v_{m-1}(\phi_{m-1}) = \lambda_{m-1}, v_m(\phi_m) = \lambda_m]$ is a MacLane pseudo-valuation, then $D_v = D(\phi_m, \lambda_m)$.*

Proof. If $v \in \mathbb{V}_M$, then the lemma follows from [Rüt, Lemma 4.55]. Suppose v is an infinite MacLane pseudo-valuation. Then $\lambda_m = \infty$. Clearly $D_v \subseteq D(\phi_m, \lambda_m)$. Let $r \in D(\phi_m, \lambda_m)$, i.e. r is a root of ϕ_m. Let $g \in K[x]$. We want to show that $v_K(g(r)) \geq v(g)$. If $\phi_m \mid g$, then $g(r) = 0$ and $v(g) = \infty$, so $v_K(g(r)) = v(g)$. If $\phi_m \nmid g$, then there is a sufficiently large $\lambda \in \mathbb{Q}$ such that $w(g) = v(g)$, with $w = [v_{m-1}, w(\phi_m) = \lambda]$. Since $w \in \mathbb{V}_M$, we have $D(\phi_m, \lambda) = D_w$. But $r \in D(\phi_m, \lambda)$, and so $v_K(g(r)) \geq w(g) = v(g)$. $\square$

Theorem 4.2.21 *The map $\hat{\mathbb{V}}_M \to \mathcal{D}_K$ taking $v \mapsto D_v$ is well-defined, bijective, and inverts partial orders, i.e. for any $v, w \in \hat{\mathbb{V}}_M$ we have*

$$w \geq v \quad \textit{if an only if} \quad D_w \subseteq D_v.$$

Given a discoid D, then $D = D_v$, where v is the MacLane pseudo-valuation given by $v(g) = \inf_{r \in D} v_K(g(r))$ for all $g \in K[x]$.

Proof. The result follows from [Rüt, Theorem 4.56], [KW, Remark 2.3]. $\square$

Lemma 4.2.22 *Let $v \in \hat{\mathbb{V}}_M$ and $D_v = D(g, \lambda)$ the associated discoid. Then $\deg v \leq \deg g$ and $v(g) \geq \lambda$.*

Proof. Theorem 4.2.21 implies that $\inf_{r\in D_v} v_K(g(r)) = v(g)$. Then $v(g) \geq \lambda$. It follows that

$$D_v \subseteq D(g, v(g)) \subseteq D(g,\lambda) = D_v.$$

Then $D_v = D(g, v(g))$. Suppose $\deg v > \deg g$ and let

$$[v_0, v_1(\phi_1) = \lambda_1, \dots, v_m(\phi_m) = \lambda_m]$$

be a MacLane chain for v. Then $v_{m-1} < v$ but $v_{m-1}(g) = v(g)$. Therefore $D_v \subsetneq D_{v_{m-1}} \subseteq D(g, v(g))$, a contradiction. □

Remark 4.2.23. Lemma 4.2.22 shows that $\deg v$ is the lowest positive integer such that $D_v = D(g,\lambda)$ for some monic irreducible polynomial $g \in K[x]$ of degree $\deg g = \deg v$ and some $\lambda \in \hat{\mathbb{Q}}$.

Proposition 4.2.24 *Let $v, w \in \hat{\mathbb{V}}_M$, with $v_0 < w \leq v$. Let*

$$[v_0, v_1(\phi_1) = \lambda_1, \dots, v_n(\phi_n) = \lambda_n]$$

be a minimal MacLane chain for v. Then there exists $m \leq n$ such that $w = [v_{m-1}, w(\phi_m) = \lambda]$, for some $v_{m-1}(\phi_m) < \lambda \leq \lambda_m$.

Proof. Let $[v_0, w_1(\psi_1) = \mu_1, \dots, w_m(\psi_m) = \mu_m]$ be a minimal MacLane chain for w. Then $n \geq m$ by [Rüt, Proposition 4.35] and $v_{m-1} = w_{m-1}$ by [Rüt, Corollary 4.37]. Then $w = [v_{m-1}, w(\psi_m) = \mu_m]$. Since $v_m \leq v \geq w$, either $v_m < w$ or $w \leq v_m$ from Remark 4.2.18(2). Suppose by contradiction that $v_m < w$. Then $m < n$. Furthermore, $v_m = [v_{m-1}, v_m(\psi_m) = \lambda_m]$ and $\lambda_m < \mu_m$ by [FGMN, Lemma 7.6]. Let r be a root of ϕ_n. Then $r \in D_v$. Since $m < n$, one has $\deg \psi_m = \deg v_m < \deg v$. Therefore $v_K(\psi_m(r)) = v(\psi_m) = \lambda_m$ by [OS2, Corollary 2.8], giving a contradiction to $w \leq v$. Hence $w \leq v_m$. Thus [FGMN, Lemma 7.6] implies $w = [v_{m-1}, w(\phi_m) = \mu_m]$ and $\mu_m \leq \lambda_m$, as required. □

Lemma 4.2.25 *Let $v, w \in \hat{\mathbb{V}}_M$. Suppose $w < v$. Then $\lambda_w < \lambda_v$ and* $\deg w \leq \deg v$. *Moreover, if* $\deg w = \deg v$, *any centre ϕ of v is also a centre of w.*

Proof. The statement is trivial when $w = v_0$. Suppose $w > v_0$. Let ϕ be a centre of v. Consider a minimal augmentation chain $[v_0, \dots, v_n(\phi_n) = \lambda_n]$ for v, with $\phi_n = \phi$. By Proposition 4.2.24 there exist $m \leq n$ and $\mu_m < \lambda_m$ such that $w = [v_{m-1}, w(\phi_m) = \mu_m]$. Then $\deg w \leq \deg v$ and $\lambda_w < \lambda_v$ by [Rüt, Lemma 4.21]. Furthermore, if $\deg w = \deg v$ then $n = m$, since the key polynomials ϕ_i have strictly increasing degrees. This concludes the proof as $\phi_m = \phi_n = \phi$ could be any centre of v. □

Lemma 4.2.26 *Let $v \in \mathbb{V}_M$. For any monic non-constant $g \in K[x]$ of degree* $\deg g \leq \deg v$ *we have $v(g) \leq \lambda_v$, with $v(g) = \lambda_v$ only if* $\deg g = \deg v$.

Proof. We prove the lemma by induction on $\deg v$. Let

$$[v_0, \dots, v_{m-1}(\phi_{m-1}) = \lambda_{m-1}, v_m(\phi_m) = \lambda_m]$$

be a minimal MacLane chain for v. Recall $\lambda_v = \lambda_m$. If $\deg v = 1$, then $\deg g = \deg v$. By definition $v(g) = \min\{v(\phi_m), v(g-\phi_m)\} \le \lambda_v$. Suppose $\deg v > 1$. If $\deg g = \deg v$ then $v(g) \le \lambda_v$ as above. If $\deg g < \deg v$ then $v(g) = v_{m-1}(g) \le \lambda_{m-1} < \lambda_m$. □

Recall the following result from [FGMN].

Theorem 4.2.27 ([FGMN, Theorem 3.10]) *Let $v \in \mathbb{V}_M$. For any monic non-constant $g \in K[x]$ one has*

$$\frac{v(g)}{\deg g} \le \frac{\lambda_v}{\deg v},$$

and the equality holds if and only if g is v-minimal.

Lemma 4.2.28 *Let $g_1, g_2 \in K[x]$ monic and non-constant. Then $g_1 \cdot g_2$ is v-minimal if and only if both g_1 and g_2 are v-minimal.*

Proof. Suppose g_1 is not v-minimal. Then there exists $a \in K[x]$, $\deg a < \deg g_1$ such that $g_1 \mid_v a$. Hence $g_1 g_2 \mid_v a g_2$ and $\deg(a g_2) < \deg(g_1 g_2)$. So $g_1 \cdot g_2$ is not v-minimal. Similarly for g_2. Suppose both g_1 and g_2 are v-minimal. Theorem 4.2.27 implies that

$$v(g_1 \cdot g_2)\deg v = (v(g_1) + v(g_2))\deg v = \lambda_v(\deg g_1 + \deg g_2) = \lambda_v \deg(g_1 \cdot g_2),$$

and so $g_1 \cdot g_2$ is v-minimal. □

Lemma 4.2.29 *Let $v, w \in \mathbb{V}_M$ satisfying $w \ge v$. Let $g \in O_K[x]$ monic and non-constant. Suppose g is w-minimal. Then g is v-minimal.*

Proof. By [Rüt, Remark 4.36] we can write

$$w = [v_0, v_1(\phi_1) = \lambda_1, \ldots, v_m(\phi_m) = \lambda_m, \ldots, v_n(\phi_n) = \lambda_n],$$

with $v = v_m$. Let $i = m, \ldots, n-1$. By recursion it suffices to show that g is v_i-minimal if it is v_{i+1}-minimal. We can suppose g irreducible by Lemma 4.2.28. Since ϕ_{i+1} is v_i-minimal, by Theorem 4.2.27 we have

$$\frac{v_i(g)}{\deg g} \le \frac{\lambda_i}{\deg \phi_i} = \frac{v_i(\phi_{i+1})}{\deg \phi_{i+1}} < \frac{\lambda_{i+1}}{\deg \phi_{i+1}} = \frac{v_{i+1}(g)}{\deg g}.$$

Therefore $v_{i+1}(g) > v_i(g)$ that is equivalent to $\phi_{i+1} \mid_{v_i} g$ by [Rüt, Lemma 4.13]. [FGMN, Theorem 6.2] implies that $v_i(g) = \deg g \cdot \frac{v_i(\phi_{i+1})}{\deg \phi_{i+1}}$. But then Theorem 4.2.27 shows that g is v_i-minimal. □

Lemma 4.2.30 *Let $v \in \mathbb{V}_M$ and let ϕ be a centre of v. Let $g \in K[x]$ monic, non-constant and v-minimal. Then*

(i) $\deg v \mid \deg g$.

(ii) $g \sim_w \phi^{\deg g/\deg v}$ *for any $w \in \mathbb{V}_M$, $w < v$.*

Proof. (i) follows from [FGMN, Lemma 2.10]. For proving (ii) we can suppose without loss of generality that $\phi \in \mathrm{KP}(w)$ by Proposition 4.2.24 and Lemma 4.2.29. Equivalently, $v = [w, v(\phi) = \lambda]$ for some $\lambda \in \mathbb{Q}$, $\lambda > w(\phi)$. Let $d = \deg g / \deg \phi$ and expand

$$g = \sum_{j=0}^{d} a_j \phi^j, \qquad \text{where } a_j \in K[x], \deg a_j < \deg \phi,$$

and $v(a_d) = w(a_d) = 0$. Note that $v(g) = v(\phi^d)$ by Theorem 4.2.27. Therefore

$$\begin{aligned} w(\phi^d) = v(g) - d(\lambda - w(\phi)) &\le v(a_j\phi^j) - d(\lambda - w(\phi)) \\ &< v(a_j\phi^j) - j(\lambda - w(\phi)) = w(a_j\phi^j), \end{aligned}$$

for all $j < d$. Thus $g \sim_w \phi^d$ as required. □

The following two results come from [OS2].

Proposition 4.2.31 ([OS2, Proposition 2.5]) *Let $\phi \in O_K[x]$ be a monic irreducible polynomial. There exists a unique MacLane valuation v_ϕ over which ϕ is a strong key polynomial.*

Proposition 4.2.32 ([OS2, Proposition 2.7]) *Let $v \in \mathbb{V}_M$ and ϕ a proper key polynomial over v. Let $w = [v, w(\phi) = \lambda]$, for some $\lambda > v(\phi)$ and let $r \in D_w$. For any $g \in K[x]$ such that $v(g) = w(g)$, we have $v_K(g(r)) = v(g)$.*

Lemma 4.2.33 *Let $v \in \hat{\mathbb{V}}_M$ given by a MacLane chain*

$$[v_0, v_1(\phi_1) = \lambda_1, \ldots, v_{m-1}(\phi_{m-1}) = \lambda_{m-1}, v_m(\phi_m) = \lambda_m].$$

Suppose $m > 0$. The ramification index $e_{v_{m-1}}$ equals $[\Gamma_{\phi_m}(v) : \mathbb{Z}]$, where

$$\Gamma_{\phi_m}(v) = \{v(a) \mid a \in K[x], a \neq 0, \deg a < \deg \phi_m\}.$$

In particular, it is independent of the chosen MacLane chain.

Proof. First note that if we restrict to minimal MacLane chains, the result is trivial. By Remark 4.2.12(2) it suffices to prove that if $m > 1$ and $\deg \phi_{m-1} = \deg \phi_m$, then $e_{v_{m-2}} = e_{v_{m-1}}$. We have

$$v_{m-1}(\phi_m - \phi_{m-1}) = \lambda_{m-1}.$$

since $\phi_{m-1} \not\sim_{v_{m-1}} \phi_m$. But $\deg(\phi_m - \phi_{m-1}) < \deg \phi_{m-1}$, so

$$\lambda_{m-1} = v_{m-1}(\phi_m - \phi_{m-1}) = v_{m-2}(\phi_m - \phi_{m-1}) \in \Gamma_{v_{m-2}}.$$

Thus $\Gamma_{v_{m-2}} = \Gamma_{v_{m-1}}$, as required. □

Definition 4.2.34 Let $v \in \hat{\mathbb{V}}_M$ given by a MacLane chain

$$[v_0, \ldots, v_{m-1}(\phi_{m-1}) = \lambda_{m-1}, v_m(\phi_m) = \lambda_m].$$

Define $\epsilon_v = e_{v_{m-1}}$ if $m > 0$, and $\epsilon_v = 1$ otherwise.

For any monic irreducible polynomial $\phi \in K[x]$, define $K_\phi = K[x]/(\phi)$, finite extension of K. Let O_ϕ be the ring of integers of K_ϕ and k_ϕ the residue field. Recall $\deg\phi = e_\phi f_\phi$, where e_ϕ and f_ϕ are respectively the ramification index and the residual degree of the extension K_ϕ/K.

Let $v \in \hat{\mathbb{V}}_M$ with centre ϕ. Then [FGMN, Proposition 1.9(2)] shows that $e_\phi = [\Gamma_\phi(v) : \mathbb{Z}]$, and so $e_\phi = \epsilon_v$ by Lemma 4.2.33. It follows that $f_\phi = \deg v/e_\phi$ is independent of the choice of the centre ϕ.

Notation 4.2.35 Given $v \in \hat{\mathbb{V}}_M$ with centre ϕ, denote $f_v = f_\phi$.

Let $f \in K[x]$, $v \in \mathbb{V}_M$ and $\phi \in \mathrm{KP}(v)$. Write

$$f = \sum_{t=0}^{d} a_t \phi^t, \qquad \text{where } \deg a_t < \deg\phi.$$

The *Newton polygon*, $N_{v,\phi}(f)$ of f is

$$N_{v,\phi}(f) = \text{lower convex hull}(\{(t, v(a_t)) \mid a_t \neq 0\}) \subset \mathbb{R}^2.$$

Notation 4.2.36 Let $\lambda \in \mathbb{Q}$, $\lambda > v(\phi)$ and $w = [v, w(\phi) = \lambda]$. We denote by $L_w(f)$ the intersection of $N_{v,\phi}(f)$ with the line of slope $-\lambda$ which first touches it from below:

$$L_w(f) := \{(t,u) \in N_{v,\phi}(f) \mid u + \lambda t \text{ is minimal}\}.$$

Therefore if $N_{v,\phi}(f)$ has an edge L of slope $-\lambda$ then $L_w(f) = L$, otherwise $L_w(f)$ is one of the vertices of $N_{v,\phi}(f)$.

Notation 4.2.37 Let $\lambda \in \hat{\mathbb{Q}}$, $\lambda > v(\phi)$ and $w = [v, w(\phi) = \lambda]$. If $\lambda < \infty$ denote by $(t_w^0, u_w^0), (t_w, u_w)$ the two endpoints of $L_w(f)$ (equal if $L_w(f)$ is a vertex), where $t_w^0 \leq t_w$. If $\lambda = \infty$, set $t_w^0 = 0$, $u_w^0 = \infty$, and denote by (t_w, u_w) the left-most vertex of $N_{v,\phi}(f)$.

4.3 MacLane chains invariants and residual polynomials

Let $f \in K[x]$ and let $-\lambda$ be the slope of an edge L of the Newton polygon of f. From §2.2, given the 1-dimensional MacLane valuation $v = [v_0, v(x) = \lambda]$, there is a natural way to define a reduction $f|_v$ as $f|_L$. Our purpose is to extend this definition to compute reductions of polynomials with respect to any MacLane valuation. Part of the current section can be found in [FGMN, §3].

Let $v \in \mathbb{V}_M$ given by a MacLane chain

$$[v_0, v_1(\phi_1) = \lambda_1, \ldots, v_n(\phi_n) = \lambda_n]. \tag{4.3}$$

Note that most of the objects and quantities we define in this section are attached to the MacLane chain (4.3) rather than v itself, starting from the following data.

Definition 4.3.1 Set $v_{-1} = v_0$, $\pi_{-1} = \pi$, $\phi_0 = x$, $\lambda_0 = 0$ and for all $0 \le i \le n$ define

$$e_i = e_{v_i}/e_{v_{i-1}}, \qquad h_i = e_{v_i}\lambda_i, \qquad f_{i-1} = f_{v_i}/f_{v_{i-1}}.$$

Fix ℓ_i, ℓ'_i, such that $\ell_i h_i + \ell'_i e_i = 1$, with $0 \le \ell_i < e_i$. Then inductively define

$$\gamma_i = \phi_i^{e_i}\pi_{i-1}^{-h_i}, \qquad \pi_i = \phi_i^{\ell_i}\pi_{i-1}^{\ell'_i}.$$

Remark 4.3.2. Let $0 \le i < n$. Then $\deg v_{i+1} = e_i f_i \deg v_i$ and $v_i(\phi_n) \in \Gamma_{v_{i-1}}$.

Lemma 4.3.3 *For any $1 \le i \le n$ and any $j > i$, we have*

- $v_j(\gamma_i) = v_i(\gamma_i) = 0$*;*
- $v_j(\pi_i) = v_i(\pi_i) = \frac{1}{e_{v_i}}$*. So π_i is a uniformiser for v_i.*

Proof. The lemma follows by induction and the equality $v_j(\phi_i) = v_i(\phi_i)$. □

Notation 4.3.4 We will denote by $b_v, h_v, \ell_v, \ell'_v$ the quantities $e_n, h_n, \ell_n, \ell'_n$ respectively. They are independent of the chosen MacLane chain for v.

Lemma 4.3.5 *For any $0 \le i \le n-1$ there exists a polynomial $S_i \in K[x]$ such that $S_i \sim_{v_i} \pi_i$. Furthermore, there exists a polynomial $S'_i \in K[x]$ such that $v_i(S'_i) = -v_i(\pi_i)$ and $(S'_i)^{-1} \sim_{v_{i+1}} \pi_i$.*

Proof. First note that if S_i exists, then $S_i \sim_{v_{i+1}} \pi_i$ as $v_i(\pi_i) = v_{i+1}(\pi_i)$ by Lemma 4.3.3. Now we prove the lemma by induction on i. When $i = 0$, we can choose $S_i = \pi = \pi_i$ and $S'_i = \pi^{-1}$. Suppose $i > 0$. Define $S_i \in K[x]$ by

$$S_i = \begin{cases} \phi_i^{\ell_i} S_{i-1}^{\ell'_i} & \text{if } \ell'_i \ge 0, \\ \phi_i^{\ell_i}(S'_{i-1})^{-\ell'_i} & \text{if } \ell'_i < 0. \end{cases}$$

By inductive hypothesis, $S_{i-1} \sim_{v_i} \pi_{i-1}$ and $(S'_{i-1})^{-1} \sim_{v_i} \pi_{i-1}$. Therefore $S_i \sim_{v_i} \pi_i$. Finally, [Rüt, Lemma 4.24] shows the existence of S'_i. □

Lemma 4.3.6 *For any $0 \le i \le n$, we have $\phi_i = \gamma_i^{\ell'_i}\pi_i^{h_i}$ and $\pi_{i-1} = \gamma_i^{-\ell_i}\pi_i^{e_i}$.*

Proof. The lemma follows from direct computation. □

Lemma 4.3.7 *For any $0 \le i \le n$, we have*

$$\pi_i = \phi_i^{m'_i} \cdots \phi_1^{m'_1} \cdot \pi^{m'_0} \quad \textit{and} \quad \pi_i^{e_{v_i}} = \gamma_i^{m_i} \cdots \gamma_1^{m_1} \cdot \pi^{m_0},$$

where

$$m'_j = \begin{cases} \ell'_1 \cdot \ell'_i & \textit{if } j = 0, \\ \ell_j \ell'_{j+1} \cdots \ell'_i & \textit{if } j > 0. \end{cases} \quad \textit{and} \quad m_j = \begin{cases} 1 & \textit{if } j = 0, \\ e_1 \cdots e_{j-1}\ell_j & \textit{if } j > 0. \end{cases}$$

Note that

$$\gamma_i = \phi_i^{e_i} \cdot \phi_{i-1}^{-h_i m'_{i-1}} \cdots \phi_1^{-h_i m'_1} \pi^{-h_i m'_0}, \quad \phi_i^{e_{v_i}} = \gamma_i^{e_{v_{i-1}}} \cdot \gamma_{i-1}^{h_i m_{i-1}} \cdots \gamma_1^{h_i m_1} \pi^{h_i}.$$

Proof. The proof follows by mathematical induction and Lemma 4.3.6. $\square$

Let $i = 0,\dots,n$. Recall the definition of the residue map H_{v_i} of v_i. From [FGMN, Lemma 2.9] a polynomial $f \in K[x]$ belongs to U_{v_i} if $v_{i-1}(f) = v_i(f)$. Therefore ϕ_j, π_j, γ_j are units of P_{v_i}, for all $j = 0,\dots,i-1$. It follows that $\phi_i, \pi_i, \gamma_i \in P_{v_i}$, domain of H_{v_i}. Denote

$$x_i = H_{v_i}(\phi_i), \quad p_i = H_{v_i}(\pi_{i-1}), \quad y_i = H_{v_i}(\gamma_i).$$

Note that by [FGMN, Lemma 2.9] the set of units $A_{v_i}^\times$ of A_{v_i} coincides with the image of the canonical homomorphism $A_{v_{i-1}} \to A_{v_i}$.

We recall the following from [Rüt, §4.1.3] and [FGMN, §3.4]. There exist a sequence of simple field extensions

$$k = k_0 \subseteq k_1 \subseteq k_2 \subseteq \cdots \subseteq k_n,$$

with $k_i \simeq k_{\phi_i}$, such that for all $i = 0,\dots,n$ there are isomorphisms of k-algebras $\bar{H}_i : A_{v_i} \to k_i[X_i]$. One can see that $\bar{H}_i$ is the unique homomorphism satisfying:

(i) $\bar{H}_i(y_i) = X_i$;

(ii) $\bar{H}_i(u) = \bar{H}_{i-1}(u)$ when $i > 0$ and $u \in A_{v_{i-1}}$, where we canonically see $u \in A_{v_i}$ via $A_{v_{i-1}} \to A_{v_i}$ and $\bar{H}_{i-1}(u) \in k_i$ via the natural map $k_{i-1}[X_{i-1}] \to k_i$ taking X_{i-1} to the generator of k_i over k_{i-1}.

By [FGMN, Proposition 3.9], the canonical embedding $A_{v_i} \hookrightarrow \mathbb{F}_{v_i}$ induces an isomorphism between the field of fractions of A_{v_i} and $\mathbb{F}_{v_i}$. Therefore we can consider the largest subring $F_{v_i} \subset K(x)$ such that the isomorphism $\bar{H}_i$ lifts to a surjective homomorphism

$$H_i : F_{v_i} \to k_i[X_i^{\pm 1}],$$

satisfying $H_i(f) = H_i(g)$ if $f \sim_{v_i} g$. In particular, $P_{v_i}(0) \subseteq F_{v_i}$ and $H_i = \bar{H}_i \circ H_{v_i}$ on $P_{v_i}(0)$. Furthermore, note that $\gamma_i^{-1} \in F_{v_i}$ from (i).

Definition 4.3.8 Let $\alpha \in \Gamma_{v_i}$. Define

(i) $F_{v_i}(\alpha) = F_{v_i} \cdot P_{v_i}(\alpha) \subset K(x)$.

(ii) $H_{i,\alpha} : F_{v_i}(\alpha) \to k_i[X_i^{\pm 1}]$ given by $H_{i,\alpha}(f) = H_i\big(f/\pi_i^{e_{v_i}\alpha}\big)$.

The map $H_{i,\alpha}$ in (ii) is well-defined since $\pi_i^{-1} \in F_{v_i}(-\alpha)$.

Definition 4.3.9 For $0 \le i \le n$ and $\alpha \in \Gamma_{v_i}$, let $t_i(\alpha), u_i(\alpha) \in \mathbb{Z}$ such that $u_i(\alpha)e_i + t_i(\alpha)h_i = e_{v_i}\alpha$, with $0 \le t_i(\alpha) < e_i$. Define

(i) $\varphi_i(\alpha) = x_i^{t_i(\alpha)} p_i^{u_i(\alpha)} \in A_{v_i}(\alpha)$;

(ii) $c_i(\alpha) = \ell_i' t_i(\alpha) - \ell_i u_i(\alpha) \in \mathbb{Z}$.

Let $\alpha \in \Gamma_{v_i}$. Let $R_{i,\alpha} : O_{v_i}(\alpha) \to k_i[X_i]$ be the map defined in [FGMN, Definition 3.13], where we replaced the variable y with X_i. By [FGMN, Theorem 4.1], we have $A_{v_i}(\alpha) = \varphi_i(\alpha) A_{v_i}$ and $R_{i,\alpha}$ is the lift of the map

$$\bar{R}_{i,\alpha} : A_{v_i}(\alpha) \to k_i[X_i]$$

given by $\bar{R}_{i,\alpha}(\varphi_i(\alpha) \cdot a) = \bar{H}_i(a)$. Since $e_{v_i}\alpha = u_i(\alpha)e_i + t_i(\alpha)h_i$, by Lemma 4.3.6, we have

$$\pi_i^{e_{v_i}\alpha} \gamma_i^{c_i(\alpha)} = \phi_i^{t_i(\alpha)} \pi_{i-1}^{u_i(\alpha)}.$$

Therefore for any $f \in A_{v_i}(\alpha)$ we have

$$H_{i,\alpha}(f) = X_i^{c_i(\alpha)} \cdot R_{i,\alpha}(f). \tag{4.4}$$

We extend $R_{i,\alpha}$ through (4.4).

Definition 4.3.10 Let $\alpha \in \Gamma_{v_i}$. The *residual polynomial operator* $R_{i,\alpha}$ is the map $F_{v_i}(\alpha) \to k_i[X_i^{\pm 1}]$ given by $R_{i,\alpha}(f) = X_i^{-c_i(\alpha)} \cdot H_{i,\alpha}(f)$.

Remark 4.3.11. Let $0 \le i < n$ and $\alpha_i = v_i(\phi_{i+1}) = f_i e_i \lambda_i$. By [FGMN, Corollary 5.5(2)] the field k_{i+1} is isomorphic to $k_i[X_i]/(R_{i,\alpha_i}(\phi_{i+1}))$. Furthermore, $k_{i+1} \simeq k_i[X_i^{\pm 1}]/(H_{i,\alpha_i}(\phi_{i+1}))$ by definition.

Notation 4.3.12 We denote by k_v the field k_n. In fact, it does not depend on the radius of v.

Definition 4.3.13 Let $\alpha \in \Gamma_v$. For any $f \in F_v(\alpha)$, define $f|_{v,\alpha} \in k_v[X]$ by $f|_{v,\alpha}(X) = R_{n,\alpha}(f)(X)$.

Let $f \in K[x]$. Let $\alpha = v(f)$. Denote by $N_n(f)$ the Newton polygon N_{v_{n-1},ϕ_n}. If $n > 0$, consider the edge $L_v(f)$ of $N_n(f)$. Let $(t_v^0, u_v^0), (t_v, u_v)$ be the two endpoints of $L_v(f)$, with $t_v^0 \le t_v$. Note that $t_v^0 - t_n(\alpha) = e_n \cdot \lfloor t_v^0 / e_n \rfloor$.

Definition 4.3.14 ([FGMN, Definition 3.15]) The *reduction of* f *with respect to* v is

$$f|_v = \begin{cases} f|_{v,\alpha} & \text{if } n = 0, \\ f|_{v,\alpha} / X^{\lfloor t_v^0 / e_n \rfloor} & \text{if } n > 0. \end{cases}$$

Remark 4.3.15. Note that $f|_{v,\alpha}$ and $f|_v$ do depend on the chosen MacLane chain for v.

Note that

$$H_{n,\alpha}(f)(X) = X^{\lfloor t_v^0/e_n \rfloor + c(\alpha)} f|_v = X^{t_v^0/e_n - \ell_n e_{v_{n-1}} \alpha} f|_v. \tag{4.5}$$

Lemma 4.3.16 *Expand* $f = \sum_t a_t \phi_n^t$, $\deg a_t < \deg \phi_n$. *If* $n > 0$, *then*

$$f|_v = \sum_{j \ge 0} H_{n-1,\alpha_j}(a_{t_j}) X^j,$$

where $t_j = t_v^0 + j e_n$ *and* $\alpha_j = \alpha - t_j \lambda_n$.

Proof. There exists $f' \in K[x]$ such that $f \sim_v f'$ and $f' = \sum_t a'_t \phi_n^t$, where either $a'_t = 0$ or $a'_t = a_t$ and $v(a'_t) = \alpha - t\lambda_n$. If $a'_t \neq 0$, then $(t, v(a_t)) \in L_v(f)$. Since

$$L_v(f) \cap \left(\mathbb{Z} \times \frac{1}{e_{v_{n-1}}}\mathbb{Z}\right) = (t_v^0, \alpha - t_v^0\lambda_n) + (e_n, -\lambda_n)\mathbb{Z},$$

we have $f' = \sum_{j\geq 0} a'_{t_j} \phi_n^{t_j}$. It follows that

$$f' = \phi_n^{t_n(\alpha)} \pi_{n-1}^{u_n(\alpha)} \gamma_n^{\lfloor t_v^0/e_n \rfloor} \sum_{j\geq 0} \frac{a'_{t_j}}{\pi_{n-1}^{e_{v_{n-1}}\alpha_j}} \gamma_n^j.$$

Therefore

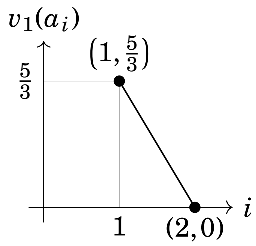
$$f|_{v,\alpha} = f'|_{v,\alpha} = X^{\lfloor t_v^0/e_n \rfloor} \sum_{j\geq 0} H_{n-1,\alpha_j}\left(a'_{t_j}\right) X^j. \tag{4.6}$$

Finally, note that $a'_{t_j} = 0$ if and only if $v(a_{t_j}) > e_{v_{n-1}}\alpha_j$. Thus in (4.6) we can replace $H_{n-1,\alpha_j}\left(a'_{t_j}\right)$ with $H_{n-1,\alpha_j}(a_{t_j})$. □

Example 4.3.17 Let $f = (x^3 - 2p)^2 - px^2(x^3 - 2p) \in \mathbb{Q}_p[x]$ $(p \neq 2)$ and

$$v = v_2 = [v_0, v_1(x) = 1/3, v_2(x^3 - 2p) = 5/3]$$

The Newton polygon $N_2(f)$ is



Then $\pi_0 = p$, $\pi_1 = x$, $\pi_2 = x, \gamma_1 = x^3p^{-1}$ and $k_1 = k_0 = \mathbb{F}_p$. Since $x^3 - 2p = p^{-1}(\gamma_1 - 2)$, then $R_{1,1}(x^3 - 2p) = X_1 - 2$. It follows that $k_2 = \mathbb{F}_p[X_1]/(X_1 - 2) \simeq \mathbb{F}_p$. Via Lemma 4.3.16 compute

$$f|_v = X + H_{1,5/3}(-px^2) = X + H_1\left(\frac{-px^2}{x^5}\right) = X + \bar{H}_1(-y_1^{-1}) = X - 2^{-1}.$$

Proposition 4.3.18 ([FGMN, Corollary 4.9, Corollary 4.11]) *Suppose $n > 0$. Following the notation above, we have:*

(i) the j-th coefficient of $f|_{v,\alpha}$ is non-zero if and only if $v_{n-1}(a_{t_j}) = \alpha_j$;

(ii) $\deg f|_{v,\alpha} = \lfloor t_v/b_v \rfloor$ and $\operatorname{ord}_X(f|_{v,\alpha}) = \lfloor t_v^0/b_v \rfloor$;

(iii) $\deg f|_v = (t_v - t_v^0)/b_v$ and $f|_v(0) \neq 0$;

(iv) $fh|_v = f|_v h|_v$ for all $h \in K[x]$.

Proposition 4.3.19 ([FGMN, Corollary 4.10]) *For non-zero $f, h \in K[x]$, the following conditions are equivalent:*

(i) $f \sim_v h$,

(ii) $v(f) = v(h)$ *and* $f|_v = h|_v$,

(iii) $L_v(f) = L_v(h)$ *and* $f|_v = h|_v$.

Lemma 4.3.20 ([FGMN, Lemma 5.1]) *A polynomial $f \in K[x]$ is v-irreducible if and only if either*

- $t_v^0 = t_v = 1$ *or*
- $t_v^0 = 0$ *and $f|_v$ is irreducible in $k_n[X]$.*

Lemma 4.3.21 ([FGMN, Lemma 5.2]) *Suppose $n > 0$. A monic $f \in K[x]$ is a key polynomial over v if and only if one of the two following conditions is satisfied:*

(1) $\deg f = \deg v$ *and* $f \sim_v \phi_n$;

(2) $t_v^0 = 0$, $\deg f = t_v \deg v$ *and $f|_v$ is irreducible.*

In case (2), $\deg f = b_v \deg v \cdot \deg f|_v$, $N_n(f) = L_v(f)$ *and $f|_v$ is monic.*

4.4 MacLane clusters

Let $f \in K[x]$ be a separable polynomial and let $c_f \in K$ be its leading term. Assume $f/c_f \in O_K[x]$ and write $\mathfrak{R}$ for the sets of roots of f in $\bar{K}$. If C/K is a hyperelliptic curve, it is always given by an equation $y^2 = f(x)$, where $f \in K[x]$ is as above.

Definition 4.4.1 A *MacLane cluster* (for f) is a pair $(\mathfrak{s}, v)$ where $\mathfrak{s} \subseteq \mathfrak{R}$, and v is a MacLane pseudo-valuation such that

1. $\mathfrak{s} = D_v \cap \mathfrak{R} \neq \varnothing$;
2. if $\mathfrak{s} = D_w \cap \mathfrak{R}$ for a MacLane valuation $w > v$ then $\deg w > \deg v$.

If v is a MacLane valuation then $(\mathfrak{s}, v)$ is said *proper MacLane cluster*. The degree of $(\mathfrak{s}, v)$ is $\deg v$. The *degree*, a *centre* and the *radius* of a MacLane cluster $(\mathfrak{s}, v)$ are the degree, a centre and the radius of v, respectively.

Remark 4.4.2. Let $(\mathfrak{s}, v)$ be a MacLane cluster. Note that by definition

(i) $\mathfrak{s}$ is G_K-invariant,

(ii) v determines $\mathfrak{s}$.

Definition 4.4.3 The *MacLane cluster picture* of f is the combinatorial data consisting of the collection of all MacLane clusters for f together with their radii. We will denote by Σ_f^M the set of all MacLane clusters for f.

Definition 4.4.4 We say that a MacLane pseudo-valuation $v \in \hat{\mathbb{V}}_M$ *defines a MacLane cluster* $(\mathfrak{s}, w) \in \Sigma_f^M$, if $w = v$ (and $\mathfrak{s} = D_v \cap \mathfrak{R}$).

Definition 4.4.5 We write $(\mathfrak{t}, w) \subseteq (\mathfrak{s}, v)$ if $w \geq v$. If $(\mathfrak{t}, w) \subsetneq (\mathfrak{s}, v)$ is maximal, we write $(\mathfrak{t}, w) < (\mathfrak{s}, v)$ and $v = P(w)$, and refer to $(\mathfrak{t}, w)$ as *a child* of $(\mathfrak{s}, v)$, and to $(\mathfrak{s}, v)$ as *the parent* of $(\mathfrak{t}, w)$. A proper MacLane cluster $(\mathfrak{s}, v)$ with no proper child of degree $\deg v$ is said *degree-minimal*.

Lemma 4.4.6 *Let $(\mathfrak{s}, v), (\mathfrak{t}, w) \in \Sigma_f^M$ such that $\mathfrak{s} \subsetneq \mathfrak{t}$. Then $(\mathfrak{s}, v) \subsetneq (\mathfrak{t}, w)$.*

Proof. Since $\mathfrak{s} \subseteq D_v \cap D_w$ either $D_v \subsetneq D_w$ or $D_w \subseteq D_v$. But

$$D_v \cap \mathfrak{R} = \mathfrak{s} \subsetneq \mathfrak{t} = D_w \cap \mathfrak{R},$$

so $D_v \subsetneq D_w$. Thus $w > v$. □

[KW, Proposition 2.26] shows that the meet of any two MacLane pseudo-valuations v and w exists; it will be denoted by $v \wedge w$. Hence $v \wedge w$ is the maximal MacLane pseudo-valuation $\leq v$ and $\leq w$. In other words, $\hat{\mathbb{V}}_M$ with $\leq$ forms a meet-semilattice.

Lemma 4.4.7 *Let $(\mathfrak{s}, v)$, $(\mathfrak{t}, w) \in \Sigma_f^M$, and $\mathfrak{s} \wedge \mathfrak{t} = D_{v \wedge w} \cap \mathfrak{R}$. Then $(\mathfrak{s} \wedge \mathfrak{t}, v \wedge w)$ is the smallest MacLane cluster containing $(\mathfrak{s}, v)$ and $(\mathfrak{t}, w)$.*

Proof. We only need to show that $(\mathfrak{s} \wedge \mathfrak{t}, v \wedge w)$ is a MacLane cluster. Suppose not. Then there exists a MacLane valuation $v' > v \wedge w$, with $\mathfrak{s} \wedge \mathfrak{t} = D_{v'} \cap \mathfrak{R}$ and $\deg v' \leq \deg(v \wedge w)$. Then $v' \not\leq v$ or $v' \not\leq w$, from the definition of $v \wedge w$. Without loss of generality we can assume that $v' \not\leq v$.

If $v \not< v'$, then $D_{v'} \not\subset D_v$ and $D_v \not\subseteq D_{v'}$ so $D_{v'} \cap D_v = \varnothing$ by Remark 4.2.18(2). But this contradicts

$$D_v \cap \mathfrak{R} = \mathfrak{s} \subseteq \mathfrak{s} \wedge \mathfrak{t} = D_{v'} \cap \mathfrak{R}.$$

If $v < v'$, then

$$\mathfrak{s} \subseteq \mathfrak{s} \cup \mathfrak{t} \subseteq \mathfrak{s} \wedge \mathfrak{t} = D_{v'} \cap \mathfrak{R} \subseteq D_v \cap \mathfrak{R} = \mathfrak{s}.$$

But then $\mathfrak{s} = D_{v'} \cap \mathfrak{R}$, $v' > v$ and $\deg v' \leq \deg(v \wedge w) \leq \deg v$ by Lemma 4.2.25, which contradicts the definition of MacLane cluster for $(\mathfrak{s}, v)$. □

Let $F \in K[x]$ be a monic irreducible factor of f. Let v_F be the MacLane pseudo-valuation with $D_{v_F} = D(F, \infty)$ (Theorem 4.2.21). We also denote v_F by v_r where $r \in \mathfrak{R}$ is any root of F. For any non-empty G_K-invariant subset $\mathfrak{s} \subseteq \mathfrak{R}$, define $g_\mathfrak{s} = \prod_{r \in \mathfrak{s}} (x - r) \in K[x]$. Then $g_\mathfrak{s} \mid f$. Let $F_1, \ldots, F_m$ be the irreducible monic factors of $g_\mathfrak{s}$. Define $v_\mathfrak{s} \in \hat{\mathbb{V}}_M$ by

$$v_\mathfrak{s} = v_{F_1} \wedge \cdots \wedge v_{F_m}.$$

Lemma 4.4.8 *Let $v \in \hat{\mathbb{V}}_M$ and let $\mathfrak{s} = D_v \cap \mathfrak{R} \neq \varnothing$. Then $v \leq v_{\mathfrak{s}}$. In particular, $(\mathfrak{s}, v_{\mathfrak{s}})$ is a MacLane cluster.*

Proof. The set $\mathfrak{s}$ is G_K-invariant, as so are D_v and $\mathfrak{R}$. Let $F_1, \dots, F_m$ be the irreducible factors of $g_{\mathfrak{s}}$ as above. Let $\mathfrak{s}_i$ be the set of roots of F_i. Note that $D_{v_{F_i}} = \mathfrak{s}_i$ for all i. Then $D_{v_{\mathfrak{s}}} \supseteq \bigcup_{i=1}^m D_{v_{F_i}} = \mathfrak{s}$. Suppose $w \in \hat{\mathbb{V}}_M$ with $\mathfrak{s} = D_w \cap \mathfrak{R}$. Then $D_{v_{F_i}} \subseteq D_w$, so $w \leq v_{F_i}$ for all i. By definition of $v_{\mathfrak{s}}$ we have $w \leq v_{\mathfrak{s}}$. Since $w \leq v_{\mathfrak{s}}$ for any w with $\mathfrak{s} = D_w \cap \mathfrak{R}$, it only remains to show that $\mathfrak{s} = D_{v_{\mathfrak{s}}} \cap \mathfrak{R}$. Since $v \leq v_{\mathfrak{s}}$ from above, we have

$$\mathfrak{s} \subseteq D_{v_{\mathfrak{s}}} \cap \mathfrak{R} \subseteq D_v \cap \mathfrak{R} = \mathfrak{s},$$

that implies $D_{v_{\mathfrak{s}}} \cap \mathfrak{R} = \mathfrak{s}$. Thus $(\mathfrak{s}, v_{\mathfrak{s}})$ is a MacLane cluster. □

Lemma 4.4.9 *Let $\mathfrak{s} = D_v \cap \mathfrak{R} \neq \varnothing$, for some $v \in \hat{\mathbb{V}}_M$. Let*

$$[v_0, v_1(\phi_1) = \lambda_1, \dots, v_n(\phi_n) = \lambda_n]$$

be a minimal MacLane chain for $v_{\mathfrak{s}}$. Then there exists $i = 0, \dots, n$ such that $v \leq v_i$, $\deg v = \deg v_i$ and $(\mathfrak{s}, v_i)$ is a cluster. In particular, if $(\mathfrak{s}, v)$ is a MacLane cluster, then $v = v_i$.

Proof. Let $w \in \hat{\mathbb{V}}_M$ such that $D_w \cap \mathfrak{R} = \mathfrak{s}$. Then $w \leq v_{\mathfrak{s}}$ by Lemma 4.4.8. Proposition 4.2.24 implies that $w \leq v_i$, $\deg w = \deg v_i$ for some $i = 0, \dots, n$.

The argument above holds in particular when $w = v$. It only remains to show that $\mathfrak{s} = D_{v_i} \cap \mathfrak{R}$. We have

$$\mathfrak{s} = D_{v_{\mathfrak{s}}} \cap \mathfrak{R} \subseteq D_{v_i} \cap \mathfrak{R} \subseteq D_v \cap \mathfrak{R} = \mathfrak{s},$$

that implies $\mathfrak{s} = D_{v_i} \cap \mathfrak{R}$, as required. □

Proposition 4.4.10 *The set Σ_f^M under the partial order $\supseteq$ forms a rooted tree.*

Proof. Let $V_f^M = \{v \in \hat{\mathbb{V}}_M \mid (D_v \cap \mathfrak{R}, v) \in \Sigma_f^M\}$. By Remark 4.4.2(ii) there is a natural bijection from V_f^M to Σ_f^M taking $v \mapsto (D_v \cap \mathfrak{R}, v)$ inverting partial orders by definition. Hence it suffices to show that V_f^M is a rooted tree. First note that $V_f^M \neq \varnothing$ since $v_F \in V_f^M$ for any monic irreducible factor F of f. Then V_f^M is a rooted tree by [KW, Corollary 2.8] and Lemma 4.4.7. □

Lemma 4.4.11 *Let $(\mathfrak{s}, v)$ be a MacLane cluster. Then $|\mathfrak{s}| \geq \deg v$. Furthermore, $|\mathfrak{s}| > \deg v$ if and only if $(\mathfrak{s}, v)$ is proper.*

Proof. First note that $v \leq v_{\mathfrak{s}}$ by Lemma 4.4.8. Then Lemma 4.2.25 implies

$$\deg v \leq \deg v_{\mathfrak{s}} \leq \min_{r \in \mathfrak{s}} \deg v_r = \min_{r \in \mathfrak{s}} |G_K \cdot r| \leq |\mathfrak{s}|.$$

If $|\mathfrak{s}| = \deg v$, then $\mathfrak{s} = G_K \cdot r$ for some (any) $r \in \mathfrak{s}$, and $\deg v = \deg v_{\mathfrak{s}}$. It follows from Lemma 4.4.9 that $v = v_{\mathfrak{s}} = v_r$. Hence $(\mathfrak{s}, v)$ is not proper.

If $(\mathfrak{s}, v)$ is not proper, that is $v \notin \mathbb{V}_M$, then $v = v_{\mathfrak{s}} = v_r$ for some (any) $r \in \mathfrak{s}$. In particular, $\mathfrak{s} = \deg v_r = \deg v$. □

Remark 4.4.12 (Alternative definition for MacLane clusters). Let Σ be the set of pairs $(\mathfrak{s},n)$, where $n\in\mathbb{Z}_+$ and $\mathfrak{s}=D_v\cap\mathfrak{R}\neq\varnothing$ for some MacLane pseudo-valuation v of degree n. It follows from Remark 4.2.18(2) and Theorem 4.2.21 that the map $\Sigma_f^M\to\Sigma$, taking $(\mathfrak{s},v)\mapsto(\mathfrak{s},\deg v)$ is bijective.

Lemma 4.4.13 *Let $(\mathfrak{s},v)$ be a MacLane cluster. Then $\lambda_v=\min_{r\in\mathfrak{s}}v_K(\phi(r))$ for any centre ϕ of v.*

Proof. Let $\lambda=\min_{r\in\mathfrak{s}}v_K(\phi(r))$. Since $\mathfrak{s}\subset D_v=D(\phi,\lambda_v)$, we have $\lambda\geq\lambda_v$. Suppose $\lambda>\lambda_v$. Let $w=[v,w(\phi)=\lambda]$. Then $w>v$ and $\deg w=\deg\phi=\deg v$. But $\mathfrak{s}=\mathfrak{R}\cap D_w$ for our choice of λ. This contradicts the fact that $(\mathfrak{s},v)$ is a MacLane cluster (Definition 4.4.12). $\square$

Notation 4.4.14 Let $\mathcal{P}\subset K[x]$ to be the subset of monic irreducible polynomials. For any $d\in\mathbb{Z}$, denote by $\mathcal{P}_{\leq d}$ the set $\{g\in\mathcal{P}\mid \deg g\leq d\}$.

Lemma 4.4.15 *Let $(\mathfrak{s},v)$ be a proper MacLane cluster. Then*

$$\lambda_v=\max_{g\in\mathcal{P}_{\leq\deg v}}\min_{r\in\mathfrak{s}}v_K(g(r)).$$

Proof. Let $d=\deg v$. By Lemma 4.4.13 we only need to show that $\lambda_v\geq\max_{g\in\mathcal{P}_{\leq d}}\min_{r\in\mathfrak{s}}v_K(g(r))$. Suppose not. Then there exists a polynomial $g\in\mathcal{P}_{\leq d}$ such that $\lambda:=\min_{r\in\mathfrak{s}}v_K(g(r))>\lambda_v$. Let $w\in\hat{\mathbb{V}}_M$ such that $D_w=D(g,\lambda)$ (Theorem 4.2.21). Then $\mathfrak{s}\subseteq D_w\cap\mathfrak{R}$. By Lemma 4.2.22 we have $\deg w\leq\deg g\leq\deg v$ and $w(g)\geq\lambda$. Since $\mathfrak{s}\subset D_w\cap D_v$, either $D_v\subseteq D_w$ or $D_w\subsetneq D_v$ by Remark 4.2.18(2). If $D_w\subsetneq D_v$, then $w>v$ and $\mathfrak{s}=D_w\cap\mathfrak{R}$, a contradiction, since $(\mathfrak{s},v)$ is a MacLane cluster. So $D_v\subseteq D_w$, that is $v\geq w$. Hence $v(g)\geq w(g)\geq\lambda>\lambda_v$. This gives a contradiction since $v(g)\leq\lambda_v$ by Lemma 4.2.26. $\square$

Lemma 4.4.16 *Let $v\in\hat{\mathbb{V}}_M$ and $\mathfrak{s}=D_v\cap\mathfrak{R}$. Then $(\mathfrak{s},v)\in\Sigma_f^M$ if and only if*

$$\lambda_v=\max_{g\in\mathcal{P}_{\leq\deg v}}\min_{r\in\mathfrak{s}}v_K(g(r)).$$

Proof. One implication follows from Lemma 4.4.15. Suppose

$$\lambda_v=\max\nolimits_{g\in\mathcal{P}_{\leq\deg v}}\min\nolimits_{r\in\mathfrak{s}}v_K(g(r)).$$

By Lemma 4.4.9, there exists a MacLane pseudo-valuation $w\geq v$ with $\deg w=\deg v$ such that $(\mathfrak{s},w)\in\Sigma_f^M$. Let λ_w be the radius of w. Then Lemma 4.4.15 implies $\lambda_w=\lambda_v$. But this is possible only if $w=v$, by Lemma 4.2.25. $\square$

Lemma 4.4.17 *Let $v\neq v_0$ be a MacLane valuation, ϕ a strong key polynomial over v and $\lambda\in\hat{\mathbb{Q}}$, $\lambda>v(\phi)$. Set $w=[v,w(\phi)=\lambda]$, $\mathfrak{s}=D_v\cap\mathfrak{R}$, $\mathfrak{t}=D_w\cap\mathfrak{R}$. If $\mathfrak{t}\neq\varnothing$, then $(\mathfrak{s},v)$ is a MacLane cluster.*

Proof. First note that $\mathfrak{t}\subseteq\mathfrak{s}$. Let $g\in K[x]$ be any monic irreducible polynomial of degree $\deg g\leq\deg v$. Then $\deg g<\deg\phi$ and so $w(g)=v(g)$. Hence Proposition 4.2.32 implies that

$$v(g)=w(g)=\min_{r\in\mathfrak{t}}v_K(g(r))\geq\min_{r\in\mathfrak{s}}v_K(g(r))\geq v(g).$$

As g was arbitrary, $(\mathfrak{s},v)$ is a MacLane cluster by Lemmas 4.2.26 and 4.4.16. $\square$

Lemma 4.4.18 *Let $(\mathfrak{s}, v)$ be a MacLane cluster and let $(\mathfrak{t}, w)$ be its parent. Then $v = [w, v(\phi) = \lambda_v]$ for any centre ϕ of v.*

Proof. The lemma follows from Proposition 4.2.24 and Lemma 4.4.17. □

Proposition 4.4.19 *Let $F \in O_K[x]$ monic and irreducible. Let $v, w \in \hat{\mathbb{V}}_M$ such that $v \le v_F$, e.g. when $v \in \mathbb{V}_M$ and $F \in \mathrm{KP}(v)$. Then*

$$(v \wedge w)(F) = \min\{v(F), w(F)\}$$

In particular, if $v \not< w$, then $w(F) = (v \wedge w)(F)$.

Proof. The first part of the statement follows from the proof of [KW, Proposition 2.26], defining $w \wedge v$. Suppose $v \not< w$. If $v = w$, then $(v \wedge w)(F) = w(F)$. If $v \not\le w$, then $v \wedge w < v \le v_F$. This implies $v(F) > (v \wedge w)(F)$ by [KW, Lemma 2.22]. Thus $(v \wedge w)(F) = w(F)$. □

Lemma 4.4.20 *Let $v \in \hat{\mathbb{V}}_M$. Then*

$$v(f) = v_K(c_f) + \sum_{F \in \mathcal{P}, F|f} \deg F \cdot \frac{\lambda_{v \wedge v_F}}{\deg(v \wedge v_F)} = v_K(c_f) + \sum_{r \in \mathfrak{R}} \frac{\lambda_{v \wedge v_r}}{\deg(v \wedge v_r)}.$$

Proof. Recall $f/c_f \in O_K[x]$. Then $f = c_f \cdot \prod_{F \in \mathcal{P}, F|f} F$ and the factors F in the product belong to $O_K[x]$. It suffices to show that $v(F) = \deg F \cdot \frac{\lambda_{v \wedge v_F}}{\deg(v \wedge v_F)}$ for all $F \in \mathcal{P} \cap O_K[x]$. Let $F \in \mathcal{P} \cap O_K[x]$. By Lemma 4.2.29, the polynomial F is w-minimal, for any MacLane valuation $w < v_F$. In particular, F is $(v \wedge v_F)$-minimal. Hence

$$\frac{(v \wedge v_F)(F)}{\deg F} = \frac{\lambda_{v \wedge v_F}}{\deg(v \wedge v_F)}.$$

by Theorem 4.2.27. Since $v_F \not< v$, Proposition 4.4.19 shows $v(F) = (v \wedge v_F)(F)$ and so concludes the proof. □

4.4.1 Newton polygons

Let v be a MacLane valuation and $\phi \in \mathrm{KP}(v)$. Recall the definition of the Newton polygon $N_{v,\phi}(f)$.

Definition 4.4.21 The *principal Newton polygon* $N^-_{v,\phi}(f)$ is formed by the edges of $N_{v,\phi}(f)$ with slope $< -v(\phi)$.

For any edge L of $N^-_{v,\phi}(f)$ with slope $-\lambda$, define the MacLane valuation $v_L = [v, v_L(\phi) = \lambda]$. Then $L = L_{v_L}(f)$ (Notation 4.2.36). Denote by λ_L the radius of v_L.

The aim of this subsection is proving the following result, that gives a correspondence between MacLane clusters and edges of certain Newton polygons attached to f. It can be viewed as a generalisation of Lemma 2.3.38. Since the statement of the theorem may be not easy to digest, let us briefly present its main consequence. Let $(\mathfrak{s}, \mu)$ be a degree-minimal MacLane cluster with centre ϕ. Suppose that $v = v_0$ or that v defines a MacLane cluster (e.g. ϕ is a strong key polynomial over v). Then there is a 1-to-1 correspondence between the proper MacLane clusters

$(\mathfrak{t}, w)$ of degree $\deg \mu$ satisfying $v < w \leq \mu$ and the edges of the principal Newton polygon $N^-_{v,\phi}(f)$. Moreover, the radii of the MacLane clusters are the opposites of the slopes of the edges.

The generality of Theorem 4.4.22 allows us to use it as one of the key results to construct proper MacLane clusters algorithmically from f (see Remark 4.4.31).

Theorem 4.4.22 *Let $v \in \mathbb{V}_M$ and $\phi \in \mathrm{KP}(v)$.*

(i) If $(\mathfrak{t}, w')$ is a MacLane cluster with centre $\phi' \sim_v \phi$ satisfying $w'(\phi) < \infty$, then $N^-_{v,\phi}(f)$ has an edge L of slope $-w'(\phi)$ and $t_{v_L} = |\mathfrak{t}|/\deg \phi$.

(ii) Conversely, for every edge L of $N^-_{v,\phi}(f)$ there is a MacLane cluster $(\mathfrak{t}, w_L)$ with $w_L \geq v_L$, $\deg w_L = \deg \phi$, $w_L(\phi) = \lambda_L$ and $|\mathfrak{t}| = t_{v_L} \deg \phi$.

In case (ii), if there exists a proper $(\mathfrak{s}, w) \in \Sigma^M_f$ with $w = [v, w(\phi) = \lambda]$, $\lambda \geq \lambda_L$, then $w_L = v_L$.

We first recall the following result from [FGMN].

Theorem 4.4.23 ([FGMN, Theorem 6.2]) *Let $F \in O_K[x]$ be a monic irreducible polynomial and $r \in \bar{K}$ a root of F. Then $\phi \mid_v F$ if and only if $v_K(\phi(r)) > v(\phi)$. Moreover, if this condition holds, one also has:*

1. *Either $F = \phi$, or $N_{v,\phi}(F)$ consists of one edge of slope $-v_K(\phi(r))$.*
2. *$d := \deg F / \deg \phi \in \mathbb{Z}_+$ and $F \sim_v \phi^d$.*

Lemma 4.4.24 *Let $w = [v, w(\phi) = \lambda]$ be an augmentation of v. Let $\mathfrak{s}_\lambda$ be the set of roots r of f satisfying $v_K(\phi(r)) = \lambda$. Then $|\mathfrak{s}_\lambda| / \deg \phi = t_w - t^0_w$.*

Proof. Without loss of generality we can suppose f monic. If $\lambda = \infty$, then $|\mathfrak{s}_\lambda| = \mathrm{ord}_\phi(f)$ and the equality $|\mathfrak{s}_\lambda| / \deg \phi = t_w - t^0_w$ follows from the definition of t^0_w, t_w. Hence suppose $\lambda < \infty$.

We first show the statement for $f = F$ irreducible. In this case either $\mathfrak{s}_\lambda = \varnothing$ or $\mathfrak{s}_\lambda = \mathfrak{R}$. Suppose $\mathfrak{s}_\lambda = \mathfrak{R}$, which means $v_K(\phi(r)) = \lambda > v(\phi)$ for any (some) $r \in \mathfrak{R}$. Since $F \neq \phi$ (otherwise $\phi(r) = 0$), Theorem 4.4.23 implies that $L_w(F) = N_{v,\phi}(F)$, $t^0_w = 0$ and $t_w = \deg F / \deg \phi = |\mathfrak{s}_\lambda| / \deg \phi$. Now suppose that $L_w(F)$ is an edge of $N_{v,\phi}(F)$. So $t_w \geq 1$. We want to show $\mathfrak{s}_\lambda \neq \varnothing$. Let $t = t_w$. Expand

$$F = \sum_{j=0}^{d} a_j \phi^j, \qquad a_j \in K[x], \deg a_j < \deg \phi, a_d \neq 0.$$

By definition of $L_w(f)$ we have $w(a_j\phi^j) \geq w(a_t\phi^t)$ for all j. Therefore

$$v(a_t\phi^t) = w(a_t\phi^t) - t(\lambda - v(\phi)) < w(a_j\phi^j) - j(\lambda - v(\phi)) = v(a_j\phi^j)$$

for all $j < t$. In particular, $v(a_0) > v(F)$, so $\phi \mid_v F$. Theorem 4.4.23 then implies that $-\lambda = -v_K(\phi(r))$ for any $r \in \mathfrak{R}$. Therefore $\mathfrak{s}_\lambda \neq \varnothing$.

Let $f \in O_K[x]$ be any monic separable polynomial. Write $f = F_0 \cdots F_t$, with $F_j \in O_K[x]$ monic irreducible. Denote by $\mathfrak{R}_j$ the set of roots of F_j and by $\mathfrak{s}_{\lambda,j}$ the elements $r \in \mathfrak{R}_j$ satisfying $v_K(\phi(r)) = \lambda$. Clearly $\mathfrak{s}_\lambda = \bigsqcup_j \mathfrak{s}_{\lambda,j}$. Moreover, from [FGMN, Corollary 2.7], we have

$$L_w(f) = L_w(F_0) + \cdots + L_w(F_t)$$

(see before [FGMN, Corollary 2.7] for a definition of $+$). The lemma then follows from the first part of the proof. $\square$

Proposition 4.4.25 *Let $w = [v, w(\phi) = \lambda]$ be an augmentation of v and let $\mathfrak{s} = D_w \cap \mathfrak{R}$. Then $t_w = |\mathfrak{s}|/\deg\phi$.*

Proof. By definition $t_w = \sum_{\lambda' \geq \lambda}(t_{w'} - t^0_{w'})$, where $w' = [v, w'(\phi) = \lambda']$. Lemma 4.4.24 implies

$$t_w \deg\phi = \sum_{\lambda' \geq \lambda} |\mathfrak{s}_{\lambda'}| = \Big|\bigcup_{\lambda' \geq \lambda} \mathfrak{s}_{\lambda'}\Big| = |\mathfrak{s}|,$$

where $\mathfrak{s}_{\lambda'} \subseteq \mathfrak{R}$ is the set of roots r of f satisfying $v_K(\phi(r)) = \lambda'$. $\square$

Now we are ready to prove Theorem 4.4.22.

Proof of Theorem 4.4.22. (i). Let $(\mathfrak{t}, w)$ be a cluster with centre $\phi' \sim_v \phi$ and $w(\phi) < \infty$. In particular, $\deg\phi = \deg\phi'$. Let $\lambda_\mathfrak{t} = \min_{r\in\mathfrak{t}} v_K(\phi(r)) \geq w(\phi)$. Consider the MacLane valuation $w_\mathfrak{t} = [v, w_\mathfrak{t}(\phi) = \lambda_\mathfrak{t}]$. Then $\mathfrak{t} \subseteq D_{w_\mathfrak{t}} \cap \mathfrak{R}$. By Remark 4.2.18(2) and Theorem 4.2.21, either $w_\mathfrak{t} > w$ or $w_\mathfrak{t} \leq w$. By definition of MacLane cluster we have $w_\mathfrak{t} \leq w$. But then $\lambda_\mathfrak{t} \leq w(\phi)$. Thus $\lambda_\mathfrak{t} = w(\phi)$. Furthermore,

$$\mathfrak{t} \subseteq D_{w_\mathfrak{t}} \cap \mathfrak{R} \subseteq D_w \cap \mathfrak{R} = \mathfrak{t},$$

and so $\mathfrak{t} = D_{w_\mathfrak{t}} \cap \mathfrak{R}$. Then Lemma 4.4.24 implies that $L_{w_\mathfrak{t}}(f)$ is an edge of $N_{v,\phi}(f)$. The equality $t_{w_\mathfrak{t}} \deg\phi = |\mathfrak{t}|$ follows from Proposition 4.4.25.

(ii). Let L be an edge of $N^-_{v,\phi}(f)$. Let $\mathfrak{t} = D_{v_L} \cap \mathfrak{R}$. From Lemma 4.4.24 and Proposition 4.4.25 it follows that

$$|\mathfrak{t}| = t_{v_L} \cdot \deg\phi \quad \text{and} \quad \min_{r\in\mathfrak{t}} v_K(\phi(r)) = \lambda_L.$$

By Lemma 4.4.9 there exists a unique MacLane pseudo-valuation $w_L \geq v_L$ such that $\deg w_L = \deg v_L = \deg\phi$ and $(\mathfrak{t}, w_L)$ is a cluster. In particular, $w_L(\phi) = \lambda_L$ as

$$\lambda_L = v_L(\phi) \leq w_L(\phi) \leq \min_{r\in\mathfrak{t}} v_K(\phi(r)) = \lambda_L.$$

there exists a proper MacLane cluster $(\mathfrak{s}, w)$ with $w = [v, w(\phi) = \lambda]$, $\lambda \geq \lambda_L$. Then $w \geq v_L$ and so $\mathfrak{s} \subseteq \mathfrak{t}$. Furthermore, $\deg w_L = \deg v_L = \deg w$; hence, by definition of cluster, if $\mathfrak{s} = \mathfrak{t}$ then $w = v_L = w_L$. So suppose $\mathfrak{s} \subsetneq \mathfrak{t}$. It follows from Lemma 4.4.6 that $(\mathfrak{s}, w) \subsetneq (\mathfrak{t}, w_L)$. Since ϕ is centre of w, Lemma 4.2.25 implies that ϕ is also a centre of w_L. But we have already showed $w_L(\phi) = \lambda_L$, so $w_L = v_L$ as required. $\square$

4.4.2 Residual polynomials

In this subsection we will see that there is a close relationship between certain children $(\mathfrak{t}, w) < (\mathfrak{s}, v)$ and multiple irreducible factors of $f|_v$. We will need the following result.

Theorem 4.4.26 ([FGMN, Theorem 6.4]) *Let $v \in \mathbb{V}_M$ and let $\phi \in K[x]$ be a proper key polynomial over v. Every monic $g \in O_K[x]$ factorises into a product of monic polynomials in $O_K[x]$*

$$g = g_0 \cdot \phi^{\mathrm{ord}_\phi(g)} \prod_{\lambda,h} g_{\lambda,h},$$

where $-\lambda$ runs on the slopes of $N^-_{v,\phi}(g)$ and $h \in k_{w_\lambda}[X]$ runs on the monic irreducible factors of $g|_{w_\lambda}$, where $w_\lambda = [v, w_\lambda(\phi) = \lambda]$. Let $g = F_1, \dots, F_s$ be the factorisation of g in monic irreducible polynomials $F_j \in O_K[x]$. Then g_0 is the product of all F_j such that $\phi \nmid_v F_j$, while $g_{\lambda,h}$ is the product of all F_j with $N_{v,\phi}(F_j)$ one-sided of slope $-\lambda$ and $F_j|_{w_\lambda} = h^l$ for some l. In particular,

$$\deg g_0 = \deg g - l(N^-_{v,\phi}(g)) \deg\phi, \qquad \deg g_{\lambda,h} = b_{w_\lambda} \cdot \mathrm{ord}_h(g|_{w_\lambda}) \cdot \deg h \cdot \deg\phi,$$

where b_{w_λ} (Notation 4.3.4) equals the denominator of $e_v\lambda$.

Consider a MacLane valuation v. Assume $v \neq v_0$. Let ϕ_v be a centre of v. By Proposition 4.2.31 there exists a unique MacLane valuation v' over which ϕ_v is a strong key polynomial. Then $v = [v', v(\phi_v) = \lambda_v]$. Let $\mathfrak{s} = D_v \cap \mathfrak{R}$. We decompose

$$f/c_f = f_0 \phi_v^{\mathrm{ord}_{\phi_v}(f)} \prod_{\lambda,h} f_{\lambda,h}, \tag{4.7}$$

as in Theorem 4.4.26 with respect to the principal Newton polygon $N^-_{v',\phi_v}(f)$. Recall $\epsilon_v = e_{v'}$ and b_v equals the denominator of $\epsilon_v\lambda_v$.

Lemma 4.4.27 *If $\phi \in \mathrm{KP}(v)$ such that $\phi|_v$ is a multiple irreducible factor of $f|_v$, then $N^-_{v,\phi}(f)$ has an edge.*

Proof. By Theorem 4.4.26 it suffices to show that f has a monic irreducible factor $F \neq \phi$ that v-divisible by ϕ. Let $h = \phi|_v$. Since $f_{\lambda_v,h}|_v = h^{\mathrm{ord}_h(f|_v)}$, one has $f_{\lambda_v,h} \neq \phi$. As f is separable, there exists a monic irreducible factor F of $f_{\lambda_v,h}$ different from ϕ. Thus $\phi \mid_v F$ by [FGMN, Theorem 5.3]. $\square$

Lemma 4.4.28 *Let $w = [v, w(\phi) = \lambda]$ be an augmentation of v. Suppose $(\mathfrak{t}, w)$ is a proper MacLane cluster. If $\phi|_v$ is irreducible*[4]*, then $\mathrm{ord}_{\phi|_v}(f|_v) > 1$.*

Proof. Let $h = \phi|_v$. Lemma 4.3.21 implies $\phi \not\sim_v \phi_v$ and

$$\deg\phi = b_v \deg h \deg\phi_v. \tag{4.8}$$

[4]Note that $\phi|_v$ is irreducible if and only if ϕ is not a centre of v, by Lemma 4.3.21.

Then by Theorem 4.4.26 it suffices to show that $\deg f_{\lambda_v,h} > \deg\phi$. Since $\phi \not\sim_v \phi_v$ one has $w(\phi_v) = \lambda_v$ by [Rüt, Lemmas 4.13,4.14]. Let $r \in \mathfrak{t}$ and let $F \in O_K[x]$ be the minimal polynomial of r. Then

$$v_K(\phi_v(r)) = w(\phi_v) = v(\phi_v) = \lambda_v > v'(\phi_v),$$

where the first equality follows from Proposition 4.2.32. Then either $F = \phi_v$ or $N_{v',\phi_v}(F)$ consists of one edge of slope $-\lambda_v$ by Theorem 4.4.23. On the other hand

$$v_K(\phi(r)) \geq w(\phi) = \lambda > v(\phi).$$

Again by Theorem 4.4.23 we have $F \sim_v \phi^l$, for some $l \in \mathbb{Z}_+$. In particular, $F \neq \phi_v$ and $F|_v = h^l$ by Propositions 4.3.19 and 4.3.18(iv). It follows from Theorem 4.4.26 that $F \mid f_{\lambda_v,h}$. Thus $|\mathfrak{t}| \leq \deg f_{\lambda_v,h}$. Then Lemma 4.4.11 concludes the proof. □

Theorem 4.4.29 *Suppose $(\mathfrak{s}, v)$ is a proper MacLane cluster with $v \neq v_0$.*

(i) Let $h \in k_v[X]$ monic and irreducible such that $\mathrm{ord}_h(f|_v) > 1$. There exists a proper child $(\mathfrak{t}, w) < (\mathfrak{s}, v)$ with centre ϕ such that $\phi|_v = h$.

(ii) Conversely, for any proper child $(\mathfrak{t}, w) < (\mathfrak{s}, v)$ with centre ϕ such that $\phi|_v$ is irreducible, one has $\mathrm{ord}_{\phi|_v}(f|_v) > 1$.

In either case, $f_{\lambda_v,\phi|_v} = \prod_{r\in\mathfrak{t}}(x-r)$ and $\mathrm{ord}_{\phi|_v}(f|_v) = |\mathfrak{t}|/\deg w$.

Proof. Without loss of generality assume f monic. Let $v' \in \mathbb{V}_M$ and $\phi_v \in \mathrm{KP}(v)$ as above and consider the factorisation (4.7) of f.

(i). Suppose that the monic irreducible polynomial $h \in k_v[X]$ is a multiple factor of $f|_v$. Then

$$f_{\lambda_v,h}|_v = h^{\mathrm{ord}_h(f|_v)} \qquad \text{where } \mathrm{ord}_h(f|_v) > 1.$$

By [FGMN, Theorem 5.7] there exists $\phi \in \mathrm{KP}(v)$ such that $\phi|_v = h$. Let $\mathfrak{R}_h$ be the set of roots of $f_{\lambda_v,h}$ and set

$$\lambda = \min_{r\in\mathfrak{R}_h} v_K(\phi(r)).$$

Now ϕ is a proper key polynomial over v since $\phi|_v$ is irreducible. Then [FGMN, Theorem 5.13] implies that $\phi \mid_v F$ for any irreducible monic factor F of $f_{\lambda_v,h}$. Hence $\lambda > v(\phi)$ by Theorem 4.4.23. Therefore $w = [v, w(\phi) = \lambda]$ is an augmentation of v. Let $\mathfrak{t} = D_w \cap \mathfrak{R}$. From the definition of λ we have $\mathfrak{R}_h \subseteq \mathfrak{t}$. The pair $(\mathfrak{t}, w)$ may not be a MacLane cluster. However, by Lemma 4.4.9, we can find a MacLane pseudo-valuation $w' \geq w$ with $\deg w' = \deg w$ such that $(\mathfrak{t}, w')$ is an MacLane cluster. Let ψ be a centre of w'. Then ψ is a centre of w by Lemma 4.2.25. It follows from Lemma 4.2.16 that $\psi \in \mathrm{KP}(v)$ and $\psi \sim_v \phi$. Hence $\psi|_v = \phi|_v = h$ by Proposition 4.3.19. Therefore, by replacing ϕ with ψ and w with w' if necessary, we can assume $(\mathfrak{t}, w)$ is a MacLane cluster. Furthermore,

$$|\mathfrak{t}| \geq |\mathfrak{R}_h| = \deg f_{\lambda_v,h} > b_v \deg h \deg v = \deg\phi$$

by Theorem 4.4.26 and Lemma 4.3.21. Lemma 4.4.11 implies that $(\mathfrak{t}, w)$ is proper.

The MacLane cluster $(\mathfrak{t}, w)$ may not be a child of $(\mathfrak{s}, v)$. Suppose there exists a (proper) MacLane cluster $(\mathfrak{t}', w')$ such that $(\mathfrak{t}, w) \subsetneq (\mathfrak{t}', w') \subsetneq (\mathfrak{s}, v)$. We want to show that ϕ is a centre of w'. Suppose $\deg w > \deg w'$. Then for any centre ϕ' of w', $\deg \phi' < \deg \phi$ and so $w(\phi') = v(\phi')$. On the other hand, $w' > w$ and $w'(\phi') > v(\phi')$, so $w(\phi') \geq w'(\phi') > v(\phi')$, which gives a contradiction. Hence Lemma 4.2.25 implies that ϕ is also a centre of $(\mathfrak{t}', w')$.

(ii). Let $(\mathfrak{t}, w) < (\mathfrak{s}, v)$ proper with centre ϕ such that $\phi|_v$ is irreducible. Then $w > v$. Proposition 4.2.24 and Lemma 4.4.17 implies that $w = [v, w(\phi) = \lambda]$ for some $\lambda > v(\phi)$, since $(\mathfrak{t}, w)$ is a child of $(\mathfrak{s}, v)$. Lemma 4.4.28 concludes the proof of (ii).

In the proof of Lemma 4.4.28 we showed that $|\mathfrak{t}| \leq \deg f_{\lambda_v, \phi|_v}$. Then $\mathfrak{t} = \mathfrak{R}_h$ from above. Finally, $\mathrm{ord}_{\phi|_v}(f|_v) = |\mathfrak{t}|/\deg w$ by Theorem 4.4.26 and (4.8). $\square$

Proposition 4.4.30 *Suppose $-\lambda_v$ is the minimum slope of $N^-_{v',\phi_v}(f)$. Then $(\mathfrak{s}, v)$ is not a degree-minimal MacLane cluster if and only if $b_v = 1$ and $f|_v$ has a multiple factor $h \in k_v[X]$ of degree 1.*

Proof. Suppose $(\mathfrak{s}, v)$ is a degree-minimal MacLane cluster. Suppose that $b_v = 1$ and that $f|_v$ has a multiple irreducible factor $h \in k_v[X]$. Theorem 4.4.29 implies that there exists a proper child $(\mathfrak{t}, w) < (\mathfrak{s}, v)$ with centre ϕ such that $\phi|_v = h$. Then $\deg \phi > \deg v$. Hence $\deg h > 1$ by Lemma 4.3.21.

Now suppose $(\mathfrak{s}, v)$ is not a degree-minimal MacLane cluster. Then there exists $w > v$ with $\deg v = \deg w$ such that $(\mathfrak{t}, w)$ is a proper MacLane cluster, for some $\mathfrak{t} \subseteq \mathfrak{R}$. Proposition 4.2.24 implies that $w = [v, w(\phi) = \lambda]$ for some $\phi \in \mathrm{KP}(v)$ and $\lambda > v(\phi)$. In particular, w is also an augmentation of v'. If $\phi \sim_v \phi_v$, then

$$w(\phi_v) = \min\{\lambda, v(\phi_v - \phi)\} > \lambda_v.$$

Hence $N^-_{v',\phi_v}(f)$ would have a slope $-w(\phi_v)$ smaller than $-\lambda_v$ by Theorem 4.4.22(i), contradicting our assumptions. Hence $\phi \not\sim_v \phi_v$. It follows that

$$\lambda_v = v(\phi_v) = v(\phi - \phi_v) = v'(\phi - \phi_v) \in \Gamma_{v'},$$

and so $b_v = 1$. By Lemma 4.3.21 the polynomial $\phi|_v$ is irreducible and $\deg \phi|_v = 1$. Therefore $\mathrm{ord}_{\phi|_v}(f|_v) > 1$ by Lemma 4.4.28. $\square$

Remark 4.4.31. In §4.3 we showed how to compute the reduction $f|_v$ algorithmically for any $v \in \mathbb{V}_M$, knowing a MacLane chain for v (see also [FGMN, §3]). Assume $v_K(r) > 0$ for any $r \in \mathfrak{R}$ (in the next section we will see that we can always require this condition for our purpose). Suppose we know how to factorise polynomials in $k[X]$, e.g. k is finite. Then we can algorithmically find MacLane chains for all MacLane valuations defining MacLane clusters, starting from the Newton polygon $N_{v_0,x}(f)$ and using the results 4.4.22, 4.4.27, 4.4.28, 4.4.30, 4.4.29.

4.5 Model construction

Suppose $\mathrm{char}(k) \neq 2$. Let C/K be a hyperelliptic curve of genus $g \geq 1$. We can find a separable polynomial $f = c_f \prod_{r \in \mathfrak{R}}(x-r) \in K[x]$, where $v_K(r) > 0$ for any $r \in \mathfrak{R}$, such that $C/K : y^2 = f(x)$. Given any proper MacLane cluster $(\mathfrak{s}, v) \in \Sigma_f^M$ we want to fix a canonical choice of a MacLane chain for v. It will be called *cluster chain* and defined in Definition 4.5.1. But first, let us fix a centre for each proper MacLane cluster.

Let $(\mathfrak{s}_1, \mu_1), \ldots, (\mathfrak{s}_n, \mu_n)$ be all degree-minimal MacLane clusters for f. Note that if $r \in \mathfrak{s}_i$ has minimal polynomial $F \in K[x]$ of degree $\deg \mu_i$, then F is a centre of μ_i by Lemma 4.2.25, as $v_F \geq \mu_i$. Choose centres $\psi_1, \ldots, \psi_n$ of $\mu_1, \ldots, \mu_n$ respectively, with the following property:

(4.9) If possible, choose ψ_i equal to the minimal polynomial of some root $r \in \mathfrak{s}_i$ of K-degree $\deg \mu_i$.

Thanks to Lemma 4.2.25, for any proper MacLane cluster $(\mathfrak{s}, v) \in \Sigma_f^M$ we inductively choose a centre ϕ_v as follows:

(i) If $(\mathfrak{s}, v)$ is degree-minimal, that is $(\mathfrak{s}, v) = (\mathfrak{s}_i, \mu_i)$ for some $1 \leq i \leq n$, fix $\phi_v = \psi_i$.

(ii) If $(\mathfrak{s}, v)$ has children of degree $\deg v$, choose one of them, say $(\mathfrak{t}, w)$, and fix $\phi_v = \phi_w$.

Definition 4.5.1 Let $(\mathfrak{s}, v)$ be a proper MacLane cluster. A *cluster chain for v* is MacLane chain

$$[v_0, v_1(\phi_1) = \lambda_1, \ldots, v_m(\phi_m) = \lambda_m]$$

for v, where $\{\phi_w \mid (\mathfrak{t}, w) \supseteq (\mathfrak{s}, v)\} = \{\phi_1, \ldots, \phi_m\}$.

The next results show that every MacLane valuation defining a MacLane cluster has a unique cluster chain (Lemma 4.5.2).

Lemma 4.5.2 *Let $(\mathfrak{s}, v) \in \Sigma_f^M$ proper and let $[v_0, \ldots, v_m(\phi_m) = \lambda_m]$ be a cluster chain for v. Consider the chain of proper MacLane clusters*

$$(\mathfrak{t}_1, w_1) \supsetneq (\mathfrak{t}_2, w_2) \supsetneq \cdots \supsetneq (\mathfrak{t}_s, w_s) = (\mathfrak{s}, v)$$

satisfying:

(a) $(\mathfrak{t}_1, w_1) \supseteq (\mathfrak{t}, w)$ *for any proper MacLane cluster* $(\mathfrak{t}, w) \supseteq (\mathfrak{s}, v)$.

(b) $\phi_{w_i} \neq \phi_{w_{i+1}}$ *for all* $1 \leq i < s$.

(c) For any $1 \leq i < s$, the MacLane cluster $(\mathfrak{t}_i, w_i)$ is the smallest MacLane cluster containing $(\mathfrak{t}_{i+1}, w_{i+1})$ and satisfying (b).

Then $m = s$, $\phi_i = \phi_{w_i}$ and $v_i = w_i$.

Proof. Clearly $\{\phi_w \mid (\mathfrak{t}, w) \supseteq (\mathfrak{s}, v)\} = \{\phi_{w_1}, \dots, \phi_{w_s}\}$, with the centres ϕ_{w_i} all distinct. By definition of cluster chain $m \geq s$. However, if $m > s$, then $\phi_i = \phi_j$ for some $i < j$. This is not possible, as $v(\phi_i) = \lambda_i < \lambda_j = v(\phi_j)$ by [Rüt, Lemmas 4.21,4.22]. Hence $m = s$.

Clearly $v_m = w_m$. Suppose there exists $i < m$ such that $\phi_i = \phi_v$. It follows that

$$\lambda_i = v_i(\phi_i) = v(\phi_i) = \lambda_v = \lambda_m,$$

a contradiction by [Rüt, Lemma 4.21]. Therefore $\phi_m = \phi_{w_m}$. Let $\sigma \in S_{m-1}$ be the permutation such that $\phi_{w_i} = \phi_{\sigma(i)}$. For any $i = 1, \dots, m-1$, either $(\mathfrak{t}_i, w_i)$ is degree-minimal or there exists a child $(\mathfrak{s}', v') < (\mathfrak{t}_i, w_i)$ not containing $(\mathfrak{t}_{i+1}, w_{i+1})$ such that $\phi_{w_i} = \phi_{v'}$ by (c).

Suppose $(\mathfrak{t}_i, w_i)$ is degree-minimal. Let

$$j_i = \max\{j = 1, \dots, m \mid \deg \phi_j = \deg \phi_{w_i}\}.$$

Lemma 4.4.17 implies that v_{j_i} defines a proper MacLane cluster of degree $\deg w_i$ and so $v_{j_i} = w_i$. In fact, ϕ_{j_i} must equal ϕ_{w_i} since $(\mathfrak{t}_i, w_i)$ is degree-minimal, for our choice of centres. Therefore $\sigma(i) = j_i$ and so $w_i = v_{\sigma(i)}$.

Suppose $(\mathfrak{t}_i, w_i)$ is not degree-minimal and let $(\mathfrak{s}', v') < (\mathfrak{t}_i, w_i)$ as above. Note that $(\mathfrak{s}', v')$ does not contain in $(\mathfrak{s}, v)$ and $(\mathfrak{s}' \wedge \mathfrak{s}, v' \wedge v) = (\mathfrak{t}_i, w_i)$. Hence $w_i(\phi_{w_i}) = v(\phi_{\sigma(i)}) = \lambda_{\sigma(i)}$ by Proposition 4.4.19. It follows that

$$D_{w_i} = D(\phi_{w_i}, w_i(\phi_{w_i})) = D(\phi_{\sigma(i)}, \lambda_{\sigma(i)}) = D_{v_{\sigma(i)}},$$

and so $w_i = v_{\sigma(i)}$ from Theorem 4.2.21.

We showed that $w_i = v_{\sigma(i)}$ for any $i = 1, \dots, m-1$. Since $v_1 < \cdots < v_m$ and $w_1 < \cdots < w_m$ the permutation σ must be the identity. □

Notation 4.5.3 Let $(\mathfrak{R}, w_{\mathfrak{R}})$ denote the root of $(\Sigma_f^M, \supseteq)$ (Proposition 4.4.10).

Lemma 4.5.4 *The pseudo-valuation $w_{\mathfrak{R}}$ is a degree 1 MacLane valuation. Furthermore, $w_{\mathfrak{R}} > v_0$.*

Proof. Let w be the maximal element of

$$\{w' \in \hat{\mathbb{V}}_M \mid D_{w'} \cap \mathfrak{R} = \mathfrak{R}, \deg w' = 1\}.$$

Note that the set is non-empty as v_0 belongs to it. If w is not a valuation, then $|\mathfrak{R}| \leq 1$ by Lemma 4.4.11, a contradiction. Hence $(\mathfrak{R}, w)$ is a proper MacLane cluster and so $w_{\mathfrak{R}} \leq w$. But then $w = w_{\mathfrak{R}}$ by definition of MacLane cluster since $\deg w = 1$. Finally,

$$\lambda_{w_{\mathfrak{R}}} \geq \min_{r \in \mathfrak{R}} v_K(r) > 0,$$

by Lemma 4.4.16, and so $w_{\mathfrak{R}} > v_0$. □

Lemma 4.5.5 *Let $(\mathfrak{s}, v)$ be a proper MacLane cluster. There exists a unique cluster chain for v. Furthermore, $v > v_0$.*

Proof. The uniqueness follows by Lemma 4.5.2. Moreover, $v > v_0$ by Lemma 4.5.4. We construct a cluster chain of v recursively to prove the existence. First let $(\mathfrak{R}, w_{\mathfrak{R}})$ as above. Then $w_{\mathfrak{R}} = [v_0, w_{\mathfrak{R}}(\phi_{\mathfrak{R}}) = \lambda_{\mathfrak{R}}]$ is a cluster chain for $(\mathfrak{R}, w_{\mathfrak{R}})$. Now let $(\mathfrak{s}, v)$ be any MacLane cluster different from $(\mathfrak{R}, w_{\mathfrak{R}})$ and consider its parent $(\mathfrak{t}, w)$. By recursion we can assume that w is equipped with a cluster chain

$$[v_0, \dots, v_{m-1}(\phi_{m-1}) = \lambda_{m-1}, v_m(\phi_m) = \lambda_m]$$

So $\phi_m = \phi_w$ from Lemma 4.5.2. If $\phi_w = \phi_v$, then

$$[v_0, \dots, v_{m-1}(\phi_{m-1}) = \lambda_{m-1}, v(\phi_m) = \lambda_v]$$

is a cluster chain for v. If $\phi_w \neq \phi_v$, Lemma 4.4.18 implies that

$$[v_0, \dots, v_{m-1}(\phi_{m-1}) = \lambda_{m-1}, v_m(\phi_m) = \lambda_m, v(\phi_v) = \lambda_v]$$

is an augmentation chain for v. Proving it is a MacLane chain would conclude the proof. Suppose by contradiction that $\phi_v \sim_w \phi_w$. Then $\deg \phi_v = \deg \phi_w$. In particular, $(\mathfrak{t}, w)$ is not degree-minimal. As $\phi_v \neq \phi_w$, there exists a child $(\mathfrak{s}', v') < (\mathfrak{t}, w)$ such that $\phi_w = \phi_{v'}$. Hence $(\mathfrak{s} \wedge \mathfrak{s}', v \wedge v') = (\mathfrak{t}, w)$. Set

$$w' = [w, w'(\phi_w) = \min\{\lambda_{v'}, \lambda_v, w(\phi_v - \phi_w)\}].$$

Therefore $w < w' \leq v'$. Moreover $v(\phi_w) = \min\{\lambda_v, w(\phi_w - \phi_v)\}$, and so $v \geq w'$. But then $w < w' \leq v \wedge v'$ which gives a contradiction. □

Thanks to cluster chains, the Newton polytopes needed for the construction of the model can be defined without ambiguity. Let $h = 1, \dots, n$ and consider the MacLane valuation μ_h of the degree-minimal cluster $(\mathfrak{s}_h, \mu_h)$. Let

$$[v_0, v_1(\phi_1) = \lambda_1, \dots, v_{m-1}(\phi_{m-1}) = \lambda_{m-1}, v_m(\phi_m) = \lambda_m] \tag{4.10}$$

be a cluster chain for μ_h. Then $\phi_m = \psi_h$. Denote $\phi = \psi_h$ and $v = v_{m-1}$. Denote by ϵ_h the ramification index $e_v = \epsilon_{\mu_h}$. Let $g(x, y) = y^2 - f(x)$ and expand

$$g = \sum_{i,j} a_{ij} \phi^i y^j, \qquad a_{ij} \in K[x], \deg a_{i,j} < \deg \phi.$$

Define the Newton polytopes

$$\Delta_h = \text{convex hull}\left(\{(i,j) : a_{ij} \neq 0\}\right) \subset \mathbb{R}^2,$$
$$\tilde{\Delta}_h = \text{lower convex hull}\left(\{(i,j,v(a_{ij})) : a_{ij} \neq 0\}\right) \subset \mathbb{R}^3.$$

Consider the homeomorphic projection $s_h : \tilde{\Delta}_h \to \Delta_h$. Above every point $P \in \Delta_h$ there is a unique point $(P, \tilde{\mu}_h(P)) \in \tilde{\Delta}_h$. This defines a piecewise affine function $\tilde{\mu}_h : \Delta_h \to \mathbb{R}$, and the pair $(\Delta_h, \tilde{\mu}_h)$ determines $\tilde{\Delta}_h$. Let $\tilde{F}$ be any 2-dimensional (open) face of $\tilde{\Delta}_h$ and let $F = s_h(\tilde{F})$. Define $\tilde{v}_{F,h} : \mathbb{R}^2 \to \mathbb{R}$ to be the unique affine function coinciding with $\tilde{\mu}_h$ on F. Let $\lambda_F = \tilde{v}_{F,h}(0,0) - \tilde{v}_{F,h}(1,0)$. Define

$\tilde{\Delta}_h^- \subseteq \tilde{\Delta}_h$ as the sub-polytope consisting of (the closure of) all 2-dimensional faces $\tilde{F}$ of $\tilde{\Delta}_h$ with $\lambda_F > v(\phi)$. Clearly

$$\tilde{\Delta}_h^- = \text{lower convex hull}\Big(\{(i,0,u) : (i,u) \in N_{v,\phi}^-(f)\} \cup \{(0,2,0)\}\Big) \subset \mathbb{R}^3.$$

where $N_{v,\phi}^-(f)$ is the principal Newton polygon of f with respect to v, ϕ. The image of $\tilde{\Delta}_h^-$ under s_h will be denoted by Δ_h^-. The images of the 0-,1- and 2-dimensional (open) faces of the polytope $\tilde{\Delta}_h^-$ under s_h are called h-vertices, h-edges and h-faces. Finally, a $*$-vertex, $*$-edge, $*$-face is respectively an h-vertex, h-edge, h-face for some $h = 1, \dots, n$.

Definition 4.5.6 Let G be a h-vertex, h-edge or h-face.

(a) Denote by $\tilde{G}$ the inverse image of G under s_h.

(b) Denote by $\bar{G}$ the closure of G in $\mathbb{R}^2$.

(c) Denote by $G_{\mathbb{Z}}$ the set of points P of G with $\epsilon_h \tilde{\mu}_h(P) \in \mathbb{Z}$.

(d) Denote by $G_{\mathbb{Z}}(\mathbb{Z})$ the intersection $G_{\mathbb{Z}} \cap \mathbb{Z}$.

Finally, define the *denominator* of G, denoted δ_G, as the common denominator of $\epsilon_h \tilde{\mu}_h(P)$ for every $P \in \bar{G}(\mathbb{Z})$.

Let $(\mathfrak{s}, w)$ be a proper MacLane cluster centre $\phi_w = \psi_h$. Lemma 4.5.2 implies that the cluster chain for w is

$$[v_0, v_1(\phi_1) = \lambda_1, \dots, v_{m-1}(\phi_{m-1}) = \lambda_{m-1}, w(\psi_h) = \lambda_w]$$

where v_i, ϕ_i, λ_i are as in (4.10). Theorem 4.4.22 implies that there is a 1-to-1 correspondence between proper MacLane clusters and $*$-faces. Given a proper MacLane cluster $(\mathfrak{s}, w)$ we will denote by F_w the corresponding $*$-face. If $\phi_w = \psi_h$, then F_w is an h-face. Then F_w has 3 edges:

(1) An h-edge, denoted L_w, linking the points $(t_w^0, 0)$ and $(t_w, 0)$.

(2) An h-edge, denoted V_w, linking the points $(t_w, 0)$ and $(0, 2)$.

(3) An h-edge, denoted V_w^0, linking the points $(t_w^0, 0)$ and $(0, 2)$.

Definition 4.5.7 For any proper MacLane cluster $(\mathfrak{s}, w)$ and any $l = 1, \dots, n$, define $\tilde{w}_l : \mathbb{R}^2 \to \mathbb{R}$ by

$$\tilde{w}_l(x, y) = -w(\psi_l)x - \tfrac{w(f)}{2}y + w(f)$$

and $\tilde{w} : \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$\tilde{w}(x_1, \dots, x_n, y) = -(w(\psi_1)x_1 + \dots + w(\psi_n)x_n) - \tfrac{w(f)}{2}y + w(f).$$

Finally define $e_{\tilde{w}} = (\tilde{\Gamma}_w : \mathbb{Z})$, where $\tilde{\Gamma}_w = \tilde{w}(\mathbb{Z}^{n+1})$.

Let $(\mathfrak{s}, w)$ be a proper MacLane cluster with centre $\phi_w = \psi_h$. Then $\tilde{w}_h = \tilde{v}_{F_w,h}$. We will denote $(\mathfrak{s}_{F_w}, v_{F_w}) = (\mathfrak{s}, w)$.

Definition 4.5.8 Let E be an h-edge. We say E is *inner* if $E = V_w$ for some proper MacLane cluster $(\mathfrak{s}, w) \neq (\mathfrak{R}, w_{\mathfrak{R}})$. In this case we say that E *bounds* F_w and $F_{P(w)}$. In all other cases E is said *outer* and *bounds* only the h-face whose it is an edge.

4.5.1 Matrices

Let $(\mathfrak{s}, v)$ be a proper cluster with centre $\phi_v = \psi_h$. Let

$$[v_0, v_1(\phi_1) = \lambda_1, \ldots, v_{m-1}(\phi_{m-1}) = \lambda_{m-1}, v_m(\phi_m) = \lambda_m] \tag{4.11}$$

be the unique cluster chain for v. Construct the invariants and the rational functions attached to (4.11) in §4.3. Denote $v_- = v_{m-1}$. Recall $e_{v_-} = \epsilon_h$.

Let E be either L_v or V_v or V_v^0 if $(\mathfrak{s}, v)$ is degree-minimal. Let $v_E = [v_-, v_E(\psi_h) = \infty]$ if $E = V_v^0$, and $v_E = v$ otherwise.

Definition 4.5.9 Let $o = 1, \ldots, n$, $o \neq h$. Define $\gamma_{o,E} = \gamma_j$ if $\psi_o = \phi_j$, while

$$\gamma_{o,E} = \begin{cases} \psi_o \cdot \psi_h^{-\deg\psi_o/\deg\psi_h} & \text{if } \mu_o \geq v_E, \\ \psi_o \cdot \pi_{m-1}^{-e_{v_{m-1}} v(\psi_o)} & \text{otherwise,} \end{cases}$$

if $\psi_o \neq \phi_j$, for all $j = 1, \ldots, m$.

Lemma 4.5.10 *Let $o = 1, \ldots, n$, $o \neq h$. Then $\gamma_{o,E}$ is a well-defined element of $K(x)$ satisfying $v_F(\gamma_{o,E}) = 0$ for any $*$-face F bounded by E.*

Proof. Let F be any $*$-face bounded by E. Then $(\mathfrak{s}, v) \leq (\mathfrak{s}_F, v_F)$. Lemma 4.5.2 implies that $v_F(\phi_j) = v(\phi_j)$ for all $j < m$. So the statement is trivial if $\psi_o = \phi_j$, for some $j < m$. Suppose $\psi_o \neq \phi_j$, for all $j < m$. Then $\mu_o \not\leq v$. In particular, $\mu_o \not< v_F$ and so $v_F(\phi_o) = (v_F \wedge \mu_o)(\phi_o)$ by Proposition 4.4.19.

Suppose $v_E \leq \mu_o$. Then $v_F \leq \mu_o$ and so ψ_o is v_F-minimal by Lemma 4.2.29. It follows that $\deg v \mid \deg\psi_o$ by Lemma 4.2.30. Theorem 4.2.27 implies that

$$\frac{v(\psi_o)}{\deg\psi_o} = \frac{\lambda_v}{\deg v} \qquad \text{and} \qquad \frac{v_F(\psi_o)}{\deg\psi_o} = \frac{v_F(\psi_h)}{\deg v},$$

since $v_F = v$ when $F = F_v$. Therefore $v_F(\gamma_{o,L}) = 0$.

Suppose $v_E \not\leq \mu_o$. First we want to show that

$$v_E \wedge \mu_o = v_F \wedge \mu_o. \tag{4.12}$$

Note that either $v_E \wedge \mu_0 \leq v_F$ or $v_E \wedge \mu_0 > v_F$ since $v_E \geq v_F$. If $E = V_v^0$ (and so v is degree-minimal), then $v_F = v$. If $v_E \wedge \mu_o \leq v$, then (4.12) follows. Suppose $v_E \wedge \mu_o > v$. Then $\mu_o > v$ and so

$\mu_o(\psi_h) = v(\psi_h)$ by Lemma 4.5.2. Furthermore, ψ_h is a centre of $v_E \wedge \mu_o \le v_E$. But then Lemma 4.2.25 and Proposition 4.4.19 imply that

$$\mu_o(\psi_h) = (v_E \wedge \mu_o)(\psi_h) = \lambda_{v_E \wedge \mu_o} > \lambda_v = v(\psi_h) = \mu_o(\psi_h),$$

a contradiction. If $E \neq V_v^0$, then $v_E = v$. Since $v \wedge \mu_o < v$ defines a MacLane cluster by Lemma 4.4.7, we have $v \wedge \mu_o \le v_F$. Hence (4.12).

It follows from (4.12) and Proposition 4.4.19 that

$$v_E(\psi_o) = (v_E \wedge \mu_o)(\psi_o) = (v_F \wedge \mu_o)(\psi_o) = v_F(\psi_o). \tag{4.13}$$

Hence it suffices to show that $v(\psi_o) \in \Gamma_{v_{m-1}}$. By Proposition 4.2.24 write

$$v \wedge \mu_o = [v_{a-1}, (v \wedge \mu_o)(\phi_a) = \lambda'_a],$$

for some $a \le m$ and $\lambda'_a \le \lambda_a$.

If $v \le \mu_o$, then v is degree-minimal. It follows that $v \wedge \mu_o$ appears in the cluster chain for μ_o by Lemma 4.5.2. Therefore $v(\psi_o) \in \Gamma_{v_{a-1}} \subseteq \Gamma_{v_{m-1}}$ by Remark 4.3.2.

If $v \not\le \mu_o$, then $v \wedge \mu_o < v$. By Lemma 4.4.7, the valuation $v \wedge \mu_o$ defines a proper MacLane cluster $(\mathfrak{s}', v \wedge \mu_o) \supsetneq (\mathfrak{s}, v)$. Let $(\mathfrak{t}, w) \in \Sigma_f^M$ such that

$$(\mathfrak{s}, v) \subseteq (\mathfrak{t}, w) < (\mathfrak{s}', v \wedge \mu_o).$$

Since $\mu_o \not\ge w$, if $\psi_w = \psi_{v \wedge \mu_o}$, then $v \wedge \mu_o$ appears in the cluster chain for μ_o by Lemma 4.5.2. Therefore $v(\psi_o) \in \Gamma_{v_{m-1}}$ as above. Finally, if $\psi_w \neq \psi_{v \wedge \mu_o}$, then $v \wedge \mu_o$ appears in the cluster chain for v again by Lemma 4.5.2. Since $v \wedge \mu_o < v$, one has $(v \wedge \mu_o)(g) \in \Gamma_{v_{m-1}}$ for any $g \in K[x]$. In particular, $v(\psi_o) \in \Gamma_{v_{m-1}}$ from (4.13). □

Let E_v^* be the unique affine function $\mathbb{Z}^2 \to \mathbb{Z}$ with $E_v^*|_E = 0$ and $E_v^*|_{F_v} \ge 0$. Choose $P_0, P_1 \in \mathbb{Z}^2$ such that $E_v^*(P_0) = 0$ and $E_v^*(P_1) = 1$.

Definition 4.5.11 Define the slopes $[s_1^E, s_2^E]$, at E to be

$$s_1^E = \delta_E \epsilon_h (\tilde{v}_h(P_1) - \tilde{v}_h(P_0)),$$

$$s_2^E = \begin{cases} \delta_E \epsilon_h (\tilde{w}_h(P_1) - \tilde{w}_h(P_0)) & \text{if E inner, with } (\mathfrak{s}, v) < (\mathfrak{t}, w), \\ \lfloor s_1^E - 1 \rfloor & \text{if E outer.} \end{cases}$$

Let $\delta = \delta_E$. Pick fractions $\frac{n_i}{d_i} \in \mathbb{Q}$ such that

$$s_1^E = \frac{n_0}{d_0} > \cdots > \frac{n_{r_E+1}}{d_{r_E+1}} = s_2^E, \quad \text{with} \quad \begin{vmatrix} n_i & n_{i+1} \\ d_i & d_{i+1} \end{vmatrix} = 1.$$

Let $r = r_E$. Redefine $n_{r+1} = -1$, $d_{r+1} = 0$ if E is outer.

Write $\tilde{E} = \tilde{P}_0 + v\mathbb{R}$, with $\delta v = (\delta a_x, \delta a_y, \delta a_z) \in \mathbb{Z}^2 \times \frac{1}{\epsilon_h}\mathbb{Z}$ primitive and such that (a_x, a_y) goes counterclockwise along ∂F_v. Let $o \neq h$. By Definition 4.5.9 and Lemma 4.3.7, we can uniquely write

$$\gamma_{o,E} = \psi_1^{m_{1o}} \cdots \psi_n^{m_{no}} \cdot \pi^{m_{(n+2)o}}$$

Define $v_o \in \mathbb{R}^{n+2}$ by $v_o = (m_{1o}, \dots, m_{no}, 0, m_{(n+2)o})$.

Now consider the embedding $\iota_h : \mathbb{R}^3 \hookrightarrow \mathbb{R}^{n+2}$ given by

$$(x_h, y, z) \mapsto (0, \dots, 0, x_h, 0, \dots, 0, y, z),$$

where x_h is the h-th coordinate in $\mathbb{R}^{n+2}$. Define $v_h^{\mathbb{R}} = \iota_h(\delta v)$. Write $P_1 - P_0 = (b_x, b_y)$ and define $\omega_i^{\mathbb{R}} = \iota_h\big(d_i b_x, d_i b_y, \frac{n_i}{\delta\epsilon_h}\big) \in \mathbb{R}^{n+2}$ for any $i = 0, \dots, r+1$. The vectors above define hyperplanes in $\mathbb{R}^{n+2}$,

$$\mathcal{P}_{E,i} = v_1\mathbb{R} + \cdots + v_n\mathbb{R} + \omega_i\mathbb{R} \qquad i = 0, \dots, r+1.$$

Let $M_{E,i}^{\mathbb{R}} \in M_{n+2}(\mathbb{R})$ be the matrix given by

$$M_{E,i}^{\mathbb{R}} = (v_1, \dots, v_{h-1}, v_h^{\mathbb{R}}, v_{h+1}, \dots, v_n, \omega_i^{\mathbb{R}}, -\omega_{i+1}^{\mathbb{R}})$$

where the vectors represent the columns of $M_{E,i}^{\mathbb{R}}$. Then[5]

$$\det M_{E,i}^{\mathbb{R}} = \textstyle\prod_{o=1}^{m-1} e_o \cdot \frac{1}{e_{v_{m-1}}} = 1.$$

Moreover, all entries of $M_{E,i}^{\mathbb{R}}$ are integers except possibly $\delta a_z \in \frac{1}{\epsilon_h}\mathbb{Z}$, and $\frac{n_i}{\delta\epsilon_h}$, $-\frac{n_{i+1}}{\delta\epsilon_h}$, rational numbers in $\frac{1}{\delta\epsilon_h}\mathbb{Z}$. Pick k_i with

$$k_i \equiv -n_i(\delta\epsilon_h a_z)^{-1} \mod \delta.$$

This is possible as δv is primitive in $\mathbb{Z}^2 \times \frac{1}{\epsilon_h}\mathbb{Z}$. Let $\tau \in S_{n+2}$ be a permutation such that $\phi_o = \psi_{\tau(o)}$ for all $o = 1, \dots, m$ and $\tau(n+1) = n+1$, $\tau(n+2) = n+2$. Define the vectors

$$v_h = v_h^{\mathbb{R}} + \sum_{o=1}^{m-1} c_o v_{\tau(o)} \delta a_z, \quad \omega_i = \omega_i^{\mathbb{R}} + k_i \frac{v_h^{\mathbb{R}}}{\delta} + \sum_{o=1}^{m-1} c_o v_{\tau(o)} \big(\tfrac{n_i}{\delta\epsilon_h} + k_i a_z\big),$$

where $c_o = e_{v_{o-1}} \ell_o$. The next lemma shows that they belong to $\mathbb{Z}^{n+2}$.

Lemma 4.5.12 *Write $\sum_{o=1}^{m-1} c_o v_{\tau(o)} = (a_1, \dots, a_{n+2})$. Then*

$$a_{\tau(j)} = \begin{cases} \epsilon_h \ell_j \ell'_{j+1} \cdots \ell'_{m-1} & \text{if } j < m, \\ 0 & \text{if } m \le j \le n+1, \\ \epsilon_h \ell'_1 \cdots \ell'_{m-1} - 1 & \text{if } j = n+2. \end{cases}$$

In particular, $v_h, \omega_i \in \mathbb{Z}^{n+2}$.

[5] See Appendix C.2 for more details.

Proof. Recall $\gamma_{E,\tau(o)} = \gamma_o$ for any $o = 1,\dots,m$. If $j < m$ Lemma 4.3.7 implies

$$\begin{aligned} a_{\tau(j)} &= c_j e_j - \textstyle\sum_{o=j+1}^{m-1} c_o h_o \ell_j \ell'_{j+1} \cdots \ell'_{o-1} \\ &= e_{v_j}\ell_j - \textstyle\sum_{o=j+1}^{m-1} e_{v_{o-1}}(\ell_o h_o)\ell_j \ell'_{j+1}\cdots\ell'_{o-1} \\ &= \ell_j\big(e_{v_j} + \textstyle\sum_{o=j+1}^{m-1} e_{v_o}\ell'_{j+1}\cdots\ell'_o - \sum_{o=j+1}^{m-1} e_{v_{o-1}}\ell'_{j+1}\cdots\ell'_{o-1}\big) \\ &= \ell_j\left(e_{v_j} + e_{v_{m-1}}\ell'_{j+1}\cdots\ell'_{m-1} - e_{v_j}\right) = e_{v_{m-1}}\ell_j\ell'_{j+1}\cdots\ell'_{m-1}, \end{aligned}$$

where we used $\ell_o h_o + \ell'_o e_o = 1$. If $m \le j \le n+1$, then the $\tau(j)$-th coordinate of $v_{\tau(o)}$ is 0 for all $o = 1,\dots,m-1$; so $a_{\tau(j)} = 0$. Finally

$$\begin{aligned} a_{n+2} &= -\textstyle\sum_{o=1}^{m-1} c_o h_o \ell'_1\cdots\ell'_{o-1} = \sum_{o=1}^{m-1} e_{v_o}\ell'_1\cdots\ell'_o - \sum_{o=1}^{m-1} e_{v_{o-1}}\ell'_1\cdots\ell'_{o-1} \\ &= e_{v_{m-1}}\ell'_1\cdots\ell'_{m-1} - 1, \end{aligned}$$

as required. □

Define $M_{E,i} = (v_1,\dots,v_n,\omega_i,-\omega_{i+1}) \in M_{(n+2)}(\mathbb{Z})$, where the vectors represent the columns of $M_{E,i}$. Note that $\det M_{E,i} = \det M^{\mathbb{R}}_{E,i} = 1$. Let us describe $M_{E,i}$ as product of simpler matrices. Let $\varepsilon_1,\dots,\varepsilon_{n+2} \in \mathbb{R}^{n+2}$ be the standard basis of $\mathbb{R}^{n+2}$. Define $\kappa_i = \frac{k_i}{\delta}\varepsilon_h$ and $\xi = \sum_{o=1}^{m-1} c_o\varepsilon_{\tau(o)}$. Define

$$\begin{aligned} T_h &= (\varepsilon_1,\dots,\varepsilon_{h-1},\varepsilon_h + \delta a_z\cdot\xi,\varepsilon_{h+1},\dots,\varepsilon_n,\varepsilon_{n+1} + \tfrac{n_i}{\delta\epsilon_h}\xi,\varepsilon_{n+2} - \tfrac{n_{i+1}}{\delta\epsilon_h}\xi), \\ T &= (\varepsilon_1,\dots,\varepsilon_n,\varepsilon_{n+1}+\kappa_i,\varepsilon_{n+2}-\kappa_{i+1}). \end{aligned}$$

Then $M_{E,i} = M^{\mathbb{R}}_{E,i}\cdot T_h\cdot T$. Now we want to describe $M^{-1}_{E,i}$. It follows from before that $M^{-1}_{E,i} = T^{-1}\cdot T_h^{-1}\cdot(M^{\mathbb{R}}_{E,i})^{-1}$, where

$$\begin{aligned} T_h^{-1} &= (\varepsilon_1,\dots,\varepsilon_{h-1},\varepsilon_h - \delta a_z\cdot\xi,\varepsilon_{h+1},\dots,\varepsilon_n,\varepsilon_{n+1} - \tfrac{n_i}{\delta\epsilon_h}\xi,\varepsilon_{n+2} + \tfrac{n_{i+1}}{\delta\epsilon_h}\xi), \\ T^{-1} &= (\varepsilon_1,\dots,\varepsilon_n,\varepsilon_{n+1}-\kappa_i,\varepsilon_{n+2}+\kappa_{i+1}), \end{aligned}$$

It remains to describe $(M^{\mathbb{R}}_{E,i})^{-1}$. First note that the h-th, $(n+1)$-th and $(n+2)$-th columns of $(M^{\mathbb{R}}_{E,i})^{-1}$ are respectively

$$\begin{aligned} &\iota_h\big((b_y/\delta, n_{i+1}a_y - \delta\epsilon_h d_{i+1}a_z b_y, n_i a_y - \delta\epsilon_h d_i a_z b_y)\big), \\ &\iota_h\big((-b_x/\delta, -n_{i+1}a_x + \delta\epsilon_h d_{i+1}a_z b_x, -n_i a_x + \delta\epsilon_h d_i a_z b_x)\big), \\ &\iota_h\big((0,\delta\epsilon_h d_{i+1},\delta\epsilon_h d_i)\big). \end{aligned}$$

Let $o = 1,\dots,n$. Lemma 4.3.7 and Definition 4.5.9 imply that we can write

$$\psi_o^{\epsilon_h} = \gamma_{1,E}^{\alpha_{1o}}\cdots\gamma_{h-1,E}^{\alpha_{(h-1)o}}\cdot\psi_h^{\alpha_{ho}}\cdot\gamma_{h+1,E}^{\alpha_{(h+1)o}}\cdots\gamma_{n,E}^{\alpha_{no}}\cdot\pi^{\alpha_{\pi o}}, \tag{4.14}$$

for some unique $\alpha_{1o},\dots,\alpha_{no},\alpha_{\pi o} \in \mathbb{Z}$. Let $\tilde{\alpha}_{oj} = \alpha_{oj}/\epsilon_h$. Define

$$\tilde{v}_o = \begin{cases} (\tilde{\alpha}_{o1},\dots,\tilde{\alpha}_{on},0,0), & \text{if } o \ne h \\ \frac{1}{\delta}(\tilde{\alpha}_{h1}b_y,\dots,\tilde{\alpha}_{hn}b_y,b_x,0) & \text{if } o = h \end{cases}$$

Finally, define

$$\tilde{\omega}_i = \delta\epsilon_h d_i\Big(\big(\tfrac{n_i}{\delta\epsilon_h d_i}a_y - a_z b_y\big)\tilde{\alpha}_{h1} + \tilde{\alpha}_{\pi 1}, \dots \\ \dots, \big(\tfrac{n_i}{\delta\epsilon_h d_i}a_y - a_z b_y\big)\tilde{\alpha}_{hn} + \tilde{\alpha}_{\pi n}, -\tfrac{n_i}{\delta\epsilon_h d_i}a_x + a_z b_x, 1\Big). \tag{4.15}$$

From the definition of $M^{\mathbb{R}}_{E,i}$ it follows that

$$(M^{\mathbb{R}}_{E,i})^{-1} = \begin{pmatrix} \tilde{v}_1 \\ \vdots \\ \tilde{v}_n \\ \tilde{\omega}_{i+1} \\ \tilde{\omega}_i \end{pmatrix},$$

where the vectors are the rows of the matrix. Lemma 4.3.7 gives an explicit of $(M^{\mathbb{R}}_{E,i})^{-1}$. Note also that for the structure of T^{-1} and T_h^{-1} the $\tau(o)$-th row of $M^{-1}_{E,i}$ coincides with the $\tau(o)$-th row of $(M^{\mathbb{R}}_{E,i})^{-1}$, when $o > m$. Define

$$\mathcal{P}^{\perp +}_{E,i} = \tilde{\omega}_i \mathbb{R}_+,$$

ray perpendicular to the hyperplane $\mathcal{P}_{E,i}$.

Remark 4.5.13. Note that $\tilde{v}_{\tau(o)} = \varepsilon_{\tau(o)}$ for $m < o \le n$.

Lemma 4.5.14 *Suppose E is inner, with $(\mathfrak{s}, v) < (\mathfrak{t}, w)$. Then $\tilde{v}_h|_E = \tilde{w}_h|_E$.*

Proof. Recall that $E = V_v$. If F_w is an h-face, the result trivially follows, as $E = V^0_w$.

Suppose F_w is not an h-face. By definition of cluster chain we have $w = v_-$. The polynomial ψ_h is w-minimal, hence $\frac{\lambda_w}{\deg w} = \frac{w(\psi_h)}{\deg v}$ by Theorem 4.2.27. From Lemma 4.4.20 and Proposition 4.4.25 it follows that

$$\tilde{v}_h(t_v, 0) = v(f) - \tfrac{|\mathfrak{s}|}{\deg v}\cdot\lambda_v = w(f) - \tfrac{|\mathfrak{s}|}{\deg w}\cdot\lambda_w = w(f) - \tfrac{|\mathfrak{s}|}{\deg v}\cdot w(\psi_h) = \tilde{w}_h(t_v, 0).$$

This concludes the proof since $\tilde{v}_h(0,2) = 0 = \tilde{w}_h(0,2)$. □

Lemma 4.5.15 *We have*

$$\tilde{\omega}_0 = e_{\tilde{v}}(v(\psi_1), \dots, v(\psi_n), \tfrac{v(f)}{2}, 1).$$

Let $r = r_E$. Then

$$\tilde{\omega}_{r+1} = \begin{cases} e_{\tilde{w}}(w(\psi_1), \dots, w(\psi_n), \frac{w(f)}{2}, 1) & \textit{if } E \textit{ inner, with } (\mathfrak{s}, v) < (\mathfrak{t}, w), \\ (-a_y\tilde{\alpha}_{h1}, \dots, -a_y\tilde{\alpha}_{hn}, a_x, 0) & \textit{if } E \textit{ outer.} \end{cases}$$

Proof. Note that $\delta_{F_v} = \delta_E d_0$ and $\delta_{F_v}\epsilon_h = e_{\tilde{v}}$. Recall $\tilde{v}_{F_v,h} = \tilde{v}_h$ and

$$\tilde{v}_h(x,y) = -\lambda_v x - \tfrac{v(f)}{2}y + v(f).$$

Then since ν and $(b_x, b_y, \frac{n_0}{\delta\epsilon_h d_0})$ generate $\tilde{F}_v$ (face of $\tilde{\Delta}_h$), we have

$$\tfrac{n_0}{\delta\epsilon_h d_0}a_y - a_z b_y = \lambda_v \quad \text{and} \quad -\tfrac{n_0}{\delta\epsilon_h d_0}a_x + a_z b_x = \tfrac{v(f)}{2}. \tag{4.16}$$

By (4.14) and Lemmas 4.3.3 and 4.5.10, we have $v(\psi_o) = \lambda_v \tilde{\alpha}_{ho} + \tilde{\alpha}_{\pi o}$ for any $o = 1, \dots, n$. Hence the description of $\tilde{\omega}_0$ follows from (4.16).

Suppose that E is inner, with $(\mathfrak{s}, v) < (\mathfrak{t}, w)$. Then either $w = v_-$ or $w = [v_-, w(\psi_h) = \lambda_w]$. In either case, $\delta_E d_{r+1} \epsilon_h = e_{\tilde{w}}$. We have

$$\tilde{w}_h(x,y) = -w(\psi_h)x - \tfrac{w(f)}{2}y + w(f).$$

Since $\tilde{w}_h|_E = \tilde{v}_h|_E$ by Lemma 4.5.14 and $\frac{n_{r+1}}{\delta\epsilon_h d_{r+1}} = \tilde{w}_h(P_1) - \tilde{w}_h(P_0)$, the vectors ν and $(b_x, b_y, \frac{n_{r+1}}{\delta\epsilon_h d_{r+1}})$ generate the plane $z = \tilde{w}_h(x,y)$ in $\mathbb{R}^3$. Hence

$$\tfrac{n_{r+1}}{\delta\epsilon_h d_{r+1}} a_y - a_z b_y = v_F(\psi_h) \quad \text{and} \quad \tfrac{n_{r+1}}{\delta\epsilon_h d_{r+1}} a_x - a_z b_x = \tfrac{v_F(f)}{2}.$$

Similarly to before, by (4.14) and Lemmas 4.3.3 and 4.5.10, we have $v_F(\psi_o) = v_F(\psi_h)\tilde{\alpha}_{ho} + \tilde{\alpha}_{\pi o}$ for any $o = 1, \dots, n$. The description of $\tilde{\omega}_{r+1}$ follows, for E inner.

Finally, suppose that E is outer. Then $n_{r+1} = -1$ and $d_{r+1} = 0$. The description of $\tilde{\omega}_{r+1}$ follows directly from the definition. □

4.5.2 Toroidal embedding

Let us start this subsection with the following notation.

Notation 4.5.16 Let A be a ring and let $a_1, \dots, a_n \in A^\times$, for some $n \in \mathbb{Z}_+$. For any matrix $M = (m_{ij}) \in \mathrm{SL}_n(\mathbb{Z})$ denote by $(a_1, \dots, a_n) \bullet M$ the vector

$$(a_1^{m_{11}} \cdots a_n^{m_{n1}}, \dots, a_1^{m_{1n}} \cdots a_n^{m_{nn}}).$$

Denote by m_{**} and $\tilde{m}_{**}$ the entries of $M_{E,i}$ and $M_{E,i}^{-1}$ respectively. Note that $\tilde{m}_{(n+1)(n+2)} \geq 0$ and $\tilde{m}_{(n+2)(n+2)} \geq 0$. Then the coordinate transformation

$$(X_1, \dots, X_n, Y, Z) = (x_1, \dots, x_n, y, \pi) \bullet M_{E,i},$$
$$(x_1, \dots, x_n, y, \pi) = (X_1, \dots, X_n, Y, Z) \bullet M_{E,i}^{-1}$$

gives the ring isomorphism

$$K[x_1^{\pm1}, \dots, x_n^{\pm1}, y^{\pm1}] \overset{M_{E,i}}{\simeq} \frac{O_K[X_1^{\pm1}, \dots, X_n^{\pm1}, Y^{\pm1}, Z^{\pm1}]}{(\pi - X_1^{\tilde{m}_{1(n+2)}} \cdots X_n^{\tilde{m}_{n(n+2)}} Y^{\tilde{m}_{(n+1)(n+2)}} Z^{\tilde{m}_{(n+2)(n+2)}})}.$$

Define

$$R = \frac{O_K[X_1^{\pm1}, \dots, X_n^{\pm1}, Y, Z]}{(\pi - X_1^{\tilde{m}_{1(n+2)}} \cdots X_n^{\tilde{m}_{n(n+2)}} Y^{\tilde{m}_{(n+1)(n+2)}} Z^{\tilde{m}_{(n+2)(n+2)}})}.$$

For any h-edge E, define cones in $\mathbb{R}^{n+1} \times \mathbb{R}_+$

0-dimensional cone	$\sigma_0 = \{0\}$,	
1-dimensional cones	$\sigma_{E,i} = \mathcal{P}_{E,i}^{\perp+}$	$(0 \leq i \leq r+1)$,
2-dimensional cones	$\sigma_{E,i,i+1} = \mathcal{P}_{E,i}^{\perp+} + \mathcal{P}_{E,i+1}^{\perp+}$	$(0 \leq i \leq r)$.

The set of all such cones from all E is a fan Σ from Appendix C.3. Recall

$$\begin{aligned}
\mathcal{P}_{E,i} &= v_1\mathbb{R} + \cdots + v_n\mathbb{R} + \omega_i\mathbb{R} = m_{*1}\mathbb{R} + \cdots + m_{*n}\mathbb{R} + m_{*(n+1)}\mathbb{R},\\
\mathcal{P}_{E,i+1} &= v_1\mathbb{R} + \cdots + v_n\mathbb{R} + \omega_{i+1}\mathbb{R} = m_{*1}\mathbb{R} + \cdots + m_{*n}\mathbb{R} + m_{*(n+2)}\mathbb{R},\\
\mathcal{P}_{E,i} \cap \mathcal{P}_{E,i+1} &= v_1\mathbb{R} + \cdots + v_n\mathbb{R} = m_{*1}\mathbb{R} + \cdots + m_{*n}\mathbb{R},
\end{aligned}$$

$$\sigma_{E,i} = \tilde{m}_{(n+2)*}\mathbb{R}_+, \quad \sigma_{E,i+1} = \tilde{m}_{(n+1)*}\mathbb{R}_+,$$

$$\sigma_{E,i,i+1} = \tilde{m}_{(n+1)*}\mathbb{R}_+ + \tilde{m}_{(n+2)*}\mathbb{R}_+.$$

The monomial exponents from the dual cone are

$$\begin{aligned}
\sigma_{E,i}^\vee \cap \mathbb{Z}^{n+2} &= m_{*1}\mathbb{Z} + \cdots + m_{*n}\mathbb{Z} + m_{*(n+1)}\mathbb{Z} + m_{*(n+2)}\mathbb{Z}_+,\\
\sigma_{E,i+1}^\vee \cap \mathbb{Z}^{n+2} &= m_{*1}\mathbb{Z} + \cdots + m_{*n}\mathbb{Z} + m_{*(n+1)}\mathbb{Z}_+ + m_{*(n+2)}\mathbb{Z},\\
\sigma_{E,i,i+1}^\vee \cap \mathbb{Z}^{n+2} &= m_{*1}\mathbb{Z} + \cdots + m_{*n}\mathbb{Z} + m_{*(n+1)}\mathbb{Z}_+ + m_{*(n+2)}\mathbb{Z}_+.
\end{aligned}$$

The toric scheme

$$T_\Sigma = \bigcup_{\sigma\in\Sigma} T_\sigma, \qquad T_\sigma = \operatorname{Spec} O_K[\sigma^\vee \cap \mathbb{Z}^{n+2}],$$

associated with Σ ([K²MS]) is then obtained by glueing $T_{\sigma_{E,i,i+1}} = \operatorname{Spec} R$ for varying E and i, along their common opens. Note that

$$T_{\sigma_0} = \operatorname{Spec} R[Y^{-1}, Z^{-1}], \quad T_{\sigma_{E,i}} = \operatorname{Spec} R[Y^{-1}], \quad T_{\sigma_{E,i+1}} = \operatorname{Spec} R[Z^{-1}].$$

Note that $\deg \psi_{\tau(1)} = 1$ by Lemmas 4.5.4 and 4.5.2. Let

$$C_0 = \operatorname{Spec} \frac{K[x][x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y^{\pm 1}]}{(y^2 - f(x), x_1 - \psi_1(x), \ldots, x_n - \psi_n(x))}.$$

Then $C_0 \subseteq C$. Furthermore it canonically embeds in T_{σ_0} via the isomorphism given by $M_{E,i}$ and the isomorphism given by

$$\frac{K[x][x_{\tau(1)}^{\pm 1}]}{\big(x_{\tau(1)} - \psi_{\tau(1)}(x)\big)} \simeq K\big[x_{\tau(1)}^{\pm 1}\big].$$

We define $\mathcal{C}$ as the closure of C_0 in T_Σ. Then $\mathcal{C}$ is integral and also separated since so is T_Σ. Furthermore, $\mathcal{C}$ is flat by [Liu4, Corollary 3.10]. We will explicitly describe $\mathcal{C}$ and show it is a proper regular model of C with strict normal crossing.

4.5.3 Charts

Keep the notation of §4.5.1. From now on we suppose without loss of generality that the permutation τ is the identity.

Let $1 \le o \le h$. By [Mac, Theorem 16.1] every polynomial $g \in K[x]$ can be uniquely written as a sum

$$g = \sum_s a_s \cdot \psi_1^{n_{1s}} \cdots \psi_o^{n_{os}},$$

where $a_s \in K$ and $n_{js} < \deg\psi_{j+1}/\deg\psi_j$ for any $j < o$. Let $u_s \in O_K^\times$ such that $a_s = u_s \cdot \pi^{v_K(a_s)}$. Then we denote by $g^{(o)}$, the polynomial

$$g^{(o)} = \sum_s u_s \cdot \pi^{v_K(a_s)} \cdot x_1^{n_{1s}} \cdots x_o^{n_{os}} \in K[x_1, \dots, x_o].$$

Consider $M_{E,i}$. Recall $\tilde{m}_{(n+1)(n+2)}, \tilde{m}_{(n+2)(n+2)} \geq 0$. Define

$$\Pi(X_1, \dots, X_n, Y, Z) = \pi - X_1^{\tilde{m}_{1(n+2)}} \dots X_n^{\tilde{m}_{n(n+2)}} Y^{\tilde{m}_{(n+1)(n+2)}} Z^{\tilde{m}_{(n+2)(n+2)}},$$

Via $M_{E,i}$ we have the following isomorphism

$$\frac{K[x][x_1^{\pm1}, \dots, x_n^{\pm1}, y^{\pm1}]}{(y^2 - f(x), x_1 - \psi_1(x), \dots, x_n - \psi_n(x))} \overset{M_{E,i}}{\simeq} \frac{O_K[X_1^{\pm1}, \dots, X_n^{\pm1}, Y^{\pm1}, Z^{\pm1}]}{(\Pi, \mathcal{F}_1, \dots, \mathcal{F}_n)},$$

where $\mathcal{F}_1, \dots, \mathcal{F}_n \in O_K^\times[X_1^{\pm1}, \dots, X_n^{\pm1}, Y, Z]$ satisfying $Y \nmid \mathcal{F}_j$, $Z \nmid \mathcal{F}_j$, and

$$\begin{aligned} y^2 - f^{(h)}(x_1, \dots, x_h) &\overset{M_{E,i}}{=} Y^{n_{Y,1}} Z^{n_{Z,1}} \mathcal{F}_1(X_1, \dots, X_n, Y, Z), \\ x_j - \psi_j^{(j-1)}(x_1, \dots, x_{j-1}) &\overset{M_{E,i}}{=} Y^{n_{Y,j}} Z^{n_{Z,j}} \mathcal{F}_j(X_1, \dots, X_n, Y, Z) \quad \text{for } 2 \leq j \leq h, \\ x_j - \psi_j^{(h)}(x_1, \dots, x_h) &\overset{M_{E,i}}{=} Y^{n_{Y,j}} Z^{n_{Z,j}} \mathcal{F}_j(X_1, \dots, X_n, Y, Z) \quad \text{for } h < j \leq n, \end{aligned}$$

for some $n_{Y,j}, n_{Z,j} \in \mathbb{Z}$. Then we define the affine O_K-scheme

$$U_{E,i} = \operatorname{Spec} \frac{O_K[X_1^{\pm1}, \dots, X_n^{\pm1}, Y, Z]}{(\Pi, \mathcal{F}_1, \dots, \mathcal{F}_n)}.$$

In the next lemma we will describe the special fibre of $U_{E,i}$. In particular, we will show that it has dimension 1. Then the next lemma implies that $U_{E,i} = \mathcal{C} \cap T_{\sigma_{E,i,i+1}}$.

Lemma 4.5.17 *If the special fibre of $U_{E,i}$ is of dimension ≤ 1, then $U_{E,i} = \mathcal{C} \cap T_{\sigma_{E,i,i+1}}$.*

Proof. By construction the generic fibre of $U_{E,i}$ is isomorphic to $C_\eta \cap T_{\sigma_{E,i,i+1}}$. Then it suffices to show that $U_{E,i}$ is the closure of its generic fibre in $T_{\sigma_{E,i,i+1}}$. Suppose not. Then $U_{E,i}$ has an irreducible component U entirely contained in its special fibre. Since $O_K[X_1^{\pm1}, \dots, X_n^{\pm1}, Y, Z]$ is regular, $\dim U \geq 2$ by Krull's height theorem. □

4.5.4 Special Fibre

In this section we want to study the special fibre of $U_{E,i}$. Now, $U_{E,i} \subset T_{\sigma_{E,i,i+1}}$ and the special fibre of the latter has underlying reduced subscheme $Z = 0$ if E is outer and $i = r$, or $YZ = 0$ otherwise.

Notation 4.5.18 Let $g \in K[x_1^{\pm1}, \dots, x_n^{\pm1}, y^{\pm1}]$. Let $\mathcal{G} \in O_K^\times[X_1^{\pm1}, \dots, X_n^{\pm1}, Y^{\pm1}, Z^{\pm1}]$ given by

$$g\big((X_1, \dots, X_n, Y, Z) \bullet M_{E,i}^{-1}\big) = \mathcal{G}(X_1, \dots, X_n, Y, Z).$$

Denote by $\operatorname{ord}_Z(g)$ [resp. $\operatorname{ord}_Y(g)$] the integer $\operatorname{ord}_Z(\mathcal{G})$ [resp. $\operatorname{ord}_Y(\mathcal{G})$].

We want to study $U_{E,i} \cap \{Z=0\}$. Let $w_{E,i} : K[x] \to \hat{\mathbb{Q}}$ be the valuation given in (C.1). Then $\mathrm{ord}_Z(x_j) = w_{E,i}(\psi_j)\mathrm{ord}_Z(\pi)$ for all $1 \le j \le n$. Let $w_j = v_j$ for all $j < h$ and $w_h = w_{E,i}$.

Lemma 4.5.19 *Let $g \in K[x]$. For all $1 \le j \le h$,*

$$\mathrm{ord}_Z\left(g^{(j)}\right) = w_j(g)\mathrm{ord}_Z(\pi)$$

Proof. If $w_{E,i}$ is MacLane then the equality follows from [Mac, Theorem 16.1]. Suppose $w_{E,i}$ is not MacLane. Then $(\mathfrak{s}, v)$ is maximal, $E = V_v$ and $1 \le i \le r$. But then $h = 1$ and $\deg\psi_1 = 1$. Expand $g = \sum_t a_t \psi_1^t$, where $a_t \in K$. Then $g^{(1)} = \sum_t a_t x_1^t$. It follows that

$$\mathrm{ord}_Z\left(g^{(1)}\right) = \min_t \left(v_K(a_t)\mathrm{ord}_Z(\pi) + t \cdot \mathrm{ord}_Z(x_1)\right) = w_{E,i}(g)\mathrm{ord}_Z(\pi)$$

as $\mathrm{ord}_Z(x_1) = w_{E,i}(\psi_1)\mathrm{ord}_Z(\pi)$. □

Notation 4.5.20 For any $\mathcal{G} \in O_K[X_1^{\pm1}, \dots, X_n^{\pm1}, Y, Z]$ denote

$$\bar{\mathcal{G}}_Y = \mathcal{G}(X_1, \dots, X_n, 0, Z), \qquad \bar{\mathcal{G}}_Z = \mathcal{G}(X_1, \dots, X_n, Y, 0),$$

and $\bar{\mathcal{G}} = \mathcal{G}(X_1, \dots, X_n, 0, 0)$.

Definition 4.5.21 Define $p_0 = \pi \in O_K$. Let $1 \le j \le h$ and recursively define $p_j \in K[x_1^{\pm1}, \dots, x_j^{\pm1}]$ by $p_j = x_j^{\ell_j} p_{j-1}^{\ell'_j}$. Then $p_j(\psi_1, \dots, \psi_j) = \pi_j$.

Define $\Pi_j \in O_K[X_1^{\pm1}, \dots, X_n^{\pm1}]$ by

$$Y^* Z^* \cdot \Pi_j \overset{M_{E,i}}{=} p_j.$$

Note that

$$(\psi_1, \dots, \psi_n, y, \pi) \bullet M_{E,i} = (\gamma_1, \dots, \gamma_{h-1}, \psi_h^{\delta a_x} y^{\delta a_y} \pi_{h-1}^{\delta a_z}, \dots), \tag{4.17}$$

and that $\tilde{\alpha}_{hj} = 0$, $\tilde{\alpha}_{\pi j} = \lambda_j$ for any $j < h$.

Lemma 4.5.22 *Let $1 \le j \le h$. Then $X_j \overset{M_{E,i}}{=} x_j^{e_j} p_{j-1}^{-h_j}$ if $j < h$ or $E = L_v$.*

Proof. When $j < h$, then $X_j = x_j^{e_j} p_{j-1}^{-h_j}$ from (4.17). If $j = h$, then $X_j = x_j^{\delta a_x} y^{\delta a_y} p_{j-1}^{\delta a_z}$. If $E = L_v$, then $w_{E,i} = v$. Since L_v corresponds to the edge $L_v(f)$ of $N^-_{v_-,\psi_h}(f)$, one has $\delta = e_h$, $a_x = 1$, $a_y = 0$, $a_z = -\lambda_h$. It follows that $X_j = x_j^{e_j} p_{j-1}^{-h_j}$, as required. □

Lemma 4.5.23 *Let $1 \le j \le h$. Then $\Pi_j \in O_K[X_1^{\pm1}, \dots, X_h^{\pm1}]$.*

Proof. If $o > h$ then $\tilde{m}_{jo} = 0$ for $j \ne o$. The lemma follows. □

Recall the definition of the fields k_j, $j = 1, \dots, h$, given in §4.3. Note that $k_1 = k_0$ since $\deg\psi_1 = 1$ (Remark 4.3.11). The ring homomorphisms $k_o[X_o^{\pm1}] \to k_{o+1}$, $1 \le o < j$, taking X_o to the generator of k_{o+1} over k_o, induce a surjective homomorphism

$$\mathcal{R}_j : O_K[X_1^{\pm1}, \dots, X_n^{\pm1}, Y, Z] \to k_j[X_j^{\pm1}, \dots, X_n^{\pm1}, Y, Z].$$

Lemma 4.5.24 *Let $1 \leq j \leq h$ and let $g \in K[x]$. Fix a polynomial $\mathcal{G} \in O_K^\times[X_1^{\pm 1},\dots,X_n^{\pm 1},Y,Z]$, $Y \nmid \mathcal{G}$, $Z \nmid \mathcal{G}$, such that*

$$g^{(j)}(x_1,\dots,x_j) \overset{M_{E,i}}{=} Y^{n_Y} Z^{n_Z} \mathcal{G}(X_1,\dots,X_n,Y,Z),$$

for some $n_Y, n_Z \in \mathbb{Z}$. If either $E = L_v$ or $j < h$, then

$$\mathcal{R}_j(\bar{\mathcal{G}}_Z) = Y^* \cdot \mathcal{R}_j(\Pi_j)^{e_{v_j}\alpha} \cdot H_{j,\alpha}(g)(X_j)$$

where $\alpha = v_j(g)$.

Proof. We prove the lemma by induction on j. Suppose either $E = L_v$ or $j < h$, so that $w_j = v_j$. Let $j = 1$. Expand $g = \sum_s a_s \phi_1^j$, where $a_s \in K$. Then $g^{(1)}(x_1) = \sum_s a_s x_1^s$. Lemma 4.5.19 implies that $\mathrm{ord}_Z(a_s x_1^s) = n_Z$ if and only if $(s, v_K(a_s))$ is a point of the edge $L_{v_1}(g)$ of the Newton polygon $N_{v_0,\psi_1}(g)$. Therefore we can assume $g^{(1)}(x_1) = \sum_{s\geq 0} a_{t_1+se_1} x_1^{t_1+se_1}$, where $t_1 = t_1(\alpha_1)$ and $\alpha_1 = v_1(g)$. Then

$$\frac{g^{(1)}}{p_1^{e_{v_1}\alpha_1}} = \sum_{s\geq 0} \left(\frac{a_{t_1+se_1}}{\pi^{u_1-sh_1}}\right)(x_1^{e_1}\pi^{-h_1})^{s+c_1(\alpha_1)},$$

where $u_1 = u_1(\alpha_1)$. Then we obtain the required equality by Lemma 4.5.22.

Now suppose $j > 1$. Expand

$$g = \sum_{s\geq 0} a_s \psi_j^s, \qquad \text{where } \deg a_s < \deg \psi_j.$$

Note that $g^{(j)} = \sum_s a_s^{(j-1)} x_j^s$ by definition. Similarly to before, by Lemma 4.5.19 we have that $\mathrm{ord}_Z\big(a_s^{(j-1)} x_j^s\big) = n_Z$ if and only if $(s, v_{j-1}(a_s))$ is a point of the edge $L_{v_j}(g)$ of the Newton polygon $N_{v_{j-1},\psi_j}(g)$. Therefore we can assume

$$g^{(j)} = \textstyle\sum_s a_{t_{j,s}}^{(j-1)} x_j^{t_{j,s}},$$

where $t_{j,s} = t_j(\alpha_j) + se_j$ and $\alpha_j = v_j(g)$. Then

$$\frac{g^{(j)}}{p_j^{e_{v_j}\alpha_j}} = \sum_s \frac{a_{t_{j,s}}^{(j-1)}}{p_{j-1}^{u_{j,s}}} \left(x_j^{e_j} p_{j-1}^{-h_j}\right)^{s+c_j(\alpha_j)},$$

where $u_{j,s} = u_j(\alpha_j) - sh_j$. Lemma 4.5.22 and the inductive hypothesis conclude the proof. □

Lemma 4.5.25 *Let $1 \leq j \leq h$. Then* $\ker(\mathcal{R}_j) = (\bar{\mathcal{F}}_{2,Z},\dots,\bar{\mathcal{F}}_{j,Z},\pi)$.

Proof. We prove the lemma by induction on j. Suppose $j = 1$. Since $\deg\psi_1 = 1$, we have $k_1 = k$, and so $\ker(\mathcal{R}_1) = (\pi)$. Let $j > 1$. It follows from Lemma 4.5.19 that

$$\mathrm{ord}_Z(x_j) = \mathrm{ord}_Z\big(\psi_j^{(j)}\big) > \mathrm{ord}_Z\big(\psi_j^{(j-1)}\big).$$

Then Lemma 4.5.24 implies that

$$\mathcal{R}_{j-1}(\bar{\mathcal{F}}_{j,Z}) = \mathcal{R}_{j-1}(\Pi_{j-1})^{e_{v_{j-1}}\alpha} \cdot H_{j-1,\alpha}(\psi_j),$$

where $\alpha = v_{j-1}(\psi_j)$. Since $k_j \simeq k_{j-1}[X_{j-1}]/(H_{j-1,\alpha}(\psi_j))$ by Remark 4.3.11 and $\mathcal{R}_{j-1}(\Pi_{j-1})$ is invertible by Lemma 4.5.23, we have

$$\ker(\mathcal{R}_j) = \ker(\mathcal{R}_{j-1}) + (\bar{\mathcal{F}}_{j,Z}).$$

The inductive hypothesis concludes the proof. □

Let $h < j \le n$. Then

$$\mathrm{ord}_Z(x_j) = \mathrm{ord}_Z\big(\psi_j^{(h)}\big)$$

by Lemma 4.5.19. Since $\tilde{m}_{jj} = 1$ and $\tilde{m}_{oj} = 0$ for all $1 \le o \le n$, $o \ne j$, there exists a Laurent polynomial $\mathcal{T}_j \in O_K[X_1^{\pm 1}, \ldots, X_h^{\pm 1}, Y, Z]$ such that $\mathcal{F}_j$ equals $X_j - \mathcal{T}_j$ up to some unit. Let $\mathcal{R} = \mathcal{R}_h$ and $\mathcal{T} = \prod_{h<j\le n} \mathcal{T}_j$. Denote $\mathcal{F} = \mathcal{F}_1$. Lemma 4.5.25 implies that $U_{E,i} \cap \{Z = 0\}$ is isomorphic to

$$\mathrm{Spec}\ \frac{k_v[X_h^{\pm 1}, Y, \mathcal{R}(\bar{\mathcal{T}}_Z)^{-1}]}{(\mathcal{R}(\bar{\mathcal{F}}_Z))}.$$

Similar computations (using $w_{E,i+1}$ instead of $w_{E,i}$) show that if $i < r$ or E is inner, then $U_{E,i} \cap \{Y = 0\}$ is isomorphic to

$$\mathrm{Spec}\ \frac{k_v[X_h^{\pm 1}, Z, \mathcal{R}(\bar{\mathcal{T}}_Y)^{-1}]}{(\mathcal{R}(\bar{\mathcal{F}}_Y))}.$$

Let $g(x,y) = y^2 - f(x)$ and expand

$$g = \sum_{j,o} a_{jo} \psi_h^j y^o, \qquad a_{jo} \in K[x], \deg a_{jo} < \deg \psi_h.$$

Then $y^2 - f^{(h)} = \sum_{j,o} a_{jo}^{(h-1)} x_h^j y^o$. Recall the notation $w_{E,i}(y)$ from Appendix C.3. Let ξ_i be the plane with normal vector $(w_{E,i}(\psi_h), w_{E,i}(y), 1)$ and on which $\tilde{E}$ lies. We have

$$\mathrm{ord}_Z(a_{jo}^{(h-1)} x_h^j y^o) = \mathrm{ord}_Z(y^2 - f^{(h)}) \quad \text{if and only if} \quad (j, o, v_-(a_{jo})) \in \xi_i.$$

More precisely, $(X_h, Y, Z) = (x_h, y, p_{h-1}) \bullet M$ with

$$M = \begin{pmatrix} \delta a_x & d_i b_x + k_i a_x & -d_{i+1} b_x - k_{i+1} a_x \\ \delta a_y & d_i b_y + k_i a_y & -d_{i+1} b_y - k_{i+1} a_y \\ \delta \epsilon_h a_z & \frac{n_i}{\delta} + \epsilon_h k_i a_z & -\frac{n_{i+1}}{\delta} - \epsilon_h k_{i+1} a_z \end{pmatrix} \in \mathrm{SL}_3(\mathbb{Z}). \tag{4.18}$$

Let $\phi : \mathbb{Z}^2 \to \mathbb{Z}^2$ given by

$$\phi(s,t) = P_0 + (\delta a_x, \delta a_y)s + (d_i b_x + k_i a_x, d_i b_y + k_i a_y)t.$$

Lemma 4.5.24 implies that, up to units, $\mathcal{R}(\bar{\mathcal{F}}_Z)$ equals

$$\sum_{(s,t)\in\mathbb{Z}^2} H_{h-1,\alpha_{s,t}}\big(a_{\phi(s,t)}\big) X_h^s Y^t,$$

where $\alpha_{s,t} = \delta a_z s + \frac{n_i}{\delta \epsilon_h} t + k_i a_z t$. In particular, $U_{E,i} \cap \{Z = 0\}$ is of dimension 1, and similarly $U_{E,i} \cap \{Y = 0\}$ when $i < r$ or E is inner. It follows that the special fibre of $U_{E,i}$ is 1-dimensional.

4.5.5 Components

We want to describe $U_{E,i} \cap \{Z=0\}$ and $U_{E,i} \cap \{Y=0\}$ explicitly.

Remark 4.5.26. Let $h < j \le n$ such that $v = \mu_j \wedge \mu_h$. Let $(\mathfrak{s}_j, \mu_j) \subseteq (\mathfrak{t}, w) < (\mathfrak{s}, v)$. Lemma 4.2.30 implies that ψ_j is v-equivalent to ϕ_w^d, where $d = \deg \mu_j / \deg w$. Thus $\psi_j|_v$ is a power of $\phi_w|_v$ by Proposition 4.3.19.

Lemma 4.5.27 *Let $\mu_1, \mu_2 \in \mathbb{V}_M$ such that $\mu_1 \ge v \not\ge \mu_2$. Suppose ϕ_n is a centre of μ_1 and let $\phi \in K[x]$ be a centre of μ_2. If $v \ne \mu_1 \wedge \mu_2$ then $\phi|_v$ is a unit.*

Proof. Let $w = \mu_1 \wedge \mu_2$. Then $\mu_1(\phi) = w(\phi)$ by Proposition 4.4.19. Since $w \le \mu_1$ and $v \le \mu_1$, either $w < v$ or $w \ge v$ by Theorem 4.2.21 and Remark 4.2.18(2).

Suppose $w < v$. Proposition 4.2.24 implies that there exists $w' \ge w$ such that $v = [w', v(\phi_n) = \lambda_n]$. In particular, $w'(\phi) = v(\phi)$. From [FGMN, Lemma 2.9] it follows that ϕ is v-equivalent to some polynomial of degree $< \deg v$. Hence $\phi|_v$ is a unit by Proposition 4.3.19.

Suppose $w > v$. Then the polynomial ϕ_n is a centre of w and so

$$\phi \sim_v \phi_n^{\deg\phi/\deg\phi_n}$$

by Lemmas 4.2.29 and 4.2.30. Then $\phi|_v$ is a unit by Proposition 4.3.19. □

Lemma 4.5.28 *Let $h < j \le n$. If $v \ne \mu_h \wedge \mu_j$ then $\psi_j|_v$ is a unit.*

Proof. The lemma follows from Lemma 4.5.27. □

Lemma 4.5.29 *Let $h < j \le n$.*

1. *Suppose $E = L_v$. Then, up to units, $\mathcal{R}(\bar{\mathcal{T}}_{j,Z})$ equals $\psi_j|_v(X_h)$, and, similarly, $\mathcal{R}(\bar{\mathcal{T}}_{j,Y})$ equals $\psi_j|_v(X_h)$ when $i < r$.*

2. *Suppose $E = V_v$ or $E = V_v^0$. Then $\bar{\mathcal{T}}_j$ is a unit. Furthermore, $\bar{\mathcal{T}}_j = \bar{\mathcal{T}}_{j,Z}$ if $i > 0$ and $\bar{\mathcal{T}}_j = \bar{\mathcal{T}}_{j,Y}$ if $i < r$.*

Proof. Suppose $E = L_v$. Then Lemma 4.5.24 implies that $\mathcal{R}(\bar{\mathcal{T}}_{j,Z})$ equals $\psi_j|_v(X_h)$ up to units. Similarly for $\mathcal{R}(\bar{\mathcal{T}}_{j,Y})$ when $i < r$.

Expand

$$\psi_j = \sum_{t=0}^{d} a_t \psi_h^t, \qquad a_t \in K[x], a_d \ne 0, \deg a_t < \deg \psi_h.$$

Then $\psi_j^{(h)} = \sum_t a_t^{(h-1)} x_h^t$.

Suppose $(\mathfrak{s}, v)$ maximal and $E = V_v$. Then $h = 1$ and $\deg \psi_h = 1$. Lemma C.3.2 implies that $w_{E,i}(\psi_j) = d \cdot w_{E,i}(\psi_1)$ for any $i = 0, \ldots, r$. In fact, for all $i = 1, \ldots, r$ we have

$$w_{E,i}(\psi_j - \psi_1^d) > w_{E,i}(\psi_j) \tag{4.19}$$

since $w_{E,i}(\psi_h) < w_{E,0}(\psi_h)$. Recall

$$\mathrm{ord}_Z(\psi_j^{(1)}) = w_{E,i}(\psi_j)\mathrm{ord}_Z(\pi)$$

from Lemma 4.5.19, and similarly, $\mathrm{ord}_Y(\psi_j^{(1)}) = w_{E,i+1}(\psi_j)\mathrm{ord}_Y(\pi)$ when $i < r$. The inequality (4.19 implies that $\bar{\mathcal{T}}_j$ is a unit, $\bar{\mathcal{T}}_j = \bar{\mathcal{T}}_{j,Z}$ when $i > 0$ and $\bar{\mathcal{T}}_j = \bar{\mathcal{T}}_{j,Y}$ when $i < r$.

Suppose $E = V_v$ inner. Then $w_{E,i}$ is a MacLane valuation with centre ψ_h and satisfying $v \geq w_{E,i} \geq w$. In particular, $w_{E,i}(\psi_j) = w_{E,i}(a_{t_v}\psi_h^{t_v})$. Lemma 4.5.19 implies that $\bar{\mathcal{T}}_{j,Z}$ is a unit if and only if $\psi_j|_{w_{E,i}}$ is a unit. But then $\bar{\mathcal{T}}_{j,Z} = \bar{\mathcal{T}}_j$ is a unit when $i > 0$ by Lemma 4.5.27. Similarly $\bar{\mathcal{T}}_{j,Y} = \bar{\mathcal{T}}_j$ is a unit when $i < r$.

Suppose $(\mathfrak{s}, v)$ degree-minimal and $E = V_v^0$. Then $w_{E,i}$ is a MacLane valuation with centre ψ_h and satisfying $v \leq w_{E,i}$. In particular, $w_{E,i}(\psi_j) = w_{E,i}(a_{t_v^0}\psi_h^{t_v^0})$. Similarly to the previous case, Lemmas 4.5.19, 4.5.27 conclude the proof. □

Suppose $E = L_v$. Fix $P_0 = (t_v, 0)$, $P_1 = \left(\left\lfloor \frac{t_v - 1}{2} \right\rfloor, 1\right)$. Then

$$s_1^E = e_v\left(\lambda_v(\lfloor t_v/2 \rfloor + 1) - \tfrac{v(f)}{2}\right), \tag{4.20}$$

and $s_2^E = \lfloor s_1^E - 1 \rfloor$. The h-edge L_v corresponds to the edge $L_v(f)$ of $N_{v_-,\psi_h}(f)$. In particular, $\delta_E = e_h$ and $\nu = (1, 0, -\lambda_h)$. Therefore, up to units,

$$\begin{aligned} \mathcal{R}(\bar{\mathcal{F}}_Z) &= f|_v(X_h) \quad \text{for } 0 < i \leq r, \\ \mathcal{R}(\bar{\mathcal{F}}_Y) &= f|_v(X_h) \quad \text{for } 0 \leq i < r. \end{aligned}$$

Fix

$$k_j = \ell_h n_j + \ell'_h e_h d_j(\lfloor t_v/2 \rfloor + 1), \quad \text{for } j = 0, \ldots, r+1.$$

Then $k_j \equiv n_j(\delta_E \epsilon_h a_z)^{-1} \mod \delta_E$, as required. Let $i = 0$ and let M be the matrix of (4.18). Then

$$M^{-1} = \begin{pmatrix} \ell'_h & 0 & -\ell_h \\ d_1 e_v \lambda_v & d_1 e_v \frac{v(f)}{2} + \frac{1}{d_0} & d_1 e_h \\ d_0 e_v \lambda_v & d_0 e_v \frac{v(f)}{2} & d_0 e_h \end{pmatrix}.$$

Hence $y^2 p_h^{-e_v v(f)} = Y^{2/d_0}$. Lemma 4.5.24 then implies that $R_h(\bar{\mathcal{F}}_Z)$ equals

$$Y^{2/d_0} - H_{h,v(f)}(f)(X_h)$$

up to units. The quantity $n_v := 2/d_0$ equals 1 if $e_v v(f)$ is odd and 2 if $e_v v(f)$ is even. Recall

$$H_{h,v(f)}(f)(X) = X^{t_v^0/e_h - \ell_h \epsilon_h v(f)} f|_v,$$

from (4.5). Note that $t_v^0 = t_w$ if $(\mathfrak{s}, v)$ has a child $(\mathfrak{t}, w)$ with centre ψ_h and $t_v^0 = 0$ otherwise.

Suppose $E = V_v$. We can choose $P_0 = (t_v, 0)$, $P_1 = \left(\left\lfloor \frac{t_v - 1}{2} \right\rfloor, 1\right)$ so that

$$s_1^E = \delta_E \epsilon_h \left(\lambda_v(\lfloor t_v/2 \rfloor + 1) - \tfrac{v(f)}{2}\right),$$

If $(\mathfrak{s},v)\neq(\mathfrak{R},w_{\mathfrak{R}})$ and $(\mathfrak{s},v)<(\mathfrak{t},w)$, then

$$s_2^E=\delta_E\epsilon_h\big(\lambda_w(\lfloor t_v/2\rfloor+1)-\tfrac{w(f)}{2}\big),$$

while $s_2^E=\lfloor s_1^E-1\rfloor$ otherwise. Up to units

$$\mathcal{R}(\bar{\mathcal{F}}_Z)=X_h^b-H_{h-1,\alpha}(a_{t_v})\quad\text{for } 0<i\leq r,$$
$$\mathcal{R}(\bar{\mathcal{F}}_Y)=X_h^b-H_{h-1,\alpha}(a_{t_v})\quad\text{for } 0\leq i<r,$$

where $b=E_{\mathbb{Z}}(\mathbb{Z})+1$ and $\alpha=v_-(a_{t_v})$. Let $u=c_f\prod_{r'\notin\mathfrak{s}}(x-r')\in K[x]$ and let $u_h=u-\psi_hq$ for some $q\in K[x]$ such that $\deg u_h<\deg\psi_h$. From Theorem 4.4.26, one has $H_{h-1,\alpha}(a_{t_v})=H_{h-1,v_-(u_h)}(u_h)$.

Suppose $v=\mu_h$ and $E=V_v^0$. Fix $P_0=(0,2)$, $P_1=(1,1)$, so

$$s_1^E=-\delta_E\epsilon_h\big(\lambda_v-\tfrac{v(f)}{2}\big),$$

and $s_2^E=\lfloor s_1^E-1\rfloor$. Then up to units

$$\mathcal{R}(\bar{\mathcal{F}}_Z)=X_h^{-b}-H_{h-1,\alpha}(a_{t_v^0})\quad\text{for } 0<i\leq r,$$
$$\mathcal{R}(\bar{\mathcal{F}}_Y)=X_h^{-b}-H_{h-1,\alpha}(a_{t_v^0})\quad\text{for } 0\leq i<r,$$

where $b=E_{\mathbb{Z}}(\mathbb{Z})+1$ and $\alpha=v_-(a_{t_v^0})$. Let $\mathfrak{R}_h$ be the set of roots of ψ_h. Let $u^0=c_f\prod_{r'\in\mathfrak{R}\setminus\mathfrak{R}_h}(x-r')\in K[x]$ and let $u_h^0=u^0-\psi_hq$ for some $q\in K[x]$ such that $\deg u_h^0<\deg\psi_h$. One has $H_{h-1,\alpha}(a_{t_v^0})=H_{h-1,v_-(u_h^0)}(u_h^0)$.

4.5.6 Regularity

If $(\mathfrak{s},v)$ has a proper child with centre $\phi_w\neq\phi_v$, then $\phi_w|_v$ is irreducible by Lemmas 4.5.2 and 4.3.21. Let $E=L_v$. By Remark 4.5.26 and Lemmas 4.5.28, 4.5.29, the subscheme $U_{E,i}\cap\{Z=0\}$ is isomorphic to

$$\operatorname{Spec}\frac{k_v\big[X_h^{\pm1},Y,\prod_{(\mathfrak{t},w)<(\mathfrak{s},v)}(\phi_w|_v(X_h))^{-1}\big]}{(\mathcal{R}(\bar{\mathcal{F}}_Z))},\tag{4.21}$$

where the product runs through all proper children of $(\mathfrak{s},v)$. Similarly for $U_{E,i}\cap\{Y=0\}$ when $i<r$.

Notation 4.5.30 We denote by $\mathring{\Gamma}_v$ the scheme $U_{L_v,0}\cap\{Z=0\}$.

Theorem 4.5.31 *The model $\mathcal{C}/O_K$ is regular.*

Proof. We want to prove that $U_{E,i}$ is regular, for any h-edge E, $h=1,\dots,n$, and any $i=0,\dots,r_E$. In fact, for the definition of Π, it suffices to show that the subschemes $U_{E,i}\cap\{Z=0\}$ and $U_{E,i}\cap\{Y=0\}$ are regular, where the latter is considered only if $i<r$. From the description given in §4.5.5 we only need to consider the case $E=L_v$, for some proper MacLane cluster $(\mathfrak{s},v)$ for f. Let $r=r_E$. For the explicit description of $\mathcal{R}(\bar{\mathcal{F}}_Z)$ and $\mathcal{R}(\bar{\mathcal{F}}_Y)$ it suffices to prove that all multiple irreducible factors of $f|_v$ are of the form $\phi_w|_v$ for some proper child $(\mathfrak{t},w)$ of $(\mathfrak{s},v)$. But this follows from Theorem 4.4.29. □

4.5.7 Properness

Let $\mathcal{C}_s^{\mathrm{red}}$ be the underlying reduced subscheme of the special fibre of $\mathcal{C}$. In the previous subsections we showed that $\mathcal{C}_s^{\mathrm{red}}$ consist of 1-dimensional subschemes Γ_v for each proper MacLane cluster $(\mathfrak{s},v)$, closures of $\mathring{\Gamma}_v$ (Notation 4.5.30) in $\mathcal{C}$, and chains of $\mathbb{P}^1$. In this subsection we will show that Γ_v is projective for any proper $(\mathfrak{s},v)\in\Sigma_f^M$. By [Liu4, Remark 3.28] the properness of $\mathcal{C}$ will follow.

Let $(\mathfrak{s},v)$ be a proper MacLane cluster and recall the notation introduced in previous subsections. Let C_v be the regular projective curve with ring of rational functions

$$k_v(X)[Y]/\big(Y^{n_v}-X^{t_v^0/e_h-\ell_h\epsilon_h v(f)}f|_v\big).$$

From (4.21) we have a natural birational map $\Gamma_v \dashrightarrow C_v$ defined on the dense open $\mathring{\Gamma}_v$. It extends to a morphism $\iota:\Gamma_v\to C_v$ by [EGA, II.7.4.9]. Zariski's Main Theorem implies that ι is an open immersion, since Γ_v is separated and regular. By point counting we can prove that ι is an isomorphism.

Let $\mathring{C}_v=\iota(\mathring{\Gamma}_v)$. By Theorem 4.4.29 we have

$$\mathrm{ord}_{\phi_w|_v}(f|_v)=|\mathfrak{t}|/\deg w,$$

for any proper child $(\mathfrak{t},w)<(\mathfrak{s},v)$ with $\phi_w\neq\phi_v$. The set $C_v(\bar{k})\setminus\mathring{C}_v(\bar{k})$ is finite and consists of:

(1) $\gcd(n_v, t_v^0/e_h-\ell_h\epsilon_h v(f)+\deg(f|_v))$ points at infinity;

(2) $\gcd(n_v, t_v^0/e_h-\ell_h\epsilon_h v(f))$ points on $X=0$;

(3) $\gcd(n_v,|\mathfrak{t}|/\deg w)$ points on $Y=0$ ($X\neq 0$) for each proper child $(\mathfrak{t},w)<(\mathfrak{s},v)$ with $\phi_w\neq\phi_v$.

(1) Let $E=V_v$. The scheme Γ_v has $(|E_{\mathbb{Z}}(\mathbb{Z})|+1)$ $\bar{k}$-points in

$$U_{E,0}\cap\{Y=Z=0\}$$

not contained in $\mathring{\Gamma}_v$. Note that $|E_{\mathbb{Z}}(\mathbb{Z})|$ equals 1 if t_v and $\epsilon_h(v(f)-t_v\lambda_v)$ are both even, while it equals 0 otherwise. In fact,

$$\big(\epsilon_h(v(f)-t_v\lambda_v),t_v\big)=\big(e_v v(f),t_v/e_h-\ell_h\epsilon_h v(f)\big)\cdot\left(\begin{smallmatrix}\ell'_h & \ell_h\\ -h_h & e_h\end{smallmatrix}\right),$$

and so

$$|E_{\mathbb{Z}}(\mathbb{Z})|+1=\gcd(n_v,t_v^0/e_h-\ell_h\epsilon_h v(f)+\deg(f|_v)),$$

since $\deg(f|_v)=(t_v-t_v^0)/e_h$.

(2) Let $E=V_v^0$. Let $(\mathfrak{t},w)<(\mathfrak{s},v)$ such that $E=V_w$ if $(\mathfrak{s},v)$ is not degree-minimal. Let $U=U_{V_v^0,0}$ if $(\mathfrak{s},v)$ is degree-minimal and $U=U_{V_w,r_E+1}$ otherwise. The scheme Γ_v has $(|E_{\mathbb{Z}}(\mathbb{Z})|+1)$ $\bar{k}$-points in

$$U\cap\{Y=Z=0\}$$

not visible on $\mathring{\Gamma}_v$. Note that $|E_{\mathbb{Z}}(\mathbb{Z})|$ equals 1 if t_v^0 and $\epsilon_h(v(f)-t_v^0\lambda_v)$ are both even, while equals 0 otherwise. Similarly to the case above,

$$\left(\epsilon_h(v(f)-t_v^0\lambda_v), t_v^0\right) = \left(e_v v(f), t_v^0/e_h - \ell_h\epsilon_h v(f)\right)\cdot\left(\begin{smallmatrix}\ell_h' & \ell_h \\ -h_h & e_h\end{smallmatrix}\right),$$

and so

$$|E_{\mathbb{Z}}(\mathbb{Z})|+1 = \gcd(n_v, t_v^0/e_h - \ell_h\epsilon_h v(f)).$$

(3) Let $(\mathfrak{t}, w) < (\mathfrak{s}, v)$ be a proper child such that $\phi_w \neq \phi_v$. Let $E = V_w$. The scheme Γ_v has $(|E_{\mathbb{Z}}(\mathbb{Z})|+1)$ $\bar{k}$-points in

$$U_{E,0}\cap\{Y=Z=0\}$$

not visible on $\mathring{\Gamma}_v$. Note that $|E_{\mathbb{Z}}(\mathbb{Z})|$ equals 1 if t_w and $e_v(v(f)-t_w\lambda_v)$ are both even, while it equals 0 otherwise. Since $t_w = |\mathfrak{t}|/\deg w$ by Proposition 4.4.25, we can compute

$$|E_{\mathbb{Z}}(\mathbb{Z})|+1 = \gcd(n_v, |\mathfrak{t}|/\deg w).$$

Thus $|\Gamma_v(\bar{k})\setminus\mathring{\Gamma}_v(\bar{k})| = |C_v(\bar{k})\setminus\mathring{C}_v(\bar{k})|$, and so $\Gamma_v \simeq C_v$.

Remark 4.5.32. If k_v is perfect, Γ_v is a generalised Baker's model of the curve $\mathring{\Gamma}_v\cap\mathbb{G}^2_{m,k_v}$ according to [Mus2].

4.6 Main result

Let C/K be a hyperelliptic curve of genus $g\geq 1$. Choose a separable polynomial $f\in K[x]$ as in the previous section so that $C/K : y^2 = f(x)$. Then $v_K(r) > 0$ for every root $r\in\bar{K}$ of f. Denote by $\mathfrak{R}$ the set of roots of f as before. Consider the MacLane cluster picture of f and fix a centre ϕ_v for all proper MacLane clusters $(\mathfrak{s}, v)\in\Sigma_f^M$ as we did at the beginning of §4.5. Denote by Σ the set of proper MacLane clusters for f.

Definition 4.6.1 Let $(\mathfrak{s}, v)\in\Sigma$. Consider its cluster chain

$$[v_0, v_1(\phi_1)=\lambda_1, \ldots, v_{m-1}(\phi_{m-1})=\lambda_{m-1}, v_m(\phi_m)=\lambda_m].$$

Define the following quantities:

$$
\boxed{
\begin{aligned}
\epsilon_v &= e_{v_{m-1}}\\
b_v &= e_v/\epsilon_v\\
\ell_v &= \ell_m\\
k_v &= k_m\\
f_v &= [k_v : k]\\
\nu_v &= v(f)\\
n_v &= 1 \text{ if } e_v\nu_v \text{ odd, } 2 \text{ if } e_v\nu_v \text{ even}\\
m_v &= 2e_v/n_v\\
t_v &= |\mathfrak{s}|/\deg v\\
p_v &= 1 \text{ if } t_v \text{ is odd, } 2 \text{ if } t_v \text{ is even}\\
s_v &= \tfrac{1}{2}(t_v\lambda_v + p_v\lambda_v - \nu_v)\\
\gamma_v &= 2 \text{ if } t_v \text{ is even and } \epsilon_v(\nu_v - t_v\lambda_v) \text{ is odd, 1 otherwise}\\
\delta_v &= 1 \text{ if } (\mathfrak{s}, v) \text{ is degree-minimal, 0 otherwise}\\
p_v^0 &= 1 \text{ if } \delta_v = 1 \text{ and } \deg v = \min_{r\in\mathfrak{s}}[K(r):K], \text{ 2 otherwise}\\
s_v^0 &= -\nu_v/2 + \lambda_v\\
\gamma_v^0 &= 2 \text{ if } p_v^0 = 2 \text{ and } \epsilon_v\nu_v \text{ is an odd integer, 1 otherwise}
\end{aligned}
}
$$

Define

$$\tilde{v} = \left\{(\mathfrak{t}, w) \in \Sigma \mid (\mathfrak{t}, w) < (\mathfrak{s}, v) \text{ and } \tfrac{f_v|\mathfrak{t}|}{f_w b_v \deg v} - \ell_v\nu_v\epsilon_w \notin 2\mathbb{Z}\right\}.$$

Let $c_v^0 = 1$ if $\frac{2-p_v^0}{b_v} - \ell_v\nu_v\epsilon_v \notin 2\mathbb{Z}$, and $c_v^0 = 0$ otherwise. Define

$$u_v = \frac{|\mathfrak{s}| - \sum_{(\mathfrak{t},w)<(\mathfrak{s},v)}|\mathfrak{t}| - (2-p_v^0)\deg v}{e_v} + \sum_{(\mathfrak{t},w)\in\tilde{v}} \frac{f_w}{f_v} + \delta_v c_v^0.$$

The *genus* $g(v)$ of $(\mathfrak{s}, v)$ is defined as follows:

- if $n_v = 1$, then $g(v) = 0$;
- if $n_v = 2$, then $g(v) = \max\{\lfloor (u_v - 1)/2 \rfloor, 0\}$.

We say that $(\mathfrak{s}, v)$ is *übereven* if $u_v = 0$.

Recall the definition of $H_{m-1,\alpha}$, for $\alpha \in \Gamma_{v_{m-1}}$, from Definition 4.3.8(ii). Define $\overline{g_v} \in k_v[y]$, and $\overline{g_v^0} \in k_v[y]$ if $\delta_v = 1$, by

$$\overline{g_v}(y) = y^{p_v/\gamma_v} - H_{m-1,v_{m-1}(u)}(u), \qquad u = c_f \prod_{r\in\mathfrak{R}\setminus\mathfrak{s}}(x-r) \mod \phi_v,$$
$$\overline{g_v^0}(y) = y^{p_v^0/\gamma_v^0} - H_{m-1,v_{m-1}(u)}(u), \qquad u = c_f \prod_{r\in\mathfrak{R}\setminus\mathfrak{R}_v}(x-r) \mod \phi_v,$$

where $\mathfrak{R}_v$ is the set of roots of ϕ_v.

Define $f'_v \in K[x]$ by

$$\phi_v^{2-p_v^0} f'_v(x) = \prod_{r\in\mathfrak{s}\setminus\bigcup_{(\mathfrak{t},w)<(\mathfrak{s},v)}\mathfrak{t}} (x-r),$$

where the union runs through all proper children $(\mathfrak{t}, w) < (\mathfrak{s}, v)$.

Define $\overline{f_v}, \tilde{f}_v \in k_v[x]$ by

$$\begin{aligned} \overline{f_v}(x) &= H_{m-1,v_{m-1}(u)}(u) \cdot f'_v|_v(x), \qquad u = c_f \prod_{r\in\mathfrak{R}\setminus\mathfrak{s}}(x-r) \mod \phi_v, \\ \tilde{f}_v(x) &= \overline{f_v}(x) \cdot x^{\delta_v c_v^0} \cdot \prod_{(\mathfrak{t},w)\in\tilde{v}} \phi_w|_v(x). \end{aligned}$$

Finally, define the k_v-schemes

- $X_v : \{\overline{f_v} = 0\} \subset \mathbb{G}_{m,k_v}$;
- $Y_v : \{\overline{g_v} = 0\} \subset \mathbb{G}_{m,k_v}$;
- $Y_v^0 : \{\overline{g_v^0} = 0\} \subset \mathbb{G}_{m,k_v}$ if $(\mathfrak{s}, v)$ is degree-minimal.

Recall Notations 2.4.16, 2.4.17 from Chapter 2.

Notation 4.6.2 Let $a, b \in K[x]$, $b \neq 0$. We denote by $a \mod b$ the remainder of the division of a by b.

In the next theorem we describe the special fibre of the scheme $\mathcal{C}$ constructed in §4.5.

Theorem 4.6.3 (Regular SNC model) *The scheme $\mathcal{C} \to O_K$ constructed in §4.5 is a regular model of C with strict normal crossings; its special fibre $\mathcal{C}_s/k$ is described as follows:*

(1) Every $(\mathfrak{s}, v) \in \Sigma$ gives a 1-dimensional closed subscheme Γ_v of multiplicity m_v. The ring of rational functions of Γ_v is isomorphic to $k_v(x)[y]/(y^{n_v} - \tilde{f}_v(x))$. If $n_v = 2$, $u_v = 0$, and $\tilde{f}_v \in k_v^2$, then $\Gamma_v \simeq \mathbb{P}^1_{k_v} \sqcup \mathbb{P}^1_{k_v}$, otherwise Γ_v is irreducible of genus $g(v)$.

(2) Every $(\mathfrak{s}, v) \in \Sigma$ with $n_v = 1$ gives the closed subscheme $X_v \times_k \mathbb{P}^1_k$, of multiplicity e_v, where $X_v \times_k \{0\} \subset \Gamma_v$ (the $\mathbb{P}^1_k$s are open-ended).

(3) Every non-maximal $(\mathfrak{s}, v) \in \Sigma$, with $(\mathfrak{s}, v) < (\mathfrak{t}, w)$, gives the closed subscheme

$$Y_v \times_k \mathbb{P}^1\big(\epsilon_v\gamma_v, s_v, s_v - \tfrac{p_v}{2}(\lambda_v - \tfrac{\deg v}{\deg w}\lambda_w)\big),$$

where $Y_v \times_k \{0\} \subset \Gamma_v$ and $Y_v \times_k \{\infty\} \subset \Gamma_w$.

(4) Every degree-minimal $(\mathfrak{s}, v) \in \Sigma$ gives the closed subscheme $Y_v^0 \times_k \mathbb{P}^1(\epsilon_v\gamma_v^0, -s_v^0)$, where $Y_v^0 \times_k \{0\} \subset \Gamma_v$ (the chains are open-ended).

(5) Finally, the maximal element $(\mathfrak{s}, v) \in \Sigma$ gives the closed subscheme $Y_v \times_k \mathbb{P}^1(\epsilon_v\gamma_v, s_v)$, where $Y_v \times_k \{0\} \subset \Gamma_v$ (the chains are open-ended).

If Γ_v is reducible, the two points in $Y_v \times_k \{0\}$ (and $Y_v^0 \times_k \{0\}$ if $(\mathfrak{s}, v)$ is degree-minimal) belong to different irreducible components of Γ_v. Similarly, if $(\mathfrak{s}, v)$ is not maximal with $(\mathfrak{s}, v) < (\mathfrak{t}, w)$, and Γ_w is reducible, then the two points of $Y_v \times_k \{\infty\}$ belong to different irreducible components of Γ_w.

Proof. The description of the special fibre of $\mathcal{C}$ follows from its explicit construction developed in §4.5 (see especially §4.5.5). We highlight the key points.

(1) Each proper MacLane cluster $(\mathfrak{s},v)$ gives the 1-dimensional closed subscheme Γ_v of $\mathcal{C}_s$, coming from the $*$-face F_v. The open subscheme $\mathring{\Gamma}_v$ (Notation 4.5.30) of Γ_v is isomorphic to

$$\operatorname{Spec}\frac{k_v\left[X^{\pm 1},Y,\prod_{(\mathfrak{t},w)<(\mathfrak{s},v)}(\phi_w|_v)^{-1}\right]}{(Y^{n_v}-X^{t_v^0/b_v-\ell_v\epsilon_v\nu_v}f|_v)},$$

where the product runs through all proper children of $(\mathfrak{s},v)$. The multiplicity of Γ_v in $\mathcal{C}_s$ is given by e_vd_0, where d_0 is the denominator of the slope $s_1^{L_v}$. We noticed in §4.5.5 that $n_v=2/d_0$. If $n_v=1$, then $\Gamma_v\simeq\mathbb{P}^1_{k_v}$. Suppose $n_v=2$. We want to show that the ring of rational functions of Γ_v is

$$k_v(X)[Y]/(Y^{n_v}-\tilde{f}_v(X)). \tag{4.22}$$

If $(\mathfrak{s},v)$ is degree-minimal, then $t_v^0=2-p_v^0$ from (4.9). If $(\mathfrak{s},v)$ is not degree-minimal, then there exists a child $(\mathfrak{t},w)<(\mathfrak{s},v)$ with $\phi_w=\phi_v$; in particular, $t_v^0=t_w$, $f_w=f_v$ and so

$$t_v^0/b_v-\ell_v\epsilon_v\nu_v=\tfrac{f_v|\mathfrak{t}|}{f_wb_v\deg v}-\ell_v\nu_v\epsilon_w.$$

Now let $(\mathfrak{t},w)<(\mathfrak{s},v)$ with $\phi_w\neq\phi_v$. Theorem 4.4.29 implies that

$$\operatorname{ord}_{\phi_w|_v}(f|_v)=|\mathfrak{t}|/\deg w.$$

Note that $\epsilon_w=e_v$ and $f_v\deg w=f_wb_v\deg v$ by Lemma 4.3.21. Then

$$\tfrac{|\mathfrak{t}|}{\deg w}\notin 2\mathbb{Z}\quad\text{ if and only if }\quad\tfrac{f_v|\mathfrak{t}|}{f_wb_v\deg v}-\ell_v\nu_v\epsilon_w\notin 2\mathbb{Z}.$$

Let $[v_0,\dots,v_h(\phi_h)=\lambda_h]$ be the cluster chain for v. Let $f_\mathfrak{s}=\prod_{r\in\mathfrak{R}\setminus\mathfrak{s}}(x-r)$. The Newton polygon $N_{v_{h-1},\phi_v}(f_\mathfrak{s})$ has only slopes $>-\lambda_v$. Then $f_\mathfrak{s}|_v=u|_v$, where $u=f_\mathfrak{s}\mod\phi_v$.

The observations above, together with Proposition 4.3.18(iv), imply that (4.22) is the ring of rational functions of Γ_v.

The subscheme given by $(\mathfrak{s},v)\in\Sigma$ in (2) is the closure of

$$\textstyle\bigcup_{i=1}^{r_E}\left(U_{E,i}\cap\{Z=0\}\right), \tag{4.23}$$

when $E=L_v$. The subscheme given by $(\mathfrak{s},v)\in\Sigma$ in (3) or (5) is the closure of (4.23) when $E=V_v$. Note that $(V_v)_\mathbb{Z}(\mathbb{Z})+1=p_v/\gamma_v$. The subscheme given by a degree-minimal $(\mathfrak{s},v)\in\Sigma$ in (4) is the closure of (4.23) when $E=V_v^0$. Note that $(V_v^0)_\mathbb{Z}(\mathbb{Z})+1=p_v^0/\gamma_v^0$. □

Remark 4.6.4. Let $(\mathfrak{s},v)\in\Sigma$. Note that

(i) if Γ_v is reducible then $p_v/\gamma_v=2$.

(ii) if $(\mathfrak{s},v)<(\mathfrak{t},w)$ and Γ_w is reducible, then $p_v/\gamma_v=2$.

(iii) if $(\mathfrak{s},v)$ is degree-minimal and Γ_v is reducible then $p_v^0/\gamma_v^0=2$.

APPENDIX A

RATIONAL CLUSTER PICTURE AND BASE EXTENSIONS

In this appendix we introduce two auxiliary results for Chapter 2. In §A.1 we study the choice of a rational centre of a proper cluster. In §A.2 we show how the dualising sheaf behaves under finite Galois extension of the base field. Note that this second result holds for every geometrically connected, smooth, projective curve.

A.1 Rational centres over tame extensions

Let C/K be a hyperelliptic curve given by $y^2 = f(x)$.

Lemma A.1.1 *Let L/K be a field extension. Consider the base extended curve C_L/L and its associated cluster picture Σ_{C_L}. Let $\mathfrak{s} \in \Sigma_{C_L}$ be a proper cluster $G_\mathfrak{s} = \mathrm{Stab}_{G_K}(\mathfrak{s})$, and $K_\mathfrak{s} = (K^s)^{G_\mathfrak{s}}$. If $L/L \cap K_\mathfrak{s}$ is tamely ramified, then $\mathfrak{s}$ has a rational centre $w_\mathfrak{s} \in L \cap K_\mathfrak{s}$.*

Proof. This proof takes ideas from [D²M², Lemma B.1]. Let $w_\mathfrak{s} \in L$ be a rational centre of $\mathfrak{s}$ and let $\rho_\mathfrak{s} = \max_{w \in L} \min_{r \in \mathfrak{s}} v(r - w)$ be its radius. Recall the rationalisation $\mathfrak{s}^{\mathrm{rat}} \in \Sigma_{C_L}^{\mathrm{rat}}$ of $\mathfrak{s}$ (Definition 2.3.11). Denote $\mathfrak{t} = \mathfrak{s}^{\mathrm{rat}}$ and define $G_\mathfrak{t} = \mathrm{Stab}_{G_K}(\mathfrak{t})$. Since $\mathfrak{s} \subseteq \mathfrak{t}$ we have $G_\mathfrak{s} \subseteq G_\mathfrak{t}$. Furthermore, $\mathrm{Gal}(K^s/L) \subseteq G_\mathfrak{t}$. Let $F_\mathfrak{s} = L \cap K_\mathfrak{s}$. Then $\mathrm{Gal}(K^s/F_\mathfrak{s}) \subseteq G_\mathfrak{t}$. Since $L/F_\mathfrak{s}$ is tamely ramified, we can consider a maximal tamely ramified extension $F_\mathfrak{s}^{\mathrm{t}}$ of $F_\mathfrak{s}$ extending L. Write $F_\mathfrak{s}^{nr}$ for the maximal unramified extension of $F_\mathfrak{s}$ in $F_\mathfrak{s}^t$. Fix a uniformiser $\pi_\mathfrak{s}$ of $F_\mathfrak{s}$. Since $L/F_\mathfrak{s}$ is tamely ramified and $w_\mathfrak{s} \in L$, for a sufficiently large b fix a choice of $\sqrt[b]{\pi_\mathfrak{s}}$ such that $w_\mathfrak{s} \in F_\mathfrak{s}^{nr}(\sqrt[b]{\pi_\mathfrak{s}})$. Write the v-adic expansion of $w_\mathfrak{s}$ as

$$w_\mathfrak{s} = u_t \sqrt[b]{\pi_\mathfrak{s}}^t + u_{t+1} \sqrt[b]{\pi_\mathfrak{s}}^{t+1} + \ldots$$

for a suitable $t \in \mathbb{Z}$, with $u_l \in F_\mathfrak{s}^{nr}$. Define

$$w = \sum_{l < e_{F_\mathfrak{s}/K} b \rho_\mathfrak{s}} u_l \sqrt[b]{\pi_\mathfrak{s}}^l.$$

We first show that $w \in F_{\mathfrak{s}}^{\mathrm{t}}$. It trivially follows if $w = 0$. Suppose $0 \neq w \notin F_{\mathfrak{s}}^{\mathrm{t}}$, and that $u_{l_0} \sqrt[b]{\pi_{\mathfrak{s}}}^{l_0}$ is the lowest valuation term of the expansion which is not in $F_{\mathfrak{s}}^{\mathrm{t}}$. Let $w' = \sum_{l<l_0} u_l \sqrt[b]{\pi_{\mathfrak{s}}}^{l}$. Note that $w' \in F_{\mathfrak{s}}^{\mathrm{t}}$ for our assumption on l_0. As $v(w - w_{\mathfrak{s}}) \geq \rho_{\mathfrak{s}}$, we have $v(w_{\mathfrak{s}} - w') = v(w - w') = l_0/e_{F_{\mathfrak{s}}/K} b$. Since $L \subseteq F_{\mathfrak{s}}^{\mathrm{t}}$, we have $w_{\mathfrak{s}} - w' \in F_{\mathfrak{s}}^{\mathrm{t}}$ and so the denominator of l_0/b is not divisible by p. But then $u_{l_0} \sqrt[b]{\pi_{\mathfrak{s}}}^{l_0} \in F_{\mathfrak{s}}^{\mathrm{t}}$ as $u_{l_0} \in F_{\mathfrak{s}}^{nr} \subseteq F_{\mathfrak{s}}^{\mathrm{t}}$ and $\sqrt[b]{\pi_{\mathfrak{s}}}^{l_0} \in F_{\mathfrak{s}}^{\mathrm{t}}$.

Let $\mathcal{D}_{\mathfrak{t}} = \{x \in K^{\mathrm{s}} \mid v(x - w_{\mathfrak{s}}) \geq \rho_{\mathfrak{s}}\}$ be the smallest disc in K^{s} cutting out $\mathfrak{t}$. Note that $\mathrm{Stab}_{G_K}(\mathcal{D}_{\mathfrak{t}}) = G_{\mathfrak{t}}$. Since $w \in \mathcal{D}_{\mathfrak{t}}$, for $\sigma \in \mathrm{Gal}(K^{\mathrm{s}}/F_{\mathfrak{s}}) \subseteq G_{\mathfrak{t}}$ we have $\sigma(w) \in \mathcal{D}_{\mathfrak{t}}$ and so $v(\sigma(w) - w_{\mathfrak{s}}) \geq \rho_{\mathfrak{s}}$. Therefore the terms in the v-adic expansions of $\sigma(w)$ and w agree up to $\sqrt[b]{\pi_{\mathfrak{s}}}^{e_{F_{\mathfrak{s}}/K} b \rho_{\mathfrak{s}}}$ (excluded). Furthermore, if $w \in L$, then w is a rational centre of $\mathfrak{s}$. Indeed, for any $r \in \mathfrak{s}$ one has

$$v(r - w) \geq \min\{v(r - w_{\mathfrak{s}}), v(w - w_{\mathfrak{s}})\} \geq \rho_{\mathfrak{s}}.$$

We showed $w \in F_{\mathfrak{s}}^{\mathrm{t}}$. It remains to prove that $w \in F_{\mathfrak{s}}$, i.e. it is $\mathrm{Gal}(K^{\mathrm{s}}/F_{\mathfrak{s}})$-invariant. Suppose not, and that $u_l \sqrt[b]{\pi_{\mathfrak{s}}}^{l}$ is the lowest valuation term of the expansion which is not $\mathrm{Gal}(K^{\mathrm{s}}/F_{\mathfrak{s}})$-invariant. Note that the denominator of l/b is not divisible by p since $w \in F_{\mathfrak{s}}^{\mathrm{t}}$. If $b \nmid l$, then there is some element σ of tame inertia of $F_{\mathfrak{s}}$ which fixes $u_l \in F_{\mathfrak{s}}^{nr}$ and maps $\sqrt[b]{\pi_{\mathfrak{s}}}^{l}$ to $\zeta \sqrt[b]{\pi_{\mathfrak{s}}}^{l}$, where $\zeta \neq 1$ is a root of unity; this contradicts the fact that $\sigma(w) \equiv w \mod \sqrt[b]{\pi_{\mathfrak{s}}}^{e_{F_{\mathfrak{s}}/K} b \rho_{\mathfrak{s}}}$. If $b \mid l$, then we must have $u_l \notin F_{\mathfrak{s}}$. Then there exists some element $\sigma \in \mathrm{Gal}(F_{\mathfrak{s}}^{nr}/F_{\mathfrak{s}})$ so that $\sigma(u_l) \neq u_l$; this contradicts $\sigma(w) \equiv w \mod \sqrt[b]{\pi_{\mathfrak{s}}}^{e_{F_{\mathfrak{s}}/K} b \rho_{\mathfrak{s}}}$ similarly to before. □

A.2 Dualising sheaf under base extensions

Let F/K be a finite Galois extension and let O_F be the ring of integers of F.

Lemma A.2.1 *Let M be a flat O_K-module and $M_F := M \otimes_{O_K} O_F$. Then*

$$M \simeq M_F^{\mathrm{Gal}(F/K)} = \{m \in M_F \mid \sigma(m) = m \textit{ for every } \sigma \in \mathrm{Gal}(F/K)\}.$$

Proof. As M is flat, the functor $M \otimes_{O_K} -$ is (left) exact. From the isomorphism $O_K \simeq O_F^{\mathrm{Gal}(F/K)}$ it follows that

$$M \otimes_{O_K} O_K \simeq M \otimes_{O_K} O_F^{\mathrm{Gal}(F/K)},$$

that is $M \simeq M_F^{\mathrm{Gal}(F/K)}$, as required. □

Proposition A.2.2 *Let C be a geometrically connected, smooth, projective curve of genus $g \geq 1$ and let $\mathcal{C}$ be a regular model of C over O_K. Denote by C_F and $\mathcal{C}_{O_F}$ the base extended schemes. Then $H^0(\mathcal{C}_F, \omega_{\mathcal{C}_F/O_F}) \simeq H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K}) \otimes_{O_K} O_F$ and*

$$H^0(\mathcal{C}, \omega_{\mathcal{C}/O_K}) \simeq H^0(\mathcal{C}_F, \omega_{\mathcal{C}_F/O_F})^{\mathrm{Gal}(F/K)}.$$

Proof. The lemma follows from the following results: [Liu4, Proposition 10.1.17], [Liu4, Theorem 6.4.9(b)], [Liu4, Exercise 6.4.6], [Liu4, Corollary 5.2.27] and the previous lemma. □

APPENDIX B

Smooth completion and Baker's model

The content of this appendix is particularly related to Chapter 3. In §B.1, as a corollary of a more general result on varieties, we show that every smooth projective curve has a dense open subscheme which is isomorphic to a smooth plane curve. In §B.2 we show that not every smooth projective curve C admits a Baker's model.

B.1 Birational smooth hypersurface of a variety

Let k be a perfect field. Recall that an algebraic variety Z over k, denoted Z/k, is a scheme $Z \to \operatorname{Spec} k$ of finite type.

Lemma B.1.1 *Let Z/k be a geometrically reduced algebraic variety, pure of dimension n. Suppose either $n > 0$ or k infinite. Then there exists a separable polynomial $f \in k(x_1, \dots, x_n)[y]$, such that $k(Z) = k(x_1, \dots, x_n)[y]/(f)$.*

Proof. Let $Z_1, \dots, Z_m$ be the irreducible components of Z. From [Liu4, Proposition 7.1.15], [Liu4, Lemma 7.5.2(a)] it follows that $k(Z) \simeq \bigoplus_{i=1}^m k(Z_i)$. Let $i = 1, \dots, m$. As Z is pure, $\dim Z_i = \dim Z = n$. Since Z_i is geometrically reduced and integral, it follows from [Liu4, Proposition 3.2.15] that the field of functions $k(Z_i)$ is a finite separable extension of a purely trascendental extension $k(x_1, \dots, x_n)$. Hence there exists a monic irreducible separable polynomial $f_i \in k(x_1, \dots, x_n)[y]$ such that

$$k(Z_i) \simeq k(x_1, \dots, x_n)[y]/(f_i).$$

We want to show that we can inductively choose the polynomials f_i above such that f_i and f_j are coprime for all $j < i$. Suppose we have fixed $f_1, \dots, f_{i-1}$ for some $i \geq 1$, and let $g_i \in k(x_1, \dots, x_n)[y]$ be any monic irreducible polynomial such that $k(Z_i) \simeq k(x_1, \dots, x_n)[y]/(g_i)$. Since

$k(x_1,\ldots,x_n)$ is infinite, there exists $c \in k(x_1,\ldots,x_n)$ such that $\tau_c g_i \neq f_j$ for any $j < i$, where $\tau_c g_i$ is the polynomial defined by $\tau_c g_i(y) = g_i(y-c)$. But $\tau_c g_i$ and f_j are irreducible monic polynomials, so $\gcd(\tau_c g_i, f_j) = 1$. Moreover, $\tau_c g_i$ is separable and

$$k(x_1,\ldots,x_n)[y]/(g_i) \simeq k(x_1,\ldots,x_n)[y]/(\tau_c g_i)$$

via the map taking $y \mapsto y-c$. Then choose $f_i = \tau_c g_i$.

Thus assume $\gcd(f_i, f_j) = 1$ for any $i,j = 1,\ldots,m$. From the Chinese Remainder Theorem it follows that

$$k(Z) \simeq \bigoplus_{i=1}^m k(Z_i) \simeq \bigoplus_{i=1}^m \frac{k(x_1,\ldots,x_n)[y]}{(f_i)} \simeq \frac{k(x_1,\ldots,x_n)[y]}{(f)},$$

where $f = \prod_{i=1}^m f_i$. □

The following result is a variant of [BMS, Theorem 5.7].

Theorem B.1.2 *Let Z/k be a geometrically reduced, separated algebraic variety, pure of dimension n. Suppose either $n > 0$ or k infinite. Then there exists a smooth affine hypersurface V in $\mathbb{A}_k^{n+1}$ birational to Z.*

Proof. Lemma B.1.1 shows that there exists a separable polynomial $f \in k(x_1,\ldots,x_n)[y]$ such that $k(Z) \simeq k(x_1,\ldots,x_n)[y]/(f)$. Rescaling f by an element of $k(x_1,\ldots,x_n)$ if necessary, we can assume that f is a polynomial in $k[x_1,\ldots,x_n,y]$ with no irreducible factors in $k[x_1,\ldots,x_n]$. Hence the total quotient ring of $k[x_1,\ldots,x_n,y]/(f)$ is $k(x_1,\ldots,x_n)[y]/(f)$. It follows that there exists a birational map $Z \dashrightarrow Z_0$, where Z_0 is the affine hypersurface defined by $f(x_1,\ldots,x_n,y) = 0$. Let $A = k[x_1,\ldots,x_n,y]/(f)$ be the coordinate ring of Z_0. If Z_0 is smooth then we are done. Suppose Z_0 is not smooth. Then there exists $h \in J \cap k[x_1,\ldots,x_n]$, where $J \subset k[x_1,\ldots,x_n,y]$ is the ideal defining the singular locus of Z_0.

The rest of the proof follows the spirit of [BMS, Theorem 5.7]. Expand $f = \sum_{i=0}^d c_i y^i$, where $c_i \in k[x_1,\ldots,x_n]$, and $c_0 \neq 0$. Via the change of variable $(hc_0^2)y' = y$ we get $f = \sum_{i=0}^d c_i (hc_0^2)^i (y')^i$. Dividing by c_0, we define $f' = 1 + \sum_{i=1}^d c_i c_0^{i-1}(hc_0 y')^i$ and $Z_0' = \operatorname{Spec} k[x_1,\ldots,x_n,y']/(f')$. Then via the homomorphism $y \mapsto (hc_0^2)y'$ we see that Z_0' is isomorphic to the smooth dense open subvariety $D(hc_0)$ of Z_0. Thus Z_0' is a smooth affine hypersurface in $\mathbb{A}_k^{n+1}$ birational to Z. □

Lemma B.1.3 *If a smooth affine curve C_0/k is birational to a smooth projective curve C/k, then C is isomorphic to the smooth completion of C_0. Equivalently, there exists an open immersion with dense image $C_0 \hookrightarrow C$.*

Proof. Since C is complete and C_0 is smooth and separated (as affine), the birational map $C_0 \dashrightarrow C$ uniquely extends to a separated birational morphism $\iota : C_0 \to C$. Denoting by $\tilde{C}$ the smooth completion of C_0 note that ι decomposes into the canonical open immersion $C_0 \hookrightarrow \tilde{C}$ and the morphism $\tilde{\iota} : \tilde{C} \to C$ extending the rational map given by ι. Therefore it suffices to prove that $\tilde{\iota}$ is an isomorphism.

First note that $\tilde{\imath}$ is proper by [Liu4, Proposition 3.3.16(e)] since $\tilde{C}$ and C are complete. Furthermore, both $\tilde{C}$ and C are smooth, so they are geometrically reduced and have irreducible connected components. For any connected component $\tilde{U}$ of $\tilde{C}$ there is a connected component U of C such that $\tilde{\imath}$ restricts to a morphism $\iota_U : \tilde{U} \to U$. Note that ι_U is a proper birational morphism, as $\tilde{U}$ is a closed subscheme of $\tilde{C}$ and $\tilde{\imath}$ is proper birational. Since both $\tilde{U}$ and U are integral and smooth of dimension 1, and so normal, [Liu4, Corollary 4.4.3(b)] implies that $\iota_U : \tilde{U} \to U$ is an isomorphism. It follows that $\tilde{\imath} : \tilde{C} \to C$ is an isomorphism. □

Corollary B.1.4 *Every smooth projective curve C/k has a dense affine open which is isomorphic to a smooth plane curve.*

Proof. From Theorem B.1.2 there exists a smooth affine plane curve C_0 birational to C. Then Lemma B.1.3 concludes the proof. □

B.2 Existence of a Baker's model

Let k be a perfect field. We say that a curve C/k is *nice* if it is geometrically connected, smooth and projective over k. In this appendix we slightly extend some results in [CV1, CV2] for studying the existence of a Baker's model of a nice curve. Define the *index* of a nice curve C/k to be the smallest extension degree of a field K/k such that $C(K) \neq \varnothing$.

Lemma B.2.1 *Let C be a nice curve of genus* 1*. Then C admits a Baker's model if and only if C has index at most* 3.

Proof. Suppose C has index at most 3. Then by [CV1, Lemma 4.1] the curve C is nondegenerate. Hence C has an outer regular Baker's model.

Suppose now that C admits a Baker's model. Then there exists a smooth curve $C_0 \hookrightarrow C$ defined in $\mathbb{G}^2_{m,k}$ by $f \in k[x^{\pm 1}, y^{\pm 1}]$ such that the completion C_1 of C_0 with respect to the Newton polygon Δ of f is regular. We follow the spirit of the proof of [CV1, Lemma 4.1]. Since the arithmetic genus of C is 1 there is exactly 1 interior integer point of Δ. There are 16 equivalence classes of integral polytopes with this condition (see [CV1, Appendix]). Then without loss of generality we can assume Δ is in this list. Note that there is an edge $\ell \subseteq \partial\Delta$ such that $\#(\ell \cap \mathbb{Z}^2) \leq 4$. Let v be the normal vector of ℓ and $\alpha = (v, ()) \in \Sigma_1$. Then $f|_\alpha$ has at most 3 roots in $\bar{k}^\times$ by Proposition 3.4.1. Therefore the splitting field K of $f|_\alpha$ has degree ≤ 3 over k. Furthermore, by definition C_1 has at least one point defined over K visible on C_α. Thus C_1, and so C, has index at most 3. □

Remark B.2.2. The lemma above implies that there are nice curves which does not have a Baker's model. Indeed, if k is a number field, [Cla] proves there exist nice curves of genus 1 of any index.

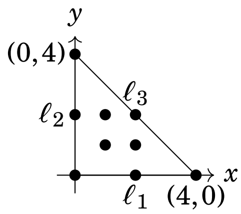
Theorem B.2.3 *Let C be a nice curve of genus $g \leq 3$. If k is finite or $C(k) \neq \varnothing$ then C admits a Baker's model.*

Proof. The first theorem in [CV1] and [CV2, Proposition 3.2] show C is nondegenerate except when C is birational to a curve C_0 given in $\mathbb{G}^2_{m,k}$ by

$$f^{(2)} = (x+y)^4 + (xy)^2 + xy(x+y+1) + (x+y+1)^2, \qquad \text{with } k = \mathbb{F}_2, \text{ or}$$
$$f^{(3)} = (x^2+1)^2 + y - y^3, \qquad \text{with } k = \mathbb{F}_3.$$

Recall that if C is nondegenerate then it has an outer regular Baker's model. Therefore it suffices to show that in the two exceptional cases above the completion C_1 of the curve C_0 with respect to its Newton polygon is smooth. We use the notation of §3.1.3.

Suppose $k = \mathbb{F}_2$ and $C_0 : f^{(2)} = 0$ over $\mathbb{G}^2_{m,\mathbb{F}_2}$. Note that C_0 is smooth. Denote $f = f^{(2)}$. The Newton polygon Δ of f is

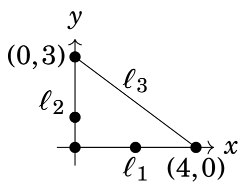


where the normal vectors of the edges ℓ_1, ℓ_2, ℓ_3 of Δ are respectively $\beta_1 = (0,1)$, $\beta_2 = (1,0)$, $\beta_3 = (-1,-1)$. Then by fixing $\delta_{\beta_1} = (1,0)$, $\delta_{\beta_2} = (-1,-1)$, $\delta_{\beta_3} = (0,1)$ we have

$$f_{\ell_i}(X,Y) = (X^2+X+1)^2 + X(X+1)Y + (X^2+X+1)Y^2 + Y^4,$$

for every $i = 1,2,3$. Note that the points on $Y = 0$ are regular points of C_{ℓ_i}. Thus C_ℓ is smooth for any edge ℓ of Δ and so C_1 is smooth.

Suppose $k = \mathbb{F}_3$ and $C_0 : f^{(3)} = 0$ over $\mathbb{G}^2_{m,\mathbb{F}_3}$. Note that C_0 is smooth. Denote $f = f^{(3)}$. The Newton polygon Δ of f is



where the normal vectors of the edges ℓ_1, ℓ_2, ℓ_3 of Δ are respectively $\beta_1 = (0,1)$, $\beta_2 = (1,0)$, $\beta_3 = (-3,-4)$. We can choose $\delta_{\beta_1} = (1,0)$ so that

$$f_{\ell_1}(X,Y) = (X^2+1)^2 + Y - Y^3.$$

The points on $Y = 0$ are regular points of C_{ℓ_1} and so C_{ℓ_1} is smooth. Furthermore, up to a power of X the polynomials $f|_{\ell_2}$ and $f|_{\ell_3}$ equal $X^3 + X^2 - 1$ and $-X + 1$ respectively. It follows that the charts C_{ℓ_2} and C_{ℓ_3} of C_1 are regular. Thus C_1 is smooth. $\square$

APPENDIX C

PSEUDO-VALUATIONS AND AN EXPLICIT TOROIDAL EMBEDDING

In this appendix we cover some definitions and results for Chapter 4. In §C.1 we give the definition of pseudo-valuation and of the associated objects. In §C.2 and §C.3, we explicitly describe the toroidal embedding introduced in §4.5.2.

C.1 Pseudo-valuations

Let A be an integral domain (with identity). Let $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. The ordering and the group law on $\mathbb{Q}$ are canonically extended to the set $\hat{\mathbb{Q}}$.

Definition C.1.1 A map $v : A \to \hat{\mathbb{Q}}$ is called *pseudo-valuation* (*of* A) if

(a) $v(ab) = v(a) + v(b)$,

(b) $v(a+b) \geq \min\{v(a), v(b)\}$,

for any $a, b \in A$. A pseudo-valuation v is said *valuation* if it also satisfies

(c) $v(a) = \infty$ if and only if $a = 0$;

we call it *infinite pseudo-valuation* otherwise.

Definition C.1.2 Let $v : A \to \hat{\mathbb{Q}}$ be a pseudo-valuation.

- The *valuation group* of a pseudo-valuation $v : A \to \hat{\mathbb{Q}}$, denoted Γ_v, is the subgroup generated by the subset $v(A) \cap \mathbb{Q}$ of $\mathbb{Q}$. Note that if $\mathbb{Z} \subseteq v(A)$, then $\Gamma_v = v(A) \cap \mathbb{Q}$.
- v is *discrete* if there exists $e \in \mathbb{Z}_+$ such that $e\Gamma_v = \mathbb{Z}$. If that happens, then $e_v = e$ is said ramification index of v.

- The *valuation ring* O_v of a pseudo-valuation $v : A \to \hat{\mathbb{Q}}$ is the set of $a \in A$ with $v(a) \geq 0$.

- The *residue ring* of v is the quotient of O_v by the prime ideal O_v^+ consisting of the elements $a \in A$ with $v(a) > 0$.

- If v is a valuation, the *residue field* of v is the residue ring of the valuation of $\mathrm{Frac}(A)$ that v induces.

C.2 Explicit matrices

In this section we explicitly describe the matrices introduced in §4.5.1. Recall the notation of §4.5.1. Suppose the permutation τ equals the identity. Let m'_j, for $j = 0, \dots, h$, be the quantities defined in Lemma 4.3.7. Then

$$M_{E,i}^{\mathbb{R}} = \begin{pmatrix}
e_1 & -h_2 m'_1 & \dots & -h_{h-1} m'_1 & 0 & -\beta_{h+1} m'_1 & \dots & \dots & -\beta_n m'_1 & 0 & 0 \\
0 & e_2 & \ddots & -h_{h-1} m'_2 & 0 & -\beta_{h+1} m'_2 & \dots & \dots & -\beta_n m'_2 & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & e_{h-1} & 0 & -\beta_{h+1} m'_{h-1} & \dots & \dots & -\beta_n m'_{h-1} & 0 & 0 \\
\vdots & \vdots & \ddots & 0 & \delta a_x & -\beta'_{h+1} & \dots & \dots & -\beta'_n & d_i b_x & -d_{i+1} b_x \\
\vdots & \vdots & \ddots & \vdots & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & 0 & 1 & \ddots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \ddots & 1 & 0 & 0 \\
0 & 0 & \dots & 0 & \delta a_y & 0 & 0 & \ddots & 0 & d_i b_y & -d_{i+1} b_y \\
-h_1 m'_0 & -h_2 m'_0 & \dots & -h_{h-1} m'_0 & \delta a_z & -\beta_{h+1} m'_0 & \dots & \dots & -\beta_n m'_0 & \frac{n_i}{\delta \epsilon_v} & -\frac{n_{i+1}}{\delta \epsilon_v}
\end{pmatrix}$$

where for any $o = h+1, \dots, n$ we have

$$\beta_o = \begin{cases} 0 & \text{if } \mu_o > v_E, \\ \epsilon_v v(\psi_o) & \text{otherwise,} \end{cases} \qquad \beta'_o = \begin{cases} v(\psi_o)/\lambda_v & \text{if } \mu_o > v_E, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\det M_{E,i}^{\mathbb{R}} = \begin{vmatrix}
e_1 & & & & & \\
& e_2 & & & * & \\
& & \ddots & & & \\
& & & e_{h-1} & & \\
& & & & 1 & \\
& \mathbf{0} & & & & \ddots \\
& & & & & & 1
\end{vmatrix} \cdot \begin{vmatrix} \delta v_x & d_i w_x & -d_{i+1} w_x \\ \delta v_y & d_i w_y & -d_{i+1} w_y \\ \delta v_z & \frac{n_i}{\delta \epsilon_v} & -\frac{n_{i+1}}{\delta \epsilon_v} \end{vmatrix} = 1$$

Furthermore, T_h and T equal respectively

$$\begin{pmatrix} 1 & & \delta a_z c_1 & & \frac{n_i c_1}{\delta \epsilon_v} & -\frac{n_{i+1} c_1}{\delta \epsilon_v} \\ & \ddots & \vdots & & \vdots & \vdots \\ & 1 & \delta a_z c_{h-1} & & \frac{n_i c_{h-1}}{\delta \epsilon_v} & -\frac{n_{i+1} c_{h-1}}{\delta \epsilon_v} \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 1 & & \frac{k_i}{\delta} & -\frac{k_{i+1}}{\delta} \\ & & & & \ddots & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix},$$

and T_h^{-1} and T^{-1} are respectively

$$\begin{pmatrix} 1 & & -\delta a_z c_1 & & -\frac{n_i c_1}{\delta \epsilon_v} & \frac{n_{i+1} c_1}{\delta \epsilon_v} \\ & \ddots & \vdots & & \vdots & \vdots \\ & 1 & -\delta a_z c_{h-1} & & -\frac{n_i c_{h-1}}{\delta \epsilon_v} & \frac{n_{i+1} c_{h-1}}{\delta \epsilon_v} \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 1 & & -\frac{k_i}{\delta} & \frac{k_{i+1}}{\delta} \\ & & & & \ddots & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix},$$

where all missing entries are 0s.

Finally, the vectors $\tilde{v}_o$, for $1 \le o \le n$, first n rows of the matrix $M_{E,i}^{\mathbb{R}}$, are

$$\tilde{v}_o = \begin{cases} \left(0,\dots,0,\frac{1}{e_o},\frac{h_{o+1}m_o}{e_{v_{o+1}}},\dots,\frac{h_{h-1}m_o}{e_{v_{h-1}}},0,\frac{\beta_{h+1}m_o}{\epsilon_v}\dots,\frac{\beta_n m_o}{\epsilon_v},0,0\right) & \text{if } 1 \le o < h, \\ \frac{1}{\delta}(0,\dots,0,b_y,\beta'_{h+1}b_y,\dots,\beta'_n b_y,b_x,0) & \text{if } o = h, \\ (0,\dots,0,1,0,\dots,0) = \varepsilon_o & \text{if } h < o \le n, \end{cases}$$

C.3 MacLane clusters fan

We want to show that the cones constructed in §4.5.2 form a fan. Let $h = 1,\dots,n$. Recall the degree-minimal MacLane cluster $(\mathfrak{s}_h,\mu_h)$. Let

$$[v_0, v_1(\phi_1) = \lambda_1, \dots, v_{m-1}(\phi_{m-1}) = \lambda_{m-1}, v_m(\phi_m) = \lambda_m]$$

be the cluster chain for μ_h. Let $c \in \mathbb{R}$, with $c > \lambda_{m-1}$ if $m > 1$. Define the valuation $v_{h,c} : K(x) \to \hat{\mathbb{R}}$ given on $K[x]^*$ by

$$v_{h,c}\left(\textstyle\sum_j c_j \psi_h^j\right) = \min_j \left(v_{m-1}(c_j) + jc\right), \quad c_j \in K[x], \deg(c_j) < \deg(\psi_h).$$

Note that when $m = 1$, then $\deg \psi_h = 1$ and so $v_0(c_j) = v_K(c_j)$.

Let $(\mathfrak{s}, v)$ be a proper MacLane cluster with centre $\phi_v = \psi_h$, and let E be the h-edge L_v, V_v, or V_v^0 if $(\mathfrak{s}, v)$ is degree-minimal. Recall the notation of §4.5.1. Let $r = r_E$ and $\delta = \delta_E$.

Lemma C.3.1 *For any $i = 0,\dots,r+1$, there exist $\alpha, \beta \in \mathbb{Q}_{\ge 0}$, such that*

$$\tilde{\omega}_i = \alpha \tilde{\omega}_0 + \beta \tilde{\omega}_{r+1}.$$

Proof. If $i=0$ or $i=r+1$, the statement is trivial. Then assume $1\le i\le r$. Since $n_0d_i>n_id_0$ and $n_id_{r+1}>n_{r+1}d_i$, there exist $\alpha_i,\beta_i\in\mathbb{Q}_+$ such that

$$\alpha_in_id_0+\beta_in_id_{r+1}=\alpha_in_0d_i+\beta_in_{r+1}d_i.$$

Define $e=\frac{d_i}{d_0\alpha_i+d_{r+1}\beta_i}$, $\alpha=e\alpha_i$, $\beta=e\beta_i$. The lemma follows from (4.15). □

Lemma C.3.2 *Let $c\in\mathbb{R}$ and $v_{h,c}:K[x]\to\hat{\mathbb{R}}$ as above. If*

(i) $(\mathfrak{s},v)<(\mathfrak{t},w)$, $E=V_v$, and $w(\psi_h)<c<\lambda_v$, or

(ii) $(\mathfrak{s},v)$ maximal, $E=V_v$, and $c<\lambda_v$, or

(iii) $(\mathfrak{s},v)$ degree-minimal, $E=V_v^0$, and $c>\lambda_v$,

then $v_{h,c}(\gamma_{j,E})=0$ for any $j=1,\dots,n$, $j\neq h$.

Proof. Let $j=1,\dots,n$, $j\neq h$. Expand

$$\psi_j=\sum_{t=1}^{d}c_t\psi_h^t,\quad c_t\in K[x], c_d\neq 0, \deg c_t<\deg\psi_h.$$

If $j=\tau(o)$ for some $o<m$, then $v_{h,c}(\psi_j)=v_{m-1}(\psi_j)$. It follows from Lemma 4.3.3 that $v_{h,c}(\gamma_{j,E})=0$. Hence assume $j\neq\tau(o)$ for all $o<m$.

(i) Assume $(\mathfrak{s},v)<(\mathfrak{t},w)$, $E=V_v$, and $w(\psi_h)<c<\lambda_v$. Suppose $\mu_j\geq v$. Lemma 4.5.10 implies that $c_d=1$ and $v(\psi_j)=v(\psi_h^d)=d\lambda_v$. Since $c<\lambda_v$ we have $v_{h,c}(\psi_j)=dc$, by definition. Then $v_{h,c}(\gamma_{j,E})=0$. Suppose $\mu_j\ngeq v$. Therefore

$$v(\psi_j)=w(\psi_j)\le v_{h,c}(\psi_j)\le v(\psi_j).$$

where the first equality follows from Lemma 4.5.10. Hence $v_{h,c}(\gamma_{j,E})=0$.

(ii) Assume $(\mathfrak{s},v)$ maximal, $E=V_v$, and $c<\lambda_v$. Then $\mu_j\geq v$. Lemma 4.5.10 implies that $c_d=1$ and $v(\psi_j)=v(\psi_h^d)=d\lambda_v$. It follows that $v_{h,c}(\psi_j)=dc$ as $c<\lambda_v$. Therefore $v_{h,c}(\gamma_{j,E})=0$.

(iii) Assume $(\mathfrak{s},v)$ degree-minimal, $E=V_v^0$, and $c>\lambda_v$. Recall the definition of v_E. Then $\mu_j\ngeq v_E$ and $v_E(\psi_j)\geq v_{h,c}(\psi_j)\geq v(\psi_j)$. It follows from (4.13) that $v_{h,c}(\psi_j)=v(\psi_j)$. Thus $v_{h,c}(\gamma_{j,E})=0$. □

Lemma C.3.3 *For any $\tilde{\omega}\in\sigma_{E,i,i+1}\setminus\sigma_{E,i+1}$, there exists $c\in\mathbb{R}$, with $c>\lambda_{m-1}$ if $m>1$, so that*

$$\tilde{\omega}=e(v_{h,c}(\psi_1),\dots,v_{h,c}(\psi_n),C,1),$$

for some $e\in\mathbb{R}_+$, $C\in\mathbb{R}$. In particular,

(i) if $(\mathfrak{s},v)<(\mathfrak{t},w)$ and $E=V_v$, then $w(\psi_h)<c<\lambda_v$;

(ii) if $(\mathfrak{s},v)$ maximal and $E=V_v$, then $c<\lambda_v$;

(iii) if $(\mathfrak{s},v)$ degree-minimal and $E=V_v^0$, then $c>\lambda_v$;

(iv) *if $E = L_v$, then $c = \lambda_v$.*

Proof. From Lemma 4.5.15, the statement is true for $\tilde{\omega} = \tilde{\omega}_0$. So suppose $\tilde{\omega} \neq \tilde{\omega}_0$. Lemma C.3.1 implies that $\tilde{\omega} = \alpha\tilde{\omega}_0 + \beta\tilde{\omega}_{r+1}$ for some $\alpha, \beta \in \mathbb{R}_+$. Let $e \in \mathbb{R}_+$, $c \in \mathbb{R}$ as follows

$$e = \alpha\delta\epsilon_h d_0 + \beta\delta\epsilon_h d_{r+1}, \qquad c = \frac{\alpha n_0 a_y + \beta n_{r+1} a_y}{e} - a_z b_y.$$

From the definition of $\tilde{\omega}_0$ and $\tilde{\omega}_{r+1}$ in (4.15) we have

$$\tilde{\omega} = e(c\tilde{\alpha}_{h1} + \tilde{\alpha}_{\pi 1}, \dots, c\tilde{\alpha}_{hn} + \tilde{\alpha}_{\pi n}, C, 1),$$

for some $C \in \mathbb{R}$. Furthermore, c satisfies the inequalities of cases (i)-(iv) by Lemma 4.5.15. In particular, $c > \lambda_{m-1}$ if $m > 1$. From (4.14), Lemma C.3.2 concludes the proof. □

Remark C.3.4. Note that the element $c \in \mathbb{R}$ in Lemma C.3.3 is uniquely determined by the vector $\tilde{\omega}$. Indeed, c equals the division of the h-th coordinate of $\tilde{\omega}$ by its last coordinate.

Let $i = 0, \dots, r+1$, with $i \leq r$ if E is outer. Let $c_i = \frac{n_i}{\delta e_{v_-} d_i} a_y - a_z b_y$. We define the valuation $w_{E,i} : K[x] \to \hat{\mathbb{Q}}$ by $w_{E,i}(g) = v_{h,c_i}(g)$ for any $g \in K[x]^*$. In other words, $w_{E,i}$ is given on $K[x]^*$ by

$$w_{E,i}\left(\textstyle\sum_j a_j \psi_h^j\right) = \min_j \left(v_-(a_j) + jc_i\right), \tag{C.1}$$

where $a_j \in K[x]$, $\deg(a_j) < \deg(\psi_h)$. In fact, $w_{E,i}$ is the MacLane valuation

$$w_{E,i} = \left[v_-, w_{E,i}(\psi_h) = \tfrac{n_i}{\delta e_{v_-} d_i} a_y - a_z b_y\right],$$

except possibly when $(\mathfrak{s}, v)$ is maximal, $E = V_v$ and $1 \leq i \leq r$. Lemma C.3.3 implies that

$$\tilde{\omega}_i = \delta e_{v_-} d_i (w_{E,i}(\psi_1), \dots, w_{E,i}(\psi_n), C, 1),$$

for some $C \in \mathbb{Q}$. We denote C by $w_{E,i}(y)$.

Theorem C.3.5 *The set of cones Σ defined in §4.5.2 is a fan.*

Proof. For any $*$-edge E let

$$\sigma_{E,0,r_E+1} = \bigcup_{i=0}^{r_E} \sigma_{E,i,i+1}.$$

By Lemma C.3.1 it suffices to show that the set

$$\Sigma' = \{\sigma_0\} \cup \bigcup_{E\ *\text{-edge}} (\sigma_{E,0} \cup \sigma_{E,r_E+1} \cup \sigma_{E,0,r_E+1})$$

is a fan. But this follows from Lemma C.3.3. □

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