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Automatization of Ternary Boolean Algebras

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Summary. The main aim of this article is to introduce formally ternary Boolean algebras (TBAs) in terms of an abstract ternary operation, and to show their connection with the ordinary notion of a Boolean algebra, already present in the Mizar Mathematical Library [2]. Essentially, the core of this Mizar [1] formalization is based on the paper of A.A. Grau "Ternary Boolean Algebras" [7]. The main result is the single axiom for this class of lattices [12]. This is the continuation of the articles devoted to various equivalent axiomatizations of Boolean algebras: following Huntington [8] in terms of the binary sum and the complementation useful in the formalization of the Robbins problem [5], in terms of Sheffer stroke [9]. The classical definition ([6], [3]) can be found in [15] and its formalization is described in [4].

Similarly as in the case of recent formalizations of WA-lattices [14] and quasilattices [10], some of the results were proven in the Mizar system with the help of Prover9 [13], [11] proof assistant, so proofs are quite lengthy. They can be subject for subsequent revisions to make them more compact.

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0. Introduction

Ternary Boolean algebras (TBA for short) were introduced in the paper by A.A. Grau [7] in 1947. There the corresponding algebraic structure is

$$\langle T, \mathrm{cmpl}, \mathrm{trn}\rangle,$$

where T is a set, $\mathrm{trn} : T^3 \to T$ is a ternary operation on T, and $\mathrm{cmpl} : T \to T$ plays a role of the complementation operator.

The set of axioms: distributivity, idempotence, and absorption is given by definitions (Def. 3) – (Def. 7) in Sect. 2. The definition of the type "Ternary Boolean algebra" concludes this section.

Section 3 is devoted to formal correspondence between the usual definition of a Boolean algebra and TBAs. It is enough to choose arbitrary element $0 \in T$ and set

$$a \sqcup b = \mathrm{trn}(a, 0, b);$$

$$a \sqcap b = \mathrm{trn}(a, \mathrm{cmpl}(0), b).$$

In order to have all the operations (binary, unary, and ternary) available in the common framework, we introduced `LattTBAStr`. The Mizar functor converting ordinary Boolean algebras into TBAs is given in Sect. 4 (actually, `BA2TBA` in (Def. 13) returns TBA structure and `BA2TBAA` (Def. 14) – merged TBA and lattice structure). The ternary operation and usual binary lattice operations satisfy the equation

$$trn(a, b, c) = (a \sqcap b) \sqcup (b \sqcap c) \sqcup (c \sqcap a).$$

We call it the rosetta operation, hence `RosTrn` is used in the Mizar source (see Sect. 5). In Sect. 6 it is proven that the structure obtained in this way satisfy classical lattice axioms and, furthermore `BA2TBAA` is indeed a Boolean algebra (Sect. 7). Section 8 presents the single axiom for TBAs (Def. 15) and concluding cluster registrations show that TBAs defined in Sect. 2 satisfy also this single axiom.

1. Preliminaries

We consider TBA structures which extend ComplStr and are systems

$$\langle \text{a carrier, a complement operation, a ternary operation} \rangle$$

where the carrier is a set, the complement operation is a unary operation on the carrier, the ternary operation is a ternary operation on the carrier.

We consider TBA lattice structures which extend TBA structures and lattice structures and are systems

$$\langle \text{a carrier}, \text{a join operation}, \text{a meet operation}, \text{a complement operation},$$

$$\text{a ternary operation}\rangle$$

where the carrier is a set, the join operation and the meet operation are binary operations on the carrier, the complement operation is a unary operation on the carrier, the ternary operation is a ternary operation on the carrier.

The functor op3 yielding a ternary operation on $\{0\}$ is defined by

(Def. 1) $\quad it(0,\ 0,\ 0) = 0$.

Let us observe that there exists a TBA structure which is trivial and non empty.

2. Axiomatization of Ternary Boolean Algebras

Let T be a non empty TBA structure and a, b, c be elements of T. The functor $\mathrm{T}(a, b, c)$ yielding an element of T is defined by the term

(Def. 2) $\quad$ (the ternary operation of T)$(a,\ b,\ c)$.

We say that T is ternary-distributive if and only if

(Def. 3) $\quad$ for every elements a, b, c, d, e of T, $\mathrm{T}(\mathrm{T}(a, b, c), d, \mathrm{T}(a, b, e)) = \mathrm{T}(a, b, \mathrm{T}(c, d, e))$.

We say that T is ternary-left-idempotent if and only if

(Def. 4) $\quad$ for every elements a, b of T, $\mathrm{T}(b, b, a) = b$.

We say that T is ternary-right-idempotent if and only if

(Def. 5) $\quad$ for every elements a, b of T, $\mathrm{T}(a, b, b) = b$.

We say that T is ternary-left-absorbing if and only if

(Def. 6) $\quad$ for every elements a, b of T, $\mathrm{T}(b^{\mathrm{c}}, b, a) = a$.

We say that T is ternary-right-absorbing if and only if

(Def. 7) $\quad$ for every elements a, b of T, $\mathrm{T}(a, b, b^{\mathrm{c}}) = a$.

One can check that every non empty TBA structure which is trivial is also ternary-distributive, ternary-left-idempotent, ternary-right-idempotent, ternary-left-absorbing, and ternary-right-absorbing.

A ternary Boolean algebra is a ternary-distributive, ternary-left-idempotent, ternary-right-idempotent, ternary-left-absorbing, ternary-right-absorbing, non empty TBA structure.

3. Converting TBAs into Ordinary Binary Boolean Algebras

Let T be a ternary Boolean algebra and x be an element of T. The functors: $\mathrm{JoinTBA}(T,x)$ and $\mathrm{MeetTBA}(T,x)$ yielding binary operations on the carrier of T are defined by conditions

(Def. 8) for every elements a, b of T, $\mathrm{JoinTBA}(T,x)(a,b) = \mathrm{T}(a,x,b)$,

(Def. 9) for every elements a, b of T, $\mathrm{MeetTBA}(T,x)(a,b) = \mathrm{T}(a,x^{\mathrm{c}},b)$,

respectively. The functor $\mathrm{TBA2BA}(T,x)$ yielding a non empty lattice structure is defined by the term

(Def. 10) $\langle$the carrier of $T, \mathrm{JoinTBA}(T,x), \mathrm{MeetTBA}(T,x)\rangle$.

4. Basic Properties of Ternary Operation

From now on T denotes a ternary Boolean algebra, a, b, c, d, e denote elements of T, and x, y, z denote elements of T. Now we state the propositions:

(1) $\mathrm{T}(a,b,a) = a$.

(2) $\mathrm{T}(\mathrm{T}(a,b,c),b,a) = \mathrm{T}(a,b,c)$.

(3) $\mathrm{T}(a,b,\mathrm{T}(c,b,d)) = \mathrm{T}(\mathrm{T}(a,b,c),b,d)$. The theorem is a consequence of (2).

(4) $\mathrm{T}(b^{\mathrm{c}},b,a) = \mathrm{T}(a,b,b^{\mathrm{c}})$.

(5) $\mathrm{T}(a,b^{\mathrm{c}},b) = a$.

(6) $(a^{\mathrm{c}})^{\mathrm{c}} = a$. The theorem is a consequence of (5).

(7) $\mathrm{T}(a,b,a^{\mathrm{c}}) = b$. The theorem is a consequence of (6).

(8) $\mathrm{T}(a,b,c) = \mathrm{T}(a,c,b)$. The theorem is a consequence of (7) and (1).

(9) $\mathrm{T}(a,b,c) = \mathrm{T}(b,c,a)$. The theorem is a consequence of (7).

(10) $\mathrm{T}(a,b,c) = \mathrm{T}(c,b,a)$. The theorem is a consequence of (8) and (9).

(11) Let us consider an element x of T. Then $\mathrm{T}(a,b,c) = \mathrm{T}(\mathrm{T}(\mathrm{T}(a,x,b),x^{\mathrm{c}},\mathrm{T}(b,x,c)),x^{\mathrm{c}},\mathrm{T}(c,x,a))$. The theorem is a consequence of (8), (10), (7), (9), and (3).

5. The Rosetta Operation

Let L be a Boolean lattice and a, b, c be elements of L. The functor $\mathrm{Ros}(a,b,c)$ yielding an element of L is defined by the term

(Def. 11) $((a \sqcap b) \sqcup (b \sqcap c)) \sqcup (c \sqcap a)$.

Let B be a Boolean lattice. The functor $\mathrm{RosTrn}(B)$ yielding a ternary operation on the carrier of B is defined by

(Def. 12) for every elements a, b, c of B, $it(a,\ b,\ c) = \mathrm{Ros}(a,b,c)$.

Let B be a Boolean lattice. The functor BA2TBA(B) yielding a TBA structure is defined by the term

(Def. 13) $\langle$the carrier of B, comp B, RosTrn(B)$\rangle$.

The functor BA2TBAA(B) yielding a TBA lattice structure is defined by the term

(Def. 14) $\langle$the carrier of B, the join operation of B, the meet operation of B, comp B, RosTrn(B)$\rangle$.

Let us note that BA2TBA(B) is non empty and BA2TBAA(B) is non empty.

6. Proof that TBA2BA Satisfy Lattice Axioms

In the sequel T denotes a ternary Boolean algebra.

Let us consider T. Let x be an element of T. Let us observe that JoinTBA(T, x) is commutative and JoinTBA(T, x) is associative and MeetTBA(T, x) is commutative.

From now on x denotes an element of T.

Let us consider T. Let x be an element of T. Note that MeetTBA(T, x) is associative.

Let T be a ternary Boolean algebra and p be an element of T. One can verify that the lattice structure of TBA2BA(T, p) is lattice-like.

7. Proof that BA2TBAA Returns Standard Example of TBA

Let B be a Boolean lattice. One can verify that BA2TBAA(B) is lattice-like.

Now we state the propositions:

(12) Let us consider a Boolean lattice B, an element x of B, and an element xx of BA2TBA(B). If $xx = x$, then $x^{\mathrm{c}} = xx^{\mathrm{c}}$.

(13) Let us consider a Boolean lattice B, an element x of B, and an element xx of BA2TBAA(B). If $xx = x$, then $x^{\mathrm{c}} = xx^{\mathrm{c}}$.

Let B be a Boolean lattice. One can verify that BA2TBA(B) is ternary-left-idempotent, ternary-right-idempotent, ternary-left-absorbing, and ternary-right-absorbing and BA2TBAA(B) is ternary-left-idempotent, ternary-right-idempotent, ternary-left-absorbing, and ternary-right-absorbing.

In the sequel B denotes a Boolean lattice and v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_{103}, v_{100}, v_{102}, v_{104}, v_{105}, v_{101} denote elements of BA2TBAA(B).

Now we state the propositions:

(14) Suppose for every v_1 and v_0, $\mathrm{T}(v_0, v_0, v_1) = v_0$ and for every v_2, v_1, and v_0, $\mathrm{T}(v_0, v_1, v_2) = \mathrm{T}(v_2, v_0, v_1)$ and for every v_2, v_1, and v_0, $\mathrm{T}(v_0, v_1, v_2) =$

$\mathrm{T}(v_0, v_2, v_1)$ and for every v_3, v_2, v_1, and v_0, $\mathrm{T}(\mathrm{T}(v_0, v_1, v_2), v_1, v_3) = \mathrm{T}(v_0, v_1, \mathrm{T}(v_2, v_1, v_3))$. $\mathrm{T}(\mathrm{T}(v_1, v_2, v_3), v_4, \mathrm{T}(v_1, v_2, v_5)) = \mathrm{T}(v_1, v_2, \mathrm{T}(v_3, v_4, v_5))$.

(15) Suppose for every v_2, v_1, and v_0, $\mathrm{T}(v_0, v_1, v_2) = ((v_0 \sqcup v_1) \sqcap (v_1 \sqcup v_2)) \sqcap (v_0 \sqcup v_2)$ and for every v_0, v_2, and v_1, $v_0 \sqcup (v_1 \sqcap v_2) = (v_0 \sqcup v_1) \sqcap (v_0 \sqcup v_2)$ and for every v_0, v_2, and v_1, $v_0 \sqcap (v_1 \sqcup v_2) = (v_0 \sqcap v_1) \sqcup (v_0 \sqcap v_2)$ and for every v_2, v_1, and v_0, $(v_0 \sqcup v_1) \sqcup v_2 = v_0 \sqcup (v_1 \sqcup v_2)$ and for every v_2, v_1, and v_0, $(v_0 \sqcap v_1) \sqcap v_2 = v_0 \sqcap (v_1 \sqcap v_2)$. $\mathrm{T}(\mathrm{T}(v_1, v_2, v_3), v_2, v_4) = \mathrm{T}(v_1, v_2, \mathrm{T}(v_3, v_2, v_4))$.

(16) Let us consider a Boolean lattice B, elements v_0, v_1 of BA2TBAA(B), and elements a, b of B. If $a = v_0$ and $b = v_1$, then $v_0 \sqcup v_1 = a \sqcup b$.

Let B be a Boolean lattice. Observe that BA2TBAA(B) is ternary-distributive.

Let T be a ternary Boolean algebra and p be an element of T. Let us note that the lattice structure of TBA2BA(T, p) is distributive and the lattice structure of TBA2BA(T, p) is bounded.

Let us consider a ternary Boolean algebra T and an element p of T. Now we state the propositions:

(17) $\top_\alpha = p$, where α is the lattice structure of TBA2BA(T, p).

(18) $\bot_\alpha = p^{\mathrm{c}}$, where α is the lattice structure of TBA2BA(T, p).

Let T be a ternary Boolean algebra and p be an element of T. Note that the lattice structure of TBA2BA(T, p) is complemented.

Let us consider T. Observe that the lattice structure of TBA2BA(T, p) is Boolean.

8. Single Axiom for TBA

In the sequel T denotes a non empty TBA structure and v_0, v_1, v_2, v_3, v_4, v_5, v_6, u, w, v, v_{100}, v_{101}, v_{102}, v_{103}, v_{104} denote elements of T.

Let T be a non empty TBA structure. We say that T is satisfying TBA_1 if and only if

(Def. 15) for every elements x, y, z, u, v, v_6, w of T, $\mathrm{T}(\mathrm{T}(x, x^{\mathrm{c}}, y), \mathrm{T}(\mathrm{T}(z, u, v), w, \mathrm{T}(z, u, v_6))^{\mathrm{c}}, \mathrm{T}(u, \mathrm{T}(v_6, w, v), z)) = y$.

Now we state the proposition:

(19) Suppose for every v_4, v_3, v_2, v_1, and v_0, $\mathrm{T}(\mathrm{T}(v_0, v_1, v_2), v_3, \mathrm{T}(v_0, v_1, v_4)) = \mathrm{T}(v_0, v_1, \mathrm{T}(v_2, v_3, v_4))$ and for every v_1 and v_0, $\mathrm{T}(v_0, v_1, v_1) = v_1$ and for every v_1 and v_0, $\mathrm{T}(v_0, v_1, v_1{}^{\mathrm{c}}) = v_0$ and for every v_1 and v_0, $\mathrm{T}(v_0, v_0, v_1) = v_0$. Let us consider elements x, y, z, u, v, v_6, w of T. Then $\mathrm{T}(\mathrm{T}(x, x^{\mathrm{c}}, y), \mathrm{T}(\mathrm{T}(z, u, v), w, \mathrm{T}(z, u, v_6))^{\mathrm{c}}, \mathrm{T}(u, \mathrm{T}(v_6, w, v), z)) = y$.

Let T be a non empty TBA structure. We say that T is TBA-like if and only if

(Def. 16) T is ternary-distributive, ternary-left-idempotent, ternary-right-idempotent, ternary-left-absorbing, and ternary-right-absorbing.

Note that every non empty TBA structure which is ternary-distributive, ternary-left-idempotent, ternary-right-idempotent, ternary-left-absorbing, and ternary-right-absorbing is also TBA-like and every non empty TBA structure which is TBA-like is also ternary-distributive, ternary-left-idempotent, ternary-right-idempotent, ternary-left-absorbing, and ternary-right-absorbing and every non empty TBA structure which is TBA-like is also satisfying TBA_1.

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