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Improper Integral. Part II

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Summary. In this article, using the Mizar system [2], [3], we deal with Riemann's improper integral on infinite interval [1]. As with [4], we referred to [6], which discusses improper integrals of finite values.

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1. Properties of Extended Riemann Integral on Infinite Interval

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(1) If f is divergent in $-\infty$ to $+\infty$, then f is not convergent in $-\infty$ and f is not divergent in $-\infty$ to $-\infty$.

(2) If f is divergent in $-\infty$ to $-\infty$, then f is not convergent in $-\infty$ and f is not divergent in $-\infty$ to $+\infty$.

(3) If f is divergent in $+\infty$ to $+\infty$, then f is not convergent in $+\infty$ and f is not divergent in $+\infty$ to $-\infty$.

(4) If f is divergent in $+\infty$ to $-\infty$, then f is not convergent in $+\infty$ and f is not divergent in $+\infty$ to $+\infty$.

(5) Suppose f is convergent in $-\infty$. Then

(i) there exists a real number r such that $f{\restriction}]-\infty, r[$ is lower bounded, and

(ii) there exists a real number r such that $f{\restriction}]-\infty, r[$ is upper bounded.

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PROOF: Consider g being a real number such that for every real number g_1 such that $0 < g_1$ there exists a real number r such that for every real number r_1 such that $r_1 < r$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < g_1$. Consider r being a real number such that for every real number r_1 such that $r_1 < r$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < 1$. For every object r_1 such that $r_1 \in \operatorname{dom}(f \restriction]-\infty, r[)$ holds $-1 + g < (f \restriction]-\infty, r[)(r_1)$. Consider r being a real number such that for every real number r_1 such that $r_1 < r$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < 1$. For every object r_1 such that $r_1 \in \operatorname{dom}(f \restriction]-\infty, r[)$ holds $(f \restriction]-\infty, r[)(r_1) < g + 1$. □

(6) Suppose f is convergent in $+\infty$. Then

(i) there exists a real number r such that $f \restriction]r, +\infty[$ is lower bounded, and

(ii) there exists a real number r such that $f \restriction]r, +\infty[$ is upper bounded.

PROOF: Consider g being a real number such that for every real number g_1 such that $0 < g_1$ there exists a real number r such that for every real number r_1 such that $r < r_1$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < g_1$. Consider r being a real number such that for every real number r_1 such that $r < r_1$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < 1$. For every object r_1 such that $r_1 \in \operatorname{dom}(f \restriction]r, +\infty[)$ holds $-1 + g < (f \restriction]r, +\infty[)(r_1)$. Consider r being a real number such that for every real number r_1 such that $r < r_1$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < 1$. For every object r_1 such that $r_1 \in \operatorname{dom}(f \restriction]r, +\infty[)$ holds $(f \restriction]r, +\infty[)(r_1) < g + 1$. □

(7) Suppose f is divergent in $-\infty$ to $+\infty$. Then there exists a real number r such that $f \restriction]-\infty, r[$ is lower bounded.
PROOF: Consider r being a real number such that for every real number r_1 such that $r_1 < r$ and $r_1 \in \operatorname{dom} f$ holds $1 < f(r_1)$. For every object r_1 such that $r_1 \in \operatorname{dom}(f \restriction]-\infty, r[)$ holds $1 < (f \restriction]-\infty, r[)(r_1)$. □

(8) Suppose f is divergent in $-\infty$ to $-\infty$. Then there exists a real number r such that $f \restriction]-\infty, r[$ is upper bounded.
PROOF: Consider r being a real number such that for every real number r_1 such that $r_1 < r$ and $r_1 \in \operatorname{dom} f$ holds $f(r_1) < 1$. For every object r_1 such that $r_1 \in \operatorname{dom}(f \restriction]-\infty, r[)$ holds $(f \restriction]-\infty, r[)(r_1) < 1$. □

(9) Suppose f is divergent in $+\infty$ to $+\infty$. Then there exists a real number r such that $f \restriction]r, +\infty[$ is lower bounded.
PROOF: Consider r being a real number such that for every real number r_1 such that $r < r_1$ and $r_1 \in \operatorname{dom} f$ holds $1 < f(r_1)$. For every object r_1 such that $r_1 \in \operatorname{dom}(f \restriction]r, +\infty[)$ holds $1 < (f \restriction]r, +\infty[)(r_1)$. □

(10) Suppose f is divergent in $+\infty$ to $-\infty$. Then there exists a real number r such that $f \restriction]r, +\infty[$ is upper bounded.

PROOF: Consider r being a real number such that for every real number r_1 such that $r < r_1$ and $r_1 \in \operatorname{dom} f$ holds $f(r_1) < 1$. For every object r_1 such that $r_1 \in \operatorname{dom}(f \restriction]r, +\infty[)$ holds $(f \restriction]r, +\infty[)(r_1) < 1$. □

Let us consider partial functions f_1, f_2 from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(11) Suppose f_1 is divergent in $-\infty$ to $-\infty$ and for every real number r, there exists a real number g such that $g < r$ and $g \in \operatorname{dom}(f_1 + f_2)$ and there exists a real number r such that $f_2 \restriction]-\infty, r[$ is upper bounded. Then $f_1 + f_2$ is divergent in $-\infty$ to $-\infty$.

(12) Suppose f_1 is divergent in $+\infty$ to $-\infty$ and for every real number r, there exists a real number g such that $r < g$ and $g \in \operatorname{dom}(f_1 + f_2)$ and there exists a real number r such that $f_2 \restriction]r, +\infty[$ is upper bounded. Then $f_1 + f_2$ is divergent in $+\infty$ to $-\infty$.

(13) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number d. Suppose $]-\infty, d] \subseteq \operatorname{dom} f$ and f is extended Riemann integrable on $-\infty$, d. Let us consider real numbers b, c. Suppose $b < c \leqslant d$. Then f is right extended Riemann integrable on b, c and left extended Riemann integrable on b, c.

(14) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number a. Suppose $[a, +\infty[\subseteq \operatorname{dom} f$ and f is extended Riemann integrable on a, $+\infty$. Let us consider real numbers b, c. Suppose $a \leqslant b < c$. Then f is right extended Riemann integrable on b, c and left extended Riemann integrable on b, c.

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a real number a, and a real number b. Now we state the propositions:

(15) Suppose $]-\infty, a] \subseteq \operatorname{dom} f$ and f is extended Riemann integrable on $-\infty$, a. Then if $b \leqslant a$, then f is extended Riemann integrable on $-\infty$, b.

(16) Suppose $[a, +\infty[\subseteq \operatorname{dom} f$ and f is extended Riemann integrable on a, $+\infty$. Then if $a \leqslant b$, then f is extended Riemann integrable on b, $+\infty$.

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers a, b. Now we state the propositions:

(17) Suppose $a \leqslant b$ and $]-\infty, b] \subseteq \operatorname{dom} f$ and f is integrable on $[a, b]$ and $f \restriction [a, b]$ is bounded and f is extended Riemann integrable on $-\infty$, a. Then

(i) f is extended Riemann integrable on $-\infty$, b, and

(ii) $(R^{<}) \int\limits_{-\infty}^{b} f(x)dx = (R^{<}) \int\limits_{-\infty}^{a} f(x)dx + \int\limits_{a}^{b} f(x)dx$.

PROOF: For every real number c such that $c \leqslant b$ holds f is integrable on

$[c, b]$ and $f{\restriction}[c, b]$ is bounded. Consider I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I = \left]-\infty, a\right]$ and for every real number x such that $x \in \operatorname{dom} I$ holds $I(x) = \int\limits_{x}^{a} f(x)dx$ and I is convergent in $-\infty$. Reconsider $B = \left]-\infty, b\right]$ as a non empty subset of $\mathbb{R}$. Define $\mathcal{F}(\text{element of } B) = (\int\limits_{\$_1}^{b} f(x)dx)(\in \mathbb{R})$. Consider I_1 being a function from B into $\mathbb{R}$ such that for every element x of B, $I_1(x) = \mathcal{F}(x)$. For every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{x}^{b} f(x)dx$. For every real number r, there exists a real number g such that $g < r$ and $g \in \operatorname{dom} I_1$. Consider G being a real number such that for every real number g_1 such that $0 < g_1$ there exists a real number r such that for every real number r_1 such that $r_1 < r$ and $r_1 \in \operatorname{dom} I$ holds $|I(r_1) - G| < g_1$. Set $G_1 = G + \int\limits_{a}^{b} f(x)dx$. For every real number g_1 such that $0 < g_1$ there exists a real number r such that for every real number r_1 such that $r_1 < r$ and $r_1 \in \operatorname{dom} I_1$ holds $|I_1(r_1) - G_1| < g_1$. □

(18) Suppose $a \leqslant b$ and $\left[a, +\infty\right[\subseteq \operatorname{dom} f$ and f is integrable on $[a, b]$ and $f{\restriction}[a, b]$ is bounded and f is extended Riemann integrable on b, $+\infty$. Then

(i) f is extended Riemann integrable on a, $+\infty$, and

(ii) $(R^{>}) \int\limits_{a}^{+\infty} f(x)dx = (R^{>}) \int\limits_{b}^{+\infty} f(x)dx + \int\limits_{a}^{b} f(x)dx$.

Proof: For every real number c such that $a \leqslant c$ holds f is integrable on $[a, c]$ and $f{\restriction}[a, c]$ is bounded. Consider I being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I = \left[b, +\infty\right[$ and for every real number x such that $x \in \operatorname{dom} I$ holds $I(x) = \int\limits_{b}^{x} f(x)dx$ and I is convergent in $+\infty$. Reconsider $A = \left[a, +\infty\right[$ as a non empty subset of $\mathbb{R}$. Define $\mathcal{F}(\text{element of } A) = (\int\limits_{a}^{\$_1} f(x)dx)(\in \mathbb{R})$. Consider I_1 being a function from A into $\mathbb{R}$ such that for every element x of A, $I_1(x) = \mathcal{F}(x)$. For every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{a}^{x} f(x)dx$. For every real number r, there exists a real number g such that $r < g$ and $g \in \operatorname{dom} I_1$. Consider G being a real number such that for every real number g_1 such that $0 < g_1$ there exists a real number r such that for every real number r_1 such that $r < r_1$ and $r_1 \in \operatorname{dom} I$ holds $|I(r_1) - G| < g_1$. Set $G_1 = G + \int_a^b f(x)dx$. For every real number g_1 such that $0 < g_1$ there exists a real number r such that for every real number r_1 such that $r < r_1$ and $r_1 \in \operatorname{dom} I_1$ holds $|I_1(r_1) - G_1| < g_1$ by [5, (17)]. □

(19) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $\operatorname{dom} f = \mathbb{R}$. Then f is ∞-extended Riemann integrable if and only if for every real number a, f is extended Riemann integrable on a, $+\infty$ and extended Riemann integrable on $-\infty$, a. The theorem is a consequence of (16), (17), (18), and (15).

2. Improper Integral on Infinite Interval

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and b be a real number. We say that f is improper integrable on $]-\infty, b]$ if and only if

(Def. 1) for every real number a such that $a \leqslant b$ holds f is integrable on $[a, b]$ and $f \restriction [a, b]$ is bounded and there exists a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int_x^b f(x)dx$ and (I_1 is convergent in $-\infty$ or divergent in $-\infty$ to $+\infty$ or I_1 is divergent in $-\infty$ to $-\infty$).

Let a be a real number. We say that f is improper integrable on $[a, +\infty[$ if and only if

(Def. 2) for every real number b such that $a \leqslant b$ holds f is integrable on $[a, b]$ and $f \restriction [a, b]$ is bounded and there exists a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 = [a, +\infty[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int_a^x f(x)dx$ and (I_1 is convergent in $+\infty$ or divergent in $+\infty$ to $+\infty$ or I_1 is divergent in $+\infty$ to $-\infty$).

Let b be a real number. Assume f is improper integrable on $]-\infty, b]$. The functor $\int_{-\infty}^b f(x)dx$ yielding an extended real is defined by

(Def. 3) there exists a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 =]-\infty, b]$

and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_x^b f(x)dx$ and (I_1 is convergent in $-\infty$ and $it = \lim_{-\infty} I_1$ or I_1 is divergent in $-\infty$ to $+\infty$ and $it = +\infty$ or I_1 is divergent in $-\infty$ to $-\infty$ and $it = -\infty$).

Let a be a real number. Assume f is improper integrable on $[a, +\infty[$. The functor $\int\limits_a^{+\infty} f(x)dx$ yielding an extended real is defined by

(Def. 4) there exists a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 = [a, +\infty[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_a^x f(x)dx$ and (I_1 is convergent in $+\infty$ and $it = \lim_{+\infty} I_1$ or I_1 is divergent in $+\infty$ to $+\infty$ and $it = +\infty$ or I_1 is divergent in $+\infty$ to $-\infty$ and $it = -\infty$).

Now we state the propositions:

(20) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose f is extended Riemann integrable on $-\infty$, b. Then f is improper integrable on $]-\infty, b]$.

(21) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number a. Suppose f is extended Riemann integrable on a, $+\infty$. Then f is improper integrable on $[a, +\infty[$.

(22) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose f is improper integrable on $]-\infty, b]$. Then

  (i) f is extended Riemann integrable on $-\infty$, b and $\int\limits_{-\infty}^b f(x)dx = (R^<) \int\limits_{-\infty}^b f(x)dx$, or

  (ii) f is not extended Riemann integrable on $-\infty$, b and $\int\limits_{-\infty}^b f(x)dx = +\infty$, or

  (iii) f is not extended Riemann integrable on $-\infty$, b and $\int\limits_{-\infty}^b f(x)dx = -\infty$.

The theorem is a consequence of (1) and (2).

(23) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose there exists a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) =$

$\int_x^b f(x)dx$ and I_1 is divergent in $-\infty$ to $+\infty$ or divergent in $-\infty$ to $-\infty$. Then f is not extended Riemann integrable on $-\infty$, b. The theorem is a consequence of (1) and (2).

(24) Let us consider partial functions f, I_1 from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose f is improper integrable on $]-\infty, b]$ and $\operatorname{dom} I_1 =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int_x^b f(x)dx$ and I_1 is convergent in $-\infty$. Then $\int_{-\infty}^b f(x)dx = \lim_{-\infty} I_1$. The theorem is a consequence of (22).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers b, c. Now we state the propositions:

(25) Suppose $b \leqslant c$ and $]-\infty, c] \subseteq \operatorname{dom} f$ and f is improper integrable on $]-\infty, c]$. Then

(i) f is improper integrable on $]-\infty, b]$, and

(ii) if $\int_{-\infty}^c f(x)dx = (R^<) \int_{-\infty}^c f(x)dx$, then $\int_{-\infty}^b f(x)dx = (R^<) \int_{-\infty}^b f(x)dx$, and

(iii) if $\int_{-\infty}^c f(x)dx = +\infty$, then $\int_{-\infty}^b f(x)dx = +\infty$, and

(iv) if $\int_{-\infty}^c f(x)dx = -\infty$, then $\int_{-\infty}^b f(x)dx = -\infty$.

The theorem is a consequence of (22).

(26) Suppose $b \leqslant c$ and $]-\infty, c] \subseteq \operatorname{dom} f$ and $f\restriction[b, c]$ is bounded and f is improper integrable on $]-\infty, b]$ and f is integrable on $[b, c]$. Then

(i) f is improper integrable on $]-\infty, c]$, and

(ii) if $\int_{-\infty}^b f(x)dx = (R^<) \int_{-\infty}^b f(x)dx$, then

$$\int_{-\infty}^c f(x)dx = \int_{-\infty}^b f(x)dx + \int_b^c f(x)dx,$$ and

(iii) if $\int\limits_{-\infty}^{b} f(x)dx = +\infty$, then $\int\limits_{-\infty}^{c} f(x)dx = +\infty$, and

(iv) if $\int\limits_{-\infty}^{b} f(x)dx = -\infty$, then $\int\limits_{-\infty}^{c} f(x)dx = -\infty$.

The theorem is a consequence of (22).

(27) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose f is improper integrable on $[b, +\infty[$. Then

(i) f is extended Riemann integrable on b, $+\infty$ and $\int\limits_{b}^{+\infty} f(x)dx = (R^{>}) \int\limits_{b}^{+\infty} f(x)dx$, or

(ii) f is not extended Riemann integrable on b, $+\infty$ and $\int\limits_{b}^{+\infty} f(x)dx = +\infty$, or

(iii) f is not extended Riemann integrable on b, $+\infty$ and $\int\limits_{b}^{+\infty} f(x)dx = -\infty$.

The theorem is a consequence of (3) and (4).

(28) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose there exists a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$ such that $\operatorname{dom} I_1 = [b, +\infty[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{b}^{x} f(x)dx$ and I_1 is divergent in $+\infty$ to $+\infty$ or divergent in $+\infty$ to $-\infty$. Then f is not extended Riemann integrable on b, $+\infty$. The theorem is a consequence of (3) and (4).

(29) Let us consider partial functions f, I_1 from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose f is improper integrable on $[b, +\infty[$ and $\operatorname{dom} I_1 = [b, +\infty[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{b}^{x} f(x)dx$ and I_1 is convergent in $+\infty$. Then $\int\limits_{b}^{+\infty} f(x)dx = \lim\limits_{+\infty} I_1$. The theorem is a consequence of (27).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and real numbers b, c. Now we state the propositions:

(30) Suppose $b \geqslant c$ and $[c, +\infty[\subseteq \operatorname{dom} f$ and f is improper integrable on $[c, +\infty[$. Then

(i) f is improper integrable on $[b, +\infty[$, and

(ii) if $\int\limits_{c}^{+\infty} f(x)dx = (R^{>}) \int\limits_{c}^{+\infty} f(x)dx$, then $\int\limits_{b}^{+\infty} f(x)dx = (R^{>}) \int\limits_{b}^{+\infty} f(x)dx$, and

(iii) if $\int\limits_{c}^{+\infty} f(x)dx = +\infty$, then $\int\limits_{b}^{+\infty} f(x)dx = +\infty$, and

(iv) if $\int\limits_{c}^{+\infty} f(x)dx = -\infty$, then $\int\limits_{b}^{+\infty} f(x)dx = -\infty$.

The theorem is a consequence of (27).

(31) Suppose $b \geqslant c$ and $[c, +\infty[\subseteq \operatorname{dom} f$ and $f \restriction [c, b]$ is bounded and f is improper integrable on $[b, +\infty[$ and f is integrable on $[c, b]$. Then

(i) f is improper integrable on $[c, +\infty[$, and

(ii) if $\int\limits_{b}^{+\infty} f(x)dx = (R^{>}) \int\limits_{b}^{+\infty} f(x)dx$, then $\int\limits_{c}^{+\infty} f(x)dx = \int\limits_{b}^{+\infty} f(x)dx + \int\limits_{c}^{b} f(x)dx$, and

(iii) if $\int\limits_{b}^{+\infty} f(x)dx = +\infty$, then $\int\limits_{c}^{+\infty} f(x)dx = +\infty$, and

(iv) if $\int\limits_{b}^{+\infty} f(x)dx = -\infty$, then $\int\limits_{c}^{+\infty} f(x)dx = -\infty$.

The theorem is a consequence of (27).

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. We say that f is improper integrable on $\mathbb{R}$ if and only if

(Def. 5) there exists a real number r such that f is improper integrable on $]-\infty, r]$ and f is improper integrable on $[r, +\infty[$ and it is not true that $\int\limits_{-\infty}^{r} f(x)dx = -\infty$ and $\int\limits_{r}^{+\infty} f(x)dx = +\infty$ and it is not true that $\int\limits_{-\infty}^{r} f(x)dx = +\infty$ and $\int\limits_{r}^{+\infty} f(x)dx = -\infty$.

Now we state the propositions:

(32) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose f is improper integrable on $\mathbb{R}$. Then there exists a real number b such that $\int_{-\infty}^{b} f(x)dx = (R^{<}) \int_{-\infty}^{b} f(x)dx$ and $\int_{b}^{+\infty} f(x)dx = (R^{>}) \int_{b}^{+\infty} f(x)dx$ or $\int_{-\infty}^{b} f(x)dx + \int_{b}^{+\infty} f(x)dx = +\infty$ or $\int_{-\infty}^{b} f(x)dx + \int_{b}^{+\infty} f(x)dx = -\infty$. The theorem is a consequence of (22) and (27).

(33) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose $\operatorname{dom} f = \mathbb{R}$ and f is improper integrable on $]-\infty, b]$ and f is improper integrable on $[b, +\infty[$ and it is not true that $\int_{-\infty}^{b} f(x)dx = -\infty$ and $\int_{b}^{+\infty} f(x)dx = +\infty$ and it is not true that $\int_{-\infty}^{b} f(x)dx = +\infty$ and $\int_{b}^{+\infty} f(x)dx = -\infty$. Let us consider a real number b_1. Suppose $b_1 \leqslant b$. Then $\int_{-\infty}^{b} f(x)dx + \int_{b}^{+\infty} f(x)dx = \int_{-\infty}^{b_1} f(x)dx + \int_{b_1}^{+\infty} f(x)dx$. The theorem is a consequence of (22), (27), and (31).

(34) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose $\operatorname{dom} f = \mathbb{R}$ and f is improper integrable on $]-\infty, b]$ and f is improper integrable on $[b, +\infty[$ and it is not true that $\int_{-\infty}^{b} f(x)dx = -\infty$ and $\int_{b}^{+\infty} f(x)dx = +\infty$ and it is not true that $\int_{-\infty}^{b} f(x)dx = +\infty$ and $\int_{b}^{+\infty} f(x)dx = -\infty$. Let us consider a real number b_2. Suppose $b \leqslant b_2$. Then $\int_{-\infty}^{b} f(x)dx + \int_{b}^{+\infty} f(x)dx = \int_{-\infty}^{b_2} f(x)dx + \int_{b_2}^{+\infty} f(x)dx$. The theorem is

a consequence of (27), (30), (31), and (22).

(35) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $\operatorname{dom} f = \mathbb{R}$ and f is improper integrable on $\mathbb{R}$. Let us consider real numbers b_1, b_2. Then $\int_{-\infty}^{b_1} f(x)dx + \int_{b_1}^{+\infty} f(x)dx = \int_{-\infty}^{b_2} f(x)dx + \int_{b_2}^{+\infty} f(x)dx$. The theorem is a consequence of (33) and (34).

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Assume $\operatorname{dom} f = \mathbb{R}$ and f is improper integrable on $\mathbb{R}$. The functor $\int_{-\infty}^{+\infty} f(x)dx$ yielding an extended real is defined by

(Def. 6) there exists a real number c such that f is improper integrable on $]-\infty, c]$ and f is improper integrable on $[c, +\infty[$ and $it = \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx$.

Now we state the proposition:

(36) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose $\operatorname{dom} f = \mathbb{R}$ and f is improper integrable on $\mathbb{R}$. Then

(i) f is improper integrable on $]-\infty, b]$, and

(ii) f is improper integrable on $[b, +\infty[$, and

(iii) $\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{b} f(x)dx + \int_{b}^{+\infty} f(x)dx$.

The theorem is a consequence of (25), (31), (35), (26), and (30).

3. Linearity of Improper Integral on Infinite Interval

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a real number b, and a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(37) Suppose f is improper integrable on $]-\infty, b]$ and $\int_{-\infty}^{b} f(x)dx = +\infty$. Then suppose $\operatorname{dom} I_1 =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int_{x}^{b} f(x)dx$. Then I_1 is divergent in $-\infty$ to $+\infty$.

(38) Suppose f is improper integrable on $]-\infty, b]$ and $\int_{-\infty}^{b} f(x)dx = -\infty$.

Then suppose $\operatorname{dom} I_1 =]-\infty, b]$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{x}^{b} f(x)dx$. Then I_1 is divergent in $-\infty$ to $-\infty$.

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, a real number a, and a partial function I_1 from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(39) Suppose f is improper integrable on $[a, +\infty[$ and $\int\limits_{a}^{+\infty} f(x)dx = +\infty$. Then suppose $\operatorname{dom} I_1 = [a, +\infty[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{a}^{x} f(x)dx$. Then I_1 is divergent in $+\infty$ to $+\infty$.

(40) Suppose f is improper integrable on $[a, +\infty[$ and $\int\limits_{a}^{+\infty} f(x)dx = -\infty$. Then suppose $\operatorname{dom} I_1 = [a, +\infty[$ and for every real number x such that $x \in \operatorname{dom} I_1$ holds $I_1(x) = \int\limits_{a}^{x} f(x)dx$. Then I_1 is divergent in $+\infty$ to $-\infty$.

(41) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers b, r. Suppose $]-\infty, b] \subseteq \operatorname{dom} f$ and f is improper integrable on $]-\infty, b]$. Then

(i) $r \cdot f$ is improper integrable on $]-\infty, b]$, and

(ii) $\int\limits_{-\infty}^{b} (r \cdot f)(x)dx = r \cdot \int\limits_{-\infty}^{b} f(x)dx$.

PROOF: For every real number d such that $d \leqslant b$ holds $r \cdot f$ is integrable on $[d, b]$ and $(r \cdot f)\restriction[d, b]$ is bounded. □

(42) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and real numbers a, r. Suppose $[a, +\infty[\subseteq \operatorname{dom} f$ and f is improper integrable on $[a, +\infty[$. Then

(i) $r \cdot f$ is improper integrable on $[a, +\infty[$, and

(ii) $\int\limits_{a}^{+\infty} (r \cdot f)(x)dx = r \cdot \int\limits_{a}^{+\infty} f(x)dx$.

PROOF: For every real number d such that $a \leqslant d$ holds $r \cdot f$ is integrable on $[a, d]$ and $(r \cdot f)\restriction[a, d]$ is bounded. □

(43) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose $]-\infty, b] \subseteq \operatorname{dom} f$ and f is improper integrable on $]-\infty, b]$. Then

(i) $-f$ is improper integrable on $]-\infty, b]$, and

(ii) $\int\limits_{-\infty}^{b}(-f)(x)dx = -\int\limits_{-\infty}^{b} f(x)dx$.

The theorem is a consequence of (41).

(44) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$, and a real number a. Suppose $[a, +\infty[\subseteq \operatorname{dom} f$ and f is improper integrable on $[a, +\infty[$. Then

(i) $-f$ is improper integrable on $[a, +\infty[$, and

(ii) $\int\limits_{a}^{+\infty}(-f)(x)dx = -\int\limits_{a}^{+\infty} f(x)dx$.

The theorem is a consequence of (42).

(45) Let us consider partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose $]-\infty, b] \subseteq \operatorname{dom} f$ and $]-\infty, b] \subseteq \operatorname{dom} g$ and f is improper integrable on $]-\infty, b]$ and g is improper integrable on $]-\infty, b]$ and it is not true that $\int\limits_{-\infty}^{b} f(x)dx = +\infty$ and $\int\limits_{-\infty}^{b} g(x)dx = -\infty$ and it is not true that $\int\limits_{-\infty}^{b} f(x)dx = -\infty$ and $\int\limits_{-\infty}^{b} g(x)dx = +\infty$. Then

(i) $f + g$ is improper integrable on $]-\infty, b]$, and

(ii) $\int\limits_{-\infty}^{b}(f+g)(x)dx = \int\limits_{-\infty}^{b} f(x)dx + \int\limits_{-\infty}^{b} g(x)dx$.

Proof: For every real number d such that $d \leqslant b$ holds $f + g$ is integrable on $[d, b]$ and $(f + g)\restriction[d, b]$ is bounded. □

(46) Let us consider partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$, and a real number a. Suppose $[a, +\infty[\subseteq \operatorname{dom} f$ and $[a, +\infty[\subseteq \operatorname{dom} g$ and f is improper integrable on $[a, +\infty[$ and g is improper integrable on $[a, +\infty[$ and it is not true that $\int\limits_{a}^{+\infty} f(x)dx = +\infty$ and $\int\limits_{a}^{+\infty} g(x)dx = -\infty$ and it is not true that $\int\limits_{a}^{+\infty} f(x)dx = -\infty$ and $\int\limits_{a}^{+\infty} g(x)dx = +\infty$. Then

(i) $f + g$ is improper integrable on $[a, +\infty[$, and

(ii) $\int\limits_{a}^{+\infty}(f+g)(x)dx = \int\limits_{a}^{+\infty} f(x)dx + \int\limits_{a}^{+\infty} g(x)dx$.

PROOF: For every real number d such that $a \leqslant d$ holds $f+g$ is integrable on $[a,d]$ and $(f+g)\restriction[a,d]$ is bounded. □

(47) Let us consider partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$, and a real number b. Suppose $]-\infty,b] \subseteq \operatorname{dom} f$ and $]-\infty,b] \subseteq \operatorname{dom} g$ and f is improper integrable on $]-\infty, b]$ and g is improper integrable on $]-\infty, b]$ and it is not true that $\int\limits_{-\infty}^{b} f(x)dx = +\infty$ and $\int\limits_{-\infty}^{b} g(x)dx = +\infty$ and it is not true that $\int\limits_{-\infty}^{b} f(x)dx = -\infty$ and $\int\limits_{-\infty}^{b} g(x)dx = -\infty$. Then

(i) $f-g$ is improper integrable on $]-\infty, b]$, and

(ii) $\int\limits_{-\infty}^{b}(f-g)(x)dx = \int\limits_{-\infty}^{b} f(x)dx - \int\limits_{-\infty}^{b} g(x)dx$.

The theorem is a consequence of (43) and (45).

(48) Let us consider partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$, and a real number a. Suppose $[a,+\infty[\subseteq \operatorname{dom} f$ and $[a,+\infty[\subseteq \operatorname{dom} g$ and f is improper integrable on $[a, +\infty[$ and g is improper integrable on $[a, +\infty[$ and it is not true that $\int\limits_{a}^{+\infty} f(x)dx = +\infty$ and $\int\limits_{a}^{+\infty} g(x)dx = +\infty$ and it is not true that $\int\limits_{a}^{+\infty} f(x)dx = -\infty$ and $\int\limits_{a}^{+\infty} g(x)dx = -\infty$. Then

(i) $f-g$ is improper integrable on $[a, +\infty[$, and

(ii) $\int\limits_{a}^{+\infty}(f-g)(x)dx = \int\limits_{a}^{+\infty} f(x)dx - \int\limits_{a}^{+\infty} g(x)dx$.

The theorem is a consequence of (44) and (46).

Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$ and a real number r. Now we state the propositions:

(49) Suppose $\operatorname{dom} f = \mathbb{R}$ and f is improper integrable on $\mathbb{R}$. Then

(i) $r \cdot f$ is improper integrable on $\mathbb{R}$, and

(ii) $\int\limits_{-\infty}^{+\infty}(r \cdot f)(x)dx = r \cdot \int\limits_{-\infty}^{+\infty} f(x)dx$.

The theorem is a consequence of (36), (41), and (42).

(50) Suppose $\operatorname{dom} f = \mathbb{R}$ and f is improper integrable on $\mathbb{R}$. Then

(i) $-f$ is improper integrable on $\mathbb{R}$, and

(ii) $\int\limits_{-\infty}^{+\infty}(-f)(x)dx = -\int\limits_{-\infty}^{+\infty}f(x)dx.$

The theorem is a consequence of (49).

Let us consider partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(51) Suppose $\operatorname{dom} f = \mathbb{R}$ and $\operatorname{dom} g = \mathbb{R}$ and f is improper integrable on $\mathbb{R}$ and g is improper integrable on $\mathbb{R}$ and it is not true that $\int\limits_{-\infty}^{+\infty}f(x)dx = +\infty$ and $\int\limits_{-\infty}^{+\infty}g(x)dx = -\infty$ and it is not true that $\int\limits_{-\infty}^{+\infty}f(x)dx = -\infty$ and $\int\limits_{-\infty}^{+\infty}g(x)dx = +\infty$. Then

(i) $f + g$ is improper integrable on $\mathbb{R}$, and

(ii) $\int\limits_{-\infty}^{+\infty}(f+g)(x)dx = \int\limits_{-\infty}^{+\infty}f(x)dx + \int\limits_{-\infty}^{+\infty}g(x)dx.$

The theorem is a consequence of (25), (26), (31), (30), (36), (45), and (46).

(52) Suppose $\operatorname{dom} f = \mathbb{R}$ and $\operatorname{dom} g = \mathbb{R}$ and f is improper integrable on $\mathbb{R}$ and g is improper integrable on $\mathbb{R}$ and it is not true that $\int\limits_{-\infty}^{+\infty}f(x)dx = +\infty$ and $\int\limits_{-\infty}^{+\infty}g(x)dx = +\infty$ and it is not true that $\int\limits_{-\infty}^{+\infty}f(x)dx = -\infty$ and $\int\limits_{-\infty}^{+\infty}g(x)dx = -\infty$. Then

(i) $f - g$ is improper integrable on $\mathbb{R}$, and

(ii) $\int\limits_{-\infty}^{+\infty}(f-g)(x)dx = \int\limits_{-\infty}^{+\infty}f(x)dx - \int\limits_{-\infty}^{+\infty}g(x)dx.$

The theorem is a consequence of (50) and (51).

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