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RESEARCH ARTICLE

The minimal Hilbert basis of the Hammond order cone

Ramses H. Abul Naga[1,2,3]

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Abstract
We characterize the minimal Hilbert basis of the Hammond order cone, and present several novel applications of the resulting basis. From the basis, we extract an invertible matrix, that provides a numerical representation of the Hammond order relation. The basis also enables the construction of a space—that we call the Hammond order lattice—where order-extensions of the Hammond order (i.e. more complete relations) may be derived. Finally, we introduce a class of maximal linearly independent Hilbert bases, in which the specific results derived in relation to the Hammond order cone, are shown to hold more generally.

Keywords Measurement of social welfare · Order relations induced by convex cones · Hammond order · Hilbert bases

1 Introduction

The last decade has witnessed an increased interest among social scientists in the distributional analysis of ordered response data, such as self-assessed health and happiness. One major methodological contribution in the field is the work of Gravel et al. (2021), who introduce a social welfare ordering founded on Hammond's equity concept (Hammond 1976).[1]

We owe to Magdalou (2021) the first study of integral Hilbert bases of cones associated with abstract inequality or social welfare order relations defined on univariate

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✉ Ramses H. Abul Naga
r.h.abulnaga@uma.es

1 Departamento de Teoría e Historia Económica, Universidad de Málaga, Málaga, Spain

2 Business School, University of Aberdeen, Aberdeen, UK

3 Pan African Scientific Research Council, Princeton, USA

[1] Other related approaches include Apouey et al. (2020) as well as Seth and Yalonetzky (2020).

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or multivariate discrete distributions. An important contribution of Magdalou (2021) is to demonstrate the fundamental role Hilbert bases perform in characterizing the set of order-preserving functions of the underlying relation of interest. Specifically, the general equivalence theorem of Magdalou requires that the set of welfare improving transformations of the distribution of interest (the so-called set of transfers) contains an integral Hilbert basis of the underlying cone ordering.

When we set out to derive the minimal Hilbert basis of the Hammond order cone (that is, the cone associated with the welfare order relation introduced by Gravel et al. 2021), we find that when the variable of interest is defined on k ordered socioeconomic states, the minimal basis consists of $k-1$ vectors, that are linearly independent. In turn, the derivation of this result enables us to extend Gravel et al. (2021) and Magdalou (2021) in several directions. Specifically, the linear independence property enables us to introduce several novel applications of the minimal Hilbert basis of the Hammond order cone, and more generally, of a class of minimal Hilbert bases that share the same linear independence properties.

Firstly, we show that the minimal Hilbert basis can be directly used to identify the numerical implementation criterion (the so-called partial sums) that enable a researcher to conclude that a pair of distribution are ordered. This result is of practical relevance: to date, there is no simple method of deriving these partial sums. The method proposed here is simple, in that it consists of deriving the partial sums by inverting a matrix extracted from the minimal Hilbert basis. The same inversion method is used to illustrate how the well known partial sums associated with the first order stochastic dominance relation, are readily obtained from the minimal Hilbert basis associated with this cone ordering. Likewise, the inversion method is used to obtain the partial sums of the Hammond order, previously derived by Gravel et al. (2021) from an entirely different perspective.

A second area of application of the minimal Hilbert basis that is proposed in the paper is to introduce a new space—that we call the Hammond order lattice—where order-extensions of the Hammond order (i.e. more complete relations) may be derived. Defining this space is useful, as it enables the researcher to better understand how various order relations compare pairs of distributions in the context of socioeconomic surveys. The Hammond order lattice then provides a straightforward method of deriving relations that may be more, or less, complete than the Hammond order. This lattice is founded on the linear independence property of the vectors that constitute the minimal Hilbert basis, and is thus easily generalizable in other contexts.

Finally, in deriving the minimal Hilbert basis of the Hammond order cone, we present a result due to Giles and Pulleyblank (1979) that enables the construction of an integral Hilbert basis of a general pointed rational cone. The resulting Hilbert basis is the set of integral vectors[2] contained in a set, called the *parallelotope* associated with the cone. We note that having a method of constructing an integral Hilbert basis of a rational cone is important, in that it enables a wider application of the equivalence theorem of Magdalou (2021) in various contexts.

More generally, because minimal Hilbert bases provide parsimonious representations of the set of integral vectors that belong to a cone, they are important to

[2] An integral vector is a vector whose components are all integers.

characterize in a variety of contexts. In the context of distributional analysis, a minimal Hilbert basis enables the researcher to identify the smallest set of vectors that generate the order relation (the so-called *irreducible transfers*) in contrast with the set of composite transfers (vectors constructed using positive integer combinations of the irreducible transfers). In integer programming for instance, minimal Hilbert bases may be used to identify sequences of vectors that are feasible, given the constraints of the underlying problem, and that improve the value of the objective function. Consider statistical inference for order relations defined on convex cones. When undertaking Monte Carlo simulation (drawing random vectors inside a cone), there is in this context a substantial computational gain from working with a minimal Hilbert basis: any randomly generated integral vector can be constructed in a parsimonious fashion by taking a weighted positive integer sum of the vectors of the basis. As such, minimal Hilbert bases may well take on a prominent role in the exploration of statistical properties of tests for order relations defined on convex cones.

A word of clarification is due regarding the terminology of minimal Hilbert bases. The resulting integral Hilbert basis constructed from the parallelotope method of Giles and Pulleyblank (1979) is in general a superset of the Hilbert basis concept discussed in Magdalou (2021). For this reason, the present paper follows a well-established literature in the mathematical sciences (e.g. Gruber 2007) of distinguishing between a general integral Hilbert basis of a convex cone, and a *minimal* Hilbert basis—the basis concept that underlies the fundamental equivalence theorem of Magdalou (2021).

After reviewing key concepts and definitions in Sect. 2, we turn in Sect. 3 to the characterization of the minimal Hilbert basis of the Hammond order cone. We then introduce in Sect. 4 a class of maximal linearly independent Hilbert bases. There, we discuss the method of extracting the partial sums, the numerical implementation criterion, from the specific minimal Hilbert basis. The results of this section, together with their limitations, are then illustrated in the context of the Hammond order cone, together with two other order relations introduced in Gravel et al. (2021). Section 5 discusses the Hammond order lattice and Sect. 6 concludes. An appendix gathers proofs of various results.

2 The Hammond order cone

The approach we will pursue in this section is to define a general relation $\succeq$ on a convex cone $\mathcal{C}$, the associated parallelotope, and Hilbert basis of the set of integral points of this cone. Subsequently, we shall specialize the relation to a rational cone[3] associated with the Hammond order, and each integral vector of the rational cone will take the form of a difference between two distributions pertaining to a variable defined on k ordered socioeconomic states. The purpose of starting from a general perspective is to enable a distinction between properties that are specific to any convex cone, and those that are specific to the cone associated with the Hammond order.

[3] A cone is said to be a *rational* cone if it is positively spanned by a set of rational vectors. See Definition 1 below.

In what follows the sets $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ respectively denote the integers, rationals and real numbers. We let $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$ denote the non-negative integers, and we likewise define the sets $\mathbb{Q}_+$ and $\mathbb{R}_+$. We begin this section by recalling a few concepts pertaining to order relations. A relation $\succeq$ on $\mathbb{R}^d$ is called a *preorder* if it is transitive and reflexive, and a *partial ordering* if it is transitive, reflexive and antisymmetric[4]. A relation $\succeq$ is additive if for all $x, y, z \in \mathbb{R}^d$, $x \succeq y$ implies $x + z \succeq y + z$. Finally, the relation $\succeq$ is scale invariant if for all $x, y \in \mathbb{R}^d$, and for all $\lambda > 0$, there holds $x \succeq y$ implies $\lambda x \succeq \lambda y$. Following Marshall et al. (1967), an additive and scale invariant partial order relation $\succeq$ may be associated with a pointed convex cone[5] $\mathcal{C} \subseteq \mathbb{R}^d$, whereby $x \succeq y$ if and only if $x - y$ is a vector that belongs to the convex cone $\mathcal{C}$. Under such circumstances, we more simply refer to the relation $\succeq$ as an order induced by a convex cone, or a cone ordering.

In this paper, we shall characterize a minimal Hilbert basis of a rational cone $\mathcal{C}$ in relation to a *positive spanning set* :

Definition 1 Let $\mathcal{V} := \left\{v^1, \dots, v^q\right\}$ denote a finite set of rational vectors in d - dimensional space $\mathbb{Q}^d$. Then,

(i) The positive span of $\mathcal{V}$ is the set of all positive linear combinations of $v^1, \dots, v^q$:

$$\operatorname{pos}(\mathcal{V}) := \left\{\lambda_1 v^1 + \cdots + \lambda_q v^q : \lambda_1, \dots, \lambda_q \in \mathbb{R}_+\right\} \tag{2.1}$$

(ii) The set $\mathcal{V}$ is said to positively span a rational cone $\mathcal{C}$ if $\operatorname{pos}(\mathcal{V}) = \mathcal{C}$.

Note in particular from (i) that any finite set of rational vectors $\mathcal{V}$ is associated with a rational cone $\operatorname{pos}(\mathcal{V})$.

For the purpose of characterizing those integral vectors that belong to the rational cone $\mathcal{C}$, we introduce the following notions of a Hilbert basis.

Definition 2 A Hilbert basis of a finitely generated cone $\mathcal{C} \subseteq \mathbb{R}^d$ is a set of vectors $\left\{h^1, \dots, h^m\right\} \subseteq \mathcal{C}$ such that each vector $z \in \mathcal{C} \cap \mathbb{Z}^d$ is expressible in the form of a positive integer combination $z = \theta_1 h^1 + \cdots + \theta_m h^m$, with $\theta_1, \dots, \theta_m \in \mathbb{Z}_+$. A Hilbert basis $\left\{h^1, \dots, h^m\right\}$ is said to be integral if $\left\{h^1, \dots, h^m\right\} \subseteq \mathbb{Z}^d$. A Hilbert basis $\left\{h^1, \dots, h^m\right\}$ is *minimal* if it is not a superset of any other Hilbert basis of the cone $\mathcal{C}$.

Let $\mathbf{0}_d$ denote a vector of zeroes in $\mathbb{R}^d$. The following result provides the link between the concepts of integral Hilbert basis and positive spanning set of a rational cone (Giles and Pulleyblank 1979; Gruber 2007 p. 349–350).

Lemma 1 *Let $\mathcal{C} \subseteq \mathbb{R}^d$ be a pointed rational cone, and let $\mathcal{V} := \left\{v^1, \dots, v^q\right\}$ be a set of vectors such that pos$(\mathcal{V}) = \mathcal{C}$. Associate with $\mathcal{V}$ a bounded set*

$$\mathcal{P} := \left\{\lambda_1 v^1 + \cdots + \lambda_q v^q : 0 \leq \lambda_1, \dots, \lambda_q \leq 1\right\} \tag{2.2}$$

[4] A relation $\succeq$ on $\mathbb{R}^d$ is called *transitive* if $x \succeq y$ and $y \succeq z$ imply $x \succeq z$ for all $x, y, z \in \mathbb{R}^d$, *reflexive* if $x \succeq x$ for all $x \in \mathbb{R}^d$, and *antisymmetric* if $x \succeq y$ and $y \succeq x$ imply $x = y$ for all $x, y \in \mathbb{R}^d$.

[5] Let $\mathbf{0}_d \in \mathbb{R}^d$ denote a vector of zeroes. A cone $\mathcal{C}$ in $\mathbb{R}^d$ is said to be pointed if for all $x \in \mathcal{C}$ such that $x, -x \in \mathcal{C}$ there holds $x = \mathbf{0}_d$.

(i) *Define* $\left\{h^1, \ldots, h^m\right\}$ *as follows:*

$$\left\{h^1, \ldots, h^m\right\} := \mathcal{P} \cap \mathbb{Z}^d,$$

then the set $\left\{h^1, \ldots, h^m\right\}$ *is an integral Hilbert basis of the cone* $\mathcal{C}$.

(ii) *The set*

$$\left\{h \in \mathcal{C} \cap \mathbb{Z}^d \setminus \{\mathbf{0}_d\} : h \textit{ is not a sum of integral vectors from } \mathcal{C} \cap \mathbb{Z}^d \setminus \{\mathbf{0}_d\}\right\}$$

is the unique minimal Hilbert basis of the cone $\mathcal{C}$.

The set $\mathcal{P}$ is often referred to as a *parallelotope*. The result outlined in Lemma 1 thus constructs the integral Hilbert basis of the rational cone $\mathcal{C}$ as the set of integral vectors contained in the associated parallelotope. Because $\mathcal{P}$ is bounded, it contains a finite set of integral vectors, and accordingly, every pointed rational cone is associated with a (finite) Hilbert basis. Vectors h in part (ii) of the lemma, that are not sums of non-zero integral vectors, will be referred to as *irreducible*. The minimal Hilbert basis can thus be interpreted as the smallest set of integral vectors that is required in order to positively span the entire set $\mathcal{C} \cap \mathbb{Z}^d$.

To give a simple example in two dimensional space, consider a cone $\mathcal{C} := \text{pos}\left(\left\{v^1, v^2\right\}\right)$, where $v^1 := (1, 0)'$ and $v^2 := (0, 2)'$. Then $\mathcal{P} := \{\lambda_1 v^1 + \lambda_2 v^2 : 0 \leq \lambda_1, \lambda_2 \leq 1\}$ is the parallelotope associated with $\mathcal{C}$, and the integral Hilbert basis of this cone, $\mathcal{P} \cap \mathbb{Z}^2$, is given by the set of integral vectors $\left\{\mathbf{0}_2, h^1, h^2, h^3, h^4, h^5\right\}$, where $h^1 := v^1$, $h^2 := \frac{1}{2}v^2$, $h^3 := v^2$, $h^4 := v^1 + \frac{1}{2}v^2$ and $h^5 := v^1 + v^2$. The irreducible vectors associated with this basis are h^1 and h^2, and accordingly the minimal Hilbert basis of $\mathcal{C}$ is given by the subset $\left\{h^1, h^2\right\}$.

We now turn our attention to the comparison of certain types of integral vectors in $\mathbb{R}^k$, that we shall refer to as *distributions* . Let $\mathbb{D}_n^k$ denote the set of distributions of counts pertaining to n data points, defined on k ordered socioeconomic states:

$$\mathbb{D}_n^k := \left\{x \in \mathbb{Z}_+^k : x_1 + \cdots + x_k = n\right\}, \tag{2.3}$$

where $i = 1$ denotes the worst socioeconomic state, and $i = k$ indexes the highest state. For instance, the European statistical agency EUROSTAT collects data on self-assessed health, asking respondents in each participating country to choose one of five possible assessments: *very bad, bad, average, good, or very good*. The state $i = 1$ then corresponds to a very bad health, while $i = k$ pertains to a state of being in very good health. For example, $y = (1, 2, 0, 0, 97)$ is an element of $\mathbb{D}_n^k$, where $k = 5$, $n = 100$, one person rates herself to be in *very bad* health, two rate themselves to be in *bad* health, and 97 respondents rate their health as *very good*.

Consider the following subspace of $\mathbb{R}^k$:

$$\mathbb{S}^k := \left\{s \in \mathbb{R}^k : s_k = -(s_1 + \cdots + s_{k-1})\right\} \tag{2.4}$$

As it is the case that for each pair of distributions x and y in $\mathbb{D}_n^k$, $x-y$ is an integral vector in the space $\mathbb{S}^k$, this space will play a prominent role in our discussion. It is important to observe for results to follow that the maximum size of a linearly independent set in $\mathbb{S}^k$ is equal to $k-1$; that is, the dimension of $\mathbb{S}^k$ is equal to $k-1$.

Consider a social planner whose preferences are defined by a relation $\succeq_G$, associated with a cone $\mathcal{C}_G \subseteq \mathbb{R}^k$. A social welfare function $W : \mathbb{R}^k \to \mathbb{R}$ is order-preserving for the relation $\succeq_G$. Specifically, when used to compare pairs of distributions $x, y \in \mathbb{D}_n^k$, $x \succeq_G y$ implies that social welfare is higher under the dominant distribution: $W(x) \geq W(y)$. At an abstract level, the set of transformations of a distribution $y \in \mathbb{D}_n^k$ that a social planner considers to improve social welfare, defines the set of transfers. Following Magdalou (2021), a finite set of vectors $\mathcal{T}_G := \{g^1, \ldots, g^q\}$ is a set of transfers if for all $g \in \mathcal{T}_G$,

[T1] g can be written as the difference between two distributions in $\mathbb{D}_n^k$, and
[T2] $g \in \mathcal{T}_G$ implies $-g \notin \mathcal{T}_G$.

Observe from [T1] that each $g \in \mathcal{T}_G$ is a rational vector, and from [T2] that the cone $\mathcal{C}_G := \text{pos}(\mathcal{T}_G)$ is pointed. It follows therefore from [T1] and [T2] that the set of transfers $\mathcal{T}_G$ positively spans a pointed rational cone $\mathcal{C}_G := \text{pos}(\mathcal{T}_G)$, associated with the relation $\succeq_G$. Finally, let x and y be two distributions in $\mathbb{D}_n^k$, such that for $\lambda_1, \ldots, \lambda_q \in \mathbb{Z}_+$ and vectors $g^1, \ldots, g^q \in \mathcal{T}_G$, we can write $x - y = \sum_{s=1}^{q} \lambda_s g^s$. Then it is the case that $x - y$ is an integral point of the rational cone $\mathcal{C}_G$, and that $x \succeq_G y$.

We now describe the set of transfers associated with the Hammond order, introduced by Gravel et al. (2021)[6]. In the context of this specific relation, $k \geq 3$ and there are two types of transformations of the distribution of counts $x = (x_1, \ldots, x_k)'$ that may be taken to improve social welfare: *increments* capture the Paretian property, and *Hammond transfers* capture the egalitarian property of the social welfare function (see also Hammond 1976). Let x, y be two distributions in $\mathbb{D}_n^k$. We say that $x = (x_1, \ldots, x_k)'$ is obtained from $y = (y_1, \ldots, y_k)'$ via an increment if for some index $i \in \{1, \ldots, k-1\}$, there holds $x_i = y_i - 1$, $x_{i+1} = y_{i+1} + 1$ and $x_j = y_j$ for all $j \neq i, i+1$. We say that x is obtained from y via an egalitarian Hammond transfer if for indices $h < i \leq j < l$ in the index set $\{1, \ldots, k\}$ there holds $x_h = y_h - 1$, $x_i = y_i + 1$, $x_j = y_j + 1$, $x_l = y_l - 1$ and $x_m = y_m$ for all $m \neq h, i, j, l$. When $i = j$, this definition specializes a Hammond transfer to the form $x_h = y_h - 1$, $x_i = y_i + 2$, $x_l = y_l - 1$ and $x_m = y_m$ for all $m \neq h, i, l$.

For example, if $y = (1, 2, 0, 0, 97)'$ and $x = y + (-1, 1, 0, 0, 0)' = (0, 3, 0, 0, 97)'$, then x is obtained from y via a single increment. On the other hand, if $x = y + (-1, 1, 0, 0, 0)' + (0, -1, 0, 2, -1)'$, that is, $x = (0, 2, 0, 2, 96)'$, we say that x is obtained from y via an increment and a progressive Hammond transfer.

Let $\mathcal{T}_I$ denote the set of increments and $\mathcal{T}_E$ the set of Hammond progressive transfers. We define the set of transfers $\mathcal{T}_H$ associated with the Hammond order as $\mathcal{T}_H := \mathcal{T}_I \cup \mathcal{T}_E$.

[6] Two further relations founded on Hammond's equity principle, that are introduced in Gravel et al. (2021), are discussed more briefly in Sect. 4.2 of this paper.

Example 1 (the set of welfare improving transfers) We describe the set of welfare improving transfers, that is all vectors in $\mathcal{T}_H = \mathcal{T}_I \cup \mathcal{T}_E$, in the context of $k = 4$ socioeconomic states.

The set of increments is given by the following three vectors:

$$\mathcal{T}_I = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

The set of egalitarian Hammond transfers is given by the following five vectors:

$$\mathcal{T}_E = \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \\ -1 \end{pmatrix} \right\}.$$

We shall return to this example in Sect. 3. $\diamondsuit$

We may now define the Hammond order cone as the positive span of the set of transfers $\mathcal{T}_H$:

$$\mathcal{C}_H := \left\{ \sum_{s=1}^{q} \theta_s \tau^s : \theta_1, \ldots, \theta_q \in \mathbb{R}_+, \tau^1, \ldots, \tau^q \in \mathcal{T}_H \right\} \tag{2.5}$$

Because the set of transfers $\mathcal{T}_H$ satisfies the defining properties [T1] and [T2], there results that the Hammond order cone is a pointed rational cone. Let x and y be two distributions in $\mathbb{D}_n^k$, such that $x \succeq_H y$. Then following Gravel et al. (2021), we define x to be the dominant distribution, and the statement $x \succeq_H y$ is definitionally equivalent to x being obtained from y via a *number* of incremental and egalitarian transfers; that is all transformations in $\mathcal{T}_H = \mathcal{T}_I \cup \mathcal{T}_E$.

Because each transfer vector $\tau \in \mathcal{T}_H$ is an integral vector in $\mathbb{S}^k$, we furthermore have that

$$\mathcal{T}_H \subseteq \mathbb{S}^k \cap \mathbb{Z}^k. \tag{2.6}$$

An important question that arises when the spanning vectors in $\mathcal{T}_G$ are integral, is whether $\mathcal{T}_G$ contains an integral Hilbert basis of the cone $\mathcal{C}_G$. For the purpose of investigating this property, we borrow from Magdalou (2021, Definition 3) the following concept of a *minimal set of transfers*:

Definition 3 Let the set of vectors $\mathcal{T}_G := \{g^1, \ldots, g^q\}$ positively span the cone $\mathcal{C}_G$. We shall say that the set of vectors $\mathcal{T}_G$ is minimal if $\mathcal{T}_G$ contains an integral Hilbert basis of $\mathcal{C}_G$.

Following Gruber (2007), we shall call the set of points $\mathcal{C}_H \cap \mathbb{Z}^k$, the integral points of the Hammond order cone. Amongst establishing other properties, the characterization of the minimal Hilbert basis of the Hammond order cone will enable us to study the relation between the set of integral points $\mathcal{C}_H \cap \mathbb{Z}^k$ on the one hand, and between the set of pairs of distributions $x, y \in \mathbb{D}_n^k$ such that $x = y + \sum_{s=1}^{q} \mu_s \tau^s$, where $\mu_1, \ldots, \mu_q \in \mathbb{Z}_+$, and $\tau^1, \ldots, \tau^q \in \mathcal{T}_H$.[7] We shall see in the next section of the paper that the minimality of $\mathcal{T}_H$ is fundamental in clarifying the relation between these two sets.

3 Minimal Hilbert basis

From Definition 2, it follows that the integral points of the Hammond order cone are expressible using various positive integer combinations of the set of vectors that constitute the Hilbert basis. It is possible, therefore, to define every such point $a \in \mathcal{C}_H \cap \mathbb{Z}^k$ as the image of a map from a set of positive integers into the set of distributions $\mathbb{D}_n^k$.

For non-negative integers $\gamma_1, \ldots, \gamma_{k-1}$, consider then the mapping $z_H : \mathbb{Z}_+^{k-1} \rightarrow \mathbb{S}^k \cap \mathbb{Z}^k$ defined as follows:

$$z_H(\gamma_1, \ldots, \gamma_{k-1}) := (-\gamma_1, 2\gamma_1 - \gamma_2, \ldots, 2\gamma_{k-2} - \gamma_{k-1}, \gamma_{k-1} - (\gamma_1 + \cdots + \gamma_{k-2})) \tag{3.1}$$

Consider furthermore the family of vectors

$$\mathcal{Z}_H := \{z_H(\gamma_1, \ldots, \gamma_{k-1}) : \gamma_1, \ldots, \gamma_{k-1} \in \mathbb{Z}_+\}. \tag{3.2}$$

In Lemma 2 below, we shall show that every vector τ in the set of transfers $\mathcal{T}_H$, can be written as the image of some point $(\gamma_1, \ldots, \gamma_{k-1})$ by the map $z_H()$. In Proposition 3, we shall show that the minimal Hilbert basis of the Hammond order cone is given by $k-1$ such vectors $z_H(\gamma_1, \ldots, \gamma_{k-1})$ in the set $\mathcal{Z}_H$. We begin the task of constructing the minimal Hilbert basis by studying some properties of the map $z_H()$ of (3.1).

It is readily verified that for any integer $\alpha \in \mathbb{Z}_+$ and for any vectors $\mu, \theta \in \mathbb{Z}_+^{k-1}$, the following two properties hold:

$$z_H(\alpha\mu_1, \ldots, \alpha\mu_{k-1}) = \alpha z_H(\mu_1, \ldots, \mu_{k-1}) \tag{L1}$$

$$z_H(\mu_1 + \theta_1, \ldots, \mu_{k-1} + \theta_{k-1}) = z_H(\mu_1, \ldots, \mu_{k-1}) + z_H(\theta_1, \ldots, \theta_{k-1}). \tag{L2}$$

From these, it follows in turn that the set $\mathcal{Z}_H$ is generated by $k-1$ elements, namely $z_H(1, 0, \ldots, 0)$, $z_H(0, 1, 0, \ldots, 0), \ldots,$ and $z_H(0, \ldots, 0, 1)$. The family of vectors $\mathcal{Z}_H$ will simplify our task of constructing the minimal Hilbert basis of the Hammond order cone.

Lemma 2 *For each vector τ in the set of transfers $\mathcal{T}_H$, there exist positive integers $\mu_1, \ldots, \mu_{k-1}$ such that $\tau = z_H(\mu_1, \ldots, \mu_{k-1}) \in \mathcal{Z}_H$.*

[7] See Magdalou (2021) for a thorough discussion of the relation between these two sets in a general abstract setting.

That is, for example, we can construct the egalitarian transfer vector $\tau = (-1, 1, 0, 1, -1)'$ of $\mathcal{T}_H \subseteq \mathbb{S}^5 \cap \mathbb{Z}^5$ as follows: $\tau = z_H(\mu_1, \ldots, \mu_4)$, where $(\mu_1, \ldots, \mu_4) = (1, 1, 2, 3)$. We next characterize the minimal Hilbert basis of the Hammond order cone.

Proposition 3 *Let $\mathcal{C}_H$ denote the Hammond order cone (2.5) and $\mathcal{Z}_H$ the family of vectors (3.2).*

(i) *The minimal Hilbert basis of the Hammond order cone consists of the set of $k-1$ vectors $\mathcal{B}_H = \{t^1, \ldots, t^{k-1}\} \subseteq \mathbb{S}^k \cap \mathbb{Z}^k$, where*

$$\begin{aligned} t^1 &:= z_H(1, 0, \ldots, 0), \\ t^2 &:= z_H(0, 1, 0, \ldots, 0), \\ &\vdots \\ t^{k-1} &:= z_H(0, \ldots, 0, 1). \end{aligned} \tag{3.3}$$

(ii) *The set of integral vectors in the Hammond order cone is the family of vectors $\mathcal{Z}_H$.*

The proof of (i) of this proposition consists in first constructing an integral Hilbert basis of the Hammond order cone by identifying the integral vectors of the associated parallelotope, and secondly in associating the minimal Hilbert basis with the subset of non-zero irreducible vectors. Statement (ii) of the proposition is then shown to follow from (i).

More generally, consider a set of transfers $\mathcal{T}_G \subseteq \mathbb{S}^k \cap \mathbb{Z}^k$ and the associated convex cone $\mathcal{C}_G := \text{pos}(\mathcal{T}_G)$. In this general context, the minimal Hilbert basis and mapping $z_G()$ may be obtained by proceeding as follows. First, characterize the set of integral points $\{a^1, \ldots, a^m\}$ in the parallelotope $\mathcal{P}_G$ associated with the set of transfers $\mathcal{T}_G$. From Lemma 1, the set of points $\{a^1, \ldots, a^m\}$ is an integral Hilbert basis of the cone ordering $\mathcal{C}_G$. Next, characterize the subset $\{b^1, \ldots, b^l\}$ of irreducible elements from the integral Hilbert basis $\{a^1, \ldots, a^m\}$. Again, from Lemma 1, the vectors $b^1, \ldots, b^l$ jointly constitute the minimal Hilbert basis of the cone ordering $\mathcal{C}_G$. Then it is possible to construct a mapping $z_G : \mathbb{Z}^l_+ \longrightarrow \mathbb{S}^k \cap \mathbb{Z}^k$ as

$$z_G(\theta_1, \ldots, \theta_l) := \theta_1 b^1 + \cdots + \theta_l b^l, \tag{3.4}$$

and to equate the set of integral points of the cone $\mathcal{C}_G$ with the set of points $\mathcal{Z}_G := \{z_G(\theta_1, \ldots, \theta_l) : \theta_1, \ldots, \theta_l \in \mathbb{Z}_+\}$. The set $\mathcal{Z}_G$ is generated by the l vectors $z_G(1, 0, \ldots, 0), \ldots, z_G(0, \ldots, 0, 1)$ that define the minimal Hilbert basis of the cone $\mathcal{C}_G$.

One property that emerges from Proposition 3, is that the minimal Hilbert basis of the Hammond order cone takes the form of a set of $k-1$ linearly independent vectors (see the illustrative example that follows for further detail). The linear independence property will be put to good use in extending the results of this paper to a general class of cone orderings (see Sect. 4). This independence property will further prove useful in Sect. 5, where we introduce a space that we call the *Hammond order lattice*.

Example 1 (continued) Returning to the context $k = 4$ of Example 1, we may illustrate the result of Proposition 3 as follows. First define the three spanning vectors $z_H(1,0,0) = (-1,2,0,-1)' := t^1$, $z_H(0,1,0) = (0,-1,2,-1)' := t^2$, and $z_H(0,0,1) = (0,0,-1,1)' := t^3$.

It is routinely verified that these three vectors are linearly independent, and therefore irreducible. For the remaining five vectors of the set of transfers associated with the Hammond order cone, we obtain:

$$\begin{gathered}(-1,1,0,0)' = t^1 + t^2 + 2t^3 \\ (0,-1,1,0)' = t^2 + t^3 \\ (-1,1,1,-1)' = t^1 + t^2 + t^3 \\ (-1,2,-1,0)' = t^1 + t^3 \\ (-1,0,2,-1)' = t^1 + 2t^2 + 2t^3\end{gathered} \tag{3.5}$$

Thus, while the eight vectors in the set $\mathcal{T}_H$ jointly characterize an integral Hilbert basis of the Hammond order cone of Example 1, the unique minimal Hilbert basis is given by the set $\mathcal{B}_H = \left\{t^1, t^2, t^3\right\}$. ◊

One immediate application of the minimal Hilbert basis of the Hammond order relation is to enable a distinction between the irreducible transfers (the vectors of the minimal Hilbert basis) and those other transfers that arise as positive integer combinations of vectors of the minimal Hilbert basis. Returning to Example 1, $t^1 = (-1,2,0-1)'$ and $t^2 = (0,-1,2,-1)'$ are examples of irreducible transfers. On the other hand, $\tau = (-1,1,1,-1) \in \mathcal{T}_H$ is a combination of irreducible transfers, in the sense that $\tau = t^1 + t^2 + t^3$. We call τ, and other positive integer combinations of irreducible transfers, *composite transfers*.

From Proposition 3, it emerges that the minimal Hilbert basis of the Hammond order cone arises as a subset of the set of transfers. That is, the set of transfers $\mathcal{T}_H$ is *minimal* in the sense of Magdalou (2021) and Definition 3. In turn, it is therefore possible to express the integral vectors of the Hammond order cone as integer combinations of the elements of the set of transfers:

$$\mathcal{C}_H \cap \mathbb{Z}^k = \left\{\sum_{s=1}^{q} \mu_s \tau^s : \mu_1, \ldots, \mu_q \in \mathbb{Z}_+, \tau^1, \ldots, \tau^q \in \mathcal{T}_H\right\} \tag{3.6}$$

The minimality of the set of transfers $\mathcal{T}_H$ then enables us to equate the integral points of the Hammond order cone $\mathcal{C}_H$ with pairs of distributions (x, y) such that x has higher social welfare than y.

4 The partial sums of the Hammond order relation

To render a cone ordering implementable on survey data, a criterion is needed to enable the data analyst to deduce which (if any) of two distributions x and y exhibits higher social welfare. Such an implementable criterion has been derived in Theorem 3 of Gravel et al. (2021), where the authors show that x dominates y if and only if $k - 1$

partial sums inequalities are satisfied:

$$x \succeq_H y \Longleftrightarrow \sum_{i=1}^{j} 2^{j-i}(x_i - y_i) \leq 0 \quad \text{for all } j = 1, \ldots, k-1. \tag{4.1}$$

Proposition 4 below shows that this numerical representation of the Hammond order relation is readily available from the minimal Hilbert basis. The result linking the minimal Hilbert basis to the implementation criterion, is further generalized in Proposition 5 in the context of a family of cone orderings.

For any vector $a \in \mathbb{Z}^k$, associate a with a vector $\widehat{a} = (a_1, \ldots, a_{k-1})' \in \mathbb{Z}^{k-1}$. Via this transformation, it will be meant that $\widehat{a}$ is the projection of a on its first $k-1$ coordinates.

Proposition 4 *Let x and y be two distributions in $\mathbb{D}_n^k$, and let $\widehat{x}$, $\widehat{y}$, respectively denote the projection of x and y on their first $k-1$ coordinates. For each vector t^i in the minimal Hilbert basis $\mathcal{B}_H$, likewise define $\widehat{t^i}$ as the projection of t^i on its first $k-1$ coordinates, and construct the matrix $B \in \mathbb{Z}^{(k-1)\times(k-1)}$ as $B := \left(\widehat{t}^1, \cdots, \widehat{t}^{k-1}\right)$. Then, there holds $x \succeq_H y$ if and only if $-B^{-1}(\widehat{x} - \widehat{y}) \leq \mathbf{0}_{k-1}$.*

Proof Let x and y denote two distributions in $\mathbb{D}_n^k$, such that

$$x - y = \sum_{s=1}^{q} \theta_s \tau^s$$

with $\theta_1, \ldots, \theta_q \in \mathbb{Z}_+$ and such that $\tau^1, \ldots, \tau^q \in \mathcal{T}_H$. From Proposition 3, the vectors $t^1, \ldots, t^{k-1}$ constitute a minimal Hilbert basis for the Hammond order cone $\mathcal{C}_H$, so that it is also the case that for some $\mu_1, \ldots, \mu_{k-1} \in \mathbb{Z}_+$ we have $x - y = \sum_{j=1}^{k-1} \mu_j t^j$.

Observe from Lemma 2 and Proposition 3 that the elements b_{ij} of the matrix B all take values in the set $\{-1, 0, 2\}$. Therefore, B of the form

$$B = \begin{pmatrix} -1 & & & & \\ 2 & -1 & & & \\ 0 & 2 & -1 & & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 2 & -1 \end{pmatrix}, \tag{4.2}$$

where the above diagonal blank entries of the matrix are all zero elements, so that B is lower-triangular. It is readily verified that the matrix B is invertible, and furthermore that $B^{-1} = -A$, where A is the matrix of the form

$$A = \begin{pmatrix} 1 & & & & \\ 2 & 1 & & & \\ 4 & 2 & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \\ 2^{k-2} & \cdots & 4 & 2 & 1 \end{pmatrix} \tag{4.3}$$

Gathering the positive integers in a vector $\mu := (\mu_1, \ldots, \mu_{k-1})'$, we obtain the following equivalent statements:

$$\begin{aligned} x - y = \sum_{j=1}^{k-1} \mu_j t^j &\Updownarrow \\ \widehat{x} - \widehat{y} = B\mu &\Updownarrow \\ B^{-1}(\widehat{x} - \widehat{y}) \geq \mathbf{0}_{k-1} &\Updownarrow \\ A(\widehat{x} - \widehat{y}) \leq \mathbf{0}_{k-1} &\Updownarrow \\ \sum_{i=1}^{j} 2^{j-i}(x_i - y_i) \leq 0 \text{ for all } j = 1, \ldots, k-1 &\Longleftrightarrow x \succeq_H y. \end{aligned}$$

where the last equivalence is the result (4.1) from Gravel et al. (2021), Theorem 3, (a)$\Longleftrightarrow$ (c). □

4.1 A class of maximal linearly independent Hilbert bases

In order to better understand the specific properties of the Hammond cone that underlie the above result relating the partial sums to the vectors of the minimal Hilbert basis, we first take a closer look at a simple example: the order $\succeq_I$ induced by the set of increments $\mathcal{T}_I$, otherwise known as first order stochastic dominance in the context of a variable defined on k ordered socioeconomic states. We then generalize the discussion to a more general class of cone orderings.

Returning to Example 1, denote the three vectors defining the set $\mathcal{T}_I$ of increments as follows: $p^1 := (-1, 1, 0, 0)'$, $p^2 := (0, -1, 1, 0)'$, and $p^3 := (0, 0, -1, 1)'$. We then associate the relation $\succeq_I$ with a pointed rational cone $\mathcal{C}_I := \mathrm{pos}(\mathcal{T}_I)$. It is readily verified in this simple case that the minimal Hilbert basis of the set of discrete points $\mathcal{C}_I \cap \mathbb{Z}^4$ coincides with the set of transfers: $\mathcal{B}_I = \{p^1, p^2, p^3\}$. As in Proposition 4, we let the vector $\widehat{p}$ denote the first $k-1$ components of p, and proceed to construct the 3×3 integral matrix $P_I := (\widehat{p}^1\ \widehat{p}^2\ \widehat{p}^3)$. We thus deduce that P_I is invertible, and that $Q_I = -P_I^{-1}$ is of the form

$$Q_I = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{pmatrix}. \tag{4.4}$$

In the context of discrete first order stochastic dominance, it is a well known result that for two distributions x and y in $\mathbb{D}_n^k$, there holds $x \succeq_I y$ if and only if $Q_I(\widehat{x} - \widehat{y}) \leq \mathbf{0}_{k-1}$ where Q_I takes the form of a $(k-1)$-dimensional lower triangular matrix of ones, or equivalently, in this example, if and only if $x_1 \leq y_1$, $x_1 + x_2 \leq y_1 + y_2$ and $x_1 + x_2 + x_3 \leq y_1 + y_2 + y_3$. That is, in the case of the cone ordering $\succeq_I$, as is the case in the context of the Hammond ordering, inversion of a matrix easily extracted from the minimal Hilbert basis also produces the desired numerical representation of the order relation.

The common property the two minimal Hilbert bases $\mathcal{B}_I$ and $\mathcal{B}_H$ share, is that they belong to a class of bases constructed from sets of maximal linearly independent vectors. We define this class as follows:

Definition 4 A minimal Hilbert basis $\mathcal{B}$ is said to belong to the set of maximal linearly independent Hilbert bases of $\mathbb{S}^k \cap \mathbb{Z}^k$ if $\mathcal{B}$ belongs to the set

$$\mathcal{M} := \Big\{ \big\{ b^1, \ldots, b^{k-1} \big\} \text{ is a minimal Hilbert basis in } \mathbb{S}^k \cap \mathbb{Z}^k : \\ b^1, \ldots, b^{k-1} \text{ are linearly independent.} \Big\} \tag{4.5}$$

The set of bases we consider therefore has three defining properties: (i) each vector b^i of the basis is an integral vector of the $(k-1)$ -dimensional subspace $\mathbb{S}^k$, (ii) the basis $\mathcal{B}$ consists of $k-1$ vectors, and (iii) the vectors of the basis $\mathcal{B}$ are linearly independent. Proposition 4 is generalized below in relation to a minimal Hilbert basis that belongs to the set $\mathcal{M}$, but is otherwise not explicitly specified.

Proposition 5 *Let $\succeq_M$ be a cone ordering associated with a pointed rational cone $\mathcal{C}_M \subseteq \mathbb{R}^k$. Assume that the set of integral vectors $\mathcal{C}_M \cap \mathbb{Z}^k$ is associated with a minimal Hilbert basis $\mathcal{B}_M = \{p^1, \ldots, p^{k-1}\} \subseteq S^k \cap \mathbb{Z}^k$, such that $\mathcal{B}_M$ is an element of the set $\mathcal{M}$ of maximal linearly independent Hilbert bases. Then, defining the matrix $P := \left(\widehat{p}^1 \cdots \widehat{p}^{k-1}\right) \in \mathbb{Z}^{(k-1)\times(k-1)}$,*

(i) *the matrix P is invertible, and*
(ii) *for all distributions x and y in $\mathbb{D}^k_n$, there holds $x \succeq_M y$ if and only if $-P^{-1}(\widehat{x} - \widehat{y}) \leq \mathbf{0}_{k-1}$.*

4.2 Hammond order trilogy

In order to better understand the crucial role played by the linear independence property of the class $\mathcal{M}$ of Hilbert bases of $\mathbb{S}^k \cap \mathbb{Z}^k$, we examine two further cone orderings that were introduced in Gravel et al. (2021) together with the Hammond order $\succeq_H$[8].

Let x, y be two distributions in $\mathbb{D}^k_n$. We say that $x = (x_1, \ldots, x_k)'$ is obtained from $y = (y_1, \ldots, y_k)'$ via a decrement if for some index $i \in \{2, \ldots, k\}$, there holds $x_i = y_i - 1, x_{i-1} = y_{i-1} + 1$ and $x_j = y_j$ for all $j \neq i, i-1$. Let $\mathcal{T}_D \subseteq \mathbb{S}^k \cap \mathbb{Z}^k$ denote the set of decrements. Together with the Hammond order cone $\mathcal{C}_H$ =pos$(\mathcal{T}_I \cup \mathcal{T}_E)$, we consider two further cones

$$\mathcal{C}_E := \text{pos}(\mathcal{T}_E) \tag{4.6}$$
$$\mathcal{C}_F := \text{pos}(\mathcal{T}_D \cup \mathcal{T}_E). \tag{4.7}$$

$\mathcal{C}_E$ is the cone spanned by the set of egalitarian Hammond transfers. The cone $\mathcal{C}_F$, spanned by the union of the set of decrements and Hammond transfers, has a structure that is very similar to that of the Hammond order cone, as we shall see in Example 2

[8] I thank a reviewer for suggesting the discussion around these two order relations, as well as Example 2 below.

below. Associate with each of $\mathcal{C}_E$ and $\mathcal{C}_F$, order relations $\succeq_E$ and $\succeq_F$ on $\mathbb{R}^k$. Observe then, as discussed in Gravel et al. (2021), that the cone ordering $\succeq_E$ is the intersection of the relations $\succeq_H$ and $\succeq_F$.

One application of the class $\mathcal{M}$ of Hilbert bases (4.5) consists in deriving the partial sums associated with the order relation $\succeq_F$, a result obtained in Theorem 4 of Gravel et al. (2021). Specifically, let x and y denote two distributions in $\mathbb{D}_n^k$. The authors show that x dominates y if and only if the following $k-1$ partial sums inequalities are satisfied:

$$x \succeq_F y \Longleftrightarrow \sum_{i=j}^{k} 2^{i-j}(x_i - y_i) \leq 0 \quad \text{for all } j = 2, \ldots, k. \tag{4.8}$$

We propose a different derivation of this result, that arises as an application of Proposition 5. First, we derive the minimal Hilbert bases of the cones $\mathcal{C}_E$ and $\mathcal{C}_F$, via the parallelotope method outlined in Lemma 1.

Proposition 6 (a) *Let $\mathcal{C}_E = pos\ (\mathcal{T}_E)$ denote the cone of egalitarian Hammond transfers. The minimal Hilbert basis $\mathcal{B}_E$ of the cone $\mathcal{C}_E$ consists of the set of vectors of the form*

$$e := (\mathbf{0}_{j-1}, -1, \mathbf{0}_{h-1}, 2, \mathbf{0}_{l-1}, -1, \mathbf{0}_{k-(j+h+l)})' \tag{4.9}$$

where j, h and l are strictly positive integers, such that $j + h + l \leq k$. For $k \geq 3$, there are $k(k-1)(k-2)/6$ such vectors in the minimal Hilbert basis.

(b) *Let $\mathcal{C}_F = pos\ (\mathcal{T}_D \cup \mathcal{T}_E)$ denote the cone of decrements and egalitarian Hammond transfers. The minimal Hilbert basis $\mathcal{B}_F$ of the cone $\mathcal{C}_F$ consists of the following $k-1$ linearly independent vectors:*

$$\begin{aligned} f^1 &:= (1, -1, 0, \ldots, 0)' \\ f^2 &:= (-1, 2, -1, 0, \ldots, 0)' \\ f^3 &:= (-1, 0, 2, -1, 0, \ldots, 0)' \\ &\vdots \\ f^{k-1} &:= (-1, 0, \ldots, 0, 2, -1)'. \end{aligned} \tag{4.10}$$

Let $l := k(k-1)(k-2)/6$ (l is the number of vectors in the minimal Hilbert basis of the cone $\mathcal{C}_E$). It is then possible to construct a mapping $z_E : \mathbb{Z}_+^l \longrightarrow \mathbb{S}^k \cap \mathbb{Z}^k$ as

$$z_E(\theta_1, \ldots, \theta_l) := \theta_1 e^1 + \cdots + \theta_l e^l, \tag{4.11}$$

and to equate the set of integral points of the cone $\mathcal{C}_E$ with the set of points $\mathcal{Z}_E := \{z_E(\theta_1, \ldots, \theta_l) : \theta_1, \ldots, \theta_l \in \mathbb{Z}_+\}$. The set $\mathcal{Z}_E$ is generated by the l vectors $z_E(1, 0, \ldots, 0) = e^1, \ldots, z_E(0, \ldots, 0, 1) = e^l$ that define the minimal Hilbert basis of the cone $\mathcal{C}_E$. For the cone $\mathcal{C}_F$, we likewise construct the mapping $z_F : \mathbb{Z}_+^{k-1} \longrightarrow \mathbb{S}^k \cap \mathbb{Z}^k$ and the associated set of integral vectors $\mathcal{Z}_F := \{z_F(\theta_1, \ldots, \theta_{k-1}) : \theta_1, \ldots, \theta_{k-1} \in \mathbb{Z}_+\}$ generated by the $k-1$ vectors

$z_F(1, 0, \ldots, 0) = f^1, \ldots, z_F(0, \ldots, 0, 1) = f^{k-1}$. We illustrate these results with the help of the following example.

Example 2 Return to the context of $k = 4$ socioeconomic states, discussed in Example 1. Then in the context of the order relation $\succeq_F$, the set of welfare improving transfers is the set $\mathcal{T}_F = \mathcal{T}_D \cup \mathcal{T}_E$. The following three vectors constitute the set of decrements:

$$\mathcal{T}_D = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}. \tag{4.12}$$

The set of egalitarian Hammond transfers is the set of five vectors $\mathcal{T}_E$ given in Example 1. The same five vectors of $\mathcal{T}_E$ define the entire set of transfers underlying the order relation $\succeq_E$.

The set of vectors that constitute the minimal Hilbert basis of the cone $\mathcal{C}_F$ specializes (4.10) to the following three vectors:

$$\begin{aligned} f^1 &= (1, -1, 0, 0)' \\ f^2 &= (-1, 2, -1, 0)' \\ f^3 &= (-1, 0, 2 - 1)'. \end{aligned} \tag{4.13}$$

and the map $z_F()$ specializes to $z_F(\theta_1, \theta_2, \theta_3) = (\theta_1 - (\theta_2 + \theta_3), -\theta_1 + 2\theta_2, -\theta_2 + 2\theta_3, -\theta_3)$. The remaining five vectors of the set of transfers $\mathcal{T}_F = \mathcal{T}_D \cup \mathcal{T}_E$ are composite transfers in the sense that they arise as positive integer combinations of f^1, f^2, and f^3:

$$\begin{aligned} (0, 0, 1, -1)' &= 2f^1 + f^2 + f^3 \\ (0, 1, -1, 0)' &= f^1 + f^2 \\ (-1, 1, 1, -1)' &= f^1 + f^2 + f^3 \\ (-1, 2, 0, -1)' &= 2f^1 + 2f^2 + f^3 \\ (0, -1, 2, -1)' &= f^1 + f^3 \end{aligned} \tag{4.14}$$

In the context of $k = 4$ socioeconomic states, the minimal Hilbert basis $\mathcal{B}_E$ of the cone of egalitarian Hammond transfers specializes to the following set of four vectors:

$$\begin{aligned} e^1 &:= (-1, 2, -1, 0)' = f^2 \\ e^2 &:= (-1, 2, 0, -1)' = t^1 \\ e^3 &:= (-1, 0, 2, -1)' = f^3 \\ e^4 &:= (0, -1, 2, -1)' = t^2. \end{aligned} \tag{4.15}$$

The map $z_E()$ specializes to $z_E(\theta_1, \theta_2, \theta_3, \theta_4) = (-\theta_1 - \theta_2 - \theta_3, 2\theta_1 + 2\theta_2 - \theta_4, -\theta_1 + 2\theta_3 + 2\theta_4, -\theta_2 - \theta_3 - \theta_4)$ and we observe that the vector $(-1, 1, 1, -1)$ of the set of transfers $\mathcal{T}_E$ is composite, in the sense that $(-1, 1, 1, -1) = e^1 + e^4$. ◇

Because the minimal Hilbert basis $\mathcal{B}_F = \left(f^1, \ldots, f^{k-1}\right)$ consists of $k - 1$ linearly independent integral vectors of the subspace $\mathbb{S}^k$, $\mathcal{B}_F$ belongs to the class of minimal

Hilbert bases of Definition 4. As such, it is possible to apply Proposition 5 in order to derive the partial sums associated with the relation $\succeq_F$.

Note that the set of partial sums of Gravel et al. (2021), in the context of the order relation $\succeq_F$, is expressed in terms of variables $x_2, \ldots, x_k$. Consider first the following $(k-1) \times (k-1)$ upper triangular matrix:

$$P_F = \begin{pmatrix} -1 & 2 & 0 & \cdots & 0 \\ & -1 & 2 & \ddots & \vdots \\ & & -1 & \ddots & 0 \\ & & & \ddots & 2 \\ & & & & -1 \end{pmatrix}. \tag{4.16}$$

Observe that P_F is a matrix extracted from the associated minimal Hilbert basis $\mathcal{B}_F$, in the sense that column j of the matrix P_F is of the form $\widetilde{f}^j := (f_2^j, \ldots, f_k^j)'$. That is, $\widetilde{f}^j$ the projection of the irreducible vector f^j of the minimal Hilbert basis on its second, to k-th, components[9]. If we invert P_F, and multiply by -1 the resulting matrix, we obtain a matrix $Q_F := -P_F^{-1}$, of the form

$$Q_F = \begin{pmatrix} 1 & 2 & 2^2 & \cdots & 2^{k-2} \\ & 1 & 2 & \cdots & 2^{k-3} \\ & & 1 & & \vdots \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}. \tag{4.17}$$

The resulting partial sums, for $x, y \in \mathbb{D}_n^k$ are then of the form $Q_F(\tilde{x} - \tilde{y}) \leq \mathbf{0}_{k-1}$, that is the expression (4.8) of Gravel et al. (2021).

On the other hand, as illustrated in Example 2, the minimal Hilbert basis $\mathcal{B}_E$ of the cone of egalitarian Hammond transfers is not made of $k-1$ linearly independent vectors. As such $\mathcal{B}_E$ is not an element of the class $\mathcal{M}$ of minimal Hilbert bases, and it is not possible to extract an invertible matrix from $\mathcal{B}_E$ that produces a set of partial sums for the order relation $\succeq_E$.

It is shown in Theorem 5 of Gravel et al. (2021) that for a pair of distributions x, y in $\mathbb{D}_n^k$, $x \succeq_E y$ if and only if the two sets of partial sum inequalities (4.1) and (4.8) are satisfied. This result arises because $x \succeq_E y$ implies that both x can be constructed from y via a sequence of Hammond transfers and increments (that is $x \succeq_H y$) and that x can be constructed from y via a sequence of Hammond transfers and decrements (that is $x \succeq_F y$). As such, it must be possible to express any vector $e \in \mathcal{C}_E$ using either the $k-1$ vectors $t^1, \ldots, t^{k-1}$ or $f^1, \ldots, f^{k-1}$ that define respectively the minimal

[9] We note that it is also possible to obtain an alternative set of partial sums by constructing P_F using the vectors $\widehat{f}^1, \ldots, \widehat{f}^{k-1}$, where $\widehat{f}^j := (f_1^j, \ldots, f_{k-1}^j)$. When comparing a pair of distributions $x, y \in \mathbb{D}_n^k$, this latter construction would entail a set of $k-1$ partial sums to be applied to the vector $(\widehat{x} - \widehat{y}) = (x_1 - y_1, \ldots, x_{k-1} - y_{k-1})$. We have opted instead for the construction based on $\widetilde{f}^j := (f_2^j, \ldots, f_k^j)$ in order to obtain the partial sums of Gravel et al. (2021).

Hilbert basis associated with the cones $\mathcal{C}_H$ or $\mathcal{C}_F$. In the appendix, we further explore this point in an extension of Example 2.

5 Hammond order lattice

Our final application of the minimal Hilbert basis of the Hammond order cone is to introduce a space in which more complete relations (order extensions, in technical jargon) of the Hammond order $\succeq_H$ may be defined. Formally, a partial ordering $\succeq_G$ on $\mathbb{R}^k$ is an order extension of $\succeq_H$, if it is the case that for all x and y such that $x \succeq_H y$, there also holds $x \succeq_G y$[10].

A geometric lattice $\mathcal{L}$ in $\mathbb{R}^k$ is the set of all integer (positive or negative) combinations of k linearly independent vectors $b^1, \dots, b^k \in \mathbb{R}^k$.[11] Because the maximum size of a linearly independent set in $\mathbb{S}^k$ is equal to $k-1$, we may define a lattice in association with the minimal Hilbert basis of the Hammond order cone as follows:

Definition 5 The Hammond order lattice $\mathcal{L}_H \subseteq \mathbb{R}^k$ is the set of all integer (positive and negative) combinations of the $k-1$ linearly independent vectors of the minimal Hilbert basis $\mathcal{B}_H$:

$$\mathcal{L}_H := \left\{ \gamma_1 t^1 + \cdots + \gamma_{k-1} t^{k-1} : \gamma_1, \dots, \gamma_{k-1} \in \mathbb{Z}, t^1, \dots, t^{k-1} \in \mathcal{B}_H \right\} \tag{5.1}$$

A square matrix U is said to be unimodular if it is integral, and $\det(U) \in \{-1, 1\}$. If $(\mathcal{B}_H)$ is the matrix whose columns are the $k-1$ vectors defining the minimal Hilbert basis of the Hammond order lattice, then every other basis $\mathcal{D}_H$ of the Hammond order lattice relates to $\mathcal{B}_H$ via the equality $(\mathcal{D}_H) = (\mathcal{B}_H)\,U$, for some unimodular matrix $U \in \mathbb{Z}^{(k-1)\times(k-1)}$. That is, all bases of the Hammond order lattice are unique up to multiplication by a unimodular matrix.

It is clear from Proposition 3 that the set of integral points of the Hammond order cone, $\mathcal{C}_H \cap \mathbb{Z}^k$, is a subset of the Hammond order lattice. The study of the lattice however is particularly useful, since all order extensions of the Hammond order cone are subsets of $\mathcal{L}_H$. Let $\mathcal{T}_G \subseteq \mathbb{S}^k \cap \mathbb{Z}^k$ denote a set of transfers, and associate with $\mathcal{T}_G$ a cone $\mathcal{C}_G := \text{pos}(\mathcal{T}_G)$ as well as an order relation $\succeq_G$ on $\mathbb{R}^k$.

Proposition 7 *Let the cone ordering $\succeq_G$ on $\mathbb{R}^k$ be an order extension of the Hammond order relation $\succeq_H$. Then,*

(i) *the set of integral points of the cone $\mathcal{C}_H \cap \mathbb{Z}^k$ is a subset of $\mathcal{C}_G \cap \mathbb{Z}^k$. Furthermore,*
(ii) the set of integral points $\mathcal{C}_G \cap \mathbb{Z}^k$ is a subset of the Hammond order lattice $\mathcal{L}_H$.

As an illustrative example, consider the Hammond order $\succeq_H$ in the simple case where $k = 3$. Then, from Proposition 3, we can use the definition of the Hilbert basis to write $\mathcal{C}_H = \text{pos}\left\{t^1, t^2\right\}$, where $t^1 := (-1, 2, -1)'$ and $t^2 := (0, -1, 1)'$. A pointed

[10] For example, both relations $\succeq_H$ and $\succeq_F$ are order extensions of the relation $\succeq_E$ associated with the set of egalitarian Hammond transfers $\mathcal{T}_E$.

[11] See chapter 21 of Gruber (2007) for a general discussion of geometric lattices and their bases.

cone $\mathcal{C}_G$ associated with an order relation $\succeq_G$ may be defined as $\mathcal{C}_G := \text{pos}\left\{g^1, g^2\right\}$, where $g^1 := (-1, 3, -2)'$ and $g^2 := t^2$. That is, a social planner with preferences given by $\succeq_G$ views a movement of one person from the bottom socioeconomic class toward the middle class, together with a movement of two individuals from the top class toward the middle, as welfare improving. By observing that $t^1 = g^1 + g^2$, it may be readily verified that $\mathcal{C}_H \subseteq \mathcal{C}_G$, and that the relation $\succeq_G$ is an order extension of the Hammond order relation.

For example, take a pair of distributions x^1 and y^1 in $\mathbb{D}_n^3$ such that $x^1 - y^1 = (-1, 1, 0)'$. Then we have $x^1 - y^1 = t^1 + t^2$, and furthermore $x^1 - y^1 = g^1 + 2g^2$; hence it is the case that $x^1 \succeq_H y^1$ and $x^1 \succeq_G y^1$. Next consider a pair of distributions x^2 and y^2 such that $x^2 - y^2 = (-1, 3, -2)'$. Then, there holds $x^2 - y^2 = g^1$, but $x^2 - y^2 = t^1 - t^2$. In this second example, we then have $x^2 \succeq_G y^2$, while x^2 and y^2 are not comparable according to the relation $\succeq_H$, in accordance with statement (i) of Proposition 7. Observe finally that because the minimal Hilbert basis of $\mathcal{C}_G$ is given by the vectors $g^1 = t^1 - t^2$ and $g^2 = t^2$, any vector $a \in \mathcal{C}_G \cap \mathbb{Z}^3$ may be written in the form $a = \gamma_1 g^1 + \gamma_2 g^2 = \gamma_1 t^1 + (\gamma_2 - \gamma_1)t^2$ where $\gamma_1, \gamma_2 \in \mathbb{Z}_+$. That is, the set of integral points $\mathcal{C}_G \cap \mathbb{Z}^3$ are contained in the Hammond order lattice $\mathcal{L}_H$, in accordance with (ii) of Proposition 7.

6 Conclusions

In this paper we have characterized the minimal Hilbert basis of the Hammond cone, and we have discussed several novel applications of this basis. We have shown how to derive the implementation criterion of the Hammond order relation, the $k - 1$ partial sums derived by Gravel et al. (2021), via the inversion of a matrix that is extracted from the Hilbert basis. We have furthermore introduced a class of maximal linearly independent Hilbert bases, from which it is similarly possible to derive the implementation criterion from the minimal Hilbert basis. The basis also enabled us to introduce a new space—that we have called the Hammond order lattice—where order-extensions of the Hammond order may be derived.

We conclude by mentioning some limitations of cone orderings and their associated Hilbert bases. One criticism that may be formulated, is that defining the set of transfers in the form of a minimal Hilbert basis is lacking in transparency. Returning to Example 1, while three vectors span the entire set of integral vectors, there is certainly some work required in reconstructing the entire set of transfers (three increments and five progressive Hammond transfers) from this basis. When one considers it a priority to clarify the value judgements underlying a particular relation used to order distributions, the set of transfers provides clarity that the minimal Hilbert basis does not possess. More importantly, one must note that order relations induced by cones are additive. As such, the results obtained in this paper do not apply in the context of several important contributions in the field, for instance Chateauneuf and Moyes (2006).

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Appendix

Proof of Lemma 2 We divide the proof in three parts. Firstly, letting $\mathbf{0}_n$ denote an n-dimensional zero vector, we show that each increment vector of the form $\tau^1 = (\mathbf{0}_{j-1}, -1, 1, \mathbf{0}_{k-(j+1)}) \in \mathcal{T}_H$ is an element of the family $\mathcal{Z}_H$. Then, we consider Hammond transfers of the form $\tau^2 = (\mathbf{0}_{j-1}, -1, \mathbf{0}_{h-1}, 1, \mathbf{0}_{l-1}, 1, \mathbf{0}_{m-1}, -1, \mathbf{0}_{k-(j+h+l+m)})$, where j, h, k, l and m are arbitrary non-negative integers. Finally we consider Hammond transfers of the form $\tau^3 = (\mathbf{0}_{j-1}, -1, \mathbf{0}_{h-1}, 2, \mathbf{0}_{l-1}, 1, \mathbf{0}_{k-(j+h+l)})$.

(i) Let $\tau^1 = (\mathbf{0}_{j-1}, -1, 1, \mathbf{0}_{k-(j+1)})$. Then $\tau^1 = z_H(\mu_1, \ldots, \mu_{k-1}) \in \mathcal{Z}_H$ where,

$$\mu_i = \begin{cases} 0, & i = 1, \ldots, j-1 \\ 1, & i = j \\ 2^{i-(j+1)}, & i = j+1, \ldots, k-1 \end{cases} \tag{6.1}$$

(ii) Consider next a generic Hammond transfer of the form τ^2. Define the following positive constants:

$$\begin{aligned} \omega_1 &= 2^h - 1 \\ \omega_2 &= 2^l \omega_1 - 1 \\ \omega_3 &= 2^m \omega_2 + 1 \end{aligned} \tag{6.2}$$

Then $\tau^2 = z_H(\mu_1, \ldots, \mu_{k-1}) \in \mathcal{Z}_H$ where,

$$\mu_i = \begin{cases} 0, & i = 1, \ldots, j-1 \\ 2^{i-j}, & i = j, \ldots, j+h-1 \\ 2^{i-(j+h)}\omega_1, & i = j+h, \ldots, j+h+l-1 \\ 2^{i-(j+h+l)}\omega_2, & i = j+h+l, \ldots, j+h+l+m-1 \\ 2^{i-(j+h+l+m)}\omega_3, & i = j+h+l+m, \ldots, k-1 \end{cases} \tag{6.3}$$

(iii) Finally, consider a generic Hammond transfer of the form τ^3. Define the following positive constants:

$$\begin{aligned} \kappa_1 &= 2^h - 2 \\ \kappa_2 &= 2^l \kappa_1 + 1 \end{aligned} \tag{6.4}$$

Then $\tau^3 = z_H(\mu_1, \ldots, \mu_{k-1}) \in \mathcal{Z}_H$ where,

$$\mu_i = \begin{cases} 0, & i = 1, \ldots, j-1 \\ 2^{i-j}, & i = j, \ldots, j+h-1 \\ 2^{i-(j+h)}\kappa_1, & i = j+h, \ldots, j+h+l-1 \\ 2^{i-(j+h+l)}\kappa_2, & i = j+h+l, \ldots, k-1 \end{cases} \tag{6.5}$$

□

Proof of Proposition 3 (i) Since $\mathcal{C}_H = \text{pos}(\mathcal{T}_H)$, it follows from (i) of Lemma 1 that a Hilbert basis of the Hammond order cone is given by the set of integral vectors $\{h^1, \ldots, h^m\}$ contained in the parallelotope

$$\mathcal{P}_H := \left\{\lambda_1\tau^1 + \cdots + \lambda_q\tau^q : 0 \leq \lambda_1, \ldots, \lambda_q \leq 1\right\} \tag{6.6}$$

where $\tau^1, \ldots, \tau^q \in \mathcal{T}_H$. That is, $\{h^1, \ldots, h^m\} = \mathcal{P}_H \cap \mathbb{Z}^k$.

In particular, because each transfer vector is integral, it follows that each vector $\tau \in \mathcal{T}_H$ is an element of the Hilbert basis, and furthermore, for each vector $(\lambda_1, \ldots, \lambda_q) \in \{0, 1\}^q$, the vector $f := \lambda_1\tau^1 + \cdots + \lambda_q\tau^q$ is also an element of the Hilbert basis.

We next characterize a vector of the general form

$$a := \lambda_1\tau^1 + \cdots + \lambda_q\tau^q \quad 0 \leq \lambda_1, \ldots, \lambda_q \leq 1. \tag{6.7}$$

such that a is an integral vector of the parallelotope $\mathcal{P}_H$. Let $\{i_1, \ldots, i_h\}$ and $\{j_1, \ldots, j_l\}$ denote two subsets from the index set $\{1, \ldots, q\}$, with $i_1 < i_2 < \cdots < i_h$ and, likewise, $j_1 < j_2 < \cdots < j_l$. On the basis of Lemma 2, there are integral vectors $(\theta_1, \ldots, \theta_{k-1})$ and $(\gamma_1, \ldots, \gamma_{k-1})$ in $\mathbb{Z}_+^{k-1}$, such that $\sum_{s=i_1}^{i_h} \tau^s = z_H(\theta_1, \ldots, \theta_{k-1})$ and $\sum_{s=j_1}^{j_l} \tau^s = z_H(\gamma_1, \ldots, \gamma_{k-1})$. Let $\mathcal{N}$ denote a set of positive integers, and $\gcd(\mathcal{N})$ denote the greatest common divisor of $\mathcal{N}$. Define the greatest common divisors $\bar{\theta} := \gcd(\{\theta_1, \ldots, \theta_{k-1}\})$ and $\bar{\gamma} := \gcd(\{\gamma_1, \ldots, \gamma_{k-1}\})$. If $\bar{\theta} > 1$, then the vector

$$g^1 := \frac{1}{\bar{\theta}}\left(\tau^{i_1} + \cdots + \tau^{i_h}\right) \tag{6.8}$$

is a vector of the form (6.7), and accordingly g^1 is an element of the Hilbert basis. If $\{i_1, \ldots, i_h\} \cap \{j_1, \ldots, j_l\} = \emptyset$ (i.e. the sets of indices are non-overlapping) and $\bar{\gamma} > 1$, then both

$$\begin{aligned} g^2 &:= \frac{1}{\bar{\gamma}}\left(\tau^{j_1} + \cdots + \tau^{j_l}\right) \\ g^3 &:= g^1 + g^2 \end{aligned} \tag{6.9}$$

are likewise vectors of the form (6.7) and belong to the Hilbert basis[12]. Sums of three or more vectors of the form (6.7) can likewise be constructed in a similar fashion.

We next characterize the subset of $\mathcal{P}_H \cap \mathbb{Z}^k$ that constitutes the minimal Hilbert basis. From (ii) of Lemma 1, the minimal Hilbert basis is the subset of irreducible elements from the set $\{h^1, \ldots, h^m\} \setminus \{\mathbf{0}_k\}$. Observe that

$$\begin{aligned} t^1 &= (-1, 2, 0, \ldots, 0, -1)', \\ t^2 &= (0, -1, 2, 0, \ldots, 0, -1)', \\ &\vdots \\ t^{k-1} &= (0, \ldots, 0, -1, 1)'; \end{aligned} \tag{6.10}$$

that is, $t^1, \ldots, t^{k-1}$ are all elements of the set of transfers $\mathcal{T}_H$. It is readily verified that the vectors $t^1, \ldots, t^{k-1}$ are linearly independent, and therefore irreducible. To show that these vectors are the only non-zero irreducible elements of the Hilbert basis, we recall from (3.3) that, by construction, $t^1 = z_H(1, 0, \ldots, 0)$, $t^2 = z_H(0, 1, 0, \ldots, 0), \ldots, t^{k-1} = z_H(0, \ldots, 0, 1)$. From Lemma 2, each integral vector in the Hammond order cone is a vector of the form $z_H(\mu_1, \ldots, \mu_{k-1})$ with $\mu_1, \ldots, \mu_{k-1} \in \mathbb{Z}_+$. From the properties [L1] and [L2], it therefore follows that $t^1, \ldots, t^{k-1}$ jointly generate the set of vectors $z_H(\mu_1, \ldots, \mu_{k-1})$, and accordingly constitute the entire set of non-zero irreducible elements of the Hilbert basis.

(ii) We must show that $C_H \cap \mathbb{Z}^k = \mathcal{Z}_H$. Let ζ be an integral vector in the Hammond order cone. Then from statement (i) of the proposition, ζ can be expressed as a positive integer combination of the vectors in the minimal Hilbert basis. Therefore, $\zeta \in C_H \cap \mathbb{Z}^k \Leftrightarrow \zeta = \mu_1 t^1 + \cdots + \mu_{k-1} t^{k-1}$ with $\mu_1, \ldots, \mu_{k-1} \in \mathbb{Z}_+ \Leftrightarrow \zeta = \mu_1 z_H(1, 0, \ldots, 0) + \cdots + \mu_{k-1} z_H(0, \ldots, 0, 1) \Leftrightarrow_{[L1]} \zeta = z_H(\mu_1, 0, \ldots, 0) + \cdots + z_H(0, \ldots, 0, \mu_{k-1}) \iff_{[L2]} \zeta = z_H(\mu_1, \ldots, \mu_{k-1}) \in \mathcal{Z}_H$. That is, we have shown that $C_H \cap \mathbb{Z}^k = \mathcal{Z}_H$, as was required. □

Proof of Proposition 5 (i) Assume to the contrary that $p^1, \ldots, p^{k-1}$ are linearly independent yet P is not an invertible matrix. Then it must be that the columns of P are linearly dependent, and (say) that for scalars $(\alpha_2, \ldots, \alpha_{k-1}) \neq (0, \ldots, 0)$

$$\widehat{p}^1 = \alpha_2 \widehat{p}^2 + \cdots + \alpha_{k-1} \widehat{p}^{k-1}$$

For $l = 1, \ldots, k-1$, let $\left(p_1^l, \ldots, p_k^l\right)$ denote the elements of the vector p^l, and define the scalar $-q := \alpha_2 \left(p_1^2 + \cdots + p_{k-1}^2\right) + \cdots + \alpha_{k-1} \left(p_1^{k-1} + \cdots + p_{k-1}^{k-1}\right)$. Then, we would have that the k-dimensional vector $\begin{pmatrix} \widehat{p}^1 \\ q \end{pmatrix}$ is a linear combination of the vectors $p^2, \ldots, p^{k-1}$. But, by construction, $\begin{pmatrix} \widehat{p}^1 \\ q \end{pmatrix} = p^1$, and from the assumption that the

[12] In Example 1 for instance,

$$a := \frac{1}{2}(-1\ 0\ 2\ -1)' + \frac{1}{2}(-1\ 2\ 0\ -1)'$$

is an element of the Hilbert basis.

set of vectors $p^1, \ldots, p^{k-1}$ in the minimal Hilbert basis are linearly independent, p^1 cannot be linearly spanned by $p^2, \ldots, p^{k-1}$. Therefore, we conclude that P is a full rank matrix, so that P^{-1} exists.

(ii) Given from (i) that the matrix P is invertible, the proof proceeds using the same steps outlined in the proof of Proposition 4. □

Proof of Proposition 6 (a) Since $\mathcal{C}_E = \text{pos}(\mathcal{T}_E)$, it follows from (i) of Lemma 1 that an integral Hilbert basis of the cone of Hammond egalitarian transfers is given by the set of integral vectors $\{h^1, \ldots, h^m\}$ contained in the parallelotope

$$\mathcal{P}_E := \left\{\lambda_1\sigma^1 + \cdots + \lambda_s\sigma^s : 0 \leq \lambda_1, \ldots, \lambda_s \leq 1\right\} \tag{6.11}$$

where $\sigma^1, \ldots, \sigma^s \in \mathcal{T}_E$. That is, $\{h^1, \ldots, h^m\} = \mathcal{P}_E \cap \mathbb{Z}^k$.

We follow the steps of the proof of Proposition 3, and obtain an integral Hilbert basis of $\mathcal{C}_E$. The basis $\{h^1, \ldots, h^m\}$ is made of the set of transfers $\mathcal{T}_E$ together with sums of elements from $\mathcal{T}_E$.

We complete the proof of (a) by proving the three statements (i), (ii) and (iii) below.

(i) Vectors (4.9), i.e. vectors of the form $e = (\mathbf{0}_{j-1}, -1, \mathbf{0}_{h-1}, 2, \mathbf{0}_{l-1}, 1, \mathbf{0}_{k-(j+h+l)})'$, are irreducible elements of the integral Hilbert basis of $\mathcal{C}_E$.

(ii) Let j, h, l and r be arbitrary non-negative integers such that $j + h + l + r \leq k$. Vectors of the form

$$\tau = (\mathbf{0}_{j-1}, -1, \mathbf{0}_{h-1}, 1, \mathbf{0}_{l-1}, 1, \mathbf{0}_{r-1}, -1, \mathbf{0}_{k-(j+h+l+r)})' \tag{6.12}$$

are composite transfers of the integral Hilbert basis of $\mathcal{C}_E$.

(iii) There are $k(k-1)(k-2)/6$ irreducible vectors in the minimal Hilbert basis $\mathcal{B}_E$.

To establish (i), we show on the basis of Lemma 1, that if $e = a + b$, where $a, b \in \mathcal{P}_E \cap \mathbb{Z}^k$, then either $a = e$ and $b = \mathbf{0}_k$, or $b = e$ and $a = \mathbf{0}_k$.

Let $a = (a_1, \ldots, a_k)'$ and $b = (b_1, \ldots, b_k)'$. Observe first that $e_i = 0$, for indices $i = 1, \ldots, j-1$. It then follows from (4.9) and the equality $e = a+b$ that $a_i = b_i = 0$. For $i = j$, there holds $a_j + b_j = -1$. Given that a and b are integral vectors in $\mathcal{C}_E$, it follows that one of two cases can hold, that we denote Case I and Case II respectively. Under Case I, we have $(a_j, b_j) = (-1, 0)$. Similarly, in Case II we obtain that $(a_j, b_j) = (0, -1)$.

Consider Case I. Then given that $e_i = 0$ for all indices $i = j+1, \ldots, j+h-1$, it follows that $a_i = b_i = 0$ for all $i = j+1, \ldots, j+h-1$. For $i = j+h$, there holds $a_i + b_i = 2$. Under Case I, it follows that $a_{j+h} \in \{0, 2\}$ while $b_{j+h} \in \{-1, 0\}$. We conclude that $a_{j+h} = 2$ and $b_{j+h} = 0$. Then given that $e_i = 0$ for all indices $i = j+h+1, \ldots, j+h+l-1$, it follows that $a_i = b_i = 0$ for all $i = j+h+1, \ldots, j+h+l-1$.

For $i = j+h+l$, we have that $a_i + b_i = -1$. Under Case I, it follows that $a_{j+h+l} \in \{-1, 0\}$ and $b_{j+h+l} \in \{-1, 0\}$. We show that $b_{j+h+l} = 0$. Assume to the contrary that $b_{j+h+l} = -1$ and $a_{j+h+l} = 0$. Then, given that $e_i = 0$ for all $i = j+h+l+1, \ldots, k$, it follows similarly that $a_i = b_i = 0$ for all $i = j+h+l+1, \ldots, k$. But then this implies that $a_1+\cdots+a_k = 1$ and $b_1+\cdots+b_k = -1$. This contradicts the assumption that a and b are integral vectors of the cone $\mathcal{C}_E$, since for all $u \in \mathcal{C}_E$, we require that $u_1 + \cdots + u_k = 0$.

We conclude that under Case I, $a = e$ and $b = \mathbf{0}_k$. Under Case II, we arrive, using the same argument, to the conclusion that $b = e$ and $a = \mathbf{0}_k$. This concludes the proof of statement (i), namely that the vector e of (4.9) is irreducible.

Consider statement (ii). We show that there are two integral vectors a and b in the cone $\mathcal{C}_E \setminus \{0_k\}$ such that $\tau = a + b$, where τ is the vector (6.12).

Construct $a = (a_1, \ldots, a_k)'$ as follows:

$$a_i = \begin{cases} -1, & i = j, j+h+l \\ 2 & i = j+h \\ 0 & \text{otherwise} \end{cases} \tag{6.13}$$

and construct $b = (b_1, \ldots, b_k)'$ as follows:

$$b_i = \begin{cases} -1, & i = j+h, j+h+l+r \\ 2 & i = j+h+l \\ 0 & \text{otherwise.} \end{cases} \tag{6.14}$$

Then it follows that $\tau = a + b$, and accordingly τ is a composite transfer. This concludes the proof of (ii).

To show statement (iii), we observe from (i) and (ii) that the minimal Hilbert basis consists only of vectors of the form (4.9). For $k \geq 3$, these vectors are constructed by inserting $k-3$ zeroes to the vector $(-1, 2, -1)$ in order to construct a k -dimensional vector. There are $\dfrac{k!}{(k-3)!3!} = k(k-1)(k-2)/6$ such ways to insert $k-3$ zero elements in k cells (think of the different ways $k-3$ students can occupy a classroom containing k seats).

We conclude that the minimal Hilbert basis of the cone of egalitarian Hammond transfers consists of vectors of the form (4.9), and that $\mathcal{B}_E$ contains $k(k-1)(k-2)/6$ such vectors. This completes the proof of part (a) of the proposition.

(b) The derivation of this result follows the same steps as the proof of Proposition 3, and accordingly the details are omitted. □

Proof of Proposition 7 (i) Let the partial order $\succeq_G$ be an order extension of the Hammond order relation $\succeq_H$. Then for every pair of distributions x and y such that $x \succeq_H y$, there also holds, by definition, $x \succeq_G y$. Define the vector $\varepsilon := x - y$. Then ε is a vector in the set $\mathcal{C}_H \cap \mathbb{Z}^k$, and therefore from the assumption that $\succeq_G$ is an order extension of $\succeq_H$, it also follows that ε is an element of the discrete cone $\mathcal{C}_G \cap \mathbb{Z}^k$. Hence we have that $\mathcal{C}_H \cap \mathbb{Z}^k \subseteq \mathcal{C}_G \cap \mathbb{Z}^k$ as required.

(ii) Observe that a basis for the space $\mathbb{S}^k$ is given by the following set of $k-1$ vectors:

$$\mathcal{B}_{\mathbb{S}^k} := \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ -1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\} \tag{6.15}$$

It is clear that $\mathcal{C}_G \cap \mathbb{Z}^k \subseteq \mathbb{S}^k \cap \mathbb{Z}^k$. To show that the set of integral points $\mathcal{C}_G \cap \mathbb{Z}^k$ is contained in the geometric lattice $\mathcal{L}_H$, we shall show that $\mathcal{L}_H = \mathbb{S}^k \cap \mathbb{Z}^k$. Equivalently, we shall show that the bases matrices $\left(\mathcal{B}_{\mathbb{S}^k}\right)$ and $(\mathcal{B}_H)$ are equal up to multiplication by a unimodular matrix. It is readily verified that $\left(\mathcal{B}_{\mathbb{S}^k}\right) = -(\mathcal{B}_H)\, A$, where A is the lower triangular matrix defined in (4.3). As A is a unimodular matrix, and $\left(\mathcal{B}_{\mathbb{S}^k}\right)$ is a basis for the lattice $\mathbb{S}^k \cap \mathbb{Z}^k$ we conclude that $\mathcal{L}_H = \mathbb{S}^k \cap \mathbb{Z}^k$. Since the discrete cone $\mathcal{C}_G \cap \mathbb{Z}^k$ is a set of points in $\mathbb{S}^k \cap \mathbb{Z}^k$, we have therefore shown that $\mathcal{C}_G \cap \mathbb{Z}^k$ is a subset of the Hammond Hilbert lattice $\mathcal{L}_H$, as required. □

Example 2 (continued from Sect. 4.2) We explore the possibility of expressing any integral vector $e \in \mathcal{C}_E$ using the $k-1$ vectors $t^1, \ldots, t^{k-1}$ that define the minimal Hilbert basis associated with the cone $\mathcal{C}_H$. The important point to observe, is that while $\mathcal{C}_E$ is a subset of $\mathcal{C}_H$, one cannot use the three generators t^1, t^2, and t^3 of $\mathcal{C}_H$ to generate any vector in $\mathcal{C}_E$, without imposing restrictions on the integer combinations of t^1, t^2, and t^3. Note first, on the basis of Proposition 6, that any integral vector in the cone $\mathcal{C}_E$ is a subset of the set of vectors in the Hammond order cone, the subset being defined as follows:

$$\mathcal{C}_E \cap \mathbb{Z}^4_+ = \{a \in \mathcal{C}_H, \theta_1, \ldots, \theta_4 \in \mathbb{Z}_+ : a = \theta_1 e^1 + \cdots + \theta_4 e^4\} \tag{6.16}$$

Because the Hammond order cone is generated by the three vectors t^1, t^2, and t^3, it is certainly the case that for any integral vector a in the cone $\mathcal{C}_E$, we have that there also exist three positive integers μ_1, μ_2, and μ_3 such that

$$\mu_1 t^1 + \mu_2 t^2 + \mu_3 t^3 = a = \theta_1 e^1 + \cdots + \theta_4 e^4. \tag{6.17}$$

It is then possible to solve for each of μ_1, μ_2, and μ_3 as functions of $(\theta_1, \ldots, \theta_4) \in \mathbb{Z}^4_+$. This produces the desired restrictions any vector a constructed as a combination of Hammond transfers must satisfy:

$$\begin{aligned} \mu_1(\theta_1, \ldots, \theta_4) &= \theta_1 + \theta_3 + \theta_4 \\ \mu_2(\theta_1, \ldots, \theta_4) &= \theta_2 + 2\theta_4 \\ \mu_3(\theta_1, \ldots, \theta_4) &= \theta_3 + 2\theta_4. \end{aligned} \tag{6.18}$$

Observe for instance that unlike vectors of the Hammond order cone, it is not possible for an integral vector $a \in \mathcal{C}_E$ to have the form $a = t^3$[13].

[13] That is, if $\theta = (0, 0, 1, 0)$ then both $\mu_3(\theta_1, \ldots, \theta_4) = 1$ and $\mu_1(\theta_1, \ldots, \theta_4) = 1$. Likewise, if $\theta = (0, 0, 0, 1)$ then $\mu_3(\theta_1, \ldots, \theta_4) = 2$, $\mu_2(\theta_1, \ldots, \theta_4) = 2$ and $\mu_1(\theta_1, \ldots, \theta_4) = 1$, etc.

Similarly, it is possible to express any integral vector a in the cone of egalitarian Hammond transfers using positive integer combinations of the vectors of the minimal Hilbert basis of the cone $\mathcal{C}_F$. $\Diamond$

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