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Fuzzy closure systems: Motivation, definition and properties

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ABSTRACT

The aim of this paper is to extend closure systems from being crisp sets with certain fuzzy properties to proper fuzzy sets. The presentation of the paper shows a thorough discussion on the different alternatives that could be taken to define the desired fuzzy closure systems. These plausible alternatives are discarded if they are proven impossible to be in a bijective correspondence with closure operators. Finally, a definition of fuzzy closure system is established and a one-to-one relation with closure operators is proved.

1. Introduction

Fuzzy logic gives a framework where working with uncertainty is within reach. Recently, the research in fuzzy mathematics is increasing and these mathematical tools are being used in order to model real world problems. For example, fuzzy tools have been used to solve transport problems, which are modelled with boundary problems in the field of differential equations [20], or, in the study and prevention of seismic behaviour [23]. Moreover, in the times of Big Data, some problems use fuzzy concepts, such as fuzzy-directed graphs or fuzzy relation equations, as data structures. Therefore, the study of these structures is fruitful for the field of data management. Some advances in this line can be consulted in [10,13]. This is the line of work followed in this article. The study of fuzzy structures is done in order to be used later in tools for data analysis, such as, for instance, formal concept analysis [17]. In particular, this work is focused on closure operators and closure systems.

Closure operators were introduced by E. H. Moore in 1910 [27]. They are key elements in several branches of mathematics, such as algebra, topology, analysis and computer science. [11]. Fuzzy closure operators [2,6] appear in several areas of fuzzy logic, just to list a few we mention: fuzzy mathematical morphology [15,26], fuzzy relational equations [14], approximate reasoning [5,12] and fuzzy logic in narrow sense [19]. But also their applications such as fuzzy logic programming [24] or formal concept analysis of data with fuzzy attributes [31].

In the classical case, there is a bijection which relates the notions of closure operator and closure system. Thus, closure systems also play an important role in the aforementioned applications. In many cases, these closure systems are defined in the powerset. Consequently, it is not surprising that the fuzzification of this notion most widespread in the literature is

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defined on the $\mathbb{L}$-powerset. This definition of fuzzy closure system was introduced by Bělohlávek [2] and is the cornerstone of the Fuzzy Formal Concept Analysis [1]. However, even in the classical case, there are many applications that require working on other ordered structures and, particularly, on lattice-type structures. For example, one of the extensions of Formal Concept Analysis that is most fruitful from the point of view of applications, introduced by Ganter and Kuznetsov and known as *pattern structures* [16], replaces the powerset by a semilattice. It is therefore of interest to study the notion of fuzzy closure system in other ordered structures.

In the literature, several distinct definitions of closure system can be found, depending on the underlying order on which the closure operator is defined. Thus, as a consequence, there are definitions of fuzzy closure system on $\mathbb{L}$-ordered sets [18], on fuzzy preposets [8] and fuzzy preordered structures [9].

The aim of this paper is to study closure systems on complete fuzzy lattices. This is an extension of the work done in [30], where the problem is similar but the underlying structure is that of Heyting algebras, instead of general residuated lattices.

There are different extensions of lattice and complete lattice to the fuzzy framework in the literature. Among all the existing extensions, two of them are considered to be the main ones. On the one hand, some authors have defined fuzzy lattice as a fuzzy subset of a (crisp) lattice $\mathbb{L}$ which is compatible with the lattice structure in one way or another [32,34]. This particular extension may be called that of *fuzzy sublattices*. Closure systems were defined on fuzzy sublattices by Šešelja and Tepavčević in [33]. On the other hand, a fuzzy lattice has been defined as a (classical) set endowed with a fuzzy order relation which satisfies certain conditions related to the existence of fuzzy infimum and supremum. The concept of *complete fuzzy lattice* considered in this paper belongs to this class. It was introduced by Bělohlávek in [4] with the name of *completely lattice $\mathbb{L}$-ordered set*, although it has received other names in the literature, for instance, *fuzzy complete lattice* in [22]. One of the most important examples of a complete fuzzy lattice is the $\mathbb{L}$-powerset with the so-called subsethood degree relation. Closure systems on $\mathbb{L}$-powerset lattices were introduced by Bělohlávek in [2].

The aim of this paper is to extend the results shown in [29], where some results are only stated but not proved. The present work begins with a section of preliminaries where the main concepts and results used throughout the paper are presented in order to make the paper as self-contained as possible. The core of this work is developed in the next section. Its aim is a written out analysis on why the definition of fuzzy closure system introduced in [29] seems to be the most adequate. Even though the results are already known we find the discussion of the different options interesting on its own. A similar analysis was made in [30] where the underlying structure was a Heyting algebra. In this case some proofs are analogous but some need different strategies to be proved. In this section, several possible definitions of fuzzy closure system are introduced but are discarded and narrowed down according to how appropriate their properties are in the sense of being in a bijective relation with closure operators. Due to our initial idea, the fuzzy sets studied are $\Phi_{\mathrm{c}}(x) = \rho(\mathrm{c}(x), x)$ where c is a closure operator. In order to find the definition of fuzzy closure system as the $\mathbb{L}$-sets in one-to-one correspondence with closure operators we must study the properties of the sets Φ_{c}. Among them, it is remarkable that extensionality resurfaces, since it was already used by Bělohlávek in his definition of $\mathbb{L}$-closure $\mathbb{L}$-systems introduced in [2]. When extensionality is introduced as a required property some issues were solved whereas some new appeared and the key to solving them was found by considering fuzzy closure systems as extensional hulls of closure systems. In addition, this definition can be characterized by the existence of minima of certain sets and, if the framework is that of powerset lattices, fuzzy closure systems are equivalent to $\mathbb{L}$-closure $\mathbb{L}$-systems. Finally, some hints of further research are presented.

2. Preliminaries

In this section, a brief presentation of the concepts needed to follow the results is written. This paper is a natural continuation of the work in [29]. Specifically, we recall the notions of complete residuated lattices [3,19], fuzzy poset and fuzzy complete lattice, and some basics results that will be needed throughout the paper [4,22].

Throughout this paper, let $\mathbb{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ be a complete residuated lattice, which is an algebra where

- $(L, \wedge, \vee, 0, 1)$ is a complete lattice with 0 and 1 being the least and the greatest elements of L, respectively,
- $(L, \otimes, 1)$ is a commutative monoid (i.e., $\otimes$ is commutative, associative, and 1 is neutral with respect to $\otimes$), and
- $\otimes$ and $\to$ satisfy the so-called *adjointness property*: for all $a, b, c \in L$, we have that $a \otimes b \le c$ iff $a \le b \to c$.

This structure is utilized in mathematical fuzzy logics and their applications as structures of truth degrees with $\otimes$ and $\to$ used as truth functions of *fuzzy conjunction* and *fuzzy implication*, respectively [19]. The unit interval with the Łukasiewicz, Gödel and Goguen pairs of t-norms and implications are examples of residuated complete lattices.

Among the properties of complete residuated lattices, we summarize only those that will be used:

$$a \le b \text{ implies } a \otimes c \le b \otimes c, \qquad \text{for all } a, b, c \in L \tag{1}$$

$$a \le b \text{ implies } c \to a \le c \to b, \qquad \text{for all } a, b, c \in L \tag{2}$$

$$a \le b \text{ implies } b \to c \le a \to c, \qquad \text{for all } a, b, c \in L \tag{3}$$

$$a \to b = 1 \text{ iff } a \le b, \qquad \text{for all } a, b \in L \tag{4}$$

$$a \to b \le a \otimes c \to b \otimes c, \qquad \text{for all } a, b, c \in L \tag{5}$$

$$a \le (a \to b) \to b, \qquad \text{for all } a, b \in L \tag{6}$$

$$a \otimes \bigvee_{b \in B} b = \bigvee_{b \in B} (a \otimes b), \qquad \text{for all } a \in L \text{ and } B \subseteq L \tag{7}$$

$$\Big(\bigvee_{b \in B} b\Big) \to a = \bigwedge_{b \in B} (b \to a), \qquad \text{for all } a \in L \text{ and } B \subseteq L \tag{8}$$

An $\mathbb{L}$-*set* is a mapping $X \colon U \to L$ from the universe set U to the membership values set L, where $X(u)$ means the degree in which u belongs to X. The set of $\mathbb{L}$-sets on the universe U is denoted by L^U. A crisp set is considered to be a particular case of $\mathbb{L}$-set by using its characteristic mapping $X \colon U \to \{0, 1\}$ with $X(u) = 1$ iff $u \in X$.

Operations with $\mathbb{L}$-sets are defined element-wise. For instance, $A \otimes B \in L^U$ is defined as $(A \otimes B)(u) = A(u) \otimes B(u)$ for all $u \in U$. The so-called *subsethood degree relation* is defined as $S \colon L^U \times L^U \to L$ where

$$S(A, B) = \bigwedge_{x \in U} \Big(A(x) \to B(x)\Big).$$

Obviously, $S(A, B) = 1$ if and only if $A(x) \le B(x)$ for all $x \in U$. In this case, A is said to be a fuzzy subset of B and is denoted by $A \subseteq B$.

Binary $\mathbb{L}$-relations (binary fuzzy relations) on a set U can be thought of as $\mathbb{L}$-sets on the universe $U \times U$. That is, a binary $\mathbb{L}$-relation on U is a mapping $\rho \in L^{U \times U}$ assigning to each $x, y \in U$ a truth degree $\rho(x, y) \in L$ (a degree to which x and y are related by ρ).

For ρ being a binary $\mathbb{L}$-relation in U, we say that

- ρ is *reflexive* if $\rho(x, x) = 1$ for all $x \in U$.
- ρ is *symmetric* if $\rho(x, y) = \rho(y, x)$ for all $x, y \in U$.
- ρ is *antisymmetric* if $\rho(x, y) \otimes \rho(y, x) = 1$ implies $x = y$ for all $x, y \in U$.
- ρ is *transitive* if $\rho(x, y) \otimes \rho(y, z) \le \rho(x, z)$ for all $x, y, z \in U$.

Definition 1. Given a non-empty set A and a binary $\mathbb{L}$-relation ρ on A, the pair $\mathbb{A} = (A, \rho)$ is said to be a

- *fuzzy preposet* if ρ is a *fuzzy preorder*, i.e. if ρ is reflexive and transitive;
- *fuzzy poset* if ρ is a *fuzzy order*, i.e. if ρ is reflexive, antisymmetric and transitive.

A typical example of fuzzy poset is (L^U, S).

As in the crisp case, any order $\mathbb{L}$-relation implicitly defines an equivalence $\mathbb{L}$-relation that is called *symmetric kernel relation*. In the fuzzy case, this equivalence $\mathbb{L}$-relation usually replaces the notion of equality in the fuzzy poset.

Definition 2. Given a fuzzy preposet $\mathbb{A} = (A, \rho)$, the *symmetric kernel relation* is defined as $\approx \colon A \times A \to L$ where $(a \approx b) = \rho(a, b) \otimes \rho(b, a)$ for all $a, b \in A$.

Proposition 3. *Given a fuzzy preposet $\mathbb{A} = (A, \rho)$, the symmetric kernel relation $\approx$ is a fuzzy equivalence relation, that is, it is a reflexive, symmetric and transitive fuzzy relation.*

A usual way to define fuzzy algebras is to consider as an underlying structure a pair which consists of a set and a tolerance or equivalence relation on it. Thus, an alternative definition of fuzzy poset that can be found in the literature [9] is given by a tuple $(A, \approx, \rho)$ where $\approx$ is a fuzzy equivalence relation on A and ρ is a fuzzy order that is compatible with $\approx$. In [35], it is shown that both definitions of fuzzy poset are equivalent.

To present the notion of fuzzy lattice we need to generalize those of upper (lower) bound and supremum (infimum).

Definition 4. Given a fuzzy preposet $\mathbb{A} = (A, \rho)$ and a fuzzy set $X \in L^A$, we define the *down-cone* of X and the *up-cone* of X, respectively, as the fuzzy sets $X_\rho, X^\rho \in L^A$ where, for all $a \in A$,

$$X_\rho(a) = \bigwedge_{x \in A} \big(X(x) \to \rho(a, x)\big) \qquad \text{and}$$

$$X^\rho(a) = \bigwedge_{x \in A} \big(X(x) \to \rho(x, a)\big).$$

Thus, $X^\rho(a)$ and $X_\rho(a)$ can be seen as the degree to which a is an upper bound and lower bound of X, respectively.

Notice that for singletons, $X = \{x\}$, we will omit the brackets for simplicity of the notation, thus $x^\rho(a) = \rho(x, a)$ and $x_\rho(a) = \rho(a, x)$.

Definition 5. Let $\mathbb{A} = (A, \rho)$ be a fuzzy preposet and $X \in L^A$. An element $a \in A$ is said to be *infimum* (resp. *supremum*) of X if the following conditions hold:

1. $X_\rho(a) = 1$ (resp. $X^\rho(a) = 1$).
2. $X_\rho(x) \leq \rho(x, a)$ (resp. $X^\rho(x) \leq \rho(a, x)$), for all $x \in A$.

The following propositions, which are straightforward, provide useful characterizations of infimum, supremum, minimum and maximum.

Proposition 6. *Let $\mathbb{A} = (A, \rho)$ be a fuzzy preposet and $X \in L^A$. An element $a \in A$ is infimum (resp. supremum) of X if and only if, for all $x \in A$,*

$$\rho(x, a) = X_\rho(x) \qquad (\text{resp. } \rho(a, x) = X^\rho(x)).$$

Proposition 7. *Let $\mathbb{A} = (A, \rho)$ be a fuzzy preposet and $X \in L^A$. An element $m \in A$ is a* minimum *(resp.* maximum*) of X if and only if m is an infimum (resp. a supremum) of X and $X(m) = 1$.*

Notice that the supremum (resp. infimum) needs not be unique in an arbitrary preposet, but if $\mathbb{A}$ is a poset and due to antisymmetry, it is not difficult to see that, if a supremum (resp. infimum) of X exists, it is unique. In such case, we will denote it by $\bigsqcup X$ (resp. $\bigsqcap X$).

Definition 8 *([4]).* We say that a fuzzy poset (A, ρ) is a complete fuzzy lattice if every fuzzy subset $X \in L^A$ has supremum and infimum.

The pair (L^U, S) is an example of complete fuzzy lattice, which is called the *L-powerset lattice of U*. This fact follows easily from [3, Theorem 5.63].

Notice that, if (A, ρ) is a complete fuzzy lattice, then $(A, \leq)$, with $a \leq b$ iff $\rho(a, b) = 1$, is a complete lattice. Therefore, there exist elements that are minimum and maximum, which we denote by $\perp$ and $\top$ respectively.

Other key concepts used in this paper are closure structures. Thus, the main definitions of the structures are needed to fully comprehend the results presented.

Definition 9. Given a fuzzy preposet $\mathbb{A} = (A, \rho)$, a mapping $c\colon A \to A$ is said to be a *closure operator* on $\mathbb{A}$ if the following conditions hold:

1. $\rho(a, b) \leq \rho(c(a), c(b))$, for all $a, b \in A$
2. $\rho(a, c(a)) = 1$, for all $a \in A$
3. $\rho(c(c(a)), c(a)) = 1$, for all $a \in A$.

Conditions 1 and 2 are well-known and are called isotony and inflationarity, respectively. Observe that condition 3 could be replaced by $(c(c(a)) \approx c(a)) = 1$, and, thus, if $\mathbb{A}$ is a fuzzy poset, a closure operator is idempotent in a classical sense, i.e., $c(c(a)) = c(a)$ for all $a \in A$.

On the other hand, there is not a unique extension of the notion of closure system to a fuzzy setting. The definition we will use in this paper is the following one, introduced in [29].

Definition 10. Let (A, ρ) be a fuzzy complete lattice. A crisp subset $\mathcal{F} \subseteq A$ is said to be a closure system if $\bigsqcap X \in \mathcal{F}$ for any fuzzy subset X of $\mathcal{F}$.

Using this definition, closure systems are in one-to-one relation with closure operators. This is illustrated in the following theorem.

Theorem 11 *([29]). Let $\mathbb{A} = (A, \rho)$ be a complete fuzzy lattice.*

1. *If $\mathcal{F}$ is a closure system on $\mathbb{A}$, then the mapping $c_\mathcal{F}\colon A \to A$ defined as $c_\mathcal{F}(x) = \bigsqcap(x^\rho \cap \mathcal{F})$ is a closure operator on $\mathbb{A}$.*
2. *If $c\colon A \to A$ is a closure operator on $\mathbb{A}$, then $\mathcal{F}_c = \{x \in A \mid c(x) = x\}$ is a closure system on $\mathbb{A}$.*
3. *If $\mathcal{F}$ is a closure system on $\mathbb{A}$, then $\mathcal{F}_{c_\mathcal{F}} = \mathcal{F}$.*
4. *If $c\colon A \to A$ is a closure operator on $\mathbb{A}$, then $c_{\mathcal{F}_c} = c$.*

Next result is a characterization of closure systems in terms of the minima of certain sets.

Proposition 12 *([29]). Let $\mathbb{A} = (A, \rho)$ be a complete fuzzy lattice. A crisp set $\mathcal{F} \subseteq A$ is a closure system if and only if the element $\min(x^\rho \cap \mathcal{F})$ exists for all $x \in A$.*

In [29], the closure system section was written out with all the motivation and proofs. On the other hand, the fuzzy closure system section was only stated.

3. Fuzzy closure systems

The general aim of this section is to extend Theorem 11 by fuzzifying the notion of closure system. This result ensures that, if c is a closure operator then, the set of closed elements $\mathcal{F}_\mathrm{c} = \{a \in A \mid \mathrm{c}(a) = a\}$ is a closure system. The natural extension of that would be to construct a fuzzy set with the membership function being the degree to which that element is closed. For this aim, the equality between the element and its closure is replaced by the so-called symetric kernel relation. Thus, the degree in which an element $a \in A$ is closed is $\mathrm{c}(a) \approx a$ and, since every closure operator is inflationary, it is equal to $\rho(\mathrm{c}(a), a)$. Thus, we will define the fuzzy set Φ_c as $\Phi_\mathrm{c}(a) = \rho(\mathrm{c}(a), a)$ for all $a \in A$.

Remark 1. Recall that in this paper the similarity relation is defined as $a \approx b = \rho(a, b) \otimes \rho(b, a)$, making use of the t-norm instead of the usual $\rho(a, b) \wedge \rho(b, a)$ where the infimum is used.

The first goal, in order to give a definition of fuzzy closure system, is to obtain a characterization theorem of those fuzzy sets Ψ for which there exists a closure operator c with $\Psi = \Phi_\mathrm{c}$.

Extending the definition of closure system introduced in Definition 2 in [30], a candidate for such characterization could be one of the following properties:

$$X \subseteq \mathrm{Core}(\Psi) \text{ implies } \bigsqcap X \in \mathrm{Core}(\Psi), \qquad \text{for all } X \in L^A. \tag{9}$$

$$X \subseteq \Psi \text{ implies } \bigsqcap X \in \mathrm{Core}(\Psi), \qquad \text{for all } X \in L^A. \tag{10}$$

$$S(X, \Psi) \leq \Psi(\bigsqcap X), \qquad \text{for all } X \in L^A. \tag{11}$$

The latter one was proposed by Liu and Lu in [25]. These equations are not independent properties. The relationship among them is shown in the next proposition.

Proposition 13. *Let $\mathbb{A} = (A, \rho)$ be a complete fuzzy lattice and $\Phi \in L^A$.*

- *Condition (11) implies Condition (10).*
- *Condition (10) implies Condition (9).*
- *If $\mathbb{L}$ is a complete Heyting algebra and Φ is extensional with respect to $\approx$, then Conditions (9), (10) and (11) are equivalent.*

Proof. The two first items are straightforward. Assuming that $\mathbb{L}$ is a complete Heyting algebra, to prove the third one, it is enough to see that, if Φ satisfies Condition (9), then it satisfies Condition (11).

$$\begin{aligned}
S(X, \Phi) &= \bigwedge_{x \in A} (X(x) \to \Phi(x)) = \bigwedge_{x \in A} \left(X(x) \to \Phi_{\mathrm{c}_\Phi}(x)\right) \\
&\overset{\text{Heyting}}{\leq} \bigwedge_{x \in A} (X(x) \to (X(x) \wedge \rho(\mathrm{c}_\Phi(x), x))) \\
&\leq \bigwedge_{x \in A} \left(X(x) \to \left(\rho\left(\bigsqcap X, x\right) \wedge \rho(\mathrm{c}_\Phi(x), x)\right)\right) \\
&\leq \bigwedge_{x \in A} \left(X(x) \to \left(\rho\left(\mathrm{c}_\Phi\left(\bigsqcap X\right), \mathrm{c}_\Phi(x)\right) \wedge \rho(\mathrm{c}_\Phi(x), x)\right)\right) \\
&\leq \bigwedge_{x \in A} \left(X(x) \to \rho\left(\mathrm{c}_\Phi\left(\bigsqcap X\right), x\right)\right) = X_\rho\left(\mathrm{c}_\Phi\left(\bigsqcap X\right)\right) \\
&\overset{(*)}{=} \rho\left(\mathrm{c}_\Phi\left(\bigsqcap X\right), \bigsqcap X\right) = \Phi_{\mathrm{c}_\Phi}\left(\bigsqcap X\right) = \Phi\left(\bigsqcap X\right).
\end{aligned}$$

The assumption of Heyting and extensionality imply, by Theorem 12 in [30], that $\Phi_{\mathrm{c}_\Phi} = \Phi$ and the step $(*)$ is a consequence of Proposition 6. □

However, these conditions are not equivalent in the general case. The following example shows some sets that satisfy Condition (9) but do not satisfy Condition (11).

Example 1. Let $\mathbb{L} = (\{0, 0.5, 1\}, \wedge, \vee, \otimes, \to, 0, 1)$ be the three-valued Łukasiewicz residuated lattice, and (A, ρ) be the fuzzy lattice where $A = \{\bot, a, b, c, d, e, \top\}$ and $\rho\colon A \times A \to L$ is the fuzzy relation order described by the following table:

ρ	$\bot$	a	b	c	d	e	$\top$
$\bot$	1	1	1	1	1	1	1
a	0.5	1	0.5	1	1	1	1
b	0.5	0.5	1	1	1	1	1
c	0.5	0.5	0.5	1	1	1	1
d	0	0.5	0	0.5	1	0.5	1
e	0	0	0.5	0.5	0.5	1	1
$\top$	0	0	0	0.5	0.5	0.5	1

Consider the closure operator $\mathrm{c}(x) = c$ if $\rho(x, c) = 1$ and $\top$ otherwise. The fuzzy set associated to c is $\Phi_{\mathrm{c}} = \{\bot/0.5, a/0.5, b/0.5, c/1, d/0.5, e/0.5, \top/1\}$. By Theorem 11, $\mathcal{F}_{\mathrm{c}} = \mathrm{Core}(\Phi_{\mathrm{c}}) = \{c, \top\}$ is a closure operator, hence Φ_{c} satisfies Condition (9). Besides, consider $X = \{a/0.5\} \subseteq \Phi_{\mathrm{c}}$, then $\bigsqcap X = d \notin \mathrm{Core}(\Phi_{\mathrm{c}})$. In addition, $S(X, \Phi_{\mathrm{c}}) = 1 \not\leq \Phi_{\mathrm{c}}(\bigsqcap X) = \Phi_{\mathrm{c}}(d) = 0.5$.

The previous example also shows that there exists a closure operator c such that Φ_{c} does not satisfy either Condition (10) nor Condition (11). Thus, these Conditions cannot be the properties in bijective correspondence with closure operators.

The case is different with Condition (9), this condition is necessary because, if there exists a closure operator c such that $\Psi = \Phi_{\mathrm{c}}$, then, by antisymmetry, $\mathrm{Core}(\Psi) = \{a \in A \mid \mathrm{c}(a) = a\} = \mathcal{F}_{\mathrm{c}}$ and, by Theorem 11, it is a closure system.

In order to study if Condition (9) is sufficient, the properties of Φ_{c} must be examined. In this line, the next technical result is useful. The theorem below shows that the construction of a closure operator given in Theorem 11 coincides using either Φ_{c} or its core.

Theorem 14. *Let (A, ρ) be a complete fuzzy lattice and c be a closure operator on $\mathbb{A}$. Then,* $\mathrm{Core}(\Phi_{\mathrm{c}}) = \mathcal{F}_{\mathrm{c}}$ *and, for all $x \in A$,*

$$\bigsqcap(x^{\rho} \otimes \Phi_{\mathrm{c}}) = \bigsqcap(x^{\rho} \otimes \mathcal{F}_{\mathrm{c}})$$

The proof is omitted as it can be found in [29].

As a consequence of the theorem above and Theorem 11, any closure operator c satisfies that $\mathrm{c}(a) = \mathrm{c}_{\mathcal{F}_{\mathrm{c}}}(a) = \bigsqcap(a^{\rho} \otimes \mathcal{F}_{\mathrm{c}}) = \bigsqcap(a^{\rho} \otimes \Phi_{\mathrm{c}})$, for all $a \in A$. Thus, given a fuzzy set Ψ, if there exists a closure operator c such that $\Psi = \Phi_{\mathrm{c}}$, then necessarily $\mathrm{c}(a) = \bigsqcap(a^{\rho} \otimes \Phi_{\mathrm{c}}) = \bigsqcap(a^{\rho} \otimes \Psi)$ for all $a \in A$. This leads us to consider the mapping $\mathrm{c}_{\Psi}(a) = \bigsqcap(a^{\rho} \otimes \Psi)$ for a given fuzzy set Ψ, which is not a closure operator in general, not even when its core is a closure system. This is shown in the following example.

Example 2. Consider the fuzzy lattice in Example 1 and the fuzzy set $\Psi = \{a/1, b/0.5, d/1, \top/1\}$. The operator c_{Ψ} is not a closure operator since $\mathrm{c}_{\Psi}(b) = c \neq \mathrm{c}_{\Psi}(\mathrm{c}_{\Psi}(b)) = d$. In addition, the core of this set is $\mathrm{Core}(\Psi) = \{a, d, \top\}$ which is a closure system because for any fuzzy subset X of $\{a, d, \top\}$, it can be checked that if $X(a) = 1$ then $\bigsqcap X = a$; if $X(a) = 0.5$ then $\bigsqcap X = d$; if $X(a) = 0$ and $X(d) = 1$ then $\bigsqcap X = d$; if $X(a) = 0$ and $X(d) \neq 1$ then $\bigsqcap X = \top$. That is, the fuzzy set Ψ satisfies Condition (9).

The example above shows that, although Condition (9) is necessary, it is not sufficient. To get the desired characterization, we look for properties of Ψ that ensure that c_{Ψ} is a closure operator and $\Phi_{\mathrm{c}_{\Psi}} = \Psi$. The following lemma shows that one of the inclusions always holds.

Lemma 15. *Let (A, ρ) be a complete fuzzy lattice. The inclusion $\Psi \subseteq \Phi_{\mathrm{c}_{\Psi}}$ holds for every fuzzy set $\Psi \in L^A$.*

Proof. Due to the properties of the infimum and reflexivity, for all $x \in A$, we have that

$$\Psi(x) = (x^{\rho} \otimes \Psi)(x) \leq \rho\left(\bigsqcap(x^{\rho} \otimes \Psi), x\right) = \rho(\mathrm{c}_{\Psi}(x), x) = \Phi_{\mathrm{c}_{\Psi}}(x).$$

Therefore, $\Psi \subseteq \Phi_{\mathrm{c}_{\Psi}}$. $\square$

The inclusion above could be strict, even if c_{Ψ} were a closure operator and Condition (9) held, as is illustrated in the following example.

Example 3. Let us consider $A = \{a_1, a_2\}$ and the relation ρ given in the following table

ρ	a_1	a_2
a_1	1	1
a_2	0.6	1

The pair (A, ρ) is a fuzzy lattice over the Heyting algebra $[0, 1]$. For the set $\Psi = \{^{a_1}/_{0.8}, ^{a_2}/_{1}\}$ we have that c_Ψ is the identity mapping and $\Psi \subsetneq \Phi_{c_\Psi} = A$. Notice that c_Ψ is a closure operator and Ψ satisfies Condition (9), i.e. $\mathrm{Core}(\Psi) = \{a_2\}$ is a closure system.

Remark 2. Since $\mathrm{Core}(\Psi)$ is a closure system and Ψ is an extensional set, the fact that the inclusion $\Psi \subseteq \Phi_{c_\Psi}$ is strict might seem to contradict Theorem 12 in [30]. However, notice that the definition of weak fuzzy closure system there is different and Ψ does not satisfy it, e.g. $X = \{^{a_1}/_{0.8}\} \subseteq \Psi$ but $\bigsqcap X \notin \mathrm{Core}(\Psi)$. Hence, this is not a counterexample of that result.

As has been stated, the properties of Ψ cannot depend only on the core, but on every element of the set. Looking for an additional property for Ψ, extensionality naturally arises. As a matter of fact, the set Φ_c is not just extensional, but also uniquely determined by its core.

Theorem 16. *Let $\mathbb{A} = (A, \rho)$ be a complete fuzzy lattice and c a closure operator on $\mathbb{A}$. Then Φ_c is extensional with respect to $\approx$. In fact, Φ_c is the extensional hull of* $\mathrm{Core}(\Phi_c)$.

Proof. Assume c is a closure operator and $\Phi_c(x) = \rho(c(x), x)$, for all $x \in A$. Let $a, b \in A$, then by isotony of c and transitivity,

$$\begin{aligned}\Phi_c(a) \otimes (a \approx b) &= \rho(c(a), a) \otimes \rho(a, b) \otimes \rho(b, a)\\ &\leq \rho(c(b), c(a)) \otimes \rho(c(a), a) \otimes \rho(a, b)\\ &\leq \rho(c(b), b) = \Phi_c(b).\end{aligned}$$

Hence, Φ_c is extensional with respect to $\approx$.

Now we search for an explicit expression for Φ_c. On the one hand, for every element $x \in \mathrm{Core}(\Phi_c)$, we have that

$$(a \approx x) = \rho(a, x) \otimes \rho(x, a) \leq \rho(c(a), c(x)) \otimes \rho(c(x), x) \otimes \rho(x, a) \leq \Phi_c(a).$$

Thus, $\bigvee_{x \in \mathrm{Core}(\Phi_c)} (a \approx x) \leq \Phi_c(a)$.

On the other hand, for all $a \in A$ we have that $c(a) \in \mathrm{Core}(\Phi_c)$ and $\Phi_c(a) = \rho(c(a), a) = \rho(c(a), a) \otimes \rho(a, c(a)) = (a \approx c(a))$. Therefore, for all $a \in A$, we get,

$$\Phi_c(a) = \bigvee_{x \in \mathrm{Core}(\Phi_c)} (a \approx x)$$

which is the extensional hull of $\mathrm{Core}(\Phi_c)$ with respect to $\approx$, as proved in [21]. $\square$

The expected characterization is proved in Theorem 18. The next lemma is a technical result that will be used in the theorem's proof.

Lemma 17. *Let Ψ be an extensional fuzzy set such that $\mathcal{F} = \mathrm{Core}(\Psi)$ is a closure system. Then, $\Phi_{c_\mathcal{F}} \subseteq \Psi$.*

Proof. First, for all $a \in A$, we have that $c_\mathcal{F}(a) \in \mathcal{F}_{c_\mathcal{F}} = \mathcal{F}$, by Theorem 11. Then, since Ψ is extensional, we have that

$$\Phi_{c_\mathcal{F}}(a) = \rho(c_\mathcal{F}(a), a) = (c_\mathcal{F}(a) \approx a) = \Psi(c_\mathcal{F}(a)) \otimes (c_\mathcal{F}(a) \approx a) \leq \Psi(a).$$

Therefore, $\Phi_{c_\mathcal{F}} \subseteq \Psi$. $\square$

Theorem 18. *Let Ψ be a fuzzy set. There exists a closure operator c such that $\Psi = \Phi_c$ if and only if $\mathcal{F} = \mathrm{Core}(\Psi)$ is a closure system and Ψ is the extensional hull of $\mathcal{F}$ with respect to $\approx$.*

Proof. Theorem 14 together with Theorem 16 ensure the direct implication.

For the contrary one, assume Ψ is a fuzzy set such that $\mathcal{F} = \mathrm{Core}(\Psi)$ is a closure system and $\Psi(a) = \bigvee_{x \in \mathcal{F}} (a \approx x)$. From Lemmas 15 and 17, we have that $\Phi_{c_\mathcal{F}} \subseteq \Psi \subseteq \Phi_{c_\Psi}$. The proof ends by proving that $c_\mathcal{F} = c_\Psi$ and, therefore, the two inclusions are in fact equalities $\Phi_{c_\mathcal{F}} = \Psi = \Phi_{c_\Psi}$.

By Theorem 11, $c_\mathcal{F}$ is a closure operator and $\mathcal{F} = \mathcal{F}_{c_\mathcal{F}}$. Besides, by Theorem 14, $\mathrm{Core}(\Phi_{c_\mathcal{F}}) = \mathcal{F}_{c_\mathcal{F}} = \mathcal{F}$.

Since Ψ is the extensional hull of $\mathcal{F}$, by Theorem 16, we have that $\Psi(a) = \bigvee_{x \in \mathcal{F}} (a \approx x) = \Phi_{c_\mathcal{F}}(a)$ and, by Theorem 14, we get

$$c_\Psi = c_{\Phi_{c_\mathcal{F}}} = c_{\mathrm{Core}(\Phi_{c_\mathcal{F}})} = c_\mathcal{F}.$$

Hence, $c_\mathcal{F}$ is a closure operator such that $\Psi = \Phi_{c_\mathcal{F}}$. $\square$

Last theorem hints what would be a plausible definition of fuzzy closure system. Indeed, as initially stated in [29], the definition is as follows.

Definition 19. Let (A, ρ) be a complete fuzzy lattice. A fuzzy set $\Psi \in L^A$ is said to be a fuzzy closure system if its core is a closure system and Ψ is the smallest extensional fuzzy set with respect to $\approx$ which contains its core.

Notice that there exist extensional sets whose core is a closure system but are not fuzzy closure systems.

Example 4. Let us consider (A, ρ) as in Example 3. Consider the fuzzy set $\Psi = \{a_1/0.8, a_2/1\}$ which is extensional and satisfies that $\mathcal{F} = \mathrm{Core}(\Psi) = \{a_2\}$ is a closure system. Then, we have that $\Phi_{c_{\mathcal{F}}} = \{a_1/0.6, a_2/1\} \subsetneq \Psi \subsetneq \Phi_{c_\Psi} = A$ and Ψ is not the extensional hull of $\mathcal{F}$, which is $\Phi_{c_{\mathcal{F}}}$.

Once the definition has been established, the first goal of the paper is achieved. Now, we put our focus on the next one, which is studying the existence of a one-to-one relation between fuzzy closure systems and closure operators. The proof of this is now a direct consequence of the results proved so far, namely, Theorem 11, and Theorems 14 and 16 in this section.

Theorem 20. *Let $\mathbb{A} = (A, \rho)$ be a complete fuzzy lattice. The following assertions hold:*

1. *If c is a closure operator on $\mathbb{A}$, the fuzzy set Φ_c defined as $\Phi_c(a) = \rho(c(a), a)$ is a fuzzy closure system.*
2. *If Φ is a fuzzy closure system, the mapping $c_\Phi \colon A \to A$ defined as $c_\Phi(a) = \bigsqcap(a^\rho \otimes \Phi)$ is a closure operator on $\mathbb{A}$.*
3. *If $c \colon A \to A$ is a closure operator on $\mathbb{A}$, then $c_{\Phi_c} = c$.*
4. *If Φ is a fuzzy closure system, then $\Phi = \Phi_{c_\Phi}$.*

Proof. The first item is straightforward from Theorem 18.

For the third item, assume that c is a closure operator. Then, by Theorem 14 in this paper and Theorem 11, we have that $c_{\Phi_c} = c_{\mathcal{F}_c} = c$.

Now, for the second item, if Φ is a fuzzy closure system, by Theorem 18, there exists a closure operator c such that $\Phi = \Phi_c$ and, by item 3, we have that $c_\Phi = c_{\Phi_c} = c$ which is a closure operator. Moreover, $\Phi = \Phi_c = \Phi_{c_\Phi}$ which proves item 4. $\square$

Example 4 shows how Condition (9) and extensionality are not sufficient in order to be a fuzzy closure system. Nevertheless, Proposition 22 shows that replacing Condition (9) by a stronger property suffices to get the expected results. First, let us show that such condition implies Condition (9).

Lemma 21. *Let $\mathbb{A} = (A, \rho)$ be a complete fuzzy lattice and $\Phi \in L^A$. If, for all $a \in A$, there exists $\min(a^\rho \otimes \Phi)$, then $c_\Phi \colon A \to A$ defined as $c_\Phi(a) = \bigsqcap(a^\rho \otimes \Phi)$ is a closure operator and $\mathrm{Core}(\Phi)$ is a closure system.*

Proof. By the definition of infimum, $\rho(a, c_\Phi(a)) = 1$, for all $a \in A$ (i.e., c_Φ is inflationary), which implies that the existence of $\min(a^\rho \otimes \Phi)$ is equivalent to $c_\Phi(a) \in \mathrm{Core}(\Phi)$, for all $a \in A$.

For any $x, y \in A$, since c_Φ is inflationary, it follows,

$$\rho(x, y) = \rho(x, y) \otimes \rho(y, c_\Phi(y)) \leq \rho(x, c_\Phi(y))$$

Due to the properties of infima $X(a) \leq \rho(\bigsqcap X, a)$, for all $a \in A$ and $X \in L^A$. In particular, for $X = x^\rho \otimes \Phi$ and $c_\Phi(y) \in \mathrm{Core}(\Phi)$, we have that

$$\rho(x, c_\Phi(y)) = (x^\rho \otimes \Phi)(c_\Phi(y)) \leq \rho(c_\Phi(x), c_\Phi(y)) \tag{12}$$

Therefore, c_Φ is isotone.

From Lemma 15 and the hypothesis, we deduce that c_Φ is idempotent:

$$\rho(c_\Phi(c_\Phi(x)), c_\Phi(x)) = \Phi_{c_\Phi}(c_\Phi(x)) \geq \Phi(c_\Phi(x)) = 1.$$

Now, $\mathrm{Core}(\Phi_{c_\Phi}) = \mathrm{Core}(\Phi)$ is proved. The inclusion $\mathrm{Core}(\Phi) \subseteq \mathrm{Core}(\Phi_{c_\Phi})$ follows from Lemma 15. Besides, if $x \in \mathrm{Core}(\Phi_{c_\Phi})$ we have $\rho(c_\Phi(x), x) = 1$, that is $x = c_\Phi(x) \in \mathrm{Core}(\Phi)$. Hence, $\mathrm{Core}(\Phi_{c_\Phi}) \subseteq \mathrm{Core}(\Phi)$.

Finally, by Theorem 14 in this paper and Theorem 11, $\mathrm{Core}(\Phi_{c_\Phi})$ is a closure system and then, $\mathrm{Core}(\Phi)$ is a closure system. $\square$

Proposition 22. *Let $\mathbb{A} = (A, \rho)$ be a complete fuzzy lattice and $\Phi \in L^A$. The following statements are equivalent:*

1. *Φ is a fuzzy closure system.*

2. *Φ is extensional and* $\min(a^\rho \otimes \Phi)$ *exists for all* $a \in A$.

Proof. Assume that Φ is a fuzzy closure system. Then, $\Phi = \Phi_{c_\Phi}$, being c_Φ a closure operator, by Theorem 20, hence Φ is extensional by Theorem 16. Furthermore, applying now Theorem 14, we have that $\mathrm{Core}(\Phi) = \mathrm{Core}(\Phi_{c_\Phi})$ is a (crisp) closure system which coincides with $\mathcal{F}_{c_\Phi}$ and, for all $a \in A$, it holds:

$$\bigsqcap(a^\rho \otimes \Phi) = \bigsqcap(a^\rho \otimes \Phi_{c_\Phi}) = \bigsqcap(a^\rho \otimes \mathcal{F}_{c_\Phi})$$

By Proposition 12, we also obtain that

$$\bigsqcap(a^\rho \otimes \Phi) = \bigsqcap(a^\rho \otimes \mathcal{F}_{c_\Phi}) \in \mathcal{F}_{c_\Phi} = \mathrm{Core}(\Phi).$$

This implies that $\min(a^\rho \otimes \Phi)$ exists, for all $a \in A$.

Conversely, assume Φ is extensional and $\min(a^\rho \otimes \Phi)$ exists for all $a \in A$.

By hypothesis, $\Phi(c_\Phi(a)) = 1$, for all $a \in A$. Also, by Lemma 21, c_Φ is a closure operator. Observe that $\Phi_{c_\Phi} = \Phi$: one inclusion always holds (Lemma 15) and

$$\Phi_{c_\Phi}(a) = \rho(c_\Phi(a), a) = (c_\Phi(a) \approx a) = \Phi(c_\Phi(a)) \otimes (c_\Phi(a) \approx a) \le \Phi(a)$$

for all $a \in A$, because Φ is extensional.

Finally, Theorem 18 implies that Φ is a fuzzy closure system. $\square$

After all, fuzzy closure systems rely mainly on satisfying Condition (9). However, the next example illustrates that, even though Condition (9) is necessary, it is not sufficient.

Example 5. Consider the complete lattice (A, ρ) as in Example 1. The fuzzy set $\Psi = \{{}^{\perp}\!/0, {}^{a}\!/0, {}^{b}\!/0, {}^{c}\!/1, {}^{d}\!/0.5, {}^{e}\!/0, {}^{\top}\!/1\}$ satisfies Condition (9) because $\mathrm{Core}(\Psi) = \{c, \top\}$ is a closure system. However, it is not a fuzzy closure system since $\Psi(c) \otimes (c \approx \perp) \not\le \Psi(\perp)$. Indeed, as it is not a fuzzy closure system, $\Phi_{c_\Psi} \ne \Psi$. In this case, $c_\Psi(x) = c$ if $\rho(x, c) = 1$ and $\top$ otherwise. And, consequently,

$$\Psi = \{{}^{c}\!/1, {}^{d}\!/0.5, {}^{\top}\!/1\} \subsetneq \{{}^{\perp}\!/0.5, {}^{a}\!/0.5, {}^{b}\!/0.5, {}^{c}\!/1, {}^{d}\!/0.5, {}^{e}\!/0.5, {}^{\top}\!/1\} = \Phi_{c_\Psi}.$$

The problem of closure systems as fuzzy sets has been faced before in the literature. For example, the approach by Liu and Lu [25] has been cited before in this paper. One of the most cited approaches to this topic was introduced by Bělohlávek in [2]. During the rest of the paper we focus on putting his and our approach together in the same framework and studying the relationship between both definitions, looking for similarities and differences.

First, let us remind the reader that the product and residuum of $\mathbb{L}$ can be extended to external operations on the powerset lattice as follows: for any pair of elements $\ell \in L, u \in U$ and any fuzzy set $X \in L^U$,

$$\begin{aligned} (\ell \otimes X)(u) &= \ell \otimes X(u) \\ (\ell \to X)(u) &= \ell \to X(u). \end{aligned} \tag{13}$$

Notice that these operations involve elements of L and elements of L^U, hence they are particularly defined on the powerset lattice.

The following definition of closure system in a fuzzy setting was given by Bělohlávek in [2], on the fuzzy lattice (L^U, S).

Definition 23 *([2]).* Let K be a filter in $\mathbb{L}$ and U be a universe set. A system $\Phi \in L^{L^U}$ is called an $\mathbb{L}_K$-closure $\mathbb{L}$-system if for every $X, Y \in L^U$ we have

$$\Phi\left(\bigcap_{X_i \in L^U, S(X, X_i) \in K} ((\Phi(X_i) \otimes S(X, X_i)) \to X_i)\right) = 1$$
$$\text{and } \Phi(X) \otimes (X \approx Y) \le \Phi(Y),$$

where the external operation introduced in (13) is used. For $K = L$ the subscript will be omitted.

Remark 3. Notice that in this definition the formula $(\Phi(X_i) \otimes S(X, X_i)) \to X_i$ is specific of the powerset lattice, since $\Phi(X_i) \otimes S(X, X_i) \in L$ and $X_i \in L^U$. Thus, this definition can only be done in the powerset lattice, whilst Definition 19 can be used in any general complete fuzzy lattice.

Next result relates Definition 19 and Definition 23 when they are used in the same framework.

Theorem 24. *Let (L^U, S) be an $\mathbb{L}$-powerset lattice. An $\mathbb{L}$-set $\Phi \subseteq L^{L^U}$ is a fuzzy closure system if and only if Φ is an $\mathbb{L}$-closure $\mathbb{L}$-system. In addition, if Φ is a fuzzy closure system then, for all $X \in L^U$,*

$$\min(X^S \otimes \Phi) = \bigcap_{Y \in L^U} ((\Phi(Y) \otimes S(X, Y)) \to Y). \tag{14}$$

Proof. Assume $\Phi \subseteq L^{L^U}$ is a fuzzy closure system. Then, for all $X \in L^U$, there exists $M_X = \min(X^S \otimes \Phi)$. On the one hand,

$$\bigcap_{Y \in L^U} ((\Phi(Y) \otimes S(X, Y)) \to Y) \subseteq (\Phi(M_X) \otimes S(X, M_X)) \to M_X = M_X.$$

On the other hand, since $(X^S \otimes \Phi)(Z) \leq S(M_X, Z)$, for all $Z \in L^U$,

$$M_X \subseteq S(M_X, Z) \to Z \subseteq (X^S \otimes \Phi)(Z) \to Z = (\Phi(Z) \otimes S(X, Z)) \to Z.$$

Thus, equality (14) holds and Φ is an $\mathbb{L}$-closure $\mathbb{L}$-system.

Conversely, assume $\Phi \subseteq L^{L^U}$ is an $\mathbb{L}$-closure $\mathbb{L}$-system. Then, for all $X \in L^U$, $F_X = \bigcap_{Y \in L^U} ((\Phi(Y) \otimes S(X, Y)) \to Y) \in \mathrm{Core}(\Phi)$. Notice that, for all $Y \in L^U$ we have

$$X \subseteq S(X, Y) \to Y \subseteq (\Phi(Y) \otimes S(X, Y)) \to Y.$$

Hence $(X^S \otimes \Phi)(F_X) = 1$.

Lastly, since $F_X \leq \Phi(Y) \otimes S(X, Y) \to Y$ we get for all $x \in U$,

$$\Phi(Y) \otimes S(X, Y) \otimes F_X(x) \leq Y(x),$$

which implies

$$\Phi(Y) \otimes S(X, Y) \leq F_X(x) \to Y(x).$$

Thus, $(X^S \otimes \Phi)(Y) \leq S(F_X, Y)$, for all $Y \in L^U$.

Therefore, $F_X = \min(X^S \otimes \Phi)$, for all $X \in L^U$ and Φ is a fuzzy closure system. □

Remark 4. Notice that the proof shows the equality between the minimum and the intersection of these particular family of sets, which is true even without extensionality. However, none of the concepts is useful to the theory without extensionality. That is why the theorem says that $\mathbb{L}$-closure $\mathbb{L}$-systems and fuzzy closure systems are equivalent instead of just stating the equivalence between the two formulas.

4. Conclusions and further work

In this paper, the starting point has been the work done on closure systems as crisp sets in [29]. The core of this work has been a thorough discussion on different possible extensions of closure systems to the fuzzy framework in order to find one that maintains the desirable property of being in bijective correspondence with closure operators. The search for an appropriate definition follows the study of the properties of fuzzy sets associated to closure operators and vice versa. Once a proper definition is established, it is proved to be in one-to-one relation with closure operators.

It follows from the results of the paper that every other structure in one-to-one correspondence with closure operator will also be in one-to-one relation with fuzzy closure systems defined here. However, the equivalence between fuzzy closure systems and those structures remains an open problem.

From the results presented in this paper it follows that, from the usual definition of closure operator in the fuzzy framework, the definition of closure system provided is unique, up to equivalences. All this assuming that (as we pointed out at the beginning of Section 3) the semantics of the term 'closure system' corresponds to 'set of closed elements', which in the fuzzy framework translates to *the degree of membership of an element to the set coincides with the degree to which the element is closed*. Obviously, other sets can be found that are in one-to-one correspondence with the closure operators, but they diverge from the semantics of the term 'closure system'.

Some ideas for future research are extending the concept of closure operator to some relational structure instead of a crisp function, this approach is similar to the one done previously in [7], and discuss whether there is a relation between fuzzy closure systems and these relations and, if that were the case, study its properties. Besides, this mostly theoretical study is a technical step necessary to continue the research started in [28] in order to study minimality of implicational bases using the concept of pseudo-closed element in the fuzzy framework.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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