A Class of Predefined-Time Stabilizing Controllers for Nonholonomic Systems

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***Abstract*— The design of a class of predefined-time stabilizing controller for a class uncertain nonholonomic systems in chained form is investigated in this paper. First, some modifications to the classical fixed-time algorithms for first and second order systems are introduced. These modified algorithms, which are developed under the concept of predefined-time stability, reduce the settling time overestimation drawback suffered by the classical fixed-time algorithm. Unlike current finite-time and fixed-time schemes, an upper bound of the settling time is easily tunable through a simple selection of the parameters of the controllers. Then, based on the developed first and second-order algorithms, a switching control strategy is designed to guarantee the predefined-time stability of the chained-form nonholonomic system. Finally, a simulation example is presented to show the effectiveness of the proposed method.**

I. INTRODUCTION

The study of the stabilization of nonholonomic systems has received a lot of attention during the recent years due to its wide range of applications, such as wheeled vehicles [1], [2], underwater vehices [3], satellites and others. On the other hand, the stabilization of nonholonomic systems represent a significant challenge due to the nonintegrable constraints from the point of view of control design. For instance, the Brockett's Theorem [4] says that this class of systems cannot be stabilized using smooth (or even continuous) state feedback controllers [5]. This problem has conducted to investigate different control strategies based on smooth time-verying feedback control [6] and discontinuous controllers [7], [8].

However, most of the mentioned works only induce asymptotic stability, whereas in practice, control schemes which induce finite-time stability are preferred [9], [10], given that they allow a faster convergence and better disturbance rejection properties [11]. The main issue of finite-time schemes is that the convergence time (settling time) grows unboundedly as the initial conditions of the system deviate from the equilibrium point. Hence, as an extension of the finite-time stability concept, the fixed-time stability concept has been developed mainly in [12], [13], allowing to eliminate the unboundedness of the settling time. Although the fixed-time stability concept represents a significant advantage over finite-time stability, it is often complicated to find an explicit and direct relationship between the system tunable parameters and the settling time (for instance, for fixed-time stability algorithms designed through the homogeneity in the bi-limit property, an upped bound estimate of the settling time is not obtained [12]).

To overcome this mentioned drawback, the predefined-time stability concept has been studied in [14], [15], [16], [17], [18]. For these systems, an upper bound (sometimes the least upper bound) of the settling time is set explicitly as a function of their parameters. A predefined-time controller for nonholonomic systems was developed in [19].

Thus, this paper deals with the design of a class of predefined-time stabilizing controller for chained-form nonholonomic systems. The main contributions of this paper are the following:

(i) based on modifications to classical fixed-time controllers [13], predefined-time algorithms for first and second-order systems are introduced. These algorithms reduce the settling time overestimation drawback suffered by the classical fixed-time algorithms;

(ii) based on the developed first and second-order algorithms, a switching control strategy is designed to guarantee the predefined-time stability of the chained-form nonholonomic system.

II. NOTATION, PRELIMINARIES AND PROBLEM STATEMENT

Some preliminaries, necessary for the main contribution of this work such as the notation to be used throughout this document and some basic definitions and results concerning the Incomplete Beta function, and the predefined-time stability concept, are introduced in this section. Moreover, the predefined-time stabilization problem for nonholonomic chained-form systems is stated.

A. Notation

We use the following notation throughout the paper: $\mathbb{R}$ is the set of real numbers and $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$ is the set of positive real numbers. For $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x}^T$ denotes its transpose and $||\boldsymbol{x}|| = \sqrt{\boldsymbol{x}^T\boldsymbol{x}}$ denotes its Euclidean norm. $B_r(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{R}^n : ||\boldsymbol{y} - \boldsymbol{x}|| < r\}$ is the open ball with radius $r \in \mathbb{R}_+$ and centered in $\boldsymbol{x} \in \mathbb{R}^n$. $\mathbb{W}^n_{\mathcal{I}} = \{\boldsymbol{y} : \mathcal{I} \to \mathbb{R}^n : \boldsymbol{y} \text{ is continuous}\}$ is the set of vector valued

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continuous trajectories which map the interval $\mathcal{I} \subseteq \mathbb{R}_+ \cup \{0\}$ to $\mathbb{R}^n$. For any real number h, the function $x \to \lfloor x \rceil^h$ is defined as $\lfloor x \rceil = |x|^h \text{sign}(x)$ for any $x \in \mathbb{R}$ if $h > 0$, and for any $x \in \mathbb{R} \setminus \{0\}$ if $h \leq 0$.

B. On the Incomplete Beta function

First of all, recall the definition of the Beta function.

Definition 1 (Beta function [20]): Let $\alpha, \beta \in \mathbb{R}_+$. The *Beta function* is defined as

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\text{d}t. \quad (1)$$

Splitting the integral (1) at a point $0 \leq r \leq 1$, two incomplete beta functions are obtained. This motivates the following definition.

Definition 2 (Incomplete Beta function [21]): Let $\alpha, \beta \in \mathbb{R}_+$ and $0 \leq r \leq 1$. The *Incomplete Beta function* and the *regularized Incomplete Beta function* are defined as

$$b(\alpha, \beta, r) = \int_0^r t^{\alpha-1}(1-t)^{\beta-1}\text{d}t, \text{ and} \quad (2)$$

$$I(\alpha, \beta, r) = \frac{b(\alpha, \beta, r)}{B(\alpha, \beta)}, \quad (3)$$

respectively.

Clearly, the Incomplete Beta function satisfies $b(\alpha, \beta, 1) = B(\alpha, \beta)$. Thus, the regularized Incomplete Beta function complies to $I(\alpha, \beta, 1) = 1$.

The following proposition related to the Incomplete Beta function will be useful for an illustrative predefined-time stablitiy example presented in the next subsection.

Proposition 1: Let $\rho_2, \rho_3, \rho_4, \rho_5, \rho_6 \in \mathbb{R}_+$ be parameters satisfying $0 < \rho_4\rho_5 < 1 < \rho_4\rho_6$. Let $r \in \mathbb{R}_+$, hence,

$$\int_0^r \frac{\text{d}z}{(\rho_2 z^{\rho_5} + \rho_3 z^{\rho_6})^{\rho_4}} = \frac{1}{\rho_2^{m_{\rho_6}} \rho_3^{m_{\rho_5}} (\rho_6 - \rho_5)} b\left(m_{\rho_5}, m_{\rho_6}, \frac{\rho_3 r^{\rho_6-\rho_5}}{\rho_2 + \rho_3 r^{\rho_6 - \rho_5}}\right), \quad (4)$$

where $m_{\rho_5} = \frac{1-\rho_4\rho_5}{\rho_6-\rho_5} > 0$ and $m_{\rho_6} = \frac{\rho_4\rho_6 - 1}{\rho_6 - \rho_5} > 0$.

Proof: The left side of (4) can be rewritten as

$$\int_0^r \frac{\text{d}z}{(\rho_2 z^{\rho_5} + \rho_3 z^{\rho_6})^{\rho_4}} = \int_0^r \left(\frac{\rho_3 z^{\rho_6-\rho_5}}{\rho_2 + \rho_3 z^{\rho_6-\rho_5}}\right)^{\rho_4} \frac{\text{d}z}{\beta^{\rho_4} z^{\rho_4\rho_6}},$$

which in turn, through the map $t = \frac{\rho_3 z^{\rho_6-\rho_5}}{\rho_2+\rho_3 z^{\rho_6-\rho_5}} =: \eta(z)$, takes the form

$$\int_0^r \frac{\text{d}z}{(\rho_2 z^{\rho_5} + \rho_3 z^{\rho_6})^{\rho_4}} = \frac{1}{\rho_2^{m_{\rho_6}} \rho_3^{m_{\rho_5}} (\rho_6 - \rho_5)} \int_0^{\eta(r)} t^{m_{\rho_5}-1}(1-t)^{m_{\rho_6}-1}dt.$$

By Definition 2, of the Incomplete Beta function (2), the result follows. ∎

C. On predefined-time stability

Consider the nonlinear system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}; \boldsymbol{\rho}), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \quad (5)$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the system state, the vector $\boldsymbol{\rho} \in \mathbb{R}^b$ stands for the system (5) parameters, which are assumed to be constant, i.e., $\dot{\boldsymbol{\rho}} = 0$. The function $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be nonlinear and may be discontinuous, so the solutions of (5) are understood in the sense of Filippov [22]. The origin is assumed to be an equilibrium point of system (5).

Although under the above assumptions the solutions of (5) may be non-unique, this study is concerned only with the case when the predefined-time stability property holds for all solutions. Let $\mathcal{S}(\boldsymbol{x}_0)$ be the set of all solutions $\boldsymbol{x}(t, \boldsymbol{x}_0)$ of (5) starting from $\boldsymbol{x}_0$. Hence, all conditions presented in the definitions below are assumed to be held for all $\boldsymbol{x}(t, \boldsymbol{x}_0) \in \mathcal{S}(\boldsymbol{x}_0)$.

Definition 3 (Lyapunov stability [23]): The origin of system (5) is said to be *Lyapunov stable* if for all $\epsilon \in \mathbb{R}_+$, there exists $\delta := \delta(\epsilon) \in \mathbb{R}_+$ such that for all $\boldsymbol{x}_0 \in B_\delta(\boldsymbol{0})$

(i) any solution $\boldsymbol{x}(t, \boldsymbol{x}_0) \in \mathcal{S}(\boldsymbol{x}_0)$ of (5) exists for all $t \geq 0$, and

(ii) $\boldsymbol{x}(t, \boldsymbol{x}_0) \in B_\epsilon(\boldsymbol{0})$ for all $t \geq 0$.

Now, consider the functional $T_0 : \mathbb{W}^n_{\mathbb{R}_+ \cup \{0\}} \to \bar{\mathbb{R}}_+ \cup \{0\}$ defined by

$$T_0(\boldsymbol{y}) = \inf_\tau \{\tau \geq 0 : \ \boldsymbol{y}(t) = 0 \ \forall t \geq \tau\}.$$

Note that if $\boldsymbol{y}(\tau) \neq 0 \ \forall \tau \in \mathbb{R}_+ \cup \{0\}$, then $T_0(\boldsymbol{y}) = +\infty$.

Definition 4 (Settling-time function [23]): The *settling-time function* of system (5) is defined as

$$T(\boldsymbol{x}_0) = \sup_{\boldsymbol{x}(t,\boldsymbol{x}_0) \in \mathcal{S}(\boldsymbol{x}_0)} T_0(\boldsymbol{x}(t, \boldsymbol{x}_0)). \quad (6)$$

Definition 5 (Finite-time stability [23]): The origin of system (5) is said to be *globally finite-time stable* if it is Lyapunov stable and the settling-time function $T(\boldsymbol{x}_0)$ is finite on $\mathbb{R}^n$, i.e., $T(\boldsymbol{x}_0) < +\infty$ for $\boldsymbol{x}_0 \in \mathbb{R}^n$.

Lemma 1: (Finite-time stability characterization for scalar systems [10]) Let $n = 1$ in (5) (scalar system). The origin of this system is globally finite-time stable if and only if for all $x \in \mathbb{R} \setminus \{0\}$

(i) $xf(x; \boldsymbol{\rho}) < 0$, and

(ii) $\int_x^0 \frac{\text{d}z}{f(z;\boldsymbol{\rho})} < +\infty$.

Sketch of the proof: A rigorous proof of Lemma 1 can be found in [24]. Intuitively, condition *(i)* implies Lyapunov stability (consider the Lyapunov function candidate $V(x) = \frac{1}{2}x^2$ and apply the Lyapunov theorem [25]). Moreover, under the conditions of Lemma 1, one can note that the settling time function is $T(x_0) = \int_0^{T(x_0)} \text{d}t$. Since first-order systems do not oscillate, the solution $x(\cdot, x_0) : [0, T(x_0)) \to [x_0, 0)$ of system (5) as a function of t defines a bijection. Using it as a variable change, the above integral equals (note that $\frac{1}{f(x;\boldsymbol{\rho})}$ is defined for all $x \in \mathbb{R}^n \setminus \{0\}$ from condition *(i)*)

$$T(x_0) = \int_0^{T(x_0)} \text{d}t = \int_{x_0}^0 \frac{\text{d}x}{f(x; \boldsymbol{\rho})}. \quad (7)$$

Thus, condition *(ii)* of Lemma 1 refers to the settling-time function being finite. ■

Definition 6 (Fixed-time stability [23]): The origin of system (5) is said to be *globally fixed-time stable* if it is globally finite-time stable and the settling-time function $T(\boldsymbol{x}_0)$ is bounded on $\mathbb{R}^n$, i.e., there exists $T_{\max} \in \mathbb{R}_+$ such that $T(\boldsymbol{x}_0) \leq T_{\max}$ for all $\boldsymbol{x}_0 \in \mathbb{R}^n$.

Assuming that the origin of (5) is fixed-time stable, the bound $T_{\max}$ in Definition 6 is trivially non-unique.

Definition 7 ([26]): Let the origin be fixed-time stable for system (5). The set of all the bounds of the settling-time function is defined as

$$\mathcal{T} = \{T_{\max} \in \mathbb{R}_+ : T(\boldsymbol{x}_0) \leq T_{\max} \ \forall \, \boldsymbol{x}_0 \in \mathbb{R}^n\}.$$

Remark 1: For several applications such as state estimation, dynamic optimization or fault detection, a desirable property would be that the trajectories of system (5) reach the origin within a time $T_c \in \mathcal{T}$, which can be defined in advance as function of the system parameters $\boldsymbol{\rho}$, i.e., $T_c = T_c(\boldsymbol{\rho})$.

Apparently, this could be a direct application of the fixed-time stability concept. However, the important work [12] shows that fixed-time stability is guaranteed if the vector field of system (5) is homogeneous in the bi-limit. Nevertheless, although it is assured that the settling-time function is bounded, an upper bound estimate $T_{\max}$ is usually not obtained using this property.

To distinguish this case to the one where the designer can actually set a settling time-function bound T_c in advance as a function of system parameters ρ, the concept of predefined-time stability is introduced.

Definition 8 (Predefined-time stability[14]): For the parameter vector $\boldsymbol{\rho}$ of the system (5) and an arbitrarily selected constant $T_c := T_c(\boldsymbol{\rho}) > 0$, the origin of (5) is said to be *predefined-time stable* if it is fixed-time stable and the settling-time function $T : \mathbb{R}^n \to \mathbb{R}$ is such that

$$T(\boldsymbol{x}_0) \leq T_c, \quad \forall \boldsymbol{x}_0 \in \mathbb{R}^n.$$

If this is the case, T_c is called a *predefined time*.

Consider the system

$$\begin{aligned} \dot{x} &= f(x; \boldsymbol{\rho}) \\ &= -\frac{\gamma(\rho_2, \rho_3, \rho_4, \rho_5, \rho_6)}{\rho_1} \left\lfloor \rho_2 \lfloor x \rceil^{\rho_5} + \rho_3 \lfloor x \rceil^{\rho_6} \right\rceil^{\rho_4}, \end{aligned} \tag{8}$$

where $x \in \mathbb{R}$ is the state of the system, $\boldsymbol{\rho} = [\rho_1, \ldots, \rho_6]^T \in \mathbb{R}^6$ is the vector of parameters, which comply to $\rho_1, \rho_2, \rho_3, \rho_4 > 0$ and $0 < \rho_4\rho_5 < 1 < \rho_4\rho_6$, and

$$\gamma(\rho_2, \rho_3, \rho_4, \rho_5, \rho_6) := \gamma = \frac{B(m_{\rho_5}, m_{\rho_6})}{\rho_2^{m_{\rho_6}} \rho_3^{m_{\rho_5}} (\rho_6 - \rho_5)}, \tag{9}$$

with $m_{\rho_5} = \frac{1-\rho_4\rho_5}{\rho_6-\rho_5} > 0$ and $m_{\rho_6} = \frac{\rho_4\rho_6-1}{\rho_6-\rho_5} > 0$.

Pretty similar systems have been studied in [12], [13] under the concept of fixed-time stability. The following lemma states that its origin is predefined-time stable.

Lemma 2 (Predefined-time stability example [13], [18]): The origin $x = 0$ of the system (8) is predefined-time stable, with predefined time $T_c(\boldsymbol{\rho}) = \rho_1$.

Proof: Note that the product

$$x f(x; \boldsymbol{\rho}) = -\frac{\gamma}{\rho_1} \left(\rho_2 |x|^{\rho_5} + \rho_3 |x|^{\rho_6}\right)^{\rho_4} < 0,$$

fulfills the hypothesis *(i)* of Lemma 1. On the other hand, from (7), the settling-time function

$$T(x_0) = -\frac{\rho_1}{\gamma} \int_{x_0}^{0} \frac{\operatorname{sign}(x) \mathrm{d}x}{\left(\rho_2 |x|^{\rho_5} + \rho_3 |x|^{\rho_6}\right)^{\rho_4}},$$

can be rewritten, through the map $z = |x|$, as

$$T(x_0) = \frac{\rho_1}{\gamma} \int_{0}^{|x_0|} \frac{\mathrm{d}z}{\left(\rho_2 z^{\rho_5} + \rho_3 z^{\rho_6}\right)^{\rho_4}}.$$

Using Proposition 1 and by the definition of γ, the above yields

$$\begin{aligned} T(x_0) &= \frac{\rho_1}{B(m_{\rho_5}, m_{\rho_6})} b\left(m_{\rho_5}, m_{\rho_6}, \frac{\rho_3 |x_0|^{\rho_6-\rho_5}}{\rho_2 + \rho_3 |x_0|^{\rho_6-\rho_5}}\right) \\ &= \rho_1 I\left(m_{\rho_5}, m_{\rho_6}, \frac{\rho_3 |x_0|^{\rho_6-\rho_5}}{\rho_2 + \rho_3 |x_0|^{\rho_6-\rho_5}}\right). \end{aligned}$$

Additionally, note that the settling-time function complies to

$$T(x_0) \leq \sup_{x_0 \in \mathbb{R}} T(x_0) = \lim_{|x_0| \to \infty} T(x_0) = \rho_1 < +\infty$$

satisfying the hypothesis *(ii)* of Lemma 1. Thus the origin $x = 0$ of the system (8) is globally finite-time stable. Indeed, it is predefined-time stable since the settling-time function is globally bounded by the arbitrary predefined time $T_c(\boldsymbol{\rho}) = \rho_1$. ■

D. Problem statement

Consider the following chained-form system

$$\begin{aligned} \dot{x}_0 &= u_0 + \Delta_0(t) \\ \dot{x}_1 &= x_2 u_0 \\ \dot{x}_2 &= u_1 + \Delta_1(t), \end{aligned} \tag{10}$$

where $\boldsymbol{x} = [x_0, \, x_1, \, x_2]^T \in \mathbb{R}^3$ is the state of the system and $\boldsymbol{u} = [u_0, \, u_1]^T \in \mathbb{R}^2$ is the control input. For $i = 0, 1$, the terms $\Delta_i : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}$ unknown but bounded perturbations, i.e., $\sup_{t \in \mathbb{R}_+ \cup \{0\}} |\Delta_i(t)| \leq \delta_i$ with δ_i a known positive constant.

Remark 2: The chained-form system (10) may represent a large class of physical systems (wheeled mobile robots, autonomous underwater vehicles, unmanned aerial vehicles, hopping robots, etc.). Indeed, many mechanical and electrical systems with first order nonholonomic constraints can be locally or globally modeled as (10). Formally, any kinematic model of first-order nonholonomic systems can be transformed into (10) as long as the state space dimension is three and the input space dimension is two [27].

The main objective is to design a controller $\boldsymbol{u}$ which guarantees the predefined-time stability of the origin $\boldsymbol{x} = \mathbf{0}$ of the system (10).

III. MAIN RESULT

The main result of this document, i.e. the solution to the predefined-time stabilization problem for nonholonomic chained-form systems, is carried out in this section. To this end, the system (10) is split into the following two coupled subsystems

$$\textstyle\sum_1: \quad \left\{ \begin{array}{rcl} \dot{x}_0 &=& u_0 + \Delta_0(t) \end{array} \right. \tag{11}$$

$$\textstyle\sum_2: \quad \left\{ \begin{array}{rcl} \dot{x}_1 &=& x_2 u_0 \\ \dot{x}_2 &=& u_1 + \Delta_1(t) \end{array} \right. \tag{12}$$

Using this decomposition, the controller $\boldsymbol{u}$ design will be performed under the following sequential strategy:

1) First, on the one hand, a constant control input $u_0 = \rho_8$ is applied. In this case, under the transformation $\xi_1 = x_1$, $\xi_2 = \rho_8 x_2$ and $v_1 = \rho_8 u_1$, the subsystem Σ_2 becomes:
$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= v_1 + \rho_8 \Delta_1(t). \end{aligned} \tag{13}$$
Hence, on the other hand, u_1 is designed as a predefined-time stabilizing controller for the transformed system (13), to ensure that its origin is predefined-time stable in spite of the matched perturbation term $a\Delta_1(t)$, with predefined time T_1. This is, to achieve the sub-objective
$$\left\{ \begin{array}{rcl} x_1(t) &=& 0 \\ x_2(t) &=& 0 \end{array} \right., \qquad \forall t \geq T_1 \tag{14}$$
2) Now, for $t \geq T_1$, the control input u_1 is designed such that x_2 remains zero in spite of the presence of perturbation term $\Delta_1(t)$. Notice, from (12), that if x_1 is initially zero and x_2 remains zero, x_1 remains also zero. The control input u_0 is designed such that the origin of the uncertain first-order subsystem Σ_1 is predefined-time stable.

It should be clear from the proposed strategy that the first step requires the design of a robust predefined-time stabilizing controller for second-order systems subject to matched perturbation terms, while the second step requires the design of a robust predefined-time stabilizing controller for first-order systems subject to matched perturbation terms. Hence, in the rest of this section, some predefined-time controllers are proposed for a first and second order systems subject to matcher perturbations; then, the switching strategy that guarantees predefined-time stability of the origin of the closed-loop nonholonomic system (10) is presented.

A. Robust predefined-time stabilization of first-order systems with matched perturbations

Consider the controlled first-order system

$$\dot{x} = u + \Delta(t) \tag{15}$$

where $x \in \mathbb{R}$ is the system state, $u \in \mathbb{R}$ is the control input and $\Delta : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}$ is an unknown perturbation term bounded by $\sup_{t\in\mathbb{R}_+\cup\{0\}} |\Delta(t)| \leq \delta$, with $0 \leq \delta < \infty$ a known constant.

Lemma 3 ([18]): Let $\boldsymbol{\rho} = [\rho_0, \ldots, \rho_6]^T \in \mathbb{R}^7$ be a vector of parameters such that $\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6 > 0$, $0 < \rho_4\rho_5 < 1 < \rho_4\rho_6$ and $\rho_0 \geq \delta$, and let $\gamma = \gamma(\rho_2, \rho_3, \rho_4, \rho_5, \rho_6)$ be as in (9). If the control input u is selected as

$$\begin{aligned} u &= \phi_1(x; \boldsymbol{\rho}) \\ &= -\left[\frac{\gamma}{\rho_1}\left(\rho_2 |x|^{\rho_5} + \rho_3 |x|^{\rho_6}\right)^{\rho_4} + \rho_0\right] \text{sign}(x), \end{aligned} \tag{16}$$

then, the origin $x = 0$ of system (15) is predefined-time stable with predefined time $T_c(\boldsymbol{\rho}) = \rho_1$.

Proof: Let $z = |x|$. The time-derivative of the variable $z \in \mathbb{R}_+ \cup \{0\}$ satisfies

$$\begin{aligned} \dot{z} &= -\frac{\gamma}{\rho_1}\left(\rho_2 z^{\rho_5} + \rho_3 z^{\rho_6}\right)^{\rho_4} - \rho_0 + \text{sign}(x)\Delta(t) \\ &\leq -\frac{\gamma}{\rho_1}\left(\rho_2 z^{\rho_5} + \rho_3 z^{\rho_6}\right)^{\rho_4} - \rho_0 + |\Delta(t)| \\ &\leq -\frac{\gamma}{\rho_1}\left(\rho_2 z^{\rho_5} + \rho_3 z^{\rho_6}\right)^{\rho_4} - (\rho_0 - \delta) \\ &\leq -\frac{\gamma}{\rho_1}\left(\rho_2 z^{\rho_5} + \rho_3 z^{\rho_6}\right)^{\rho_4}. \end{aligned}$$

By the comparison lemma [25] and Lemma 2, the trajectories of the variable $z(t) = |x(t)|$ over time are majored by the solutions of (8), i.e. $z(t) = 0$ for $t \geq \rho_1$. Hence, the origin $x = 0$ of the closed-loop system (15)-(16) is predefined-time stable with predefined time $T_c(\boldsymbol{\rho}) = \rho_1$. ■

B. Robust predefined-time stabilization of second-order systems with matched perturbations

Consider the controller second-order system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + \Delta(t), \end{aligned} \tag{17}$$

where $x_1, x_2 \in \mathbb{R}$ are the state variables, $u \in \mathbb{R}$ is the control input and $\Delta(t) \in \mathbb{R}$ is an unknown but bounded perturbation term of the form $\sup_{t\in\mathbb{R}_+\cup\{0\}} |\Delta(t)| \leq \delta$, with $0 \leq \delta < \infty$ a known constant.

Lemma 4: Let $\boldsymbol{\rho}^1 = \left[\rho_1^1, \ldots, \rho_6^1\right]^T \in \mathbb{R}^6$ and $\boldsymbol{\rho}^2 = \left[\rho_1^2, \ldots, \rho_6^2\right]^T \in \mathbb{R}^6$ be vectors of parameters such that $\rho_1^i, \rho_2^i, \rho_3^i, \rho_4^i, \rho_5^i, \rho_6^i > 0$, $0 < \rho_4^i\rho_5^i < 1 < \rho_4^i\rho_6^i$, for $i = 1, 2$. Additionally, let $\rho_7 \geq \delta$, and let $\gamma_i = \gamma_i(\rho_2^i, \rho_3^i, \rho_4^i, \rho_5^i, \rho_6^i)$ be as in (9), for $i = 1, 2$. Select the control input as

$$\begin{aligned} u &= \phi_2(x_1, x_2; \boldsymbol{\rho}^1, \boldsymbol{\rho}^2, \rho_7) \\ &= -\frac{\gamma_2}{\rho_1^2}\left(\rho_2^2 |\sigma|^{\rho_5^2} + \rho_3^2 |\sigma|^{\rho_6^2}\right)^{\rho_4^2} \text{sign}(\sigma) - \rho_7 \text{sign}(\sigma) - \\ &\quad \frac{2\gamma_1^2 \rho_4^1 |\xi(x_1)|^{2\rho_4^1 - 1}}{(\rho_1^1)^2}\left(\rho_2^1\rho_5^1 |x_1|^{\rho_5^1 - 1} + \rho_3^1\rho_6^1 |x_1|^{\rho_6^1 - 1}\right) \\ &\quad \text{sign}(\sigma), \end{aligned} \tag{18}$$

where the sliding variable σ is defined as

$$\sigma = x_2 + \lfloor \varphi(x_1, x_2) \rceil^{1/2}, \tag{19}$$

with $\varphi(x_1,x_2) = \lfloor x_2 \rceil^2 + \frac{2\gamma_1^2}{(\rho_1^1)^2}\lfloor \xi(x_1) \rceil^{2\rho_4^1}$ and $\xi(x_1) = \rho_2^1 \lfloor x_1 \rceil^{\rho_5^1} + \rho_3^1 \lfloor x_1 \rceil^{\rho_6^1}$. Then, the origin $[x_1 \quad x_2]^T = \mathbf{0} \in \mathbb{R}^2$ of the closed-loop system (17)-(18) is predefined-time stable with predefined time $T_c(\boldsymbol{\rho}^1, \boldsymbol{\rho}^2, \rho_7) = \rho_1^1 + \rho_1^2$.

Proof: The time-derivative of the sliding variable σ (19) is

$$\begin{aligned}
\dot{\sigma} &= u + \Delta + \frac{|x_2|(u+\Delta)}{|\varphi(x_1,x_2)|^{1/2}} + \\
&\frac{\frac{2\gamma_1^2\rho_4^1|\xi(x_1)|^{2\rho_4^1-1}}{(\rho_1^1)^2}\left(\rho_2^1\rho_5^1|x_1|^{\rho_5^1-1} + \rho_3^1\rho_6^1|x_1|^{\rho_6^1-1}\right)x_2}{|\varphi(x_1,x_2)|^{1/2}} \\
&= -\frac{\gamma_2}{\rho_1^2}\left(\rho_2^2|\sigma|^{\rho_5^2} + \rho_3^2|\sigma|^{\rho_6^2}\right)^{\rho_4^2}\text{sign}(\sigma) - \rho_7\text{sign}(\sigma) + \Delta - \\
&\frac{2\gamma_1^2\rho_4^1|\xi(x_1)|^{2\rho_4^1-1}}{(\rho_1^1)^2}\left(\rho_2^1\rho_5^1|x_1|^{\rho_5^1-1} + \rho_3^1\rho_6^1|x_1|^{\rho_6^1-1}\right)\text{sign}(\sigma) - \\
&\frac{\frac{\gamma_2}{\rho_1^2}\left(\rho_2^2|\sigma|^{\rho_5^2} + \rho_3^2|\sigma|^{\rho_6^2}\right)^{\rho_4^2} + \rho_7 - \Delta\text{sign}(\sigma)}{|\varphi(x_1,x_2)|^{1/2}}\text{sign}(\sigma) - \\
&\frac{\frac{2\gamma_1^2\rho_4^1|\xi(x_1)|^{2\rho_4^1-1}}{(\rho_1^1)^2}\left(\rho_2^1\rho_5^1|x_1|^{\rho_5^1-1} + \rho_3^1\rho_6^1|x_1|^{\rho_6^1-1}\right)}{|\varphi(x_1,x_2)|^{1/2}} \cdot \\
&\qquad (|x_2| - x_2\text{sign}(\sigma))\,\text{sign}(\sigma).
\end{aligned}$$

Then, the time-detivative of the nonnegative variable $s = |\sigma|$ complies to

$$\dot{s} \le -\frac{\gamma_2}{\rho_1^2}\left(\rho_2^2 s^{\rho_5^2} + \rho_3^2 s^{\rho_6^2}\right)^{\rho_4^2}.$$

By the comparison lemma [25] and Lemma 2, the trajectories of the variable $s(t) = |\sigma(t)|$ over time are majored by the solutions of (8), i.e. $s(t) = 0$ for $t \ge \rho_1^2$. Furthermore, once on the manifold $\sigma = 0$, one gets the following reduced-order dynamics for x_1

$$\dot{x}_1 = -\frac{\gamma_1}{\rho_1^1}\left\lfloor \rho_2^1 \lfloor x_1 \rceil^{\rho_5^1} + \rho_3^1 \lfloor x_1 \rceil^{\rho_6^1} \right\rceil^{\rho_4^1}.$$

Hence, the origin $x_1 = 0$ of this reduced order system is predefined-time stable with predefined time ρ_1^1. Moreover, from (19), if $x_1 = 0$ and $\sigma = 0$, it must be that $x_2 = 0$ also. Thus, it is concluded that the origin $[x_1 \quad x_2]^T = \mathbf{0} \in \mathbb{R}^2$ of system (17) is predefined-time stable with predefined time $T_c(\boldsymbol{\rho}^1, \boldsymbol{\rho}^2, \rho_7) = \rho_1^1 + \rho_1^2$. ■

C. A predefined-time stabilizing controller for nonholonomic systems

Based on the above results for uncertain first and second order systems, the switching strategy described previously is introduced in the following theorem to guarantee predefined-time stability of the closed-loop system for a class of uncertain chained-form nonholonomic systems.

Theorem 1: Let $\rho_8 \in \mathbb{R}$ and $\rho_9 \ge \delta_1$. Additionally, let $\boldsymbol{\rho}$ be as in Lemma 3 with $\rho_0 \ge \delta_0$, and $\boldsymbol{\rho}^1, \boldsymbol{\rho}^2$ be as in Lemma 4 with $\rho_7 \ge |\rho_8|\,\delta_1$. Selecting the switching controller for system (10)

$$\begin{aligned}
u_0 &= \begin{cases} \rho_8, & \text{if } t \le \rho_1^1 + \rho_1^2 \\ \phi_1(x_0; \boldsymbol{\rho}), & \text{else} \end{cases} \\
u_1 &= \begin{cases} \frac{\phi_2(x_1, \rho_8 x_2; \boldsymbol{\rho}^1, \boldsymbol{\rho}^2, \rho_7)}{\rho_8}, & \text{if } t \le \rho_1^1 + \rho_1^2 \\ -\rho_9\text{sign}(x_2), & \text{else} \end{cases}
\end{aligned} \tag{20}$$

where the functions ϕ_1 and ϕ_2 are defined in (16) (Lemma 3) and (18) (Lemma 4), then, the origin of closed-loop system (10)-(20) is predefined-time stable with predefined-time $T_c(\boldsymbol{\rho}^1, \boldsymbol{\rho}^2, \boldsymbol{\rho}) = \rho_1^1 + \rho_1^2 + \rho_1$.

Proof: Let us divide the proof into two steps.

- For $t < \rho_1^1 + \rho_1^2$, a constant control input $u_0 = \rho_8$ is applied. Hence, following the decomposition introduced in (11)-(12), subsystem Σ_2 can be reduced to (13) with $\xi_1 = x_1$, $\xi_2 = \rho_8 x_2$ and $v_1 = \rho_8 u_1$. From Lemma 4, one can conclude that x_1 and x_2 converge to zero in predefined-time $\rho_1^1 + \rho_1^2$ in spite of the presence of perturbation Δ_1.
- For $t \ge \rho_1^1 + \rho_1^2$, the control input u_1 is designed to keep $x_2(t) = 0$. Indeed, considering the candidate Lyapunov function $V = |x_2|$. Its time derivative is given by
$$\dot{V} \le -|x_2|(\gamma - \delta_1)$$
It means that $x_2(t) = 0$ for all $t \ge \rho_1^1 + \rho_1^2$ in spite of the presence of perturbation Δ_1. Moreover, from (10), x_1 and x_2 remain zero no matter what the control input u_0 is. Finally, from Lemma 3, x_0 converge to zero in predefined-time $\rho_1^1 + \rho_1^2 + \rho_1$ in spite of the presence of perturbation Δ_0.

■

IV. SIMULATION RESULTS

Consider a unicycle-type mobile robot as in [19]. Under the nonholonomic constraints, the kinematics of the wheeled-mobile robot is

$$\begin{aligned}
\dot{x} &= v\cos\theta \\
\dot{y} &= v\sin\theta \\
\dot{\theta} &= w
\end{aligned} \tag{21}$$

where (x, y) is the center of mass, θ is the heading angle, and v (resp. w) is the linear (resp. angular) velocity.

The nonsingular transformation given by $x_0 = x$, $x_1 = y$, $x_2 = \tan\theta$, $u_0 = v\cos\theta$ and $u_1 = w\sec^2\theta$, transforms (21) into (10). To test the robustness of the proposed scheme, the perturbations $\Delta_0(t) = 0.1\sin t$ and $\Delta_1(t) = 0.3\sin t$ have been added in the following simulations.

The control objective is the predefined-time stabilization problem, i.e. parking problem, of the unicycle-type mobile robot. In the simulation, the following control parameters are selected: $\boldsymbol{\rho} = [0.1, 2, 1, 1, 1, 0.9, 1.1]^T$, $\boldsymbol{\rho}^1 = [0.5, 1, 1, 0.6, 1, 3]^T$, $\boldsymbol{\rho}^2 = [5, 1, 1, 1, 0.9, 1.1]^T$, $\rho_7 = 0.03$, $\rho_8 = -0.1$, and $\rho_9 = 0.3$.

From Theorem 1, the controller (20) guarantees the predefined-time stability of the closed-loop system, with predefined-time $T_c = 7.5$ s, in spite of the perturbations, as can be seen in Fig. 1.

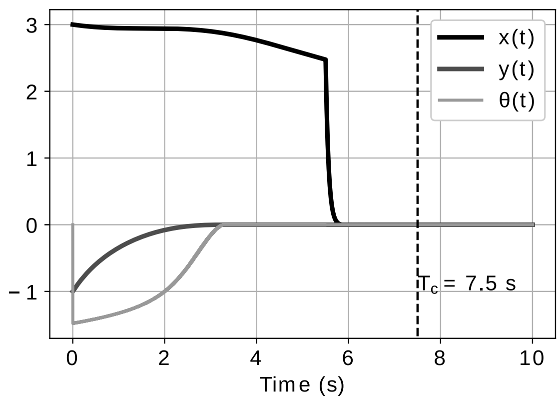

Fig. 1. State variables of wheeled-mobile robot.

V. CONCLUSIONS

This work was devoted to giving one solution to the predefined-time stabilization of a chained-form class of nonholonomic systems problem. In this path, we proposed modifications to the classical fixed-time stabilization controllers for first and second-order systems proposed, mainly, in [12], [13]. These modifications allowed us to achieve two intentions:

(i) to reduce the settling time overestimation issues presented by the classical fixed-time algorithms;

(ii) since the modified algorithms were developed under the predefined-time stability concept, an upper bound of the settling time is easily tuned through a simple selection of the controller parameters.

Finally, the solution to the stated problem was developed as a switching contoller, which was designed based on the proposed first and second-order controllers. A numerical simulation was carried out to show the performance of the proposed scheme.

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