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# Fuzzy closure relations

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**Abstract**

The concept of closure operator is key in several branches of mathematics. In this paper, closure operators are extended to relational structures, more specifically to fuzzy relations in the framework of complete fuzzy lattices. The core of the work is the search for a suitable definition of (strong) fuzzy closure relation, that is, a fuzzy relation whose relation with fuzzy closure systems is one-to-one. The study of the properties of fuzzy closure systems and fuzzy relations helps narrow down this exploration until an appropriate definition is settled.


*Keywords:* Fuzzy relation; Closure operator; Complete lattice; Fuzzy logic; Extensional

## 1. Introduction

Fuzzy mathematics were introduced by Lotfi A. Zadeh in 1965 [28]. In that paper fuzzy sets and fuzzy relations appeared for the first time in the literature. Since then, the research on fuzzy structures has grown extensively. From the original papers using maximum and minimum on the unit interval, research is now focused on structures over a partially ordered structure, normally a residuated lattice, and the minimum has been replaced by the more general family of operations called t-norms, usually denoted by $\otimes$, see e.g., [19]. The work with minimum and maximum, currently infimum and supremum due to the lack of a linearly ordered structure, is still present in the so-called Gödel logic and structures that receive the name of Heyting algebras [16]. However, the framework in this paper is a complete fuzzy lattice over a general residuated lattice.

The concept of *complete fuzzy lattice* considered in this paper was originally introduced by Bělohlávek in [3] with the name of *completely lattice* $\mathbb{L}$*-ordered set*. One of the most important examples of a complete fuzzy lattice is the $\mathbb{L}$-powerset with the so-called subsethood degree relation. Closure operators and systems on $\mathbb{L}$-powerset lattices were introduced by Bělohlávek in [1].

The other key concept to be used in this paper is that of closure operator. Originally introduced by E. H. Moore in [24], these structures seem to be ubiquitous in mathematics, together with their counterpart closure systems. In

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particular, fuzzy closure operators [1,5] appear in several areas of fuzzy logic such as fuzzy mathematical morphology [14,23], fuzzy relational equations [12], approximate reasoning [4,11] and fuzzy logic in narrow sense [16]. But also its applications such as fuzzy logic programming [21] or formal concept analysis of data with fuzzy attributes [26].

Even though the definition of fuzzy closure operator seems to be the same for most authors, i.e., a crisp mapping with some fuzzy extension of inflationarity, isotonicity and idempotency, the situation is quite different for fuzzy closure systems. There are several distinct definitions of this concept in the literature, e.g., [1,6,15,22,25].

In [25], closure systems were extended to the fuzzy framework in two levels, first as crisp sets, called closure systems, which are crisp sets closed under fuzzy infima; and then as fuzzy sets, called fuzzy closure systems, which are the extensional hulls of closure systems, that is, the smallest extensional set that contains them. These definitions were proved to behave properly with the notion of closure operator due to the existence of a bijective correspondence among them. Along with the main results mentioned above, some characterization theorems were proved as well that relate the research done in complete fuzzy lattices with papers that manage these concepts in weaker structures, such as posets and preposets. Finally, when these structures are considered in the $\mathbb{L}$-powerset lattice, the definition of closure system is equivalent to that of $\mathbb{L}$-closure system and fuzzy closure systems are equivalent to $\mathbb{L}$-closure $\mathbb{L}$-systems, which are the most cited definitions that played the role of closure systems in the fuzzy framework, both introduced by Bělohlávek in [1].

As stated above, in [25] a definition of (fuzzy) closure system was presented. The analysis there was straightforward. First, we worked with a crisp object endowed with fuzzy properties and then directly with a fuzzy object. This very discussion can be extended to the other side of the problem since closure operators are crisp mappings with certain fuzzy properties. The main goal of this paper is to extend closure operators to fuzzy structures with fuzzy properties. Fuzzy closure relations have been defined previously in the literature but there is no one-to-one relation between them and fuzzy closure systems. The paper starts with a section of preliminaries where the basic concepts needed in order to follow the paper are presented and referenced. This paper is self-contained, even though the readership is assumed to know the basic concepts of fuzzy sets, fuzzy relations and residuated lattices. If they are not, the sources where the results used are introduced are properly cited and easy to find. Next, the core of the paper is presented as follows. Fuzzy closure relations are defined in the literature as inflationary, isotone and idempotent relations. However, this definition is not in a one-to-one relation with the definition of fuzzy closure system. Hence, a stronger definition of fuzzy closure relation is needed. The search for a suitable definition is the main goal of next section. The idea of fuzzy closure relation induced by a fuzzy closure system helps narrow down the search by adding requirements for the desired *strong fuzzy closure relation* definition. One of the key properties needed for this bijection to hold is again extensionality, which already appeared in [25]. Finally, a suitable definition is set using extensionality and some sort of minimality condition. This definition turns out to be equivalent with several diverse properties, one of the most noticeable being an explicit expression for every strong fuzzy closure relation. Another interesting characterization is the one related to perfect fuzzy functions, introduced by Demirci [13]. Perfect fuzzy functions are fuzzy relations which satisfy certain properties that relate them with functions, e.g., the partial property ($\mu(a, b) \otimes \mu(a, c) \leq (b \approx c)$) reflects that every element has a unique image up to fuzzy equivalence. The main theorem in this section proves the one-to-one relation between strong fuzzy closure relations and fuzzy closure systems. Last, we present our conclusions and some ideas of future work.

## 2. Preliminaries

In this section, a brief presentation of the concepts needed to follow the results is written. This paper is a natural continuation of the work in [25]. Specifically, we recall the notions of complete residuated lattices [2,16], fuzzy poset and fuzzy complete lattice, and some basics results that will be needed throughout the paper [3,20].

Throughout this paper, let $\mathbb{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ be a complete residuated lattice, which is an algebra where

- $(L, \wedge, \vee, 0, 1)$ is a complete lattice with 0 and 1 being the least and the greatest elements of $L$, respectively,
- $(L, \otimes, 1)$ is a commutative monoid (i.e., $\otimes$ is commutative, associative, and 1 is neutral with respect to $\otimes$), and
- $\otimes$ and $\to$ satisfy the so-called *adjointness property*: for all $a, b, c \in L$, we have that $a \otimes b \leq c$ iff $a \leq b \to c$.

This structure is utilized in mathematical fuzzy logics and their applications as structures of truth degrees with $\otimes$ and $\to$ used as truth functions of *fuzzy conjunction* and *fuzzy implication*, respectively [16]. The unit interval with the Łukasiewicz, Gödel and Goguen pairs of t-norms and implications are examples of residuated complete lattices.

Among the properties of complete residuated lattices, we summarize only those that will be used:

$$a \le b \text{ implies } a \otimes c \le b \otimes c, \quad \text{for all } a, b, c \in L \tag{1}$$

$$a \le b \text{ implies } c \to a \le c \to b, \quad \text{for all } a, b, c \in L \tag{2}$$

$$a \le b \text{ implies } b \to c \le a \to c, \quad \text{for all } a, b, c \in L \tag{3}$$

$$a \to b = 1 \text{ iff } a \le b, \quad \text{for all } a, b \in L \tag{4}$$

$$a \to b \le a \otimes c \to b \otimes c, \quad \text{for all } a, b, c \in L \tag{5}$$

$$a \le (a \to b) \to b, \quad \text{for all } a, b \in L \tag{6}$$

$$a \otimes \bigvee_{b \in B} b = \bigvee_{b \in B} (a \otimes b), \quad \text{for all } a \in L \text{ and } B \subseteq L \tag{7}$$

$$\Big(\bigvee_{b \in B} b\Big) \to a = \bigwedge_{b \in B} (b \to a), \quad \text{for all } a \in L \text{ and } B \subseteq L \tag{8}$$

An $\mathbb{L}$-*set* is a mapping $X\colon U \to L$ from the universe set $U$ to the membership values set $L$, where $X(u)$ means the degree in which $u$ belongs to $X$. The set of $\mathbb{L}$-sets on the universe $U$ is denoted by $L^U$. A crisp set is considered to be a particular case of $\mathbb{L}$-set by using its characteristic mapping $X\colon U \to \{0, 1\}$ with $X(u) = 1$ iff $u \in X$.

Operations with $\mathbb{L}$-sets are defined element-wise. For instance, $A \otimes B \in L^U$ is defined as $(A \otimes B)(u) = A(u) \otimes B(u)$ for all $u \in U$. The so-called *subsethood degree relation* is defined as $S\colon L^U \times L^U \to L$ where

$$S(A, B) = \bigwedge_{x \in U} \Big(A(x) \to B(x)\Big).$$

Obviously, $S(A, B) = 1$ if and only if $A(x) \le B(x)$ for all $x \in U$. In this case, $A$ is said to be a fuzzy subset of $B$ and is denoted by $A \subseteq B$.

Binary $\mathbb{L}$-relations (binary fuzzy relations) on a set $U$ can be thought of as $\mathbb{L}$-sets on the universe $U \times U$. That is, a binary $\mathbb{L}$-relation on $U$ is a mapping $\rho \in L^{U \times U}$ assigning to each $x, y \in U$ a truth degree $\rho(x, y) \in L$ (a degree to which $x$ and $y$ are related by $\rho$).

For $\rho$ being a binary $\mathbb{L}$-relation in $U$, we say that

- $\rho$ is *reflexive* if $\rho(x, x) = 1$ for all $x \in U$.
- $\rho$ is *symmetric* if $\rho(x, y) = \rho(y, x)$ for all $x, y \in U$.
- $\rho$ is *antisymmetric* if $\rho(x, y) \otimes \rho(y, x) = 1$ implies $x = y$ for all $x, y \in U$.
- $\rho$ is *transitive* if $\rho(x, y) \otimes \rho(y, z) \le \rho(x, z)$ for all $x, y, z \in U$.

**Definition 1.** Given a non-empty set $A$ and a binary $\mathbb{L}$-relation $\rho$ on $A$, the pair $\mathbb{A} = (A, \rho)$ is said to be a

- *fuzzy preposet* if $\rho$ is a *fuzzy preorder*, i.e. if $\rho$ is reflexive and transitive;
- *fuzzy poset* if $\rho$ is a *fuzzy order*, i.e. if $\rho$ is reflexive, antisymmetric and transitive.

A typical example of fuzzy poset is $(L^U, S)$.

As in the crisp case, any order $\mathbb{L}$-relation implicitly defines an equivalence $\mathbb{L}$-relation that is called *symmetric kernel relation*. In the fuzzy case, this equivalence $\mathbb{L}$-relation usually replaces the notion of equality in the fuzzy poset.

**Definition 2.** Given a fuzzy preposet $\mathbb{A} = (A, \rho)$, the *symmetric kernel relation* is defined as $\approx\colon A \times A \to L$ where $(a \approx b) = \rho(a, b) \otimes \rho(b, a)$ for all $a, b \in A$.

**Proposition 3.** *Given a fuzzy preposet $\mathbb{A} = (A, \rho)$, the symmetric kernel relation $\approx$ is a fuzzy equivalence relation, that is, it is a reflexive, symmetric and transitive fuzzy relation.*

A usual way to define fuzzy algebras is to consider as an underlying structure a pair which consists of a set and a tolerance or equivalence relation on it. Thus, an alternative definition of fuzzy poset was originally introduced by

Höhle and Blachard in the eighties [17] and later rediscovered independently by Bělohlávek and Bodenhofer [3,7]. It is given by a tuple $(A, \approx, \rho)$ where $\approx$ is a fuzzy equivalence relation on $A$ and $\rho$ is a fuzzy order that is compatible with $\approx$. In [27], it is shown that both definitions of fuzzy poset are equivalent.

To present the notion of fuzzy lattice we need to generalize those of upper (lower) bound and supremum (infimum).

**Definition 4.** Given a fuzzy preposet $\mathbb{A} = (A, \rho)$ and a fuzzy set $X \in L^A$, we define the *down-cone* of $X$ and the *up-cone* of $X$, respectively, as the fuzzy sets $X_\rho, X^\rho \in L^A$ where, for all $a \in A$,

$$X_\rho(a) = \bigwedge_{x\in A} \big(X(x) \to \rho(a, x)\big) \qquad \text{and}$$
$$X^\rho(a) = \bigwedge_{x\in A} \big(X(x) \to \rho(x, a)\big).$$

Thus, $X^\rho(a)$ and $X_\rho(a)$ can be seen as the degree to which $a$ is an upper bound and lower bound of $X$, respectively.

Notice that for singletons, $X = \{x\}$, we will omit the brackets for simplicity of the notation, thus $x^\rho(a) = \rho(x, a)$ and $x_\rho(a) = \rho(a, x)$.

**Definition 5.** Let $\mathbb{A} = (A, \rho)$ be a fuzzy preposet and $X \in L^A$. An element $a \in A$ is said to be *infimum* (resp. *supremum*) of $X$ if the following conditions hold:

1. $X_\rho(a) = 1$ (resp. $X^\rho(a) = 1$).
2. $X_\rho(x) \le \rho(x, a)$ (resp. $X^\rho(x) \le \rho(a, x)$), for all $x \in A$.

The following propositions, which are straightforward, provide useful characterizations of infimum, supremum, minimum and maximum.

**Proposition 6.** *Let $\mathbb{A} = (A, \rho)$ be a fuzzy preposet and $X \in L^A$. An element $a \in A$ is infimum (resp. supremum) of $X$ if and only if, for all $x \in A$,*

$$\rho(x, a) = X_\rho(x) \qquad (\text{resp. } \rho(a, x) = X^\rho(x)).$$

**Proposition 7.** *Let $\mathbb{A} = (A, \rho)$ be a fuzzy preposet and $X \in L^A$. An element $m \in A$ is a* minimum *(resp.* maximum*) of $X$ if and only if $m$ is an infimum (resp. a supremum) of $X$ and $X(m) = 1$.*

Notice that the supremum (resp. infimum) needs not be unique in an arbitrary preposet, but if $\mathbb{A}$ is a poset and due to antisymmetry, it is not difficult to see that, if a supremum (resp. infimum) of $X$ exists, it is unique. In such case, we will denote it by $\bigsqcup X$ (resp. $\bigsqcap X$).

**Definition 8** *([3]).* We say that a fuzzy poset $(A, \rho)$ is a complete fuzzy lattice if every fuzzy subset $X \in L^A$ has supremum and infimum.

The pair $(L^U, S)$ is an example of complete fuzzy lattice, which is called the *$L$-powerset lattice of $U$*. This fact follows easily from [2, Theorem 5.63].

Notice that, if $(A, \rho)$ is a complete fuzzy lattice, then $(A, \le)$, with $a \le b$ iff $\rho(a, b) = 1$, is a complete lattice. Therefore, there exist elements that are minimum and maximum, which we denote by $\bot$ and $\top$ respectively.

Given a fuzzy relation $\mu$ between $A$ and $B$, i.e., a crisp mapping $\mu\colon A \times B \to L$, and $a \in A$, the *afterset* $a^\mu$ is the fuzzy set $a^\mu\colon B \to L$ given by $a^\mu(b) = \mu(a, b)$. A fuzzy relation $\mu$ is said to be *total* if the aftersets $a^\mu$ are normal fuzzy sets, for all $a \in A$. The composition of two fuzzy relations $\mu_1\colon A \times B \to L$ and $\mu_2\colon B \times C \to L$ is defined as $(\mu_1 \circ \mu_2)(x, y) = \bigvee_{z\in B}(\mu_1(x, z) \otimes \mu_2(z, y))$.

The so-called full fuzzy powering $\rho_\propto$ is a fuzzy relation between two powersets that has been used in previous works [8,9]. Its direct extension to fuzzy powersets is as follows: for all $X, Y \in L^A$,

$$\rho_\propto(X, Y) = \bigwedge_{x,y\in A} (X(x) \otimes Y(y)) \to \rho(x, y).$$

A fuzzy set $X \in L^A$ is said to be a clique if $\rho_\propto(X, X) = 1$.

Other key concepts used in this paper are closure structures. Thus, the main definitions of the structures are needed to fully comprehend the results presented.

**Definition 9.** Given a fuzzy preposet $\mathbb{A} = (A, \rho)$, a mapping $\mathrm{c}\colon A \to A$ is said to be a *closure operator* on $\mathbb{A}$ if the following conditions hold:

1. $\rho(a, b) \leq \rho(\mathrm{c}(a), \mathrm{c}(b))$, for all $a, b \in A$
2. $\rho(a, \mathrm{c}(a)) = 1$, for all $a \in A$
3. $\rho(\mathrm{c}(\mathrm{c}(a)), \mathrm{c}(a)) = 1$, for all $a \in A$.

Conditions 1 and 2 are well-known and are called isotonicity and inflationarity, respectively. Observe that condition 3 could be replaced by $(\mathrm{c}(\mathrm{c}(a)) \approx \mathrm{c}(a)) = 1$, and, thus, if $\mathbb{A}$ is a fuzzy poset, a closure operator is idempotent in a classical sense, i.e., $\mathrm{c}(\mathrm{c}(a)) = \mathrm{c}(a)$ for all $a \in A$.

On the other hand, there is not a unique extension of the notion of closure system to a fuzzy setting. The definition we will use in this paper is the following one, introduced in [25].

**Definition 10.** Let $(A, \rho)$ be a complete fuzzy lattice. We say that an $\mathbb{L}$-set $\Phi \in L^A$ is a *fuzzy closure system* if it is the minimal extensional set such that $\bigsqcap(a^\rho \otimes \Phi) \in \mathrm{Core}(\Phi)$ for all $a \in A$.

Fuzzy closure systems and fuzzy closure operators are related concepts. In fact, similarly to the classical case, there is a one-to-one relation among them.

**Theorem 11** *([25]). Let $\mathbb{A} = (A, \rho)$ be a complete fuzzy lattice. The following assertions hold:*

1. *If $\Phi$ is a fuzzy closure system, the mapping $\mathrm{c}_\Phi\colon A \to A$ defined as $\mathrm{c}_\Phi(a) = \bigsqcap(a^\rho \otimes \Phi)$ is a closure operator.*
2. *If $\mathrm{c}$ is a closure operator, the fuzzy set $\Phi_\mathrm{c}$ defined as $\Phi_\mathrm{c}(a) = \rho(\mathrm{c}(a), a)$ is a fuzzy closure system.*
3. *If $\Phi$ is a fuzzy closure system, then $\Phi = \Phi_{\mathrm{c}_\Phi}$.*
4. *If $\mathrm{c}\colon A \to A$ is a closure operator, then $\mathrm{c}_{\Phi_\mathrm{c}} = \mathrm{c}$.*

The following result is a characterization of fuzzy closure systems which is useful in the proofs.

**Proposition 12.** *Let $(A, \rho)$ be a complete fuzzy lattice. A fuzzy set $\Phi \in L^A$ is a fuzzy closure system if and only if $\Phi$ is extensional and $\min(a^\rho \otimes \Phi)$ exists for all $a \in A$.*

## 3. Fuzzy closure relations

The use of fuzzy relations as closure structures is not groundbreaking, these structures have been used in the literature before. In [9], for instance, they were defined in the general setting of partial ordered set as follows.

**Definition 13.** Let $\mathbb{A} = (A, \rho)$ be a fuzzy preposet. A total fuzzy relation $\kappa\colon A \times A \to L$ is a *fuzzy closure relation* if the following properties hold:

- $\kappa$ is *inflationary*, i.e. $\rho_\propto(a, a^\kappa) = 1$ for all $a \in A$.
- $\kappa$ is *isotone*, i.e. $\rho(a_1, a_2) \leq \rho_\propto(a_1^\kappa, a_2^\kappa)$ for all $a_1, a_2 \in A$.
- $\kappa$ is *idempotent*, i.e. $\rho_\propto(a^{\kappa\circ\kappa}, a^\kappa) = 1$ for all $a \in A$.

Notice that, in this definition, since $\kappa$ is inflationary and idempotent, we have that $\rho_\propto(a^\kappa, a^{\kappa\circ\kappa}) = 1$ for all $a \in A$. Therefore, the condition of $\kappa$ being idempotent can be replaced for $a^\kappa(x) \otimes a^{\kappa\circ\kappa}(y) \leq (x \approx y)$ for all $a, x, y \in A$, where the relation $\approx$ is defined from $\rho$ as follows: $(x \approx y) = \rho(x, y) \otimes \rho(y, x)$ for all $x, y \in A$.

**Proposition 14.** *Let $\mathbb{A} = (A, \rho)$ be a complete fuzzy lattice. Let $\kappa \colon A \times A \to L$ be a fuzzy closure relation. Then,* $\mathrm{Core}(\kappa)$ *is a closure operator and $a^\kappa$ is a normal clique for all $a \in A$.*

**Proof.** First, since $\kappa$ is isotone, we have that for all $a, b \in A$ the inequality $\rho(a, b) \leq \rho_\propto(a^\kappa, b^\kappa)$ holds. Using the reflexivity of $\rho$ we get

$$1 = \rho(a, a) \leq \rho_\propto(a^\kappa, a^\kappa).$$

Thus, $a^\kappa$ is a clique for all $a \in A$. Also, since $\kappa$ is total, for all $a \in A$ there exists $a^* \in A$ such that $\kappa(a, a^*) = 1$. Hence, $a^\kappa$ is a normal clique.

Let now $x, y \in \mathrm{Core}(a^\kappa)$. Since $a^\kappa$ is a clique we get,

$$1 = \big(a^\kappa(x) \otimes a^\kappa(y) \to \rho(x, y)\big) = \rho(x, y)$$
$$1 = \big(a^\kappa(y) \otimes a^\kappa(x) \to \rho(x, y)\big) = \rho(y, x)$$

Thus, by antisymmetry, $(x \approx y) = \rho(x, y) \otimes \rho(y, x) = 1$, then $x = y$. Therefore, since $\kappa$ is total and $a^\kappa$ is a clique, for every $a \in A$ there exists a unique element $a^* \in A$ such that $\kappa(a, a^*) = 1$. Therefore $\mathrm{Core}(\kappa)$ is a mapping. Let us denote it as $\mathtt{c}$, i.e., $\mathtt{c} \colon A \to A$ is defined by $\mathtt{c}(a) = a^*$, and prove that it is a closure operator. Since $\kappa$ is inflationary we have $\rho_\propto(a, a^\kappa) = 1$ for all $a \in A$. Thus,

$$1 = \rho_\propto(a, a^\kappa) = \bigwedge_{x \in A} a^\kappa(x) \to \rho(a, x) \overset{x=a^*}{\leq} a^\kappa(a^*) \to \rho(a, a^*) = \rho(a, \mathtt{c}(a)).$$

Similarly it is proved that $\mathtt{c}$ is isotone,

$$\begin{aligned} \rho(a, b) \leq \rho_\propto(a^\kappa, b^\kappa) &= \bigwedge_{x, y \in A} a^\kappa(x) \otimes b^\kappa(y) \to \rho(x, y) \\ &\leq a^\kappa(a^*) \otimes b^\kappa(b^*) \to \rho(a^*, b^*) = \rho(a^*, b^*) = \rho(\mathtt{c}(a), \mathtt{c}(b)), \end{aligned}$$

where the inequality holds by taking $x = a^*$ and $y = b^*$.

Finally, $\mathtt{c}$ is idempotent. Since $\kappa$ is idempotent $\rho_\propto(a^{\kappa \circ \kappa}, a^\kappa) = 1$, we get

$$\begin{aligned} 1 = \rho_\propto(a^{\kappa \circ \kappa}, a^\kappa) &= \bigwedge_{x, y \in A} \Big(\bigvee_{z \in A} a^\kappa(z) \otimes z^\kappa(x)\Big) \otimes a^\kappa(y) \to \rho(x, y) \\ &= \bigwedge_{x, y, z \in A} a^\kappa(z) \otimes z^\kappa(x) \otimes a^\kappa(y) \to \rho(x, y) \\ &\leq a^\kappa(a^*) \otimes (a^*)^\kappa(a^{**}) \otimes a^\kappa(a^*) \to \rho(a^{**}, a^*) \\ &= \rho(a^{**}, a^*) = \rho(\mathtt{c}(\mathtt{c}(a)), \mathtt{c}(a)), \end{aligned}$$

where the second equality uses (8) and the inequality holds by taking $z = y = a^*$ and $x = a^{**}$. Therefore, since $\mathtt{c}$ is an inflationary, isotone and idempotent mapping, it is a closure operator. □

**Remark 1.** Notice that in the proof, the fuzzy relation $\rho$ is required to be just reflexive and antisymmetric. Moreover, for $\mathrm{Core}(\kappa)$ to be a mapping, the fuzzy relation $\kappa$ needs to be only total and isotone.

Following the spirit of the work done in [25], the main goal of this paper is to find two mappings which relate fuzzy closure systems and fuzzy closure relations and are the inverse of one another. In the cited paper, one of the mappings related any closure operator $\mathtt{c}$ to a fuzzy set defined as $\Phi_\mathtt{c}(a) = \rho(\mathtt{c}(a), a)$. In the terminology of fuzzy relations, for any closure relation $\kappa$, the assigned fuzzy set should represent in which degree the closure of an element is below the element itself. A plausible option, similar to the one showed above, would be the fuzzy set $\Phi_\kappa(a) = \rho_\propto(a^\kappa, a)$. The reciprocal mapping was $\mathtt{c}_\Phi(a) = \prod(a^\rho \otimes \Phi)$, again, in the terminology of relations, we need to define $\kappa_\Phi(a, b)$ as the degree to which the element $b$ is the closure of $a$. Following this idea, the following relation comes up, $\kappa_\Phi(a, b) = \big(\prod(a^\rho \otimes \Phi) \approx b\big)$.

The following is a technical result that will be used in the rest of the paper. This result was originally proved in [10] on fuzzy T-digraphs. Since complete fuzzy lattices are T-digraphs the result applies to the framework of this paper.

**Proposition 15.** *Let $\mathbb{A} = (A, \rho)$ be a complete fuzzy lattice. Let $X \in L^A$ be normal cliques and $y_0 \in A$. For all $x_0 \in \mathrm{Core}(X)$ it holds*

$$\rho_\propto(X, y_0) = \rho(x_0, y_0).$$

**Proof.** Let $X \in L^A$ be normal cliques and $x_0 \in \mathrm{Core}(X)$, $y_0 \in A$. Then,

$$\begin{aligned}\rho_\propto(X, y_0) &= \bigwedge_{x\in A} X(x) \to \rho(x, y_0)\\ &\le X(x_0) \to \rho(x_0, y_0)\\ &= \rho(x_0, y_0),\end{aligned}$$

by taking $x = x_0$.

Conversely,

$$\begin{aligned}\rho(x_0, y_0) &= 1 \to \rho(x_0, y_0)\\ &\overset{(i)}{\le} \rho(x, x_0) \to \rho(x, x_0) \otimes \rho(x_0, y_0)\\ &\overset{(ii)}{\le} X(x) \otimes X(x_0) \to \rho(x, y_0)\\ &= X(x) \to \rho(x, y_0)\end{aligned}$$

where the inequality $(i)$ holds by (5) and the inequality $(ii)$ holds by (2), (3), transitivity of $\rho$ and $X$ being a clique.

Thus, $\rho_\propto(x_0, y_0) \le \bigwedge_{x\in A} X(x) \to \rho(x, y_0) = \rho_\propto(X, y_0)$.

Therefore, $\rho_\propto(X, y_0) = \rho(x_0, y_0)$. □

Next, some properties of fuzzy closure relations are proved. The theorem below shows how close fuzzy closure relations are to being in a one-to-one relation with fuzzy closure systems.

**Lemma 16.** *Let $\mathbb{A} = (A, \rho)$ be a complete fuzzy lattice. Let $\mathrm{c}\colon A \to A$ be a closure operator. Then, the fuzzy relation $\kappa_{\mathrm{c}}(a, b) = (\mathrm{c}(a) \approx b)$ is a fuzzy closure relation.*

**Proof.** The proof is a comprobation of inflationarity, isotonicity and idempotency. Let $a \in A$,

$$\begin{aligned}\rho_\propto(a, a^{\kappa_{\mathrm{c}}}) &= \bigwedge_{x\in A} \kappa_{\mathrm{c}}(a, x) \to \rho(a, x)\\ &= \bigwedge_{x\in A} (\mathrm{c}(a) \approx x) \to \rho(a, x)\\ &\ge \bigwedge_{x\in A} \rho\,(\mathrm{c}(a), x) \to \rho(a, x)\\ &= \rho(a, \mathrm{c}(a)) = 1,\end{aligned}$$

where the second to last equality is by adjointness and transitivity and the last one due to inflationarity of $\mathrm{c}$.

Therefore, $\kappa_{\mathrm{c}}$ is inflationary. Next, let $a, b \in A$,

$$\begin{aligned}\rho_\propto(a^{\kappa_{\mathrm{c}}}, b^{\kappa_{\mathrm{c}}}) &= \bigwedge_{x,y\in A} a^{\kappa_{\mathrm{c}}}(x) \otimes b^{\kappa_{\mathrm{c}}}(y) \to \rho(x, y)\\ &= \bigwedge_{x,y\in A} (\mathrm{c}(a) \approx x) \otimes (\mathrm{c}(b) \approx y) \to \rho(x, y)\\ &= \bigwedge_{x,y\in A} (x \approx \mathrm{c}(a)) \otimes (\mathrm{c}(b) \approx y) \to \rho(x, y)\\ &\overset{(i)}{\ge} \bigwedge_{x,y\in A} (\rho\,(\mathrm{c}(a), \mathrm{c}(b)) \to \rho(x, y)) \to \rho(x, y)\end{aligned}$$

$$\overset{(ii)}{\geq} \bigwedge_{x,y\in A} \rho\left(\mathrm{c}(a), \mathrm{c}(b)\right)$$
$$= \rho\left(\mathrm{c}(a), \mathrm{c}(b)\right) \geq \rho(a, b),$$

where $(i)$ is a consequence of

$$(x \approx \mathrm{c}(a)) \otimes \rho(\mathrm{c}(a), \mathrm{c}(b)) \otimes (\mathrm{c}(b) \approx y) \leq \rho(x, y)$$

being true by transitivity, and applying adjointness

$$(x \approx \mathrm{c}(a)) \otimes (\mathrm{c}(b) \approx y) \leq \rho(\mathrm{c}(a), \mathrm{c}(b)) \to \rho(x, y),$$

$(ii)$ is the direct use of property (6) and the last inequality is due to $\mathrm{c}$ being isotone. Hence, $\kappa_{\mathrm{c}}$ is isotone.

And finally idempotency is proved.

$$\rho_{\propto}(a^{\kappa_{\mathrm{c}}\circ\kappa_{\mathrm{c}}}, a^{\kappa_{\mathrm{c}}}) = \bigwedge_{x,y\in A} (\kappa_{\mathrm{c}} \circ \kappa_{\mathrm{c}})(a, x) \otimes \kappa_{\mathrm{c}}(a, y) \to \rho(x, y) =$$
$$= \bigwedge_{x,y\in A} \left(\left(\bigvee_{z\in A} \kappa_{\mathrm{c}}(a, z) \otimes \kappa_{\mathrm{c}}(z, x)\right) \otimes \kappa_{\mathrm{c}}(a, y)\right) \to \rho(x, y)$$
$$= \bigwedge_{x,y\in A} \left(\bigvee_{z\in A} \kappa_{\mathrm{c}}(a, z) \otimes \kappa_{\mathrm{c}}(z, x) \otimes \kappa_{\mathrm{c}}(a, y)\right) \to \rho(x, y)$$
$$= \bigwedge_{x,y\in A} \left(\bigvee_{z\in A} (x \approx \mathrm{c}(z)) \otimes (z \approx \mathrm{c}(a)) \otimes (\mathrm{c}(a) \approx y)\right) \to \rho(x, y)$$
$$\geq \bigwedge_{x,y\in A} \left(\bigvee_{z\in A} (x \approx \mathrm{c}(z)) \otimes (\mathrm{c}(z) \approx \mathrm{c}(\mathrm{c}(a))) \otimes (\mathrm{c}(a) \approx y)\right) \to \rho(x, y)$$
$$\overset{(iii)}{\geq} \bigwedge_{x,y\in A} \left(\bigvee_{z\in A} \rho(x, y)\right) \to \rho(x, y) = \bigwedge_{x,y\in A} \rho(x, y) \to \rho(x, y) = 1,$$

where $(iii)$ holds by the idempotency of $\mathrm{c}$ and applying transitivity,

$$(x \approx \mathrm{c}(z)) \otimes (\mathrm{c}(z) \approx \mathrm{c}(\mathrm{c}(a))) \otimes (\mathrm{c}(a) \approx y)$$
$$= (x \approx \mathrm{c}(z)) \otimes (\mathrm{c}(z) \approx \mathrm{c}(\mathrm{c}(a))) \otimes \rho(\mathrm{c}(\mathrm{c}(a)), \mathrm{c}(a)) \otimes (\mathrm{c}(a) \approx y)$$
$$\leq \rho(x, y).$$

Therefore, $\kappa_{\mathrm{c}}(a, b) = \left(\prod(a^{\rho} \otimes \mathrm{c}\right) \approx b) = (\mathrm{c}(a) \approx b)$ is a fuzzy closure relation. □

**Theorem 17.** *Let $\mathbb{A} = (A, \rho)$ be a complete fuzzy lattice. The following statements hold:*

1. *If $\kappa$ is a fuzzy closure relation, the fuzzy set $\Phi_\kappa$ defined as $\Phi_\kappa(a) = \rho_{\propto}(a^{\kappa}, a)$ is a fuzzy closure system.*
2. *If $\Phi$ is a fuzzy closure system, the relation $\kappa_\Phi \colon A \times A \to L$ defined as $\kappa_\Phi(a, b) = \left(\prod(a^{\rho} \otimes \Phi\right) \approx b)$ is a fuzzy closure relation.*
3. *If $\Phi$ is a fuzzy closure system, then $\Phi = \Phi_{\kappa_\Phi}$.*

**Proof.** By Proposition 14, $\mathrm{Core}(\kappa)$ is a closure operator, thus, throughout the proof, it will be managed with the standard mapping notation and will be denoted by $\mathrm{c}$.

For the first item, the main goal is to prove $\Phi_\kappa = \Phi_{\mathrm{c}}$. Since, $\mathrm{c}(a) = \mathrm{Core}(a^{\kappa})$, using Proposition 15, we get $\Phi_{\mathrm{c}}(a) = \rho(\mathrm{c}(a), a) = \rho_{\propto}(a^{\kappa}, a) = \Phi_\kappa(a)$, for all $a \in A$. By Theorem 11, $\Phi_{\mathrm{c}}$ is a fuzzy closure system. Hence, $\Phi_\kappa$ is a fuzzy closure system.

For the second item, by Theorem 11, $\mathrm{c}_\Phi(a) = \prod(a^{\rho} \otimes \Phi)$ is a closure operator, then using Lemma 16, $\kappa_\Phi$ is a fuzzy closure relation.

For the third item, let $\Phi \in L^A$ be a fuzzy closure system. Then,

$$\begin{aligned}\Phi_{\kappa_\Phi}(a) = \rho_\propto(a^{\kappa_\Phi}, a) &= \bigwedge_{x\in A} \kappa_\Phi(a, x) \to \rho(x, a)\\ &= \bigwedge_{x\in A} \rho(\mathtt{c}_\Phi(a), x) \otimes \rho(x, \mathtt{c}_\Phi(a)) \to \rho(x, a)\\ &\geq \bigwedge_{x\in A} \rho(x, \mathtt{c}_\Phi(a)) \to \rho(x, a)\\ &= \rho(\mathtt{c}_\Phi(a), a) = \Phi_{\mathtt{c}_\Phi}(a) = \Phi(a),\end{aligned}$$

where the last equality holds due to Theorem 11.

Conversely,

$$\begin{aligned}\Phi_{\kappa_\Phi}(a) = \rho_\propto(a^{\kappa_\Phi}, a) &= \bigwedge_{x\in A} (\mathtt{c}_\Phi(a) \approx x) \to \rho(x, a)\\ &\leq (\mathtt{c}_\Phi(a) \approx \mathtt{c}_\Phi(a)) \to \rho(\mathtt{c}_\Phi(a), a)\\ &= \rho(\mathtt{c}_\Phi(a), a) = \Phi_{\mathtt{c}_\Phi}(a) = \Phi(a),\end{aligned}$$

where the inequality holds by choosing $x = \mathtt{c}_\Phi(a)$. □

Even though Definition 13 seems to be the natural relational extension of a closure operator and is interesting on its own, notice that in the previous result $\kappa_{\Phi_\kappa} = \kappa$ was not proved because it does not necessarily hold, as the following example shows:

**Example 1.** Consider the Gödel unit interval as the underlying algebra of truth values and the complete fuzzy lattice $(A, \rho)$ where $A = \{a, b\}$ and

| $\rho$ | $a$ | $b$ |
|---|---|---|
| $a$ | 1 | 1 |
| $b$ | 0.8 | 1 |

Consider the following two fuzzy relations $\kappa_1$ and $\kappa_2$.

| $\kappa_1$ | $a$ | $b$ |
|---|---|---|
| $a$ | 1 | 0.3 |
| $b$ | 0.4 | 1 |

| $\kappa_2$ | $a$ | $b$ |
|---|---|---|
| $a$ | 1 | 0.1 |
| $b$ | 0.1 | 1 |

These relations are indeed fuzzy closure relations, the proof is tedious but simple, we omit it since it is not the point of the example.

The fuzzy sets induced by these relations are the following. By Proposition 14, $a^{\kappa_1}$ and $a^{\kappa_2}$ are normal cliques for all $a \in A$, calculations are simplified by Proposition 15,

$$\begin{aligned}\Phi_{\kappa_1}(a) &= \rho_\propto(a^{\kappa_1}, a) = \rho(a, a) = 1,\\ \Phi_{\kappa_1}(b) &= \rho_\propto(b^{\kappa_1}, b) = \rho(b, b) = 1,\\ \Phi_{\kappa_2}(a) &= \rho_\propto(a^{\kappa_2}, a) = \rho(a, a) = 1,\\ \Phi_{\kappa_2}(b) &= \rho_\propto(b^{\kappa_2}, b) = \rho(b, b) = 1.\end{aligned}$$

Thus, $\Phi_{\kappa_1} = \Phi_{\kappa_2} = A$. Furthermore, for each $x, y \in A$,

$$\begin{aligned}\kappa_{\Phi_{\kappa_1}}(x, y) = \kappa_{\Phi_{\kappa_2}}(x, y) = \kappa_A(x, y) &= \left(\bigsqcap(x^\rho \otimes A) \approx y\right)\\ &= \left(\bigsqcap(x^\rho) \approx y\right) = (x \approx y) = \rho(x, y) \otimes \rho(y, x).\end{aligned}$$

Hence, we have

| $\kappa_A$ | $a$ | $b$ |
|---|---|---|
| $a$ | 1 | 0.8 |
| $b$ | 0.8 | 1 |

Notice how $\kappa_{\Phi_{\kappa_1}} \neq \kappa_1$ and $\kappa_{\Phi_{\kappa_2}} \neq \kappa_2$. Therefore, the definition of fuzzy closure relation used in the literature is not strong enough to get a bijection with fuzzy closure systems.

## 4. Strong fuzzy closure relations

Since Definition 13 cannot ensure the one-to-one correspondence we are looking for, some other definitions shall be considered. In order to find a suitable definition for these relations, first the two mappings in play $\Phi \mapsto \kappa_\Phi(a,b) = \left(\prod(a^\rho \otimes \Phi) \approx b\right)$ and $\kappa \mapsto \Phi_\kappa(a) = \rho_\propto(a^\kappa, a)$ must be studied.

We focus now on extensionality, which is related to previous work done on closure systems [25]. A fuzzy relation $\kappa\colon A \times A \to L$ is said to be extensional if it is extensional when considered as a fuzzy subset of $A \times A$ with the similarity relation defined pointwise as

$$(a_1, b_1) \approx_{A\times A} (a_2, b_2) = (a_1 \approx a_2) \otimes (b_1 \approx b_2).$$

It is easy to see that, due to the reflexivity of $\approx$, extensionality in $A \times A$ is equivalent to satisfying left and right extensionality:

$$\kappa(a_1, b) \otimes (a_1 \approx a_2) \leq \kappa(a_2, b), \qquad \text{for all } a_1, a_2, b \in A.$$
$$\kappa(a, b_1) \otimes (b_1 \approx b_2) \leq \kappa(a, b_2), \qquad \text{for all } a, b_1, b_2 \in A.$$

The use of this binary fuzzy equivalence is a common approach which can be seen, for example, in [13, Remark 3.3].

The next example shows that there exist fuzzy closure relations which are not extensional.

**Example 2.** Consider the complete fuzzy lattice from Example 1 and consider the fuzzy closure relation $\kappa_1$. This fuzzy relation is not extensional.

$$\kappa_1(a, a) \otimes (a \approx a) \otimes (a \approx b) = 0.8 \not\leq 0.3 = \kappa_1(a, b).$$

Nevertheless, next proposition shows that the fuzzy relation $\kappa_\Phi$ is extensional for any $\Phi \in L^A$.

**Proposition 18.** *Let $\mathbb{A} = (A, \rho)$ be a complete fuzzy lattice. Let $\Phi \in L^A$ be a fuzzy closure system, then $\kappa_\Phi$ is extensional.*

**Proof.** Let $\Phi \in L^A$ be a fuzzy closure system, then, $c_\Phi$ is a closure operator and,

$$\begin{aligned}
&\kappa_\Phi(a, b) \otimes (a \approx x) \otimes (b \approx y)\\
&\quad = (c_\Phi(a) \approx b) \otimes (a \approx x) \otimes (b \approx y)\\
&\quad \overset{\text{isot.}c_\Phi}{\leq} (c_\Phi(x) \approx c_\Phi(a)) \otimes ((c_\Phi(a)) \approx b) \otimes (b \approx y)\\
&\quad \overset{\text{trans.}}{\leq} (c_\Phi(x) \approx y) = \kappa_\Phi(x, y). \quad \square
\end{aligned}$$

These properties hint that the fuzzy closure relations which are in one-to-one relation with fuzzy closure systems must be extensional. In the general case, extensional fuzzy closure relations do not behave as expected.

**Example 3.** Let $(A, \rho)$ be the complete fuzzy lattice, the set $\{0, 0.5, 1\}$ with Łukasiewicz logic as the underlying algebra of truth values, where the universe is $A = \{\bot, a, b, c, d, e, \top\}$ and $\rho$ is the fuzzy order defined by the following table,

| $\rho$ | $\bot$ | $a$ | $b$ | $c$ | $d$ | $e$ | $\top$ |
|---|---|---|---|---|---|---|---|
| $\bot$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $a$ | 0.5 | 1 | 0.5 | 1 | 1 | 1 | 1 |
| $b$ | 0.5 | 0.5 | 1 | 1 | 1 | 1 | 1 |
| $c$ | 0.5 | 0.5 | 0.5 | 1 | 1 | 1 | 1 |
| $d$ | 0 | 0.5 | 0 | 0.5 | 1 | 0.5 | 1 |
| $e$ | 0 | 0 | 0.5 | 0.5 | 0.5 | 1 | 1 |
| $\top$ | 0 | 0 | 0 | 0.5 | 0.5 | 0.5 | 1 |

Let $\mu\colon A \times A \to L$ be the fuzzy relation defined by the following table,

| $\mu$ | $\bot$ | $a$ | $b$ | $c$ | $d$ | $e$ | $\top$ |
|---|---|---|---|---|---|---|---|
| $\bot$ | 0.5 | 0.5 | 1 | 0.5 | 0 | 0.5 | 0 |
| $a$ | 0 | 0 | 0.5 | 0.5 | 0.5 | 1 | 0.5 |
| $b$ | 0.5 | 0.5 | 1 | 0.5 | 0 | 0.5 | 0 |
| $c$ | 0 | 0 | 0.5 | 0.5 | 0.5 | 1 | 0.5 |
| $d$ | 0 | 0 | 0 | 0.5 | 0.5 | 0.5 | 1 |
| $e$ | 0 | 0 | 0.5 | 0.5 | 0.5 | 1 | 0.5 |
| $\top$ | 0 | 0 | 0 | 0.5 | 0.5 | 0.5 | 1 |

The relation $\mu$ is an extensional fuzzy closure relation in $A$ and the induced fuzzy closure system is $\Phi_\mu = \{{}^{\bot}/0.5, {}^{a}/0, {}^{b}/1, {}^{c}/0.5, {}^{d}/0.5, {}^{e}/1, {}^{\top}/1\}$; the proof is long and tedious but simple, hence it is omitted. However, $\kappa_{\Phi_\mu} \neq \mu$, as shown below,

$$\kappa_{\Phi_\mu}(\bot, a) = \left(\prod(\bot^\rho \otimes \Phi_\mu) \approx a\right) = \left(\prod(\Phi_\mu) \approx a\right) = (b \approx a) = 0 \neq \mu(\bot, a).$$

Thus, extensional fuzzy closure relations are not in a one-to-one relation with fuzzy closure systems.

However, these properties together with a minimality condition do achieve the expected results.

**Definition 19.** Let $\mathbb{A} = (A, \rho)$ be a complete fuzzy lattice. A fuzzy closure relation $\kappa\colon A \times A \to L$ is said to be strong if it is minimal among the extensional fuzzy closure relations, i.e. $\kappa$ is extensional and, for all extensional fuzzy closure relation $\kappa_1\colon A \times A \to L$, one has that $\kappa_1 \leq \kappa$ implies $\kappa_1 = \kappa$.

Not all fuzzy closure relations are strong fuzzy closure relations. As was proved in Example 2, $\kappa_1$ is not extensional, so it is not a strong fuzzy closure relation, even though it is a fuzzy closure relation.

The following proposition gives several characterizations of the concept of strong fuzzy closure relation. The second one is particularly useful for the proofs of the subsequent results in this paper since it gives an explicit expression for strong fuzzy closure relations.

**Proposition 20.** *Let $\mathbb{A} = (A, \rho)$ be a complete fuzzy lattice. Let $\kappa\colon A \times A \to L$. The following assertions are equivalent:*

1. *$\kappa$ is a strong fuzzy closure relation.*
2. *There exists a closure operator $\mathrm{c}\colon A \to A$ such that $\kappa(a, b) = (\mathrm{c}(a) \approx b)$ for all $a, b \in A$.*
3. *$\kappa$ is the extensional hull of a closure operator $\mathrm{c}\colon A \to A$.*
4. *$\kappa$ is a minimal extensional fuzzy relation such that* Core($\kappa$) *is a closure operator.*
5. Core($\kappa$) *is a closure operator, and $\kappa$ is the smallest extensional fuzzy closure relation whose core is* Core($\kappa$).

**Proof.** First, let us prove 1 implies 2. Assume the fuzzy relation $\kappa\colon A \times A \to L$ is a strong fuzzy closure relation. Then, by Proposition 14, the crisp relation Core($\kappa$) is a closure operator which from this point on will be managed as a mapping denoted by $\mathrm{c}$.

Consider the fuzzy relation $\mu(a, b) = (\mathrm{c}(a) \approx b)$. By Lemma 16, $\mu$ is a fuzzy closure relation. In addition, $\mu$ is extensional,

$$\begin{aligned}\mu(a,b)\otimes(a\approx x)\otimes(b\approx y) &= (\mathtt{c}(a)\approx b)\otimes(a\approx x)\otimes(b\approx y)\\ &= (x\approx a)\otimes(\mathtt{c}(a)\approx b)\otimes(b\approx y)\\ &\le (\mathtt{c}(x)\approx \mathtt{c}(a))\otimes(\mathtt{c}(a)\otimes b)\otimes(b\approx y)\\ &\le (\mathtt{c}(x)\approx y) = \mu(x,y).\end{aligned} \tag{9}$$

Then, by $\mathrm{Core}(\kappa)=\mathtt{c}$ and by extensionality we get

$$\mu(a,b) = (\mathtt{c}(a)\approx b) = \kappa(a,\mathtt{c}(a))\otimes(a\approx a)\otimes(\mathtt{c}(a)\approx b)\le \kappa(a,b) \tag{10}$$

Hence, $\mu$ is an extensional fuzzy closure relation and $\mu\le\kappa$. Then, by minimality of $\kappa$, we get $\mu=\kappa$.

Second, let us prove 2 implies 3. Assume there is a closure operator $\mathtt{c}$ such that $\kappa(a,b)=(\mathtt{c}(a)\approx b)$. Let $\hat{\mathtt{c}}$ be the extensional hull of $\mathtt{c}$. The extensional hull of a set was explicitly described in [18]. Considering $\mathtt{c}$ as a fuzzy subset of $A\times A$, its extensional hull satisfies the following formula

$$\hat{\mathtt{c}}(a,b) = \bigvee_{x,y\in A}\mathtt{c}(x,y)\otimes(x\approx a)\otimes(y\approx b) = \bigvee_{x\in A}(x\approx a)\otimes(\mathtt{c}(x)\approx b)$$

To prove this item it suffices to prove

$$(\mathtt{c}(a)\approx b) = \bigvee_{x\in A}(x\approx a)\otimes(\mathtt{c}(x)\approx b) \tag{11}$$

On the one hand, by choosing $x=a$ we get

$$\hat{\mathtt{c}}(a,b) = \bigvee_{x\in A}(x\approx a)\otimes(\mathtt{c}(x)\approx b)\ge(a\approx a)\otimes(\mathtt{c}(a)\approx b) = (\mathtt{c}(a)\approx b) = \kappa(a,b)$$

On the other hand, by isotonicity of $\mathtt{c}$ and transitivity of $\approx$, we have for all $x\in A$

$$(x\approx a)\otimes(\mathtt{c}(x)\approx b)\le(\mathtt{c}(x)\approx\mathtt{c}(a))\otimes(\mathtt{c}(x)\approx b)\le(\mathtt{c}(a)\approx b)$$

Thus, $\hat{\mathtt{c}}(a,b)=\bigvee_{x\in A}(x\approx a)\otimes(\mathtt{c}(x)\approx b)\le\kappa(a,b)$. Hence, $\kappa=\hat{\mathtt{c}}$ is the extensional hull of the closure operator $\mathtt{c}$.

Third, let us prove 3 implies 4. Assume $\kappa$ is the extensional hull of the closure operator $\mathtt{c}$ and $(a,b)\in\mathrm{Core}(\kappa)$ then, by (11),

$$1=\kappa(a,b)=\bigvee_{x\in A}(x\approx a)\otimes(\mathtt{c}(x)\approx b)=(\mathtt{c}(a)\approx b).$$

Thus, $\mathtt{c}(a)=b$ which yields to $\mathrm{Core}(\kappa)=\mathtt{c}$. Now, let $\mu$ be an extensional fuzzy relation such that $\mathrm{Core}(\mu)$ is a closure operator and $\mu\le\kappa$. Let us show that $\mathrm{Core}(\mu)=\mathrm{Core}(\kappa)$. On the one hand, the inclusion $\mu\le\kappa$ implies $\mathrm{Core}(\mu)\le\mathrm{Core}(\kappa)$. Since $\mathrm{Core}(\kappa)$ is a mapping, then for each $a\in A$ there exists a unique $a^*\in A$ such that $\kappa(a,a^*)=1$. Similarly, for each $a\in A$ there exists one unique $a^\bullet\in A$ such that $\mu(a,a^\bullet)=1$. Since $\mu\le\kappa$ we have $1=\mu(a,a^\bullet)\le\kappa(a,a^\bullet)$. Thus, $a^\bullet=a^*$ for all $a\in A$, and $\mathrm{Core}(\mu)=\mathrm{Core}(\kappa)=\mathtt{c}$.

Since $\mu$ is extensional and $\mathtt{c}\le\mu$, by the definition of extensional hull we have $\kappa\le\mu$.

Fourth, let us prove 4 implies 5. Assume $\kappa$ is a minimal extensional fuzzy relation such that $\mathrm{Core}(\kappa)$ is a closure operator. Let $\kappa_1$ be an extensional fuzzy relation such that $\mathtt{c}=\mathrm{Core}(\kappa)=\mathrm{Core}(\kappa_1)$ and prove $\kappa\le\kappa_1$. Consider the fuzzy relation $\mu$ defined as $\mu(a,b)=(\mathtt{c}(a)\approx b)$. By extensionality of $\kappa$, and using the same reasoning of (10), we have that $\mu\le\kappa$ and, since $\mu$ is extensional by (9) and $\kappa$ is minimal, $\mu=\kappa$. Thus, by Lemma 16, $\kappa$ is a fuzzy closure relation. In addition, by extensionality of $\kappa_1$ and using the same reasoning of (10) again, we get $\kappa=\mu\le\kappa_1$.

Finally, let us prove 5 implies 1. Assume $\mathtt{c}=\mathrm{Core}(\kappa)$ is a closure operator, and $\kappa$ is the smallest extensional fuzzy closure relation whose core is $\mathtt{c}$.

Let $\kappa_1\colon A\times A\to L$, be an extensional fuzzy closure relation such that $\kappa_1\le\kappa$ and prove $\kappa_1=\kappa$. Since $\mathrm{Core}(\kappa_1)\le\mathtt{c}$ and, by Lemma 16, $\mathrm{Core}(\kappa_1)$ is a mapping, then $\mathrm{Core}(\kappa_1)=\mathtt{c}$. Therefore, by hypothesis, $\kappa_1=\kappa$. □

In the previous result, given a closure operator $\mathtt{c}$, we have considered the fuzzy relation defined as $\mu(a,b)=(\mathtt{c}(a)\approx b)$. This construction resembles the perfect fuzzy functions introduced by Demirci [13]; particularly, the *vague description* of crisp functions. In our framework, a fuzzy relation $\mu\colon A\times A\to L$ is said to be a perfect fuzzy function if it is total, extensional and satisfies the following property:

$$\mu(a,b_1)\otimes\mu(a,b_2)\le(b_1\approx b_2),\ \text{for all } a,b_1,b_2\in A \tag{12}$$

We now study the relationship between fuzzy closure relations and perfect fuzzy functions.

Condition (12) implies that every $a^\mu$ is a clique. The converse holds in Heyting algebras (where $\otimes=\wedge$): since $\mu(a,b_1)\otimes\mu(a,b_2)\le\rho(b_1,b_2)$ and $\mu(a,b_1)\otimes\mu(a,b_2)\le\rho(b_2,b_1)$, then, $\mu(a,b_1)\otimes\mu(a,b_2)\le\rho(b_1,b_2)\wedge\rho(b_2,b_1)=(b_1\approx b_2)$. The following example shows that this does not hold in general.

**Example 4.** Let $\rho,\mu\colon A\times A\to L$ be two fuzzy relations defined by

| $\rho$ | $a$ | $b$ |
|---|---|---|
| $a$ | 1 | 0.5 |
| $b$ | 0.5 | 1 |

| $\mu$ | $a$ | $b$ |
|---|---|---|
| $a$ | 1 | 0.5 |
| $b$ | 0.5 | 1 |

Consider the fuzzy poset $(A,\rho)$ with $A=\{a,b\}$ with the three-valued Łukasiewicz logic. Then, even though $a^\mu$ and $b^\mu$ are cliques (only proved for $a^\mu$ due to symmetry)

$$\mu(a,x)\otimes\mu(a,y)=\left\{\begin{array}{ll}1=\rho(a,a) & \text{, if } x=y=a\\ 0.5=\rho(a,b) & \text{, if } x=a, y=b\\ 0.5=\rho(b,a) & \text{, if } x=b, y=a\\ 0\le 1=\rho(b,b) & \text{, if } x=y=b\end{array}\right\}\le\rho(x,y)$$

But does not satisfy (12) since

$$\mu(a,a)\otimes\mu(a,b)=1\otimes 0.5=0.5\not\le 0=\rho(a,b)\otimes\rho(b,a)=(a\approx b)$$

Next result proves the relationship between strong fuzzy closure relations and perfect fuzzy functions.

**Proposition 21.** *Let $\mathbb{A}=(A,\rho)$ be a complete fuzzy lattice. The following statements are equivalent:*

1. *$\kappa$ is a strong fuzzy closure relation.*
2. *$\kappa$ satisfies condition (12) and is a extensional fuzzy closure relation.*
3. *$\kappa$ is a perfect fuzzy function whose core is a closure operator.*

**Proof.** First, we prove 1. implies 2. Assume $\kappa$ is a strong fuzzy closure relation. By hypothesis $\kappa$ is an extensional fuzzy closure relation, so it suffices to prove that it satisfies condition (12). By item 2 of Proposition 20, there exists a closure operator $\mathrm{c}$ such that $\kappa(a,b)=(\mathrm{c}(a)\approx b)$. Thus,

$$a^\kappa(x)\otimes a^\kappa(y)=(\mathrm{c}(a)\approx x)\otimes(\mathrm{c}(a)\approx y)\le(x\approx y).$$

Now, for 2. implies 3. Assume that $\kappa$ is an extensional fuzzy closure relation such that condition (12) holds. Then, $\kappa$ satisfies being total, extensionality and condition (12). Hence, $\kappa$ is a perfect fuzzy function. In addition, by Proposition 14, $\mathrm{Core}(\kappa)$ is a closure operator.

3 implies 1 is proved as follows. Assume $\kappa$ is a perfect fuzzy function and let $\mathrm{c}=\mathrm{Core}(\kappa)$, then

$$\kappa(a,b)=\kappa(a,\mathrm{c}(a))\otimes\kappa(a,b)\le(\mathrm{c}(a)\approx b) \text{ and}$$
$$(\mathrm{c}(a)\approx b)=(\mathrm{c}(a)\approx b)\otimes\kappa(a,\mathrm{c}(a))\le\kappa(a,b),$$

where the first line uses the condition (12) and the second one uses extensionality.

Hence, $\kappa(a,b)=(\mathrm{c}(a)\approx b)$ and, by Proposition 20, $\kappa$ is a strong fuzzy closure relation. □

In Example 3, it was shown that extensional fuzzy closure relations are not necessarily strong fuzzy closure relations. However, if the underlying structure of truth values is a Heyting algebra, strong fuzzy closure relations are exactly the extensional fuzzy closure relations.

**Corollary 22.** *Let $(A,\rho)$ be a complete fuzzy lattice over a Heyting algebra $\mathbb{L}$. Then, every extensional fuzzy closure relation is a strong fuzzy closure relation.*

**Proof.** Let $\kappa$ be an extensional fuzzy closure relation. In a Heyting algebra, every extensional fuzzy closure relation is a perfect fuzzy function: extensionality is given by hypothesis, perfection follows from being a total relation and condition (12) follows from the aftersets being cliques. Also, by Proposition 14, $\mathrm{Core}(\kappa)$ is a closure operator. Therefore, applying Proposition 21, we get that $\kappa$ is a strong fuzzy closure relation. □

The main theorem in this paper shows that strong fuzzy closure relations and fuzzy closure systems are indeed in one-to-one correspondence, that is, for any fuzzy closure system there is a unique strong fuzzy closure relation and vice versa.

**Theorem 23.** *Let $\mathbb{A} = (A, \rho)$ be a complete fuzzy lattice. The following statements hold:*

1. *If $\kappa$ is a strong fuzzy closure relation, the fuzzy set $\Phi_\kappa$ defined as $\Phi_\kappa(a) = \rho_\propto(a^\kappa, a)$ is a fuzzy closure system.*
2. *If $\Phi$ is a fuzzy closure system, the relation $\kappa_\Phi \colon A \times A \to L$ defined as $\kappa_\Phi(a, b) = \left(\bigsqcap(a^\rho \otimes \Phi\right) \approx b)$ is a strong fuzzy closure relation.*
3. *If $\kappa \colon A \times A \to L$ is a strong fuzzy closure relation, then $\kappa_{\Phi_\kappa} = \kappa$.*
4. *If $\Phi$ is a fuzzy closure system, then $\Phi = \Phi_{\kappa_\Phi}$.*

**Proof.** Most of the proof was already done in Theorem 17, so we focus on items 2 and 3.

For the second item, assume $\Phi$ is a fuzzy closure system. Then, by Theorem 11, the mapping $c_\Phi(a) = \bigsqcap(a^\rho \otimes \Phi)$ is a closure operator and by Proposition 20, we obtain $\kappa_\Phi(a, b) = (c_\Phi(a) \approx b))$ is a strong fuzzy closure relation.

For the third item, assume $\kappa \colon A \times A \to L$ is a strong fuzzy closure relation. By item 1, $\Phi_\kappa = \Phi_c$ holds. Then,

$$\kappa_{\Phi_\kappa}(a, b) = \left(\bigsqcap(a^\rho \otimes \Phi_c) \approx b\right) = (c_{\Phi_c}(a) \approx b) = (c(a) \approx b) = \kappa(a, b).$$

In second to last equality, $c_{\Phi_c} = c$ is used, this holds due to Theorem 11. The last equality is due to Proposition 20. □

## 5. Conclusions and future work

This paper presents an extension of the concept of closure operator as a fuzzy relation. Following the idea of closure operators, i.e., inflationarity, isotonicity and idempotency, we have introduced a definition of fuzzy closure relation which is interesting on its own. Most of the results concerning a one-to-one relation with fuzzy closure systems can be proved using this definition.

However, the existing definitions were not in bijective correspondence with fuzzy closure systems. The one-to-one relation among these structures is one of the key properties of closure. Thus, we have looked for a stronger definition of fuzzy closure relation, one that could assure the existence of a bijection with fuzzy closure systems. In that, the two candidate mappings have been studied, specifically the focus has been on the properties of their images. One of the main properties found has been the extensionality of relations. However, extensionality was not strong enough and some minimality conditions needed to be added to the definition. Finally, the definition was found and this structure has been proved to be in bijective correspondence with fuzzy closure systems. These fuzzy relations have been called strong fuzzy closure relations. We also proved that the extension of closure operators to a relational framework in order to maintain the one-to-one relation has a deep connection with perfect fuzzy functions. In some sense, we have justified that closure operators cannot be extended any further in the relational framework while maintaining the one-to-one relation with fuzzy closure systems.

From the applications point of view, the one-to-one relation can be dropped. In fact, in Theorem 17, it can be seen that fuzzy closure relations satisfy interesting properties even though the one-to-one relation does not hold. In the short-term, we could look for other extensions of closure operator which are interesting from the application point of view, but dropping the one-to-one relation restriction. Besides, it would be worth to study the relation of $\kappa_\Phi(a, b) = (\bigsqcap(a^\rho \otimes \Phi) \approx b)$ with $\mu_\Phi(a, b) = m(a^\rho \otimes \Phi)(b)$, where $m$ is the fuzzy set of minima. The relationship between these two relations is key to extend this work to weaker structures since the infimum is only assured to exist in complete fuzzy lattices, but the set $m$ can be worked with in posets and preposets as well.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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