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ORIGINAL PAPER

Analytical solutions for fractional partial delay differential-algebraic equations with Dirichlet boundary conditions defined on a finite domain

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Abstract

In this paper, we investigate the solution of multi-term time-space fractional partial delay differential-algebraic equations (MTS-FPDDAEs) with Dirichlet boundary conditions defined on a finite domain. We use Laplace transform method to give the solutions of multi-term time fractional delay differential-algebraic equations (MTS-FDDAEs). Then, the technique of spectral representation of the fractional Laplacian operator is used to convert the MTS-FPDDAEs into the MTS-FDDAEs. By applying our obtained solutions to the resulting MTS-FDDAEs, the desired analytical solutions of the MTS-FPDDAEs are obtained. Finally, we give the solutions of some special cases.

Keywords Fractional partial differential-algebraic equations with delays · Dirichlet boundary condition · Fractional Laplacian operator · Spectral representation · Analytical solution

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1 Introduction

In the last few years, the interest of the scientific community towards fractional calculus experienced an exceptional boost, so that its applications can now be found in a great variety of science fields, for example, anomalous diffusion [1], solute transport [2], random and disordered media [3, 4], electrical circuits [5], and so on. The reason of the success of the fractional calculus in modeling natural phenomena is that the fractional calculus operators are nonlocal operators, that makes them suitable to describe the long memory or nonlocal effects characterizing most of the physical phenomena.

Fractional partial differential equations are an important class of differential equations. They can model the dynamics of complex systems. At the same time, the necessity of a powerful technique for solving these new types of equations came up, becoming one of the main research objects in both the fields of theoretical and applied sciences, In the available literature, there exist various methods for solving fractional partial differential equations, such as analytical methods and numerical algorithms, [6–10].

Analytical solutions of fractional partial equations are of fundamental importance in describing and understanding the physical phenomena, since all the parameters are expressed in a mathematically closed form and therefore the influence of individual parameters on natural phenomena can be easily examined. Also, the analytical solutions make it easy to study asymptotic behaviors of the solutions, which are usually difficult to obtain through numerical calculations. Besides, the analytical solutions may serve as tools in assessing the computational performance and accuracy of numerical solutions.

The analytical solutions of fractional partial equations have been reported in literature, see e.g. [11–23]. Integral transforms, such as the Laplace transform and the Mellin transform, have been widely applied to develop the analytical solutions to fractional differential equations (for example, see [11, 12]). But, one of the disadvantages of the integral transform method is that the inverse transform is mostly performed based on the complex functions, thus limiting the types of the analytical solutions of the equations. Accordingly, some researchers developed a spectral representation technique to obtain the desired solutions [20–22].

Fractional partial delay differential-algebraic equations often arise in many important physical problems, [24–27]. To the authors' knowledge, the analytical solutions of the fractional partial delay differential-algebraic equations have not been reported in literature yet. In [28], Zaczkiewicz applied Laplace transform for investigate of linear stationary fractional differential-algebraic equations with delays and to obtain analytical representation of solutions in the form of series in power of solutions to the equations. In this paper, we consider the following MTS-FPDDAEs defined on a finite domain of the form

$$\begin{cases} P(D_t^*)u(x,t) = -k_{p_1}(-\triangle)^{\frac{p_1}{2}}u(x,t) - k_{p_2}(-\triangle)^{\frac{p_2}{2}}v(x,t) + f(x,t), \\ v(x,t) = -k_{q_1}(-\triangle)^{\frac{q_1}{2}}u(x,t) - k_{q_2}(-\triangle)^{\frac{q_2}{2}}v(x,t-h) + g(x,t), \end{cases} \tag{1.1}$$

with the nonhomogeneous Dirichlet boundary conditions

$$u(0,t) = u_0(t), \quad u(L,t) = u_L(t), \quad 0 \leqslant t \leqslant T, \tag{1.2}$$

$$v(0,t) = v_0(t), \quad v(L,t) = v_L(t), \quad -h \leqslant t \leqslant T, \tag{1.3}$$

where $(x, t) \in [0, L] \times [0, T]$ (L and T are constants), the operator $P(D_t^*)u(x, t)$ is defined as

$$P(D_t^*)u(x,t) = \left(D_t^\alpha + \sum_{i=1}^{p} a_i D_t^{\alpha_i}\right)u(x,t), 0 \leqslant \alpha_p < \cdots < \alpha_1 < \alpha \leqslant 2, a_i \geqslant 0,$$

and assume that $0 < \alpha - \alpha_1 \leqslant 1$, D_t^o (o stands for α or α_i) is the Caputo fractional derivative of order o with respect to t, and the Laplacian operator is defined as $(-\triangle) = -\frac{\partial^2}{\partial x^2}$. The space fractional derivative $(-\triangle)^{\frac{\kappa}{2}}$ ($\kappa = p_1, p_2, q_1, q_2$, and $1 < p_1, q_2 \leqslant 2$, $0 < q_1, p_2 \leqslant 1$) is a fractional Laplacian operator defined through the eigenfunction expansion on a finite domain. The detailed definitions of the Caputo fractional derivative and the fractional Laplacian operator are given in the next section (or see [20, 29]).

The rest of this paper is organized as follows. In Section 2, we give some basic definitions and useful properties, which will be used in the paper. In Section 3, we use the Laplace transform to discuss the analytical solutions of fractional delay differential-algebraic equations. In Section 4, we discuss the analytical solutions of MTS-FPDDAEs with fractional diffusion terms. In Section 5, we give the analytical solutions of MTS-FPDDAEs with fractional wave terms. In Section 6, we provide the details of the solutions of some special cases.

2 Preliminaries

In this section, we give some basic definitions about fractional calculus and important properties on Laplace transform, which will be used throughout this paper. For details, one can refer to [29].

Definition 1 Let $f \in C([0, T])$ and $\alpha > 0$. The Riemann-Liouville fractional integral of order α with respect to t is defined as

$$\mathcal{I}_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad t > 0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2 Let $f \in C^m([0, T])$ and $m - 1 < \alpha \leqslant m$, where $m \in \mathbb{N}^+$. The Caputo fractional derivative of order α with respect to t is defined as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, \quad t > 0.$$

The Laplace transform of the Caputo derivative of f is given by

$$\mathcal{L}\{(D_t^\alpha f)(t); s\}=s^\alpha F(s)-\sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0),\ m-1<\alpha\leqslant m, m\in\mathbb{N}^+, \quad (2.1)$$

where $F(s)$ denotes the Laplace transform of the function $f(t)$.

There exists the following relationship between the Riemann-Liouville fractional integral and the Caputo fractional derivative.

Property 1 ([29]) Let $m-1<\alpha\leqslant m$, where $m\in\mathbb{N}^+$. Then the relations hold:

$$(D_t^\alpha \mathcal{I}_t^\alpha f)(t)=f(t),\quad (\mathcal{I}_t^\alpha D_t^\alpha f)(t)=f(t)-\sum_{k=0}^{m-1}\frac{t^k}{k!}f^{(k)}(0),\quad t>0.$$

Definition 3 ([20]) Suppose that the Laplacian $(-\triangle)$ has a complete set of orthonormal eigenfunctions φ_n corresponding to eigenvalues μ_n^2 on a bounded region $\mathcal{D}$, i.e., $(-\triangle)\varphi_n=\mu_n^2\varphi_n$ on $\mathcal{D}$; $\mathcal{B}(\varphi_n)=0$ on ∂D, where $\mathcal{B}(\varphi_n)$ is one of the standard three homogeneous boundary conditions. Let

$$\mathcal{F}=\left\{f=\sum_{n=1}^{\infty}c_n\varphi_n,\ c_n=\langle f,\varphi_n\rangle,\ \sum_{n=1}^{\infty}|c_n|^2|\mu_n|^\alpha<\infty\right\},$$

then for any $f\in\mathcal{F}$, $(-\triangle)^{\frac{\alpha}{2}}$ is defined by

$$(-\triangle)^{\frac{\alpha}{2}}f=\sum_{n=1}^{\infty}c_n\mu_n^\alpha\varphi_n.$$

Lemma 1 ([20]) *Suppose that the one-dimensional Laplacian* $(-\triangle)$ *defined with Dirichlet boundary conditions at* $x=0$ *and* $x=L$ *has a complete set of orthonormal eigenfunctions* φ_n *corresponding to eigenvalues* μ_n^2 *on a bounded region* $[0,L]$. *If* $(-\triangle)\varphi_n=\mu_n^2\varphi_n$ *on* $[0,L]$, *and* $\varphi_n(0)=\varphi_n(L)=0$, *then, the eigenvalues are given by* $\mu_n^2=\frac{n^2\pi^2}{L^2}$, *and the corresponding eigenfunctions are* $\varphi_n(x)=\sin(n\pi x/L)$, $n=1,2,\ldots$.

Definition 4 ([29]) The three-parameter Mittag-Leffler function is defined by

$$E_{\beta,\gamma}^{\rho}(z)=\sum_{k=0}^{\infty}\frac{(\rho)_k z^k}{\Gamma(\beta k+\gamma)k!},\quad \beta,\gamma,\rho>0,\ z\in\mathbb{R},$$

where $(\rho)_0=1$, $(\rho)_k=\frac{\Gamma(\rho+k)}{\Gamma(\rho)}=\rho(\rho+1)\cdots(\rho+k-1), k=1,2,\ldots$.

This is known also as the Prabhakar function. In particular, when $\rho=1$, it coincides with the two-parameter Mittag-Leffler function, i.e., $E_{\beta,\gamma}^1(z)=E_{\beta,\gamma}(z)$; when $\rho=$

$\gamma = 1$, it coincides with the one-parameter Mittag-Leffler function, i.e., $E^1_{\beta,1}(z) = E_\beta(z)$.

The Laplace transform of the three-parameter Mittag-Leffler function is

$$\mathcal{L}\{z^{\beta-1}E^{\rho}_{\alpha,\beta}(\pm a z^{\alpha}); s\} = \int_0^{\infty} e^{-sz} z^{\beta-1} E^{\rho}_{\alpha,\beta}(\pm a z^{\alpha}) dz = \frac{s^{\alpha\rho-\beta}}{(s^{\alpha} \mp a)^{\rho}}, \tag{2.2}$$

provided that $|as^{-\alpha}| < 1$.

An important function occurring in electrical systems is the delayed unit step function

$$u_a(t) = \begin{cases} 1, & t \geqslant a, \\ 0, & t < a, \end{cases} \tag{2.3}$$

and its Laplace transform is given by

$$\mathcal{L}\{u_a(t); s\} = \frac{e^{-as}}{s}, \quad \mathrm{Re}(s) > 0. \tag{2.4}$$

If $F(s)$ is the Laplace transform of the function $f(t)$, i.e., $F(s) = \mathcal{L}\{f(t); s\}$, then

$$\mathcal{L}\{e^{at} f(t); s\} = F(s-a), \tag{2.5}$$

and

$$\mathcal{L}\{u_a(t) f(t-a); s\} = e^{-as} F(s), \quad a \geqslant 0, \tag{2.6}$$

and also we have

$$\mathcal{L}^{-1}\{e^{-as} F(s); t\} = u_a(t) f(t-a), \quad a \geqslant 0. \tag{2.7}$$

Another important function is the unit impulsive function

$$\delta(t) = \begin{cases} 0, & t \neq 0, \\ \infty, & t = 0, \end{cases} \tag{2.8}$$

and one knows that

$$\mathcal{L}\{\delta(t); s\} = 1, \quad \text{i.e.,} \quad \mathcal{L}^{-1}\{1; t\} = \delta(t). \tag{2.9}$$

With respect to the convolution of the unit impulsive function, there exist several basic properties, which will be used in the latter discussion.

Property 2 Let δ be the unit impulsive function, f be any continuous function defined on $\mathcal{D}$, and $*$ denote the convolution operation. Then, the following statements hold:

(i) $\delta * \delta = \delta$;
(ii) $\delta * f = f * \delta = f$;
(iii) $f * \delta(t - t_0) = f(t - t_0)$, where $t_0 \in \mathcal{D}$.

Finally, we introduce some notations, which will be used in the remain sections. We use C_n^m to denote the combinational number formula, $*$ stands for the convolution operation, and $r_t := \left[\frac{t}{h}\right]$, where $\left[\frac{t}{h}\right]$ denotes the largest integer less than or equal to $\frac{t}{h}$.

3 Solution representation

In this section, we investigate the solution representations of fractional differential-algebraic equations with delays.

Consider a simple fractional differential-algebraic equation with delay of the form:

$$\begin{cases} D_t^\alpha x_1(t) = \lambda_1 x_1(t) + \lambda_2 x_2(t) + u(t), \quad 0 < \alpha \leqslant 2, \\ x_2(t) = \lambda_3 x_1(t) + \lambda_4 x_2(t-h) + e(t), \quad h > 0, \ t \in [0, T], \end{cases} \tag{3.1}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$, and u, e are two known continuous functions defined on $[0, T]$.

First, we consider the solution of equation (3.1) in the case $0 < \alpha \leqslant 1$. When we discuss its solution, equation (3.1) satisfies the initial conditions

$$x_1(0) = x_0, \quad x_2(t) = \varphi(t), \ t \in [-h, 0], \tag{3.2}$$

and we assume that the initial conditions are consistent, i.e., $\varphi(0) = \lambda_3 x_1(0) + \lambda_4 \varphi(-h) + e(0)$.

For giving the solution of equation (3.1) with the initial conditions (3.2), we take the Laplace transforms on the both sides of equation (3.1) to obtain

$$\begin{aligned} X_1(s) = & \frac{\lambda_4 s^{\alpha-1} e^{-hs} x_1(0)}{(s^\alpha - \lambda_1)(\lambda_4 e^{-hs} - 1) + \lambda_2\lambda_3} - \frac{s^{\alpha-1} x_1(0)}{(s^\alpha - \lambda_1)(\lambda_4 e^{-hs} - 1) + \lambda_2\lambda_3} \\ & - \frac{\lambda_2\lambda_4 e^{-hs} \int_{-h}^0 e^{-s\tau}\varphi(\tau)d\tau}{(s^\alpha - \lambda_1)(\lambda_4 e^{-hs} - 1) + \lambda_2\lambda_3} + \frac{\lambda_4 e^{-hs} U(s)}{(s^\alpha - \lambda_1)(\lambda_4 e^{-hs} - 1) + \lambda_2\lambda_3} \\ & - \frac{U(s)}{(s^\alpha - \lambda_1)(\lambda_4 e^{-hs} - 1) + \lambda_2\lambda_3} \\ & - \frac{\lambda_2 E(s)}{(s^\alpha - \lambda_1)(\lambda_4 e^{-hs} - 1) + \lambda_2\lambda_3}, \end{aligned} \tag{3.3}$$

where $X_1(S)$, $U(s)$ and $E(s)$ denote the Laplace transforms of $x_1(t)$, $u(t)$ and $e(t)$, respectively.

Next, we consider the Laplace inverse transform of $X_1(s)$.

Let $\left|\frac{\lambda_4 s^\alpha-\lambda_1\lambda_4}{s^\alpha-\lambda_1-\lambda_2\lambda_3}\right|<1$. Then, we have

$$\begin{aligned}
&\frac{s^{\alpha-1}e^{-hs}}{(s^\alpha-\lambda_1)(\lambda_4 e^{-hs}-1)+\lambda_2\lambda_3}\\
&=-\frac{s^{\alpha-1}e^{-hs}}{s^\alpha-\lambda_1-\lambda_2\lambda_3}\cdot\frac{1}{1-(s^\alpha-\lambda_1-\lambda_2\lambda_3)^{-1}(\lambda_4 s^\alpha-\lambda_1\lambda_4)e^{-hs}}\\
&=-\frac{s^{\alpha-1}e^{-hs}}{s^\alpha-\lambda_1-\lambda_2\lambda_3}\cdot\sum_{n=0}^{\infty}\frac{(\lambda_4 s^\alpha-\lambda_1\lambda_4)^n e^{-nhs}}{(s^\alpha-\lambda_1-\lambda_2\lambda_3)^n}\\
&=-\sum_{n=0}^{\infty}\sum_{i=0}^{n}C_n^i(-\lambda_1)^{n-i}\lambda_4^n\frac{s^{i\alpha+\alpha-1}e^{-(n+1)hs}}{(s^\alpha-\lambda_1-\lambda_2\lambda_3)^{n+1}}\\
&=-\sum_{l=1}^{\infty}\sum_{i=0}^{l-1}C_{l-1}^i(-\lambda_1)^{l-i-1}\lambda_4^{l-1}\frac{s^{i\alpha+\alpha-1}e^{-lhs}}{(s^\alpha-\lambda_1-\lambda_2\lambda_3)^l}.
\end{aligned}$$

For brevity, we introduce the following two notations:

$$\lambda=\lambda_1+\lambda_2\lambda_3,\quad e_{\beta,\gamma}^{\rho;\lambda z}=t^{\gamma-1}E_{\beta,\gamma}^{\rho}(\lambda z^\alpha),\quad \beta,\gamma,\rho>0,\lambda,\ z\in\mathbb{R}. \tag{3.4}$$

Then, by (2.2) and (2.7), we have

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}e^{-hs}}{(s^\alpha-\lambda_1)(\lambda_4 e^{-hs}-1)+\lambda_2\lambda_3};t\right\}\\
&=-\sum_{l=1}^{\infty}\sum_{i=0}^{l-1}C_{l-1}^i(-\lambda_1)^{l-i-1}\lambda_4^{l-1}(t-lh)^{\alpha(l-i-1)}E_{\alpha,\alpha(l-i-1)+1}^{l}(\lambda(t-lh)^\alpha)u_{lh}(t)\\
&=-\sum_{l=1}^{r_t}\sum_{i=0}^{l-1}C_{l-1}^i(-\lambda_1)^{l-i-1}\lambda_4^{l-1}(t-lh)^{\alpha(l-i-1)}E_{\alpha,\alpha(l-i-1)+1}^{l}(\lambda(t-lh)^\alpha)\\
&=-\sum_{l=1}^{r_t}\sum_{i=0}^{l-1}C_{l-1}^i(-\lambda_1)^{l-i-1}\lambda_4^{l-1}e_{\alpha,\alpha(l-i-1)+1}^{l;\lambda(t-lh)}.
\end{aligned}\tag{3.5}$$

Using the similar arguments to (3.5), we can obtain

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{(s^\alpha-\lambda_1)(\lambda_4 e^{-hs}-1)+\lambda_2\lambda_3};t\right\}\\
&=-\sum_{n=0}^{r_t}\sum_{i=0}^{n}C_n^i(-\lambda_1)^{n-i}\lambda_4^n e_{\alpha,\alpha(n-i)+1}^{n+1;\lambda(t-nh)},
\end{aligned}\tag{3.6}$$

and

$$\mathcal{L}^{-1}\left\{\frac{e^{-hs}}{(s^{\alpha}-\lambda_1)(\lambda_4 e^{-hs}-1)+\lambda_2\lambda_3}; t\right\}$$
$$= -\sum_{l=1}^{r_t}\sum_{i=0}^{l-1} C_{l-1}^{i}(-\lambda_1)^{l-i-1}\lambda_4^{l-1} e_{\alpha,\alpha(l-i)}^{l;\lambda(t-lh)}, \tag{3.7}$$

and

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^{\alpha}-\lambda_1)(\lambda_4 e^{-hs}-1)+\lambda_2\lambda_3}; t\right\}$$
$$= -\sum_{n=0}^{r_t}\sum_{i=0}^{n} C_{n}^{i}(-\lambda_1)^{n-i}\lambda_4^{n} e_{\alpha,\alpha(n-i)+\alpha}^{n+1;\lambda(t-nh)}. \tag{3.8}$$

On the other hand, we define a new staircase function $p(t)$ on $[-h,\infty)$ such that

$$p(t)=\begin{cases} 0, & t\geqslant 0,\\ 1, & -h\leqslant t<0,\end{cases} \tag{3.9}$$

and extend the function $\varphi(t)$ to $[-h,\infty)$ such that $\varphi(t)=\varphi(0)$ for $t\geqslant 0$. Based on this extension, it has the following relationship

$$e^{-hs}\int_{-h}^{0} e^{-s\tau}\varphi(\tau)d\tau = \int_{0}^{\infty} e^{-st}\varphi(-h+t)p(-h+t)dt$$
$$= \mathcal{L}\{\varphi(-h+t)p(-h+t); s\}. \tag{3.10}$$

Therefore, from (3.3), (3.4), (3.5), (3.6), (3.7), (3.8) and (3.10) and the definition (3.9) of the function $p(t)$, we can obtain that: For $0\leqslant t<h$, the representation of the solution x_1 is given by

$$x_1(t)=E_{\alpha}(\lambda t^{\alpha})x_0+\lambda_2\lambda_4\int_{-h}^{t-h} e_{\alpha,\alpha}^{1;\lambda(t-h-\tau)}\varphi(\tau)d\tau+\int_{0}^{t} e_{\alpha,\alpha}^{1;\lambda(t-\tau)}\Big(u(\tau)+\lambda_2 e(\tau)\Big)d\tau.$$

For $h\leqslant t\leqslant T$, we have

$$x_1(t)=\Bigg(\sum_{n=0}^{r_t}\sum_{i=0}^{n} C_{n}^{i}(-\lambda_1)^{n-i}\lambda_4^{n} e_{\alpha,\alpha(n-i)+1}^{n+1;\lambda(t-nh)}$$
$$-\sum_{l=1}^{r_t}\sum_{i=0}^{l-1} C_{l-1}^{i}(-\lambda_1)^{l-i-1}\lambda_4^{l} e_{\alpha,\alpha(l-i-1)+1}^{l;\lambda(t-lh)}\Bigg)x_0$$
$$+\sum_{n=0}^{r_t}\sum_{i=0}^{n} C_{n}^{i}(-\lambda_1)^{n-i}\lambda_2\lambda_4^{n+1}\int_{-h}^{0} e_{\alpha,\alpha(n+1-i)}^{n+1;\lambda(t-(n+1)h-\tau)}\varphi(\tau)d\tau$$

$$-\sum_{l=1}^{r_t}\sum_{i=0}^{l-1} C_{l-1}^{i}(-\lambda_1)^{l-i-1}\lambda_4^{l}\int_0^t e_{\alpha,\alpha(l-i)}^{l;\lambda(t-lh-\tau)}u(\tau)d\tau$$
$$+\sum_{n=0}^{r_t}\sum_{i=0}^{n} C_{n}^{i}(-\lambda_1)^{n-i}\lambda_4^{n}\int_0^t e_{\alpha,\alpha(n+1-i)}^{n+1;\lambda(t-nh-\tau)}\Big(u(\tau)+\lambda_2 e(\tau)\Big)d\tau.$$

On the other hand, we solve the second equation of (3.1) by step-wise method to deduce that

$$x_2(t)=\sum_{n=0}^{r_t}\lambda_3\lambda_4^n x_1(t-nh)+\lambda_4^{r_t+1}\varphi(t-r_t h-h)+\sum_{n=0}^{r_t}\lambda_4^n e(t). \tag{3.11}$$

Based on the above analysis, we can establish the following theorem.

Theorem 1 *Let* $0<\alpha\leqslant 1$, $x_0\in\mathbb{R}$, *and* φ *be a continuous function on* $[-h,0]$. *Then the solutions of equation (3.1) with the initial conditions (3.2) are given by*

$$x_1(t)=\begin{cases} E_\alpha(\lambda t^\alpha)x_0+\lambda_2\lambda_4\int_{-h}^{t-h} e_{\alpha,\alpha}^{1;\lambda(t-h-\tau)}\varphi(\tau)d\tau \\ +\int_0^t e_{\alpha,\alpha}^{1;\lambda(t-\tau)}\Big(u(\tau)+\lambda_2 e(\tau)\Big)d\tau, \quad 0\leqslant t<h; \\ \Big(\sum_{n=0}^{r_t}\sum_{i=0}^{n} C_n^i(-\lambda_1)^{n-i}\lambda_4^n e_{\alpha,\alpha(n-i)+1}^{n+1;\lambda(t-nh)} \\ -\sum_{l=1}^{r_t}\sum_{i=0}^{l-1} C_{l-1}^i(-\lambda_1)^{l-i-1}\lambda_4^l e_{\alpha,\alpha(l-i-1)+1}^{l;\lambda(t-lh)}\Big)x_0 \\ +\sum_{n=0}^{r_t}\sum_{i=0}^{n} C_n^i(-\lambda_1)^{n-i}\lambda_2\lambda_4^{n+1}\int_{-h}^{0} e_{\alpha,\alpha(n+1-i)}^{n+1;\lambda(t-(n+1)h-\tau)}\varphi(\tau)d\tau \\ -\sum_{l=1}^{r_t}\sum_{i=0}^{l-1} C_{l-1}^i(-\lambda_1)^{l-i-1}\lambda_4^l\int_0^t e_{\alpha,\alpha(l-i)}^{l;\lambda(t-lh-\tau)}u(\tau)d\tau \\ +\sum_{n=0}^{r_t}\sum_{i=0}^{n} C_n^i(-\lambda_1)^{n-i}\lambda_4^n\int_0^t e_{\alpha,\alpha(n+1-i)}^{n+1;\lambda(t-nh-\tau)}\Big(u(\tau)+\lambda_2 e(\tau)\Big)d\tau, \\ h\leqslant t\leqslant T, \end{cases}$$

and

$$x_2(t)=\sum_{n=0}^{r_t}\lambda_4^n(\lambda_3 x_1(t-nh)+e(t))+\lambda_4^{r_t+1}\varphi(t-r_t h-h),\ t\in[0,T], \tag{3.12}$$

where λ *and* $e_{\alpha,\beta}^{\rho;\lambda z}$ *are defined in (3.4).*

In the following, we consider the solutions of equation (3.1) in the case $1 < \alpha \leqslant 2$. We discuss its solution that satisfies the initial conditions

$$x_1(0) = x_0, \quad x_1'(0) = x_0', \quad x_2(t) = \varphi(t), \quad t \in [-h, 0], \tag{3.13}$$

and assume that φ satisfies $\varphi(0) = \lambda_3 x_1(0) + \lambda_4 \varphi(-h) + e(0)$.

In this case, using the similar arguments as in Theorem 1, we can establish the following theorem.

Theorem 2 *Let $1 < \alpha \leqslant 2$, $x_0, x_0' \in \mathbb{R}$, and φ be a continuous function on $[-h, 0]$. Then the solutions of equation (3.1) with the initial conditions (3.13) are given by*

$$x_1(t) = \begin{cases} E_\alpha(\lambda t^\alpha)x_0 + E_{\alpha,2}(\lambda t^\alpha)x_0' + \lambda_2\lambda_4 \int_{-h}^{t-h} e_{\alpha,\alpha}^{1;\lambda(t-h-\tau)}\varphi(\tau)d\tau \\ + \int_0^t e_{\alpha,\alpha}^{1;\lambda(t-\tau)}\Big(u(\tau) + \lambda_2 e(\tau)\Big)d\tau, 0 \leqslant t < h; \\ \Big(\sum_{n=0}^{r_t}\sum_{i=0}^{n} C_n^i(-\lambda_1)^{n-i}\lambda_4^n e_{\alpha,\alpha(n-i)+1}^{n+1;\lambda(t-nh)} \\ - \sum_{l=1}^{r_t}\sum_{i=0}^{l-1} C_{l-1}^i(-\lambda_1)^{l-i-1}\lambda_4^l e_{\alpha,\alpha(l-i-1)+1}^{l;\lambda(t-lh)}\Big)x_0 \\ + \Big(\sum_{n=0}^{r_t}\sum_{i=0}^{n} C_n^i(-\lambda_1)^{n-i}\lambda_4^n e_{\alpha,\alpha(n-i)+2}^{n+1;\lambda(t-nh)} \\ - \sum_{l=1}^{r_t}\sum_{i=0}^{l-1} C_{l-1}^i(-\lambda_1)^{l-i-1}\lambda_4^l e_{\alpha,\alpha(l-i-1)+2}^{l;\lambda(t-lh)}\Big)x_0' \\ + \sum_{n=0}^{r_t}\sum_{i=0}^{n} C_n^i(-\lambda_1)^{n-i}\lambda_2\lambda_4^{n+1}\int_{-h}^{0} e_{\alpha,\alpha(n+1-i)}^{n+1;\lambda(t-(n+1)h-\tau)}\varphi(\tau)d\tau \\ - \sum_{l=1}^{r_t}\sum_{i=0}^{l-1} C_{l-1}^i(-\lambda_1)^{l-i-1}\lambda_4^l \int_0^t e_{\alpha,\alpha(l-i)}^{l;\lambda(t-lh-\tau)}u(\tau)d\tau \\ + \sum_{n=0}^{r_t}\sum_{i=0}^{n} C_n^i(-\lambda_1)^{n-i}\lambda_4^n \int_0^t e_{\alpha,\alpha(n+1-i)}^{n+1;\lambda(t-nh-\tau)}\Big(u(\tau) + \lambda_2 e(\tau)\Big)d\tau, \\ h \leqslant t \leqslant T, \end{cases}$$

and

$$x_2(t) = \sum_{n=0}^{r_t} \lambda_4^n(\lambda_3 x_1(t-nh) + e(t)) + \lambda_4^{r_t+1}\varphi(t-r_t h-h), \quad t \in [0, T], \tag{3.14}$$

where λ and $e_{\alpha,\beta}^{\rho;\lambda z}$ are defined in (3.4).

At this stage, we consider the solution of MTS-FPDDAE of the form:

$$\begin{cases} P(D_t^*)x_1(t) = \lambda_1 x_1(t) + \lambda_2 x_2(t) + u(t), \\ x_2(t) = \lambda_3 x_1(t) + \lambda_4 x_2(t-h) + e(t), \quad t \in [0, T], \end{cases} \tag{3.15}$$

where the operator $P(D_t^*)$ is defined as

$$P(D_t^*)x(t) = \left(D_t^\alpha + \sum_{i=1}^{p} a_i D_t^{\alpha_i}\right)x(t), \quad 0 \leqslant \alpha_p < \cdots < \alpha_1 < \alpha \leqslant 2, \; a_1 > 0,$$
$$a_i \geqslant 0 \; (i = 2, \ldots, p),$$

and $0 < \alpha - \alpha_1 \leqslant 1$, $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$, and u, e are two known continuous functions defined on $[0, T]$.

First, we consider the solutions of equation (3.15) in the case $0 \leqslant \alpha_p < \cdots < \alpha_1 < \alpha \leqslant 1$. When we discuss its solution, the equation satisfies the initial conditions

$$x_1(0) = x_0, \quad x_2(t) = \varphi(t), \;\; t \in [-h, 0], \tag{3.16}$$

and we assume that φ satisfies $\varphi(0) = \lambda_3 x_1(0) + \lambda_4 \varphi(-h) + e(0)$.

To provide the solution of (3.15) with the initial conditions (3.16), we need some useful lemmas.

Lemma 2 ([22]) *Let* $0 \leqslant \alpha_p < \cdots < \alpha_1 < \alpha$, $0 < \alpha - \alpha_1 \leqslant 1$, $a_1 > 0$, $a_i \geqslant 0 (i = 2, \ldots, p)$, λ *be defined in (3.4), and* $\left|\frac{\sum_{i=2}^{p} a_i s^{\alpha_i} + \lambda}{s^\alpha + a_1 s^{\alpha_1}}\right| < 1$. *Then we have*

$$\begin{aligned} \Phi(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha + \sum\limits_{i=1}^{p} a_i s^{\alpha_i} + \lambda}; t\right\} \\ &= \sum_{m=0}^{\infty} (-1)^m \sum_{k_1+k_2+\cdots+k_p=m} \frac{m!}{k_1!k_2!\cdots k_p!} \\ &\quad \times \prod_{i=2}^{p} \lambda^{k_1} a_i^{k_i} t^{(m+1)\alpha - \sum_{j=2}^{p} k_j \alpha_j - 1} E^{m+1}_{\alpha-\alpha_1, (m+1)\alpha - \sum_{j=2}^{p} k_j \alpha_j}(-a_1 t^{\alpha-\alpha_1}), \end{aligned}$$

where $k_i \geqslant 0$, $i = 1, 2, \ldots, p$.

Lemma 3 ([22]) *Let* $0 \leqslant \alpha_p < \cdots < \alpha_1 < \alpha \leqslant 1$, $a_1 > 0$, $a_i \geqslant 0 (i = 2, \ldots, p)$, λ *be defined in (3.4), and* $\left|\frac{\sum_{i=2}^{p} a_i s^{\alpha_i} + \lambda}{s^{\alpha} + a_1 s^{\alpha_1}}\right| < 1$. *Then, we have*

$$\begin{aligned}\Psi(t) =& \mathcal{L}^{-1}\left\{\frac{s^{\alpha-1} + \sum_{i=1}^{p} a_i s^{\alpha_i - 1}}{s^{\alpha} + \sum_{i=1}^{p} a_i s^{\alpha_i} + \lambda}; t\right\} \\ =& \sum_{m=0}^{\infty}(-1)^m \sum_{k_1+k_2+\cdots+k_p=m} \frac{m!}{k_1!k_2!\cdots k_p!} \\ &\times \prod_{i=2}^{p} \lambda^{k_1} a_i^{k_i} t^{m\alpha - \sum_{j=2}^{p} k_j\alpha_j} E^{m+1}_{\alpha-\alpha_1, m\alpha - \sum_{j=2}^{p} k_j \alpha_j + 1}(-a_1 t^{\alpha-\alpha_1}) \\ &+ \sum_{m=0}^{\infty}(-1)^m \sum_{k_1+k_2+\cdots+k_p=m} \frac{m!}{k_1!k_2!\cdots k_p!} \prod_{i=2}^{p} \lambda^{k_1} a_i^{k_i} \\ &\times \sum_{n=1}^{p} a_n t^{(m+1)\alpha - \alpha_n - \sum_{j=2}^{p} k_j \alpha_j} E^{m+1}_{\alpha-\alpha_1, (m+1)\alpha - \alpha_n - \sum_{j=2}^{p} k_j\alpha_j + 1}(-a_1 t^{\alpha - \alpha_1}),\end{aligned}$$

where $k_i \geqslant 0$, $i = 1, 2, \ldots, p$.

Lemma 4 ([22]) *Let* $1 < \alpha_{p_0+1} < \cdots < \alpha_1 < \alpha \leqslant 2$, $a_1 > 0$, $a_i \geqslant 0 (i = 2, \ldots, p)$, λ *be defined in (3.4), and* $\left|\frac{\sum_{i=p_0+1}^{p} a_i s^{\alpha_i} + \lambda}{s^{\alpha} + a_1 s^{\alpha_1}}\right| < 1$. *Then we have*

$$\begin{aligned}\Upsilon(t) =& \mathcal{L}^{-1}\left\{\frac{s^{\alpha-2} + \sum_{i=p_0+1}^{p} a_i s^{\alpha_i - 2}}{s^{\alpha} + \sum_{i=1}^{p} a_i s^{\alpha_i} + \lambda}; t\right\} \\ =& \sum_{m=0}^{\infty}(-1)^m \sum_{k_1+k_2+\cdots+k_p=m} \frac{m!}{k_1!k_2!\cdots k_p!} \\ &\times \prod_{i=2}^{p} \lambda^{k_1} a_i^{k_i} t^{m\alpha - \sum_{j=2}^{p} k_j\alpha_j + 1} E^{m+1}_{\alpha-\alpha_1, m\alpha - \sum_{j=2}^{p} k_j \alpha_j + 2}(-a_1 t^{\alpha-\alpha_1}) \\ &+ \sum_{m=0}^{\infty}(-1)^m \sum_{k_1+k_2+\cdots+k_p=m} \frac{m!}{k_1!k_2!\cdots k_p!} \prod_{i=2}^{p} \lambda^{k_1} a_i^{k_i} \\ &\times \sum_{n=p_0+1}^{p} a_n t^{(m+1)\alpha - \alpha_n - \sum_{j=2}^{p} k_j \alpha_j + 1} E^{m+1}_{\alpha-\alpha_1, (m+1)\alpha - \alpha_n - \sum_{j=2}^{p} k_j\alpha_j + 2}(-a_1 t^{\alpha - \alpha_1}),\end{aligned}$$

where $k_i \geqslant 0$, $i = 1, 2, \ldots, p$.

Remark 1 The series involved in Lemma 2 converges uniformly on $[0, T]$. That is to say, the function $\Phi(t)$ is well-defined. In fact, from the relationship $0 < \alpha - \alpha_1 \leqslant 1$ and

$$(m+1)\alpha - \sum_{j=2}^{p} k_j \alpha_j \geqslant (m+1)(\alpha - \alpha_1),$$

one knows that the three-parameter Mittag-Leffler functions involved in Lemma 2 are monotone decaying functions of t and so they are largest at $t = 0$. (For details, one can refer to [30]) That is to say, we have

$$\begin{aligned}
&\Bigg|\sum_{m=0}^{\infty}(-1)^m \sum_{k_1+k_2+\cdots+k_p=m} \frac{m!}{k_1!k_2!\cdots k_p!} \prod_{i=2}^{p} \lambda^{k_1} a_i^{k_i} t^{(m+1)\alpha-\sum_{j=2}^{p} k_j\alpha_j-1} \\
&\times E^{m+1}_{\alpha-\alpha_1,(m+1)\alpha-\sum_{j=2}^{p} k_j\alpha_j}(-a_1 t^{\alpha-\alpha_1})\Bigg| \\
&\leqslant \frac{1}{\Gamma(\alpha-\alpha_1)} \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k_1+\cdots+k_p=m} \frac{m!}{k_1!\cdots k_p!} \prod_{i=2}^{p} |\lambda|^{k_1} a_i^{k_i} t^{(m+1)\alpha-\sum_{j=2}^{p} k_j\alpha_j-1} \\
&= \frac{1}{\Gamma(\alpha-\alpha_1)} \sum_{m=0}^{\infty} \frac{t^{(m+1)\alpha-1}}{m!} \left(|\lambda| + \sum_{i=2}^{p} a_i t^{-\alpha_i}\right)^m. \qquad (3.17)
\end{aligned}$$

Obviously, the series of the right hand of inequality (3.17) is convergent uniformly on $[0, T]$. Hence, the series involved in Lemma 2 is convergent uniformly on $[0, T]$. Using the similar arguments, we can prove that the series involved in Lemmas 3 and 4 converge uniformly on $[0, T]$. Therefore, the function $\Psi(t)$ and $\Upsilon(t)$ are also well-defined.

Now we can give the analytical representation of the solution.

Theorem 3 *Let* $0 \leqslant \alpha_p < \cdots < \alpha_1 < \alpha \leqslant 1$, $x_0 \in \mathbb{R}$, *and* φ *be a continuous function on* $[-h, 0]$. *Then the solutions of equation (3.15) with the initial conditions (3.16) are given by*

$$\begin{aligned}
&x_1(t) \\
&= \sum_{n=0}^{r_t} \lambda_4^n \sum_{i=0}^{n} C_n^i (\lambda_2\lambda_3)^i \Big(\Psi(t) - \lambda_4 \Psi(t-h)\Big) * \underbrace{\Phi(t-h) * \cdots * \Phi(t-h)}_{i} x_0 \\
&\quad + \sum_{n=0}^{r_t} \lambda_2 \lambda_4^{n+1} \sum_{i=0}^{n} C_n^i (\lambda_2\lambda_3)^i \Phi(t) * \underbrace{\Phi(t-h) * \cdots * \Phi(t-h)}_{i} * \big(\varphi(t-h) p(t-h)\big)
\end{aligned}$$

$$+\sum_{n=0}^{r_t}\lambda_4^n\sum_{i=0}^{n}C_n^i(\lambda_2\lambda_3)^i\Big(\Phi(t)-\lambda_4\Phi(t-h)\Big)*\underbrace{\Phi(t-h)*\cdots*\Phi(t-h)}_{i}*u(t)$$

$$+\sum_{n=0}^{r_t}\lambda_2\lambda_4^n\sum_{i=0}^{n}C_n^i(\lambda_2\lambda_3)^i\Phi(t)*\underbrace{\Phi(t-h)*\cdots*\Phi(t-h)}_{i}*e(t), \tag{3.18}$$

and

$$x_2(t)=\sum_{n=0}^{r_t}\lambda_4^n(\lambda_3x_1(t-nh)+e(t))+\lambda_4^{r_t+1}\varphi(t-r_th-h),\quad t\in[0,T], \tag{3.19}$$

where λ is defined in (3.4), $p(t)$ is defined in (3.9), Φ and Ψ are defined in Lemmas 2 and 3.

Proof Using the similar proof to Theorem 1, we take the Laplace transforms on the both sides of equation (3.15) to obtain

$$\begin{aligned}X_1(s)=&\frac{\lambda_4\Big(s^{\alpha-1}+\sum_{i=1}^{p}a_is^{\alpha_i-1}\Big)e^{-hs}x_1(0)}{\Big(s^\alpha+\sum_{i=1}^{p}a_is^{\alpha_i}-\lambda_1\Big)(\lambda_4e^{-hs}-1)+\lambda_2\lambda_3}\\
&-\frac{\Big(s^{\alpha-1}+\sum_{i=1}^{p}a_is^{\alpha_i-1}\Big)x_1(0)}{\Big(s^\alpha+\sum_{i=1}^{p}a_is^{\alpha_i}-\lambda_1\Big)(\lambda_4e^{-hs}-1)+\lambda_2\lambda_3}\\
&-\frac{\lambda_2\lambda_4e^{-hs}\int_{-h}^{0}e^{-s\tau}\varphi(\tau)d\tau}{\Big(s^\alpha+\sum_{i=1}^{p}a_is^{\alpha_i}-\lambda_1\Big)(\lambda_4e^{-hs}-1)+\lambda_2\lambda_3}\\
&+\frac{\lambda_4e^{-hs}U(s)}{\Big(s^\alpha+\sum_{i=1}^{p}a_is^{\alpha_i}-\lambda_1\Big)(\lambda_4e^{-hs}-1)+\lambda_2\lambda_3}\\
&-\frac{U(s)}{\Big(s^\alpha+\sum_{i=1}^{p}a_is^{\alpha_i}-\lambda_1\Big)(\lambda_4e^{-hs}-1)+\lambda_2\lambda_3}\\
&-\frac{\lambda_2E(s)}{\Big(s^\alpha+\sum_{i=1}^{p}a_is^{\alpha_i}-\lambda_1\Big)(\lambda_4e^{-hs}-1)+\lambda_2\lambda_3}.\end{aligned}$$

Next, we consider the Laplace inverse transform of $X_1(s)$. Let

$$\left|\frac{\lambda_4\left(s^\alpha+\sum\limits_{i=1}^{p}a_is^{\alpha_i}-\lambda_1\right)}{s^\alpha+\sum\limits_{i=1}^{p}a_is^{\alpha_i}-\lambda_1-\lambda_2\lambda_3}\right|<1.$$

Then, we can deduce that

$$\begin{aligned}
&\frac{\left(s^{\alpha-1}+\sum\limits_{i=1}^{p}a_is^{\alpha_i-1}\right)e^{-hs}}{\left(s^\alpha+\sum\limits_{i=1}^{p}a_is^{\alpha_i}-\lambda_1\right)(\lambda_4e^{-hs}-1)+\lambda_2\lambda_3}\\
&=-\frac{\left(s^{\alpha-1}+\sum\limits_{i=1}^{p}a_is^{\alpha_i-1}\right)e^{-hs}}{s^\alpha+\sum\limits_{i=1}^{p}a_is^{\alpha_i}-\lambda_1-\lambda_2\lambda_3}\frac{1}{1-\frac{\lambda_4\left(s^\alpha+\sum\limits_{i=1}^{p}a_is^{\alpha_i}-\lambda_1\right)e^{-hs}}{s^\alpha+\sum\limits_{i=1}^{p}a_is^{\alpha_i}-\lambda_1-\lambda_2\lambda_3}}\\
&=-\frac{\left(s^{\alpha-1}+\sum\limits_{i=1}^{p}a_is^{\alpha_i-1}\right)e^{-hs}}{s^\alpha+\sum\limits_{i=1}^{p}a_is^{\alpha_i}-\lambda_1-\lambda_2\lambda_3}\sum_{n=0}^{\infty}\frac{\lambda_4^n\left(s^\alpha+\sum\limits_{i=1}^{p}a_is^{\alpha_i}-\lambda_1\right)^ne^{-nhs}}{\left(s^\alpha+\sum\limits_{i=1}^{p}a_is^{\alpha_i}-\lambda_1-\lambda_2\lambda_3\right)^n}.
\end{aligned}\tag{3.20}$$

On the one hand, by Lemma 3 and (2.7), one knows that

$$\mathcal{L}^{-1}\left\{\frac{\left(s^{\alpha-1}+\sum\limits_{i=1}^{p}a_is^{\alpha_i-1}\right)e^{-hs}}{s^\alpha+\sum\limits_{i=1}^{p}a_is^{\alpha_i}-\lambda_1-\lambda_2\lambda_3};t\right\}=\Psi(t-h),\tag{3.21}$$

where Ψ is defined in Lemma 3. On the other hand, since

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{\left(s^\alpha+\sum\limits_{i=1}^{p}a_is^{\alpha_i}-\lambda_1\right)}{\left(s^\alpha+\sum\limits_{i=1}^{p}a_is^{\alpha_i}-\lambda_1-\lambda_2\lambda_3\right)};t\right\}\\
&=\mathcal{L}^{-1}\left\{1+\frac{\lambda_2\lambda_3}{\left(s^\alpha+\sum\limits_{i=1}^{p}a_is^{\alpha_i}-\lambda_1-\lambda_2\lambda_3\right)};t\right\}\\
&=\delta(t)+\lambda_2\lambda_3\Phi(t),
\end{aligned}\tag{3.22}$$

where δ is the unit impulsive function, and Φ is defined in Lemma 2, it follows that

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\sum_{n=0}^{\infty}\frac{\lambda_4^n\Big(s^{\alpha}+\sum_{i=1}^{p}a_i s^{\alpha_i}-\lambda_1\Big)^n e^{-nhs}}{\Big(s^{\alpha}+\sum_{i=1}^{p}a_i s^{\alpha_i}-\lambda_1-\lambda_2\lambda_3\Big)^n};t\right\}\\
&=\sum_{n=0}^{\infty}\lambda_4^n\underbrace{\Big(\delta(t-nh)+\lambda_2\lambda_3\Phi(t-nh)\Big)*\cdots*\Big(\delta(t-nh)+\lambda_2\lambda_3\Phi(t-nh)\Big)}_{n}u_{nh}(t)\\
&=\sum_{n=0}^{r_t}\lambda_4^n\underbrace{\Big(\delta(t-nh)+\lambda_2\lambda_3\Phi(t-nh)\Big)*\cdots*\Big(\delta(t-nh)+\lambda_2\lambda_3\Phi(t-nh)\Big)}_{n}.
\end{aligned}\tag{3.23}$$

Based on the above analysis and Property 2, we can obtain

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{\Big(s^{\alpha-1}+\sum_{i=1}^{p}a_i s^{\alpha_i-1}\Big)e^{-hs}}{\Big(s^{\alpha}+\sum_{i=1}^{p}a_i s^{\alpha_i}-\lambda_1\Big)(\lambda_4 e^{-hs}-1)+\lambda_2\lambda_3};t\right\}\\
&=-\mathcal{L}^{-1}\left\{\frac{\Big(s^{\alpha-1}+\sum_{i=1}^{p}a_i s^{\alpha_i-1}\Big)e^{-hs}}{s^{\alpha}+\sum_{i=1}^{p}a_i s^{\alpha_i}-\lambda_1-\lambda_2\lambda_3};t\right\}\\
&\quad*\mathcal{L}^{-1}\left\{\sum_{n=0}^{\infty}\frac{\lambda_4^n\Big(s^{\alpha}+\sum_{i=1}^{p}a_i s^{\alpha_i}-\lambda_1\Big)^n e^{-nhs}}{\Big(s^{\alpha}+\sum_{i=1}^{p}a_i s^{\alpha_i}-\lambda_1-\lambda_2\lambda_3\Big)^n};t\right\}\\
&=-\sum_{n=0}^{r_t}\lambda_4^n\sum_{i=0}^{n}C_n^i(\lambda_2\lambda_3)^i\Psi(t-h)*\underbrace{\Phi(t-h)*\cdots*\Phi(t-h)}_{i}.
\end{aligned}\tag{3.24}$$

Similarly, we can obtain that

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{\Big(s^{\alpha-1}+\sum_{i=1}^{p}a_i s^{\alpha_i-1}\Big)}{\Big(s^{\alpha}+\sum_{i=1}^{p}a_i s^{\alpha_i}-\lambda_1\Big)(\lambda_4 e^{-hs}-1)+\lambda_2\lambda_3};t\right\}\\
&=-\sum_{n=0}^{r_t}\lambda_4^n\sum_{i=0}^{n}C_n^i(\lambda_2\lambda_3)^i\Psi(t)*\underbrace{\Phi(t-h)*\cdots*\Phi(t-h)}_{i},
\end{aligned}\tag{3.25}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-hs}}{\left(s^{\alpha}+\sum_{i=1}^{p}a_i s^{\alpha_i}-\lambda_1\right)(\lambda_4 e^{-hs}-1)+\lambda_2\lambda_3};t\right\}$$
$$=-\sum_{n=0}^{r_t}\lambda_4^n\sum_{i=0}^{n}C_n^i(\lambda_2\lambda_3)^i\underbrace{\Phi(t-h)*\cdots*\Phi(t-h)}_{i+1}, \tag{3.26}$$

and

$$\mathcal{L}^{-1}\left\{\frac{1}{\left(s^{\alpha}+\sum_{i=1}^{p}a_i s^{\alpha_i}-\lambda_1\right)(\lambda_4 e^{-hs}-1)+\lambda_2\lambda_3};t\right\}$$
$$=-\sum_{n=0}^{r_t}\lambda_4^n\sum_{i=0}^{n}C_n^i(\lambda_2\lambda_3)^i\Phi(t)*\underbrace{\Phi(t-h)*\cdots*\Phi(t-h)}_{i}. \tag{3.27}$$

From (3.24), (3.25), (3.26) and (3.27), the Laplace inverse transform $x_1(t)$ of $X_1(s)$ can be obtained. Furthermore, using the stepwise method, we can give the expression of $x_2(t)$. The proof is completed. □

In the following, we consider the solutions of equation (3.15) in the case $0 \leqslant \alpha_p < \cdots < \alpha_{p_0} \leqslant 1 < \alpha_{p_0+1} < \cdots < \alpha \leqslant 2$. We discuss its solution satisfying the initial conditions

$$x_1(0)=x_0,\quad x_1'(0)=x_0',\quad x_2(t)=\varphi(t),\quad t\in[-h,0], \tag{3.28}$$

and we assume that φ satisfies $\varphi(0)=\lambda_3 x_1(0)+\lambda_4\varphi(-h)+e(0)$.

Using similar arguments to Theorem 3, we can establish the following theorem.

Theorem 4 *Let* $0 \leqslant \alpha_p < \cdots < \alpha_{p_0} \leqslant 1 < \alpha_{p_0+1} < \cdots < \alpha \leqslant 2$, $x_0, x_0' \in \mathbb{R}$, $0 < \alpha - \alpha_1 \leqslant 1$, *and φ be a continuous function on* $[-h, 0]$. *Then the solutions of equation (3.15) with the initial conditions (3.28) are given by*

$$\begin{aligned}
&x_1(t)\\
&=\sum_{n=0}^{r_t}\lambda_4^n\sum_{i=0}^{n}C_n^i(\lambda_2\lambda_3)^i\left(\Psi(t)-\lambda_4\Psi(t-h)\right)*\underbrace{\Phi(t-h)*\cdots*\Phi(t-h)}_{i}x_0\\
&\quad+\sum_{n=0}^{r_t}\lambda_4^n\sum_{i=0}^{n}C_n^i(\lambda_2\lambda_3)^i\left(\Upsilon(t)-\lambda_4\Upsilon(t-h)\right)*\underbrace{\Phi(t-h)*\cdots*\Phi(t-h)}_{i}x_0'\\
&\quad+\sum_{n=0}^{r_t}\lambda_2\lambda_4^{n+1}\sum_{i=0}^{n}C_n^i(\lambda_2\lambda_3)^i\Phi(t)*\underbrace{\Phi(t-h)*\cdots*\Phi(t-h)}_{i}*\left(\varphi(t-h)p(t-h)\right)
\end{aligned}$$

$$+\sum_{n=0}^{r_t}\lambda_4^n\sum_{i=0}^{n}C_n^i(\lambda_2\lambda_3)^i\Big(\Phi(t)-\lambda_4\Phi(t-h)\Big)*\underbrace{\Phi(t-h)*\cdots*\Phi(t-h)}_{i}*u(t)$$

$$+\sum_{n=0}^{r_t}\lambda_2\lambda_4^n\sum_{i=0}^{n}C_n^i(\lambda_2\lambda_3)^i\Phi(t)*\underbrace{\Phi(t-h)*\cdots*\Phi(t-h)}_{i}*e(t),$$

and

$$x_2(t)=\sum_{n=0}^{r_t}\lambda_3\lambda_4^n x_1(t-nh)+\lambda_4^{r_t+1}\varphi(t-r_th-h)+\sum_{n=0}^{r_t}\lambda_4^n e(t),\quad t\in[0,T],$$

where λ *is defined in (3.4),* $p(t)$ *is defined in (3.9),* Φ, Ψ *and* Υ *are defined in Lemmas 2, 3 and 4, respectively.*

Proof We firstly take the Laplace transforms on the both sides of equation (3.15) to obtain

$$\begin{aligned}X_1(s)=&\frac{\lambda_4\Big(s^{\alpha-1}+\sum_{i=1}^{p}a_is^{\alpha_i-1}\Big)e^{-hs}x_1(0)}{\Big(s^\alpha+\sum_{i=1}^{p}a_is^{\alpha_i}-\lambda_1\Big)(\lambda_4e^{-hs}-1)+\lambda_2\lambda_3}\\
&-\frac{\Big(s^{\alpha-1}+\sum_{i=1}^{p}a_is^{\alpha_i-1}\Big)x_1(0)}{\Big(s^\alpha+\sum_{i=1}^{p}a_is^{\alpha_i}-\lambda_1\Big)(\lambda_4e^{-hs}-1)+\lambda_2\lambda_3}\\
&-\frac{\Big(s^{\alpha-2}+\sum_{i=s_0+1}^{p}a_is^{\alpha_i-2}\Big)x_1'(0)}{\Big(s^\alpha+\sum_{i=1}^{p}a_is^{\alpha_i}-\lambda_1\Big)(\lambda_4e^{-hs}-1)+\lambda_2\lambda_3}\\
&+\frac{\lambda_4\Big(s^{\alpha-2}+\sum_{i=s_0+1}^{p}a_is^{\alpha_i-2}\Big)e^{-sh}x_1'(0)}{\Big(s^\alpha+\sum_{i=1}^{p}a_is^{\alpha_i}-\lambda_1\Big)(\lambda_4e^{-hs}-1)+\lambda_2\lambda_3}\\
&-\frac{\lambda_2\lambda_4e^{-hs}\int_{-h}^{0}e^{-s\tau}\varphi(\tau)d\tau}{\Big(s^\alpha+\sum_{i=1}^{p}a_is^{\alpha_i}-\lambda_1\Big)(\lambda_4e^{-hs}-1)+\lambda_2\lambda_3}\\
&+\frac{\lambda_4e^{-hs}U(s)}{\Big(s^\alpha+\sum_{i=1}^{p}a_is^{\alpha_i}-\lambda_1\Big)(\lambda_4e^{-hs}-1)+\lambda_2\lambda_3}\end{aligned}$$

$$
-\frac{U(s)}{\left(s^{\alpha}+\sum\limits_{i=1}^{p} a_i s^{\alpha_i}-\lambda_1\right)(\lambda_4 e^{-hs}-1)+\lambda_2\lambda_3}
-\frac{\lambda_2 E(s)}{\left(s^{\alpha}+\sum\limits_{i=1}^{p} a_i s^{\alpha_i}-\lambda_1\right)(\lambda_4 e^{-hs}-1)+\lambda_2\lambda_3}.
$$

Using the similar arguments to (3.24), we have

$$
\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{\left(s^{\alpha-2}+\sum\limits_{i=p_0+1}^{p} a_i s^{\alpha_i-2}\right)e^{-hs}}{\left(s^{\alpha}+\sum\limits_{i=1}^{p} a_i s^{\alpha_i}-\lambda_1\right)(\lambda_4 e^{-hs}-1)+\lambda_2\lambda_3};t\right\}\\
&=-\sum_{n=0}^{r_t}\lambda_4^n\sum_{i=0}^{n}C_n^i(\lambda_2\lambda_3)^i\Upsilon(t-h)*\underbrace{\Phi(t-h)*\cdots*\Phi(t-h)}_{i},
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{\left(s^{\alpha-2}+\sum\limits_{i=p_0+1}^{p} a_i s^{\alpha_i-2}\right)}{\left(s^{\alpha}+\sum\limits_{i=1}^{p} a_i s^{\alpha_i}-\lambda_1\right)(\lambda_4 e^{-hs}-1)+\lambda_2\lambda_3};t\right\}\\
&=-\sum_{n=0}^{r_t}\lambda_4^n\sum_{i=0}^{n}C_n^i(\lambda_2\lambda_3)^i\Upsilon(t)*\underbrace{\Phi(t-h)*\cdots*\Phi(t-h)}_{i}.
\end{aligned}
$$

And we combine (3.25), (3.26) and (3.27) to obtain the expression of $x_1(t)$. The proof is completed. □

4 Analytical solutions for the MTS-FPDDAES with fractional diffusion terms

In this section, we consider the analytical solutions of the MTS-FPDDAE in the case $0 \leqslant \alpha_s < \cdots < \alpha_1 < \alpha \leqslant 1$ and $a_1 > 0$. In this case, equation (1.1) is a generalized time and space fractional partial delay differential-algebraic equation with fractional diffusion terms. We discuss its solution that satisfies the nonhomogeneous Dirichlet boundary condition (1.2) and the initial conditions

$$
u(x,0)=\psi(x),\quad v(x,t)=\phi(x,t),\quad 0\leqslant x\leqslant L,\quad -h\leqslant t\leqslant 0. \tag{4.1}
$$

In order to solve the equation with the nonhomogeneous Dirichlet boundary conditions, we first transform the nonhomogeneous Dirichlet boundary conditions into homogeneous Dirichlet boundary conditions. Let

$$u(x,t) = W_1(x,t) + \xi(x,t), \quad v(x,t) = W_2(x,t) + \zeta(x,t), \tag{4.2}$$

where $W_1(x,t)$, $W_2(x,t)$ are two new unknown functions, and

$$\xi(x,t) = \frac{u_L(t) - u_0(t)}{L}x + u_0(t), \quad \zeta(x,t) = \frac{v_L(t) - v_0(t)}{L}x + v_0(t).$$

Substituting (4.2) into (1.1), we get the fractional partial delay differential-algebraic equation

$$\begin{cases} P(D_t^*)W_1(x,t) = -k_{p_1}(-\triangle)^{\frac{p_1}{2}}W_1(x,t) - k_{p_2}(-\triangle)^{\frac{p_2}{2}}W_2(x,t) + f_1(x,t), \\ W_2(x,t) = -k_{q_1}(-\triangle)^{\frac{q_1}{2}}W_1(x,t) - k_{q_2}(-\triangle)^{\frac{q_2}{2}}W_2(x,t-h) + g_2(x,t), \end{cases} \tag{4.3}$$

with the homogeneous Dirichlet boundary conditions

$$W_1(0,t) = W_2(L,t) = 0, \quad 0 \leqslant t \leqslant T,$$
$$W_2(0,t) = W_2(L,t) = 0, \quad -h \leqslant t \leqslant T,$$

and the initial conditions

$$W_1(x,0) = \psi(x) - \frac{u_L(0) - u_0(0)}{L}x - u_0(0), \; 0 \leqslant x \leqslant L, \tag{4.4}$$

$$W_2(x,t) = \phi(x,t) - \frac{v_L(t) - v_0(t)}{L}x - v_0(t), \; 0 \leqslant x \leqslant L, -h \leqslant t \leqslant 0, \tag{4.5}$$

where

$$f_1(x,t) = -P(D_t^*)\xi(x,t) - k_{p_1}(-\triangle)^{\frac{p_1}{2}}\xi(x,t) - k_{p_2}(-\triangle)^{\frac{p_2}{2}}\zeta(x,t) + f(x,t),$$
$$g_1(x,t) = -\zeta(x,t) - k_{q_1}(-\triangle)^{\frac{q_1}{2}}\xi(x,t) - k_{q_2}(-\triangle)^{\frac{q_2}{2}}\zeta(x,t-h) + g(x,t).$$

According to Lemma 1, the eigenvalues $\mu_n^2 \;\; (n = 1, 2, \ldots)$ of the operator $(-\triangle)$ with the homogeneous boundary conditions is $\mu_n^2 = n^2\pi^2/L^2$, and the corresponding eigenfunctions are $\varphi_n(x) = \sin(n\pi x/L)$, $n = 1, 2, \ldots$. Then we set

$$W_1(x,t) = \sum_{n=1}^{\infty} w_{n1}(t)\sin(n\pi x/L), \; W_2(x,t) = \sum_{n=1}^{\infty} w_{n2}(t)\sin(n\pi x/L), \tag{4.6}$$

$$f_1(x,t) = \sum_{n=1}^{\infty} f_{n1}(t)\sin(n\pi x/L), \; g_1(x,t) = \sum_{n=1}^{\infty} g_{n1}(t)\sin(n\pi x/L). \tag{4.7}$$

Substituting (4.6) and (4.7) into (4.3) leads to

$$\begin{cases} P(D_t^*)w_{n1}(t) = -k_{p_1}\mu_n^{p_1}w_{n1}(t) - k_{p_2}\mu_n^{p_2}w_{n2}(t) + f_{n1}(t), \\ w_{n2}(t) = -k_{q_1}\mu_n^{q_1}w_{n1}(t) - k_{q_2}\mu_n^{q_2}w_{n2}(t-h) + g_{n1}(t), \end{cases} \tag{4.8}$$

with the initial conditions

$$w_{n1}(0) = \frac{2}{L}\int_0^L W_1(x,0)\sin(n\pi x/L)dx,$$
$$w_{n2}(t) = \frac{2}{L}\int_0^L W_2(x,t)\sin(n\pi x/L)dx, \tag{4.9}$$

where $-h \leqslant t \leqslant 0$. By Theorem 3, the solutions of equation (4.8) with the initial conditions (4.9) are

$$\begin{aligned} &w_{n1}(t)\\ &= \sum_{n=0}^{r_t}\lambda_4^n\sum_{i=0}^{n}C_n^i(\lambda_2\lambda_3)^i\Big(\Psi(t)-\lambda_4\Psi(t-h)\Big)*\underbrace{\Phi(t-h)*\cdots*\Phi(t-h)}_{i}w_{n1}(0)\\ &\quad+\sum_{n=0}^{r_t}\lambda_2\lambda_4^{n+1}\sum_{i=0}^{n}C_n^i(\lambda_2\lambda_3)^i\Phi(t)*\underbrace{\Phi(t-h)*\cdots*\Phi(t-h)}_{i}*\big(w_{n2}(t-h)p(t-h)\big)\\ &\quad+\sum_{n=0}^{r_t}\lambda_4^n\sum_{i=0}^{n}C_n^i(\lambda_2\lambda_3)^i\Big(\Phi(t)-\lambda_4\Phi(t-h)\Big)*\underbrace{\Phi(t-h)*\cdots*\Phi(t-h)}_{i}*f_{n1}(t)\\ &\quad+\sum_{n=0}^{r_t}\lambda_2\lambda_4^n\sum_{i=0}^{n}C_n^i(\lambda_2\lambda_3)^i\Phi(t)*\underbrace{\Phi(t-h)*\cdots*\Phi(t-h)}_{i}*g_{n1}(t), \end{aligned} \tag{4.10}$$

and

$$w_{n2}(t) = \sum_{n=0}^{r_t}\lambda_4^n(\lambda_3 w_{n1}(t-nh)+g_{n1}(t)) + \lambda_4^{r_t+1}w_{n2}(t-r_t h-h), \tag{4.11}$$

where

$$\lambda_1 = -k_{p_1}\mu_n^{p_1}, \lambda_2 = -k_{p_2}\mu_n^{p_2}, \lambda_3 = -k_{q_1}\mu_n^{q_1},$$
$$\lambda_4 = -k_{q_2}\mu_n^{q_2}, \lambda = \lambda_1 + \lambda_2\lambda_3, \tag{4.12}$$

and Φ, Ψ are defined in Lemmas 2 and 3, respectively.

Therefore, we obtain the solutions of equation (1.1) with the boundary condition (1.2) and the initial conditions (3.11) are

$$u(x,t) = \sum_{n=1}^{\infty} w_{n1}(t)\sin(n\pi x/L) + \frac{u_L(t)-u_0(t)}{L}x + u_0(t),$$
$$v(x,t) = \sum_{n=1}^{\infty} w_{n2}(t)\sin(n\pi x/L) + \frac{v_L(t)-v_0(t)}{L}x + v_0(t),$$

where $w_{n1}(t)$, $w_{n2}(t)$, $\lambda_i (i = 1, 2, 3, 4)$ and λ are given in (4.10), (4.11) and (4.12), respectively.

5 Analytical solutions of the MTS-FPDDAES with fractional wave terms

In this section, we consider the analytical solutions of the MTS-FPDDAE in the case $0 \leqslant \alpha_p < \cdots < \alpha_{p0} \leqslant 1 < \alpha_{p0+1} < \cdots < \alpha \leqslant 2$ and $a_1 > 0$. In this case, equation (1.1) is a generalized time and space fractional partial delay differential-algebraic equation with fractional wave terms. We discuss its solution that satisfies boundary condition (1.2) and the initial conditions

$$u(x, 0) = \psi(x), u_t(x, 0) = \chi(x), v(x, t) = \phi(x, t), 0 \leqslant x \leqslant L. \tag{5.1}$$

In order to solve the equation with the nonhomogeneous Dirichlet boundary conditions, we firstly transform the nonhomogeneous Dirichlet boundary conditions into Dirichlet homogeneous boundary conditions. Let

$$u(x, t) = W_1(x, t) + \xi(x, t), \quad v(x, t) = W_2(x, t) + \zeta(x, t), \tag{5.2}$$

where $W_1(x, t)$, $W_2(x, t)$ are two new unknown functions, and

$$\xi(x, t) = \frac{u_L(t) - u_0(t)}{L} x + u_0(t), \quad \zeta(x, t) = \frac{v_L(t) - v_0(t)}{L} x + v_0(t).$$

Substituting (5.2) into (1.1), we get the following fractional differential equation

$$\begin{cases} P(D_t^*)W_1(x, t) = -k_{p_1}(-\triangle)^{\frac{p_1}{2}} W_1(x, t) - k_{p_2}(-\triangle)^{\frac{p_2}{2}} W_2(x, t) + f_1(x, t), \\ W_2(x, t) = -k_{q_1}(-\triangle)^{\frac{q_1}{2}} W_1(x, t) - k_{q_2}(-\triangle)^{\frac{q_2}{2}} W_2(x, t-h) + g_2(x, t), \end{cases} \tag{5.3}$$

with the homogeneous Dirichlet boundary conditions

$$W_1(0, t) = W_2(L, t) = 0, \quad W_2(0, t) = W_2(L, t) = 0, 0 \leqslant t \leqslant T,$$

and the initial conditions

$$W_1(x, 0) = \psi(x) - \frac{u_L(0) - u_0(0)}{L} x - u_0(0), \quad 0 \leqslant x \leqslant L, \tag{5.4}$$

$$\left.\frac{\partial W_1(x, t)}{\partial t}\right|_{t=0} = \chi(x) - \frac{u_L'(0) - u_0'(0)}{L} x - u_0'(0), \quad 0 \leqslant x \leqslant L, \tag{5.5}$$

$$W_2(x, t) = \phi(x, t) - \frac{v_L(t) - v_0(t)}{L} x - v_0(t), 0 \leqslant x \leqslant L, -h \leqslant t \leqslant 0, \tag{5.6}$$

where

$$f_1(x, t) = -P(D_t^*)\xi(x, t) - k_{p_1}(-\triangle)^{\frac{p_1}{2}} \xi(x, t) - k_{p_2}(-\triangle)^{\frac{p_2}{2}} \zeta(x, t) + f(x, t),$$

$$g_1(x,t) = -\zeta(x,t) - k_{q_1}(-\triangle)^{\frac{q_1}{2}}\xi(x,t) - k_{q_2}(-\triangle)^{\frac{q_2}{2}}\zeta(x,t-h) + g(x,t).$$

According to Lemma 1, the eigenvalues μ_n^2 $(n = 1, 2, \ldots)$ of the operator $(-\triangle)$ with the homogeneous boundary conditions is $\mu_n^2 = n^2\pi^2/L^2$, and the corresponding eigenfunctions are $\varphi_n(x) = \sin(n\pi x/L)$, $n = 1, 2, \ldots$. Then we set

$$\begin{aligned} W_1(x,t) &= \sum_{n=1}^{\infty} w_{n1}(t)\sin(n\pi x/L), \\ W_2(x,t) &= \sum_{n=1}^{\infty} w_{n2}(t)\sin(n\pi x/L), \quad (5.7) \\ f_1(x,t) &= \sum_{n=1}^{\infty} f_{n1}(t)\sin(n\pi x/L), \\ g_1(x,t) &= \sum_{n=1}^{\infty} g_{n1}(t)\sin(n\pi x/L). \quad (5.8) \end{aligned}$$

Substituting (5.7) and (5.8) into (5.3) leads to

$$\begin{cases} P(D_t^*)w_{n1}(t) = -k_{p_1}\mu_n^{p_1}w_{n1}(t) - k_{p_2}\mu_n^{p_2}w_{n2}(t) + f_{n1}(t), \\ w_{n2}(t) = -k_{q_1}\mu_n^{q_1}w_{n1}(t) - k_{q_2}\mu_n^{q_2}w_{n2}(t-h) + g_{n1}(t), \end{cases} \quad (5.9)$$

with the initial conditions

$$w_{n1}(0) = \frac{2}{L}\int_0^L W_1(x,0)\sin(n\pi x/L)dx, \quad (5.10)$$

$$w'_{n1}(0) = \frac{2}{L}\int_0^L \frac{\partial W_1(x,t)}{\partial t}\Big|_{t=0}\sin(n\pi x/L)dx, \quad (5.11)$$

$$w_{n2}(t) = \frac{2}{L}\int_0^L W_2(x,t)\sin(n\pi x/L)dx, \quad -h \leqslant t \leqslant 0. \quad (5.12)$$

By Theorem 4, the solutions of (5.9) with the initial conditions (5.10), (5.11) and (5.12) are

$$\begin{aligned} &w_{n1}(t) \\ &= \sum_{n=0}^{r_t}\lambda_4^n\sum_{i=0}^{n}C_n^i(\lambda_2\lambda_3)^i\Big(\Psi(t) - \lambda_4\Psi(t-h)\Big) * \underbrace{\Phi(t-h) * \cdots * \Phi(t-h)}_{i} w_{n1}(0) \\ &+ \sum_{n=0}^{r_t}\lambda_4^n\sum_{i=0}^{n}C_n^i(\lambda_2\lambda_3)^i\Big(\Upsilon(t) - \lambda_4\Upsilon(t-h)\Big) * \underbrace{\Phi(t-h) * \cdots * \Phi(t-h)}_{i} w'_{n1}(0) \\ &+ \sum_{n=0}^{r_t}\lambda_2\lambda_4^{n+1}\sum_{i=0}^{n}C_n^i(\lambda_2\lambda_3)^i\Phi(t) * \underbrace{\Phi(t-h) * \cdots * \Phi(t-h)}_{i} * \big(w_{n2}(t-h)p(t-h)\big) \end{aligned}$$

$$+\sum_{n=0}^{r_t}\lambda_4^n\sum_{i=0}^{n}C_n^i(\lambda_2\lambda_3)^i\Big(\Phi(t)-\lambda_4\Phi(t-h)\Big)*\underbrace{\Phi(t-h)*\cdots*\Phi(t-h)}_{i}*f_{n1}(t)$$
$$+\sum_{n=0}^{r_t}\lambda_2\lambda_4^n\sum_{i=0}^{n}C_n^i(\lambda_2\lambda_3)^i\Phi(t)*\underbrace{\Phi(t-h)*\cdots*\Phi(t-h)}_{i}*g_{n1}(t), \tag{5.13}$$

and

$$w_{n2}(t)=\sum_{n=0}^{r_t}\lambda_4^n(\lambda_3 w_{n1}(t-nh)+g_{n1}(t))+\lambda_4^{r_t+1}w_{n2}(t-r_t h-h), \tag{5.14}$$

where λ_i $(i=1,2,3,4)$ are defined as (4.12), and Φ, Ψ and Υ are defined in Lemmas 2, 3 and 4, respectively.

Therefore, we obtain the solutions of Eq. (1.1) with the boundary condition (1.2) and the initial condition (3.11) are

$$u(x,t)=\sum_{n=1}^{\infty}w_{n1}(t)\sin(n\pi x/L)+\frac{u_L(t)-u_0(t)}{L}x+u_0(t),$$
$$v(x,t)=\sum_{n=1}^{\infty}w_{n2}(t)\sin(n\pi x/L)+\frac{v_L(t)-v_0(t)}{L}x+v_0(t),$$

where $w_{n1}(t)$ and $w_{n2}(t)$ are given in (5.13) and (5.14), respectively.

6 Special cases

In this section, we provide details of solutions for some special cases.

Case 1: $0<\alpha\leqslant 1$,**and** $a_1=\cdots=a_p=0$. In this case, the problem can be described as

$$\begin{cases} D_t^\alpha u(x,t)=-k_{p_1}(-\triangle)^{\frac{p_1}{2}}u(x,t)-k_{p_2}(-\triangle)^{\frac{p_2}{2}}v(x,t)+f(x,t),\\ v(x,t)=-k_{q_1}(-\triangle)^{\frac{q_1}{2}}u(t,x)-k_{q_2}(-\triangle)^{\frac{q_2}{2}}v(x,t-h)+g(x,t),\end{cases} \tag{6.1}$$

with the nonhomogeneous Dirichlet boundary conditions

$$u(0,t)=u_0(t),\quad u(L,t)=u_L(t),\quad 0\leqslant t\leqslant T, \tag{6.2}$$
$$v(0,t)=v_0(t),\quad v(L,t)=v_L(t),\quad -h\leqslant t\leqslant T, \tag{6.3}$$

and the initial conditions

$$u(x,0)=\psi(x),\quad v(x,t)=\phi(x,t),\quad 0\leqslant x\leqslant L,\quad -h\leqslant t\leqslant 0. \tag{6.4}$$

Following the method in Section 4, let

$$u(x,t) = W_1(x,t) + \xi(x,t), \quad v(x,t) = W_2(x,t) + \zeta(x,t), \tag{6.5}$$

where $W_1(x,t)$, $W_2(x,t)$ are two new unknown functions, and

$$\xi(x,t) = \frac{u_L(t) - u_0(t)}{L}x + u_0(t), \quad \zeta(x,t) = \frac{v_L(t) - v_0(t)}{L}x + v_0(t).$$

Substituting (6.5) into (6.1), we get the equation

$$\begin{cases} D_t^\alpha W_1(x,t) = -k_{p_1}(-\triangle)^{\frac{p_1}{2}} W_1(x,t) - k_{p_2}(-\triangle)^{\frac{p_2}{2}} W_2(x,t) + f_1(x,t), \\ W_2(x,t) = -k_{q_1}(-\triangle)^{\frac{q_1}{2}} W_1(x,t) - k_{q_2}(-\triangle)^{\frac{q_2}{2}} W_2(x,t-h) + g_2(x,t), \end{cases} \tag{6.6}$$

with the homogeneous Dirichlet boundary conditions

$$\begin{aligned} W_1(0,t) &= W_2(L,t) = 0, \quad 0 \leqslant t \leqslant T, \\ W_2(0,t) &= W_2(L,t) = 0, \quad -h \leqslant t \leqslant T, \end{aligned}$$

and the initial conditions

$$W_1(x,0) = \psi(x) - \frac{u_L(0) - u_0(0)}{L}x - u_0(0), 0 \leqslant x \leqslant L, \tag{6.7}$$

$$W_2(x,t) = \phi(x,t) - \frac{v_L(t) - v_0(t)}{L}x - v_0(t), 0 \leqslant x \leqslant L, -h \leqslant t \leqslant 0, \tag{6.8}$$

where

$$\begin{aligned} f_1(x,t) &= -D_t^\alpha \xi(x,t) - k_{p_1}(-\triangle)^{\frac{p_1}{2}} \xi(x,t) - k_{p_2}(-\triangle)^{\frac{p_2}{2}} \zeta(x,t) + f(x,t), \\ g_1(x,t) &= -\zeta(x,t) - k_{q_1}(-\triangle)^{\frac{q_1}{2}} \xi(x,t) - k_{q_2}(-\triangle)^{\frac{q_2}{2}} \zeta(x,t-h) + g(x,t). \end{aligned}$$

According to Lemma 1, the eigenvalues μ_n^2 $(n = 1, 2, \ldots)$ of the operator $(-\triangle)$ with the homogeneous boundary conditions is $\mu_n^2 = n^2\pi^2/L^2$, and the corresponding eigenfunctions are $\varphi_n(x) = \sin(n\pi x/L)$, $n = 1, 2, \ldots$. Then we set

$$\begin{aligned} W_1(x,t) &= \sum_{n=1}^{\infty} w_{n1}(t) \sin(n\pi x/L), \\ W_2(x,t) &= \sum_{n=1}^{\infty} w_{n2}(t) \sin(n\pi x/L), \\ f_1(x,t) &= \sum_{n=1}^{\infty} f_{n1}(t) \sin(n\pi x/L), \\ g_1(x,t) &= \sum_{n=1}^{\infty} g_{n1}(t) \sin(n\pi x/L). \end{aligned} \tag{6.9, 6.10}$$

Substituting (6.9) and (6.10) into (6.6) leads to

$$\begin{cases} D_t^{\alpha} w_{n1}(t) = -k_{p_1}\mu_n^{p_1} w_{n1}(t) - k_{p_2}\mu_n^{p_2} w_{n2}(t) + f_{n1}(t), \\ w_{n2}(t) = -k_{q_1}\mu_n^{q_1} w_{n1}(t) - k_{q_2}\mu_n^{q_2} w_{n2}(t-h) + g_{n1}(t), \end{cases} \tag{6.11}$$

with the initial conditions

$$w_{n1}(0) = \frac{2}{L}\int_0^L W_1(x,0)\sin(n\pi x/L)dx, \tag{6.12}$$

$$w_{n2}(t) = \frac{2}{L}\int_0^L W_2(x,t)\sin(n\pi x/L)dx, \quad -h \leqslant t \leqslant 0, \tag{6.13}$$

Then, according to Theorem 1, the solutions of (6.11) with the initial conditions (6.12) and (6.13) are

$$w_{n1}(t) = \begin{cases} E_\alpha(\lambda t^\alpha) w_{n1}(0) + \lambda_2\lambda_4 \int_{-h}^{t-h} e_{\alpha,\alpha}^{1;\lambda(t-h-\tau)} \varphi(\tau) d\tau \\ + \int_0^t e_{\alpha,\alpha}^{1;\lambda(t-\tau)} \Big(f_{n1}(\tau) + \lambda_2 g_{n1}(\tau) \Big) d\tau, 0 \leqslant t < h; \\ \Big(\sum_{n=0}^{r_t} \sum_{i=0}^{n} C_n^i (-\lambda_1)^{n-i} \lambda_4^n e_{\alpha,\alpha(n-i)+1}^{n+1;\lambda(t-nh)} \\ - \sum_{l=1}^{r_t} \sum_{i=0}^{l-1} C_{l-1}^i (-\lambda_1)^{l-i-1} \lambda_4^l e_{\alpha,\alpha(l-i-1)+1}^{l;\lambda(t-lh)} \Big) w_{n1}(0) \\ + \sum_{n=0}^{r_t} \sum_{i=0}^{n} C_n^i (-\lambda_1)^{n-i} \lambda_2 \lambda_4^{n+1} \int_{-h}^{0} e_{\alpha,\alpha(n+1-i)}^{n+1;\lambda(t-(n+1)h-\tau)} w_{n2}(\tau) d\tau \\ - \sum_{l=1}^{r_t} \sum_{i=0}^{l-1} C_{l-1}^i (-\lambda_1)^{l-i-1} \lambda_4^l \int_0^t e_{\alpha,\alpha(l-i)}^{l;\lambda(t-lh-\tau)} f_{n1}(\tau) d\tau \\ + \sum_{n=0}^{r_t} \sum_{i=0}^{n} C_n^i (-\lambda_1)^{n-i} \lambda_4^n \int_0^t e_{\alpha,\alpha(n+1-i)}^{n+1;\lambda(t-nh-\tau)} \Big(f_{n1}(\tau) + \lambda_2 g_{n1}(\tau) \Big) d\tau, \\ h \leqslant t \leqslant T, \end{cases} \tag{6.14}$$

and

$$w_{n2}(t) = \sum_{n=0}^{r_t} \lambda_4^n (\lambda_3 w_{n1}(t-nh) + g_{n1}(t)) + \lambda_4^{r_t+1} w_{n2}(t - r_t h - h), \tag{6.15}$$

where λ_i $(i = 1, 2, 3, 4)$ are defined as (4.12), and $e_{\alpha,\beta}^{\rho;\lambda z}$ is defined as (3.4).

Therefore, the solutions of equation (6.1) with the boundary conditions (6.2) and (6.3) and the initial conditions (6.4) are

$$u(x,t)=\sum_{n=1}^{\infty} w_{n1}(t)\sin(n\pi x/L)+\frac{u_L(t)-u_0(t)}{L}x+u_0(t),$$
$$v(x,t)=\sum_{n=1}^{\infty} w_{n2}(t)\sin(n\pi x/L)+\frac{v_L(t)-v_0(t)}{L}x+v_0(t),$$

where $w_{n1}(t)$ and $w_{n2}(t)$ are defined in (6.14) and (6.15), respectively.

Case 2: $1<\alpha\leqslant 2$,**and** $a_1=\cdots=a_p=0$. In this case, the problem can be described as

$$\begin{cases} D_t^{\alpha}u(x,t)=-k_{p_1}(-\triangle)^{\frac{p_1}{2}}u(x,t)-k_{p_2}(-\triangle)^{\frac{p_2}{2}}v(x,t)+f(x,t),\\ v(x,t)=-k_{q_1}(-\triangle)^{\frac{q_1}{2}}u(x,t)-k_{q_2}(-\triangle)^{\frac{q_2}{2}}v(x,t-h)+g(x,t), \end{cases} \tag{6.16}$$

with the nonhomogeneous Dirichlet boundary conditions

$$u(0,t)=u_0(t),\quad u(L,t)=u_L(t),\quad 0\leqslant t\leqslant T, \tag{6.17}$$
$$v(0,t)=v_0(t),\quad v(L,t)=v_L(t),\quad -h\leqslant t\leqslant T, \tag{6.18}$$

and the initial conditions

$$u(x,0)=\psi(x),\quad v(x,t)=\phi(x,t),\quad 0<x<L,\quad -h\leqslant t\leqslant 0. \tag{6.19}$$

Following the method in Section 5, let

$$u(x,t)=W_1(x,t)+\xi(x,t),\quad v(x,t)=W_2(x,t)+\zeta(x,t), \tag{6.20}$$

where $W_1(x,t)$, $W_2(x,t)$ are two new unknown functions, and

$$\xi(x,t)=\frac{u_L(t)-u_0(t)}{L}x+u_0(t),\quad \zeta(x,t)=\frac{v_L(t)-v_0(t)}{L}x+v_0(t).$$

Substituting (6.20) into (6.16), we get the following equation

$$\begin{cases} D_t^{\alpha}W_1(x,t)=-k_{p_1}(-\triangle)^{\frac{p_1}{2}}W_1(x,t)-k_{p_2}(-\triangle)^{\frac{p_2}{2}}W_2(x,t)+f_1(x,t),\\ W_2(x,t)=-k_{q_1}(-\triangle)^{\frac{q_1}{2}}W_1(x,t)-k_{q_2}(-\triangle)^{\frac{q_2}{2}}W_2(x,t-h)+g_2(x,t), \end{cases} \tag{6.21}$$

with the homogeneous Dirichlet boundary conditions

$$W_1(0,t)=W_2(L,t)=0,\quad W_2(0,t)=W_2(L,t)=0,\quad 0\leqslant t\leqslant T,$$

and the initial conditions

$$W_1(x,0) = \psi(x) - \frac{u_L(0) - u_0(0)}{L}x - u_0(0), \quad 0 \leqslant x \leqslant L, \tag{6.22}$$

$$\frac{\partial W_1(x,t)}{\partial t}\Big|_{t=0} = \chi(x) - \frac{u'_L(0) - u'_0(0)}{L}x - u'_0(0), \quad 0 \leqslant x \leqslant L, \tag{6.23}$$

$$W_2(x,t) = \phi(x,t) - \frac{v_L(t) - v_0(t)}{L}x - v_0(t), \; 0 \leqslant x \leqslant L, \; -h \leqslant t \leqslant 0, \tag{6.24}$$

where

$$f_1(x,t) = -D_t^{\alpha}\xi(x,t) - k_{p_1}(-\triangle)^{\frac{p_1}{2}}\xi(x,t) - k_{p_2}(-\triangle)^{\frac{p_2}{2}}\zeta(x,t) + f(x,t),$$
$$g_1(x,t) = -\zeta(x,t) - k_{q_1}(-\triangle)^{\frac{q_1}{2}}\xi(x,t) - k_{q_2}(-\triangle)^{\frac{q_2}{2}}\zeta(x,t-h) + g(x,t).$$

According to Lemma 1, the eigenvalues $\mu_n^2 \quad (n = 1, 2, \ldots)$ of the operator $(-\triangle)$ with the homogeneous boundary conditions is $\mu_n^2 = n^2\pi^2/L^2$, and the corresponding eigenfunctions are $\varphi_n(x) = \sin(n\pi x/L)$, $n = 1, 2, \ldots$. Then we set

$$W_1(x,t) = \sum_{n=1}^{\infty} w_{n1}(t)\sin(n\pi x/L), \; W_2(x,t) = \sum_{n=1}^{\infty} w_{n2}(t)\sin(n\pi x/L), \tag{6.25}$$

$$f_1(x,t) = \sum_{n=1}^{\infty} f_{n1}(t)\sin(n\pi x/L), \; g_1(x,t) = \sum_{n=1}^{\infty} g_{n1}(t)\sin(n\pi x/L). \tag{6.26}$$

Substituting (6.25) and (6.26) into (6.21) leads to

$$\begin{cases} D_t^{\alpha} w_{n1}(t) = -k_{p_1}\mu_n^{p_1} w_{n1}(t) - k_{p_2}\mu_n^{p_2} w_{n2}(t) + f_{n1}(t), \\ w_{n2}(t) = -k_{q_1}\mu_n^{q_1} w_{n1}(t) - k_{q_2}\mu_n^{q_2} w_{n2}(t-h) + g_{n1}(t), \end{cases} \tag{6.27}$$

with the initial conditions

$$w_{n1}(0) = \frac{2}{L}\int_0^L W_1(x,0)\sin(n\pi x/L)dx, \tag{6.28}$$

$$w'_{n1}(0) = \frac{2}{L}\int_0^L \frac{\partial W_1(x,t)}{\partial t}\Big|_{t=0}\sin(n\pi x/L)dx, \tag{6.29}$$

$$w_{n2}(t) = \frac{2}{L}\int_0^L W_2(x,t)\sin(n\pi x/L)dx, \quad -h \leqslant t \leqslant 0. \tag{6.30}$$

Then, according to Theorem 2, the solutions of (6.27) with the initial conditions (6.28), (6.29) and (6.30) are

$$w_{n1}(t) = \begin{cases} E_\alpha(\lambda t^\alpha) w_{n1}(0) + E_{\alpha,2}(\lambda t^\alpha) w'_{n1}(0) + \lambda_2\lambda_4 \int_{-h}^{t-h} e_{\alpha,\alpha}^{1;\lambda(t-h-\tau)} w_{n2}(\tau) d\tau \\ + \int_0^t e_{\alpha,\alpha}^{1;\lambda(t-\tau)} \Big(f_{n1}(\tau) + \lambda_2 g_{n1}(\tau) \Big) d\tau, \quad 0 \leqslant t < h; \\ \Big(\sum_{n=0}^{r_t} \sum_{i=0}^{n} C_n^i (-\lambda_1)^{n-i} \lambda_4^n e_{\alpha,\alpha(n-i)+1}^{n+1;\lambda(t-nh)} \\ - \sum_{l=1}^{r_t} \sum_{i=0}^{l-1} C_{l-1}^i (-\lambda_1)^{l-i-1} \lambda_4^l e_{\alpha,\alpha(l-i-1)+1}^{l;\lambda(t-lh)} \Big) w_{n1}(0) \\ + \Big(\sum_{n=0}^{r_t} \sum_{i=0}^{n} C_n^i (-\lambda_1)^{n-i} \lambda_4^n e_{\alpha,\alpha(n-i)+2}^{n+1;\lambda(t-nh)} \\ - \sum_{l=1}^{r_t} \sum_{i=0}^{l-1} C_{l-1}^i (-\lambda_1)^{l-i-1} \lambda_4^l e_{\alpha,\alpha(l-i-1)+2}^{l;\lambda(t-lh)} \Big) w'_{n1}(0) \\ + \sum_{n=0}^{r_t} \sum_{i=0}^{n} C_n^i (-\lambda_1)^{n-i} \lambda_2 \lambda_4^{n+1} \int_{-h}^{0} e_{\alpha,\alpha(n+1-i)}^{n+1;\lambda(t-(n+1)h-\tau)} w_{n2}(\tau) d\tau \\ - \sum_{l=1}^{r_t} \sum_{i=0}^{l-1} C_{l-1}^i (-\lambda_1)^{l-i-1} \lambda_4^l \int_0^t e_{\alpha,\alpha(l-i)}^{l;\lambda(t-lh-\tau)} f_{n1}(\tau) d\tau \\ + \sum_{n=0}^{r_t} \sum_{i=0}^{n} C_n^i (-\lambda_1)^{n-i} \lambda_4^n \int_0^t e_{\alpha,\alpha(n+1-i)}^{n+1;\lambda(t-nh-\tau)} \Big(f_{n1}(\tau) + \lambda_2 g_{n1}(\tau) \Big) d\tau, \\ h \leqslant t \leqslant T, \end{cases} \tag{6.31}$$

and

$$w_{n2}(t) = \sum_{n=0}^{r_t} \lambda_4^n (\lambda_3 x_1(t - nh) + e(t)) + \lambda_4^{r_t+1} \varphi(t - r_t h - h), \tag{6.32}$$

where λ_i $(i = 1, 2, 3, 4)$ are defined as (4.12), and $e_{\alpha,\beta}^{\rho;\lambda z}$ is defined as (3.4).

Therefore, the solutions of equation (6.16) with the boundary conditions (6.17) and (6.18) and the initial conditions (6.19) are

$$u(x, t) = \sum_{n=1}^{\infty} w_{n1}(t) \sin(n\pi x/L) + \frac{u_L(t) - u_0(t)}{L} x + u_0(t),$$
$$v(x, t) = \sum_{n=1}^{\infty} w_{n2}(t) \sin(n\pi x/L) + \frac{v_L(t) - v_0(t)}{L} x + v_0(t),$$

where $w_{n1}(t)$ and $w_{n2}(t)$ are given by (6.31) and (6.32), respectively.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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