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Positive radial solutions for Dirichlet problems via a Harnack-type inequality

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We deal with the existence and localization of positive radial solutions for Dirichlet problems involving ϕ-Laplacian operators in a ball. In particular, p-Laplacian and Minkowski-curvature equations are considered. Our approach relies on fixed point index techniques, which work thanks to a Harnack-type inequality in terms of a seminorm. As a consequence of the localization result, it is also derived the existence of several (even infinitely many) positive solutions.

KEYWORDS
compression–expansion, Dirichlet problem, fixed point index, Harnack-type inequality, mean curvature operator, Positive radial solution

MSC CLASSIFICATION
35J25, 35J60, 34B18, 35J92, 35J93

1 | INTRODUCTION

In this paper, we deal with the existence, localization, and multiplicity of positive radial solutions to the Dirichlet problem involving ϕ-Laplacian operators:

$$-\mathrm{div}\left(\psi\left(|\nabla v|\right)\nabla v\right) = f\left(|x|, v\right) \text{ in } \mathcal{B},\ \ v = 0 \text{ on } \partial\mathcal{B}, \tag{1.1}$$

where $\mathcal{B}$ is the unit open ball in $\mathbb{R}^n$ $(n \geq 3)$ centered at the origin, the function $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, and $\psi : (-a, a) \to \mathbb{R}$ is C^1, such that $\phi(s) := s\psi(s)$ is an increasing homeomorphism between two intervals $(-a, a)$ and $(-b, b)$ $(0 < a, b \leq +\infty)$.

The following particular cases are of much interest due to their corresponding models arising from physics:

(a) $\phi : \mathbb{R} \to \mathbb{R}$, $\phi(s) = |s|^{p-2}s$, where $p > 1$ (here $a = b = +\infty$), when the left side Lv in (1.1) is

$$Lv = -\mathrm{div}\left(|\nabla v|^{p-2}\nabla v\right) \ (p - \text{Laplace operator}),$$

involved in a nonlinear Darcy law for flows through porous media;[1]

(b) (*singular* homeomorphism) $\phi : (-a, a) \to \mathbb{R}$, $\phi(s) = \frac{s}{\sqrt{a^2-s^2}}$ (here $0 < a < +\infty$ and $b = +\infty$), when

$$Lv = -\mathrm{div}\left(\frac{\nabla v}{\sqrt{a^2 - |\nabla v|^2}}\right) \ (\textit{Minkowski mean curvature operator}),$$

involved in the relativistic mechanics;[2,3]

(c) (*bounded* homeomorphism) $\phi : \mathbb{R} \to (-b, b)$, $\phi(s) = \frac{bs}{\sqrt{1+s^2}}$ (here $a = +\infty$ and $0 < b < +\infty$), when

$$Lv = -b\,\mathrm{div}\left(\frac{\nabla v}{\sqrt{1 + |\nabla v|^2}}\right) \ (\textit{Euclidian mean curvature operator}),$$

associated to capillarity problems.[4,5]

Looking for radial solutions of (1.1), that is, functions of the form $v(x) = u(r)$ with $r = |x|$, the Dirichlet problem (1.1) reduces to the mixed boundary value problem

$$-\left(r^{n-1}\phi\left(u'\right)\right)' = r^{n-1}f(r, u), \qquad u'(0) = u(1) = 0. \tag{1.2}$$

Radial and nonradial solutions for the Dirichlet problem involving ϕ -Laplace operators have been intensively investigated in the literature, both by means of topological and variational methods. We refer the interested reader to the papers[6–14] and the references therein.

Our approach here is based on fixed point index theory, namely, on compression–expansion-type homotopy arguments. The most known are those from Krasnosel'skiı's compression–expansion theorem in a conical annulus defined by using the max-norm of the space. Applications to one-dimensional ϕ-Laplace equations are given in the papers.[15,16] In the radial case considered in the present paper, the absence of a Harnack-type inequality in terms of the max-norm makes Krasnosel'skiı's theorem inoperative and forces us to use instead some other homotopy conditions and properties of the fixed point index.

The first paper in radial solutions that uses the compression–expansion technique, but in a variational form and only for p-Laplacian equations, is Precup et al.[17] As explained there, the difficulty in applying the compression–expansion method consists in the necessity that, for the considered differential operator, a Harnack-type inequality be available. In the present paper, such a key inequality is established for problem (1.2) with a general homeomorphism ϕ satisfying some additional conditions. With its help, a precise localization of positive solutions is possible, allowing in a natural way to obtain multiple solutions. The results apply in particular for the p-Laplacian and the Minkowski mean curvature operator.

Our basic assumptions are as follows:

(H_ϕ) $\phi : (-a, a) \to (-b, b)$ $(0 < a, b \le +\infty)$ is an odd increasing homeomorphism such that

$$\lambda\phi(x) \ge \phi(\lambda x) \ \text{ for all } \ \lambda \in [0, 1], x \in [0, a) \ (\phi \text{ is convex on } [0, a)). \tag{1.3}$$

(H_f) $f : [0, 1] \times \mathbb{R}_+ \to [0, b)$ is continuous, with $f(\cdot, s)$ nonincreasing in $[0, 1]$ for every $s \in \mathbb{R}_+$ and $f(r, \cdot)$ nondecreasing in $\mathbb{R}_+$ for every $r \in [0, 1]$.

Note that the homeomorphisms related to the p-Laplacian for $p \ge 2$ and the Minkowski mean curvature operator both satisfy condition (H_ϕ). Contrarily, the bounded homeomorphisms with $a = +\infty$, for example, the one involved by the Euclidian mean curvature operator, are not convex on $[0, +\infty)$ and thus they do not satisfy our assumption (H_ϕ).

2 | A HARNACK-TYPE INEQUALITY

In the space of functions $u \in C^1[0,1]$ satisfying $u'(0) = u(1) = 0$, we consider the following norms:

$$\|u\|_p = \left(\int_0^1 \left(r^{n-1} \left| u'(r) \right| \right)^p dr \right)^{1/p} \text{ for } 1 \le p < \infty;$$
$$\|u\|_\infty = \sup_{r \in [0,1]} r^{n-1} \left| u'(r) \right| \text{ for } p = +\infty.$$

Let ϕ satisfy (H_ϕ) and denote

$$h_0(r) = -r^{n-1}\phi\left(u'(r)\right),$$
$$J(u) = h_0'(r) = -\left(r^{n-1}\phi(u'(r))\right)'.$$

First, we prove a Harnack-type inequality for problem (1.2), given in terms of the norm $\|\cdot\|_p$, with $1 \le p \le +\infty$.

Theorem 2.1. *Let $u \in C^1[0,1]$ be such that $u'(r) \in (-a,a)$ for all $r \in [0,1]$, $h_0 \in C^1[0,1]$ and $J(u) \ge 0$ on $[0,1]$. Then $u' \le 0$ on $[0,1]$.*

If in addition $r^{1-n}J(u)$ is nonincreasing on $(0,1]$, then

$$u(r) \ge (1-r)\, r^n \|u\|_p, \quad r \in [0,1],$$

for every $1 \le p \le +\infty$.

Proof. By assumption,

$$h_0' = J(u) \ge 0.$$

Hence, h_0 is nondecreasing in $[0,1]$. Since $h_0(0) = 0$, one has $h_0 \ge 0$ and so $u' \le 0$ on $[0,1]$. Thus,

$$h_0(r) = r^{n-1}\phi\left(\left|u'(r)\right|\right). \tag{2.1}$$

Next,

$$\phi^{-1}(h_0(r)) = \phi^{-1}\left(r^{n-1}\phi\left(\left|u'(r)\right|\right)\right) \le \phi^{-1}\left(\phi\left(\left|u'(r)\right|\right)\right) = \left|u'(r)\right|, \quad r \in [0,1]. \tag{2.2}$$

Also,

$$\phi^{-1}(h_0(1)) = \left|u'(1)\right|.$$

Since both ϕ^{-1} and h_0 are nondecreasing, using (2.2), we have

$$u(r) = \int_r^1 \left|u'(s)\right| ds \ge \int_r^1 \phi^{-1}(h_0(s))\, ds \ge (1-r)\,\phi^{-1}(h_0(r)).$$

Therefore,

$$u(r) \ge (1-r)\,\phi^{-1}(h_0(r)), \quad r \in [0,1].$$

Next, we prove the inequality

$$\Phi(r) := h_0(r) - r^n h_0(1) \ge 0 \text{ on } [0,1]. \tag{2.3}$$

One has

$$\Phi'(r) = J(u)(r) - n r^{n-1} h_0(1) = r^{n-1}\left(r^{1-n}J(u)(r) - n h_0(1)\right).$$

By assumption, $\Psi(r) := r^{1-n}J(u)(r) - n h_0(1)$ is nonincreasing. As in the proof of Precup et al.,[17, Theorem 2.1] since $\Phi(0) = \Phi(1) = 0$, we deduce (2.3).

Then

$$\phi^{-1}(h_0(r)) \ge \phi^{-1}\left(r^n h_0(1)\right) = \phi^{-1}\left(r^n\phi\left(\left|u'(1)\right|\right)\right) \ge r^n \left|u'(1)\right|,$$

where the last inequality is based on (1.3). Hence,

$$u(r) \geq (1-r)\, r^n \left|u'(1)\right|, \quad r \in [0,1].$$

Finally, by (2.1) and using again (1.3), one has $\phi^{-1}(h_0(r)) \geq r^{n-1}|u'(r)|$. Hence, we have

$$\begin{aligned}\|u\|_p^p &:= \int_0^1 \left(r^{n-1}\left|u'(r)\right|\right)^p dr \leq \int_0^1 \phi^{-1}(h_0(r))^p dr \\ &\leq \phi^{-1}(h_0(1))^p = \left|u'(1)\right|^p, \ (1 \leq p < \infty).\end{aligned}$$

Therefore,

$$u(r) \geq (1-r)\, r^n \|u\|_p \ \ (1 \leq p \leq \infty), \tag{2.4}$$

for all $r \in [0,1]$. □

A similar result has been established in Precup et al[17] for the particular case of the p-Laplacian with $p > n$, for which $\phi(s) = |s|^{p-2}s$ and $a = b = +\infty$. More exactly, it has been proved that

$$u(r) \geq \left(\frac{p-n}{p-1}\right)^{\frac{1}{p}} (1-r)\, r^{\frac{n}{p-1}} \|u\|_{1,p} \ \ (r \in [0,1]), \tag{2.5}$$

where $\|u\|_{1,p} = \left(\int_0^1 r^{n-1}|u'(r)|^p dr\right)^{\frac{1}{p}}$. In this case, since by using Hölder's inequality one has

$$\begin{aligned}|u(r)| &\leq \int_r^1 \left|u'(s)\right| ds = \int_r^1 s^{\frac{n-1}{p}} \left|u'(s)\right| s^{-\frac{n-1}{p}} ds \\ &\leq \|u\|_{1,p}\left(\int_0^1 s^{\frac{n-1}{p-1}} ds\right)^{\frac{p-1}{p}} = \left(\frac{p-1}{p-n}\right)^{\frac{p-1}{p}} \|u\|_{1,p},\end{aligned}$$

a Harnack-type inequality in terms of the max-norm $|u|_\infty = \max_{r\in[0,1]} |u(r)|$ can be immediately derived from (2.5), namely,

$$u(r) \geq \frac{p-n}{p-1} (1-r)\, r^{\frac{n}{p-1}} |u|_\infty \ \text{for } r \in [0,1].$$

It is an open problem to obtain an analog result for more general homeomorphisms ϕ satisfying (H_ϕ). At this moment we are only able to establish such an inequality in terms of a max-seminorm on $C[0,1]$. For example, taking $p = +\infty$ in (2.4), we have the following Harnack-type inequality related to a seminorm on $C[0,1]$.

Corollary 2.2. *Under the assumptions of Theorem 2.1, if for a fixed subinterval $[\eta, \nu]$ with $0 < \eta < \nu < 1$, one can define in $C[0,1]$ the seminorm $[u]_\infty = \max_{r\in[\eta,\nu]} |u(r)|$, then*

$$u(r) \geq (1-\nu)\, \eta^{2n-2} (n-2)\, [u]_\infty \ \textit{for} \ \ r \in [\eta, \nu]. \tag{2.6}$$

Proof. Clearly,

$$\begin{aligned}|u(r)| &\leq \int_r^1 \left|u'(s)\right| ds = \int_r^1 s^{n-1} \left|u'(s)\right| s^{-(n-1)} ds \\ &\leq \|u\|_\infty \int_r^1 s^{-(n-1)} ds \leq \frac{r^{2-n}}{n-2} \|u\|_\infty,\end{aligned}$$

which implies

$$[u]_\infty \leq \frac{\eta^{2-n}}{n-2} \|u\|_\infty.$$

This combined with (2.4) yields (2.6). □

In the sequel, inequality (2.6) is a key ingredient for the localization and multiplicity of positive radial solutions. We will use the main ideas in Precup[17] in order to localize the solutions in terms of a norm and a seminorm.

3 | POSITIVE RADIAL SOLUTIONS

Recall that by a (nonnegative) *solution* of (1.2), we mean a function $u \in C^1([0,1], \mathbb{R}_+)$ with $u'(0) = u(1) = 0, |u'(r)| < a$ for all $r \in [0,1]$, such that $r^{n-1}\phi(u') \in C^1[0,1]$ and (1.2) is satisfied. We will say that a nonnegative solution is *positive* if it is distinct from the identically zero function. Let X be the Banach space of continuous functions $X = C[0,1]$ and K_0 its positive cone $K_0 = \{u \in X : u \geq 0 \text{on} [0,1]\}$. It is not difficult to see that a nonnegative function u is a solution of problem (1.2) if and only if u is a fixed point of the operator $T : K_0 \to K_0$ given by

$$T(u)(r) = \int_r^1 \phi^{-1}\left(\tau^{1-n}\int_0^\tau s^{n-1} f(s, u(s))\, ds\right) d\tau. \tag{3.1}$$

As proved in earlier studies,[6,8] the operator T is completely continuous.

Let us now consider a subcone of K_0 related to the Harnack inequality (2.6), namely,

$$K = \left\{ u \in K_0 : u \text{ is nonincreasing on } [0,1] \text{ and } \min_{r\in[\eta,\nu]} u(r) \geq c[u]_\infty \right\}, \tag{3.2}$$

where $c := (1-\nu)\eta^{2n-2}(n-2)$.

Lemma 3.1. *The operator T maps the cone K into itself.*

Proof. Indeed, take $u \in K$ and let us show that $v := Tu$ belongs to K. Since f is nonnegative, $v \geq 0$, and moreover, $J(v) \geq 0$ and so $v' \leq 0$ (see Theorem 2.1), that is, v is nonincreasing on $[0,1]$. Furthermore, by the monotonicity properties of f imposed in (H_f) and the fact that u is nonincreasing in $[0,1]$, the composed function $r \mapsto f(r, u(r))$ is nonincreasing in $[0,1]$. Hence,

$$r^{1-n}J(v) = f(r,u)$$

is nonincreasing in $[0,1]$. Then Corollary 2.2 ensures that

$$v(r) \geq (1-\nu)\eta^{2n-2}(n-2)[v]_\infty \text{ for } r \in [\eta,\nu].$$

Therefore, $v \in K$, as claimed. □

Now, for any number $\alpha > 0$, consider the set

$$U_\alpha := \{u \in K : |u|_\infty < \alpha\}.$$

The operator T being completely continuous, the set $T\left(\overline{U}_\alpha\right)$ is bounded, so there is a number $\widetilde{\alpha} \geq \alpha$ such that $T\left(\overline{U}_\alpha\right) \subset \overline{U}_{\widetilde{\alpha}}$. Define the operator $\tilde{T} : \overline{U}_{\widetilde{\alpha}} \to \overline{U}_{\widetilde{\alpha}}$ by

$$\tilde{T}(u) = T\left(\min\left\{\frac{\alpha}{|u|_\infty}, 1\right\} u\right).$$

Lemma 3.2. *If*

$$T(u) \neq \lambda u \quad \textit{for} \quad u \in K \quad \textit{with} \quad |u|_\infty = \alpha \quad \textit{and} \quad \lambda \geq 1, \tag{3.3}$$

then the fixed point index $i\left(\tilde{T}, U_\alpha, \overline{U}_{\widetilde{\alpha}}\right) = 1$.

Proof. Clearly, $\overline{U}_{\widetilde{\alpha}}$ is a convex closed set and $\tilde{T}$ is a compact map. Consider the homotopy $H : [0,1] \times \overline{U}_{\widetilde{\alpha}} \to \overline{U}_{\widetilde{\alpha}}$ given by

$$H(\tau, u) = \tau\tilde{T}(u).$$

By (3.3), this homotopy is *admissible* and so

$$i\left(\tilde{T}, U_\alpha, \overline{U_{\widetilde{\alpha}}}\right) = i\left(H(1,\cdot), U_\alpha, \overline{U_{\widetilde{\alpha}}}\right) = i\left(H(0,\cdot), U_\alpha, \overline{U_{\widetilde{\alpha}}}\right) = 1,$$

where the last equality is due to the normalization property of the fixed point index, since $0 \in U_\alpha$. □

Next, for a number $\beta > 0$, consider the set

$$V_\beta := \left\{u \in \overline{U_{\widetilde{\alpha}}} \; : \; [u]_\infty < \beta\right\}.$$

It is clear that V_β is open in $\overline{U_{\widetilde{\alpha}}}$.

Lemma 3.3. *Assume that there exists a function $h \in K$ such that $|h|_\infty = \alpha$, $[h]_\infty > \beta$ and*

$$(1-\lambda)\,\tilde{T}(u) + \lambda h \neq u \text{ for } u \in K \text{ with } |u|_\infty \leq \widetilde{\alpha},\, [u]_\infty = \beta \text{ and } \lambda \in [0,1]. \tag{3.4}$$

Then $i\left(\tilde{T}, V_\beta, \overline{U_{\widetilde{\alpha}}}\right) = 0$.

Proof. Observe that $\partial V_\beta = \left\{u \in K \; : \; [u]_\infty = \beta,\, |u|_\infty \leq \widetilde{\alpha}\right\}$. Thus, (3.4) implies that

$$(1-\lambda)\,\tilde{T}(u) + \lambda h \neq u \;\text{ for }\; u \in \partial V_\beta.$$

By the homotopy property of the fixed point index, one has

$$i\left(\tilde{T}, V_\beta, \overline{U_{\widetilde{\alpha}}}\right) = i\left(h, V_\beta, \overline{U_{\widetilde{\alpha}}}\right).$$

Finally, $i\left(h, V_\beta, \overline{U_{\widetilde{\alpha}}}\right) = 0$, since $h \in \overline{U_{\widetilde{\alpha}}} \setminus \overline{V_\beta}$. □

Remark 3.1. If the operator T maps $\overline{U_\alpha}$ into itself, then $\widetilde{\alpha} = \alpha$ and condition (3.4) reduces to

$$(1-\lambda)\,T(u) + \lambda h \neq u \text{ for } u \in K \text{ with } |u|_\infty \leq \alpha,\, [u]_\infty = \beta \text{ and } \lambda \in [0,1].$$

By using the previous fixed point index computations, we deduce the following existence result.

Lemma 3.4. *Under the assumptions of Lemmas 3.2 and 3.3, the operator T has a fixed point u in $U_\alpha \setminus \overline{V_\beta}$, that is, problem (1.2) has a solution such that*

$$\beta < [u]_\infty \text{ and } |u|_\infty < \alpha.$$

Proof. One has

$$1 = i\left(\tilde{T}, U_\alpha, \overline{U_{\widetilde{\alpha}}}\right) = i\left(\tilde{T}, U_\alpha \setminus \overline{V_\beta}, \overline{U_{\widetilde{\alpha}}}\right) + i\left(\tilde{T}, U_\alpha \cap V_\beta, \overline{U_{\widetilde{\alpha}}}\right),$$
$$0 = i\left(\tilde{T}, V_\beta, \overline{U_{\widetilde{\alpha}}}\right) = i\left(\tilde{T}, V_\beta \setminus \overline{U_\alpha}, \overline{U_{\widetilde{\alpha}}}\right) + i\left(\tilde{T}, U_\alpha \cap V_\beta, \overline{U_{\widetilde{\alpha}}}\right).$$

As a result,

$$i\left(\tilde{T}, U_\alpha \setminus \overline{V_\beta}, \overline{U_{\widetilde{\alpha}}}\right) - i\left(\tilde{T}, V_\beta \setminus \overline{U_\alpha}, \overline{U_{\widetilde{\alpha}}}\right) = 1.$$

In addition $i\left(\tilde{T}, V_\beta \setminus \overline{U_\alpha}, \overline{U_{\widetilde{\alpha}}}\right) = 0$ since otherwise there would exist $v \in V_\beta \setminus \overline{U_\alpha}$ with $\tilde{T}(v) = v$, that is,

$$T\left(\frac{\alpha}{|v|_\infty}v\right) = v,$$

or equivalently, $T(w) = \lambda w$, where $w = \frac{\alpha}{|v|_\infty}v$ and $\lambda = \frac{|v|_\infty}{\alpha}$. Since $|w|_\infty = \alpha$ and $\lambda > 1$, we arrived to a contradiction with (3.3). Therefore $i\left(\tilde{T}, U_\alpha \setminus \overline{V_\beta}, \overline{U_{\widetilde{\alpha}}}\right) = 1$, which implies our conclusion. □

Now we give sufficient conditions in order to guarantee the assumptions of the previous lemmas hold.
We will use the following notation. If $b < +\infty$, denote

$$A := \frac{1}{\int_0^1 \phi^{-1}(b\tau)d\tau} \text{ and } B := \int_\eta^1 \phi^{-1}(b\tau)d\tau.$$

If $b = +\infty$, denote

$$A := \frac{1}{\int_0^1 \phi^{-1}(\tau)d\tau} \text{ and } B := \int_\eta^1 \phi^{-1}(\tau)d\tau.$$

Theorem 3.5. *Assume that $n \geq 3$ and conditions (H_ϕ) and (H_f) are fulfilled. If there exist $\alpha, \beta > 0$ with $\beta < AB\alpha$, such that*

$$\phi^{-1}(f(0,\alpha)) < \alpha, \tag{3.5}$$

$$(1-\nu)\phi^{-1}\left((\nu-\eta)\eta^{n-1}f(\nu, c\beta)\right) > \beta, \tag{3.6}$$

then problem (1.2) has at least one solution $u \in K$ such that $\beta < [u]_\infty$ and $|u|_\infty < \alpha$.

Proof. We shall apply Lemma 3.4. First, we show that (3.3) holds. Indeed, for $u \in K$ with $|u|_\infty \leq \alpha$, by the monotonicity assumptions on f, we have that

$$f(s, u(s)) \leq f(0, \alpha),$$

and thus, from (3.5),

$$\begin{aligned}|T(u)(r)| &\leq \int_0^1 \phi^{-1}\left(\tau^{1-n}\int_0^\tau s^{n-1}f(s,u(s))\,ds\right)d\tau \\ &\leq \int_0^1 \phi^{-1}\left(\int_0^\tau f(s,u(s))\,ds\right)d\tau \leq \phi^{-1}(f(0,\alpha)) < \alpha.\end{aligned}$$

Hence, $|T(u)|_\infty < \alpha$ for all $u \in K$ with $|u|_\infty \leq \alpha$, which implies (3.3). In addition, on the basis of Remark 3.1, we can take $\widetilde{\alpha} = \alpha$.

Next, we prove that (3.4) holds for the following choice of h: if $b < +\infty$,

$$h(r) = A\alpha \int_r^1 \phi^{-1}\left(\tau^{1-n}\int_0^\tau bns^{n-1}ds\right)d\tau = A\alpha\int_r^1 \phi^{-1}(b\tau)\,d\tau,$$

and, otherwise, for $b = +\infty$,

$$h(r) = A\alpha \int_r^1 \phi^{-1}\left(\tau^{1-n}\int_0^\tau ns^{n-1}ds\right)d\tau = A\alpha\int_r^1 \phi^{-1}(\tau)\,d\tau.$$

Note that $|h|_\infty = h(0) = \alpha$ and $\left[h\right]_\infty = h(\eta) = AB\alpha > \beta$. Assume that (3.4) does not hold. Then there exist $u \in K$ with $|u|_\infty \leq \alpha$, $[u]_\infty = \beta$ and $\lambda \in [0,1]$ such that

$$(1-\lambda)\,T(u) + \lambda h = u.$$

In particular, since $[u]_\infty = \max_{r\in[\eta,\nu]} u(r) = \beta$, one has

$$\begin{aligned}\beta \ge u(\eta) &= (1-\lambda)T(u)(\eta) + \lambda h(\eta)\\ &= (1-\lambda)\int_\eta^1 \phi^{-1}\left(\tau^{1-n}\int_0^\tau s^{n-1} f(s,u(s))\,ds\right)d\tau + \lambda\big[h\big]_\infty\\ &\ge (1-\lambda)\int_\nu^1 \phi^{-1}\left(\tau^{1-n}\int_\eta^\nu s^{n-1} f(s,u(s))\,ds\right)d\tau + \lambda\beta.\end{aligned}$$

Since $u\in K$ with $[u]_\infty = \beta$, we have $u(r)\ge c\beta$ for all $r\in[\eta,\nu]$. Thus, by (H_f),

$$\begin{aligned}\beta &\ge (1-\lambda)\int_\nu^1 \phi^{-1}\left(\tau^{1-n}\int_\eta^\nu s^{n-1} f(\nu,c\beta)\,ds\right)d\tau + \lambda\beta\\ &\ge (1-\lambda)(1-\nu)\phi^{-1}\left((\nu-\eta)\eta^{n-1} f(\nu,c\beta)\right) + \lambda\beta,\end{aligned}$$

that is,

$$(1-\lambda)\beta \ge (1-\lambda)(1-\nu)\phi^{-1}\left((\nu-\eta)\eta^{n-1} f(\nu,c\beta)\right),$$

which contradicts (3.6) for any $\lambda\in[0,1)$. Note that, in case $\lambda = 1$, one has the contradiction

$$\beta \ge u(\eta) = h(\eta) = \big[h\big]_\infty > \beta.$$

Finally, the conclusion follows from Lemma 3.4. □

Remark 3.2 (Asymptotic conditions). Existence of both positive numbers α and β satisfying inequalities (3.5) and (3.6) is guaranteed if the following asymptotic conditions at zero and infinity hold:

$$\limsup_{x\to 0^+}\frac{\phi^{-1}((\nu-\eta)\eta^{n-1} f(\nu,x))}{x} > \frac{1}{c(1-\nu)},\quad \liminf_{x\to+\infty}\frac{\phi^{-1}(f(0,x))}{x} < 1.$$

Obviously, if ϕ is a classical homeomorphism ($a = b = +\infty$), conditions (3.5) and (3.6) can be rewritten as

$$f(0,\alpha) < \phi(\alpha), \qquad f(\nu,c\beta) > \frac{1}{(\nu-\eta)\eta^{n-1}}\phi\left(\frac{\beta}{1-\nu}\right).$$

Hence, if we assume in addition that ϕ satisfies:

$$\limsup_{x\to 0^+}\frac{\phi(\tau x)}{\phi(x)} < +\infty \text{ for all } \tau > 0, \tag{3.7}$$

then the existence of both positive numbers α and β is guaranteed under suitable asymptotic conditions about f at zero and at infinity.

Note that assumption (3.7) holds in the case of the p-Laplacian operator and so it is commonly employed in the literature, see for instance.[8,12]

Theorem 3.6. *Assume that $n\ge 3$, conditions (H_ϕ) and (H_f) are fulfilled, and ϕ is a classical homeomorphim. If (3.7) and*

$$\limsup_{x\to 0^+}\frac{f(\nu,x)}{\phi(x)} = +\infty,\quad \liminf_{x\to+\infty}\frac{f(0,x)}{\phi(x)} < 1 \tag{3.8}$$

hold, then problem (1.2) has at least one positive solution.

Proof. By (3.7), with $\tau = 1/(c(1-\nu))$, there exists $L > 0$ so that

$$L > \limsup_{x\to 0^+}\frac{\phi(x/(c(1-\nu)))}{\phi(x)},$$

and thus, there exists $\rho > 0$ such that

$$L\phi(x) \geq \phi\left(\frac{x}{c(1-\nu)}\right) \text{ for all } x \in (0, \rho).$$

Now, by (3.8), there exists $r > 0$ (we may suppose $r < \rho$) such that

$$f(\nu, r) > \frac{1}{(\nu - \eta)\eta^{n-1}} L\phi(r),$$

which implies that

$$f(\nu, r) > \frac{1}{(\nu - \eta)\eta^{n-1}} \phi\left(\frac{r}{c(1-\nu)}\right).$$

Finally, taking $\beta = r/c$, condition (3.6) is obtained.

On the other hand, condition

$$\liminf_{x \to +\infty} \frac{f(0,x)}{\phi(x)} < 1$$

clearly implies the existence of a positive number α satisfying (3.5) and such that $\beta < AB\alpha$.

Therefore, Theorem 3.5 ensures the existence of at least one positive solution for problem (1.2). □

Corollary 3.7. *Assume that $n \geq 3$, $p \geq 2$, and (H_f) holds. If*

$$\limsup_{x \to 0^+} \frac{f(\nu,x)}{x^{p-1}} = +\infty \text{ and } \liminf_{x \to +\infty} \frac{f(0,x)}{x^{p-1}} < 1, \tag{3.9}$$

then problem

$$-div\left(|\nabla v|^{p-2}\nabla v\right) = f(|x|, v) \text{ in } \mathcal{B},\ v = 0 \text{ on } \partial\mathcal{B}, \tag{3.10}$$

has at least one positive radial solution.

Proof. It suffices to show that problem (1.2) has at least one positive solution with $\phi(x) = |x|^{p-2}x$, $p \geq 2$. Since ϕ is a classical homeomorphism which satisfies (H_ϕ) and (3.7), the conclusion follows from Theorem 3.6. □

We show the applicability of our theory with an example involving radial solutions of p-Laplacian equations.

Example 3.8. Consider the function f given by

$$f(s,x) = f(x) = x^q + \sqrt{x},$$

with $0 \leq q < p - 1$ and $p \geq 2$, which clearly satisfies condition (H_f).

It is immediate to check that

$$\lim_{x \to 0^+} \frac{f(x)}{x^{p-1}} = \lim_{x \to 0^+} \frac{x^q + \sqrt{x}}{x^{p-1}} = +\infty \text{ and } \lim_{x \to +\infty} \frac{f(x)}{x^{p-1}} = \lim_{x \to +\infty} \frac{x^q + \sqrt{x}}{x^{p-1}} = 0.$$

Therefore, problem (3.10) associated to this function f has at least one positive radial solution, as a consequence of Corollary 3.7.

Finally, we highlight that due to the asymptotic behavior of f at zero and at infinity, this problem falls outside the scope of the results in earlier studies.[12,17]

On the other hand, it is worth to mention that in the case of a *singular* homeomorphism ϕ (i.e., with $a < +\infty$, $b = +\infty$), condition (3.5) is trivially satisfied for α large enough. Hence, in that case, we only need to ensure the existence of the number β in order to obtain positive solutions for problem (1.2).

Let us assume in the rest of this section that ϕ is singular. We present an existence result inspired by those in Bereanu et al.[8]

Theorem 3.9. *Assume that $n \geq 3$, conditions (H_ϕ) and (H_f) are fulfilled, and ϕ is a singular homeomorphim. If (3.7) and*

$$\limsup_{x\to 0^+} \frac{f(v,x)}{\phi(x)} = +\infty \tag{3.11}$$

hold, then problem (1.2) has at least one positive solution.

Proof. Arguing as in the proof of Theorem 3.6, conditions (3.7) and (3.11) imply the existence of a positive number β satisfying (3.6). Therefore, Theorem 3.5 ensures the existence of at least one positive solution for problem (1.2). □

As a consequence, we derive a simple existence result concerning positive radial solutions for Dirichlet problems involving the Minkowski mean curvature operator.

Corollary 3.10. *Assume that $n \geq 3$ and condition (H_f) holds. If*

$$\limsup_{x\to 0^+} \frac{f(v,x)}{x} = +\infty, \tag{3.12}$$

then problem

$$-div\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = f(|x|,v) \text{ in } \mathcal{B},\ v=0 \text{ on } \partial\mathcal{B}, \tag{3.13}$$

has at least one positive radial solution.

Proof. If suffices to show that problem

$$-\left(r^{n-1}\frac{u'}{\sqrt{1-u\prime^2}}\right)' = r^{n-1}f(r,u), \qquad u'(0)=u(1)=0,$$

has at least one positive solution. The conclusion follows from Theorem 3.9 with $\phi(x) = x/\sqrt{1-x^2}$, $-1 < x < 1$. Note that

$$\lim_{x\to 0^+} \frac{\phi(\tau x)}{\phi(x)} = \tau \text{ for all } \tau > 0,$$

and that for this homeomorphism ϕ, condition (3.12) implies (3.11). □

Example 3.11. Consider problem (3.13) with a function f of the form

$$f(s,x) = g(s)x^q,\ s\in[0,1],\ x\geq 0,$$

where $0 \leq q < 1$ and g is a nonincreasing positive and continuous function. Clearly, f satisfies (H_f) and the asymptotic condition (3.12), so Corollary 3.10 ensures the existence of a positive radial solution.

Example 3.12. If $0 \leq q < 1 \leq p$, $\lambda > 0$ and g is a nonincreasing nonnegative continuous function, then problem (3.13) with

$$f(s,x) = \lambda x^q + g(s)x^p,\ s\in[0,1],\ x\geq 0,$$

has at least one positive radial solution. Clearly, f satisfies (H_f) and the asymptotic condition (3.12), so the claim is again a consequence of Corollary 3.10.

4 | MULTIPLICITY RESULTS

Obviously, the localization of positive solutions provided by Theorem 3.5 makes possible to obtain multiplicity results for problem (1.2) if there exist several (perhaps infinitely many) well-ordered pairs of numbers (α, β) satisfying (3.5) and (3.6). Nevertheless, a suitable computation of the fixed point index related to the operator T allows us to establish a three-solution-type result under less stringent assumptions.

First, we present the three-solution-type fixed point theorem concerning the operator T defined in (3.1).

Lemma 4.1. *Under the assumptions of Lemma 3.4, if in addition there exists a positive number α_0 with $\alpha_0 < \beta$ and*

$$T(u) \neq \lambda u \text{ for } u \in K \text{ with } |u|_\infty = \alpha_0 \text{ and } \lambda \geq 1, \tag{4.1}$$

then T has at least three fixed points u_1, u_2, and u_3 such that

$$\beta < [u_1]_\infty,\ |u_1|_\infty < \alpha;\ \ |u_2|_\infty < \alpha_0;\ \ \alpha_0 < |u_3|_\infty < \alpha,\ [u_3]_\infty < \beta.$$

Proof. By Lemma 3.4, T has a fixed point u_1 such that

$$\beta < [u_1]_\infty,\ |u_1|_\infty < \alpha.$$

Moreover, assumption (4.1) ensures that $i\left(\tilde{T}, U_{\alpha_0}, \overline{U_{\tilde{\alpha}}}\right) = 1$ and thus the operator T has a fixed point u_2 in U_{α_0}, that is, $|u_2|_\infty < \alpha_0$. Since $\alpha_0 < \beta$, one has $\overline{U}_{\alpha_0} \subset V_\beta$, and so the properties of the fixed point index together with its computation in Lemma 3.3 imply

$$i\left(\tilde{T}, V_\beta \setminus \overline{U}_{\alpha_0}, \overline{U_{\tilde{\alpha}}}\right) = i\left(\tilde{T}, V_\beta, \overline{U_{\tilde{\alpha}}}\right) - i\left(\tilde{T}, U_{\alpha_0}, \overline{U_{\tilde{\alpha}}}\right) = 0 - 1 = -1.$$

Therefore, the existence property of the fixed point index ensures that the operator T has a third fixed point u_3 located in $V_\beta \setminus \overline{U}_{\alpha_0}$. □

As a consequence, we obtain a three-solution-type result for problem (1.2).

Theorem 4.2. *Assume that $n \geq 3$ and conditions (H_ϕ) and (H_f) are fulfilled. If there exist $\alpha_0, \alpha_1, \beta > 0$ with $\alpha_0 < \beta < AB\alpha_1$, such that*

$$\phi^{-1}\left(f\left(0, \alpha_i\right)\right) < \alpha_i,\ \ i = 0, 1,$$
$$(1-\nu)\phi^{-1}\left((\nu-\eta)\eta^{n-1} f\left(\nu, c\beta\right)\right) > \beta,$$

then problem (1.2) has at least three solutions u_1, u_2, and u_3 such that

$$\beta < [u_1]_\infty,\ |u_1|_\infty < \alpha_1;\ \ |u_2|_\infty < \alpha_0;\ \ \alpha_0 < |u_3|_\infty < \alpha_1,\ [u_3]_\infty < \beta.$$

Remark 4.1. Theorem 4.2 ensures the existence of at least two positive solutions, namely, u_1 and u_3 with the localizations above. Furthermore, if $f(\cdot, 0) \neq 0$, then u_2 is also a positive solution.

A multiplicity result can be also obtained under a suitable behavior of the nonlinearity at zero and infinity.

Corollary 4.3. *Assume that $n \geq 3$ and conditions (H_ϕ) and (H_f) are fulfilled. In addition, suppose that there exists $\beta > 0$ satisfying condition (3.6) and*

$$\liminf_{x \to 0^+} \frac{\phi^{-1}(f(0,x))}{x} < 1, \qquad \liminf_{x \to +\infty} \frac{\phi^{-1}(f(0,x))}{x} < 1. \tag{4.2}$$

Then problem (1.2) has at least two positive solutions v_1 and v_2 such that $[v_1]_\infty > \beta$ and $[v_2]_\infty < \beta$.

Proof. By the asymptotic behavior of f at zero and at infinity given by (4.2), there exist $0 < \alpha_0 < \beta$ (sufficiently small) and $\alpha_1 > \beta/(AB)$ (sufficiently large) such that

$$\phi^{-1}(f(0,\alpha_i)) < \alpha_i, \ i = 0, 1.$$

Therefore, the conclusion follows from Theorem 4.2. □

Next, we emphasize the multiplicity result in the remarkable particular cases of Dirichlet problems involving the p-Laplacian and Minkowski mean curvature operators.

Corollary 4.4. *Assume that $n \geq 3$, $p \geq 2$, and condition (H_f) holds. In addition, suppose that there exists $\beta > 0$ satisfying condition (3.6) and*

$$\liminf_{x\to 0^+} \frac{f(0,x)}{x^{p-1}} < 1, \qquad \liminf_{x\to +\infty} \frac{f(0,x)}{x^{p-1}} < 1.$$

Then problem (3.10) has at least two positive radial solutions.

Corollary 4.5. *Assume that $n \geq 3$ and condition (H_f) holds. In addition, suppose that there exists $\beta > 0$ satisfying condition (3.6) and*

$$\liminf_{x\to 0^+} \frac{f(0,x)}{x} < 1.$$

Then problem (3.13) has at least two positive radial solutions.

Proof. The conclusion follows from Corollary 4.3 with $\phi(x) = x/\sqrt{1-x^2}$, $-1 < x < 1$. Since ϕ is singular, condition

$$\liminf_{x\to +\infty} \frac{\phi^{-1}(f(0,x))}{x} < 1$$

is trivially satisfied. □

We illustrate the applicability of the previous multiplicity results with the following example.

Example 4.6. Consider problem (3.13) with a function f of the form

$$f(s,x) = f(x) = \lambda x^q, \ s \in [0,1], \ x \geq 0,$$

where $q > 1$ and $\lambda > 0$.

We shall study the existence of two positive solutions for problem (1.2) with f as above and ϕ being the singular homeomorphism given by $\phi(x) = x/\sqrt{1-x^2}$, $-1 < x < 1$.

Clearly, f satisfies (H_f) and it is immediate to check that

$$\lim_{x\to 0^+} \frac{f(x)}{x} = 0.$$

Finally, taking $\eta = 1/4$ and $\nu = 3/4$, it is a simple matter to see that $\beta = 1/16$ satisfies condition (3.6) for any λ large enough (e.g., with $\lambda > 4^{(2n+1)q+n-1/2}/(n-2)^q\sqrt{15}$).

Therefore, Corollary 4.5 guarantees that problem

$$-\mathrm{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = \lambda v^q \text{ in } \mathcal{B}, \ v = 0 \text{ on } \partial\mathcal{B},$$

has at least two positive radial solutions for any $q > 1$ provided that $\lambda > 0$ is sufficiently large.

Finally, the existence of infinitely many positive solutions for (1.2) is obtained if the nonlinearity f has an oscillating behavior at zero or at infinity.

Corollary 4.7. *Assume that $n \geq 3$ and conditions (H_ϕ) and (H_f) are fulfilled.*

(a) If

$$\limsup_{x\to 0^+}\frac{\phi^{-1}((\nu-\eta)\eta^{n-1}f(\nu,x))}{x} > \frac{1}{c(1-\nu)},\quad \liminf_{x\to 0^+}\frac{\phi^{-1}(f(0,x))}{x}<1, \tag{4.3}$$

then (1.2) has a sequence of positive solutions u_k such that $|u_k|_\infty \to 0$ as $k\to\infty$.

(b) If

$$\limsup_{x\to +\infty}\frac{\phi^{-1}((\nu-\eta)\eta^{n-1}f(\nu,x))}{x} > \frac{1}{c(1-\nu)},\quad \liminf_{x\to +\infty}\frac{\phi^{-1}(f(0,x))}{x}<1, \tag{4.4}$$

then (1.2) has a sequence of positive solutions u_k such that $|u_k|_\infty \to +\infty$ as $k\to\infty$.

Proof. Let us prove claim (a) (case (b) is analogous). By (4.3), there exist two decreasing sequences $\{\alpha_i\}_{i\in\mathbb{N}}$ and $\{\beta_i\}_{i\in\mathbb{N}}$ tending to zero such that $\alpha_{i+1}<\beta_i<AB\alpha_i$ and

$$\phi^{-1}\left(f\left(0,\alpha_i\right)\right)<\alpha_i,\ (1-\nu)\phi^{-1}\left((\nu-\eta)\eta^{n-1}f\left(\nu,c\beta_i\right)\right)>\beta_i.$$

Therefore, Theorem 3.5 can be applied to each pair (β_i,α_i) and so the conclusion is immediately obtained. □

Remark 4.2. Note that case (b) in Corollary 4.7 is not possible if ϕ is a singular homeomorphism. Indeed, in that case, ϕ^{-1} is bounded and thus

$$\lim_{x\to+\infty}\frac{\phi^{-1}((\nu-\eta)\eta^{n-1}f(\nu,x))}{x}=0,$$

which makes impossible that (4.4) holds.

To finish, we provide an example concerning the existence of infinitely many positive radial solutions for a Dirichlet problem involving the *relativistic* operator.

Example 4.8. Consider the problem (3.13) associated to the function

$$f(s,x)=f(x)=x\left[\lambda+\rho\sin\left(\gamma\ln\frac{1}{x}\right)\right],\ \text{for } x>0,\ f(0)=0,$$

where $\lambda,\rho,\gamma>0$. Observe that f is continuous, and moreover, it is nondecreasing if $\lambda\geq\rho(\gamma+1)$.

Furthermore, since $\phi^{-1}(x)=x/\sqrt{1+x^2}$ and $f(0)=0$, one has

$$\limsup_{x\to 0^+}\frac{\phi^{-1}((\nu-\eta)\eta^{n-1}f(x))}{x}=\limsup_{x\to 0^+}\frac{(\nu-\eta)\eta^{n-1}f(x)}{x}=(\nu-\eta)\eta^{n-1}(\lambda+\rho)$$

and

$$\liminf_{x\to 0^+}\frac{\phi^{-1}(f(x))}{x}=\liminf_{x\to 0^+}\frac{f(x)}{x}=\lambda-\rho.$$

Then the asymptotic condition (4.3) holds if

$$(\nu-\eta)\eta^{n-1}(\lambda+\rho)>\frac{1}{c(1-\nu)}\ \text{and}\ \lambda-\rho<1.$$

Therefore, Corollary 4.7 ensures that the corresponding problem (3.13) associated to this nonlinearity f has a sequence of positive radial solutions u_k such that $|u_k|_\infty\to 0$ as $k\to\infty$ provided that

$$\lambda\geq\rho(\gamma+1),\ (\nu-\eta)\eta^{n-1}(\lambda+\rho)>\frac{1}{c(1-\nu)}\ \text{and}\ \lambda-\rho<1.$$

In particular, taking $\eta=1/3$ and $\nu=2/3$, the previous inequalities reduce to

$$\lambda\geq\rho(\gamma+1),\ \lambda+\rho>\frac{3^{3n}}{n-2}\ \text{and}\ \lambda-\rho<1.$$

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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