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A modified Lyapunov method and its applications to ODE

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Here, we propose a method to obtain local analytic approximate solutions of ordinary differential equations with variable coefficients, or even some nonlinear equations, inspired in the Lyapunov method, where instead of polynomial approximations, we use truncated Fourier series with variable coefficients as approximate solutions. In the case of equations admitting periodic solutions, an averaging over the coefficients gives global solutions. We show that, under some restrictive condition, the method is equivalent to the Picard-Lindelöf method. After some numerical experiments showing the efficiency of the method, we apply it to equations of interest in physics, in which we show that our method possesses an excellent precision even with low iterations.

MSC CLASSIFICATION
34A45, 65L99

1 | INTRODUCTION

The motivation of the present article is a contribution to the methods for approximate solutions of ordinary differential equations (ODE) either nonlinear or linear with variable coefficients. These methods have been developed in order to solve different kinds of problems that arise in physics and that need of these kinds of ODE. Solutions for the vast majority of ODE, out of trivial cases studied in textbooks, are unknown, and even in the case that an equation has known solutions, there are either incomplete or of a complexity that makes them inappropriate for a first analysis of the problem given. For the almost ubiquitous second-order linear equations with variable coefficients, not always a solution a la Frobenius is possible, a trouble experienced sometimes by theoretical physicists.

It is interesting to mention some methods to obtain approximate solutions to non-trivial ODE, although for obvious reasons, we may give only a very limited number of references on a field in which the number of publications is really enormous.[1–9] Our point of departure is the search for analytic approximate solutions for given initial values, boundary conditions, or solving the Sturm-Liouville problem for second-order ODE of the form

$$\ddot{w}(t) + p(t)\dot{w}(t) + q(t)w(t) = 0. \tag{1}$$

Here, we propose a modification of the Lyapunov method to obtain approximate solutions of equations of the type (1), which improves the standard Lyapunov method and which is also based on iterations. We introduce the method, compare it with the Picard-Lindelöf method, and show the advantages that it may have with respect to the others in terms of precision, CPU times, etc. The method is also applicable to nonlinear equations.

We support our arguments with some numerical experiments. Applications to second-order differential equations of interest in physics have been discussed. These equations are either linear with variable coefficients or nonlinear.

The modified Lyapunov method we are introducing in the present article has some important features in common with the standard Lyapunov method. Both are intended to find local approximate analytic solutions. Both are polynomial approximations. While in the standard Lyapunov method, they are given by powers, in our approach, they are truncated Fourier series with variable coefficients. This poses a clear advantage with respect to Standard Lyapunov, since after the numerical recipe of averaging over the coefficients, we obtain a global approximation valid for periodic solutions of equations that admit such solutions. As mentioned, the advantage of reducing computational times makes these techniques widely accessible using a software such as Mathematica.

This paper has the following organization: In Section 2, we introduced our modified Lyapunov method and discuss convergence properties and the construction of approximate periodic solutions after non-periodic ones. In Section 3, we compare our method to the well-established Picard-Lindelöf method and provide of some simple numerical experiments to show the advantage of our method in terms of smaller CPU times. Section 4 is devoted to the applications of the method on differential equations of interest in physics. This presentation is continued in Section 5. We pay particular attention to the precision of the method. It closes with some concluding remarks.

2 | A MODIFIED LYAPUNOV METHOD

Let us go back to Equation (1), where the functional coefficients $p(t)$ and $q(t)$ are defined on an interval $[0, T]$, on where both are continuous and, in addition, $p(t)$ admits a continuous first derivative. Let us perform the following change of variables to introduce a new indeterminate $w(t)$ as

$$w(t) = x(t)\exp\left\{-\frac{1}{2}\int_0^t p(s)\,ds\right\}. \tag{2}$$

Clearly, $w(t)$ satisfies Equation (1) if and only if $x(t)$ satisfies

$$\ddot{x}(t) + a(t)x(t) = 0, \tag{3}$$

with

$$a(t) = q(t) - \frac{1}{4}p^2(t) - \frac{1}{2}\dot{p}(t). \tag{4}$$

Note that the function $a(t)$ is continuous on the interval $[0, T]$.

The standard Lyapunov method solves Equation (3) by first considering the one parameter family $\ddot{x}(t) = \lambda a(t)x(t)$, and, after a procedure similar to what follows, one singles out the value $\lambda = -1$. Our modified Lyapunov method instead considers the following one parameter family of equations given by

$$\ddot{x}(t) + \omega^2 x(t) + \lambda b(t)x(t), \quad \text{with} \quad b(t) := a(t) - \omega^2. \tag{5}$$

Note that we recover the original (3) equation by choosing $\lambda = 1$. However, we shall yield, in principle, certain freedom in the choice of ω. In fact, if the approximation replaces to a non-periodic solution, the choice is $\omega = 1$, while if it is periodic and the period is given, then ω is well determined. On the other hand, if the solution is periodic and the period is unknown, we should determine ω through the periodicity condition.

In the original Lyapunov method,[2,3] one considers the one-parameter family $\ddot{x}(t) = \lambda a(t)x(t)$ and, then, chooses $\lambda = -1$. Here, we are using an extension of this method in (5).

Let us consider the particular solution $\phi_\lambda(t)$ of (5) with the initial conditions given by $\phi_\lambda(0)$ and $\dot{\phi}_\lambda(0)$. In order to obtain such a solution, let us consider the following span into series:

$$\phi_\lambda(t) = \sum_{k=0}^{\infty} x_k(t)\,\lambda^k. \tag{6}$$

Next, we replace (6) into (5) so as to obtain

$$\sum_{k=0}^{\infty} \ddot{x}_k(t)\,\lambda^k + \omega^2 \sum_{k=0}^{\infty} x_k(t)\,\lambda^k + \sum_{k=0}^{\infty} b(t)x_k(t)\,\lambda^{k+1} = 0. \tag{7}$$

Let us keep arbitrary the value of the parameter λ, so that we may identify the coefficient of λ^k for $k = 0, 1, 2, \dots$ on both sides of (7). The result is

$$\ddot{x}_0(t) + \omega^2 x_0(t) = 0 \tag{8}$$

for $k = 0$ and

$$\ddot{x}_k(t) + \omega^2 x_k(t) + b(t)x_{k-1}(t) = 0 \tag{9}$$

for $k = 1, 2, \dots$.

Next, let us impose initial conditions such as $x_0(0) = A$, $\dot{x}_0(0) = B$, $x_k(0) = 0$ and $\dot{x}_k(0) = 0$, $k = 1, 2, \dots, n$. We obtain

$$x_0(t) = A\cos\omega t + \frac{B}{\omega}\sin\omega t. \tag{10}$$

Also, and after the stated initial condition and using the variation of parameters method, for any $k = 1, 2, \dots$, we have that

$$x_k(t) = \frac{1}{\omega}\int_0^t b(s)x_{k-1}(s)\sin\omega(s-t)\,ds. \tag{11}$$

Consequently, we have obtained a recurrence relation to obtain the form of the series (6). The series converges if it converges absolutely. Continuous functions on the compact interval $[0, T]$ are bounded so is $b(t)$. Let us call K to an upper bound of $|b(t)z_0(t)|$ on $[0, T]$. Hence, for $k = 1, 2, \dots$, we have

$$|x_k(t)| \le \frac{1}{k!}\left(\frac{K}{\omega}t\right)^k. \tag{12}$$

Then, after (6) and (12), we have

$$\sum_{k=0}^{\infty} |x_k(t)|\,|\lambda|^k \le K_0 + \sum_{k=1}^{\infty} \frac{|\lambda|^k}{k!}\left(\frac{K}{\omega}t\right)^k \le H_0 + \exp\left\{\frac{|\lambda|K t}{\omega}\right\} \le H_0 + \exp\left\{\frac{\Delta K \beta}{\omega}\right\}, \tag{13}$$

where $H_0 = K_0 - 1$ and Δ is an upper bound of the possible values of $|\lambda|$. Recall that $t \in [0, T]$. Thus, the series (6) converges absolutely and uniformly on the interval $[0, T]$. The choice $\lambda = 1$ gives the desired solution to Equation (3).

A simple analysis of (11) shows that an approximation of the solution of (3):

$$\psi(t) = \phi_1(t) = \sum_{k=0}^{\infty} x_k(t), \tag{14}$$

up to order n, has the following form

$$\psi_n(t) = \sum_{k=0}^{m} \{p_k(t)\cos(k\omega t) + q_k(t)\sin(k\omega t)\}, \tag{15}$$

where $p_k(t)$ and $q_k(t)$ are polynomials and $m > n$. We should point out that the approximation given by (15) is local.

We need to determine the value of n in order to control the error produced by the choice of the approximation (15) with respect to the exact solution. We define this error as

$$e_n := \int_0^T (\ddot{\psi}_n(t) + a(t)\psi_n(t))^2 \, dt. \tag{16}$$

Then, we settle a desirable maximal error, $\delta > 0$, and choose n such that $e_n < \delta$.

Nevertheless, the approximation given by (15) is not periodic. We propose a global periodic approximation after (15) constructed as follows: Define

$$\overline{p}_k := \frac{1}{P}\int_0^P p_k(t)\, dt, \quad \overline{q}_k := \frac{1}{P}\int_0^P q_k(t)\, dt, \tag{17}$$

where P is the period of the searched periodic solution Then, we propose as the approximate periodic solution as follows:

$$\overline{\psi}_n(t) := \sum_{k=1}^{m} \{\overline{p}_k \cos(k\omega t) + \overline{q}_k \sin(k\omega t)\}. \tag{18}$$

Another possibility is to use $p_k(0)$ and $q_k(0)$ instead of $\overline{p}_k$ and $\overline{q}_k$, although this is just a conjecture that may be supported just by numerical precision.

Henceforth, we shall call this procedure the *modified Lyapunov method*.

2.1 | First order systems

The above method may be extended to first order systems of the form

$$\dot{x}(t) = f(t, x, y), \quad \dot{y}(t) = g(t, x, y), \tag{19}$$

where f and g are real and polynomials on the variables x and y and continuous with respect all the three variables. In this case, the system playing the role of Equation (5) has the following form:

$$\dot{x}(t) = y + \lambda(f(t, x, y) - y), \quad \dot{y}(t) = -\omega^2 x + \lambda(g(t, x, y) + \omega^2 x). \tag{20}$$

By approximation of order n, we mean the choice

$$x_n(t) := \sum_{k=1}^{n} \lambda^k u_k(t), \quad y_n(t) := \sum_{k=0}^{n} \lambda^k v_k(t). \tag{21}$$

From here, we repeat the above procedure, we determine the value of ω as did after (5), and then, we obtain the approximate periodic solutions.

3 | EQUIVALENCE BETWEEN THE MODIFIED LYUAPUNOV AND THE PICARD-LINDELÖF METHODS

Along the present short section, we show that our proposed modified Lyapunov is equivalent to the Picard-Lindelöf method *provided that we made a particular choice of the seed solution*. The Picard-Lindelöf method finds approximate solutions of equations of the form

$$\ddot{z}(t) + \omega^2 z(t) = f(t, z, \dot{z}), \qquad z(0) = A, \ \dot{z}(0) = B, \tag{22}$$

by iteration.[10] In Ramos,[10] it is shown that if $f(t,z(t),\dot{z}(t))$ is Lipschitz continuous on $t \in [0,T]$, then a fixed point iterative approximation method to solve (22) uses the following relation, valid for all $k = 1, 2, \ldots$:

$$\ddot{z}_{k+1}(t) + \omega^2 z_{k+1}(t) = f(t, z_k, \dot{z}_k), \ \ z_k(0) = A, \ \dot{z}_k(0) = B, \tag{23}$$

so that (22) admits the following iterative approximate solutions:

$$z_{k+1}(t) = z_0(t) - \frac{1}{\omega}\int_0^t f(s, z_k(s), \dot{z}_k(s)) \sin(\omega(s-t))\, ds, \tag{24}$$

where $z_k(t)$ converges uniformly to the exact solution on a compact interval $[0,T]$. The important point here is that *this method*[10] *relies on the choice of the initial seed $z_0(t)$ and that this choice is somehow arbitrary*. After n iterations, $z_n(t)$ is the approximation to the order n of the solution of (23).

Observe that (5) with $\lambda = 1$ is a particular case of (22) with $f(t,z,\dot{z}) \equiv -b(t)z(t)$. Therefore, (24) becomes

$$z_{k+1}(t) = z_0(t) + \frac{1}{\omega}\int_0^t b(s) z_k(s) \sin(\omega(s-t))\, ds. \tag{25}$$

It was proven in Ramos[10] that whenever $b(t)$ be continuous on the integration interval $[0,T]$, the sequence of approximate solutions $\{z_n(t)\}$ converges uniformly to the exact solution on $[0,T]$.

In order to determine the sequence $z_k(t)$, let us first construct a sequence of functions $\{x_k(t)\}$ in the following form: The first term of the sequence is $x_0(t)$ defined in (10), and then, write $z_0(t) \equiv x_0(t)$, so that *we choose as seed the function* (10). Then,

$$x_1(t) := \frac{1}{\omega}\int_0^t b(s) z_0(s) \sin(\omega(s-t))\, ds. \tag{26}$$

Thus, $z_1(t) = x_0(t) + x_1(t)$. Note that $z_0(t) \equiv x_0(t)$. Next, write

$$x_2(t) := \frac{1}{\omega}\int_0^t b(s) x_1(s) \sin(\omega(s-t))\, ds, \tag{27}$$

so that

$$\begin{aligned} z_2(t) &= x_0(t) + \frac{1}{\omega}\int_0^t b(s) z_1(s) \sin(\omega(s-t))\, ds, \\ &= x_0(t) + \frac{1}{\omega}\int_0^t b(s)\,[x_0(s) + x_1(s)] \sin(\omega(s-t))\, ds = x_0(t) + x_1(t) + x_2(t). \end{aligned} \tag{28}$$

We may proceed by induction and, hence,

$$z_{k+1}(t) = x_0(t) + x_1(t) + \cdots + x_k(t), \tag{29}$$

with

$$x_k(t) := \frac{1}{\omega}\int_0^t b(s) x_{k-1}(s) \sin(\omega(s-t))\, ds. \tag{30}$$

The exact solution of (22) with the given initial conditions is just[10]

$$z(t) = \lim_{k\to\infty} z_k(t) = \sum_{k=0}^{\infty} x_k(t). \tag{31}$$

This limit does exist uniformly on compact intervals.[10]

Now, let us come back to Section 2, where we shall use $\lambda = 1$ and the initial conditions $x_k(0) = A$ and $\dot{x}_k(0) = B$, for $k = 0, 1, 2, \ldots$. The seed, $x_0(t)$, is given by (10). Then, let us go to (6) with $\lambda = 1$ and (11), so as to conclude that the exact solution is

$$\psi(t) \equiv \phi_1(t) = \sum_{k=0}^{\infty} x_k(t) = x_0(t) + \frac{1}{\omega}\int_0^t b(s)\sin\omega(s-t)\sum_{k=1}^{\infty} x_k(s)\,ds, \tag{32}$$

with $x_0(t)$ as in (10), so that

$$\psi_n(t) = x_0(t) + \frac{1}{\omega}\int_0^t b(s)\sin\omega(s-t)\sum_{k=1}^{n-1} x_k(s)\,ds = x_0(t) + x_1(t) + \cdots + x_n(t), \tag{33}$$

due to (11). A comparison between (33) with (30) and (31) shows that $z_n(t)$ and $\psi_n(t)$ provide of the same approximation for the solution of (5) with $\lambda = 1$ and the same value of ω in both approximations.

Nevertheless, this coincidence is a consequence of a particular choice of the seed solution made when using the Picard-Lindelöf method.[10] This seed solution must coincide with the zero-order approximation (10) for the modified Lyapunov method, so that both methods be equivalent. Nevertheless, Picard-Lindelöf shows a bigger level of complexity than the modified Lyapunov method, and this fact may imply an advantage in favor of the latter. To begin with, the arbitrary choice of the seed solution is a factor that has a great influence on the speed of the convergence and, hence, in the precision of the nth approximation. Another origin of the bigger complexity of Picard-Lindelöf as compared with modified Lyapunov concerns on the procedure for the construction of each approximation in the first case, which relies on the generation of a sequence of partial sums, for which its convergence to a infinite series gives the exact solution. The advantage of the modified Lyapunov with respect to the Picard-Lindelöf may be shown by numerical experiments as we discuss next.

3.1 | Two numerical experiments

Numerical experiments show that the modified Lyapunov method requires shorter CPU times that the Picard-Lindelöf method. We give here two significative and simple examples. The former is just a simple linear oscillator such as

$$\ddot{x}(t) + 4x(t) = 0, \quad x(0) = \dot{x}(0) = 1. \tag{34}$$

Let us write (34) in the form (5) with $\omega = 1$. We have

$$\ddot{x}(t) + x(t) + 3\mu x(t) = 0. \tag{35}$$

Let us use the Picard-Lindelöf method, where now (23) has the following form:

$$\ddot{x}_k(t) + x_k(t) + 3x_{k-1}(t) = 0, \quad k = 1, 2, \ldots, n, \tag{36}$$

and initial conditions $x_k(0) = \dot{x}_k(0) = 1$. Now, it is possible the choice of the seed solution under the condition that the same iteration produce the same precision. This can be done with $x_0(t) = \sin t + \cos t$. We observe the following facts:

a. In order to obtain a reasonable approximation to the exact solution, which is $x_{\text{exact}}(t) = cos2t + \frac{1}{2}\sin 2t$, on the interval $[0, 2\pi]$, we need a minimum of 15 iterations, so that $n \geq 15$. Note that although the exact solution has a period equal to π, we have used the interval $[0, 2\pi]$, in order to evaluate the error. This helps to understand the quality of the obtained result, since although the solutions are local, they have nevertheless an excellent behavior over two periods. Then, the norm on $L^2[0, 2\pi]$ of the difference between the approximate solution and the exact solution is of the order of 10^{-6}. Note that both integration methods are local, i.e., valid on compact intervals.
b. We compare the variation of CPU times with the order n of approximation on both methods in Figure 1. We observe that the modified Lyapunov is more efficient that the Picard-Lindelöf method.

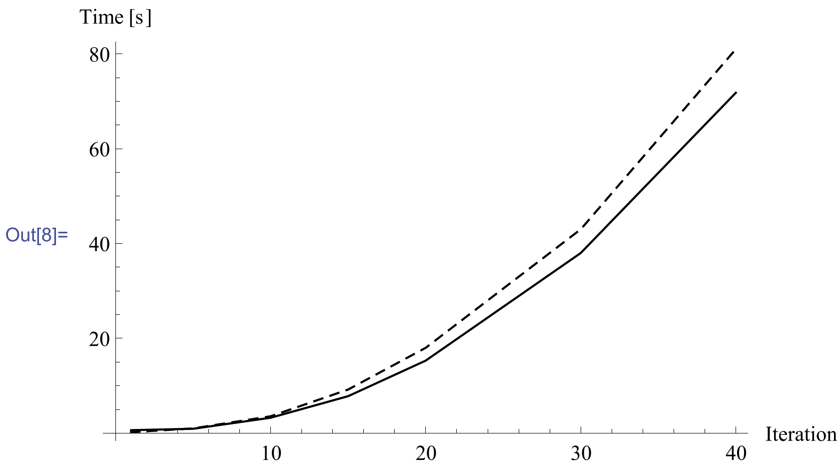

FIGURE 1 Comparison of the CPU times of the Poincaré-Lindelöf method (dashed curve) with the CPU times of the modified Lyapunov (continuous curve) for the linear oscillator (34), with the same number of iterations

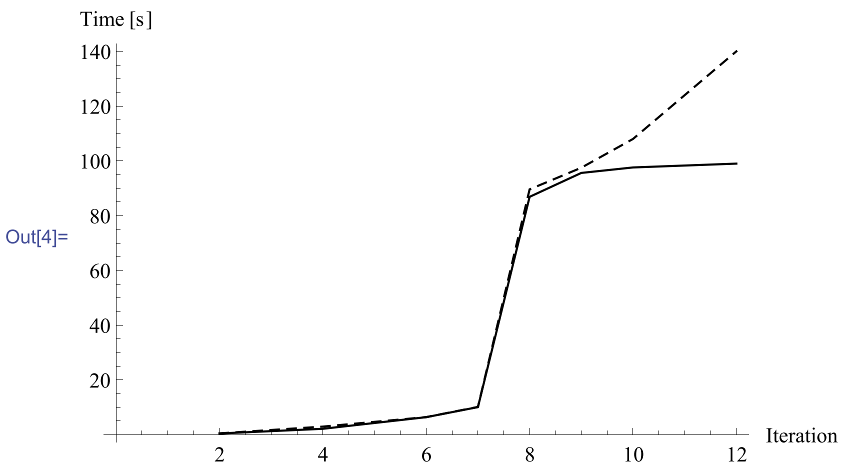

FIGURE 2 Comparison of the CPU times of the Poincaré-Lindelöf method (dashed curve) with the CPU times of the modified Lyapunov (continuous curve) for the Mathieu equation, with the same number of iterations

In the second example, we use the Mathieu equation; see below in (39). We choose as values of the parameters $a = 1$ and $q = 0.05$ (these values have been chosen by simplicity in the procedure) and, again, the initial values $x(0) = \dot{x}(0) = 1$. This equation may be rewritten as

$$\ddot{x}(t) + x(t) + \lambda(a - 1 - 2q\cos 2t)x(t) = 0. \tag{37}$$

The implementation of Poincaré-Lindelöf yields to

$$\ddot{x}_k(t) + x_k(t) + (a - 1 - 2\cos 2t)x_{k-1}(t) = 0, \qquad k = 0, 1, 2, \ldots, n, \tag{38}$$

with initial conditions $x_k(0) = \dot{x}_k(0) = 1$. We use $x_0(t) = \sin t + \cos t$ as initial seed. We compare the CPU times in Figure 2, so as to obtain similar conclusions than in the precedent example. Observe that modified Lyapunov is particularly advantageous for $n > 10$.

4 | APPLICATIONS OF THE METHOD ON ODE OF INTEREST IN PHYSICS

Along the present section, we test this extended Lyapunov method to some well-known second-order differential equations of interest in science. These are the Mathieu, which is a first order Hill equation, the Airy, and the Bratu equations.

4.1 | The Mathieu equation

The Mathieu equation has the following form[2,11]:

$$\ddot{x}(t) + (a - 2q\cos 2t)x(t) = 0, \tag{39}$$

with $a, q > 0$. The general solution is a linear combination of the so called Mathieu special functions, $C(a, q, t)$ and $S(a, q, t)$. These solutions are not, in general periodic, although periodic solutions may be found for some values of the parameters a and q. A method to approximate periodic solutions has been proposed in Gadella et al.[12]

In order to check the extended Lyapunov method, we shall use an exact solution of (39). In this case, $a = 1$, and $q = 0.05$ with initial conditions given by $x(0) = 1$ and $\dot{x}(0) = 0$. The resulting solution is

$$x(t) = (0.763507 - 0.641316\,i)C(1, 0.05, t). \tag{40}$$

This solution is not periodic.

Let us call $w(t)$ to the function, obtained through the above method, that approximates the solution $x(t)$ on the interval $[0, T]$ (usually $T = 2\pi$), $x(t)$ being either (40) or another periodic exact solution. On this interval, the error produced by the use of $w(t)$ is defined by

$$E_n := \int_0^{\beta} (x(t) - w(t))^2\, dt. \tag{41}$$

We have written E_n as we have defined on the above procedure the approximate solution by an iterative process. Here, n would be the order of the iteration necessary to achieve the approximate solution $w(t)$. On (16), we have defined the error e_n that may be applied either to our modified Lyapunov (ML) or the standard Lyapunov method (L). We have made some numerical experiments using both taking (40) as reference solution. We have used $n = 2, 4, 6$ iterations with both methods. The results are displayed in Table 1.

The information we obtain from Table 1 is clear: The approximation of non-periodic solutions on the interval $[0, 2\pi]$ by the modified Lyapunov method gives a much higher precision than the precision obtained by the traditional Lyapunov method. We have performed some other numerical experiments leading to the same result. It is already outstanding the precision obtained with the use of the second iteration.

Needless to say that the method allows finding the explicit form of the approximate solutions. Listing these solutions makes no sense as they are easily obtainable with the package Mathematica. Just to show the explicit form of one of them, the approximation to (39) obtained after the second iteration is

$$\begin{aligned} x_2(t) = {} & (1.00639 + 0.000315\,t^2)\cos 2t - 0.00640625\cos 3t \\ & + 0.0000130208\cos 5t + 0.0246875\,t\sin t - 0.00015625\,t\sin 3t. \end{aligned} \tag{42}$$

Note that this is an approximation of the type (15) with $p_2(t)$ and $q_2(t)$ polynomial, so that (42) is not periodic. Thus, we estimate the error E_n. The estimation of the error e_n comes after the use of the averaging (17), which gives periodic approximations to the solution on the interval $[0, 2\pi]$.

TABLE 1 Error estimations produced when approaching solution (39) on the interval $[0, 2\pi]$ either using the modified Lyapunov method (ML) or the traditional Lyapunov method (L)

n	e_n ML	E_n ML	e_n L	E_n L
2	$6.86\ 10^{-7}$	$1.05\,10^{-7}$	$2.72\,10^{3}$	$1.15\ 10^{3}$
4	$1.40\ 10^{-12}$	$1.21\ 10^{-14}$	$1.17\ 10^{3}$	$1.26\ 10^{2}$
6	$1.47\ 10^{-20}$	$7.99\ 10^{-22}$	$1.34\ 10$	$4.81\ 10^{-1}$

Note: We use two different definitions of errors, e_n as in (16) and E_n as in (36). In our numerical tests, $e_n < E_n$ in general. The furthermost left column gives the order of iteration.

4.1.1 | Characteristic values

As is well known, not all solutions of the Mathieu equation are periodic. Periodic solutions are labeled by some values of a for given q. In particular, for $|q| < 1$ and solutions with period π, we may single out the three first characteristic values, which are[11]

$$a_1 = 1 - q - \frac{1}{8}q^2 + \frac{1}{64}q^3 - \frac{1}{1536}q^4 + \dots ,$$
$$a_2 = 4 + \frac{5}{12}q^2 - \frac{763}{13824}q^4 + \frac{10002401}{79626240}q^6 + \dots , \tag{43}$$
$$a_3 = 9 + \frac{1}{16}q^2 - \frac{1}{64}q^3 + \frac{13}{20480}q^4 + \dots .$$

To apply the modified Lyapunov method, let us go to (15) with $\omega = 2$, which takes the form

$$x_n(t) = \sum_{k=0}^{m(n)} \{p_k(t)\cos 2kt + q_k(t)\sin 2kt\}, \tag{44}$$

and write the Mathieu equation (39) as

$$\ddot{x}(t) + 4x(t) + \lambda(a - 4 - 2q\cos 2t)x(t) = 0. \tag{45}$$

In order to single out a particular solution, we have to choose some initial conditions such as $x(0) = 0$ and $\dot{x}(0) = 1$. Note that in (45), we have to determine somehow the characteristic value a, which is not arbitrary and requires some other conditions. In fact, the values of a are determined as follows: Take a given value of n. The following equation

$$x_n(\pi) = 0, \tag{46}$$

where $x_n(\pi)$ is given in (44), is a polynomial equation on the variable a having $m(n)$ solutions.

Take the real roots on a of (46). Then, select those real roots fulfilling the condition $\dot{x}(\pi) \approx 1$. This gives a list of approximate characteristic values for the Mathieu equation. We repeat the procedure with higher values of n until a change in the value of n does not produce any substantial change on the characteristic values. The percent relative error, e_a, of the values of a is defined as the modulus of

$$e_a := \frac{a_{ML} - a_{\text{Exact}}}{a_{\text{Exact}}}, \tag{47}$$

where a_{Exact} corresponds to an exact value of the characteristic value under consideration and a_{ML} is the value obtained by the use of modified Lyapunov. In Table 2, we give the percent errors of the three first characteristic values obtained with modified Lyapunov as compared to the exact values as appeared in (43). Here, we have used $q = 0.1$.

Once we have established the approximate characteristic values, we may determine the approximate solutions. On Table 3, we compare the errors e_n and E_n produced as the consequence of applying either Modified Lyapunov of order 10 or Runge-Kutta on the same interval, once we have fixed certain values of a_1, a_2, and a_3. We recall that for each characteristic value, we have one solution and that these errors correspond to the solution provided by its characteristic value.

TABLE 2 Percent relative errors e_a in the evaluation of the three first characteristic values of the Mathieu equation

$n = 10$	a_1	a_2	a_3
Exact	0.89876	4.00416	9.00061
ML	0.89876	3.99917	9.00182
e_a	0.0	0.12	0.013

Note: In the second row, "Exact" denotes the values given by (43) with $q = 0.1$. In the third row "ML" mean the values obtained by modified Lyapunov. The values of e_a are listed on the last row.

The approximate explicit solutions have the form (15), where $p_k(t)$ and $q_k(t)$ are polynomials on the variable t. In order to obtain a periodic solution on the whole real line $\mathbb{R}$, we should take the mean of these polynomials as defined in (17). Just an example, take $n = 4$ and $a_2 = 3.99917$, which yields to the following approximate periodic solution (after having taken the averages on the values of the polynomials):

$$x_4(t) = 0.508434 \sin 2t - 4.23676\,10^{-3} \sin 4t + 1.3297 \sin 6t, \tag{48}$$

with the errors $e_n = 5.08\,10^{-16}$ and $E_n = 4.41\,10^{-15}$.

The exact solution with $q = 0.1$ and $a = 4.00416$ is here given by

$$x(t) = (0.35921 + 0.20801i)\,S(4.00416, 0.1, t), \tag{49}$$

where $S(a, q, t)$ is the second Mathieu special function.[11] If we expand (49) into Fourier series, we obtain

$$\begin{aligned} x(t) = &\,1.22378\,10^{-5} + 2.4938\,10^{-4} \cos 2t - 2.00187\,10^{6} \cos 4t + 0.508307 \sin 2t \\ &- 4.34328\,10^{-3} \sin 4t - 4.6836 \sin 6t. \end{aligned} \tag{50}$$

This approximation gives an error, $e = 9.2\,10^{-6}$. Compare to the error given on Table 3, we show that the error produced by modified Lyapunov is $5.28\,10^{-19}$, which is much smaller.

4.2 | The airy equation

The Airy equation has the following form:

$$\ddot{x}(t) + tx(t) = 0. \tag{51}$$

Take the initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$. Now, we have an exact solution that is known. This is

$$x_{\text{exact}}(t) = \frac{1}{2}\Gamma(2/3)(3^{2/3}A_i[(-1)^{1/3}t] + 3^{1/6}B_i[(-1)^{1/3}t]), \tag{52}$$

where $\Gamma(z)$ is the gamma function and $A_i(x)$ and $B_i(x)$ are the Airy functions.[11] This solution is obviously not oscillatory. In order to implement Modified Lyapunov, we first rewrite (49) as

$$\ddot{x}(t) + x(t) + \lambda(t-1)x(t) = 0. \tag{53}$$

Then, we proceed to its approximate integration on the interval $[0, 2]$ using both modified Lyapunov and Lyapunov. While with modified Lyapunov one obtains a combination of harmonics with fundamental frequency equal to one, with Lyapunov, we obtain the truncated Taylor series of $x_{\text{exact}}(t)$ on a neighborhood of $t = 0$. On Table 4, we display the errors e_n and E_n, with respect to the exact solution, produced by both approximate methods for some low values of n.

TABLE 3 Errors e_n and E_n produced when choosing the solution for the characteristic values a_1, a_2, and a_3, both for the Modified Lyapunov of order $n = 10$ and Runge-Kutta

$n=10$	a_1	a_2	a_3
e_n ML	$2.67\ 10^{-10}$	$5.28\ 10^{-19}$	$9.18\ 10^{-6}$
e_n RK	$2.04\ 10^{-8}$	$6.81\ 10^{-10}$	$1.01\ 10^{-9}$
E_n ML	$1.00\ 10^{-13}$	$5.02\ 10^{-26}$	$3.37\ 10^{-9}$
E_n RK	$8.34\ 10^{-15}$	$4.41\ 10^{-15}$	$1.02\ 10^{-15}$

TABLE 4 Errors e_n and E_n obtained when we take the approximated solutions either with modified Lyapunov or Lyapunov, with respect to the exact solution, for the values $n = 2, 4, 6$

n	e_n ML	E_n ML	e_n L	E_n L
2	$9.14\ 10^{-5}$	$2.90\ 10^{-17}$	$6.74\ 10^{-2}$	$1.48\ 10^{-4}$
4	$1.41\ 10^{-11}$	$4.06\ 10^{-15}$	$1.70\ 10^{-6}$	$5.12\ 10^{-10}$
6	$4.63\ 10^{-20}$	$3.20\ 10^{-24}$	$1.17\ 10^{-12}$	$9.34\ 10^{-17}$

Observe that we have gain in precision using modified Lyapunov with respect Lyapunov. Nevertheless, the solutions are local, so that if we enlarge the domain $[0, 2]$, we need higher values of n so as to obtain similar precision.

4.3 | Bratu equation

The modified Lyapunov method may also be used to determine approximate solutions if we replace the initial conditions by boundary conditions.[8] Take, for instance, the Bratu equation:

$$\ddot{x}(t) = -\alpha\, e^{x(t)}, \quad \alpha > 0. \tag{54}$$

In this example, we integrate (54) on the interval $[0, 1]$ with the boundary conditions $x(0) = 0$ and $x(1) = 0$. Under these conditions, the exact solution of (54) is known and is

$$x_{\text{exact}}(t) = -2 \log \left(\frac{\cosh(0.5(t-0.5)\theta)}{\cosh(0.25\,\theta)} \right), \tag{55}$$

where θ satisfies the following transcendental equation:[8]

$$\theta = \sqrt{2\alpha} \cosh(0.25\,\theta). \tag{56}$$

Equation (56) has either zero, one, or two solutions depending if $\alpha > \alpha_c$, $\alpha = \alpha_c$ or $\alpha < \alpha_c$, respectively, where the critical value α_c must satisfy the following relation:[8]

$$4 = \sqrt{2\alpha_c} \sinh(0.25\,\theta). \tag{57}$$

By using an expansion of the exponential in (54) on a neighborhood of the origin, we have the following approximation for (54):

$$\ddot{x}(t) + \alpha \left(1 + x(t) + \frac{1}{2} x^2(t) \right) = 0. \tag{58}$$

With the goal of testing our method, we make a choice on the parameters, say $\alpha = 1$. This gives $\theta = 1.57716459905$. In Hermann and Saravi,[8] we may found a calculation to obtain approximate analytic solutions using the Variational Iteration method, developed in He.[6,7] This method also gives approximate solutions by iteration. For instance, if for the solution on the interval $[0, 1]$, we impose $x(0) = 0$; we found for the second iteration

$$\begin{aligned} H_2(t) = kt - \frac{t^2}{2!} - \frac{t^3}{3!} - \frac{(k^2-1)t^4}{4!} + \frac{4kt^5}{5!} + \frac{(5k^2-3)t^6}{6!} + \frac{5k(k^2-2)t^7}{7!} \\ - \frac{25k^2t^8}{8!} - \frac{35k^3 2t^9}{9!} - \frac{35k^4t^{10}}{10!}. \end{aligned} \tag{59}$$

Values of k can be obtained using the second boundary condition $x(1) = 1$. This gives two conjugate complex solutions and two real solutions. One of these real solutions produces an enormous error on the solution. The other is $k = 0.6231399$, which we consider the only admissible.

Let us solve (58) by modified Lyapunov. Let us write

$$\ddot{x}(t) + x(t) + \lambda \left(1 + \frac{1}{2} x^2(t) \right) = 0. \tag{60}$$

We have used modified Lyapunov with initial conditions, and now, we are interested in extended the method so as to use boundary conditions instead. Then, we need a slight change of strategy. Assume, for instance, that we want to make an approximate integration at second order, $n = 2$, knowing that the boundary conditions are, say, $x(0) = 0$ and $x(1) = 0$. Then, we begin with fixing the initial conditions $x(0) = 0$ and $\dot{x}(0) = u$, where u is unknown. Thus, the solution, $x_u(t)$, that provides the chosen initial conditions depends on u. To find the solution with the given boundary conditions, we have to fix u as the real root of $x_u(1) = 0$. Thus, we have the desired approximation with the given boundary conditions.

TABLE 5 Values of the parameters e and E for second order of iteration and different methods of approximation of solutions: Modified Lyapunov (ML), Runge-Kutta (RK), and Variational Iteration (VIM)

$n=2$	ML	RK	VIM
e	$1.43\ 10^{-3}$	$7.55\ 10^{-4}$	$4.99\ 10^{-2}$
E	$3.16\ 10^{-6}$	$5.95\ 10^{-6}$	$4.03\ 10^{-4}$

Assume that by whatever method, we have obtained an approximate solution, say $z(t)$. We want to compare the efficiency of each method in relation to two parameters. One is

$$e := \int_0^1 \left(\ddot{z}(t) + 1 + z(t) + \frac{1}{2} z^2(t)\right)^2 dt. \tag{61}$$

The other one compares the approximate solution to the exact solution $x_{\text{exact}}(t)$ in (55):

$$E := \int_0^1 (z(t) - x_{\text{exact}}(t))^2 \, dt. \tag{62}$$

Results are gathered on Table 5, where we compare the values of these two parameters obtained either by modified Lyapunov (ML), Runge-Kutta (RK), and Variational Iteration.

Finally, the approximate solution for $n = 2$ by modified Lyapunov is given by

$$\begin{aligned} z_{\text{ML}}(t) = & -1.07413 + (1.09885 - 0.289098t)\cos t - 0.0247115\cos 2t \\ & + 0.636714\sin t + 0.0997301\sin 2t - 0.000841047\sin 3t. \end{aligned} \tag{63}$$

We conclude this section at this point.

5 | NON-LINEAR EQUATIONS

It is simple to see how modified Lyapunov works if the first-order system (19) is written in the form of a second-order nonlinear equation:

$$\ddot{x}(t) + f(t, x, \dot{x}) = 0, \tag{64}$$

where the dot means derivative with respect to the variable t and $f(t, x, \dot{x})$ is a polynomial on the variables x and $\dot{x}$, and, with respect its explicit dependence on t, it is continuous on a neighborhood of the origin. The one parameter family of equations that replaces to (20) is now

$$\ddot{x}(t) + \omega^2 x(t) + \lambda(f(t, x(t), \dot{x}(t)) - \omega^2 x) = 0, \tag{65}$$

where λ is a real parameter. Now, the procedure is essentially identical as in the previous situation. Solutions are written as in (6), which in the present case gives

$$\sum_{k=0}^{n} \ddot{x}_k(t)\lambda^k + \omega^2 \sum_{k=0}^{n} x_k(t)\lambda^k + \lambda(f(t, \phi_\lambda, \dot{\phi}_\lambda) - \omega^2 \sum_{k=0}^{n} x_k(t)\lambda^{k+1} = 0, \tag{66}$$

where ϕ_λ was defined in (6). Note that we have truncated the series for some value of n, so that the relation (66) is just an approximation, the higher the value of n the better. For $n = 0$, we recover (8). Otherwise, and taking into account that $f(t, x(t), \dot{x}(t))$ is a polynomial on x and $\dot{x}$, we have for $n = 1, 2, \ldots$

$$\ddot{x}_k(t) + \omega^2 x_k(t) + f(t, \phi_\lambda, \dot{\phi}_\lambda) = 0. \tag{67}$$

Finally, we define the error obtained by some approximate solution, $z(t)$, within the integration interval $[0, T]$ as

$$e_n := \int_0^T (\ddot{z}(t) + f(t, z(t), \dot{z}(t)))^2 \, dt. \tag{68}$$

Next, we analyze this procedure in two examples: The Duffing and the van der Pol equations.

5.1 | The duffing equation

The Duffing equation has been introduced to study damped oscillators, driven or not by an external force.[3,13–15] However, we search for periodic solutions. Therefore, we have chosen a simplified form of this equation as

$$\ddot{x}(t) + x(t) + x^3(t) = 0, \tag{69}$$

where we have omitted the damping term, because otherwise we cannot have periodic solutions, and the external force for simplicity. Recall that the term on $\dot{x}$ introduces a "friction," so that solutions of equations with this term decay. Equation (69) has the following first integral:

$$E = \dot{x}^2(t) + x^2(t) + \frac{1}{4}x^4(t), \tag{70}$$

so that all its solutions are periodic.

To use modified Lyapunov in (69), let us write

$$\ddot{x}(t) + \omega^2 x(t) + \lambda(1 - \omega^2 + x^2(t))x(t) = 0, \tag{71}$$

where we have introduced an extra term of the form $\omega^2 x$ in order to produce an output with harmonics with fundamental frequency $\omega = 2\pi/P$, where P is the period to be determined.

In order to determine the period of periodic solutions, we may proceed as follows. Fix some initial conditions. We establish $x(0) = 1$ and $\dot{x}(0) = 0$ for simplicity. We are in the position of using Modified Lyapunov as described in Section 2 so as to obtain the approximate solution $x(t) := \phi_{\lambda=1}(t)$, see (14). This solution depends on the period P. The periodicity condition yields to

$$(x(0) - 1)^2 + \dot{x}^2(0) = 0. \tag{72}$$

This equation gives the value of P. We wish to compare the precision of our method with the precision given by a eight order Runge-Kutta. We denote the period and the error as defined by (68) obtained using Runge-Kutta as P_{RK} and e_{RK}, respectively. We have $P_{\mathrm{RK}} = 4.768022$ and $e_{\mathrm{RK}} = 1.50\,10^{-2}$.

On Table 6, e_P is the percent relative error of P with respect to P_{RK}.

This is the approximate solution for $n = 2$:

$$\begin{aligned} x(t) = {} & 0.981824 \cos \omega t + 1.78571\,10^{-2} \cos 3\omega t + 3.18878\,10^{-4} \cos 3\omega t \\ & + 5.06201\,10^{-3} t \sin \omega t - 8.66948\,10^{-10}\, t \sin 3\omega t. \end{aligned} \tag{73}$$

Here, $e_n = 2.39\,10^{-4}$.

Next, we average the coefficients on the interval $[0, P]$ and obtain the following periodic approximate solution:

$$\begin{aligned} x(t) = {} & 0.981824 \cos \omega t + 1.78571\,10^{-2} \cos 3\omega t + 3.18878\,10^{-4} \cos 3\omega t \\ & + 1.20214\,10^{-2} \sin \omega t - 1.64194\,10^{-9} \sin 3\omega t. \end{aligned} \tag{74}$$

n	P	e_n	e_P
2	4.749641	$2.39\ 10^{-4}$	0.38
3	4.767863	$2.74\ 10^{-6}$	$4.61\ 10^{-3}$
4	4.768020	$2.21\ 10^{-7}$	$4.20\ 10^{-4}$

TABLE 6 We give the values of the period P, the percent relative error of P with respect to P_{RK}, e_P, and the error as defined in (16) for the iterations $n = 2, 3, 4$

Here, $e_n = 4.16 10^{-4}$. We see that the averaged periodic solution (74) has a very similar precision to (71). In addition, it produces the global solution by periodicity, a property that does not have (73).

5.2 | The van der Pol equation

The van der Pol equation[2,16] is given by

$$\ddot{x}(t) + \mu\dot{x}(t)(x^2(t) - 1) + x(t) = 0. \tag{75}$$

Now, we may apply the modified Lyapunov in order to obtain a function that approximates its unique limit cycle and get some approximation of the period. In order to express the solution $x(t)$ as a combination of harmonics with fundamental frequency ω, let us proceed as done in the case of the Duffing equation (69) and write

$$\ddot{x}(t) + \omega^2 x(t) + \lambda(\mu\dot{x}(t)(x^2(t) - 1) + (1 - \omega^2)x(t)) = 0. \tag{76}$$

As happens with the Duffing equation (69), the frequency $\omega = 2\pi/P$ and the position of the limit cycle on the plane $(x, \dot{x})$ are unknown. The only critical point is the origin and the limit cycle goes around it. Then, we propose the following initial conditions $x(0) = u$ and $\dot{x}(0) = 0$, where u is unknown. Since P is the period, we may determine the values of u and P as solutions of

$$x(0) = x(P), \;\; \dot{x}(0) = \dot{x}(P). \tag{77}$$

This is an algebraic system for which does not exist a unique solution. With each of the solutions of (77), we construct a solution of (76). Then, we have a set of solutions, from which we select the desired solution as a critical point (possible minimum) of the functional

$$e := \int_0^P (\ddot{x}(t) + \mu\dot{x}(t)(x^2(t) - 1) + x(t))^2 \, dt. \tag{78}$$

Observe that the expression under the integral sign is given by the left hand side of (75). This suggests that for approximate solutions $e(x) << 1$. Nevertheless, the idea of the approximate solution as the critical functions of the functional (78) is being supported by numerical experiments.

We may compare the solution given by this procedure with the numerical solution given by Runge-Kutta, which gives a value for the period $P_{\text{RK}} = 6.38116$ and an error $e_{\text{RK}} = 2.49 10^{-3}$, for the values $\mu = 0.5$ and $\mu = 0.1$. Again, we call e_P to the percent relative error of P with respect to P_{RK}. We give some results on Tables 7 and 8, corresponding to the values of the parameter $\mu = 0.5$ and $\mu = 0.1$, respectively. In both tables, e_n comes after the definition (16), and the coefficients of the approximating functions have been averaged according to (17).

TABLE 7 Values of the period, the percent relative error of the period with respect to the result obtained by using Runge-Kutta, the error e_n defined in (16), and the given value of u for the choice $\mu = 0.5$ for different iterations $n = 1, 2, 3, 4$

n	P	e_P	e_n	u
1	6.28319	1.5	$7.50\ 10^{-1}$	2.0000
2	6.39775	$2.6\ 10^{-1}$	$1.44\ 10^{-1}$	2.0065
3	6.38017	$1.6\ 10^{-2}$	$3.56\ 10^{-2}$	2.0056
4	6.38787	$1.0\ 10^{-1}$	$7.04\ 10^{-6}$	2.0084

TABLE 8 Values of the period, the percent relative error of the period with respect to the result obtained by using Runge-Kutta, the error e_n defined in (16), and the given value of u for the choice $\mu = 0.1$ for different iterations $n = 1, 2, 3, 4$

n	P	e_P	e_n	u
1	6.28319	$6.27\ 10^{-2}$	$1.21\ 10^{-3}$	2.0000
2	6.28711	$3.2\ 10^{-4}$	$1.26\ 10^{-5}$	1.9999
3	6.28712	$1.6\ 10^{-4}$	$1.37\ 10^{-7}$	2.0002
4	6.28711	$3.2\ 10^{-4}$	$1.25\ 10^{-9}$	2.0001

Another option to figure out the precision of the method is to compare its solutions to the solutions given by another well established method; in the present case, we compare the approximate solution obtained by modified Lyapunov, after the averaging procedure (17), with the approximate solution given by Lindstedt-Poincaré,[1,10] which is

$$\begin{aligned} x_{\mathrm{LP}}(t) &= 1.99865\cos\omega t + 0.001875\cos 3\omega t - 0.000520833\cos 5\omega t \\ &+ 0.0749972\sin\omega t - 0.0249991\sin 3\omega t. \end{aligned} \tag{79}$$

If we choose $\omega = 1$, we obtain for the period $P_{\mathrm{LP}} = 2\pi$, the error $e = 2.97\,10^{-5}$, and the percent error $e_P = 6.5\,10^{-2}$. We have chosen $\mu = 0.1$.

As discussed along the present article, modified Lyapunov gives as approximate solution a linear combination of sine and cosine functions with polynomial coefficients with a small variation within a period. After averaging as in (17), we obtain the following periodic solution for $n = 4$:

$$\begin{aligned} x_{\mathrm{ML}}(t) &= 1.99876\cos\omega t + 0.00172964\cos 3\omega t - 0.000521438\cos 5\omega t \\ &+ 0.0749972\sin\omega t - 0.0249991\sin 3\omega t. \end{aligned} \tag{80}$$

Here, we have obtained $\omega = 0.999375$, which is very closed to $\omega = 1$ as given by Lindelöf-Poincaré. The other resulting parameters are of a similar order. For the period, we obtain $P = 6.28711$; for the error, $e = 1.26\,10^{-5}$; and for the percent error, we have $e_P = 3.2\,10^{-4}$, which in fact are of the same order than in Lindelöf-Poincaré.

6 | CONCLUDING REMARKS

We have proposed a substantial modification of the Lyapunov method in order to find either periodic or non-periodic approximate solutions to second-order linear differential equations with variable coefficients. We obtain approximate solutions by an iteration method, and we show the absolute convergence of the resulting series on compact intervals. Using a non-periodic approximation, we may find a periodic one using averages on the time dependent coefficients of the non-periodic approximation. The numerical recipe may be extended to first order systems of two equations.

The range of equations to which our method is applicable also includes non-linear ODE. Thus, we have tested it using the Bratu, the Duffing, and the van der Pol equations.

We have found that our modified Lyapunov method is equivalent, making a particular choice of the seed solution, to the widely used Picard-Lindelöf method. The choice of this seed solution is crucial in the approximate integration by Picard-Lindelöf, although both methods are equivalent *only* with a particular choice of this seed, choice that has some ambiguity. We discuss this property with detail. At the same time, we have made a number of numerical experiments that show that, even if we find the right choice for the seed, our modified Lyapunov method is more efficient than Picard-Lindelöf. We have added two simple examples of these numerical experiments on the text with detailed explanations.

A method for approximate solutions must be applicable and easily implementable. We believe that our modified Lyapunov method fulfills these requirements. We apply it on some equations well known by physicists, such as Mathieu, Airy, Bratu, Duffing, and van der Pol equations. In all cases, we discuss the precision of the method, which is good even if we use a low number of iterations. CPU times are also quite reasonable, definitively smaller that those needed for standard Lyapunov, an interesting property for researchers not having a strong computational power.

This paper is a part of a project by the authors intending to study methods to finding approximate solutions of ordinary differential equations of interest in physics.[17]

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REFERENCES

1. Mickens RE. *Truly Non-linear Oscillations: Harmonic Balance, Parameter Expansions, Iteration and Averaging Methods*. Singapore: World Scientific; 2012.
2. Farkas M. *Periodic Motions*. New York: Springer; 1994.
3. Chicone C. *Ordinary Differential Equations With Applications*. Berlin and New York: Springer; 1999.
4. Hairer E, Norsett SP, Wanner G. *Solving Ordinary Differential Equations I, Nonstiff Problems*. New York: Springer-Verlag; 2000.
5. Strogatz R. *Nonlinear Dynamics and Chaos*. New York: Westview; 1994.
6. He JH. Variational principles for some nonlinear partial differential equations with variable coefficients. *Chaos Solitons Fractals*. 2004;19(4):847-851.
7. He JH. Some asymptotic methods for strongly nonlinear equations. *Int J Mod Phys B*. 2006;20(10):1141-1199.
8. Hermann M, Saravi S. *A First Course in Ordinary Differential Equations. Analytical and Numerical Methods*. Berlin: Springer; 2018.
9. Gilmore RJ, Steer MB. Nonlinear circuit analysis using the method of harmonic balance. A review of the art. Part I. Introductory concepts. *J Microw Mill- Wave Comput Aided Eng*. 1991;1:22-37.
10. Ramos JI. On the Picard-Lindelöf method for non-linear second order differential equations. *Appl Math Comp*. 2008;203:238-242.
11. Abramowitz M, Stegun I. *Handbook of Mathematical Functions*. New York: Dover; 1972.
12. Gadella M, Giacomini H, Lara LP. Periodic analytic approximate solutions for the Mathieu equation. *Appl Math Comp*. 2015;271:436-445.
13. Duffing G. Erzwungene Schwingungen bei veränderlicher Eigenfrequenz und ihre technische Bedeutung, Heft 41/42, Braunschweig: Vieweg, vi+134 pp. OCLC 12003652; 1918.
14. Bender CM, Orszag SA. *Advanced Mathematical Methods for Scientists and Engineers I: Asymptotic Methods and Perturbation Theory*. Berlin: Springer; 1999.
15. Estevez PG, Kuru S, Negro J, Nieto LM. Solutions of a class of Duffing oscillators with variable coefficients. *Int J Theor Phys*. 2011;50:2046-2056.
16. van der Pol B. On oscillation hysteresis in a triode generator with two degrees of freedom. *Philos Mag*. 1922;43(6):700-719.
17. Alvarez-Sánchez JJ, Gadella M, Lara LP. A method to find approximate solutions of first-order systems of nonlinear ordinary equations. *Math Meth Appl Sci*. 2021;44:10014-10031.

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