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An explanation of *or*-deletions and other paradoxical disjunctive inferences

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ABSTRACT

Some inferences of the sort: *A* or *B*; therefore *A*, which are invalid in standard logics, are sensible in life: *You can enter now or later*; therefore, *you can enter now*. That these "*or*-deletions" follow necessarily or only possibly is a by-product of a theory of mental models. Its semantics for "or" refers to conjunctions of possibilities holding in default of knowledge to the contrary. It predicts new sorts of *or*-deletion, such as: He likes to drink red wine or white wine. So, he likes to drink red wine. and: You are permitted to do only one of the following: You can enter now. You can enter later. Therefore, you are permitted to enter now. They are invalid in standard logics, and neither previous pragmatic nor semantic theories predicted them. Four experiments corroborated their occurrence.

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> … for there is nothing either good or bad, but thinking makes it so … . *Hamlet*, Act II, Sc. 2

Hamlet's remark to convince his friends that Denmark is a prison at least for him is a disjunction hinging on "or", but it has a striking property apart from its moral relativism. It implies a conjunction. Thinking can make something good, *and* thinking can make something bad. Disjunctions shouldn't imply conjunctions, and philosophers and linguists refer to such inferences as "free choice paradoxes"—"free choice" because they were first discovered for disjunctions offering choices, and "paradoxes" because they violate the semantics for disjunctions in standard logics. We refer to such inferences instead as "*or*-deletions" because in effect they delete "or" from a disjunction to allow its clauses to follow as conclusions. Standard logics treat the meaning of "or" as a function from the truth values of its clauses to the truth value of the disjunction. An *inclusive* disjunction, *A or B, or both*, is true if at least one of its clauses is true, and false if neither of them is true. An *exclusive* disjunction, *A or B, but not both*, is true if just one of its two clauses is true, and false in any other case. So, an exclusive disjunction in standard logics is equivalent to *A or B, and not both A and B*: $(A \vee B) \& \neg (A \& B)$. Hamlet's disjunction probably has an exclusive interpretation, because it is not likely that something is both good and bad for the same person. But, given their truth functions, neither an inclusive nor an exclusive disjunction implies both of its clauses. So, philosophers, formal semanticists, and linguists have proposed numerous theories of *or*-deletions, some pragmatic with the goal of saving truth functions for disjunctions, and some giving them up in favor of a different semantics (for a review, see Meyer, 2020). We outlined this literature in a previous article, proposed a semantic solution to *or*-deletions, and presented the results of the first direct experimental tests of them because previous studies focused on their indirect effects. These results corroborated our semantic theory (Johnson-Laird et al., 2021). In the present paper, we outline a more general theory also based on mental models—the "model" theory, for short—and report new phenomena that it revealed.

Most theories of *or*-deletion are couched in terms of formal semantics using a "possible worlds"

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semantics. So, an assertion such as: *It is possible that it's raining*, is true in the actual world if *it's raining* is true in at least one of the possible worlds that determine the truth values of assertions in the real world—they are known as its "accessible" worlds. Assumptions about accessibility correspond to axioms for different modal logics (Kripke, 1963). Each possible world determines the truth value of any assertion in worlds to which it is accessible, and so, as Partee (1979) wrote, possible worlds are too big to fit inside anyone's head. Our cognitive theory therefore cannot use them. The real challenge to formal semanticists, however, is to formulate an algorithm that recovers the logical forms of assertions in daily life. Despite many years of analyses (e.g. Keene, 1992), no such algorithm exists—a scandalous gap if logic is supposed to underlie rational thought. In fact, no robust evidence exists that people rely on standard logic or on its logical forms in their reasoning. And certain inferential phenomena are inexplicable if reasoners were relying on logical rules of inference, e.g. deductions are easier from exclusive disjunctions (*A or B but not both; not A; ∴ B*) than from inclusive disjunctions (*A or B or both*; *not A; ∴ B*; see Johnson-Laird et al., 1992). We, therefore, implement the model theory, not in formal rules of inference, but in working computer simulations. They have three advantages for theories of cognition. They establish that a theory is computable. They must embody realistic constraints to ensure that proposed mental processes are computationally tractable. And they make it easy to determine the status of a putative inference. The pertinent program for our studies is mModal due to David Guerth and Marco Ragni. Its source code in Python is at: https://github.com/CognitiveComputationLab/cogmods/tree/master/modal/student_projects/2019_guerth.

The present article is straightforward. It begins with a brief account of relevant previous theories. It outlines the development of the model theory of reasoning (from Johnson-Laird, 1983; Johnson-Laird & Byrne, 1991; to Johnson-Laird et al., 2021; and Khemlani & Johnson-Laird, 2021). It describes the minimum needed for readers to understand the theory's predictions about *or*-deletions. It presents four experiments that corroborated these predictions. And it concludes with a discussion of their implications.

Previous accounts of *or*-deletions

Grice (e.g. 1989) was the great defender of truth functions as the semantics of compound assertions, such as disjunctions. He argued that the conventions of discourse enable speakers to convey more than the literal meanings of their assertions. For instance, if her husband asserts:

1. Eva is in Lisbon or Madrid

he conveys a "conversational implicature" that he does not know which of the two cities his wife is in. Otherwise, granted the cooperative nature of the conversation, he would have named a single city (Grice, 1989, p. 26). One sign of a conversational implicature is that it can be canceled without contradiction (ibid., p. 44), as when her husband says:

2. Eva is in Lisbon or Madrid; I know which, but you've got to guess.

Implicatures are therefore defeasible (or "nonmonotonic" in the jargon of artificial intelligence, Marek & Truszynski, 2013) in that additional information can lead to their withdrawal without contradiction. No additional information—not even a direct contradiction—justifies the withdrawal of a valid conclusion in standard logic.

Gricean pragmatics has burgeoned in ways that often rely on possible-worlds semantics. Theorists formalized Grice's approach (Gazdar, 1979), developed it into a linguistic theory (Levinson, 2000) and transformed it into a psychological theory (e.g. Sperber & Wilson, 1995). But, Grice's account concerns social conventions, not cognitive processes, and so, as Cohen (1971) pointed out, his theory applies to utterances as a whole, not to their constituents. Hence, it doesn't explain this inference:

3. Fred knows that Eva is in Lisbon or Madrid.
   ∴ Fred doesn't know which of the two cities she is in.

However, a post-Gricean theory solved this problem. It postulated that implicatures arise from an exhaustion operator akin to "only", which grammar can insert into a sentence (Fox, 2007). This account has the advantage that it yields *or*-deletions from disjunctions that refer neither to permissions nor to possibilities, such as those containing quantifiers:

4. Some students delayed the project or never finished it.
   ∴ Some students delayed the project.

The inference fails, however, if the subject is "some student" in the singular (Klinedinst, 2007). Fox suggests the generalization that *or*-deletion is viable as long as the quantifier is existential, *some x*, and refers to a plurality. This post-Gricean approach was extended and modified (in Bar-Lev & Fox, 2020) so that it no longer relies on multiple uses of the exhaustion operator. And it accounts for *or*-deletions, such as:

5. The teacher is happy with every student either talking to Mary or to Sue;
   ∴ The teacher is happy with every student talking to Mary.

which it treats as concerning possibilities. This post-Gricean account is the most powerful application of pragmatics to *or*-deletions, but we will argue that like Gricean theories it is unable to account for all of them.

Semantic accounts of *or*-deletion propose modifications or alternatives to truth-functions for disjunctions. Zimmermann (2000) had the brilliant idea that a disjunction, such as:

6. It's cloudy or it's hot

refers to a conjunctive list of epistemic possibilities:

7. Possibly it's cloudy and possibly it's hot: ◇cloudy & ◇hot

where "◇" is a symbol for epistemic possibility, "&" is a symbol for logical conjunction, "cloudy" abbreviates *it is cloudy*, and "hot" abbreviates *it is hot*. The assertion receives an interpretation in possible worlds. For a disjunction that refers in addition to explicit possibilities, such as:

8. It may be cloudy or it may be hot

the result contains adjacent pairs of possibilities, one from "may" and the other from "or": (◇◇cloudy & ◇◇hot). Zimmermann formulated a principle in a standard modal logic that allowed these pairs to be reduced to single possibilities. He proposed an analogous treatment of deontic possibilities, i.e. permissions. Geurts (2005) followed Zimmermann, but rejected his principles for reducing pairs of possibilities into single ones. Instead, they "fuse" into one. And he pointed out a serious problem for conjunctive lists. The negation of a disjunction: *not (A or B)* is equivalent to *not-A & not-B*, but such an equivalence does not hold for the negation of a conjunction: not(A and B). We add a further problem: if one member of a conjunctive list is false then so too is the conjunction as a whole. So, consider this inference, which is valid in standard logic:

9. There is a triangle or there is a rectangle, or both.
   In fact, there cannot be a triangle.
   ∴ There is a rectangle.

It cannot be guaranteed given that the disjunctive premise is interpreted as a conjunction of possibilities. The second premise refutes this conjunction, and this inference is valid in standard modal logics:

10. (◇triangle & ◇rectangle & ◇(triangle & rectangle))
    Not (◇ triangle)
∴ Not (◇triangle & ◇rectangle & ◇(triangle & rectangle))

So, one false conjunct falsifies a conjunctive list in standard logics, and the premises in (9) are self-contradictory. In standard modal logics, they imply any conclusion whatsoever. We turn now to the model theory, which avoids these problems.

The theory of mental models

Everyday reasoning is carried out in ignorance of formal logics and of the logical conception of validity (Jeffrey, 1981, p. 1): "A valid inference is one whose conclusion is true in every case in which all its premises are true". It follows from this definition that any conclusion whatsoever is a valid inference from contradictory premises, because there are no cases in which all the premises are true, and so there can be no counterexamples to the inference. Naive individuals—those innocent of logic or its cognate disciplines—do not draw conclusions from contradictory premises. Hence, the model theory no longer uses the standard definition of "validity". Instead, it distinguishes the status of inferences in terms of alethic possibilities:-

- an inference is *necessary* if its conclusion holds in all of the possibilities to which its premises refer (and they do refer to some).

- an inference is *probable* if its conclusion holds in most of these possibilities.
- an inference is *possible* if its conclusion holds in at least one of these possibilities.
- an inference is *impossible* if its conclusion and the premises refer to disjoint possibilities, i.e. they are contrary to one another, or contradict one another.

Inferences can depend on meaning, reference, context, and knowledge. So, each inference has to be analyzed on its own terms. And each is defeasible: if people know a fact contrary to its conclusion—even one that follows necessarily from the premises—they withdraw the conclusion. They search for an explanation to resolve the inconsistency, which they judge as more likely than a mere amendment to one of its premises to restore consistency (Johnson-Laird et al., 2004; cf. Marek & Truszynski, 2013). The model theory therefore differs from an account of human reasoning founded on truth functions (e.g. Rips, 1994) or standard possible-world semantics (e.g. Montague, 1974). Some proponents of mental models have formalized a theory of them (Koralus & Mascarenhas, 2013), but the present model theory is inconsistent with any standard logic. It has been applied to all the main domains of reasoning, and descriptions of its explanantions of sentential and quantified deductions, and its computer implementation, can be found elsewhere (e.g. Johnson-Laird & Khemlani, 2020; Khemlani et al., 2018; Khemlani & Johnson-Laird, 2021).

The present studies concern *or*-deletions of the following sort:

11. A or B.
    ∴ A.
    ∴ B.

We, therefore, spell out the theory's main assumptions that together make predictions about such inferences:-

- *Disjunctions refer to conjunctions of default possibilities:* Suppose you know that there may be a circle on the blackboard, that there may be a square there, and that there may be both of these shapes there. You can express this knowledge in a disjunction, which is a valid inference from these possibilities, even in standard modal logics:

12. There is a circle or there is a square or both (on the board).

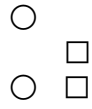
Some languages have no word corresponding to "or", and can express disjunctions only in this way, marking each clause, in effect, as possible (Mauri, 2008). Conversely, given a disjunction such as (12), individuals infer that each of the preceding possibilities follows from it (Hinterecker et al., 2016). These inferences, however, are invalid in standard modal logics. Analogous inferences of possibilities occur from conditionals (e.g. Barrouillet et al., 2000). The model theory predicts them, because it postulates that all compound assertions, including disjunctions and conditionals, refer to conjunctions of possibilities that each hold in default of knowledge to the contrary.

As the human system of comprehension uses its knowledge of grammar to parse a disjunction, such as example (12), it composes its semantic representation from the meanings of its parts and their grammatical relations (Johnson-Laird & Khemlani, 2020), e.g.:

13. ◇(circle) & ◇(square) & ◇(circle & square)

These possibilities are exhaustive, and each refers to a defeasible core possibility. The only remaining case is not possible: ¬ circle & ¬ square, where "¬" is a symbol for the negation of a proposition. Semantic representations such as (13) govern the construction and modification of mental models, which represent the situations to which assertions refer.

The theory postulates two different systems of reasoning—an idea due to the late Peter Wason (see Manktelow, 2021, pp. 2–3, and, e.g. Wason & Johnson-Laird, 1970), and that others have developed into many different sorts of "dual process" theories. The model theory's version is a rare instance of one implemented in a computer program, mModal. System 1 is intuitive, because it makes no use of a working memory for the results of intermediate computations, and does not repeat a loop of operations for more than a small number of times, depending on a parameter that can be set to characterize individual performance. The theory thus embodies a universal characteristic of human reasoning—its parsimony to ease the load on working memory. The intuitive models constructed from the meaning in (13) are as follows:



Each row in such diagrams designates a core possibility, which underlies different modal interpretations, such as alethic, deontic or epistemic ones (see below). Possibilities, such as ◇A and ◇¬A, are consistent with one another, and so the models represent a *conjunction* of possibilities, and example (12) therefore yields the following inferences (see Hinterecker et al., 2016):

14. ∴ It is possible that there is a circle.
    ∴ It is possible that there is a square.
    ∴ It is possible that there is a circle and a square.

Each model of a possibility holds in default of knowledge to the contrary. So, the discovery, say, that the circle and square cannot both occur, modulates the interpretation of the disjunction by blocking the construction of the third model above. The remaining models of the defaults still hold, and so this modulation yields an exclusive disjunction. Experiments have corroborated this and other modulatory effects of knowledge (e.g. Quelhas & Johnson-Laird, 2017). The idea that individual defaults can be refuted without affecting others is standard in object-oriented programming languages. As the example above illustrates, intuitive models represent only clauses that are true in a possibility. Intuitive models therefore yield systematic fallacies from certain premises (Johnson-Laird et al., 2000; Khemlani & Johnson-Laird, 2017).

In contrast, system 2 is used to deliberate, and it has access to a working memory of intermediate computations. It includes in each model of a possibility representations of clauses that are false, using negation to do so. Deliberation can therefore correct the fallacies that intuition yields, and its ideal performance—within the constraints of working memory—serves as a normative account of reasoning according to the model theory. The deliberative models of the disjunction:

15. There is a circle or there is a square or both are:

◯ ¬ □
¬ ◯ □
◯ □

where "¬" is the symbol for negation.

- *The negation of disjunctions:* Negation, which is better described to naive individuals as "denial", calls for the complement of the set of possibilities to which the negated assertion refers. Hence, this negation:

16. It is not the case that there is a circle or there is a square or both

calls for the complement of the preceding set of models, i.e. the one case not amongst them:

¬ ◯ ¬ □

As the theory predicts, individuals grasp the denial of an inclusive disjunction, which yields one possibility, more accurately than the denial of a conjunction, *There is a circle and there is a square*, which yields three possibilities (Khemlani et al., 2014). The theory solves the problems of negation that bedeviled Zimmermann (2000) and Geurts (2005). The negation of a single possibility in a conjunction of defaults representing a disjunction does not refute the conjunction itself, because each possibility holds in default of knowledge to the contrary, and so the negation leaves any other possibilities intact. The negation of a disjunction itself, however, yields the complement of its defaults.

Inferences depend on conjunctions of sets of models: Suppose that two premises in an inference are:

17. There is a circle or there is a square or both.
    There is not a circle.

Their intuitive models are respectively:

◯
□ and ¬ ◯
◯ □

The fundamental inferential principle is to conjoin existing models with those of the current premise using pairwise conjunctions to form a new set of possibilities. But, no attempt is made to conjoin any pair of models contradicting one another, because their conjunction would yield only the null model from which nothing follows, other than the denial of premises that yield only the null model. The pairwise conjunctions of the preceding intuitive models are as follows:

◯ and ¬ ◯ no model is formed, because of the contradiction.
□ and ¬ ◯ yields this model □ ¬ ◯
◯ □ and ¬ ◯ no model is formed, because of the contradiction.

The same principles apply to the conjunctions of sets of deliberative models. The result for both sorts of the model from this inference is, as shown above,

a model of a single possibility. And when there is only one possibility, it *is* a fact. The new conclusion that it represents follows of necessity from the premises:

18. ∴ There is a square.

The process of conjoining sets of models accommodates any number of premises bounded only by the capacity of working memory. A corresponding AI program, however, exceeds human working memory, and its algorithm for simplifying models draws its own parsimonious conclusions (see Ch. 9 of Johnson-Laird & Byrne, 1991).

When conclusions are given for individuals to evaluate in relation to a set of premises, the procedure concerns alethic modalities. Unlike a valid conclusion in logic, a necessary conclusion in the model theory holds only if the premises refer to at least one possibility, and so have at least one model that is not null. Natural language allows "or" to connect constituents of the same or similar sorts in sentences, from clauses down to morphemes, e.g. *he is un- or dis-interested*. Disjunctions of small constituents have what is known as a "narrow scope", whereas disjunctions of complete clauses have what is known as a "wide scope". A common assumption is that a narrow-scope disjunction, such as:

19. There is a circle or triangle, or both

is synonymous with its corresponding wide-scope disjunction:

20. There is a circle or there is a triangle, or both.

But, as we will show, the two are not always equivalent in daily life.

Mental models condense possibilities: In the model theory, the possibilities that "or" elicits have an underlying core meaning (Johnson-Laird, 1978; Kratzer, 1977). The contents and context of an assertion can elicit knowledge that modulates this core meaning, so it is interpreted as an alethic possibility concerning the status of inferences, a deontic possibility concerning permissions, and an epistemic possibility concerning knowledge of states of the world. Many assertions are ambiguous, so a disjunction, such as:

21. You can do it now or later

could be ambiguous between a speaker giving permission, describing what is feasible, or both (for syntactic and semantic cues to the different interpretations, see Johnson-Laird & Ragni, 2019). Interpretations can even depend on shared but otherwise idiosyncratic knowledge.

Standard modal logics treat the meaning of possibility as yielding the following valid deduction:

22. It's raining
∴ It is possible that it's raining.

The inference is not acceptable in daily life, and Karttunen (1972) treated it as evidence against standard modal logics. The model theory adopts a different interpretation of "possible", similar to one that Aristotle advocated (*De Interpretatione*, 21b35). The assertion:

23. It is possible that there's a circle presupposes:
24. It is possible that there is not a circle

and vice versa. People make both sorts of inference (Ragni & Johnson-Laird, 2020). One view is that these inferences are conversational implicatures: for all a speaker knows on asserting (24), the denial of its complement (25) is also possible (Karttunen, 1972).

A crucial consequence of the parsimony of human reasoning is the *condensation* of possibilities. A single model of a possibility, which condensation creates, is much easier to cope with than multiple models—a prediction first confirmed years ago (Johnson-Laird et al., 1992). So, when possibilities seem consistent with one another, individuals tend to condense them into one (Ragni & Johnson-Laird, 2020), e.g.:

25. It is possible that Pat is single, and it is possible that Viv is married.
∴ It is possible that Pat is single and that Viv is married.

But, individuals do not condense possibilities when they are aware of their inconsistency, e.g.:

26. It is possible that Pat is single, and it is possible that Pat is married.

We have now outlined the key ideas: conjunctions of default possibilities, the interpretations of possibilities, their intuitive and deliberative models, the combinations of models in reasoning, the scope of disjunctions, and the condensation of

possibilities. And so we can now describe the model theory of *or*-deletions.

The model theory's predictions of *or*-deletions

In most situations in daily life—from the toss of a coin to the election of a President, people can envisage a small set of exhaustive and mutually exclusive alternatives, of which only one can occur. Each alternative can occur in many ways, which formal semanticists treat as referring to different possible worlds. The model theory instead postulates that each alternative has a small finite mental model, which represents only what is common to all its differing potential realizations. The resulting set of alternatives is *primordial* in that they are the basis of possibilities and probabilities (Johnson-Laird, Quelhas, et al., 2021). A simple criterion establishes that a disjunction refers to primordial alternatives: its truth implies that one, and only one, of them can be the case. For example, this disjunction is primordial:

27. She visited Avon or she visited Bath, or both.

Granted its truth, one and only one of the three default possibilities to which it refers occurred.

Primordial disjunctions, therefore, refer to possibilities that cannot be condensed. The model theory explains *or*-deletions without further ado. They occur only with disjunctions that are not primordial.

It is probably no accident that the first instances of *or*-deletion that theorists discovered concerned permissions. A speech act giving permission for an action, and a description of an epistemic possibility, can have the same disjunctive description, such as:

28. She may visit Avon or she may visit Bath, but not both

It is not primordial, because neither of the possibilities to which it refers may occur. Hence, it yields *or*-deletions. No need exists to infer one possibility operator from an adjacent pair (as in Zimmermann, 2000), or to fuse them together (as in Geurts, 2005). Knowledge can modulate the core concept of possibility to yield an alethic, deontic, or epistemic interpretation (see Johnson-Laird, Quelhas, et al., 2021). This interpretation governs the construction of the intuitive models of the cities referred to in (29) that she may visit. The program simulating the theory uses words to describe them:

Avon

Bath

…

The ellipsis allows that she need not visit either city. Each model yields an *or*-deletion inference:

29. ∴ She may visit Avon.
∴ She may visit Bath.

The assertion of an obligation:

30. She must visit either Avon or else Bath

yields the same two models as above, but without the ellipsis. So, the same *or*-deletions in (30) follow, but she is obligated to visit one of them. However, the models do not yield the inference that she must go to Avon, because she could go to Bath instead, and vice versa. And, in the end, she may fail to fulfill her obligation.

In general, intuitive models of disjunctions of explicit possibilities, such as (29), yield *or*-deletions, and they follow whether the possibilities are alethic, deontic, or epistemic, whether the disjunction is inclusive or exclusive, and whether "or" has wide scope as in (29), or narrow scope, as in:

31. She may visit Avon or Bath, but not both

Our previous study corroborated these predictions. Contrary to the post-Gricean account (Bar-Lev & Fox, 2020), most participants accepted *or*-deletions from wide-scope disjunctions (Johnson-Laird, Quelhas, et al., 2021, Experiments 1 and 2).

A different source of *or*-deletions are quantified assertions, e.g.:

32. ∴ Some of the actors are dancers or singers, but not both.
∴ Some of the actors are dancers.

The model theory of quantified assertions treats them as establishing a particular set-theoretic relation between subject and predicate (see Khemlani & Johnson-Laird, 2021, for a description of the theory, its computer simulation, and fit to the inferences of individual reasoners). But, example (33) has a grammatical ambiguity. In its salient interpretation the disjunction has a narrow scope, and this meaning underlies the construction of a single

intuitive model that condenses the disjunctive predicate. Each individual in the situation has no more than one of the predicates:

actor dancer
actor singer
actor dancer
actor

Hence, the model yields *or*-deletions, such as:

33. ∴ Some of the actors are dancers.

The disjunctive predicate in (33) can be treated as elliptical, and parsed so that "or" has a wide-scope:

34. Some of the actors are dancers or some of the actors are singers, but not both.

This meaning yields two deliberative models, such as:

1.		2.	
actor	dancer	actor	singer
actor	dancer	actor	singer
actor	¬ dancer	actor	¬ singer

These models show that the *or*-deletions from (33) are not necessary inferences, but follow only as alethic possibilities. Analogs of (33) and (35) are not logically equivalent in standard predicate logic, and neither of them yields valid *or*-deletions.

Other quantified assertions have meanings that cannot yield an intuitive model that condenses possibilities, e.g.:

35. More than half the actors are dancers or singers but not both.

An intuitive model in which more than half the actors are dancers cannot accommodate more than half of them as singers without violating the exclusive disjunction. The assertion has only two alternative models such as those above, and they do not yield *or*-deletions. A side effect is that an explicit reference to a single individual, *one of the x*, also blocks the condensation of an exclusive disjunction of predictates. Given a disjunction of predicates, such as *dancers or singers*, the model theory implies that the quantifier has to refer to a small enough proportion of the individuals in the situation having the disjunctive predicate for it also to hold for one of its single predicates. Hence, if more than half the students are dancers or singers, the disjunctive assertion has to refer to two alternative situations, and *or*-deletion is impossible. Determiners such as: *at least some, few, less than half*, can form quantifiers referring to a single situation, whereas *all*, *most*, and *more than half*, can not do so. Our previous study (Johnson-Laird, Quelhas, et al., 2021) corroborated this distinction, and contrary to post-Gricean theory, individuals accepted *or*-deletions from assertions with a quantifier that is not existential, *few x* (pace Fox, 2007). These inferences are only possible, not necessary, because deliberation can construct a wide-scope interpretation that blocks any of them.

Condensations yield *or*-deletions. What varies are degrees to which disjunctions are condensable. Those that assert possibilities—alethic, deontic, or epistemic—are bound to be condensable. Those that are in quantified assertions can be consensable for certain quantifiers. Other disjunctions can provide cues that allow or prevent condensation. We describe the model theory's predictions for them in introducing each of the following experiments.

Experiment 1: *or*-deletions with deontic metadisjunctions

Our previous studies corroborated the occurrence of *or*-deletions from speech acts creating permissions (Johnson-Laird, Quelhas, et al., 2021). The present experiment concerned the same disjunctions but expressed as *metadisjunctions* of possibilities. A metadisjunction is an assertion that expresses "or", but relies instead on "true" and "false", or their synonyms, which are part of the metalanguage for a standard logic, but which are also part of natural language (see, e.g. Tarski, 1944). They can be used to express exclusive disjuctions, as in:

36. One of the following assertions is true, and one of them is false:

    It is possible that the defendant was trespassing in the victim's garden.

    It is possible that the defendant was on business in the victim's garden.

    Is it possible that the defendant was trespassing in the victim's garden?

The metadisjunction elicits the same models as those of ordinary disjunctions expressed with "or". They should therefore yield *or*-deletions, and

recent study corroborated their occurrence in (37) and with other epistemic possibilities (Sklarek et al., 2022).

No existing pragmatic theory appears to be able to predict *or*-deletions from metadisjunctions. Gricean and neo-Gricean theories apply to single assertions, post-Gricean theory applies to assertions and to constituents within them. None of these theories, as far as we can tell, yield implicatures from three separate assertions as in (37). Bar-Lev (p.c., 1/17/21) wrote that the post-Gricean theory is silent about metadisjunctions, but he suggested that they do not seem to apply to permissions. The model theory predicts that *or*-deletions of permission should occur from them. Hence, our Experiment 1 tested these contrasting predictions, using inferences from metadisjunctions of the following sort, which we here translate from Portuguese, the language in which all our present experiments were carried out:

37. Imagine that your professor told you that you are *permitted* to do only one of the following actions:
    You can do your homework.
    You can do the presentation slides.

    Are you *permitted* to do your homework?
    Yes ☐ No ☐ Impossible to determine ☐

The model theory predicts that participants should accept *or*-deletions from metadisjunctions of permissions to a single permission (as in 38), from metadisjunctions of obligations to a single permission, but reject the other two cases, from metadisjunctions of permissions to a single obligation, and from metadisjunctions of obligations to a single obligation.

Participants

G*Power shows that to achieve 90% power for a medium effect size with alpha < .05, sample size should be at least 80 participants. So, this experiment tested 87 psychology undergraduates (74 female, 13 male, 0 other) from ISPA-IU, in Lisbon. Their mean age was 19.2 years ($SD = 2.4$). Their participation was voluntary, but they received a course credit.

Design

The participants acted as their own controls and evaluated whether *or*-eliminations followed from 16 deontic metadisjunctions. There were two sorts of premise: an individual creates either a permission or an obligation for the subject of the sentence, "you", to carry out one of two everyday actions. These actions were described in assertions with infinitival complements (to do A, to do B). There were four sorts of questions that the participants had to answer on separate trials, depending on whether the question was about A or about B, and whether it was about an action that was permitted or obligated. The preceding manipulations yielded 8 sorts of trial, and participants carried out each sort twice with different contents for a total of 16 trials.

All our experiments were carried out in Portuguese, the participants's native language, which we translate here into English. The premises used modal auxiliaries: "may" for permissions and "must" for obligations, and the conclusions used the verbs "permitted" and "obligated". The theory predicts the acceptance of *or*-eliminations for all trials in which the conclusion is about a permissible action, and rejection of them for all trials in which the conclusion is about an obligatory action. So, to unconfound this correlation, and to inhibit a potential response bias, the participants also carried out four "filler" trials interleaved among those of the experiment proper for which the predicted evaluations were "yes" for questions about obligations, and "no" for questions about permissions. For instance, there were two trials of this sort with the predicted answer of, "yes":

38. Imagine that your uncle told you that you are *obligated* to do only one of the following actions:
    You must go to the party this Saturday.
    You must go to the party this Sunday.

Are you *obligated* to go to the party this weekend?

Their answers are so obvious that those participants who erred on more than two of them were excluded from the experimental results, on the grounds that were not paying enough attention during the experiment.

The experiment tested four separate groups of participants in order to rotate the contents over all four sorts of premise and conclusion, and to counterbalance which half of the inferences concerned A and which half concerned B. The 20 trials (16

experimental and 4 fillers) were presented to each participant in a different random order.

Materials and procedure

All the assertions in the inferences used the pronoun "you" as their subjects to enhance a deontic interpretation, and their main actions were about topics related to animals, leisure activities, travels, and studies. Appendix A in the Supplemental materials available at: https://osf.io/ajsrv/ has the full set of contents in English and in their original Portuguese, and their results for each of the four groups.

The participants were tested individually at a computer terminal but in small groups of about 20 at a time in the same room. They were told the general nature of the experiment and gave their informed consent. They received a link to access the experiment in a Qualtrics program. Its first page requested their age and gender. Its second page gave the general instructions, which included the key passage:

> You will carry out two sorts of problem: one sort is about permissions and the other sort are about obligations. In both sorts of problem, the final question will also be about either what is permissible or about what is obligatory. Please read carefully the first sentence, and then evaluate if the conclusion follows or not, or if it's impossible to determine.

The participants used the computer's mouse to tick the appropriate box: "Yes", "No", or "Impossible to determine". As the instructions stated, they could take as much time as they wanted in order to complete the experiment, and they could leave the experiment at any time.

Results and discussion

Three participants were excluded from the statistical analysis, because they got more than two out of the four filler items wrong. All of the remaining 84 participants made more predicted than non-predicted evaluations (Binomial test, $p < .5^{84}$). Likewise, all eight experimental conditions yielded more predicted than non-predicted evaluations (Binomial test, $p < .005$). Whether the conclusion referred to the first action, A, or the second action, B, had no reliable effect on the percentages of predicted evaluations (88% vs. 89%, Wilcoxon test, $z = .64$, $p > .5$, Cliff's $d = .05$). Table 1, therefore, presents their mean percentages in evaluations of the four sorts of inference. The results for the individual problems in each of the four groups are in the Supplemental materials.

As Table 1 illustrates, inferences from premises about permissions yielded more predicted evaluations (93%) than premises about obligations did (84%; Wilcoxon test, $z = 3.67$, $p < .0001$; Cliff's $d = .23$). The percentages of predicted conclusions did not differ reliably between those about permissions (88%) and those about obligations (88%; Wilcoxon test, $z = 1.18$, $p > .25$, Cliff's $d = .091$). But, participants rejected obligations more often when the premise referred to a permission than when it referred to an obligation (a difference of 16%), and they accepted permissions when the premise referred to a permission more often than when it referred to an obligation (a difference of 3%). The first effect was greater than the second effect in a reliable interaction (Wilcoxon test, $z > 2.65$, $p < .01$, Cliff's $d = .21$). This result bears out a general modal asymmetry: obligations do not follow from permissions, but permissions can follow from obligations. It also eliminates the simplistic post hoc claim that participants responded "yes" in case the predicate in the conclusion, *permitted* or *obligated*, matched the meaning of the predicate in the premise, *can* or *must*. So, the participants corroborated the model theory's predicted evaluations.

Experiment 2: *or*-deletions from condensable disjunctions

Any disjunction that is not primordial is potentially condensable, and so it should yield *or*-deletions, at least as possible inferences. One such category is disjunctions in an infinitival complement, e.g.:

39. At Christmas, Vasco loves to eat turkey or pork.

Table 1. Experiment 1 ($N = 84$): Deontic metadisjunctions, the model theory's predictions for their or-deletions, and the percentages of them, where A and B stand for infinitival predicates referring to everyday actions.

The metadisjunctive premise	The conclusion to be evaluated about *A*, or about *B*.	Predicted evaluations	Percentage of predicted evaluations
You are permitted to do only one of: *You can do A.* *You can do B.*	*Permitted to do _?*	Yes	90
	Obligated to do _?	No	96
You are obligated to do only one of: *You must do A.* *You must to B.*	*Permitted to do _?*	Yes	87
	Obligated to do _?	No	80

Lacking a tense, infinitivals are missing information that would turn them into propositions (Johnson-Laird & Ragni, 2019). Because they do not imply that anything is bound to occur, they and the use of generic verbs, such as "loves", yield a condensable disjunction. So, example (40) yields a condensation of the possibilities that Vasco loves to eat at Christmas: turkey and pork. It should yield *or*-deletions as possible inferences, such as:

40. ∴ At Christmas, Vasco loves to eat turkey.

One reason for the use of "or" in assertions such as (40) is to signal that Vasco eats them as alternatives. No condensations should occur, however, for a similar primordial disjunction, such as:

41. Last Christmas, Vasco ate turkey or pork.

It should resist *or*-deletions. The contents of both sorts of preceding disjunction suggest that the alternatives are mutually exclusive. This interpretation might inhibit *or*-deletion, and so we compared these contents with those less likely to suggest mutual exclusion, as in:

42. At Christmas, Vasco likes to eat turkey or green vegetables.

Experiment 2 tested the contrasting predictions for the two sorts of disjunction, condensable and primordial.

Participants

The experiment tested 102 new participants from the same population as before (88 female, 14 male, and 0 other), which was more than enough for the requisite power (see Experiment 1). Their mean age was 20.7 years ($SD = 4.0$).

Design

The participants acted as their own controls and evaluated *or*-deletions from condensable disjunctions and primordial disjunctions, crossed with whether the contents of the disjunctions were biased towards exclusive interpretations (e.g. *turkey or pork*) or inclusive interpretations (e.g. *turkey or green vegetables*). The experiment tested four separate groups of participants in order to counterbalance the particular conclusions referring to the first or second alternatives in the predicate, and to counterbalance the order of the two alternatives in the disjunctive predicates. Each participant carried out three trials with different contents of the four sorts of inference, and received the resulting 12 problems in a different random order.

Materials and procedure

We created 12 pairs of contents in which one assertion had a disjunction in a condensable predicate and the other disjunction had a primordial predicate. Their contents were from the same four domains used in the previous experiment, and the two noun phrases in the predicates were either from the same categories or from different categories to suggest exclusive and inclusive interpretations, respectively. The full set of contents is in Appendix B of the Supplemental materials. The procedure was identical to the one in Experiment 1.

Results and discussion

The differences in the evaluations between the exclusive and inclusive contents were not greater than 2% for either sort of disjunction, and neither their overall evaluations nor their potential interaction with the two sorts of disjunctive predicate were reliable (Wilcoxon tests, $z < 1$, $p < .3$, in both cases). Hence, Table 2 presents the mean percentages of the effects of the two sorts of disjunction, and it shows that the participants tended to accept *or*-deletions from condensable disjunctions, and to evaluate *or*-deletions from primordial disjunctions as "impossible to determine". Out of the 102 participants, 100 fit this pattern, only 2 yielded evaluations inconsistent with it, and there were no ties (Binomial test, with a prior probability of 0.5, yields a probability of less than one in ten million). Hence, as the model theory predicts, the percentage of "Yes" evaluations for *or*-deletions was reliably greater for condensable disjunctions (82%) than for primordial disjunctions (8%, Wilcoxon test, $z = 8.67$, $p < .0001$; Cliff's $d = 0.6$).

In general, the results supported the model theory's account in which condensable disjunctions yield *or*-deletions as inferences of possibilities, but primordial disjunctions do not.

The theory predicts that individuals will reject such inferences from primordials, but the

Table 2. Experiment 2 ($N = 102$): The percentages of the three evaluations of or-deletions for condensable disjunctions (e.g. He loves to do A or B) and corresponding primordial disjunctions (e.g. He did A or B). The effect of whether doing A and doing B suggested an exclusive or inclusive interpretation was negligible (no more than 2%).

	Given a disjunctive premise, *A or B*		
Evaluations of: ∴ *A*, or: ∴ *B*.	Yes	No	Impossible to determine
Condensable disjunction	82	4	14
Primordial disjunction	8	5	87

participants tended instead to respond "Impossible to determine". At first sight, this evaluation seems contrary to the model theory's prediction. But, we asked participants after the experiment why they had made this evaluation, and their typical response was that they did so because it was uncertain which action the subject of the sentence had carried out. They had tried to evaluate, not the inference, but which action the agent had carried out. The difference is subtle, not easy to distinguish in instructions without confusing participants, and so we continued to use the same instructions in the subsequent studies.

The condensable disjunctions in the experiment referred to general cases, as in:

43. Rui likes to drink red wine or white wine.

The model theory predicts that generalizations alone should elicit *or*-deletions, e.g.:

44. At lunches, Rui drinks red wine or white wine.
∴ At lunches, Rui drinks red wine.

A future task is to test this prediction. What is not in doubt, and more pertinent to the model theory, is that *or*-deletions from condensable disjunctions tend to occur even though they do not follow of necessity, whereas they seldom occur from primordial disjunctions. A more urgent task is to examine the effects of scope on *or*-deletions, because they are germane to differences among theories.

Experiment 3: scope and *or*-deletions

The previous experiment showed that condensable disjunctions tended to elicit *or*-deletions, whereas primordial disjunctions tended not to do so. *Or*-deletions from these condensable disjunctions follow only as possibilities: deliberative models can interpret the disjunctions as calling for two separate models, which prevent *or*-deletions. One factor that should affect condensation is the scope of "or". The mere adjacency of consistent predicates in a narrow-scope disjunction, as in:

45. At nights, Paulo used to watch the news or a documentary

should be more conducive to a condensation than one with wide scope:

46. At nights, Paulo used to watch the news or he used to watch a documentary.

A narrow-scope disjunction, such as (46), elicits the start of the construction of a model and then arrives at the disjunctive predicate; if it its two predicates are consistent with the start of the model, they can be added to it. The result yields *or*-deletions. In contrast, a wide-scope disjunction, such as example (47), elicits a model of the first clause and then a switch to construct a model of the second clause. Primordial disjunctions should be less likely to be effected by scope, because they should resist *or*-deletion in both cases. Those with a narrow scope, such as:

47. Last night, Paulo watched the news or a documentary

should be more conducive to condensation than those with wide scope:

48. Last night, Paulo watched the new or he watched a documentary.

Hence, the theory predicts that *or*-deletions from such disjunctions depends on an interaction between their meanings (condensable vs. primordial) and their syntax (narrow scope vs. wide scope). The present experiment tested this interaction: narrow-scope condensables should tend to elicit more *or*-deletions than wide-scope ones, whereas scope should have a little effect on primordials, which should elicit fewer *or*-deletions.

Participants

The experiment tested 112 new participants from the same population as before (92 female, 20

male, and 0 other). Their mean age was 19.4 years ($SD = 1.3$). We increased the power of the experiment in order to detect the predicted interaction.

Design

The participants acted as their own controls and evaluated *or*-deletions from condensable and primordial disjunctions, crossed with whether their scope was narrow or wide (see the four examples above). The participants were assigned at random to one of four groups, which were used to counterbalance the contents of the disjunctions. They carried out three trials with different contents for each of the four sorts of inference to yield a total of 12 trials, which were presented to each of them in a different random order. As in the previous experiment, they evaluated whether each inference followed using three categories: "yes", "no", and "impossible to determine".

Procedure and materials

The procedure was identical to the one in the previous experiments. Likewise, the contents of the disjunctions came from the same four domains, and half of them had female proper nouns as subjects, and half of them had male proper nouns as subjects. The contents for condensable disjunctions referred to a plurality of times, e.g. "On Sundays", whereas those for primordial disjunctions referred to a singular time, e.g. "On Sunday". As in Experiment 2, we assigned the 12 different contents four times to each of the four sorts of inference, so that in each set half the conclusions concerned the first predicate and half the conclusions concerned the second predicate. Appendix C in the Supplemental materials presents the full set of contents for the four versions in English and in their original Portuguese.

Results

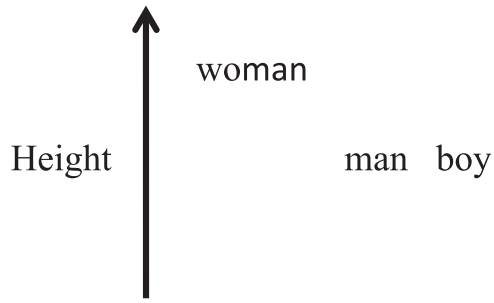
Table 3 presents the percentages of the three sorts of evaluation of *or*-deletions from the four sorts of disjunction: condensable and primordial disjunctions of narrow and wide scope. The participants were much more likely to accept *or*-deletions from condensable disjunctions than from primordial disjunctions, and to evaluate those from primordial disjunction as "impossible to determine". Out of the 112 participants, 80 fitted this pattern, 12 were inconsistent with it, and there were 20 ties (Binomial test, $p < .000005$). As the model theory predicts, the acceptances of *or*-deletions were reliably greater for condensable disjunctions (58%) than for primordial disjunctions (14%, Wilcoxon test, $z = 8.13$, $p < .001$, effect size: $r = 0.54$), and they were reliably greater, albeit only slightly so, from narrow-scope disjunctions (38%) than from wide-scope disjunctions (34%; Wilcoxon test, $z = 2.11$, $p < .035$, effect size, $r = 0.14$). But, the critical interaction was reliable: as Table 3 shows, the difference between narrow and wide scope disjunctions was greater for condensable disjunctions (14%) than for primordial disjunctions (1%; Wilcoxon test, $z = 1.70$, $p < .05$, effect size: $r = 0.11$). So, the results corroborated the model theory: condensable disjunctions tend to yield *or*-deletions as inferences of possibilities, and are more likely to do so with a narrow scope than with a wide scope, whereas primordial disjunctions seldom yield such inferences, and scope has no reliable effect on their occurrence.

Experiment 4: *or*-deletions from comparatives disjunctions

A disjunction can occur as a constituent of a ternary relation concerning a property that three entities have in common to varying degrees, e.g.:

49. The woman is taller than the man or the boy.

The representation of the meaning of this "regular" disjunction, which leaves open whether it is inclusive or exclusive, reflects its narrow scope, and yields an intuitive model in which the height of the woman is greater than that of both of the others, whose heights the model does not distinguish. It condenses them:



Models can be modified as long as the result is consistent with the representation of the

Table 3. Experiment 3 ($N = 112$): the percentages of the three sorts of evaluation of or-deletions from condensable disjunctions (x used to do A or B) and from primordial disjunctions (e.g. x did A or B), depending on whether the disjunctions had a narrow or wide scope.

	Narrow scope "or"			Wide scope "or"		
Condensable disjunctions	*x used to do A or B*			*x used to do A or x used to do B*		
	Yes	No	Impossible to determine	Yes	No	Impossible to determine
Evaluations of conclusions: ∴ *x used to do__*	62	8	31	54	7	39
Primordial disjunctions	*x did A or B*			*x did A or x did B*		
	Yes	No	Impossible to determine	Yes	No	Impossible to determine
Evaluations: of conclusions: ∴ *x did__*	14	5	81	13	9	77

assertion's meaning, and no constraint exists on the relative heights of the man and boy as long as they are each less than that of the woman. The model, therefore, yields *or*-deletions as possible inferences:

50. ∴ The woman is taller than the man.
    ∴ The woman is taller than the boy.

Deliberation, however, can construct a wide-scope representation of meaning, as from:

51. The woman is taller than the man or the woman is taller than the boy.

which is more likely to yield two models of the alternative possibilities, and thus to resist *or*-deletions. Another independent factor is that condensation should be more likely for a regular disjunction, which allows an inclusive interpretation than for an explicit exclusive disjunction, such as:

52. The woman is taller than the man or the boy, but not both.

The attempt to construct a single model is liable to halt at the point in which it cannot accommodate the two mutually exclusive objects of the relation. If it represents the woman as taller than the man, then she cannot be taller than the boy, and vice versa. So, the process has to construct two models. The two independent factors should therefore enhance *or*-deletions in such ternary comparisons: a regular disjunction as opposed to an exclusive disjunction, and a narrow scope as opposed to a wide scope. The experiment examined these two predictions.

Participants

The experiment tested 83 new participants from the same population as before (68 female, 15 male, and 0 other). Their mean age was 20.1 years ($SD = 5.9$).

Design

The participants acted as their own controls and evaluated *or*-deletions from ternary relations in disjunctions that were regular ("or") or exclusive ("but not both") crossed with scopes that were narrow or wide. Because exclusive disjunctions have more words than regular disjunctions, and wide scope disjunctions have more words than narrow-scope disjunctions, we added further inessential words to ensure that in Portuguese the four sorts of disjunction had balanced numbers of words, e.g.:

- Regular narrow-scope disjunction: The one woman in the management-course finalist group is thinner than the youngest man or the oldest boy.
- Regular wide-scope disjunction: The one man in the law firm is more honest than the intern girl or he is more honest than the woman secretary.
- Exclusive narrow-scope disjunction: The one woman in the Coimbra experimental theater is more creative than the man or than the boy, but not both.
- Exclusive wide-scope disjunction: The one man in the club is faster than the woman or he is faster than the girl, but not both.

There were two separate groups in order to counterbalance which individual in the disjunctive predicate occurred in the conclusion, and to vary the assignments of materials to the different sorts of inference. Each participant carried out three trials with different contents of the four sorts of inference, and six filler assertions to vary the nature of the comparatives. They carried out the 18 inferences in a different random order.

Materials and procedure

As the examples above illustrate, we devised comparative assertions relating persons referred to in

definite descriptions, and using additional words in some conditions to ensure that each of the four sorts of disjunction had 25 or 26 words in Portuguese. The 18 inferences counterbalanced the relations so that half referred to a physical relations, e.g. *thinner,* and *taller*, and half referred to personality, e.g. *honest*, and *creative*. To ensure that not all the disjunctions made positive comparisons, four of the six filler items used a "less than" comparison, two with simple binary comparisons and two with regular narrow-scope disjunctions, and the remaining two fillers were simple binary relations. The full set of contents in their original Portuguese and in their translations into English are shown with their results in Appendix D in the Supplemental materials. The procedure was identical to the one in the previous experiments.

Results

Table 4 presents the percentages of the three sorts of evaluations (Yes, No, and Impossible to determine) for *or*-deletions from regular disjunctions and exclusive disjunctions, depending on whether the disjunctions had a narrow or wide scope. As the model theory predicts, the percentages of *or*-deletions were reliably greater for regular disjunctions (50%) than for exclusive disjunctions (26%, Wilcoxon test, $z = 5.46$, $p < .001$, effect size: $r = .77$), and reliably greater for narrow-scope disjunctions (43%) than for wide-scope disjunctions (34%; Wilcoxon test, $z = 3.4$, $p < .001$, effect size, $r = .28$). The model theory has no grounds to predict an interaction between these two variables, and it was not significant: for regular disjunctions, the difference between narrow scope and wide scope was 10%, and for exclusive disjunctions, it was 8% (Wilcoxon test, $z = .613$, $p < .5$, effect size: $r = .08$). Overall, the results bore out the model theory's predictions.

General discussion

The four experiments examined certain inferences from disjunctions that are paradoxical in standard logics, but consequences of the model theory. The original "paradoxes" of free-choice permission are inferences that in effect delete "or" from a disjunction of deontic possibilities, i.e. permissions such as:

53. You can have a dog or a cat.
    ∴ You can have a dog.
    ∴ You can have a cat.

These *or*-deletions are invalid in standard logics (see Kamp, 1973), which defines the meaning of disjunctions with truth functions. So, an inclusive disjunction (A or B or both) is true provided that at least one of its two clauses is true. A corollary is that an inference of a sort known as *or*-introduction is valid, e.g.:

54. ∴ You can have a dog.
    ∴ You can have a dog or a hippopotamus, or both. [*or*-introduction]

If the premise is true, then so is the disjunctive conclusion, because its first clause is true. Yet, even proponents of formal rules of inference have misgivings about this sort of inference. In the model theory, the premise establishes only one of the possibilities in the conjunction of defaults to which the disjunction refers. The other default possibility—that you can have a hippopotamus—is not even mentioned in the premise, and so the alethic status of the disjunctive conclusion is no more than a possibility. In standard logics, the combination of a formal rule for *or*-deletion with one for *or*-introduction, as Kamp (1973) showed, is disastrous. It allows any conclusion to be deduced from any premise, e.g.:

55. You can have a dog.
    ∴ You can have a dog or a hippopotamus, or both. [rule for *or*-introduction]

Table 4. Experiment 4 ($N = 83$): The percentages of the three evaluations of or-deletions from comparatives between one individual and two others in regular or exclusive disjunctions with a narrow or wide scope. Conclusions referred to b or to c on equal numbers of trials.

	Narrow scope "or": *a is more Y than b or c. ∴ a is more Y than _*			Wide scope "or": *a is more Y than b or a is more Y than c. ∴ a is more Y than _*		
	Yes	No	Impossible to determine	Yes	No	Impossible to determine
Regular disjunction: "or"	55	9	36	45	3	51
Exclusive disjunction: "or … but not both""	30	6	64	22	8	70

∴ You can have a hippopotamus. [rule for *or*-deletion]

One way to maintain standard logics, however, is to postulate pragmatic principles that enable *or*-deletions to be drawn as "conversational implicatures", which can be withdrawn without contradiction, and further pragmatic principles that suppress *or*-introductions. But, here's the rub. No principles in standard logics call for the withdrawal of valid deductions. It cannot be done, not even from a true denial of the conclusion. It contradicts the conclusion, but, as readers should recall, in standard logics contradictions yield any conclusion whatsoever as a valid deduction.

In contrast, the model theory's semantics is based on principles derived independently from *or*-deletion: a disjunction refers to a conjunction of core possibilities that hold in default of knowledge to the contrary. As Grice (1989, p. 68) allowed:

> A standard (if not *the* standard) employment of "or" is in the specification of possibilities (one of which is supposed by the speaker to be realised, though he does not know which one) each of which is relevant in the same way to a given topic.

This anticipation of the model theory describes in parentheses primordial disjunctions, and quite why Grice did not introduce possibilities into his theory is a mystery. Knowledge, context, and the meaning of other aspects of assertions, can modulate the core possibilities of disjunctions so that they are interpreted as alethic, deontic, or epistemic (see the account above of the model theory). These meanings, in turn, have intuitive models of each possibility representing only those clauses in the premises that are true, and they also have deliberative models of each possibility representing in addition clauses that are false. In principle, deliberative models are normative in that they can establish conclusions that follow of necessity from premises. The theory explains the familiar paradoxes, and it also predicts new sorts of *or*-deletion. Our previous paper and the present investigations made the first direct experimental tests of *or*-deletions (for a review of earlier experimental investigations of their indirect consequences, see Johnson-Laird, Quelhas, et al., 2021). The question at issue is whether the model theory gives a better account than the pragmatic theories proposed over the years.

These theories began with Grice's "conversational implicatures" from assertions as a result of conventions governing discourse (e.g. Grice, 1989). They led to analogous neo-Gricean accounts (e.g. Kratzer & Shimoyama, 2002), and then to post-Gricean accounts to explain implicatures from the constitutents of assertions (e.g. Fox, 2007; Bar-Lev & Fox, 2020). However, these theories are all silent about *or*-deletions from metadisjunctions, such as:

56. Imagine that your landlord gave you are permission for only one of the following:
    You can have a cat.
    You can have a dog.
Are you permitted to have a cat?

Gricean and neo-Gricean theories explain inferences from single assertions, and post-Gricean theories, which use grammar to insert an operator akin to "only" into sentences, likewise cannot explain inferences that depend on three separate assertions. Bar-Lev (p.c., 1/17/21) wrote that post-Gricean theory is silent about metadisjunctions, but he did not believe that they applied to permissions. Yet, nearly everyone (in Experiment 1) responded, "Yes", to inferences similar to example (57). The intuitive models of the metadisjunction represent a conjunction of two permissible pets:

cat
dog
…

where the ellipsis represents that you are also permitted to have neither pet. These models also predict that individuals reject the inference of a conclusion concerning an obligation, say, to have a cat. Experiment 1 corroborated these predictions. Post-Griceans might argue that metadisjunctions cannot be used in speech acts that create permissions. The model theory has no such impediment, and so a speaker might give you permission in this way:

57. You have my permission for one of the following:
    You can have a dog.
    You can have a cat.

It follows that you can have a dog (and not a cat), and that you can have a cat (and not a dog).

Turning from disjunctions of explicit possibilities, the model theory draws a fundamental distinction between *primordial* disjunctions, which imply that one outcome is bound to occur (see the quotation above from Grice, 1989), and which therefore

prevent *or*-deletions, and *condensable* disjunctions, which do not imply that any outcome will occur, and which therefore allow disjunctive possibilities to be condensed into one. They yield *or*-deletions, though only as inferences of possibilities. Examples include disjunctions concerning a plurality, as in this inference of *or*-deletion (see Experiment 2):

58. On weekdays, Maria hates to wear a dress or a suit.
    ∴ On weekdays, Maria hates to wear a suit.

Deliberative models show these inferences to be possible, but not necessary. In contrast, a primordial disjunction tends not have an intuitive model that condenses the two possibilities into one, and so people tend to reject *or*-deletions, such as:

59. ∴ Yesterday, Maria wore a dress or a suit.
    ∴ Yesterday, Maria wore a suit.

The scope of a condensable disjunction has a reliable effect: those with a narrow scope are more likely to yield *or*-deletions than those with a wide scope (Experiment 3), such as:

60. On weekdays, Maria hates to wear a blouse or she hates to wear a suit.

The process of building models from a narrow-scope disjunction starts with a single model, and maintains it in the case of a condensation, whereas it constructs one model from the first clause of a wide-scope disjunction, and tends to construct an alternative model from the second clause.

Primordial disjunctions seldom yield *or*-deletions whether scope is narrow or wide.

Comparatives include another sort of condensable disjunction that yields an intuitive *or*-deletion, which deliberation can overrule. Individuals are more likely to accept *or*-deletion from a narrow-scope regular disjunction (Experiment 4):

61. The one woman in the Coimbra experimental theater is more creative than the man or the boy.

than from a wide-scope exclusive disjunction:

62. The one woman is more creative than the man or she is more creative than the boy, but not both.

The latter is likely to impede the condensation of the two possibilities into one, and instead to elicit models of distinct possibilities, which inhibit *or*-deletions. The explanation in terms of the processes of transforming a representation of the meanings of assertions into models can be found in the introduction to Experiment 4.

Readers may have difficulty in keeping track of all our findings and of their implications for pragmatic theories. Table 5, therefore, summarizes the main results from both of our articles, their status in the model theory—whether they are necessary inferences or only inferences of possibilities, and their predictability from pragmatic theories. The findings corroborate the model theory and reveal two principal phenomena. First, individuals accept necessary *or*-deletions from disjunctions of explicit modals, such as permissions or epistemic possibilities, and tend not to be much affected by whether "or" is exclusive or inclusive, whether it has a wide or narrow scope, or whether it is in a disjunction or a metadisjunction. Second, disjunctions that are not primordial refer, especially with a narrow scope, to condensable possibilities yielding *or*-deletion conclusions as possibilities.

Grice (1989) did not consider *or*-deletions, but Gricean theories can explain free-choice permissions and epistemic *or*-deletions. But, they cannot account for judgements of the truth of conditionals that embody *or*-deletions or their inference from metadisjunctions. Neo-Gricean theory can explain the former but it is silent about *or*-deletions from metadisjunctions, and it predicts—wrongly in some cases—that *or*-deletions from disjunctions of deontic or epistemic possibilities should occur from those with narrow scopes more often than from those with wide scopes. The model theory is correct in that scope has neglible effect on inferences that are necessary in the model theory, but wide scope does inhibit inferences that are only of possibilities—an emergent property from the construction of models.

Could a new sort of pragmatic theory deliver predictions akin to those of the model theory from a semantics based on truth functions? Perhaps. Yet, this semantics faces other problems. For example, as we mentioned earlier, inferences of *or*-introduction are valid in all standard logics but people tend to reject them. But, as the model theory predicts, they do accept them if the premise implies the new clause in the disjunctive conclusion (Orenes & Johnson-Laird, 2012), e.g.:

Table 5. Five earlier experiments on or-deletions listed with Roman numerals and the four present experiments listed with Arabic numerals, their key inferences, their alethic status in the model theory, and their predictability from pragmatic theories. The symbol "◇" denotes "possibly", "xor" denotes an exclusive disjunction, and "or" denotes a regular disjunction or an inclusive disjunction.

Sort of inferences	Key example of findings	Alethic status of inference in model theory	Predicatability from pragmatic theories
I. Deontic	Narrow scope only slightly more effective than wide for: ◇(*A or B*) ∴ ◇*A*	Necessary	Predictable
II. Epistemic	No reliable effect of scope: ◇(*A or B*) ∴ ◇*A*	Necessary	Wide scope results contrary to post-Gricean theory
III. Truth or falsity of epistemic conditionals	*This conditional judged true: If* ◇(*A or B*) *then* ◇*A*.	Necessary	Not predictable from Gricean theory. Predictable from post-Gricean theory
IV. Quantified disjunctions	*Few of the x did A xor B.* ∴ *Few of the x did A.*	Possible	Contrary to post-Gricean theory
V. Epistemic meta-disjunctions	*One is true & one is false*: ◇*A*. ◇*B*. ∴◇*A*:	Necessary	Not predictable from Gricean or post-Gricean theory
1. Deontic meta-disjunctions	*Permitted to do only one*: *You can do A. You can do B.* ∴ *You are permitted to do A.*	Necessary	Not predictable from Gricean or post-Gricean theory
2. General infinitivals	*x used to do A or B.* ∴ *x used to do A.*	Possible	Unknown
3. General infinitivals	Narrow scope yields more *or*-deletions than wide.	Possible	Unknown
4. Ternary relations	*a is more Y than b or c.* ∴ *a is more Y than b.* Narrow scope yields more, and *or* yields more than *xor*.	Possible	Unknown

Note: Experiments I-IV are from Johnson-Laird, Quelhas, et al., 2021; Experiment V is from Sklarek et al., 2022; and Experiments 1-4 are from the present article.

63. ∴ Lucia wore the bracelet.
∴ Lucia wore the bracelet or she wore jewelry.

Another problem for truth functions is that disjunctions can run in parallel to conditionals, and just as there are counterfactual conditionals so, too, there are counterfactual disjunctions (Byrne & Johnson-Laird, 2019):

64. Fred didn't do the burglary.

He would have got into the house by the back door or the kitchen window.

In this case, Fred did not get in by the back door, and he did not get in by the kitchen. Both clauses of the disjunction are therefore false, and so its truth-functional semantics guarantees that it is false too. Yet, the disjunction could well be true, and the model theory explains how counterfactual possibilities can be verified (Byrne & Johnson-Laird, 2019; Byrne et al., 2022). As the model theory predicts, people draw some necessary inferences that

are invalid in all standard logics, and they reject some inferences that are valid in all of them.

Conclusions

The paradoxes of free choice permission and other *or*-deletions are incompatible with all standard modal logics, which treat "or" as having meanings that are truth functions. An index of the importance of these inferences is the vast literature designed to explain them without sacrificing truth functions. In contrast, the theory of mental models was proposed, not to account for the paradoxes, but to explain reasoning of other sorts. It led to the discovery of phenomena contrary to standard logics, and it embodies a semantics of connectives that is based on possibilities. It postulates that disjunctions refer to conjunctions of possibilities that hold in default of knowledge to the contrary. Intuitive models represent the possibilities in which the premise's clauses are true, but deliberative models also represent its clauses that are false. One of the theory's side-effects is an explanation of *or*-deletions. It makes crucial predictions about new sorts of *or*-deletions, which our experiments corroborated. In sum, primordial disjunctions impede *or*-deletions, whereas disjunctions of explicit possibilities yield those that follow as necessary, and condensable disjunctions yield those that follow as possible.

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No potential conflict of interest was reported by the author(s).

Data availability and deposition

The supplemental materials for the four experiments and their datasets are available at: https://osf.io/ajsrv/.

References

Bar-Lev, M. E., & Fox, D. (2020). Free choice, simplification, and innocent inclusion. *Natural Language Semantics*, *28* (3), 1–49. https://doi.org/10.1007/s11050-020-09162-y

Barrouillet, P., Grosset, N., & Lecas, J. F. (2000). Conditional reasoning by mental models: Chronometric and developmental evidence. *Cognition*, *75*(3), 237–266. https://doi.org/10.1016/S0010-0277(00)00066-4

Byrne, R. M. J., & Johnson-Laird, P. N. (2019). If and or: Real and counterfactual possibilities in their truth and probability. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *46*(4), 760–780. https://doi.org/10.1037/xlm0000756

Byrne, R. M. J., Khemlani, S., & Johnson-Laird, P. N. (2022). Mental simulations of counterfactual possibilities bias judgments of truth and falsity (Under submission).

Cohen, L. J. (1971). The logical particles of natural language. In Y. Bar-Hillel (Ed.), *Pragmatics of natural language* (pp. 50–68). Reidel.

Fox, D. (2007). Free choice and the theory of scalar implicatures. In U. Sauerland, & P. Stateva (Eds.), *Presupposition and implicature in compositional semantics* (pp. 71–120). Palgrave Macmillan.

Gazdar, G. (1979). *Pragmatics: Implicature, presupposition, and logical form*. Academic Press.

Geurts, B. (2005). Entertaining alternatives: Disjunctions as models. *Natural Language Semantics*, *13*(4), 383–410. https://doi.org/10.1007/s11050-005-2052-4

Grice, H. P. (1989). *Studies in the way of words*. Harvard University Press.

Hinterecker, T., Knauff, M., & Johnson-Laird, P. N. (2016). Modality, probability, and mental models. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *42*(10), 1606–1620. https://doi.org/10.1037/xlm0000255

Jeffrey, R. (1981). *Formal logic: Its scope and limits* (2nd ed.). McGraw-Hill.

Johnson-Laird, P. N. (1978). The meaning of modality. *Cognitive Science*, *2*(1), 17–26. https://doi.org/10.1016/S0364-0213(78)80059-7

Johnson-Laird, P. N. (1983). *Mental models*. Harvard University Press.

Johnson-Laird, P. N., Bucciarelli, M., Mackiewicz, R., & Khemlani, S. S. (2021). Recursion in programs, thought, and language. *Psychonomic Bulletin and Review*, published online. https://doi.org/10.3758/s13423-021-01977-y.

Johnson-Laird, P. N., & Byrne, R. M. J. (1991). *Deduction*. Lawrence Erlbaum Associates.

Johnson-Laird, P. N., Byrne, R. M. J., & Schaeken, W. S. (1992). Propositional reasoning by model. *Psychological Review*, *99*(3), 418–439. https://doi.org/10.1037/0033295X.99.3.418

Johnson-Laird, P. N., Girotto, V., & Legrenzi, P. (2004). Reasoning from inconsistency to consistency. *Psychological Review*, *111*(3), 640–661. https://doi.org/10.1037/0033-295X.111.3.640

Johnson-Laird, P. N., & Khemlani, S. (2020). Mental models and the algorithms of deduction. In R. Sun (Ed.), *Cambridge handbook of computational cognitive sciences* (pp. 339–359). Cambridge University Press.

Johnson-Laird, P. N., Legrenzi, P., Girotto, P., & Legrenzi, M. S. (2000). Illusions in reasoning about consistency. *Science*, *288*(5465), 531–532. https://doi.org/10.1126/science.288.5465.531

Johnson-Laird, P. N., Quelhas, A. C., & Rasga, C. (2021). The mental model theory of free choice permissions and paradoxical disjunctive inferences. *Journal of Cognitive Psychology*, *33*(8), 951–973. https://doi.org/10.1080/20445911.2021.1967963

Johnson-Laird, P. N., & Ragni, M. (2019). Possibilities as the foundation of reasoning. *Cognition*, *193*, 130950. https://doi.org/10.1016/j.cognition.2019.04.019

Kamp, H. (1973). Free choice permission. *Proceedings of the Aristotelian Society*, *74*(1), 57–74. https://doi.org/10.1093/aristotelian/74.1.57

Karttunen, L. (1972). Possible and must. In J.P. Kimball (Ed.), *Syntax and semantics* (Vol. 1, pp. 1-20). New York: Seminar Press.

Keene, G. B. (1992). *The foundations of rational argument*. Edward Mellen Press.

Khemlani, S., Orenes, I., & Johnson-Laird, P. N. (2014). The negations of conjunctions, conditionals, and disjunctions. *Acta Psychologica*, *151*, 1–7. https://doi.org/10.1016/j.actpsy.2014.05.004

Khemlani, S. S., Byrne, R. M. J., & Johnson-Laird, P. N. (2018). Facts and possibilities: A model-based theory of sentential reasoning. *Cognitive Science*, *42*(6), 1–38. https://doi.org/10.1111/cogs.12634

Khemlani, S. S., & Johnson-Laird, P. N. (2017). Illusions in reasoning. *Minds and Machines*, *27*(1), 11–35. https://doi.org/10.1007/s11023-017-9421-x

Khemlani, S. S., & Johnson-Laird, P. N. (2021). Reasoning from quantified assertions: A computational theory. *Psychological Review*, Advance online publication https://doi.org/10.1037/rev0000240.

Klinedinst, N. W. (2007). Plurals, possibilities, and conjunctive disjunction. *UCL Working Papers in Linguistics*, *19*, 261–284.

Koralus, P., & Mascarenhas, S. (2013). The erotetic theory of reasoning: Bridges between formal semantics and the psychology of deductive inference. *Philosophical Perspectives*, *27*(1), 312–365. https://doi.org/10.1111/phpe.12029

Kratzer, A. (1977). What 'must' and 'can' must and can mean. *Linguistics and Philosophy*, *1*(3), 337–355. https://doi.org/10.1007/BF00353453

Kratzer, A. & Shimoyama, J. (2002). Indeterminate pronouns: The view from Japanese. In Y. Otsu (Ed.), *Third Tokyo conference on psycholinguistics* (pp. 1–25). Hituzi Syobo.

Kripke, S. (1963). Semantical considerations on modal logic. *Acta Philosophica Fennica*, *16*, 83–94.

Levinson, S. C. (2000). *Presumptive meanings: The theory of generalized conversational implicature*. MIT Press.

Manktelow, K. (2021). *Beyond reasoning: The life, times and work of Peter Wason. Pioneering psychologist*. Routledge.

Marek, W., & Truszynski, M. (2013). *Nonmonotonic logic*. Springer.

Mauri, C. (2008). The irreality of alternatives: Towards a typology of disjunction. *Studies in Language*, *32*(1), 22–55. https://doi.org/10.1075/sl.32.1.03mau

Meyer, M-C. (2020). Free choice disjunction: 'An apple or a pear'. In *The Wiley Blackwell companion to semantics* (pp. 1–44). Wiley Online Library.

Montague, R. (1974). *Formal philosophy: Selected papers*. Yale University Press.

Orenes, I., & Johnson-Laird, P. N. (2012). Logic, models, and paradoxical inferences. *Mind & Language*, *27*(4), 357–377. https://doi.org/10.1111/j.1468-0017.2012.01448.x

Partee, B. H. (1979). Semantics – Mathematics or psychology? In R. Bäuerle, U. Egli, & A. von Stechow (Eds.), *Semantics from different points of view* (pp. 1–14). Springer-Verlag.

Quelhas, A. C., & Johnson-Laird, P. N. (2017). The modulation of disjunctive assertions. *Quarterly Journal of Experimental Psychology*, *70*(4), 703–715. https://doi.org/10.1080/17470218.2016.1154079

Ragni, M., & Johnson-Laird, P. N. (2020). Reasoning about epistemic possibilities. *Acta Psychologica*, *208*, 103081. http://doi.org/10.1016/j.actpsy.2020.103081

Rips, L. J. (1994). *The psychology of proof*. MIT Press.

Sklarek, B., Knauff, M., & Johnson-Laird, P. N. (2022). *Disjunctions, metadisjunctions, and models*.

Sperber, D., & Wilson, D. (1995). *Relevance: Communication and cognition* (2nd ed.). Blackwell.

Tarski, A. (1944). The semantic conception of truth. *Philosophy and Phenomenological Research*, *4*(3), 341–375. https://doi.org/10.2307/2102968

Wason, P. C., & Johnson-Laird, P. N. (1970). A conflict between selecting and evaluating information in an inferential task. *British Journal of Psychology*, *61*(4), 509–515. https://doi.org/10.1111/j.2044-8295.1970.tb01270.x

Zimmermann, T. E. (2000). Free choice disjunction and epistemic possibility. *Natural Language Semantics*, *8*(4), 255–290. https://doi.org/10.1023/A:1011255819284