COMMENT

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Comment on 'Information hidden in the velocity distribution of ions and the exact kinetic Bohm criterion'

M S Benilov

Departamento de Física, Faculdade de Ciências Exatas e da Engenharia, Universidade da Madeira, Largo do Município, 9000 Funchal, Portugal
Instituto de Plasmas e Fusão Nuclear, Instituto Superior Técnico, Universidade de Lisboa, 1041 Lisboa, Portugal

E-mail: benilov@uma.pt

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Abstract
Tsankov and Czarnetzki (2017 *Plasma Sources Sci. Technol.* 055003) showed that measurements with a wall-mounted plasma monitor can allow noninvasive access to spatially resolved plasma parameters, including the ion velocity distribution. On the basis of experimental data and numerical modeling, they proposed a new form of collisionally modified kinetic Bohm criterion. In this work, we point out that the kinetic Bohm criterion may be written in a form that is exactly equivalent to the original criterion but does not pose the much-debated problem of divergence for slow ions. This form enables one to show that the kinetic Bohm criterion, as well as the Bohm criterion for monoenergetic ions, has a distinct mathematical meaning. Collisionally modified Bohm criteria do not possess the same, or any other, mathematical meaning. Hence, there are no mathematical grounds for speaking of a collisionally-modified Bohm criterion and no non-arbitrary way to introduce it. Therefore, the use of the term 'exact kinetic Bohm criterion' in the paper by Tsankov and Czarnetzki is hardly justified.

Keywords: sheath, plasma-sheath transition, Bohm criterion

1. Introduction

In the recent paper by Tsankov and Czarnetzki [1], entitled 'Information hidden in the velocity distribution of ions and the exact kinetic Bohm criterion', it is shown that measurements with a wall-mounted plasma monitor, combined with numerical modeling, allow noninvasive access to spatially resolved plasma parameters, including the ion velocity distribution. The experimental data obtained are applied, jointly with a theory and a numerical model, to the problem of the kinetic Bohm criterion. An additional term is identified which accounts for collisions and geometry effects and restores the validity of the Bohm criterion under high collisionality. It is concluded that under real conditions the Bohm criterion is never fulfilled marginally (with an equality sign).

We do not discuss the experimental technique proposed or the experimental data obtained. Rather, the legitimacy of using the term 'exact kinetic Bohm criterion' in this context is analyzed.

2. 72 generalized Bohm criteria?

In his famous 1949 work [2], David Bohm came to the conclusion that 'a stable sheath is possible only when ions reach the sheath with a kinetic energy at least half the electron mean temperature'. This was a very important result, which has showed the way to match solutions describing a quasi-neutral plasma and a positive space-charge sheath in the case where ion-neutral collisions, ionization, and geometrical effects are negligible in the sheath.

In order for this result to be workable, one has to interpret the words 'reach the sheath'. In the work [1] and the subsequent discussion [3, 4], as well as in most other works concerned with plasma-sheath transition, the assumption is made that the Bohm criterion should be applied at a certain point in space; a 'sheath edge'. However, the procedure employed in Bohm's work [2] (the expansion of the sheath equation in powers of potential measured from the plasma potential, which amounts to investigation of asymptotic behavior of solutions to this equation at large distances from the surface) is of an asymptotic nature and does not involve any particular point separating the sheath and the plasma.[1] Thus, there is no unique (non-arbitrary) way to define a sheath edge.[2,3]

The authors [1] postulated that the sheath edge is a point where the mean ion speed equals the Bohm speed $v_s = \sqrt{kT_e/m_i}$ and formulated at this point a new version of the kinetic Bohm criterion, modified in order to take into account collisionality and geometrical effects. Many other such definitions are available in the literature; a search of the Web of Science with 'modified Bohm criterion' or 'generalized Bohm criterion' in the 'Topic' field returned a list of 72 papers (Oct. 12, 2018).[4] Without going into details, we mention two examples. The first is the paper by Godyak [7], which was published in 1982 and appears at the top of the list. The sheath edge was defined in [7] by the condition $E = kT_e/e\lambda_D$ (a definition which is still in use; e.g., [8, 9]), and a collisionally modified Bohm criterion was postulated at this point. The second example is [10], where the sheath edge (or 'collisionally modified Bohm point') was identified with a removable singularity in the sheath equation derived in that work and another version of a collisionally modified Bohm criterion was applied.

The fact that there are no widely accepted definitions of the sheath edge and collisionally modified Bohm criterion, in spite of the efforts made by able researchers over decades, shows that there are no objective criteria and that such definitions can be introduced only arbitrarily.

[1] One should note that Bohm [2] explicitly speaks of a 'sheath edge' while interpreting the mathematical results, and although he admits that it cannot be given a precise definition, he suggests that a sheath begins roughly when the ions reach the speed $v_s = \sqrt{kT_e/m_i}$. On the other hand, it has been pointed out in the past [5, 6] that the terminology of the work [2] is generally not too accurate. Citing Riemann [5]: 'More important than the intention of Bohm, however, is the common interpretation of his sheath solution. This interpretation is based on his algebra, and not on his "wordy" description.'

[2] It should be stressed that 'sheath edge' means in the present context a 'boundary' limiting a sheath containing both the ions and the electrons (i.e., a boundary between a quasi-neutral plasma and the sheath). It should not be confused with the edge limiting the ion sheath, which admits a meaningful and unambiguous definition (e.g., the discussion in [6]).

[3] Of course, one can define a sheath edge 'in a natural way' as a point where the charge separation equals 1% (or 5%, or 20%, etc), but any such definition will be arbitrary.

[4] Of course, not all the papers on the list give a new generalized or modified Bohm criterion; on the other hand, [1] is not on the list, for example. Nevertheless, the number of papers on the list gives some idea.

This fact should not have surprised workers who for many years steadfastly argued against the use of the concept of a sheath edge and collisionally modified Bohm criteria. Citing Riemann [11]: 'Various attempts to derive a 'generalized' Bohm criterion accounting for collisions are inconsistent'. A paper published in 2003 by Franklin [12] has an expressive title 'There is no such thing as a collisionally modified Bohm criterion'. However, their arguments have had little effect, and one of Franklin's most recent papers [13] is entitled 'The quest to find the plasma edge and discover a collisionally modified Bohm criterion' and likens the enterprise to a 'chimera of Homeric proportions'.

3. Mathematical meaning of the Bohm criterion

The arguments of Franklin and Riemann are based on asymptotic theory and may be summarized as follows. The concepts of a quasi-neutral plasma and a space-charge sheath are meaningful only if the characteristic Debye length λ_D is small compared to the scale L of the adjacent quasi-neutral plasma, or presheath. (Note that $L = \min(\lambda_s, d, l)$, where λ_s is the mean free path for ion-neutral collisions, d is the ionization length, and l is a characteristic geometrical dimension.) Consequently, plasma-sheath transition is a two-scale problem and the appropriate means of treating it is an asymptotic approach, considering $\varepsilon = \lambda_D/L$ as a small parameter and employing a singular perturbation formalism; see Caruso and Cavaliere [14]. The appropriate version of such formalism is the method of matched asymptotic expansions [15–20], which is a standard tool for solving problems involving layers of fast variation and is a powerful alternative to intuitive approaches. The method in a natural way reveals and exploits the underlying physics of layers of fast variations, such as viscous boundary layers in fluid mechanics, shock waves in gas dynamics, skin layers in electromagnetic theory, or, in this case, positive space-charge sheaths.

The method was applied to the case of a collision-dominated sheath in [21, 22] and to the case of a collisionless sheath, treated by Bohm, in [23]; refined treatments were given in [24, 25] and [26], respectively. It is important to stress that the results of [24–26] do not involve any 'sheath edge': in the framework of the method of matched asymptotic expansions, solutions describing adjacent zones dominated by different physics are matched asymptotically, rather than joined at one point, and therefore do not involve boundaries separating different zones. By means of this approach, the Bohm criterion has been confirmed in the equality form, which was first postulated by Allen and Thonemann [27]. It was shown that the Bohm criterion represents a manifestation of one of the general scenarios of asymptotic matching; namely, matching on a constant [28]. The asymptotic approach has allowed the generalizing of the Bohm criterion in a number of aspects (e.g., in review [29]). However, no generalization is possible for the case where ion-neutral collisions play a role in the sheath.

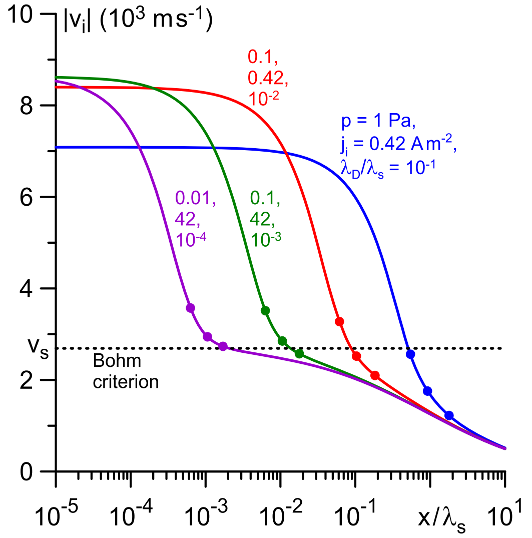

Figure 1. Distribution of ion speed near a floating wall in argon, fluid model. Circles on each curve represent points where the charge separation $(n_i - n_e)/n_i$ reaches, in the direction from the plasma to the wall, 0.1%, 1%, and 10%. Adapted from [28].

The method of matched asymptotic expansions is widely used in different areas of applied mathematics, mechanics, and physics; e.g., in books [15–20]. However, many workers interested in sheath physics view this method as a mathematical formality and there is confusion in the literature on sheath physics about some important aspects of the method; statements that the Bohm criterion is based on the asymptotic limit $\lambda_D/L = 0$ as opposite to $\lambda_D/L \ll 1$ and attempts to identify the sheath edge through a singularity in the quasi-neutral plasma solution are not rare. It is reasonable in such a situation to turn to experiment: physical, as in [1], or numerical, as in [10, 28]. In other words, let us suppose that one has access to distributions of plasma parameters in the near-surface region, obtained from an experiment of very high accuracy or a numerical solution of equations governing distributions of the positive ions and the electrons in the near-wall region jointly with the Poisson equation, computed without dividing the computation domain into a quasi-neutral plasma and a space-charge sheath. Is there a non-arbitrary way to identify in the distribution of a particular parameter (the mean speed of ions, or the mean energy of ions, or the electric field, or something else) a value separating the sheath from the quasi-neutral plasma?

An answer to this question is clear from figure 1, where distributions of ion speed in the region near a planar floating wall in argon plasma are shown for several combinations of the plasma pressure p and the ion current density to the wall j_i. The distributions have been computed numerically with the use of the well-known fluid model, which includes the ion conservation equation, the ion momentum equation written with account taken of the friction force due to collisions, the equilibrium equation for the electrons, and the Poisson equation. The electron temperature T_e in the simulations was assumed to be equal to 3 eV , which corresponds to the Bohm speed $v_s = 2.7 \times 10^3$ m s^{-1}; further details are described in [28] and are the same as in [10]. The ionization is neglected, so the presheath is represented by the Knudsen layer and the presheath scale L may be set as equal to λ_s, the characteristic ion mean free path (which was estimated for the ion speed $|v_i|$ equal to v_s). The distributions in figure 1 are plotted on the presheath scale, i.e., x, the distance from the wall, is normalized by λ_s. The parameter ε is defined as $\varepsilon = \lambda_D/\lambda_s$, where the Debye length is evaluated in terms of the charged particle density corresponding to the Bohm speed: $\lambda_D = (\varepsilon_0 k T_e v_s / e j_i)^{1/2}$. The variants shown in figure 1 are such that $\varepsilon \leqslant 10^{-1}$, i.e., the sheath is weakly collisional.

For each ε, one can see in figure 1 two regions of variation of the ion speed $v_i(x)$: x of the order λ_s and x of the order λ_D; the presheath (the Knudsen layer) and the sheath. Note that this characteristic two-scale structure is visible due to the usage of a logarithmic x-axis; an appropriate choice in multiscale problems. In the cases $\varepsilon = 10^{-4}$, 10^{-3}, the ion speed reveals a plateau in the intermediate region between the sheath and the presheath, $\lambda_D \ll x \ll \lambda_s$: the reduction of ion speed from v_s by 30% occurs as x increases by factors of 83 for $\varepsilon = 10^{-4}$ and 13 for $\varepsilon = 10^{-3}$. The plateau becomes less pronounced with the increase of ε, i.e., as ion-neutral collisions come into play, and disappear for $\varepsilon \gtrsim 10^{-2}$. (A change in slope around the Bohm speed is still visible in the case $\varepsilon = 10^{-2}$, at the beginning of the formation of the plateau.)

Thus, if collisions in the sheath are negligible, which means that the ratio λ_D/λ_s is sufficiently small, of the order of 10^{-3} or smaller, the ion speed in the intermediate region between the sheath and the presheath varies little and is approximately equal to the Bohm speed; this is the speed with which the ions 'reach the sheath'. This is the bottom *mathematical meaning of the Bohm criterion.* Any relationship that does not reflect this mathematical meaning can hardly be termed Bohm criterion, even if adjectives such as 'modified' or 'generalized' are added.

The Bohm criterion is satisfied asymptotically, rather than at just one point is space. Hence, there are no mathematical grounds for speaking of a sheath edge and no non-arbitrary way to define it.

If collisions in the sheath are non-negligible, there is no plateau in the ion speed distribution in the intermediate region and, consequently, no sense in talking of a definite speed at which ions enter the sheath. Hence, there are no mathematical grounds for speaking of a collisionally-modified Bohm criterion and no non-arbitrary way to introduce it.

4. Mathematical meaning of the kinetic Bohm criterion

The kinetic Bohm criterion was formulated by Harrison and Thompson in 1959 [30] and requires that the mean inverse kinetic energy (or, more precisely, the kinetic energy of motion in the direction to the surface) of the ions entering the sheath be smaller than or equal to $(kT_e/2)^{-1}$:

$$\frac{1}{n_i}\int_{-\infty}^{0}\frac{2}{mv_i^2}f_s(v_i)\,dv_i \leqslant \frac{2}{kT_e}, \qquad (1)$$

where v_i is the x-component of the particle velocity of the

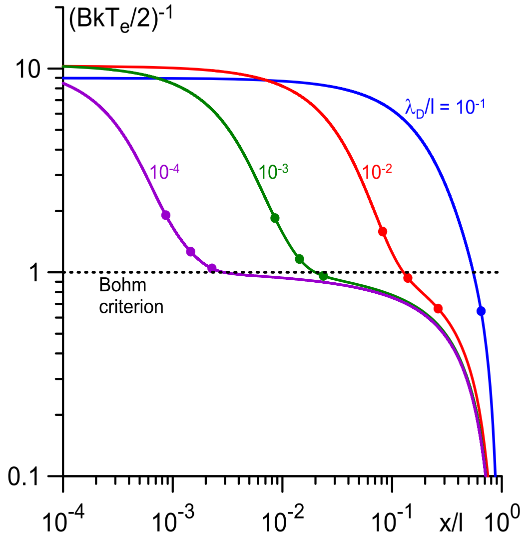

Figure 2. Distribution of the inverse of the normalized weighted inverse ion kinetic energy. Tonks–Langmuir kinetic model.

ions, the axis x is directed from the surface into the plasma, and $f_s = f_s(v_i)$ is the distribution function of the ions entering the sheath. (More precisely, $f_s(v_i)$ is the multiplier that describes a dependence of the distribution function on v_i; dependency on y- and z-components of the ion particle velocity is supposed to be described by a normalized factor.)

The kinetic Bohm criterion (1) may be derived by means of investigation of asymptotic behaviour at large distances from the surface of solutions of the Poisson equation in the sheath, i.e., in the same way in which the original Bohm criterion was derived in [2], provided that the ions entering the sheath are not assumed to be monoenergetic (all having the same velocity) as in [2], but rather are characterized by a velocity distribution. This was shown by Riemann's derivation [29, section 3.2]; a more straightforward version of this derivation is given in [31].

Similarly to the original Bohm derivation, the kinetic derivations in section 3.2 of [29] and in [31] refer to the case where ion-neutral collisions, ionization, and geometrical effects in the sheath are negligible, i.e., to the first approximation in the small parameter λ_D/L. Since cold ions cannot leave a one-dimensional, collisionless, and ionization-free sheath, $f_s(v_i) = 0$ for $v_i > 0$. Moreover, it is explicitly shown that the equality $f_s(0) = 0$, which is an obvious necessary condition for the integral in the kinetic Bohm criterion (1) to converge, is also a necessary condition for the sheath equations to admit solutions with a monotonic (non-oscillating) potential distribution.

Thus, the kinetic Bohm criterion (1) represents an inherent property of collision- and ionization-free one-dimensional sheaths. The words 'collision- and ionization-free' are very important: if ion-neutral collisions and/or ionization play a non-negligible role in the sheath, then the asymptotic behavior of the sheath electric field at large distances from the surface will be different and no analogue of the Bohm criterion will exist.

The authors [1] found that the ion distribution function does not vanish at zero velocity and concluded that the kinetic Bohm criterion must be remedied by careful treatment of the full Boltzmann equation. Note that the latter conclusion is a continuation of an old discussion, e.g., [32–36] and references therein. However, the inference that the kinetic Bohm criterion requires a remedy since the measured ion distribution function does not vanish at zero velocity is a misunderstanding. This misunderstanding once again stems from a failure to recognize the asymptotic nature of the Bohm criterion: the distribution function f_s of the ions entering the sheath in the kinetic Bohm criterion (1) is the one *evaluated in the quasi-neutral* approximation, rather than the exact one. The former function satisfies the condition $f_s(v_i) = 0$ for $v_i \geqslant 0$, which follows from matching with the first-approximation sheath solution, and satisfies the kinetic Bohm criterion (1) with the equality sign [29], section 3.3, and [34], Appendix.

Since $f_s(0) = 0$, the inequality (1) may be rewritten as

$$\frac{1}{n_i}\int_{-\infty}^{0}\frac{2}{mv_i^2}[f_s(v_i) - f_s(0)]\,dv_i \leqslant \frac{2}{kT_e}. \qquad (2)$$

This form of the kinetic Bohm criterion does not rely on any additional assumptions; we stress once again that $f_s(0) = 0$ is a general feature of the distribution function of ions entering the sheath, evaluated in the quasi-neutral approximation, which appears in the classic kinetic Bohm criterion (1). Therefore, the kinetic Bohm criterion written in the form (2) is exactly equivalent to the classic form (1).

On the other hand, the lhs of the kinetic Bohm criterion in the form (2) may be evaluated not only for an ion distribution function evaluated in the quasi-neutral approximation, $f_s(v_i)$, but also for an 'exact' distribution function $f(v_i)$, obtained from an experiment of very high accuracy or a numerical solution of a system of equations including the kinetic equation for the ions and the Poisson equation, computed without dividing the computation domain into a quasi-neutral plasma and a space-charge sheath. (For simplicity, we restrict the consideration with the usual case where the spectrum of ion velocities in the direction to the surface is limited, i.e., $f(v_i) = 0$ for $v_i < -v_{\max}$, where $v_{\max}$ is a maximum speed, and the ion energy distribution is analytic at low energies, which requires that $\partial f/\partial v_i(0) = 0$.) Moreover, the lhs of (2) may be evaluated not only for ions entering the sheath, but for arbitrary x.

Thus, one can introduce the quantity

$$B = \frac{1}{n_i}\int_{-\infty}^{0}\frac{2}{mv_i^2}[f(v_i) - f(0)]\,dv_i, \qquad (3)$$

which may be roughly termed the weighted inverse kinetic energy with which the ions move in the direction to the wall. If the ratio λ_D/L is small enough, the exact distribution function $f(v_i)$ in the region between the presheath and the sheath, $\lambda_D \ll x \ll L$, will be close to the quasi-neutral function f_s, and in particular, $f(0)$ will be small; hence B will be close to the integral on the lhs of (1) and, consequently, approximately equal to $(kT_e/2)^{-1}$. Hence, the inequality

$$B \leqslant (kT_e/2)^{-1} \qquad (4)$$

correctly reproduces the asymptotic nature of the kinetic

Bohm criterion and should be suitable for identifying this criterion in numerical or experimental results.

Figure 2 illustrates the application of the criterion (4) to the Tonks–Langmuir problem, which has served as a test case for studies of the kinetic Bohm criterion since the work of Harrison and Thompson [30]. The presheath is represented by the whole of the discharge column and the presheath scale L may be set equal to l, the half-width of the column. Therefore, x, the distance from the wall in figure 2 is normalized by l (thus $x/l = 1$ corresponds to the plane of symmetry of the discharge column) and ε is defined as $\varepsilon = \lambda_D/l$, where the Debye length λ_D is evaluated at the plane of symmetry. As in figure 1, the variants shown in figure 2 are such that $\varepsilon \leqslant 10^{-1}$, i.e., the ionization in the sheath is a weak effect. For each ε, B was evaluated in the whole discharge column by means of exact numerical solution of the Tonks–Langmuir problem (including the Poisson equation). Plotted in figure 2 is the inverse of the normalized weighted inverse kinetic energy, in order to facilitate a comparison with figure 1. The circles on the curves have the same meaning as in figure 1; for $\varepsilon = 0.1$, the charge separation at the plane of symmetry is about 4% and the only point shown is the one marking the charge separation of 10%.

As expected, figure 2 is quite similar to figure 1. In the cases $\varepsilon = 10^{-4}$ and $\varepsilon = 10^{-3}$, B^{-1} reveals a plateau in the intermediate region between the sheath and the presheath: the reduction of B^{-1} by 30% from the Bohm value of $kT_e/2$ occurs as x increases by factors of 46 and 7.5, respectively.

Thus, if ionization in the sheath is negligible, which means that the ratio λ_D/l should be of the order of 10^{-3} or smaller, the weighted inverse ion kinetic energy B is approximately constant in the region $\lambda_D \ll x \ll L$. This is similar to what happens to the ion speed in the case of monoenergetic ions, as seen in figure 1, and the numerical value of B is approximately equal to $(kT_e/2)^{-1}$; this is the weighted inverse kinetic energy with which the ions 'reach the sheath'. This is the bottom *mathematical meaning of the kinetic Bohm criterion.*

It should be stressed once again that under the conditions of the first approximation in the method of matched asymptotic expansions, which ensure the validity of the kinetic Bohm criterion (1), the latter is exactly equivalent to (2). Therefore, researchers familiar with this method will immediately recognize the validity of the criterion (2). Other researchers, including those who do not recognize the validity of the classic kinetic Bohm criterion (1) in the first place, are encouraged to consider figure 2, which demonstrates by the example of the Tonks–Langmuir problem the validity of this criterion in the form (4) for a sufficiently small value of λ_D/L (and, for that matter, provides compelling evidence of the asymptotic nature of the kinetic Bohm criterion).

Some researchers will probably criticize the kinetic Bohm criterion in the form (4) for its lack of agreement with a particular physical and/or numerical experiment. These researchers are encouraged to evaluate the λ_D/L ratio under the conditions of this experiment; most probably it is not small enough to justify application of the Bohm criterion in any form.

5. Conclusions

The kinetic Bohm criterion (1) may be written in the form (2), which is exactly equivalent to (1) but does not pose the much-debated problem of divergence for slow ions when applied to an 'exact' distribution function.

This new form enables one to show that the kinetic Bohm criterion has a distinct mathematical meaning and this meaning is the same as that of the Bohm criterion for monoenergetic ions: it is an asymptotic feature which manifests itself as a plateau in the distributions of the ion speed or weighted inverse kinetic energy of the ions in the intermediate region between the space-charge sheath and the presheath; see figures 1 and 2. However, this feature is manifested only for very low values of the ratio λ_D/L, of the order of 10^{-3} or smaller. For this reason, the Bohm criterion is hardly relevant for many conditions of practical interest.

Therefore, attempts to postulate a collisionally modified Bohm criterion are understandable, useful, as theorizing always is, and will probably continue in the future. However, collisionally modified Bohm criteria do not possess the mathematical meaning of the Bohm criterion. There is simply no sense in speaking of a definite speed or weighted inverse kinetic energy at which the ions enter the sheath if ion-neutral collisions or ionization in the sheath are non-negligible; see the lines referring to $\lambda_D/L \geqslant 10^{-2}$ in figures 1 and 2. The term 'Bohm criterion' is hardly justified in this context, even if adjectives such as 'modified' or 'generalized' are added, and is better avoided. Using Brinkmann's words [37]: 'there is nothing wrong with collisionally modified Bohm criteria; you only need to give a clear definition and choose a suitable name.'

The criterion (2) is not the only relationship which is equivalent to the classic kinetic Bohm criterion (1) while eliminating the divergence at $v_i = 0$. All such relationships are close to each other and the classic kinetic Bohm criterion (1) in the intermediate region between the sheath and the presheath.

It is interesting to note that there have been virtually no misunderstandings concerning plasma-sheath transition in the case of collision-dominated sheaths [21, 22, 24, 25, 38], where the charged particle densities and speeds in the plasma and the sheath are matched on algebraic functions rather than constants. Apparently, the confusion around the Bohm criterion—paradoxically—stems from its clear physical meaning: researchers with a powerful physical intuition tend to view it as a real physical feature, while in reality it is an asymptotic feature.

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ORCID iDs

M S Benilov https://orcid.org/0000-0001-9059-1948

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