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Space-charge sheath with ions accelerated into the plasma

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Abstract
The conventional model of near-cathode space-charge sheath with ions entering the sheath from the quasi-neutral plasma may not be applicable to discharges burning in cathode vapour, e.g. vacuum arcs, where ionization of emitted atoms may occur inside the sheath with some of the produced ions returning to the cathode and others moving into the plasma. In this connection, a simple model is considered of a sheath formed by electrons and positive ions injected into the sheath with a very low velocity and moving from the sheath into the plasma. It is shown that such a sheath is possible provided that the sheath voltage is equal to or exceeds approximately $1.256kT_e/e$. This limitation is due to the space charge in the sheath and is in this sense analogous to the limitation of ion current in a vacuum diode expressed by the Child–Langmuir law. The ions leave the sheath and enter the plasma with a velocity equal to or exceeding approximately $1.585u_B$.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The model of the space-charge sheath formed by cold ions and electrons at a negative surface is well known since the works by Langmuir [1] and Bohm [2]. A huge number of subsequent references can be found, e.g., in [3–5]. Within the framework of this model, the ions enter the sheath from the quasi-neutral plasma, are accelerated by the sheath electric field and reach the surface without suffering collisions with neutral particles; the electrons are decelerated by the sheath field and are in equilibrium with it, i.e. follow the Boltzmann distribution. This model has been widely used for the description of sheaths in discharges burning in an ambient gas, such as glow discharges and low- to high-pressure arc discharges on refractory cathodes, as well as for the description of sheaths in discharges burning in cathode vapour, such as vacuum arcs and low- to high-pressure arc discharges on volatile cathodes (e.g. [6] and references therein).

As far as discharges burning in cathode vapour are concerned, this model implies that the neutral atoms emitted from the cathode surface are ionized beyond the sheath and a part of the ions produced return to the cathode, thus forming the sheath. However, an evaluation of the characteristic length scales given in appendix shows that this condition is likely not to be fulfilled, therefore the ionization of atoms emitted by the cathode surface probably occurs in the space-charge sheath rather than in the quasi-neutral plasma.

Thus, the conventional model of space-charge sheath with ions entering the sheath from the quasi-neutral plasma may not be justified under conditions typical of discharges burning in cathode vapour and a model accounting for ionization of emitted atoms inside the sheath may be more appropriate. One could think of a model which is schematically shown in figure 1 and is characterized by the following features. The distribution of electrostatic potential in the sheath possesses a maximum. There is a maximum also in the distribution of the electric field, which means that the sheath is actually a double layer. There is a flux of atoms emitted by the cathode surface which gradually get ionized. The kinetic energy of the atoms may be assumed to be much smaller than the electron energy kT_e, hence one can treat the ions as being generated at rest. The ions produced before the maximum of potential return to the cathode surface and those produced after the maximum escape into the plasma.

The above model has some features in common with the model of a collisionless positive column of a plane glow discharge enclosed by two parallel absorbing walls, introduced in the seminal paper of Tonks and Langmuir [7] and treated in textbooks (e.g. [8, 9]). In particular, there is a potential hump in both problems with the ions generated at rest on both sides of the hump and moving away from it. On the other hand, there is a very important feature in the above model that seems to have no immediate analogue in the literature: the ions that are produced beyond the potential hump move in the

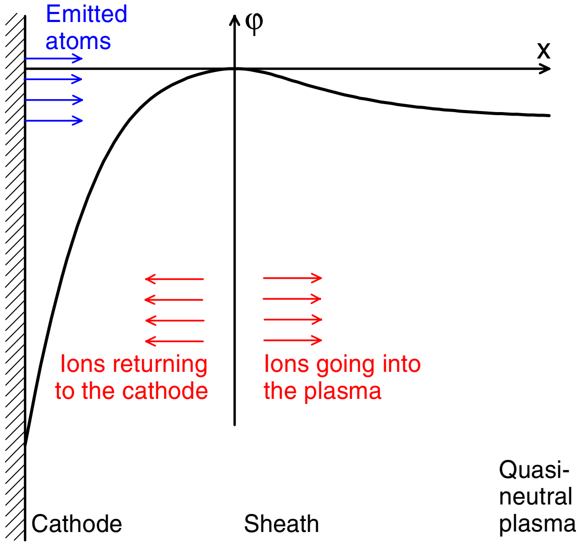

Figure 1. Schematic of a space-charge sheath on a cathode emitting neutral atoms.

direction from the sheath into the plasma, rather than the other way round as in conventional sheath models. There should be some resemblance with the problem of a double sheath on a cathode with electronic emission ([8, 10] and references therein); however, the mathematical description will not be the same. In fact, it does not seem obvious that a sheath with ions moving in the direction into the plasma is possible at all.

This work deals with treating a simple mathematical model describing such a sheath. The main questions to be answered are whether a solution exists and if it does, under what conditions. Another question which is important for obtaining numerical solution of the full problem concerns the character of the decay of the electric field into the plasma.

2. The model

A simple mathematical model describing movement of the ions beyond the potential hump into the plasma is obtained by assuming that the ionization occurs in a narrow vicinity of the point of maximum of potential, after which the ion flux remains constant. This model may be formulated as follows. Let us consider an infinite plate, positioned at $x = 0$, from which singly charged positive ions are emitted into the half-space $x \geqslant 0$ (plasma) with a very low velocity. Electrons are also present in the plasma. The electric field is directed from the plate and accelerates the ions in the direction into the plasma. The plasma far away from the plate is neutral, hence the electric field tends to zero and the potential tends to a constant value. The task is to calculate the space-charge sheath, i.e. the region separating the neutral plasma from the plate. The ions suffer no collisions while crossing the sheath and the electron density in the sheath is related to the potential distribution by the Boltzmann relation.

The system of governing equations includes the equation of motion of an ion, the equation of conservation of the ion flux, the Poisson equation, and the Boltzmann relation:

$$m_i v_i \frac{dv_i}{dx} = -e\frac{d\varphi}{dx}, \qquad j_i = e n_i v_i, \tag{1}$$

$$\varepsilon_0 \frac{d^2\varphi}{dx^2} = -e\left(n_i - n_e\right), \qquad n_e = n_{e0}\exp\frac{e\varphi}{kT_e}. \tag{2}$$

Here m_i, n_i and v_i are the particle mass, number density and velocity of the ions, n_e is the number density of the electrons, φ is the electrostatic potential, j_i is the density of electric current transported by the ions and n_{e0} is the number density of the electrons at a point where $\varphi = 0$. j_i and the electron temperature T_e are known (positive) parameters and n_{e0} is a constant to be determined.

The unknown functions $v_i(x)$ and $\varphi(x)$ satisfy the boundary conditions

$$v_i(0) = 0, \qquad \varphi(0) = 0, \qquad \varphi(\infty) = -U, \tag{3}$$

where U is the sheath voltage (a known positive parameter).

Equations (1) and (2) were introduced by Langmuir [1] and treated by Bohm [2], however, with essentially different boundary conditions: the sheath voltage U was negative in [1, 2], which means that ions were assumed to move from the neutral plasma to the surface, rather than the other way round as in this work.

3. Solution

The first equation in (1) may be integrated to give the equation of conservation of total energy of an ion. Determining the integration constant with the use of the first two boundary conditions (3), one finds $v_i = \sqrt{-2e\varphi/m_i}$. Using this result and the second equation in (1) in order to eliminate n_i from the Poisson equation in terms of φ and then substituting into the Poisson equation the second equation (2), one arrives at an equation involving only one unknown function, $\varphi(x)$. Introducing dimensionless variables, one can write this equation and the corresponding boundary conditions as

$$\frac{d^2\Phi}{d\xi^2} = \frac{\gamma}{\sqrt{2\Phi}} - e^{\chi-\Phi} \tag{4}$$

$$\Phi(0) = 0, \qquad \Phi(\infty) = \chi, \tag{5}$$

where

$$\xi = \frac{x}{\lambda_{D\infty}}, \qquad \Phi = -\frac{e\varphi}{kT_e}, \qquad \lambda_{D\infty} = \left(\frac{\varepsilon_0 kT_e}{n_{e\infty}e^2}\right)^{1/2},$$
$$\gamma = \frac{j_i}{e u_B n_{e\infty}}, \qquad \chi = \frac{eU}{kT_e}, \tag{6}$$

$n_{e\infty} = n_{e0}e^{-\chi}$ has the meaning of the charged particle density in the neutral plasma (an unknown parameter) and $u_B = \sqrt{kT_e/m_i}$ is the so-called Bohm velocity. Note that the ion velocity may be expressed as $v_i = u_B\sqrt{2\Phi}$.

The parameter γ (and, consequently, $n_{e\infty}$ and n_{e0}) may be found from the condition of neutrality of the plasma far away from the plate: $\gamma = \sqrt{2\chi}$.

Let us multiply equation (4) by $d\Phi/d\xi$, integrate over ξ and find the integration constant by taking into account that

$\mathrm{d}\Phi/\mathrm{d}\xi \to 0$ as $\Phi \to \chi$ in accordance with the last boundary condition (5). One arrives at

$$\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}\right)^2 = 2F(\Phi), \tag{7}$$

where

$$F(\Phi) = 2\sqrt{\chi\Phi} + \mathrm{e}^{\chi-\Phi} - 2\chi - 1. \tag{8}$$

Solving equation (7) for $\mathrm{d}\xi/\mathrm{d}\Phi$, integrating over Φ and finding the integration constant by means of the first boundary condition (5), one arrives at

$$\xi = \int_0^\Phi \frac{\mathrm{d}\Phi}{\sqrt{2F(\Phi)}}. \tag{9}$$

This is an implicit solution in quadratures of the present problem.

4. Discussion

At large ξ, the rhs of equation (4) may be expanded in $(\chi - \Phi)$ and this equation to a first approximation assumes the form

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\xi^2} = \frac{2\chi - 1}{2\chi}(\Phi - \chi). \tag{10}$$

If $\chi < 1/2$, then this equation admits only oscillating solutions, which are incompatible with the second boundary condition (5). Thus, a necessary condition for the present problem to be solvable reads as

$$\chi \geqslant 1/2 \tag{11}$$

or, alternatively,

$$v_\mathrm{i}(\infty) \geqslant u_\mathrm{B}. \tag{12}$$

The latter condition coincides with the conventional Bohm criterion for a sheath with a negative voltage [2] (see also [3–5] and references therein) and its mathematical meaning is similar: it ensures the absence of oscillations of potential at large distances from the plate (at the 'sheath edge'). It should be stressed, on the other hand, that in the present context the Bohm criterion (12) limits from below the speed of ions moving from the sheath into the neutral plasma, rather than in the opposite direction as in the case of a conventional sheath with a negative voltage.

The solution obtained in the preceding section corresponds to the case of a sheath with a monotonically decreasing potential distribution, in accordance with the physical model introduced in section 1. This solution exists provided that function $F(\Phi)$ is positive at all Φ in the interval $0 < \Phi < \chi$. The sign of the derivative $\mathrm{d}F/\mathrm{d}\Phi$ is governed by the difference $\sqrt{\chi}\mathrm{e}^{-\chi} - \sqrt{\Phi}\mathrm{e}^{-\Phi}$. If $\chi \leqslant 1/2$, the derivative is positive at all $0 \leqslant \Phi < \chi$, so the function $F(\Phi)$ increases. Since $F(\chi) = 0$, this means that $F < 0$ at all $0 \leqslant \Phi < \chi$ and no solution exists. If $\chi > 1/2$, the function $F(\Phi)$ in the interval $0 < \Phi < \chi$ first increases and then decreases to zero. Hence, the solution exists provided that $F(0) \geqslant 0$ or, equivalently, if $\chi \geqslant C_1$, where C_1 is the positive root of the equation $\mathrm{e}^\chi = 1 + 2\chi$, $C_1 \approx 1.256\,4312$.

Thus, the sheath voltage is limited from below:

$$U \geqslant C_1 \frac{kT_\mathrm{e}}{e}. \tag{13}$$

This limitation is due to the space charge in the sheath and is in this sense analogous to the limitation of ion current in a vacuum diode expressed by the Child–Langmuir law [11]. It is therefore of interest to compare inequality (13) with the Child–Langmuir law. The model of vacuum diode [11] is obtained from the model formulated in section 2 by assuming that there is a second (negative) plate at a finite distance d from the first plate and neglecting the presence of the electrons in the gap between the plates. The mathematical formulation is obtained from (1)–(3) by dropping the electron density term in the Poisson equation, dropping the second equation (2), and shifting the last boundary condition (3) to $x = d$, i.e. writing it as $\varphi(d) = -U$. The resulting problem is solvable provided that the diode voltage is limited from below:

$$U \geqslant \frac{3^{4/3}}{2^{5/3}}\left(\frac{m_\mathrm{i} j_\mathrm{i}^2 d^4}{e\varepsilon_0^2}\right)^{1/3}. \tag{14}$$

This is the Child–Langmuir law interpreted as a limitation from below on the diode voltage for a given ion current density (rather than a limitation from above on the ion current density for a given diode voltage, which is the usual interpretation).

It is useful to analyse dimensions in order to understand the difference between structures of the minimum voltages appearing on the rhs's of inequalities (13) and (14). There are four control parameters in the problem of vacuum diode: e/m_i, j_i/e, ε_0/e and d. There are four control parameters also in the problem of space-charge sheath, the first three parameters being the same as in the problem of vacuum diode and kT_e/e appearing in place of d. Each set of control parameters involves three independent dimensions, therefore one dimensionless parameter may be introduced in each problem, e.g. $\alpha_1 = m_\mathrm{i}\varepsilon_0 j_\mathrm{i}^2 d^7/e^4$ and $\alpha_2 = m_\mathrm{i} e^{10} j_\mathrm{i}^2/\varepsilon_0^6 (kT_\mathrm{e})^7$, respectively. A combination of the dimension of voltage which involves the first three control parameters and is therefore common for both problems is $(e^3 m_\mathrm{i} j_\mathrm{i}^2/\varepsilon_0^6)^{1/7}$. Therefore, the general combination of the dimension of voltage in each problem may be written as $f_{1,2}(e^3 m_\mathrm{i} j_\mathrm{i}^2/\varepsilon_0^6)^{1/7}$, where $f_1 = f_1(\alpha_1)$ and $f_2 = f_2(\alpha_2)$ are arbitrary functions of α_1 and α_2, respectively. One can see that the rhs's of (14) and (13) indeed have such a structure, being $f_1 = 3^{4/3} 2^{-5/3} \alpha_1^{4/21}$ and $f_2 = C_1 \alpha_2^{-1/7}$.

It follows from inequality (13) that $v_\mathrm{i}(\infty) \geqslant u_\mathrm{B}\sqrt{2C_1}$; in other words, the ions leave the sheath with the velocity equal to or exceeding approximately $1.585 u_\mathrm{B}$. Again, this limitation is due to the space charge in the sheath. Comparing this result with the Bohm criterion, inequality (12), one concludes that the Bohm criterion in this case is satisfied by a large margin, rather than with the equality sign which is usual in the case of a sheath with a negative voltage.

The mechanism of acceleration of the ions in the space-charge sheath, revealed by these results, should be responsible for the initial stage of formation of high-speed jets emitted by cathode spots of vacuum arcs. A further acceleration of the jets occurs in the quasi-neutral plasma (e.g. [6, 12] and references therein).

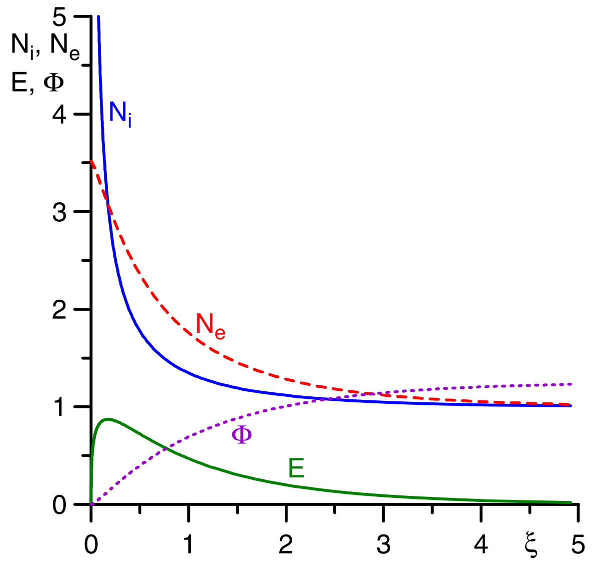

Figure 2. Distributions across the sheath of the dimensionless potential Φ, electric field $E = \mathrm{d}\Phi/\mathrm{d}\xi$, ion and electron densities $N_\mathrm{i} = \sqrt{\chi/\Phi}$ and $N_\mathrm{e} = \mathrm{e}^{\chi-\Phi}$; $\chi = C_1$.

The electric field decays at large ξ proportionally to $\exp(-\sqrt{\frac{2\chi-1}{2\chi}}\xi)$, i.e. exponentially. Note that in the usual situation in the case of a sheath with a negative voltage where the Bohm criterion is satisfied with the equality sign, the electric field decays algebraically.

It follows from the above analysis that the function $F(\Phi)$ on the rhs of equation (7) in the interval $0 < \Phi < \chi$ first increases and then decreases. Therefore, there is a maximum in the spatial distribution of the electric field. Hence, the density of the space charge changes its sign and the sheath represents a double layer, being in this respect similar to the sheath on a cathode with electronic emission ([8, 10] and references therein). Of course, the ions and the electrons are not fully interchangeable since the electrons are Boltzmann-distributed, therefore the results are quantitatively different.

The electric field at the plate is positive at $\chi > C_1$ and vanishes at $\chi = C_1$. Since the plate represents the point of maximum of potential in the context discussed in the section 1, the case $\chi = C_1$ is of main interest. The distributions of the dimensionless parameters across the sheath in this case, obtained with the use of a numerical evaluation of equation (9), are shown in figure 2. The maximum of the electric field occurs at $\xi \approx 0.17$; the ion density exceeds the electron density for $\xi \lesssim 0.17$ and vice versa for $\xi \gtrsim 0.17$. Note that the first term of asymptotic behaviour of the potential in the vicinity of the plate in the case $\chi = C_1$ is $\Phi = (\frac{3}{2}C_1^{1/4}\xi)^{4/3} \approx 1.853\xi^{4/3}$.

5. Conclusions

A simple model of a sheath formed by electrons and positive ions injected into the sheath with a very low velocity and moving from the sheath into the neutral plasma is treated. It is shown that such a sheath is possible provided that the sheath voltage is equal to or exceeds approximately $1.256kT_\mathrm{e}/e$. This limitation is due to the space charge in the sheath and is in this sense analogous to the limitation of ion current in a vacuum diode expressed by the Child–Langmuir law. The ions leave the sheath and enter the neutral plasma with a velocity equal to or exceeding approximately $1.585u_\mathrm{B}$.

The conclusion that a sheath with ions moving in the direction from the sheath into the plasma, rather than the other way round as in conventional sheath models, is possible, provides a necessary support to a more general model which was introduced in section 1. This model accounts for ionization inside the sheath of atoms emitted by the cathode surface and may be capable of predicting what fraction of the ions return to the cathode, thus contributing to a better understanding of cathode erosion in discharges burning in cathode vapour.

Acknowledgments

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Appendix. Does the ionization of emitted atoms occur in the plasma or in the sheath?

In order to answer this question, one should compare δ the scale of thickness of the space-charge sheath with l_i a characteristic distance which an atom emitted by the cathode travels before getting ionized. These two scales may be estimated by order of magnitude as, respectively, $\delta = \lambda_\mathrm{D}(eU/kT_\mathrm{e})^{3/4}$ (e.g. [4, p 6]) and $l_\mathrm{i} = C_\mathrm{a}\tau_\mathrm{i}$, where $\lambda_\mathrm{D} = \sqrt{\varepsilon_0 kT_\mathrm{e}/n_\mathrm{e}e^2}$ is the Debye length, U is the sheath voltage, T_e and n_e are characteristic temperature and number density of plasma electrons, C_a is a characteristic speed of emitted atoms and τ_i is a characteristic time which an atom takes to be ionized.

Let us assume for the purposes of estimation that vapourization is the dominating mechanism of emission of atoms by the cathode surface, then the speed of emitted atoms in the near-cathode layer is of the order of thermal speed and may be estimated as $C_\mathrm{a} = \sqrt{8kT_\mathrm{s}/\pi m_\mathrm{i}}$, where T_s is a characteristic temperature of the cathode surface inside the spot and m_i is the atomic mass of the cathode material. Let us assume that the ionization degree of the plasma in the near-cathode layer is not small, then the number density of plasma electrons may be estimated by order of magnitude as $n_\mathrm{e} = \frac{p}{k(2T_\mathrm{s}+T_\mathrm{e})}$, where p is the local plasma pressure and the temperature of heavy particles is assumed to be equal to the temperature of the cathode surface.

Ionization by electron impact is a dominating mechanism of ionization of neutral atoms. The estimates [13] show that electrons emitted by the cathode transfer their energy to the plasma electrons, and those produce ionization. Hence, one should set $\tau_\mathrm{i} = 1/k_\mathrm{i}n_\mathrm{e}$, where k_i is the rate constant of ionization by electron impact which should be evaluated in terms of T_e. In this work, this evaluation was performed by means of formulae [14], which take into account both direct and stepwise ionization.

Consider, as an example, a vacuum arc on a copper cathode. Let us set $U = 15$ V, which is the near-cathode

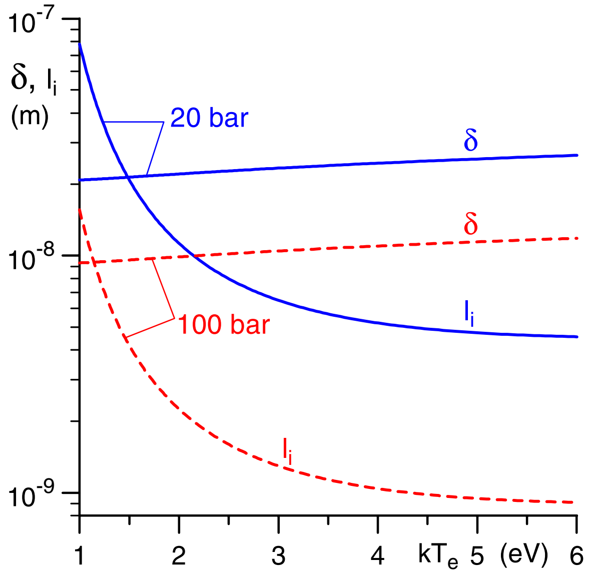

Figure 3. δ: scale of thickness of the near-cathode space-charge sheath; l_i: characteristic distance which an atom evaporated from the cathode travels before getting ionized.

voltage drop in a vacuum arc on a copper cathode cited by most authors (e.g. [15, p 229]). The values of the temperature of the surface of a copper cathode inside a spot given by different authors are somewhere between 3.5×10^3 and 5.5×10^3 K; for example, values of 3.7×10^3 and 5.1×10^3 K are cited in [15, p 231]. In this work, C_a and n_e have been estimated assuming $T_s = 4 \times 10^3$ K and the plasma pressure was set equal to 20 bar or 100 bar, which are values of the pressure of saturated copper vapour evaluated with the use of [16] for temperatures of 3.8×10^3 K and 4.6×10^3 K, respectively. The values of the temperature of plasma electrons given by different authors vary from about one to several electronvolts; for example, values of 4–6 eV; 0.9 eV and 3.5 eV; 1–6 eV are cited in, respectively, [17, p 194]; [15, p 231]; [12, p 260]. Since variations in T_e strongly affect the ionization rate constant, the estimates of this work have been performed for T_e varying between 1 and 6 eV.

The results of the evaluation are shown in figure 3. The ratio l_i/δ attains a maximum value of approximately 3.7 at $T_e = 1$ eV, $p = 20$ bar and a minimum value of approximately 0.08 at $T_e = 6$ eV, $p = 100$ bar. In other words, the ratio l_i/δ is likely to be small or comparable to unity rather than large. Hence, the ionization of atoms emitted by the cathode surface probably occurs in the space-charge sheath rather than in the quasi-neutral plasma.

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