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What is the mathematical meaning of Steenbeck's principle of minimum power in gas discharge physics?

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Abstract
It is shown that Steenbeck's principle of minimum power, or voltage, for discharges with fixed current is not a corollary of the principle of minimum entropy production, in contrast to what is frequently assumed; besides, the latter principle itself does not provide a reasonable approximation in gas discharge physics. Similarly, Steenbeck's principle is not a corollary of mathematical models of gas discharges. Hence, this principle contradicts the mathematical models. A methodically correct evaluation of the error caused by the use of Steenbeck's principle requires a comparison of a solution obtained with the use of this principle with an exact solution to the same problem, rather than with experimental results or results deemed reasonable from the point of view of common sense. Such a comparison is performed for two examples from the theory of a cylindrical arc column. The examples show that the error incurred by the usage of Steenbeck's principle is uncontrollable and may be unacceptably high.

1. Introduction

In 1932, Max Steenbeck made a conjecture [1] that parameters of a current-controlled cylindrical arc discharge, such as the arc temperature, vary in a way that the axial electric field in the arc attains a minimum value. This conjecture attracted a lot of attention in the literature and became known as Steenbeck's minimum principle. Brief comments on the history of this principle can be found in [2]. A prominent role in promoting Steenbeck's principle has been played by the work [3], where a conclusion was drawn that this principle follows from the principle of minimum entropy production, well known in non-equilibrium thermodynamics.

Nowadays, Steenbeck's principle of minimum power, or voltage, is broadly understood as follows: states that are realized in gas discharges with fixed current are those that require minimal power (voltage) for their maintenance. This principle has been extensively invoked in investigations of many gas discharge phenomena, in particular cylindrical arcs [4], effect of normal current density on glow cathodes [5], cathode spots in arc discharges [6]. Steenbeck's principle has still been used in the literature, e.g. [7–20].

The present note is concerned with discussion of mathematical meaning of application of Steenbeck's principle in theoretical models of gas discharges and of errors incurred. The outline of the paper is as follows. Section 2 is concerned with discussion of the following questions: whether Steenbeck's principle is a corollary of fundamental physical laws; whether it can be proved for particular models of gas discharges; whether a lower power (voltage) is an indication of stability of a particular mode of a discharge; what the mathematical meaning of Steenbeck's principle is. In section 3, a comparison of a solution obtained with the use of this principle with an exact solution of the same problem is shown for two examples from the theory of a cylindrical arc column, one of these examples being a simple analytical model and the other a wall-stabilized arc in air. Conclusions are summarized in section 4. Appendix A is concerned with Gouy–Stodola theorem, which has been employed in proofs of relation of Steenbeck's principle with the principle of minimum entropy production. The question whether the principle of minimum entropy production provides a reasonable approximation in discharge physics is analysed in appendix B for the example of a cylindrical arc.

2. General discussion

2.1. Is Steenbeck's principle a corollary of fundamental physical laws?

Most of the authors who make use of Steenbeck's principle seem to believe that it is related to laws of thermodynamics. This belief can be traced back to the work of Peters [3], who concluded that Steenbeck's principle for an LTE arc discharge follows from the principle of minimum entropy production, well known in non-equilibrium thermodynamics [21]. The derivation [3] may be summarized as follows. The entropy production in the arc vessel was expressed in terms of the electrical power supplied to the discharge by means of the formula

$$P = \frac{IU}{T_w}, \quad (1)$$

where I and U are the arc voltage and current and T_w is the temperature of the walls of the arc vessel. For convenience, a derivation of this formula is given in appendix A. Note that this formula represents a particular case of the Gouy–Stodola theorem, which is well known in applied thermodynamics (e.g. [22]).

According to the principle of minimum entropy production, the entropy production, of all possible states of a system, attains a minimum value in the stationary state, i.e. in a state that is governed by steady-state equations. From here the author [3] concluded, invoking the Gouy–Stodola theorem, that the stationary state corresponds to the minimum of the electrical power IU, which for discharges operated at constant I amounts to a minimum of U.

Note, however, that the Gouy–Stodola theorem (1), being valid only for stationary states, cannot be employed for evaluation of entropy production in states that are not stationary. Therefore, a transition from the principle of minimum entropy production to the principle of minimum of power with the use of this theorem is unjustified, which invalidates the reasoning [3]. The same is true for the reasoning of the works [16, 20], in which an attempt was made to derive Steenbeck's principle of minimum power from the principle of maximum entropy production.

Note that Steenbeck's principle is frequently applied to a set of stationary states, i.e. a state with a minimum voltage is sought among a family of stationary states with the same discharge current. Then the Gouy–Stodola theorem is applicable to each of these states and the application of Steenbeck's principle is equivalent to finding a state with a minimum entropy production. (An example of such a situation is found in the channel model of cylindrical arc; see section 3.1.) However, this procedure is not equivalent to application of the principle of minimum entropy production as it is understood in non-equilibrium thermodynamics [21]. The reason for this is as follows. If a family of different stationary states exists at the same discharge current, then normally these states refer to different discharge properties. Hence, a minimum in the Steenbeck procedure is sought among stationary states of *different* systems, while in the principle of minimum entropy production a minimum is sought among all possible states of the *same* system and these possible states are not stationary, i.e. do not satisfy steady-state equations.

Of course, there is also a serious problem concerning the principle of minimum entropy production itself: for continuous systems, this principle may be derived from laws of thermodynamics only in certain very special cases [21], and a question arises whether this principle provides a reasonable approximation in gas discharge physics. This question is studied in appendix B for the example of a cylindrical arc and a negative answer is found.

Thus, Steenbeck's principle cannot be derived from (i.e. is not a corollary of) laws of thermodynamics or, for that matter, of any other fundamental physical law, being in this respect fundamentally different from well-known variational principles such as the principle of least action in mechanics or the Fermat principle in geometrical optics.

2.2. Can Steenbeck's principle be proved for particular models of gas discharges?

Although Steenbeck's principle is not a corollary of fundamental physical laws, one could still think of proving it for a particular situation on the basis of a particular mathematical model describing the discharge being considered. That is, one should formulate an appropriate mathematical model of the discharge which will include all the relevant differential (or integrodifferential) equations supplemented with necessary boundary conditions, and then try to derive an inequality having the meaning of a principle of minimum power under some constraints or other. However, no accurate derivation of a principle of minimum power from a meaningful model of a gas discharge is known. On the other hand, there are counterexamples; see, e.g. section 3. One should presume therefore that Steenbeck's principle is not a corollary of mathematical models of gas discharges.

2.3. Is a lower power (voltage) indication of stability?

Some authors assume, on the basis of arguments stemming from Steenbeck's principle of minimum voltage, that if different modes of discharge are possible at the same discharge current, the mode with a lower voltage drop is the preferred (stable) one. The incorrectness of this point of view is immediately clear from the well-known experimental fact that transitions between different modes are frequently accompanied by hysteresis. In fact, stability of different modes, as predicted by an accurate linear stability theory, has nothing to do with which one of them operates at a lower voltage; see, for example, discussion of hysteresis occurring in transitions between diffuse and spot modes of current transfer to cathodes of high-pressure arc discharges in section 3.4.2 of review [23].

2.4. What is the mathematical meaning of Steenbeck's principle?

Given that Steenbeck's principle is not a corollary of mathematical models of gas discharges, the mathematical meaning of any approach making use of this principle amounts to the following: one or more equations describing the discharge physics are discarded (or not invoked from the very beginning) and replaced by an unjustified relation.

The same idea may be formulated in even stronger terms. If a complete mathematical model, which includes all relevant equations and boundary conditions, is supplemented by an additional relation, then two cases are possible: either this relation conforms to the model, i.e., represents its corollary, or it contradicts the model. Steenbeck's principle is not a corollary of mathematical models of gas discharges; therefore this principle contradicts mathematical models.

Of course, there are many arguments against legitimacy of application of such procedures not only in theoretical physics but in applied physics as well. An extreme point of view is that the introduction of an arbitrary relationship, the only criterion being an agreement of results with common sense and/or the experiment, is no better than fitting experimental results by an arbitrary formula. A strong argument against the usage of Steenbeck's principle is that in most cases it can be implemented in many different ways (which is a consequence of it not being a rigorous variational principle), therefore any subsequent researcher tends to implement Steenbeck's principle in his own way and his results are different from those obtained by his predecessors in the same problem.

However, Steenbeck's principle (or principle of minimum or maximum entropy production) is still in use in gas discharge physics, and the reasons being invoked are, as formulated in the recent work [20], the existence of modelling problems with a lack of information that often appears in practice and belief that such principles give a 'best' or 'most unbiased' estimate on the basis of the information available. Setting aside the difference in terminology ('a lack of information' in terms of [20] amounts to 'not invoking one or more equations describing the discharge physics' in terms of this work), one is led to a pragmatic question: how big is the error incurred by the use of Steenbeck's principle?

A methodically correct evaluation of this error requires a comparison of a solution obtained with the use of this principle with an exact solution to the same problem, rather than with experimental results or results deemed reasonable from the point of view of common sense. Such a comparison is the subject of the next section.

3. Examples: applying Steenbeck's principle to a cylindrical arc column

3.1. The model

The problem of the cylindrical column of a wall-stabilized arc is a classic object of the application of the minimum principle. Under LTE conditions the radial distribution of the temperature T in the column is governed by the Elenbaas–Heller equation

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\kappa\frac{\mathrm{d}T}{\mathrm{d}r}\right)+\sigma E^2=0, \tag{2}$$

$$r=0: \quad \frac{\mathrm{d}T}{\mathrm{d}r}=0; \qquad r=R: \quad T=T_{\mathrm{w}}. \tag{3}$$

Here r is the distance from the axis of the discharge tube, E is the (axial) electric field, κ and σ are the thermal and electrical conductivity coefficients and R is the tube radius. Radiative losses are neglected. Equations (2) and (3) should be supplemented with Ohm's law

$$I=2\pi E\int_0^R \sigma r\,\mathrm{d}r. \tag{4}$$

Equations (2)–(4) allow one to calculate the discharge parameters as functions of the discharge current I, provided that dependences $\kappa=\kappa(T)$ and $\sigma=\sigma(T)$ are known.

There is an approach to obtaining an approximate solution called the 'channel' arc model (e.g. [4, 5]), and it is in the framework of this approach that the minimum principle is invoked. The channel model is based on dividing the arc into an arc channel, which is a current-carrying arc region of a radius r_*, and a surrounding region $r>r_*$ which is assumed to be non-conductive. The temperature inside the arc channel is assumed to be constant, $T=T_0$, the temperature in the surrounding current-free region decreases, due to conductive heat transfer, from T_0 at $r=r_*$ to T_{w} at $r=R$. The arc parameters in the framework of this model are governed by the relations

$$I=\sigma_0 E\pi r_*^2, \qquad 2\pi S_0=IE\ln\frac{R}{r_*}, \tag{5}$$

where S is the heat flux potential: $S=S(T)=\int_{T_{\mathrm{w}}}^{T}\kappa\,\mathrm{d}T$, $\sigma_0=\sigma(T_0)$ and $S_0=S(T_0)$.

Two equations (5) involve, at a given current I, three unknown variables: E, T_0 and r_*, hence one more relation is needed. Many different ways to establish this relation have been discussed in the literature, in particular, in several-decade old Soviet works; e.g. [24, 25] and references therein. The way which is of interest in the context of this work is to invoke Steenbeck's minimum principle: the lacking relation is assumed to be the condition of the minimum of the arc power per unit length, IE, for a given current I or, equivalently, the minimum of the electric field E. The mathematical formulation of this condition can be derived as follows. One differentiates equations (5) with respect to r_*, considering the material property σ as a function of S, arc parameters E and S_0 as functions of r_* and parameter I as fixed. Setting in the obtained equations $dE/\mathrm{d}r_*=0$ and eliminating $\mathrm{d}S_0/\mathrm{d}r_*$, one arrives at the desired relation:

$$r_*^2=R^2\exp\left[-\frac{\mathrm{d}\,(\ln\sigma)}{\mathrm{d}\,(\ln S)}\right]_{S=S_0}. \tag{6}$$

In the subsequent sections, the solutions obtained by means of equations (5) and (6) are compared with the exact solutions for two particular sets of material properties.

We conclude this section with a discussion of relation between Steenbeck's principle and the principle of minimum entropy production for the particular case of arc channel model. Application of Steenbeck's principle to the arc channel model amounts to finding a state with a minimum electric field among states with different channel radii and with the same wall temperature T_{w} and the same arc current I. It was proved in [3] by evaluation of the integral (17) of appendix B that the Gouy–Stodola theorem is applicable to each of these states. This result is not surprising since all these states are stationary. (Of course, the same result follows from equation (15) of appendix A. Note that although the temperature gradient inside the arc channel is neglected, the term $\nabla \cdot \boldsymbol{q}$ in equation (15) must be retained in the arc channel.)

States with different channel radii are characterized by different values of T_0, i.e. of the temperature at which the electrical conductivity of the plasma switches from 0 to $\sigma_0 = \sigma(T_0)$. In other words, these states refer to plasmas with different material properties, and this is why stationary states with different channel radii are possible at the same arc current. Hence, application of Steenbeck's principle to the arc channel model is equivalent to finding a state with a minimum entropy production among stationary states of different systems, in contrast to the principle of minimum entropy production in non-equilibrium thermodynamics where a minimum is sought among all possible states of the same system, including states that are not stationary.

3.2. Analytical example

The most convincing example is the one which admits an exact analytical solution. When the temperature of an LTE plasma is relatively low and the ionization degree is below, say, 10^{-3}, the dependence $\sigma(T)$ is very strong (Arrhenius) and σ increases with an increase in T very rapidly. As T grows further and the ionization degree approaches 10^{-3}, the Coulomb collisions come into play and the increase in σ becomes much slower: the dependence $\sigma(T)$ follows the Spitzer formula and $\sigma \sim T^{3/2}$. If one assumes that the dependence $\kappa(T)$ may be approximated by a square-root function and takes into account that the wall temperature T_{w} is much lower than temperatures inside the arc, then one can set $S \sim T^{3/2}$. Thus, one can assume that σ is negligible at low S and equals aS at high S, where a is a given coefficient (material constant).

The above-described model has a number of features in common with the well-known model introduced by Maecker [26]. The model is merely illustrative and is not intended to provide a quantitative approximation for a particular arc. It is worth stressing once again that the task of this section is to compare an approximate solution obtained with the use of Steenbeck's principle with an exact analytical solution rather than with experimental data for a particular arc, and the model is well suited for the task.

In the framework of this model, the derivative on the right-hand side of equation (6) equals unity, i.e. does not depend on S_0. Therefore equation (6) is decoupled from equations (5) and one immediately finds $r_* = \mathrm{e}^{-1/2}R \approx 0.61R$. Thus, in the framework of this approach the arc channel radius is independent of the arc current; a surprising result.

One can try to improve the arc channel model by taking into account variation of the temperature inside the channel. Let us transform the Elenbaas–Heller equation, equation (2), to the unknown function S and set $\sigma = aS$. The obtained equation may be solved in terms of the Bessel functions and the distribution of heat flux potential inside the channel is

$$S = CJ_0(E\sqrt{a}r). \tag{7}$$

Here C is an integration constant and $J_i(x)$ here and further is the Bessel function of the first kind of order i. Equations (5) are replaced with

$$I = 2\pi C\sqrt{a}r_* J_1(x), \qquad J_0(x) = xJ_1(x)\ln\frac{R}{r_*}, \tag{8}$$

where $x = E\sqrt{a}r_*$.

Equations (8) involve three unknowns: C, E and r_*. In order to try to establish the lacking relation by means of Steenbeck's principle, one can differentiate the second equation (8) with respect to r_* and then set $\mathrm{d}E/\mathrm{d}r_* = 0$. However, the obtained equation has no meaningful roots. It follows that the dependence $E(r_*)$ is monotonic in the improved channel model and Steenbeck's principle cannot be invoked.

Let us proceed to find the exact solution. For the material functions being considered, the exact solution describes distributions with an arc channel and a surrounding current-free region. The heat flux potential distribution inside the channel is governed by equation (7) and integral parameters of the arc obey equations (8). The fact that one equation is missing means that information has been lost. Indeed, one must completely specify the dependence $\sigma(S)$ in order to have a closed statement of the problem, and this has not been done up to now: the value of S at which switching occurs from $\sigma = 0$ to $\sigma = aS$ remains unspecified. In other words, one must specify the temperature at which the ionization degree reaches the above-mentioned value around 10^{-3} where the Coulomb collisions come into play. (Note that this temperature depends only on the plasma-producing gas and its pressure and should be treated as a material constant in the present context.) Let us designate this temperature by T_*. Then $S_* = S(T_*)$ should be treated as a given parameter and the dependence $\sigma(S)$ being considered reads $\sigma = 0$ at $S < S_*$, $\sigma = aS$ at $S > S_*$. (Note that this approximation does not describe the variation of $\sigma(S)$ from very low values to aS_*, which rapidly occurs in a narrow temperature range in the vicinity of $T = T_*$. An approximation of $\sigma(S)$ which describes this variation is known [25]; however, it is unnecessary for the purposes of this work and the above-described discontinuous approximation will suffice.)

Thus, the missing boundary condition reads $CJ_0(x) = S_*$. This equation must be solved jointly with equations (8). The solution may be conveniently expressed in a parametric form, with x playing the role of a control parameter instead of I: C, r_*, E and I can be found as functions of x by means of the expressions

$$C = \frac{S_*}{J_0(x)}, \qquad r_* = R\exp\left[-\frac{J_0(x)}{xJ_1(x)}\right], \qquad E = \frac{x}{\sqrt{a}r_*}, \tag{9}$$

and of the first equation (8). The parameter x varies between 0 and the first root of the Bessel function J_0, which approximately equals 2.40. Analysis of properties of Bessel functions reveals that as x increases, I monotonically increases, E monotonically decreases and the channel radius r_* monotonically increases from 0 to R as one should have expected.

In summary, a distribution with an arc channel and a surrounding current-free region represents an exact solution

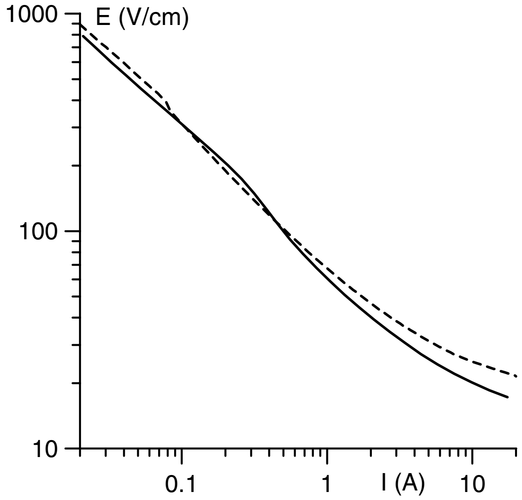

Figure 1. The electric field versus the current in an air arc: numerical solution (solid), channel model with Steenbeck's principle (dashed).

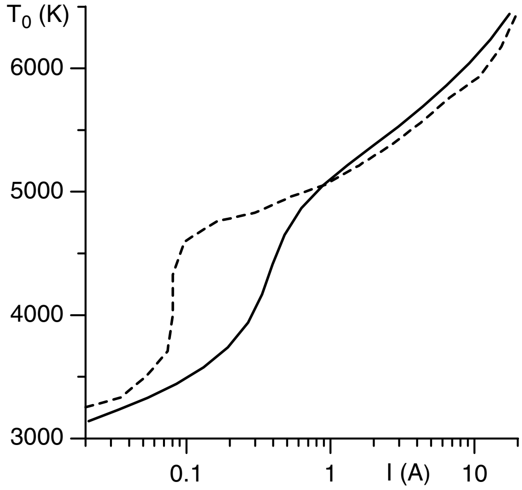

Figure 2. The temperature at the discharge axis versus the current in an air arc: numerical solution (solid), channel model with Steenbeck's principle (dashed).

in the example being considered. However, an attempt to determine the radius r_* of the arc channel by means of invoking Steenbeck's principle either fails, since the dependence $E(r_*)$ is monotonic or leads to the conclusion that r_* is independent of the arc current and equal to a fixed fraction of the arc tube radius ($0.61R$). The latter result may be viewed at best as correct in order of magnitude and is no better than to assume, without any justification, that r_* equals, say, $0.5R$.

3.3. Wall-stabilized arc in air

As the second example, we present results of calculations of arc discharges in atmospheric-pressure air in a cooled tube (at $T_w = 300$ K) with radius $R = 1$ cm. Radiation losses are neglected, which is justified in the range of discharge currents being considered, $I < 20$ A. The lower boundary of the current range was set equal to 0.02 A, as non-LTE effects come into play at lower currents [27]. The dependences $\kappa(T)$ and $\sigma(T)$ were taken from [28].

Values of the electric field E and of the temperature T_0 on the axis of the arc are shown as functions of the discharge current in figures 1 and 2. The data obtained both by means of numerical solution of the Elenbaas–Heller equation and by means of equations (5) and (6) (i.e. in the framework of the channel model supplemented with Steenbeck's minimum principle) are depicted. It is seen that the channel model with Steenbeck's principle gives a reasonable estimate of the electric field. However, the estimate of the arc temperature deviates substantially from the results of numerical solution.

Note that the electric field given by the channel model with the minimum principle is lower than the exact (numerical) E value in the current range $0.1\,\text{A} \lesssim I \lesssim 0.4\,\text{A}$ (figure 1). It is possible in this current range to choose, in the framework of the channel model (5), a value of r_* which would ensure values of E and T_0 that are much closer to the exact values than those governed by the minimum principle. As an example, the dependences $E(r_*)$ and $T_0(r_*)$, given by equations (5), are shown in figure 3 for $I = 0.2$ A. The horizontal lines

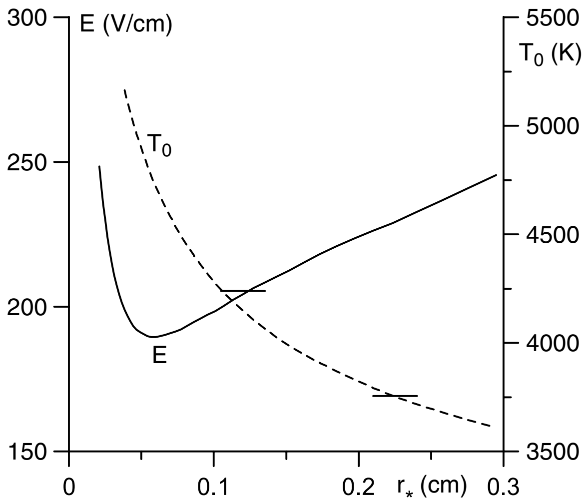

Figure 3. The electric field and the temperature at the discharge axis in the channel model versus the arc channel radius. Air arc, $I = 0.2$ A.

in this figure mark exact values of E and T_0 for this I. The minimum of E corresponds to $r_* \approx 0.05$ cm, while a much better agreement with exact E and T_0 values can be obtained by choosing r_* in the range 0.12–0.15 cm. It follows that the best choice among the solutions to equations of the channel model is in a general case not the one corresponding to the minimum principle.

4. Conclusions

Steenbeck's principle of minimum power, or voltage, is not a corollary of mathematical models of gas discharges. Hence, this principle contradicts mathematical models. The mathematical meaning of any approach making use of this principle amounts to disregarding one or more equations describing the discharge physics and replacing them by an unjustified relation. The error incurred by the usage of Steenbeck's principle is uncontrollable and may be unacceptably high, as the above examples show.

Equally unjustified is the idea that if different modes of discharge are possible at the same discharge current, the mode with a lower voltage drop is the preferred (stable) one.

A proper way to avoid the usage of Steenbeck's principle and of similar unjustified relations is, of course, to look for information where it has been lost. As far as channel models are concerned, this means an appropriate matching of solutions describing the channel and the current-free surrounding. Asymptotic treatment is a proper means of performing such matching for the channel model of a cylindrical arc column [25, 29, 30]. The same is true for the channel model of arc cathode spots; [31] and references therein. The effect of normal current density in glow discharges can be mentioned as another example: the current density inside the normal spot considerably exceeds the current density at the point of minimum of the current density–voltage characteristic, in contrast to what some authors assume on the basis of Steenbeck's principle, and approximately corresponds to the current density which occurs in the abnormal mode at the same discharge voltage [32].

Acknowledgments

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Appendix A. Gouy–Stodola theorem for arc discharges

Let us consider an arc discharge in a quasi-neutral LTE plasma with constant mole fractions of chemical elements. Pressure in the discharge vessel is given (and fixed), walls of the vessel have a given temperature T_w. Assuming that Joule heating and heat removal by heat conduction are the only mechanisms of heat generation in the arc, one can write the equation of conservation of entropy as

$$\rho T \frac{\mathrm{d}s}{\mathrm{d}t} = -\nabla\cdot\boldsymbol{q} + \boldsymbol{j}\cdot\boldsymbol{E}, \tag{10}$$

where s, ρ and T are entropy per unit mass, mass density and temperature of the plasma, $\boldsymbol{q} = -\kappa\nabla T$ and $\boldsymbol{j} = \sigma\boldsymbol{E}$ are densities of heat flux and electric current, $\boldsymbol{E}$ is the electric field and κ and σ are the thermal and electrical conductivity coefficients of the plasma.

Following the standard formalism (e.g. [21], p 24), one can rewrite this equation as

$$\rho \frac{\mathrm{d}s}{\mathrm{d}t} = -\nabla\cdot\frac{\boldsymbol{q}}{T} + \sigma_s, \tag{11}$$

where $\boldsymbol{q}/T$ has the meaning of the density of entropy flux and

$$\sigma_s = -\frac{1}{T^2}\boldsymbol{q}\cdot\nabla T + \frac{\boldsymbol{j}\cdot\boldsymbol{E}}{T} \tag{12}$$

is the entropy production per unit volume and unit time.

The total entropy production in the arc (or, more precisely, in the discharge vessel) is

$$P = \int_V \sigma_s\,\mathrm{d}V. \tag{13}$$

Here V is the volume of the discharge vessel. In the case of a cylindrical (or, more precisely, cylindrically symmetric) arc, P is the entropy production per unit length of the discharge tube and V is the corresponding volume.

Equation (12) may be rewritten as

$$\sigma_s = \nabla\cdot\left(\frac{\boldsymbol{q}}{T} - \frac{\boldsymbol{q}}{T_w}\right) + \frac{\boldsymbol{j}\cdot\boldsymbol{E}}{T_w} + \left(\frac{1}{T} - \frac{1}{T_w}\right)(-\nabla\cdot\boldsymbol{q} + \boldsymbol{j}\cdot\boldsymbol{E}). \tag{14}$$

The integral over V of the first term on the rhs vanishes. The integral of the second term equals IU/T_w, where I and U are the arc current and voltage. (In the case of a cylindrical arc, the axial electric field E appears in place of U.) Hence, equation (13) may be rewritten as

$$P = \frac{IU}{T_w} + \int_V \left(\frac{1}{T} - \frac{1}{T_w}\right)(-\nabla\cdot\boldsymbol{q} + \boldsymbol{j}\cdot\boldsymbol{E})\,\mathrm{d}V. \tag{15}$$

If the arc is in a stationary state, then the rhs of equation (10) vanishes, i.e. the temperature distribution in the arc satisfies the Elenbaas–Heller equation

$$-\nabla\cdot\boldsymbol{q} + \boldsymbol{j}\cdot\boldsymbol{E} = 0. \tag{16}$$

Then the integral on the rhs of equation (15) vanishes and this equation is reduced to the Gouy–Stodola theorem (1).

Appendix B. Does the principle of minimum entropy production provide a reasonable approximation for a cylindrical arc?

In certain situations, stationary non-equilibrium states are characterized by a minimum of the entropy production [21]. However, continuous systems do not belong to such situations except in very special cases. Therefore, a question arises whether a state of an arc discharge characterized by a minimum of the entropy production is reasonably close to the stationary state or not. The aim of this appendix is to study this question using as an example a cylindrical arc.

An expression for the entropy production per unit length of the discharge may be found by applying equations (12) and (13) to the particular case of a cylindrical arc;

$$P = 2\pi\int_0^R \left[\frac{\kappa}{T^2}\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)^2 + \frac{\sigma E^2}{T}\right] r\,\mathrm{d}r, \tag{17}$$

where all designations are the same as in section 3.

We need to find a state of the arc with a given wall temperature T_w and a given arc current I at which the entropy production attains a minimum value. In other words, we need to minimize P on a set of pairs $[T(r), E]$ with functions $T(r)$ and constants E satisfying the boundary conditions (3) and equation (4) with T_w and I given. Following the standard procedure, we represent $T(r) = T_0(r)+\varepsilon T_1(r)$, $E = E_0+\varepsilon E_1$, where $[T_0(r), E_0]$ are the (desired) temperature distribution and the electric field for which P assumes the minimum value, $T_1(r)$ is an arbitrary function, E_1 is an arbitrary constant and ε is a parameter. Then

$$\left.\frac{\mathrm{d}P}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = 0 \qquad \forall\,[T_1(r), E_1] \tag{18}$$

is the necessary condition for P to take a minimum value in the state associated with the pair $[T_0(r), E_0]$.

Substituting equation (17), one can transform equation (18) to

$$\int_0^R \left[\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{\kappa}{T^2}\right)\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)^2 + \frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{\sigma}{T}\right)E_0^2\right]_{T=T_0} T_1 r\,\mathrm{d}r$$
$$+2\int_0^R \frac{\kappa_0}{T_0^2}\frac{\mathrm{d}T_0}{\mathrm{d}r}\frac{\mathrm{d}T_1}{\mathrm{d}r} r\,\mathrm{d}r + 2E_0E_1\int_0^R \frac{\sigma_0}{T_0} r\,\mathrm{d}r = 0. \tag{19}$$

Here and further $\sigma_0 = \sigma(T_0)$, $\kappa_0 = \kappa(T_0)$.

The second integral on the lhs of equation (19) may be evaluated by means of integration by parts:

$$\int_0^R \frac{\kappa_0}{T_0^2}\frac{\mathrm{d}T_0}{\mathrm{d}r}\frac{\mathrm{d}T_1}{\mathrm{d}r} r\,\mathrm{d}r = \left[\frac{\kappa_0}{T_0^2}\frac{\mathrm{d}T_0}{\mathrm{d}r} r\,T_1\right]_{r=0}^{r=R}$$
$$-\int_0^R T_1 \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\kappa_0}{T_0^2}\frac{\mathrm{d}T_0}{\mathrm{d}r} r\right)\mathrm{d}r. \tag{20}$$

Since the set of functions $T(r)$ under consideration satisfies the boundary conditions (3), function $T_0(r)$ satisfies these conditions as well and $T_1(R) = 0$. Hence, the first term on the rhs of equation (20) vanishes.

A relation between T_1 and E_1 may be obtained by differentiating equation (4) with respect to ε:

$$E_0\int_0^R \left[\frac{\mathrm{d}\sigma}{\mathrm{d}T}\right]_{T=T_0} T_1 r\,\mathrm{d}r + E_1\int_0^R \sigma_0 r\,\mathrm{d}r = 0. \tag{21}$$

Note that the second integral on the lhs may be conveniently expressed as $I/2\pi E_0$. Solving this equation for E_1 and substituting the obtained expression and equation (20) into equation (19), one obtains

$$\int_0^R \left[r\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{\kappa}{T^2}\right)\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)^2 + r\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{\sigma}{T}\right)E_0^2\right.$$
$$\left. -2\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\kappa}{T^2}\frac{\mathrm{d}T}{\mathrm{d}r}\right) - rAE_0^2\frac{\mathrm{d}\sigma}{\mathrm{d}T}\right]_{T=T_0} T_1\,\mathrm{d}r = 0, \tag{22}$$

where

$$A = \frac{4\pi E_0}{I}\int_0^R \frac{\sigma_0}{T_0} r\,\mathrm{d}r. \tag{23}$$

Since equation (22) is valid for any function $T_1(r)$, the multiplier in the square brackets must be identically zero. The equation obtained represents the Euler equation for the variational problem being considered. After simple transformations, this equation may be written as (the index 0 is dropped)

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\kappa\frac{\mathrm{d}T}{\mathrm{d}r}\right) - \left[1 + \frac{1}{2}\frac{\mathrm{d}(\ln\kappa)}{\mathrm{d}(\ln T)}\right]\frac{\kappa}{T}\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)^2$$
$$+\frac{1}{2}\left[1 - (1 - AT)\frac{\mathrm{d}(\ln\sigma)}{\mathrm{d}(\ln T)}\right]\sigma E^2 = 0. \tag{24}$$

Thus, the temperature distribution associated with a state with a minimum entropy production, among all possible states with a given wall temperature T_w and a given arc current I, satisfies equation (24).

Alternatively, one can be interested in finding a state with a minimum entropy production among all possible states with a given wall temperature T_w and a given electric field E. It can be shown that an equation describing this state coincides with equation (24) provided that the multiplier in front of the derivative $\mathrm{d}(\ln\sigma)/\mathrm{d}(\ln T)$ in the last term on the lhs is dropped, or, equivalently, that A is formally set equal to zero.

Equation (24) would coincide with the Elenbaas–Heller equation (2) provided that three conditions are satisfied: κ is proportional to T^{-2}, σ is proportional to T^{-1} and $A = 0$. The third condition is not problematic by itself: it is perfectly legitimate to calculate an arc for a given electric field, rather than for a given arc current. However, this condition becomes problematic if the aim is to relate the principle of minimum entropy production to Steenbeck's principle of minimum power, which refers to states with the same current.

On the other hand, the first two conditions for real plasmas are not satisfied even approximately: electric conductivity of an LTE plasma is an increasing function of temperature; thermal conductivity typically is an increasing function for monoatomic gases and a non-monotonic function for molecular gases (e.g. [28]). As a consequence, the physics described by equation (24) has nothing to do with the physics described by the Elenbaas–Heller equation (2) except that heat conduction is described in both equations in the same way (the first term on the lhs). As an example, let us apply equation (24) to an arc with a given electric field burning in a gas with an increasing dependence $\kappa(T)$. Note that the derivative $\mathrm{d}(\ln\sigma)/\mathrm{d}(\ln T)$ equals approximately 1.5 at high temperatures, where the Coulomb collisions are dominating, and exceeds this value at lower T, where the dependence $\sigma(T)$ is Arrhenius-like. Therefore, the quantity in the square brackets in the last term on the lhs of equation (24) is negative (below -0.5) in the considered example, i.e. the Joule effect in equation (24) results in cooling of the plasma rather than heating. Furthermore, the quantity in the square brackets in the second term on the lhs of equation (24) is positive (and exceeds unity), hence this term describes a cooling effect as well.

Thus, equation (24) is not even approximately close to the Elenbaas–Heller equation. Hence a stationary state of a cylindrical arc, which is governed by the Elenbaas–Heller equation, is not even approximately close to a state at which the entropy production is minimal and which is governed by equation (24).

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