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The Child–Langmuir law and analytical theory of collisionless to collision-dominated sheaths

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Abstract
This paper is concerned with summarizing simple analytical models of space-charge sheaths and tracing their relation to the Child–Langmuir model of an ion sheath. The topics discussed include the Child–Langmuir law and model of a collisionless ion sheath, the Mott–Gurney law and model of a collision-dominated ion sheath, the Bohm model of a collisionless ion–electron sheath, the Su–Lam–Cohen model of a collision-dominated ion–electron sheath, ion sheaths with arbitrary collisionality, high-accuracy boundary conditions for the Child–Langmuir and Mott–Gurney models of an ion sheath and the mathematical sense of Child–Langmuir type models of an ion sheath from the point of view of modern theoretical physics.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

In his 1911 paper [1], Child treated what he called the electrostatic effect produced by ions. He considered positive ions moving without collisions between two infinite parallel plates, i.e. in a plane-parallel vacuum diode. There is an infinitely large number of ions at the positive plate and their velocity there is zero. The electric field between the plates is steady state and distorted by the space charge of the ions. Solving the equation of motion of an ion jointly with the Poisson equation, Child found an analytical solution for distributions of the electrostatic potential, electric field and the ion density between the plates. One of the results was that the current which is possible to be carried by positive ions with a given distance and a given potential difference between the plates is limited, however big the capability of ion supply from the positive plate. This limitation is caused by the ion space charge, i.e. by the mutual repulsion of the ions, and the limiting current occurs when the electric field at the positive plate is zero. The limiting current varies directly as the three-halves power of the applied voltage and inversely as the square of the distance separating the plates. In 1913, Langmuir [2] applied similar equations to the case where the conduction takes places by electrons rather than ions, as a part of his study of the effect of space charge on thermionic emission currents in high vacuum.

The works [1, 2] represent a basis of the theory of space-charge-limited currents, and the formula for the space-charge-limited current in a plane-parallel diode is known as the Child–Langmuir law, or Child's law, or the three-halves power law. The theory has different applications, and the one which is of particular interest for gas discharge and plasma physics emerged in the 1920s: in the papers [3, 4] Langmuir suggested a model of a high-voltage near-cathode sheath which is virtually electron-free and formed by positive ions that enter the sheath with energies negligible compared with those they acquire in the sheath itself, and these ion sheaths are in a natural way described by Child's theory of space-charge-limited ion current. This model has initiated a huge number of publications concerned with the analytical theory of space-charge sheaths. By now, this theory is well developed (e.g. review [5] and references therein) and includes both simple and sophisticated models, the simplest and still most well known being the Child–Langmuir model.

Nowadays, simulation methods are available that allow unified modeling of the whole volume occupied by the ionized gas, without *a priori* dividing it into a quasi-neutral plasma and space-charge sheaths. However, analytical models of space-charge sheaths, and in the first place simple ones, still retain there significance. One of the reasons for this is, of course, their methodical value. Another reason is as follows: the above-mentioned unified simulation methods are available for 'cold' plasmas but are more difficult to develop for plasmas with a high density of charged particles and, consequently, a high degree of quasi-neutrality in the bulk. The example of the latter are plasmas of high-pressure arc discharges, where methods of unified simulation of near-cathode layers are just appearing and the Child–Langmuir sheath model or models not very different from it still remain a workhorse; e.g. review [6] and references therein.

This paper is concerned with summarizing simple analytical models of space-charge sheaths and tracing their relation to the Child–Langmuir model of an ion sheath. Both collision-free and collision-dominated sheaths are treated. The outline of the paper is as follows. The Child–Langmuir model of a collisionless ion sheath and its analog for a collision-dominated sheath, the Mott–Gurney model, are described in section 2. Section 3 is concerned with sheath models accounting for the presence of ions and electrons, which are capable of providing descriptions of a sheath with a moderate voltage and of a transition from an ion sheath to a quasi-neutral plasma. The models treated in this section include the Bohm model of a collisionless sheath and its analog for a collision-dominated sheath, the Su–Lam–Cohen model. A model of an ion sheath with an arbitrary degree of collisionality, which describes a smooth transition from the Child–Langmuir model for a collisionless sheath to the Mott–Gurney model for a collision-dominated sheath, is treated in section 4. Also given in section 4 are high-accuracy boundary conditions for the Child–Langmuir and Mott–Gurney models. Another topic discussed in section 4 is the mathematical sense of models of ion sheaths from the point of view of modern theoretical physics.

2. Ion sheaths

2.1. Collisionless ion sheath: the Child–Langmuir law and sheath model

Following [1], we consider a vacuum diode consisting of two infinite parallel plates, a positive one (anode) positioned at $x = 0$ and a negative one (cathode) at $x = d$. Singly charged positive ions are emitted into the gap by the anode with a zero velocity. There is a voltage U applied to the diode. The system of governing equations includes the equation of motion of an ion, the equation of conservation of the ion flux across the gap and the Poisson equation:

$$m_i v_i \frac{dv_i}{dx} = -e\frac{d\varphi}{dx}, \qquad j_i = e n_i v_i, \qquad \varepsilon_0 \frac{d^2\varphi}{dx^2} = -e n_i. \tag{1}$$

Here m_i, n_i and v_i are the particle mass, number density and velocity of the ions, j_i is the density of electric current transported by the ions (a constant positive quantity) and φ is the electrostatic potential. While writing the first equation, the acceleration of an ion, dv_i/dt, was represented as $v_i dv_i/dx$.

The unknown functions $v_i(x)$ and $\varphi(x)$ satisfy boundary conditions

$$v_i(0) = 0, \qquad \varphi(0) = 0, \qquad \varphi(d) = -U. \tag{2}$$

The first equation in (1) may be integrated to give the equation of conservation of the total energy of an ion. Finding the integration constant with the use of the first two boundary conditions (2), one finds $v_i = \sqrt{-2e\varphi/m_i}$. Using this result and the second equation in (1) in order to eliminate n_i from the Poisson equation in terms of φ, one arrives at an equation involving only one unknown function, $\varphi(x)$. Since it involves neither the independent variable x nor the first derivative $d\varphi/dx$, it can be solved analytically. To this end, one can multiply this equation by $d\varphi/dx$ and integrate over x. The equation obtained may be written as

$$\left(\frac{e\varepsilon_0^2}{8m_i j_i^2}\right)^{1/2}\left(\frac{d\varphi}{dx}\right)^2 = \sqrt{-\varphi} + C_1. \tag{3}$$

Here and further C_1, C_2, ... are integration constants. Note that the constant C_1 is related to the electric field at the anode surface: $C_1 = \left(\frac{e\varepsilon_0^2}{8m_i j_i^2}\right)^{1/2}\left(\left.\frac{d\varphi}{dx}\right|_{x=0}\right)^2$. Solving equation (3) for $dx/d\varphi$ and integrating over φ, one arrives at the following solution for the function $\varphi(x)$:

$$\frac{4}{3}\left(\sqrt{-\varphi} + C_1\right)^{1/2}\left(\sqrt{-\varphi} - 2C_1\right) = \left(\frac{8m_i j_i^2}{e\varepsilon_0^2}\right)^{1/4}(x + C_2)\,. \tag{4}$$

A relationship between integration constants C_1 and C_2 is found by means of the second boundary condition (2):

$$C_2 = -\frac{8^{3/4}}{3}\left(\frac{e\varepsilon_0^2}{m_i j_i^2}\right)^{1/4} C_1^{3/2}. \tag{5}$$

The constant C_1 is found by means of the third boundary condition (2), which can be written in a dimensionless form as

$$(1 + b_1)^{1/2}(1 - 2b_1) + 2b_1^{3/2} = a_1, \tag{6}$$

where $a_1 = (81 m_i d^4 j_i^2/32 e\varepsilon_0^2 U^3)^{1/4}$ may be interpreted as the square root of the dimensionless ion current density and $b_1 = C_1/\sqrt{U}$ as the squared dimensionless electric field at the anode surface.

Equation (6) represents an algebraic equation relating the (positive) unknown b_1 to a (positive) parameter a_1. It admits no real positive roots at $a_1 > 1$. There is one root in the interval $0 < a_1 \leqslant 1$, which is shown in figure 1. It should be stressed that $b_1 = 0$ at $a_1 = 1$. Note that a simple approximate formula for this root may be obtained by means of a rational-fraction interpolation (Padé approximant) over a_1^2 between the two-term asymptotic expansion of the function $b_1(a_1)$ at $a_1 \to 0$ and the value $b_1 = 0$ at $a_1 = 1$:

$$b_1 = \frac{243}{16a_1^2}\frac{1 - a_1^2}{27 + 5a_1^2}. \tag{7}$$

One can see from figure 1 that this formula is quite accurate.

The solution is complete now. One can conclude that the problem being considered is solvable provided that the ion current density does not exceed a value that corresponds to $a_1 = 1$, namely,

$$j_i = \left(\frac{32 e\varepsilon_0^2 U^3}{81 m_i d^4}\right)^{1/2}. \tag{8}$$

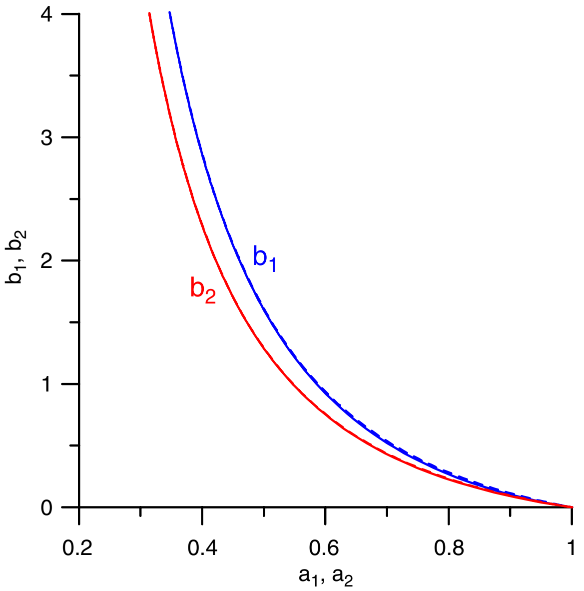

Figure 1. The square root of the dimensionless current density versus the squared dimensionless electric field at the emitting electrode in the Child–Langmuir and Mott–Gurney models. Solid: solutions to equations (6) and (15). Dashed: approximate solutions, equations (7) and (16).

If the density of the ion current emitted by the anode is below this value, then every ion emitted will reach the cathode, j_i the current density in the gap equals the ion emission current density and is treated as a known quantity, the electric field at the anode is non-zero and its value is given by formula (7) (or may be determined by numerically solving equation (6)). As the emission capability of the anode increases and the current in the gap approaches the limiting value (8), the electric field at the anode surface vanishes and the current in the gap stops increasing. This limitation is due to the space charge which is created in the gap by the charge carriers (ions).

Obviously, the above treatment is equally applicable to the case where the charge carriers are electrons emitted by the cathode rather than ions emitted by the anode. In particular, the limiting electron current is given by equation (8) with m_i replaced by the electron mass [2].

The above treatment represents a basis of the theory of space-charge-limited currents, and equation (8) is the Child–Langmuir law.

Let us apply the above treatment to a physical object other than a vacuum diode, namely, a high-voltage near-cathode sheath in a gas discharge [3, 4]. Citing Langmuir [4]: 'If the electrode is 50 volts or more negative with respect to the plasma the sheath will contain no appreciable number of electrons and the positive ions will enter the sheath with energies negligible compared to those they acquire in the sheath itself.' The positive ions are assumed to cross the sheath and reach the cathode without collisions with neutral particles. The value $x = 0$, which in the case of a vacuum diode was attributed to the anode, is now attributed to the outer edge of the sheath, i.e. a plasma–sheath boundary, and $x = d$ to the cathode surface, so d designates the thickness of the sheath. The zero of potential is attributed to the sheath edge and the sheath voltage is designated by U.

Under the above assumptions, the near-cathode sheath is described by equations (1) and boundary conditions (2). Since the electric field in the plasma is much smaller than a characteristic field inside the sheath, the electric field at the outer edge of the sheath may be set equal to zero while treating the sheath:

$$\left.\frac{d\varphi}{dx}\right|_{x=0} = 0. \tag{9}$$

A solution to the stated problem is given by equation (4) in which C_1 is set equal to zero in accordance with the boundary condition (9). (This also implies $C_2 = 0$ in accordance with equation (5).) In other words, a collisionless ion sheath is described by the same solution that describes a vacuum diode with a space charge-limited ion current. In particular, the ion current density, the sheath voltage and thickness are related by the Child–Langmuir law (8). However, the physical interpretation of the latter equation is now different. Citing Langmuir [4]: 'In the usual applications of [equation (8)] the voltage and the distance between the electrodes are known and we wish to calculate the current density that can flow when the current is limited by space charge. In the present case, however, the current density [j_i] is fixed by conditions within the plasma and since the applied voltage is usually known the equation can be used to calculate only the thickness [d] of the sheath'.

The Child–Langmuir law (8) was applied to the description of a near-cathode layer of a gas discharge in the 1923 papers by Ryde [7] and Langmuir [3]. The concept of a high-voltage electron-free near-cathode sheath, which constitutes the basis of such approach, was formulated in the latter work.

Let us introduce a characteristic value of the ion density in the sheath: $n_i^{(s)} = j_i/e\sqrt{eU/m_i}$. Eliminating from equation (8) j_i in terms of $n_i^{(s)}$, one can obtain

$$d = \frac{2^{5/4}}{3}\sqrt{\frac{\varepsilon_0 U}{n_i^{(s)} e}}. \tag{10}$$

Thus, the thickness of the ion sheath is of the order of the Debye length in which the electron thermal energy kT_e is replaced by the electrostatic energy eU.

Equation (3) with $C_1 = 0$ allows one to find the electric field at the cathode surface in terms of the ion current density j_i and the sheath voltage U (but not the sheath thickness):

$$-\left.\frac{d\varphi}{dx}\right|_{x=d} = \left(\frac{8 m_i j_i^2 U}{e\varepsilon_0^2}\right)^{1/4}. \tag{11}$$

This equation is used in the theory of arc discharges for evaluation of the effect of the electric field at the cathode surface over the electron emission current [8] and is usually called the Mackeown equation.

The above-described ion sheath model is based on the assumption that the electron contribution to the space charge in the sheath is negligible. The electrons can enter the sheath both from the adjacent plasma and on being emitted from the cathode. The above-cited condition of the sheath voltage being high enough ensures that the plasma electrons cannot overcome the retarding electric field and enter the sheath, but the question of the contribution of the emitted electrons remains. If needed, this contribution can be taken into account [7, 8]. On the other hand, a simple estimate follows from the fact that the electric current in the sheath transported by the emitted electrons is comparable to or much smaller than the ion current (in the cases of an arc or, respectively, glow discharge): since the velocity of the electrons is much higher than the ion velocity, it follows immediately that the space charge of the emitted electrons is negligible [7].

2.2. Collision-dominated ion sheath: the Mott–Gurney law and sheath model

The above theory of space charge-limited currents in a plane-parallel vacuum diode can be readily modified for the case where the interelectrode gap is filled with a gas and motion of charge carriers in the gap is collision-dominated rather than collision-free. In this case, the first equation in (1) is replaced by

$$v_{\mathrm{i}} = -\mu_{\mathrm{i}}\frac{\mathrm{d}\varphi}{\mathrm{d}x}, \tag{12}$$

where μ_{i} is the mobility of the charge carriers (ions). Equation (12) is written under the assumption that diffusion of the ions is negligible compared with drift in the electric field. (This assumption is justified provided that the ion gas is cold enough: the thermal energy of an ion should be much smaller than the electrostatic energy eU. The thermal energy of an ion equals the thermal energy of a neutral particle plus a difference of the order of work of the electric field over the ion mean free path. The thermal energy of a neutral particle is always much smaller than eU, work of the electric field over the ion mean free path in the collision-dominated case is also much smaller than eU. Hence, the above-mentioned assumption is always satisfied in the collision-dominated case.) For brevity, let us restrict consideration to the case where the frequency of ion–atom collisions does not depend on velocity, then μ_{i} may be treated as constant. The first boundary condition in (2) must be discarded.

The stated problem is rather similar to the problem of a vacuum diode considered in the preceding section and may be solved in a similar way. The Poisson equation may be rewritten as

$$\varepsilon_0\mu_{\mathrm{i}}\frac{\mathrm{d}\varphi}{\mathrm{d}x}\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} = j_{\mathrm{i}}. \tag{13}$$

Integrating equation (13) over x, solving the obtained equation for $\mathrm{d}\varphi/\mathrm{d}x$ and integrating over x once again, one arrives at the following solution for the function $\varphi(x)$:

$$\varphi = -\left(\frac{8j_{\mathrm{i}}}{9\varepsilon_0\mu_{\mathrm{i}}}\right)^{1/2}(x+C_3)^{3/2} + C_4. \tag{14}$$

The integration constant C_3 is related to the electric field at the anode surface, $C_3 = \frac{\varepsilon_0\mu_{\mathrm{i}}}{2j_{\mathrm{i}}}\left(\left.\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right|_{x=0}\right)^2$, and is governed by the equation

$$(1+b_2)^{3/2} - b_2^{3/2} = a_2^{-1}, \tag{15}$$

where $a_2 = (8d^3 j_{\mathrm{i}}/9\varepsilon_0\mu_{\mathrm{i}}U^2)^{1/2}$ and $b_2 = C_3/d$. Again, this is an algebraic equation for the (positive) unknown b_2, which admits no real positive roots at $a_2 > 1$ and has one root in the interval $0 < a_2 \leqslant 1$. This root is shown in figure 1. One can see from figure 1 that this root is accurately approximated by the simple formula

$$b_2 = \frac{32}{9a_2^2}\frac{1-a_2^2}{8+a_2^2}. \tag{16}$$

Again, this formula was obtained by means of a rational-fraction interpolation between the two-term asymptotic expansion of the function $b_2(a_2)$ at $a_2 \to 0$ and the value $b_2 = 0$ at $a_2 = 1$.

The physical meaning of the obtained solution is quite similar to that in the problem of vacuum diode. The space-charge-limited ion current is

$$j_{\mathrm{i}} = \frac{9\varepsilon_0\mu_{\mathrm{i}}U^2}{8d^3}. \tag{17}$$

This formula, which is usually called the Mott–Gurney law, represents an analog of the Child–Langmuir law, equation (8), for the collision-dominated case. It was derived in [9] in connection with conduction current in semiconductors and insulators and in [10] as an analog of the Child–Langmuir law for a plane-parallel diode filled with a high-pressure gas.

The above treatment, including solution (14) with $C_3 = C_4 = 0$ and the Mott–Gurney law, can be in a natural way applied to a collision-dominated near-cathode ion sheath, just in the same way as the Child–Langmuir law applies to a collision-free near-cathode ion sheath. Let us introduce a characteristic value of the ion density in the sheath: $n_{\mathrm{i}}^{(s)} = j_{\mathrm{i}}d/e\mu_{\mathrm{i}}U$. Eliminating j_{i} from equation (17), one again arrives at equation (10), except for the numerical coefficient $(3/2^{3/2}$ rather than $2^{5/4}/3)$. Hence, the thickness of the ion sheath again is of the order of the Debye length evaluated in terms of the electrostatic energy.

An analogue of the Mott–Gurney law for the case of constant mean free path, where the cross section of ion–atom collisions does not depend on velocity, was given in [11].

3. Sheaths formed by ions and electrons

The applicability of the model of an electron-free sheath introduced by Langmuir [3, 4] is limited to the case of high sheath voltages. A question arises regarding the models applicable at moderate sheath voltages. Obviously, such models must take into account not only positive ions but also electrons entering the sheath from the plasma against the retarding sheath electric field.

The model of an electron-free sheath represents a reasonable approximation in the bulk of a high-voltage near-cathode sheath, where the plasma electrons cannot penetrate and their density n_{e} is much smaller than n_{i} the positive ion density. On the other hand, the model clearly loses its validity in an outer section of the near-cathode sheath: if the two densities are virtually equal in the (quasi-neutral) plasma outside the sheath, then n_{e} in an outer section of the sheath, while being smaller than n_{i}, is still comparable to it. This outer section of a high-voltage near-cathode sheath can be called an ion–electron layer. A question arises of finding a solution describing this layer and matching it with the solution describing the electron-free bulk of the sheath (which can be called the ion layer). This can be done by applying the limit of high sheath voltages to a model of a moderately negative sheath mentioned in the previous paragraph. Results of such analysis must in a natural way describe both the electron-free ion layer and the ion–electron layer.

Both above-mentioned questions are addressed in this section, first for a collisionless sheath and then for a collision-dominated one.

3.1. Collisionless ion–electron sheath: the Bohm model

In order to describe a collisionless negative sheath formed by cold (monoenergetic) ions and Boltzmann-distributed plasma electrons, it is sufficient to supplement the Poisson equation in the Child–Langmuir ion sheath model with a term accounting for space charge contributed by the electrons: the third equation (1) is replaced by

$$\varepsilon_0 \frac{d^2\varphi}{dx^2} = -e(n_i - n_e), \qquad n_e = n_0 \exp\frac{e\varphi}{kT_e}, \tag{18}$$

where T_e is the electron temperature and n_0 is the electron number density at a point where $\varphi = 0$. This equation was introduced by Langmuir; cf equations (68) and (70) of [4]. Note that the assumption of cold ions is justified in this case provided that the ion temperature is much smaller than the electron temperature. The assumption of Boltzmann distribution of the density of plasma electrons is justified if the flux of the electrons from the plasma to the electrode surface is much smaller than the chaotic electron flux inside the plasma. The latter is the case if the potential of the surface at which the sheath is formed is around or below the floating potential.

This is the only amendment that has to be introduced in the equations of the electron-free collision-free ion sheath model in order to render it applicable to a collision-free ion–electron sheath; the first and second equations (1) remain unchanged. There is, however, a fundamental difference between the two models as far as boundary conditions on the plasma side are concerned. The ion sheath model breaks down at a finite distance from the cathode: a solution cannot be extended beyond a point at which the electric field vanishes, and this point represents an outer edge of the ion sheath. There is no such breakdown in the model of an ion–electron sheath: a solution can be extended to infinitely large distances from the cathode, with the ion and electron densities tending to the same constant value and the potential also tending to a constant value. In modern terms, one can say that the ion–electron sheath solution can be asymptotically matched with a solution describing the quasi-neutral plasma, in contrast to the ion sheath solution which neglects the presence of electrons and therefore cannot be matched with a plasma solution. This feature of the ion–electron sheath model and its most important consequence, the 'Bohm criterion', were described in 1949 by Bohm [12]. Indications in the same direction can be found already in the 1929 paper by Langmuir [4]; e.g. the first paragraph on p 976 and fourth paragraph on p 980.

Obviously, there is no sense in talking of a sheath edge in the Bohm model. Therefore, the definition of the axis x used up to now needs to be modified: in the current section and in the next one we will assume that the origin is positioned at the electrode (or wall) surface and the axis is directed from the electrode into the plasma. (Note that the ion velocity v_i and the ion current density j_i become negative.) The boundary condition at the electrode surface is the same as in (2):

$$\varphi = -U. \tag{19}$$

Boundary conditions on the plasma side of the sheath, i.e. at infinitely large x, read

$$n_i \to n_s, \qquad n_e \to n_s, \qquad v_i \to -v_s, \qquad \varphi \to 0, \tag{20}$$

where n_s is the density of the charged particles on the plasma side of the sheath, $v_s = (-j_i)/en_s$ is the speed with which the ions leave the plasma and enter the sheath, the potential on the plasma side of the sheath is set equal to zero. (The latter means that $n_0 = n_s$.) The words 'infinitely large x' have here the conventional theoretical-physics meaning: boundary conditions (20) apply at distances from the electrode much larger than the scale of thickness of the sheath but much smaller than a characteristic length scale in the quasi-neutral plasma.

Note that some authors, including Bohm himself, use the term 'sheath edge' in connection with the boundary conditions (20). This is shorter to write than 'region on distances from the electrode much larger than the scale of thickness of the sheath but much smaller than a characteristic length scale in the quasi-neutral plasma', but has proved confusing. We stress once again that an ion–electron sheath has no definite edge; of course, one can define a sheath edge as a point where the charge separation equals 1% (or 3%, 5%, etc), but any such definition will be arbitrary.

A solution to the stated problem may be found in quadratures in the same way as the solution to the Child–Langmuir ion sheath model was found in section 2.1. The ion velocity may be expressed as

$$v_i = -\sqrt{v_s^2 - \frac{2e\varphi}{m_i}}. \tag{21}$$

Eliminating from the first equation in (18) (the Poisson equation) n_i and n_e in terms of φ with the use of the second equation in (1), equation (21), and the second equation in (18), one arrives at an equation involving only one unknown function, $\varphi(x)$, and not involving the independent variable nor the first derivative. Multiplying this equation by $d\varphi/dx$, integrating over x and finding the integration constant by taking into account that $d\varphi/dx \to 0$ as $\varphi \to 0$ in accordance with the last boundary condition (20), one obtains

$$\frac{\varepsilon_0}{2}\left(\frac{d\varphi}{dx}\right)^2 = n_s m_i v_s^2 \left[\left(1 - \frac{2e\varphi}{m_i v_s^2}\right)^{1/2} - 1\right] + n_s k T_e \left(\exp\frac{e\varphi}{kT_e} - 1\right). \tag{22}$$

Solving this equation for $\mathrm{d}x/\mathrm{d}\varphi$ and integrating over φ, one can obtain a solution in quadratures. Without elaborating this point, we note the following. At large x, where φ is small, the right-hand side of equation (22) may be expanded in φ and assumes the form

$$\frac{n_s e^2}{2}\left(\frac{1}{kT_e}-\frac{1}{m_i v_s^2}\right)\varphi^2+\cdots. \tag{23}$$

The left-hand side of equation (22) is positive, so the first term in (23) must be non-negative. One comes to the inequality

$$v_s \geqslant \sqrt{\frac{kT_e}{m_i}}. \tag{24}$$

Thus, the ions leave the quasi-neutral plasma and enter the sheath with a velocity equal to or exceeding $u_B = \sqrt{kT_e/m_i}$; the famous Bohm criterion [12].

There is a huge number of works dedicated to the Bohm criterion and, in more general terms, different aspects of the transition from a quasi-neutral plasma to a collisionless sheath; see reviews [5, 13] and references therein. The most reliable results have been obtained by means of the method of matched asymptotic expansions, which is a standard tool for solving multi-scale problems (e.g. [14–19]) and represents a powerful alternative to intuitive approaches. Except in a few artificial cases, the Bohm criterion is satisfied with the equality sign. To a first approximation, the plasma-sheath transition is rather simple: the above-described Bohm solution for the sheath can be directly matched with a solution describing the quasi-neutral plasma and the Bohm criterion with the equality sign plays the role of a boundary condition for both the sheath and plasma solutions. The situation turns more complex in a second approximation: an intermediate region separating the sheath and the quasi-neutral plasma must be introduced [20–22]. The necessity of such a region is clear from the fact that the ion velocities in the sheath and in the quasi-neutral plasma approach the Bohm velocity from above and, respectively, below; hence a direct matching is impossible beyond the first approximation and an intermediate region must be introduced where the ions moving to the electrode pass the Bohm velocity or, as one can say (e.g. [13] and references therein), the ion acoustic sound barrier is broken.

We consider at the moment the case of a moderately negative sheath on an electrode or an insulating wall, where the sheath voltage U is of the order of kT_e/e the electron temperature measured in electronvolts. Variations of potential in the sheath are also of the order of kT_e/e. The charged-particle densities are of the order of n_s. Equation (22) with v_s replaced by the Bohm velocity u_B may be written as

$$\frac{\lambda_D^2}{2}\left(\frac{e}{kT_e}\right)^2\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2=\left(1-\frac{2e\varphi}{kT_e}\right)^{1/2}-2+\exp\frac{e\varphi}{kT_e}, \tag{25}$$

where $\lambda_D = (\varepsilon_0 kT_e/n_0 e^2)^{1/2}$ is the Debye length evaluated on the plasma side of the sheath. Assuming that in the sheath the term on the left-hand side of this equation is of the same order of magnitude as the terms on the right-hand side, one finds that the thickness (length scale) of the sheath is of the order of λ_D.

The above-described Bohm model is applicable not only to moderately negative near-electrode and near-wall sheaths but also to high-voltage near-cathode sheaths, where $U \gg kT_e/e$. Hence, the Child–Langmuir model can be derived from the Bohm model by means of the limiting transition $\chi \to \infty$, where $\chi = eU/kT_e$. An investigation of this limiting transition is of interest in order to better understand errors introduced by different approximations of the Child–Langmuir model and to eventually improve the overall accuracy. Such an investigation, performed by means of the method of matched asymptotic expansions in [23], revealed two sub-layers: the ion–electron layer, which is an outer section of the space-charge sheath where the electron and ion densities are comparable, and the ion layer, which occupies the bulk of the space-charge sheath and in which the electron density is exponentially small compared with the ion density.

Scalings in the ion–electron layer are the same as those in a moderately negative sheath, i.e. the length scale, the charged-particle densities and variations of potential are of the orders of λ_D, n_s, and kT_e/e, respectively. Let us estimate orders of magnitude of parameters in the ion layer. In the ion layer, $-\varphi$ is of the order of U and considerably exceeds kT_e/e; hence the right-hand side of equation (25) equals $(-2e\varphi/kT_e)^{1/2}$ to a first approximation and is of the order of $\chi^{1/2}$. Equating the order of magnitude of the left-hand side of equation (25) to $\chi^{1/2}$, one finds that the thickness of the ion layer is of the order of $\lambda_D\chi^{3/4}$.

It follows from equation (21) that the ion speed in the ion layer is of the order of $u_B\chi^{1/2}$. Consequently, the ion density is of the order of $n_s\chi^{-1/2}$. The electron density in the ion layer, being governed by the Boltzmann distribution (18), is exponentially small with respect to the large parameter χ.

It follows from the above asymptotic estimates that the thickness (length scale) of the ion layer substantially exceeds the thickness of the ion–electron layer. Hence, the ion layer has a more or less distinct edge. On the other hand, no unambiguous definition of this edge can be given to an accuracy better than $O(\chi^{-3/4})$, which is the order of the ratio of the thickness of the ion–electron layer to the thickness of the ion layer. The voltage drop in the ion layer is much higher than that in the ion–electron layer. The ion speed and density in the ion layer are much higher and, respectively, lower than in the ion–electron layer. The scale of ion density in the ion layer, $n_i^{(il)} = n_s\chi^{-1/2}$, and the length scale $\delta = \lambda_D\chi^{3/4}$ are related by the equation

$$\delta = \sqrt{\frac{\varepsilon_0 U}{n_i^{(il)} e}}, \tag{26}$$

which is similar to equation (10) and has the same physical meaning (the thickness of the ion layer is of the order of the Debye length evaluated in terms of the local ion density and the electrostatic energy).

The above-described asymptotic structure is shown in figure 2.

It can be shown on the basis of the above asymptotic estimates that the ion layer to a first approximation is described by the Child–Langmuir model, and the (relative) error of this approximation is of the order of $\chi^{-1/2}$. In other words,

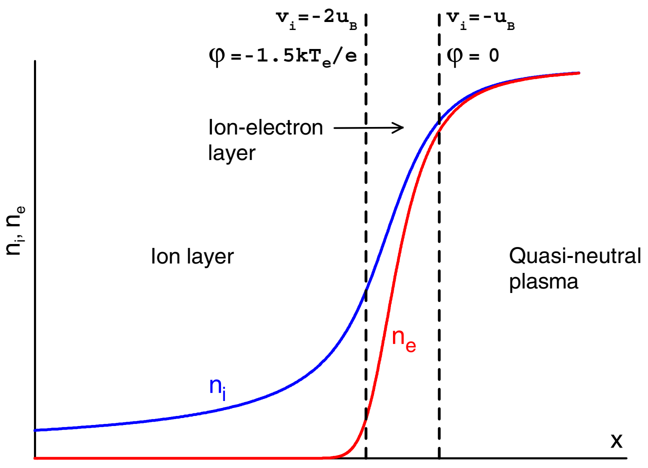

Figure 2. Asymptotic structure of the collision-free high-voltage cathode sheath.

terms which are missing from the solution, or are calculated incorrectly, are of the order of $\chi^{-1/2}$ relative to the leading terms. It should be stressed that this estimate refers to the accuracy of the Child–Langmuir model compared with the Bohm model. In other words, this is just one component of the overall error, other components being errors inherent to the Bohm model, e.g. errors originating in the neglect of ion–atom collisions in the sheath.

3.2. Collision-dominated ion–electron sheath: the Su–Lam–Cohen model

A theory of collision-dominated space-charge sheaths in a weakly ionized plasma was developed in 1963 by Su and Lam [24] and Cohen [25] in connection with a problem of a spherical electrostatic probe in a quiescent weakly ionized plasma with constant electron and heavy-particle temperatures and without ionization and recombination. The work [24] was concerned with the case of high negative probe potentials and the work [25] with the case of moderate potentials. These were the first works on the sheath theory in which matched asymptotic expansions were applied. Refined asymptotic treatments were given in [26, 27]. In this section, a summary is given of a simplified version of the theory which was developed in the work [28] and is based on the assumptions of cold ions and Boltzmann-distributed electrons.

A model of a collision-dominated sheath formed by cold ions and Boltzmann-distributed plasma electrons can be obtained from the Mott–Gurney model in the same way as the Bohm model for the collisionless case is obtained from the Child–Langmuir model. First, the Poisson equation is supplemented with a term accounting for space charge contributed by the electrons. Thus, the system of equations includes the second equation (1) and equations (12) and (18). Second, the boundary conditions at the edge of the ion sheath must be replaced by conditions of matching with the quasi-neutral plasma. These boundary conditions apply at 'infinitely large' values of x, which again means distances from the electrode much larger than the scale of thickness of the sheath but much smaller than a characteristic length scale in the quasi-neutral plasma. In order to derive these conditions, let us multiply the first equation (18) by $\mathrm{d}\varphi/\mathrm{d}x$, remove n_i by means of the second equation (1) and equation (12) and n_e by means of the second equation (18) and integrate over x. The resulting equation may be written as

$$\frac{\lambda_\mathrm{D}^2}{2}\left(\frac{e}{kT_\mathrm{e}}\right)^2\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2 = -\frac{x}{\Delta} + \exp\frac{e\varphi}{kT_\mathrm{e}} + C_5, \qquad (27)$$

where $\lambda_\mathrm{D} = (\varepsilon_0 kT_\mathrm{e}/n_0 e^2)^{1/2}$ as before and $\Delta = n_0 kT_\mathrm{e}\mu_\mathrm{i}/(-j_\mathrm{i})$. At large x where the plasma is quasi-neutral, the left-hand side of this equation is negligible and one finds the following asymptotic behavior:

$$n_\mathrm{e} = n_0\frac{x}{\Delta} + \cdots, \qquad n_\mathrm{i} = n_0\frac{x}{\Delta} + \cdots, \qquad (28)$$

$$\varphi = \frac{kT_\mathrm{e}}{e}\left[\ln\frac{x}{\Delta} + o(1)\right]. \qquad (29)$$

As usual, dots designate terms of higher orders of smallness in the asymptotic parameter; in this case, in the large parameter x and $o(1)$ designate terms that tend to zero at large x.

Expressions (28) and (29) are substantially different from the corresponding expressions in the Bohm model, equation (20): while the (first-approximation) solution in a collisionless sheath tends at large x to constant values, in a collision-dominated sheath it manifests an algebraic asymptotic behavior for charged-particle densities and a logarithmic behavior for the potential. In other words, parameters of a collisionless plasma at distances from the electrode surface much larger than the scale of thickness of the sheath but much smaller than a characteristic length scale in the quasi-neutral plasma are constant to a first approximation, and just these values are implied when one speaks of the density of the charged particles and potential on the plasma side of the sheath or the velocity with which the ions leave the plasma and enter the sheath. Nothing of the latter makes any sense in the case of a collision-dominated sheath. On the other hand, an algebraic or logarithmic intermediate asymptotic behavior is quite common in multi-scale problems and the method of matched asymptotic expansions is very well suited for such cases, so a theory of collision-dominated sheaths can be readily developed.

Since there is no way to unambiguously identify a potential on the plasma side of the sheath, a voltage drop in the sheath cannot be unambiguously defined. It is appropriate then to define a combined voltage drop across the sheath and the adjacent plasma region where the ion flux to the cathode surface is generated. In other words, let us choose a reference point for the potential at the external boundary of the plasma region adjacent to the sheath where the ion flux to the cathode surface is generated. Then n_0 signifies the charged-particle density at this boundary and U appearing in the boundary condition (19) signifies a combined voltage drop across the sheath and the adjacent plasma region. As an example, let us consider a spherical electrostatic probe in a quiescent weakly ionized plasma with constant electron and heavy-particle temperatures and without ionization and recombination, treated in [24, 25]. In this example, the ion current to the probe is generated in the whole volume of the quasi-neutral plasma up to the undisturbed plasma at distances from the probe much larger than the probe radius. Then n_0 represents the charged-particle density in the undisturbed plasma and $\varphi = 0$ corresponds to the plasma potential. The near-cathode plasma region in arc discharges can be considered as another example. In this example, the ion current to the cathode is generated in the so-called ionization layer (e.g. [6] and references therein), which is a region of quasi-neutral plasma where deviations from ionization equilibrium are localized. It is the ionization layer that plays a role of the plasma region adjacent to the sheath in this example, so n_0 represents the charged-particle density on the plasma side of the ionization layer and U is the combined voltage drop in the sheath and in the ionization layer.

The ion current to a negative electrode generated in a collision-dominated quasi-neutral plasma is governed by ambipolar diffusion and may be evaluated as $j_i = -eD_a \mathrm{d}n_c/\mathrm{d}x$, where D_a is the coefficient of ambipolar diffusion which in the case of cold ions is related to the ion mobility by the formula $D_a = kT_e\mu_i/e$ and $\mathrm{d}n_c/\mathrm{d}x$ is the derivative of the charged-particle density evaluated in the quasi-neutral approximation at the electrode surface. Now one can understand the meaning of the length Δ appearing in equations (27)–(29): $\Delta = n_0(\mathrm{d}n_c/\mathrm{d}x)^{-1}$, i.e. it is a scale of thickness of the plasma region adjacent to the sheath where the ion flux to the cathode surface is generated. (In the above-mentioned examples, Δ equals the probe radius or, respectively, the ionization length [6].) Note that $\lambda_D = (\varepsilon_0 kT_e/n_0e^2)^{1/2}$ has in this section the meaning of Debye length evaluated at the external boundary of the plasma region adjacent to the sheath where the ion flux to the cathode surface is generated. The applicability of the present analysis is obviously subject to the inequality $\lambda_D \ll \Delta$, which justifies a division of the near-electrode region into a quasi-neutral plasma and a space-charge sheath. On the other hand, λ_D must be not too small so that the sheath thickness is much larger than the mean free paths of the charged particles.

We consider at the moment the case of a moderately negative near-electrode or near-wall sheath, where the sheath voltage is of the order of kT_e/e. Variations of potential in the sheath are also of the order of kT_e/e. Assuming that in the sheath the term on the left-hand side of equation (27) and the first term on the right-hand side are of the same order of magnitude, one finds that the scale of thickness of the sheath is $\lambda_D^{2/3}\Delta^{1/3}$. Assuming that the second term on the right-hand side of equation (27) is in the sheath of the same order as the first term and the term on the left-hand side, one finds that the potential distribution in the sheath may be represented as

$$\varphi = -\frac{kT_e}{e}\left(\frac{2}{3}\ln\frac{\Delta}{\lambda_D} + \Phi\right), \tag{30}$$

where $\Phi = \Phi(X)$ is a dimensionless function of the stretched coordinate $X = x/\lambda_D^{2/3}\Delta^{1/3}$.

The electric field in the sheath is of the order of $kT_e/e\lambda_D^{2/3}\Delta^{1/3}$. It follows from the second equation (1) and equation (12) that n_i in the sheath is of the order of $n_0(\lambda_D/\Delta)^{2/3}$. It follows from equation (30) that $\exp\frac{e\varphi}{kT_e}$ is of the order of $(\lambda_D/\Delta)^{2/3}$; hence n_e in the sheath is of the order of $n_0(\lambda_D/\Delta)^{2/3}$, i.e. comparable to n_i as it should be.

The function $\Phi(X)$ is governed by equation (27) written in the dimensionless form

$$\frac{1}{2}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}X}\right)^2 = -X + \mathrm{e}^{-\Phi} + C_6, \tag{31}$$

where $C_6 = C_5(\Delta/\lambda_D)^{2/3}$. Boundary conditions are obtained from (19) and (29):

$$X = 0: \ \Phi = \Phi_w, \qquad X \longrightarrow \infty: \ \Phi = -\ln X + o(1), \tag{32}$$

where Φ_w is related to U by the formula

$$U = \frac{2}{3}\frac{kT_e}{e}\ln\frac{\Delta}{\lambda_D} + \frac{kT_e}{e}\Phi_w \tag{33}$$

and should be treated as a given positive parameter.

The boundary-value problem (31), (32) may be conveniently solved as follows. The constant C_6 is removed by introducing the new independent variable $\tilde{X} = X - C_6$. After this, equation (31) is solved for $\mathrm{d}\Phi/\mathrm{d}\tilde{X}$ and the obtained first-order differential equation is integrated numerically with the second boundary condition (32). Note that the function $\Phi(\tilde{X})$ is universal, i.e. does not depend on any control parameters. A graph of this function can be found in [28] and a number of subsequent works (e.g. [29, 30]). After the function $\Phi(\tilde{X})$ has been determined, the constant C_6 is found: $C_6 = -\tilde{X}_w$, where $\tilde{X}_w$ is the value of the argument at which this function attains the value Φ_w, i.e. the root of the equation $\Phi(\tilde{X}) = \Phi_w$.

As discussed above, U represents a combined voltage drop across the sheath and the adjacent plasma region and there is no way to unambiguously separate contributions of the sheath and the adjacent plasma region. On the other hand, the first term on the right-hand side of equation (33) involves parameters Δ and n_0, characterizing the plasma region, while the second term involves parameter Φ_w, characterizing the potential distribution in the sheath. Therefore, it is natural to assume, by convention, that these terms represent contributions of, respectively, the plasma region and the sheath.

The above analysis applies to the case of a moderately negative near-electrode or near-wall sheath, where the sheath voltage $kT_e\Phi_w/e$ is comparable to kT_e/e, i.e. Φ_w is of order unity. The asymptotic nature of the results, including the scalings, remains the same also if a finite ion temperature and deviations of the electron density from the Boltzmann distribution (i.e. transport of the electrons) are taken into account [25, 27].

In the case of a high-voltage near-cathode sheath, where the sheath voltage is considerably higher than kT_e/e, the sheath includes two sub-layers: the ion–electron layer and the ion layer. Scalings in the ion–electron layer are the same as those in a moderately negative sheath, i.e. the length scale, the charged-particle densities and variations of potential are of the orders of, respectively, $\lambda_D^{2/3}\Delta$, $n_0(\lambda_D/\Delta)^{2/3}$ and kT_e/e. Let us estimate orders of magnitude of parameters in the ion layer. For brevity, the consideration is restricted to the case where the sheath voltage substantially exceeds not only $\frac{kT_e}{e}$ but also $\frac{kT_e}{e}\ln\frac{\Delta}{\lambda_D}$. In this case, the voltage drop in the adjacent plasma region may be neglected compared with the sheath voltage and the latter to a first approximation equals U. In the ion layer, $-\varphi$ is of the order of U. Assuming that the term on the left-hand side of equation (27) and the first term on the right-hand side are of the same order of magnitude in the ion layer, one finds that the thickness of the ion layer is of the order of $\lambda_D^{2/3}\Delta^{1/3}\chi^{2/3}$. The electric field in the ion layer is of the order of $U/\lambda_D^{2/3}\Delta^{1/3}\chi^{2/3}$. It follows from the second equation (1) and equation (12) that n_i in the ion layer is of the order of $n_0(\lambda_D/\Delta)^{2/3}\chi^{-1/3}$. The electron density in the ion layer, being governed by the Boltzmann distribution (18), is exponentially small.

It follows from the above asymptotic estimates that the thickness (length scale) of the ion layer substantially exceeds the thickness of the ion–electron layer. Hence, the ion layer has a more or less distinct edge; however no unambiguous definition of this edge can be given to an accuracy better than $O(\chi^{-2/3})$. The voltage drop in the ion layer is much higher than that in the ion–electron layer. The electric field and the ion density in the ion layer are much stronger and, respectively, much lower than those in the ion–electron layer. The scale of ion density and the length scale in the ion layer are related by equation (26), the difference being that $n_i^{(il)} = n_0(\lambda_D/\Delta)^{2/3}\chi^{-1/3}$ and $\delta = \lambda_D^{2/3}\Delta^{1/3}\chi^{2/3}$ in the present case.

It can be shown on the basis of the above asymptotic estimates that the ion layer to a first approximation is described by the Mott–Gurney model, the error of this approximation being of the order of χ^{-1}. Except for different scalings and the Mott–Gurney model appearing in place of the Child–Langmuir model, all above are similar to what happens in the case of a collision-free high-voltage sheath and are schematically shown in figure 2.

The above-described ion–electron and ion layers also appear in the treatments [24, 26], in which a theory of high-voltage near-cathode sheaths with account of a finite ion temperature and transport of the electrons was developed in connection to spherical electrostatic probes. The ion drift in a strong electric field occurring in the ion layer significantly exceeds the ion diffusion caused by the ion pressure gradient; therefore the ion layer in the analysis [24, 26] is described by the Mott–Gurney model (with the curvature effect taken into account). Additionally, an ion diffusion layer adjacent to the cathode surface appears 'at the bottom' of the ion layer. In the ion diffusion layer, ion diffusion comes into play and the ion density rapidly falls in the direction of the cathode surface.

4. Advanced models of ion sheaths

Without trying to review all the works concerned with advanced models of ion sheath, we will consider three topics here. The first one is fluid modeling of ion sheaths with an arbitrary degree of collisionality, which spans the whole range of conditions from a collisionless sheath described by the Child–Langmuir model to a collision-dominated sheath described by the Mott–Gurney model. The second topic is a derivation of high-accuracy boundary conditions for the Child–Langmuir and Mott–Gurney models. The third topic is a mathematical interpretation of ion sheath models from the point of view of modern theoretical physics.

4.1. Ion sheath with arbitrary collisionality

A theory of an electron-free ion sheath with an arbitrary degree of collisionality (e.g. [31–36]) can be developed in the framework of the so-called fluid model. A fluid model is based on treating the ions and the atoms as separate fluids coexisting with, rather than diffusing in, each other; e.g. [37–39]. The fluid model is understood to ensure a sufficient accuracy in applications and is widely used in modeling of low-pressure gas discharges; e.g. [39] and references therein. In the case of cold ions, the fluid model of an electron-free ion sheath relies on equations (1) with the difference that the first equation (1) is written in the form of an equation of motion of the ion fluid:

$$m_i v_i \frac{dv_i}{dx} = -e\frac{d\varphi}{dx} - \frac{ev_i}{\mu_i}. \tag{34}$$

The term on the left-hand side of this equation describes the inertia force. If the sheath is dominated by collisions, this term is minor and equation (34) becomes identical to the corresponding equation of the Mott–Gurney model, equation (12). The terms on the right-hand side describe the electric-field force and a friction force resulting from elastic collisions of the ions with neutral particles. If the sheath is collision-free, the second term on the right-hand side is minor and equation (34) becomes identical to the corresponding equation of the Child–Langmuir model, first equation (1). Thus, the fluid model contains the Child–Langmuir and Mott–Gurney models as limiting cases for the collision-free and, respectively, collision-dominated sheaths and describes a smooth transition from one to the other at finite collisionalities.

Let us assume that the origin $x = 0$ is positioned at the sheath edge and the x-axis is directed to the cathode as in section 2. Then the boundary conditions (2) and (9) remain applicable.

As above, μ_i is treated as constant. In this case, the above-described problem admits a simple analytical solution [36], which can be conveniently found by treating the electric field as a new independent variable and v_i as an unknown function. This solution may be written in a universal form:

$$V = E + e^{-E} - 1, \tag{35}$$

$$\Phi = \frac{E^3}{3} - \frac{E^2}{2} + 1 - Ee^{-E} - e^{-E}, \tag{36}$$

$$\xi = \frac{E^2}{2} - E + 1 - e^{-E}, \tag{37}$$

where the dimensionless variables are defined as

$$\xi = \frac{ex}{m_i\mu_i v_i^{(0)}}, \qquad V = \frac{v_i}{v_i^{(0)}}, \qquad E = -\frac{\mu_i}{v_i^{(0)}}\frac{d\varphi}{dx},$$
$$\Phi = -\frac{e\varphi}{m_i v_i^{(0)2}}, \tag{38}$$

with the characteristic ion velocity $v_i^{(0)} = m_i\mu_i^2 j_i/e\varepsilon_0$. The solution (35)–(37) is parametric, the role of the parameter being played by the dimensionless electric field E.

The dimensionless coordinate ξ may be interpreted as a characteristic number of collisions that an ion suffers while

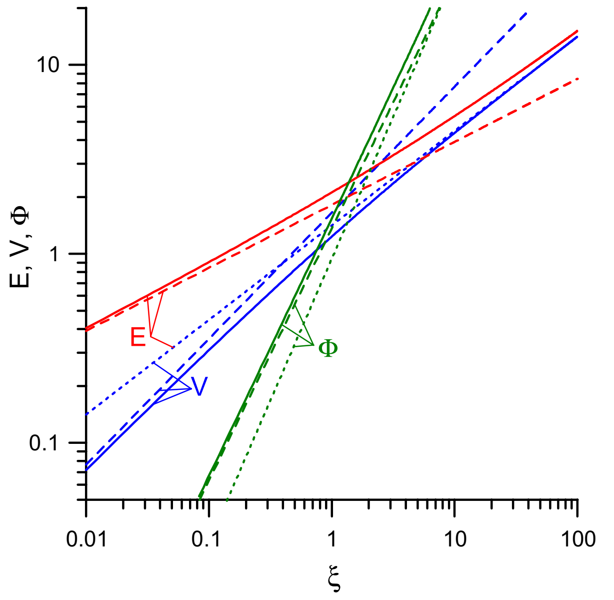

Figure 3. Solid: solution for the ion sheath with arbitrary degree of collisionality, equations (35)–(37). Dashed: the Child–Langmuir solution, equations (39). Dotted: the Mott–Gurney solution, equations (40).

crossing the distance x. Therefore, solutions that correspond to the limiting cases of collision-free and collision-dominated sheaths are obtained from equations (35)–(37) by applying the limits of small and, respectively, large ξ (or E, which is equivalent):

$$V = \frac{(3\xi)^{2/3}}{2^{1/3}}, \qquad \Phi = \frac{(3\xi)^{4/3}}{2^{5/3}}, \qquad E = (6\xi)^{1/3}. \quad (39)$$

$$V = \sqrt{2\xi}, \qquad \Phi = \frac{(2\xi)^{3/2}}{3}, \qquad E = \sqrt{2\xi}. \quad (40)$$

The second equations in (39) and (40) represent dimensionless forms of equations (4) and, respectively, (14), i.e. the Child–Langmuir and Mott–Gurney solutions, as they should.

The solution (35)–(37) is shown in figure 3. Also shown are solutions that correspond to the Child–Langmuir and Mott–Gurney models. As expected, the fluid model describes a smooth transition between the Child–Langmuir and Mott–Gurney models.

4.2. High-accuracy boundary conditions for the Child–Langmuir and Mott–Gurney models

There are two sources of errors in the Child–Langmuir and Mott–Gurney models of electron-free sheath compared with the corresponding models taking into account the presence of the electrons (the Bohm model and, respectively, the Su–Lam–Cohen model with cold ions and Boltzmann-distributed plasma electrons): the disregard of the electron density term of the Poisson equation and the trivial boundary conditions at the sheath edge. The electron density term of the Poisson equation in the bulk of the sheath is exponentially small in the large parameter $\chi = eU/kT_e$, so the error caused by dropping this term also is exponentially small. The error caused by the trivial boundary conditions is of algebraic order ($\chi^{-1/2}$ for the Child–Langmuir model and χ^{-1} for the Mott–Gurney model), i.e. essentially higher, and it is this error that limits the accuracy of the model on the whole. It is natural in such a situation to try to derive more accurate boundary conditions and thus improve the overall accuracy of the model.

There has been a significant number of works concerned with this task; see, e.g. [23, 30, 32, 33, 40–48]. Boundary conditions were found that indeed ensure the exponential accuracy of the Child–Langmuir and Mott–Gurney models [23, 43, 46]. Let us start with the Child–Langmuir model. Let us transform equation (25) to variables normalized in such a way that they are of order unity in the bulk of the sheath:

$$\frac{1}{2}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}X}\right)^2 = (\chi^{-1} + 2\Phi)^{1/2} - 2\chi^{-1/2} + \chi^{-1/2}\mathrm{e}^{-\chi\Phi}, \quad (41)$$

where $X = x/\lambda_D\chi^{3/4}$, $\Phi = -\varphi/U$. The right-hand side of equation (41) contains three terms which are small at large χ, all of them of different orders of smallness. The simplest model of an ion layer is obtained by dropping all the three terms, i.e. by retaining only the term $(2\Phi)^{1/2}$. One can readily see that this approximation corresponds to the original Child–Langmuir model, in which the ion velocity and electric field at the edge of the ion layer are neglected as well as the voltage drop in the ion–electron layer. The most accurate model is obtained by dropping only the last term on the right-hand side of equation (41), which is exponentially small, while retaining the other two terms, which are small algebraically. The resulting equation reads after transformation to the original variables

$$\frac{\lambda_D^2}{2}\left(\frac{e}{kT_e}\right)^2\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2 = \left(1 - \frac{2e\varphi}{kT_e}\right)^{1/2} - 2. \quad (42)$$

This equation, which coincides with equation (25) without the last term on the right-hand side, describing the electron density, represents the desired exponential-accuracy model of an ion layer. The procedure of its derivation is equivalent to asymptotic matching of a solution describing the region where $-\varphi \gg kT_e/e$, i.e. the ion layer, with a solution describing the region where $-e\varphi/kT_e$ is of order unity, i.e. the ion–electron layer.

Let us recast the exponential-accuracy model (42) into a form of the Child–Langmuir model with new (non-trivial) boundary conditions at the edge of the ion layer (electron-free sheath). First, one must define what is the edge of the ion layer in this new model; for example, it can be associated with a given value of the electric field, or the potential, or the ion velocity. A natural definition which is usual in multi-scale asymptotic methods is obtained by assuming that the model is used in the whole region where its solution exists; thus the edge of the ion layer is a point at which the ion-layer solution breaks down. A solution to equation (42) exists provided that the right-hand side of this equation is non-negative, i.e. at $\varphi \leqslant -3kT_e/2e$, and breaks down at a point where φ described by this solution reaches the value $-3kT_e/2e$ or, in other words, where the electric field described by this solution vanishes. It follows from equation (21) that $v_i = -2u_B$ at this point. Thus, the above definition results in the following boundary conditions for an ion-layer solution at the edge of the ion layer:

$$\frac{d\varphi}{dx} = 0, \qquad v_i = -2u_B, \qquad \varphi = -\frac{3}{2}\frac{kT_e}{e}. \tag{43}$$

Boundary conditions (43) admit a simple interpretation shown in figure 2: the ions are accelerated in the ion–electron layer from the Bohm velocity, $v_i = -u_B$, on the plasma side of the ion–electron layer, to twice the Bohm velocity, $v_i = -2u_B$, at the edge of the ion layer; the voltage drop in the ion–electron layer is $\frac{3}{2}\frac{kT_e}{e}$. We stress that these boundary conditions have a merely illustrative purpose; they have been obtained with the use of an extrapolation (the ion-layer solution loses its validity before the electric field described by this solution vanishes) and do not amount to assuming that there is a real physical point at which $d\varphi/dx = 0$ and $v_i = -2u_B$ (of course, no such point exists: the electric field at a point where $v_i = -2u_B$ is not exactly zero; however it is much smaller than the electric field inside the ion layer).

One can readily check that the Child–Langmuir equations (1) jointly with boundary conditions (43) are exactly equivalent to the exponential-accuracy model (42). In other words, boundary conditions (43) indeed ensure the exponential accuracy of the Child–Langmuir model. A further discussion of the exponential-accuracy model of a collision-free ion layer and a comparison of its results with exact results as well as results given by various approximate models can be found in [46].

An exponential-accuracy model for a collision-dominated ion layer formed by cold ions in a weakly ionized plasma may be developed in a similar way [43]. If the frequency of collisions ion–atom does not depend on velocity and the ion mobility may be treated as constant, then the exponential-accuracy model is equivalent to the Mott–Gurney model with the following boundary conditions at the edge of the ion layer:

$$\frac{d\varphi}{dx} = 0, \qquad \varphi = -\frac{2}{3}\frac{kT_e}{e}\ln\frac{\Delta}{\lambda_D} - 1.0082\frac{kT_e}{e}. \tag{44}$$

The first boundary condition represents a definition of an edge of the ion layer: it is once again defined as a point where the extrapolation of the ion-layer electric field vanishes. The first and second terms on the right hand-side of the second boundary condition may be interpreted as the voltage drop in the plasma region adjacent to the sheath where the ion flux to the cathode surface is generated and, respectively, the voltage drop in the ion–electron layer. The numerical coefficient in the second term was determined by means of numerical calculations [43, 47] and is quite close to unity, meaning that the voltage drop in the collision-dominated ion–electron layer is very close to kT_e/e.

The above-described models ensure the accuracy of several per cent for sheath voltages exceeding kT_e/e three or four times or more. Since the voltage drop across a sheath on a floating surface is no smaller than approximately $4kT_e/e$, these models are accurate enough for all near-cathode and near-wall sheaths.

4.3. Mathematical sense of Child–Langmuir type models from the point of view of modern theoretical physics

The mathematical sense of Langmuir's model of an electron-free ion sheath with the ion velocity, electric field and potential vanishing at the sheath edge became clear after the method of matched asymptotic expansions was developed in the 1950s–1960s and applied to the problem of plasma-electrode transition: the model perfectly fits into the formalism of matched asymptotic expansions in the large parameter equal to the ratio of the sheath voltage to the electron temperature, and a solution to this model represents the first term of an asymptotic expansion valid in the bulk of the space-charge sheath. Note that a similar statement can be made concerning Bohm's model of a collisionless ion–electron sheath [12]: it perfectly fits into the formalism of matched asymptotic expansions in the small parameter equal to the ratio of the Debye length to the ion mean free path, and Bohm's solution represents the first term of an asymptotic expansion describing a collisionless space-charge sheath with cold ions on a moderately negative surface. Such a fit is a manifestation, on the one hand, of the power of physical intuition of Langmuir and Bohm, and on the other hand, of the capability of the method of matched asymptotic expansions to reveal and exploit the underlying physics of layers of fast variations, such as viscous boundary layers in fluid mechanics, shock waves in gas dynamics, skin layers in electromagnetic theory or, in this case, different layers constituting a space-charge sheath or separating a sheath from the plasma.

The method of matched asymptotic expansions has been widely used for treating problems with layers of fast variation in different areas of theoretical physics, in particular, in fluid mechanics. One of the very important advantages of this method with respect to intuitive methods is that the method of matched asymptotic expansions provides an estimate of accuracy of a solution. Solutions obtained by means of this method are smooth, in contrast to solutions given by intuitive methods, which inevitably suffer discontinuities, or have discontinuous derivatives, at boundaries separating different regions. Another very important advantage is that this is a regular procedure in the sense that the results do not depend on the researcher's taste, in contrast to intuitive methods which usually lead different researchers to different solutions of the same problem.

In the course of the procedure of the method of matched asymptotic expansions, all mathematical simplifications which are justified by the underlying physics will be duly revealed and exploited. In other words, the method of matched asymptotic expansions produces relevant physical information at a minimal effort; thus the idea that this method is unnecessarily heavy mathematically, which is encountered in some works on the theory of collisionless and moderately collisional space-charge sheaths, is unjustified.

Another sometimes encountered idea is that the method of matched asymptotic expansions requires a great deal of mathematical expertise. This idea is also unjustified: one can readily verify by inspecting textbooks (e.g. [14–19]) that the formalism of this method is not complex and does not require specific mathematical skills. Equally unjustified is the idea that the method of matched asymptotic expansions produces results which are not robust enough for engineering purposes: such results always have a distinct physical sense and, if reduced to the same level of description, are no more difficult to use than results provided by any minimally reasonable intuitive approach. In fact, asymptotic results are usually easier to use, since they account for only effects that are pertinent at the level of accuracy being adopted, while intuitive results in most cases take into account some effects which are below the adopted level of accuracy and therefore unnecessarily complicate the results.

As far as space-charge sheaths are concerned, the method of matched asymptotic expansions plays a dominant role in the theory of collision-dominated sheaths. The situation in the theory of collisionless and moderately collisional sheaths for some reason is different: the intuitive approach based on patching (i.e. 'gluing' solutions at one point rather than asymptotically matching them) is still in use, and a heated discussion around the two methods has unfolded during the last decade, e.g. [49–52] and references therein.

Without trying to follow the above-mentioned discussion, we note that the most instructive way of comparing different methods is to apply them to several clear-cut examples. In [46], results given by the Child–Langmuir model, patching and the method of matched asymptotic expansions were compared in three examples. One of the examples was a collision-free dc sheath, treated in sections 3.1 and 4.2. The second example was a matrix sheath. The third example was a high-voltage collisionless capacitive RF sheath, driven by a sinusoidal current source, with the ions responding to the time-averaged electric field and the inertialess electrons responding to the instantaneous electric field. (An elegant analytical solution to this problem was given in [53–55]; note that an analytical solution for the case of a collision-dominated RF sheath was given in [56].)

In all three examples, the Child–Langmuir model and patching provide results which are accurate to the first approximation in the sheath voltage but not to the second, irrespective of details of patching. In order to illustrate this statement, let us turn to the example of an RF sheath. In [46], this problem was treated by means of the method of matched asymptotic expansions with an asymptotic large parameter κ equal to the fourth power of the dimensionless amplitude of the driving current. A two-term expansion of the dimensionless (normalized by kT_e/e) sheath voltage was found, with the first and second terms being of the orders of κ and $\kappa^{1/2}$, respectively. The first term represents an analogue for an RF sheath of the Child–Langmuir solution for a dc sheath. The solutions [53, 55] and [54] correctly describe the first term, i.e. the Child–Langmuir type solution, but not the second. Furthermore, the order of the second term in the solution [53, 55] is incorrect: $\chi^{3/4}$ instead of $\chi^{1/2}$. It follows that the Child–Langmuir type solution and the solution [54] are more accurate than the solution [53, 55]: terms missing from the Child–Langmuir type solution and calculated incorrectly in the solution [54] are of the order of $\kappa^{1/2}$, which is lower than the order of the terms calculated incorrectly in the solution [53, 55] ($\kappa^{3/4}$).

In all the three examples, the method of matched asymptotic expansions ensures a higher or considerably higher asymptotic accuracy than the Child–Langmuir model and patching. The same is true for numerical accuracy, as was demonstrated in [46] by means of a comparison with exact solutions (in the cases of dc and matrix sheaths) and the numerical simulations [57] (in the case of an RF sheath).

These and other examples clearly show that intuitive considerations can hardly improve the quality of results beyond those given by simple Child–Langmuir type models. If not satisfied with a simple Child–Langmuir type model, one should better resort to a standard tool, i.e. to the method of matched asymptotic expansions, or alternatively directly computed solutions (but one has to recognize that in the latter case the number of parameters is too large to obtain simple analytic approximations). They would give more accurate and reliable results than intuitive considerations.

One more comment on Bohm's 1949 paper [12] seems to be in place. Authors of the recent work [52] assumed that Bohm was trying to patch the plasma and sheath solutions rather than to asymptotically match them and found on these grounds misinterpretations and contradictions in Bohm's work. One should admit that not all aspects of Bohm's work [12] are presented equally clearly; in particular, the term 'sheath edge' in the context on an ion–electron sheath model is appropriate for patching and not matching. On the other hand, there are clear indications towards matching in the text of the paper [12], for example, on p 79: '... plasma fields, compared with sheath fields, are so small that they produce negligible changes of potential over distances many sheath thicknesses in extent. To a first approximation, therefore, it may be assumed that the plasma potential is constant, at least in so far as the processes involved in sheath formation are concerned. However, the plasma fields cannot be completely neglected, because over the long distances that they cover they are able to accelerate positive ions up to appreciable energies...'. These words reflect the very essence of the asymptotic multi-scale approach.

The clearest indication of the intentions of a theoretical physicist comes from his formulae. It is mentioned above that the solution [12] without a single change represents the first term of an asymptotic expansion describing a collisionless space-charge sheath with cold ions on a moderately negative surface. If Bohm's solution is correct from the point of the method of matched asymptotic expansions and contains misinterpretations and contradictions from the point of view of patching, is it not legitimate to assume that Bohm aimed at—and succeeded in—what is called today matching rather than patching?

Bohm's paper [12] was written before multi-scale asymptotic methods were developed. The term 'sheath edge', while being misleading in the framework of Bohm's model of an ion–electron sheath, is relevant to Langmuir's model of an electron-free ion sheath (see discussion in section 3.1), and this may have influenced Bohm. If this term is replaced by 'region on distances from the electrode much larger than the local Debye length but much smaller than a characteristic length scale in the quasi-neutral plasma', then the corresponding statements in [12] will become consistent and appropriate.

5. Conclusions

We have tried to trace the footprint of the Child–Langmuir model of an electron-free ion sheath with the ion velocity, electric field and potential vanishing at the sheath edge. It is no exaggeration to say that this model gave the origins of the modern theory of near-electrode and near-wall space-charge sheaths and still remains the most widely used model of a collisionless near-cathode sheath.

From the point of view of modern theoretical physics, the ion sheath model perfectly fits into the formalism of the method of matched asymptotic expansions and represents the first term of an asymptotic expansion in the large parameter equal to the ratio of the sheath voltage to the electron temperature, describing the bulk of the space-charge sheath. The method of matched asymptotic expansions has given the most accurate and reliable results in the analytical theory of sheaths and sheath–plasma transition. In particular, it has provided a better understanding and improvement of the accuracy of the ion sheath model.

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