Smashing localizations of rings of weak global dimension at most one

SMASHING LOCALIZATIONS OF RINGS OF WEAK GLOBAL DIMENSION AT MOST ONE

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Abstract. We show for a ring R of weak global dimension at most one that there is a bijection between the smashing subcategories of its derived category and the equivalence classes of homological epimorphisms starting in R. If, moreover, R is commutative, we prove that the compactly generated localizing subcategories correspond precisely to flat epimorphisms. We also classify smashing localizations of the derived category of any valuation domain, and provide an easy criterion for the Telescope Conjecture (TC) for any commutative ring of weak global dimension at most one. As a consequence, we show that the TC holds for any commutative von Neumann regular ring R, and it holds precisely for those Prüfer domains which are strongly discrete.

Contents

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Introduction

If R is a ring and $\mathbf{D}(R)$ its unbounded derived category, it is usually hopeless to try to understand all objects of $\mathbf{D}(R)$. A fruitful and recently extensively studied approach is to try to understand the inner structure of $\mathbf{D}(R)$ through various localizations of $\mathbf{D}(R)$. As demonstrated by our present paper and also elsewhere, triangulated localization theory provides a fascinating natural meeting point for abstract homotopy theory, algebraic geometry, homological algebra, module theory and other fields.

However, only compactly generated localizations of $\mathbf{D}(R)$ with R commutative are well understood in general. Going back to results of Devinatz, Hopkins and Smith [DHS88], and Neeman [Nee92a], the classification was finished by Thomason [Tho97]. These results have been recently considerably extended and further interesting applications found by Balmer [Bal05] and Benson, Iyengar and Krause [BIK08, BIK11]. For more general localizations, the situation remains not so clear. To understand all Bousfield localizations of $\mathbf{D}(R)$ is generally an extremely difficult problem as illustrated in [Nee00, DP08, Ste12].

However, there is an intermediate class of so-called smashing localizations—those where the localization functor is given by tensoring. In contrast to the present state of art in stable homotopy theory, in the case of derived categories of rings of weak global dimension ≤ 1 this is a perfectly tractable class. One of our main results is a complete classification of smashing localizations of $\mathbf{D}(R)$ for a valuation domain R. This seems to give one of a very few positive results for non-compactly generated localizations of $\mathbf{D}(R)$ with R non-noetherian.

Of course, smashing localizations are also intimately related to the Telescope Conjecture from the works of Bousfield and Ravenel [Bou79, Rav84]. The conjecture asks whether every smashing localization is compactly generated. In fact, it makes more sense to ask whether a particular triangulated category satisfies the Telescope Conjecture as there are derived categories which do not have this property [Kel94b]. Although in the original setting, for the stable homotopy category, the answer seems still unclear, for $\mathbf{D}(R)$ with $\operatorname{w.gl.dim} R \leq 1$ we are sometimes even able to provide a list of all smashing localizations which are not compactly generated. Our hope is that this new light shed on the problem will foster further research and in the end leads to better understanding of triangulated localizations.

Let us briefly list the highlights of the present paper.

(1) In Theorem 4.10 we explain the reason for the assumption of weak global dimension ≤ 1. In general, smashing localizations of $\mathbf{D}(R)$ for R not necessarily commutative are in bijection with equivalence classes of homological epimorphisms in the homotopy category of dg algebras. If $\operatorname{w.gl.dim} R \leq 1$, it suffices to study classical homological epimorphisms of rings. This often allows to study smashing localizations in the module rather than in the derived category.
(2) If, moreover, R is commutative, we will show in Theorem 7.8 that compactly generated localizations correspond precisely to flat ring epimorphisms $f\colon R \to S$.

(3) In Theorem 6.23 we classify all smashing localizations of $\mathbf{D}(R)$ with R a valuation domain. We will show that knowing $\operatorname{Spec} R$ as a topological space is in general not enough to determine the lattice of smashing localizations, but knowing in addition which prime ideals are idempotent suffices. In particular, we immediately see which of the localizations are flat and whether the Telescope Conjecture holds for $\mathbf{D}(R)$.
(4) For commutative rings R of weak global dimension ≤ 1 we are able to combine (2) and (3) in Theorem 8.2 to get a simple criterion for the Telescope Conjecture for $\mathbf{D}(R)$. In particular we show that the conjecture holds for any commutative von Neumann regular ring R, generalizing a result from [Ste12].

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1. Smashing localization of triangulated categories

Let $\mathcal{T}$ be a triangulated category with small coproducts and let us denote the suspension functor by Σ. We refer to [Nee01] for the definitions and abstract theory.

Smashing localizations and smashing subcategories arise naturally if, moreover, $\mathcal{T}$ admits a closed symmetric monoidal structure which is compatible with the triangulated structure in the sense of [HPS97, Appendix A.2]. Such categories are also called tensor triangulated; see [BF11]. This situation arises in particular when one considers the stable homotopy category of spectra with the smash product, or the derived category $\mathbf{D}(R)$ of a commutative ring together with the usual derived tensor product $\otimes^{\mathbf{L}}_R$.

Here, however, we shall mostly focus on a different branch of the theory of smashing localizations which does not require any monoidal structure on $\mathcal{T}$. Our main references are Pauksztello [Pau09] and Nicolás and Saorín [NS09].

1.1. Bousfield localization.

When $\mathcal{T}$ is a triangulated category with coproducts, we are often interested only in Verdier quotients $\mathcal{T}/\mathcal{X}$ (see [Nee01, Ch. 2]) such that the canonical functor $q^*\colon \mathcal{T} \to \mathcal{T}/\mathcal{X}$ preserves coproducts. Equivalently, the class $\mathcal{X}$ is localizing in the following sense.

Definition 1.1. A full subcategory $\mathcal{X}$ of $\mathcal{T}$ is *localizing* if it is triangulated and closed under set indexed coproducts.

Note that such an $\mathcal{X}$ is automatically closed under direct summands by [Nee01, 1.6.8]. In this situation it often happens that q^* has a right adjoint $q_*\colon \mathcal{T}/\mathcal{X} \to \mathcal{T}$. In fact, the existence of q_* is often equivalent to the fact that $\mathcal{T}/\mathcal{X}$ has small homomorphism spaces, so that it is a category in the usual sense (see for instance [Kra00, Lemma 3.5] or [Nee01, Proposition 9.1.19]).

The existence of the right adjoint q_* to the localization functor q^* is equivalent to the existence of a right adjoint functor $i^!$ to the inclusion functor $i_*\colon \mathcal{X} \to \mathcal{T}$. This is further equivalent to the existence of a so-called Bousfield localization functor $L\colon \mathcal{T} \to \mathcal{T}$ such that $\operatorname{Ker} L = \mathcal{X}$; see for instance [Kra10, Proposition 4.9.1].

Definition 1.2. A *Bousfield localization functor* is a triangulated endofunctor $L\colon \mathcal{T} \to \mathcal{T}$ together with a natural transformation $\eta : Id_{\mathcal{T}} \longrightarrow L$ with $L\eta : L \longrightarrow L^2$ being invertible and $L\eta = \eta L$. The objects in the essential image of L are called *L-local* and the objects in $\operatorname{Ker} L$ are called *L-acyclic*.

A very convenient way to describe a Bousfield localization functor L is via a triangulated analogue of a torsion pair. If $L\colon \mathcal{T} \to \mathcal{T}$ is such a functor, then the pair $(\mathcal{X}, \mathcal{Y}) = (\operatorname{Ker} L, \operatorname{Im} L)$ of full subcategories of $\mathcal{T}$ enjoys the following properties:

(1) $\mathcal{X} = \Sigma\mathcal{X}$ and $\Sigma\mathcal{Y} = \mathcal{Y}$;
(2) $\mathcal{T}(X, Y) = 0$ for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$;
(3) For each $W \in \mathcal{T}$, there is a triangle of the form
$$X \longrightarrow W \longrightarrow Y \longrightarrow \Sigma X$$
with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

It is well known that the map $X \to W$ in a triangle as in (3) is always an $\mathcal{X}$-coreflection, and the map $W \to Y$ is a $\mathcal{Y}$-reflection. Moreover, $\mathcal{X}$ and $\mathcal{Y}$ are triangulated subcategories of $\mathcal{T}$ and determine each other: $\mathcal{X} = {}^{\perp}\mathcal{Y}$ and $\mathcal{Y} = \mathcal{X}^{\perp}$. Here we use the following notation for so-called *perpendicular classes* to a class of objects $\mathcal{C} \subseteq \mathcal{T}$:

$$\mathcal{C}^{\perp} = \{X \in \mathcal{T} \mid \operatorname{Hom}_{\mathcal{T}}(\Sigma^n C, X) = 0 \text{ for all } C \in \mathcal{C} \text{ nad } n \in \mathbb{Z}\},$$
$${}^{\perp}\mathcal{C} = \{X \in \mathcal{T} \mid \operatorname{Hom}_{\mathcal{T}}(X, \Sigma^n C) = 0 \text{ for all } C \in \mathcal{C} \text{ nad } n \in \mathbb{Z}\}.$$

It is also a standard fact (see for instance [Nee01, Ch. 9] or [Kra10, §4.9]) that on the other hand a pair $(\mathcal{X}, \mathcal{Y})$ satisfying (1)–(3) determines a Bousfield localization functor L with $\operatorname{Ker} L = \mathcal{X}$, and such an L is unique up to a suitably defined natural equivalence. More precisely, if L' is another such functor together with $\eta' : Id_{\mathcal{T}} \longrightarrow L'$, then there is a unique natural equivalence $\xi\colon L \to L'$ such that $\eta' = \xi\eta$.

Needless to say that this setup has been observed in other situations. Bondal and Orlov [BO95] call such $(\mathcal{X}, \mathcal{Y})$ semiorthogonal decompositions of $\mathcal{T}$. The pair $(\mathcal{X}, \mathcal{Y})$ can also be viewed as a t-structure with a trivial heart in the sense of Beilinson, Bernstein and Deligne [BBD82].

1.2. **Smashing localization and TTF triples.** If $\mathcal{T}$ admits a closed symmetric monoidal structure $(\mathcal{T}, \otimes, \mathbb{1})$, one may ask under which conditions a Bousfield localization functor is equivalent to $- \otimes Y$ for some $Y \in \mathcal{T}$. This in particular happens for localizations generated by a small set of compact objects [HPS97, Theorem 3.3.3]:

Definition 1.3. An object $C \in \mathcal{T}$ of an additive category with arbitrary small coproducts is called *compact*, if the natural homomorphism
$$\coprod_{i \in I} \mathcal{T}(C, Y_i) \longrightarrow \mathcal{T}(C, \coprod_{i \in I} Y_i)$$

is an isomorphism for any small collection $(Y_i \mid i \in I)$ of objects of I.

A Bousfield localization functor is called *compactly generated* if there is a small set $\mathcal{C} \subseteq \mathcal{T}$ of compact objects such that the class of L-local objects is equal to $\mathcal{C}^\perp$. Equivalently we may require that the set of L-acyclic objects is the smallest localizing subcategory of $\mathcal{T}$ containing $\mathcal{C}$; see [Nee92b, Lemma 1.7] and §1.1.

One feature of a localization functor of the form $L = - \otimes Y$ is that L preserves coproducts in $\mathcal{T}$. As the latter property rather often characterizes localizations of the form $L = - \otimes Y$ (see [HPS97, Definition 3.3.2]), it was taken by Krause [Kra00, Kra05] as the definition of a smashing localization in the absence of a tensor product:

Definition 1.4. A Bousfield localization functor $L\colon \mathcal{T} \to \mathcal{T}$ is called *smashing* if it preserves coproducts. A localizing class $\mathcal{X} \subseteq \mathcal{T}$ is called *smashing* if it is the class of acyclic objects for a smashing localization functor.

If we do not wish to refer to the localization functor explicitly, we can use the following lemma:

Lemma 1.5. *Let $\mathcal{X} \subseteq \mathcal{T}$ be a localizing subcategory. Then the following are equivalent:*

(1) *$\mathcal{X}$ is smashing.*
(2) *The inclusion functor $i_*\colon \mathcal{X} \to \mathcal{T}$ admits a right adjoint $i^!$ and the perpendicular class $\mathcal{X}^\perp$ is closed under small coproducts.*

Proof. See the argument in [HPS97, Definition 3.3.2]. □

If now L is a smashing localizing functor and $\mathcal{Y} = \operatorname{Im} L$ is the class of L-local objects, it is again a localizing class. This suggests that there should exist another Bousfield localization $L'\colon \mathcal{T} \to \mathcal{T}$ such that $\mathcal{Y} = \operatorname{Ker} L'$. This is indeed often the case, assuming a technical condition on $\mathcal{T}$. To this end, it suffices that every cohomological (in the sense of [Nee01, Remark 1.1.9]) functor $F\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ which preserves small products is representable, i.e. isomorphic to $\mathcal{T}(-, E)$ for an object $E \in \mathcal{T}$. Note that any compactly generated or well generated triangulated category in the sense of [Nee01, Kra10] has this property. In particular, the unbounded derived category $\mathbf{D}(R)$ of any ring R (commutative or not) is an example, see [Kel98, §8.1.3].

Now we can give the characterization, which closely relates smashing localizations to recollements [Kra10, §4.13] (see also Remark 3.4).

Definition 1.6. A *torsion-torsion-free triple* (*TTF triple* for short) on a triangulated category $\mathcal{T}$ is a triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ of full subcategories of $\mathcal{T}$ such that both $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ enjoy properties (1)–(3) stated at the end of §1.1. Equivalently, $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ both determine t-structures on $\mathcal{T}$ in the sense of [BBD82].

Proposition 1.7. *Let $\mathcal{T}$ be a triangulated category with small coproducts such that every cohomological functor which preserves small products is representable. Let $L\colon \mathcal{T} \to \mathcal{T}$ be a Bousfield localization functor and $\mathcal{X} = \operatorname{Ker} L$. Then the following are equivalent:*

(1) $\mathcal{X}$ *is smashing;*
(2) *There is a TTF triple* $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ *on* $\mathcal{T}$.

Proof. See the proof of [Kra10, Proposition 5.5.1]. □

2. The homotopy categories of dg modules and algebras

In this section, we recall basics about homotopy categories of dg modules and, more importantly, dg algebras. One could view this material as preliminaries to Sections 3 and 4. Although the presented results are known, we will rely on precise manipulation with dg algebras and also dg bimodules which are homotopically projective from one side and it seems convenient to have the necessary background collected here.

Given a ring R, we denote by $\mathbf{C}(R)$ the category of cochain complexes of right R-modules. It is well known (see [Hov99, §2.3]) that $\mathbf{C}(R)$ carries a model structure such that:

(1) Weak equivalences are the quasi-isomorphisms.
(2) Fibrations are precisely the epimorphisms in $\mathbf{C}(R)$ (i.e. the maps of complexes which are componentwise surjective).

In particular every object is fibrant. Moreover, cofibrations are precisely monomorphisms with cofibrant cokernel, a cofibrant object has all components projective, and trivially cofibrant objects are precisely the projective objects in $\mathbf{C}(R)$. Various names are used for cofibrant objects in this context: K-projective complexes [Spa88], complexes with property (P) [Kel94a], homotopically projective complexes [Kel98] and probably several others. We will use the term *homotopically projective* here.

The *unbounded derived category* of R, denoted by $\mathbf{D}(R)$, is by definition the homotopy category $\operatorname{Ho}\mathbf{C}(R)$, in the sense of [Hir03, Hov99], of the model category $\mathbf{C}(R)$.

Suppose now that R, S, T are rings, X_R is a complex of right R-modules, ${}_RY_S$ is a complex of R-S-bimodules and Z_S is a complex of right S-modules. Then we can define the tensor product $X \otimes_R Y \in \mathbf{C}(S)$ in the usual way. That is,

$$(X \otimes_R Y)^i = \bigoplus_{p+q=i} X^p \otimes_R Y^q$$

and the differential $\partial^i_{X\otimes_R Y} \colon (X \otimes_R Y)^i \longrightarrow (X \otimes_R Y)^{i+1}$ is defined using the graded Leibniz rule, so that for $x \in X^p$ and $y \in Y^q$ with $p + q = i$ we have

$$\partial_{X\otimes_R Y}(x \otimes y) = \partial_X(x) \otimes y + (-1)^p x \otimes \partial_Y(y). \qquad (*)$$

It is straightforward to check that this is well-defined and that $\partial^2 = 0$.

Similarly we can define the internal Hom-functor. We define $\mathcal{H}om_S(Y, Z) \in \mathbf{C}(R)$ so that

$$\mathcal{H}om_S(Y, Z)^i = \prod_{p\in\mathbb{Z}} \operatorname{Hom}_S(Y^p, Z^{p+i})$$

for each $i \in \mathbb{Z}$, and the differential is defined as the graded commutator. That is, if $f = (f^p)_{p\in\mathbb{Z}} \in \prod_{p\in\mathbb{Z}} \operatorname{Hom}_S(Y^p, Z^{p+i})$ is a collection of morphisms of S-modules, we put

$$\partial_{\mathcal{H}om_S(Y,Z)}(f) = \partial_Z \circ f - (-1)^i f \circ \partial_Y.$$

It is a standard fact that there is an isomorphism of complexes of abelian groups

$$\mathcal{H}om_S(X \otimes_R Y, Z) \cong \mathcal{H}om_R(X, \mathcal{H}om_S(Y, Z))$$

which is natural in all three variables. In fact, all this can be done much more abstractly, see for instance [HPS97, Appendix A].

If we start with a commutative ring k, then the above specialize to functors

$$\otimes_k \colon \mathbf{C}(k) \times \mathbf{C}(k) \longrightarrow \mathbf{C}(k) \qquad \text{and} \qquad \mathcal{H}om_k \colon \mathbf{C}(k)^{\mathrm{op}} \times \mathbf{C}(k) \to \mathbf{C}(k)$$

and provides us with a closed symmetric monoidal structure on $\mathbf{C}(k)$ in the sense of [ML98, §VII.7]. A little care is due when defining the commutativity isomorphisms $\gamma_{X,Y} \colon X \otimes Y \to Y \otimes X$ as we need to introduce the so-called Koszul signs. If $x \in X^p$ and $y \in Y^q$, then $\gamma_{X,Y}(x \otimes y) = (-1)^{pq} y \otimes x$. The tensor unit is k itself viewed as a complex concentrated in degree zero.

A *dg algebra* A over k is defined as a monoid in $\mathbf{C}(k)$ in the sense of [ML98, §VII.3]. Strictly speaking, we should write (A, μ, η) instead of just A, where $\mu \colon A \otimes_k A \to A$ and $\eta \colon k \to A$ are morphisms in $\mathbf{C}(k)$, but we as usual view these as implicitly given. In more pedestrian terms, A is a $\mathbb{Z}$-graded k-algebra with a differential of degree 1 which satisfies the graded Leibniz rule with respect to multiplication. A left or right *dg module* M over A is a complex $M \in \mathbf{C}(k)$ together with a (left or right) action of A in the sense of [ML98, §VII.4]. A *dg A-B-bimodule* is a complex M with left dg A-module and right dg B-module structures which are compatible via the obvious associativity $(a \cdot m) \cdot b = a \cdot (m \cdot b)$ for each $a \in A$, $m \in M$ and $b \in B$.

We denote the category of dg algebras over k by $\mathrm{Dga}(k)$ and, following the notation from [Kel94a], the category of right dg modules over a given dg algebra A will be denoted by $\mathbf{C}(A)$. Note that if we view an ordinary k-algebra R as a dg algebra concentrated in degree 0, then the category of dg modules over R is none other than the category of complexes of R modules—that is $\mathbf{C}(R)$ is the same in both senses. If A, B are dg algebras, a dg A-B-bimodule can be viewed as module over $A \otimes_k B^{\mathrm{op}}$, where the multiplication in $A \otimes_k B^{\mathrm{op}}$ involves the corresponding Koszul signs.

The key point now is that both $\mathrm{Dga}(k)$ and $\mathbf{C}(A)$ for any fixed $A \in \mathrm{Dga}(k)$ again admit model structures such that

(1) Weak equivalences are the quasi-isomorphisms.
(2) Fibrations are the surjective maps of complexes over k.

For $\mathbf{C}(A)$ this is well known and covered in detail in [Bec14, Kel94a, SS00]. We will again call the cofibrant objects in $\mathbf{C}(A)$ *homotopically projective*. The following description is available for them:

Proposition 2.1. [Kel94a, §3.1] *Let A be a dg algebra over k and $X \in \mathbf{C}(A)$ be a right dg module over A. Then X is homotopically projective if and only if X is a summand in a dg module P such that P is the union $P = \bigcup_{i \geq 0} P_i$ of a chain*

$$0 = P_0 \subseteq P_1 \subseteq P_2 \subseteq P_3 \subseteq \cdots$$

of dg submodules such that P_{i+1}/P_i is for each $i \in \mathbb{N}$ a direct sum of copies of suspensions of A.

In particular, we also have:

Corollary 2.2. *Let $X \in \mathbf{C}(A)$ be homotopically projective. Then X is projective when viewed only as a $\mathbb{Z}$-graded module over A.*

For future reference, we shall also record the following basic properties of homotopically projective modules.

Lemma 2.3. *Let A, B be dg algebras over k. Then the following hold:*

(1) *Suppose that a dg module P_A can be written as a union $P = \bigcup_{i\geq 0} P_i$ of a chain of dg A-submodules such that $P_0 = 0$ and each P_{i+1}/P_i is homotopically projective in $\mathbf{C}(A)$. Then also P is homotopically projective.*

(2) *Suppose that ${}_AX_B$ is a dg A-B-bimodule and ${}_BY$ is a dg B-module. If both ${}_AX$ and ${}_BY$ are homotopically projective as left dg modules, then so is ${}_A(X \otimes_B Y)$.*

Proof. (1) This follows from the description of homotopically projective dg modules in [ŠP12, Remark 2.15].

(2) We know that Y is a summand of $P = \bigcup_{i\geq 0} P_i$ where $P_0 = 0$ and each P_{i+1}/P_i is a coproduct of copies of suspensions of ${}_BB$. Since each $P_i \subseteq P_{i+1}$ splits as a map of graded B-modules, the chain

$$0 = X \otimes_B P_0 \longrightarrow X \otimes_B P_1 \longrightarrow X \otimes_B P_2 \longrightarrow X \otimes_B P_3 \longrightarrow \cdots$$

consists of monomorphism. Moreover, each factor $X \otimes_B P_{i+1}/P_i$ is isomorphic to a coproduct of suspensions of X, hence is homotopically projective. Thus, $X \otimes_B Y$ is a homotopically projective A-module by part (1). □

The *derived category* $\mathbf{D}(A)$ of A is then by definition the homotopy category $\operatorname{Ho}\mathbf{C}(A)$ with respect to the above model structure. Either by using standard results on model structures [Hir03, Hov99] or by referring to [Kel94a], $\mathbf{D}(A)$ is equivalent to a quotient of the category of homotopically projective dg A-modules by a two-sided ideal, the ideal of so-called *null-homotopic maps*.

The knowledge of the model structure on $\mathrm{Dga}(k)$, however, does not seem to be so widely spread among algebraists. It was first described in [Jar97] and the following proposition is a specialization of [SS00, Theorem 4.1(3)] (see also [SS00, pp. 503–504]).

Proposition 2.4. [Jar97, SS00] *Let k be a commutative ring. Then* $\mathrm{Dga}(k)$ *admits a cofibrantly generated model structure such that weak equivalences are the quasi-isomorphisms of dg algebras and fibrations are the surjective morphisms of dg algebras. In particular every dg algebra is fibrant.*

Moreover, if $f\colon A \to B$ is a cofibration of dg algebras and the underlying complex of A is homotopically projective in $\mathbf{C}(k)$, then so is the underlying complex of B. If A is a cofibrant dg algebra, this in particular means that the underlying complex of A is homotopically projective in $\mathbf{C}(k)$.

Suppose now that A, B are dg algebras, X_A is a right dg A-module, ${}_AY_B$ is a dg A-B-bimodule and that Z_B is a right dg B-module. Then we can define $X \otimes_A Y \in \mathbf{C}(B)$ and $\mathcal{H}om_B(Y, Z) \in \mathbf{C}(A)$, extending the definitions for complexes over ordinary algebras. Forgetting the differential and the right action of B for the moment, the underlying $\mathbb{Z}$-graded k-module of $X \otimes_A Y$ is obtained as the tensor product $X \otimes_A Y$ of graded modules over A as a graded

algebra. The differential is again defined by formula $(*)$. In fact, $X \otimes_A Y$ is a factor of $X \otimes_k Y$ as a right dg B-module, see [ML98, Exercise VII.4.6, p. 175]. Similarly $\mathcal{H}om_B(Y,Z)$ is a k-subcomplex of $\mathcal{H}om_k(Y,Z)$ such that $\mathcal{H}om_B(Y,Z)^i$ consists of degree i graded B-module homomorphisms $Y \to Z$; see [SS00, p. 499]. Again we have an adjunction (natural isomorphism in $\mathbf{C}(k)$)

$$\mathcal{H}om_B(X \otimes_A Y, Z) \cong \mathcal{H}om_A(X, \mathcal{H}om_B(Y,Z)).$$

Suppose now that $f\colon B \to A$ is a homomorphism of dg algebras and X_A and ${}_AY$ are dg A-modules. We can also view X and Y as B-modules via f. Later on we need the following result for the derived tensor product—this is by definition the total left derived functor $\otimes_A^{\mathbf{L}}\colon \mathbf{D}(A) \times \mathbf{D}(A^{\mathrm{op}}) \to \mathbf{D}(k)$ of the functor $\otimes_A\colon \mathbf{C}(A) \times \mathbf{C}(A^{\mathrm{op}}) \to \mathbf{C}(k)$ in the sense of [Hir03, §8.4].

Lemma 2.5. *Let $f\colon B \to A$ be a homomorphism of dg algebras which is a quasi-isomorphism, and let X_A and ${}_AY$ be dg A-modules. Then there is a natural isomorphisms $X \otimes_A^{\mathbf{L}} Y \cong X \otimes_B^{\mathbf{L}} Y$ in $\mathbf{D}(k)$.*

Proof. The key claim is that if $p\colon P_B \to X_B$ is a homotopically projective replacement of X as a B-module, then the composition of $p \otimes_B A\colon P \otimes_B A \to X \otimes_B A$ with the multiplication $\mu\colon X \otimes_B A \to X$ is a homotopically projective replacement of X as an A-module. Indeed, since p is surjective, so is $\mu \circ (p \otimes_B A)$, hence $\mu \circ (p \otimes_B A)$ is a fibration in $\mathbf{C}(A)$. Further, $P \otimes_B A$ is homotopically projective over A by Lemma 2.3(2).

It remains to prove that $\mu \circ (p \otimes_B A)$ is a quasi-isomorphism. To this end, note that p can be written as the composition

$$\begin{array}{ccc} P & \xrightarrow{\quad p \quad} & X \\ {\scriptstyle P\otimes_B f}\downarrow & & \uparrow{\scriptstyle \mu} \\ P \otimes_B A & \xrightarrow{p \otimes_B A} & X \otimes_B A. \end{array}$$

Since p is a quasi-isomorphism to start with, our task is equivalent to proving that $P \otimes_B f$ is a quasi-isomorphism. However, the latter follows from the fact that f is a quasi-isomorphism of left dg B-modules and P_B is homotopically projective as a right dg B-module. This proves the claim.

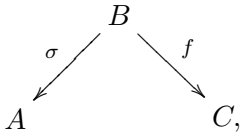
Now we have $X \otimes_B^{\mathbf{L}} Y = P \otimes_B Y$ and $X \otimes_A^{\mathbf{L}} Y = (P \otimes_B A) \otimes_A Y$ (see [Hir03, Proposition 8.4.8]) and the right hand sides are naturally isomorphic. □

3. Homological epimorphisms for dg algebras

Next we shall consider smashing localizations at the level of models, which for us means dg algebras and dg modules over them. The advantage is that building on Pauksztello's definition of a homological epimorphism of dg algebras and the results in [NS09], a smashing localization turns out to be always given by a certain tensor product, even if the triangulated category in question is not tensor triangulated. Later on we will show that for derived categories of rings of weak global dimension at most one everything simplifies to the notion of classical (homological) epimorphisms of ordinary rings.

To start with, note that every morphism $A \to C$ in the homotopy category $\mathrm{Ho}\,\mathrm{Dga}(k)$ of dg algebras over k is by Proposition 2.4 represented by a

fraction



where $\sigma\colon B \to A$ is a surjective quasi-isomorphism of dg algebras and B is cofibrant. Generalizing Pauksztello's [Pau09, Definition 3.10], we say that:

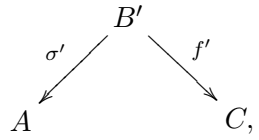
Definition 3.1. A morphism $f\sigma^{-1}\colon A \to C$ in $\operatorname{Ho}\operatorname{Dga}(k)$ (with $\sigma\colon B \to A$ and $f\colon B \to C$ as above) is a *homological epimorphism* if the canonical map $C \otimes^{\mathbf{L}}_B C \to C$ coming from the multiplication morphism is a quasi-isomorphism.

Remark 3.2. The symbol $C \otimes^{\mathbf{L}}_B C$ may cause some confusion since $- \otimes_B -$ may be viewed as a functor $\mathbf{C}(B) \times \mathbf{C}(B^{\mathrm{op}}) \to \mathbf{C}(k)$, but also $\mathbf{C}(B) \times \mathbf{C}(B^{\mathrm{op}} \otimes_k C) \to \mathbf{C}(C)$ and in other similar contexts. Firstly, it turns out that it actually does not matter for Definition 3.1 which of these functors we derive. Secondly, we need that $C \otimes^{\mathbf{L}}_B C \to C$ is a quasi-isomorphism when we view $- \otimes_B -$ as a functor $\mathbf{C}(B) \times \mathbf{C}(B^{\mathrm{op}} \otimes_k C) \to \mathbf{C}(C)$ because this is the interpretation in [NS09] and yields crucial Proposition 3.5.

We also need to prove that our notion of a homological epimorphism is well defined in the following sense:

Lemma 3.3. *The definition of a homological epimorphism in* $\operatorname{Ho}\operatorname{Dga}(k)$ *is independent of the particular choice of the representing fraction* $f\sigma^{-1}$.

Proof. If $f'(\sigma')^{-1}$ is another fraction such that $f\sigma^{-1} = f'(\sigma')^{-1}$ in $\operatorname{Ho}\operatorname{Dga}(k)$,



there is a quasi-isomorphism $\sigma''\colon B \to B'$ of dg algebras such that $\sigma = \sigma'\sigma''$ and $f \sim f'\sigma''$ where $\sim$ stands for the homotopy relation [Hov99, 1.2.4]. Then there is a cylinder object

$$B \amalg B \xrightarrow{i_0+i_1} D \xrightarrow{\tau} B$$

and a map $h\colon D \to C$ such that $hi_0 = f$ and $hi_1 = f'\sigma''$.

Since also i_0, i_1 are quasi-isomorphisms, Lemma 2.5 provides us with isomorphisms in $\mathbf{D}(k)$:

$$C \otimes^{\mathbf{L}}_{B'} C \cong C \otimes^{\mathbf{L}}_B C \cong C \otimes^{\mathbf{L}}_D C \cong C \otimes^{\mathbf{L}}_B C.$$

Here, C is viewed as a dg D-bimodule via the map h. Our final comment is regarding the double occurrence of $C \otimes^{\mathbf{L}}_B C$ in the chain of isomorphisms—this is because the first copy is taken with respect to the morphism $f'\sigma''\colon B \to C$, while the second one is with respect to $f\colon B \to C$. □

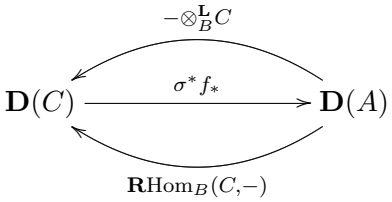
Remark 3.4. Note that if $f\sigma^{-1}\colon A \to C$ represents a homomorphism in $\operatorname{Ho}\operatorname{Dga}(k)$, then the quasi-isomorphism $\sigma\colon B \to A$ induces a triangle equivalence $\sigma_*\colon \mathbf{D}(A) \to \mathbf{D}(B)$, whose quasi-inverse is $\sigma^* = - \otimes^{\mathbf{L}}_B A\colon \mathbf{D}(B) \to$

$\mathbf{D}(A)$ (see [Kel94a, Lemma 6.1 (a)]). Hence we have a triangle functor

$$\mathbf{D}(C) \xrightarrow{\quad\sigma^* f_*\quad} \mathbf{D}(A),$$

which takes the role of the functor induced by the restriction of scalars. Then [Pau09, Theorem 3.9] says that $\sigma^* f_*$ is fully faithful if and only if $\sigma^* f_*$ is a homological epimorphism in the sense of Definition 3.1.

It is not difficult to convince oneself that then $-\otimes^{\mathbf{L}}_B C$ and $\mathbf{R}\mathrm{Hom}_B(C,-)$ are, respectively, left and right adjoint of the functor $\sigma^* f_*$. The situation can be described by the following diagram:



Denoting $\mathcal{X} = \mathrm{Ker}(-\otimes^{\mathbf{L}}_B C)$, $\mathcal{Y} = \mathrm{Im}\,\sigma^* f_*$ and $\mathcal{Z} = \mathrm{Ker}\big(\mathbf{R}\mathrm{Hom}_B(C,-)\big)$, where Ker and Im stand for the kernel on objects and the essential image, respectively, we obtain a torsion-torsion-free triple, that is a triple $(\mathcal{X},\mathcal{Y},\mathcal{Z})$ in $\mathbf{D}(A)$ as in Definition 1.6.

Following a suggestion of Pedro Nicolás, we now provide a slight improvement of the main result of [NS09] which basically says that the converse of the latter observation is true. That is, every torsion-torsion-free triple in $\mathbf{D}(A)$ occurs in this way.

Proposition 3.5. *Let A be a dg algebra over a commutative ring k and $\mathcal{X} \subseteq \mathbf{D}(A)$ be a smashing localizing class. Then there is a homological epimorphism $g = f\sigma^{-1}\colon A \to C$ in $\mathrm{Ho}\,\mathrm{Dga}(k)$, represented by homomorphisms of dg algebras $\sigma\colon B \to A$ and $f\colon B \to C$ as in Definition 3.1, such that*

$$\sigma^* f_*\colon \mathbf{D}(C) \longrightarrow \mathbf{D}(A).$$

is fully faithful, its essential image coincides with $\mathcal{X}^\perp$ and $\mathcal{X} = \{X \in \mathbf{D}(A) \mid X \otimes^{\mathbf{L}}_B C = 0\}$.

Conversely, if $g = f\sigma^{-1}\colon A \to C$ is a homological epimorphism in the category $\mathrm{Ho}\,\mathrm{Dga}(k)$, then $\mathcal{X} = \{X \in \mathbf{D}(A) \mid X \otimes^{\mathbf{L}}_B C = 0\}$ is a smashing localizing class in $\mathbf{D}(A)$.

Proof. Suppose that $\mathcal{X} \subseteq \mathbf{D}(A)$ is smashing localizing and let $\sigma\colon B \to A$ be a cofibrant replacement of A. Then B is homotopically projective in $\mathbf{C}(k)$ by Proposition 2.4 and the restriction functor $\sigma_*\colon \mathbf{D}(A) \to \mathbf{D}(B)$ is a triangle equivalence with $\sigma^* = -\otimes^{\mathbf{L}}_B A$ as a quasi-inverse. Let now $\mathcal{X}' = \mathrm{Im}\,\sigma_*$ be the essential image. It is clearly a smashing subcategory of $\mathbf{D}(B)$, so by [NS09, Theorem in §4] there exists a morphism of dg algebras $f\colon B \to C$ such that $C \otimes^{\mathbf{L}}_B C \to C$ is a quasi-isomorphism and $\mathrm{Im}\, f_* = (\mathcal{X}')^\perp$. Hence $g = f\sigma^{-1}\colon A \to C$ is a homological epimorphism in $\mathrm{Ho}\,\mathrm{Dga}(k)$ and $\mathcal{X}^\perp$ is the essential image of the fully faithful functor $\sigma^* f_*\colon \mathbf{D}(C) \longrightarrow \mathbf{D}(A)$. It also follows from [NS09, §4] that

$$\mathcal{X}' = \{X' \in \mathbf{D}(B) \mid X' \otimes^{\mathbf{L}}_B C = 0\}.$$

Transferring this along the triangle equivalence σ^* provides the formula for $\mathcal{X}$. The last part follows from Remark 3.4 and [NS09, Theorem in §5]. □

Hence there is a surjective correspondence from the class of homological epimorphisms in $\operatorname{Ho}\operatorname{Dga}(k)$ originating in A to the set (not a proper class, see [Kra00]) of smashing localizing classes in $\mathbf{D}(A)$. One might ask when exactly two homological epimorphisms $g\colon A \to C$ and $g'\colon A \to C'$ induce the same smashing localizing class. The answer is given by the following result which we will prove in a special case in the next section.

Proposition 3.6. [NS13] *Two homological epimorphisms $g\colon A \to C$ and $g'\colon A \to C'$ induce the same smashing localizing class in $\mathbf{D}(A)$ if and only if there exists an isomorphism $\varphi\colon C \to C'$ in $\operatorname{Ho}\operatorname{Dga}(k)$ such that $g' = \varphi g$.*

4. Homological epimorphisms for rings of weak dimension one

The main objects of interest in our paper are smashing localizations of the derived category $\mathbf{D}(R)$ of a ring of weak global dimension at most one. As it turns out, the situation in this case is extremely favorable in that for studying smashing localizations of $\mathbf{D}(R)$ it will be enough to consider homological epimorphisms of ordinary algebras (Definition 4.2) instead of homological epimorphisms of dg algebras (Definition 3.1). The aim of the section is to explain this reduction, which has been already known for hereditary algebras [KŠ10].

4.1. **Basics on homological epimorphisms of rings.** We start by recalling some standard facts which we will need. Let R, S be associative and unital algebras over a fixed base commutative ring k. This is no restriction at all since $k = \mathbb{Z}$ is a legal choice, but in some cases it may be convenient to take other base rings. We will denote by Mod-R and Mod-S the categories of right R-modules and S-modules, respectively. An algebra homomorphism $f\colon R \to S$ is an epimorphism if it is an epimorphism in the category of k-algebras. Ring (and algebra) epimorphisms have been investigated in [Sil67, Ste75, GdlP87, Laz69].

An algebra homomorphism $f\colon R \to S$ is an epimorphism if and only if $S \otimes_R S \cong S$, if and only if $1_R \otimes x = x \otimes 1_R$ in $S \otimes_R S$ for every $x \in S$, if and only if the restriction functor $f_*\colon \text{Mod-}S \to \text{Mod-}R$ is fully faithful (or the same holds for left modules). A direct way to present elements of S in terms of elements of R, which is essentially due to Mazet [Maz68], will be discussed later in §6.2.

Two algebra epimorphisms $f\colon R \to S$ and $f'\colon R \to S'$ are said to be *equivalent* if there exists a k-algebra isomorphism $\varphi\colon S \to S'$ such that $f' = \varphi f$. Equivalently, the essential images of f_* and f'_* in Mod-R coincide.

The following results will be useful in the sequel.

Proposition 4.1. *Let R be a commutative k-algebra and $f\colon R \to S$ an algebra homomorphism. The following hold true:*

(1) [Sil67, Corollary 1.2] *If f is an epimorphism, then S is a commutative algebra.*
(2) [Laz69, Lemma 1.1] *f is an epimorphism if and only if $f_{\mathfrak{p}}\colon R_{\mathfrak{p}} \to S \otimes_R R_{\mathfrak{p}}$ is an epimorphism for every prime ideal $\mathfrak{p}$ of R.*

Definition 4.2. A k-algebra epimorphism $f\colon R \to S$ is a *homological epimorphism* if $\operatorname{Tor}_i^R(S,S) = 0$ for every $i \geq 1$.

Homological algebra epimorphisms have been introduced and characterized by Geigle and Lenzing in [GL91], see Proposition 4.3 below. While an algebra epimorphism $R \to S$ implies that the category of S-modules is equivalent to a full subcategory of the category of R-modules, homological epimorphisms are characterized by the analogous property for derived categories.

An algebra epimorphism $f\colon R \to S$ with S flat as a left or right R-module is clearly a homological epimorphism. It is called a *flat epimorphism*.

Proposition 4.3. [GL91, 4.4] *Let R, S be k-algebras. An algebra homomorphism $f\colon R \to S$ is a homological ring epimorphism if and only if one of the following equivalent conditions holds:*

1. *$S \otimes_R S \cong {}_S S_S$ and $\operatorname{Tor}_i^R(S,S) = 0$ for every $i \geq 1$ (i.e. the natural map $S \otimes^{\mathbf{L}}_R S \to S$ is an isomorphism).*
2. *For every right S-module N and a left S-module M, the natural map $\operatorname{Tor}_i^R(N,M) \to \operatorname{Tor}_i^S(N,M)$ is an isomorphism for every $i \geq 0$ (i.e. the natural map $N \otimes^{\mathbf{L}}_R M \to N \otimes^{\mathbf{L}}_S M$ is an isomorphism).*
3. *For every S-modules M, M', the natural morphism $\operatorname{Ext}^i_S(M,M') \to \operatorname{Ext}^i_R(M,M')$ is an isomorphism for every $i \geq 0$ (i.e. the natural morphism $\mathbf{R}\operatorname{Hom}_S(M,M') \to \mathbf{R}\operatorname{Hom}_R(M,M')$ is an isomorphism).*
4. *The induced functor $f_*\colon \mathbf{D}(S) \longrightarrow \mathbf{D}(R)$ is a full embedding of triangulated categories.*

In the coming lemma we collect some easy observations about homological epimorphisms.

Lemma 4.4.

1. *The composition of homological epimorphisms is a homological epimorphism.*

If, moreover, R is a commutative ring, then also the following hold:

2. *If Σ is a multiplicative subset of R, then $R \to R\Sigma^{-1}$ is a flat epimorphism.*
3. *A ring homomorphism $f\colon R \to S$ is a homological epimorphism if and only if $f_{\mathfrak{p}}\colon R_{\mathfrak{p}} \to S \otimes_R R_{\mathfrak{p}}$ is a homological epimorphism for every prime ideal $\mathfrak{p} \in \operatorname{Spec} R$.*

Proof. (1) Let $f\colon R \to S$ and $g\colon S \to T$ be two homological epimorphisms. Clearly gf is an epimorphism and, since T is an S-bimodule, $\operatorname{Tor}_i^R(T,T) \cong \operatorname{Tor}_i^S(T,T) = 0$.

(2) Obvious.

(3) Let $i \geq 1$; then $\operatorname{Tor}_i^R(S,S) = 0$ if and only if $\operatorname{Tor}_i^R(S,S) \otimes_R R_{\mathfrak{p}} = 0$, for every prime ideal $\mathfrak{p}$ of R and $\operatorname{Tor}_i^R(S,S) \otimes_R R_{\mathfrak{p}} \cong \operatorname{Tor}_i^{R_{\mathfrak{p}}}(S \otimes_R R_{\mathfrak{p}}, S \otimes_R R_{\mathfrak{p}})$, for every prime ideal $\mathfrak{p}$ of R, since $- \otimes_R R_{\mathfrak{p}}$ is an exact functor (see also [EJ00, Theorem 2.1.11]). Thus the conclusion follows by Proposition 4.1(2). □

4.2. **An application of Künneth's theorem.** Now we specialize to not necessarily commutative k-algebras R of weak global dimension (w. gl. dim)

at most 1. That is, we require by definition that $\operatorname{Tor}_2^R(-,-) \equiv 0$, or equivalently that submodules of flat modules are flat. We start with collecting some easy facts about homological ring epimorphisms in this case. Compare also with [NS09, Example 4].

Lemma 4.5. *Let R be an algebra with $\operatorname{w.gl.dim} R \leq 1$ and let $f\colon R \to S$ be a homological epimorphism. Then the following hold:*

(1) $\operatorname{Ker} f$ *is an idempotent two-sided ideal of R and* $\operatorname{w.gl.dim} S \leq 1$.
(2) *The canonical projection $\pi\colon R \to R/\operatorname{Ker} f$ and the induced homomorphism $\overline{f}\colon R/\operatorname{Ker} f \to S$ are homological ring epimorphisms.*

Moreover, for any two sided ideal I, the canonical projection $R \to R/I$ is a homological ring epimorphism if and only if I is an idempotent two-sided ideal of R.

Proof. (1) Let $I = \operatorname{Ker} f$ and apply the functors $S \otimes_R -$ and $- \otimes_R R/I$ to the exact sequence

$$0 \longrightarrow R/I \xrightarrow{\overline{f}} S \longrightarrow S/f(R) \longrightarrow 0$$

to get $0 = \operatorname{Tor}_2^R(S, S/f(R)) \to \operatorname{Tor}_1^R(S, R/I) \to \operatorname{Tor}_1^R(S,S) = 0$ and $0 = \operatorname{Tor}_2^R(S/f(R), R/I) \to \operatorname{Tor}_1^R(R/I, R/I) \to \operatorname{Tor}_1^R(S, R/I) = 0$. Consequently $\operatorname{Tor}_1^R(R/I, R/I) = 0$. Now consider the exact sequence $0 \to I \to R \to R/I \to 0$ and apply the functor $R/I \otimes_R -$ to obtain the exact sequence

$$0 \longrightarrow R/I \otimes_R I \cong I/I^2 \longrightarrow R/I \longrightarrow R/I \otimes_R R/I \longrightarrow 0,$$

which yields $I^2 = I$, since $R/I \otimes_R R/I \cong R/I$.

By Proposition 4.3(2), $\operatorname{Tor}_2^S(-,-) \cong \operatorname{Tor}_2^R(-,-)$, hence $\operatorname{w.gl.dim} S \leq 1$.

(2) Let $I = \operatorname{Ker} f$. From the proof of part (1) $\operatorname{Tor}_1^R(R/I, R/I) = 0$, thus, π is a homological ring epimorphism. In particular, $\operatorname{Tor}_1^{R/I}(S,S) \cong \operatorname{Tor}_1^R(S,S) = 0$, since S is an R/I-bimodule. Moreover, $\overline{f}$ is clearly an algebra epimorphism, so also homological.

The previous arguments show that a two-sided ideal is idempotent if and only if $\operatorname{Tor}_1^R(R/I, R/I) = 0$, hence the last statement follows immediately. □

In order to relate this to smashing localizations of $\mathbf{D}(R)$ and homological epimorphisms of dg algebras, we state a version of Künneth's theorem.

Proposition 4.6. *Let R be an algebra with $\operatorname{w.gl.dim} R \leq 1$. Let X be a complex of right R modules and Z a complex of left R-modules. Then, the following are equivalent:*

(1) $X \otimes_R^{\mathbf{L}} Z = 0$ *in* $\mathbf{D}(\mathrm{Ab})$*;*
(2) $\operatorname{Tor}_i^R\big(H^p(X), H^q(Z)\big) = 0$ *for every $p, q \in \mathbb{Z}$ and every $i \geq 0$;*
(3) $H^p(X) \otimes_R^{\mathbf{L}} H^q(Z) = 0$*, for every $p, q \in \mathbb{Z}$.*

Proof. Let $P \to X$ be a homotopically projective replacement of X in $\mathbf{C}(R)$ in the sense of §2, so that the morphism is a quasi-isomorphism and P is homotopically projective. We have $X \otimes_R^{\mathbf{L}} Z = 0$ if and only if $H^n(P \otimes_R Z) = 0$ for every $n \in \mathbb{Z}$. The complex P has projective terms, so the coboundary module $\partial(P^n)$, where $\partial\colon P^n \to P^{n+1}$ is the differential of P, is

a flat submodule of P^{n+1} for every $n \in \mathbb{Z}$. By Künneth's theorem [CE56, Ch. VI, Theorem 3.1] there is an exact sequence:

$$0 \longrightarrow \bigoplus_{p+q=n} H^p(P) \otimes_R H^q(Z) \longrightarrow H^n(P \otimes_R Z) \longrightarrow$$
$$\longrightarrow \bigoplus_{p+q=n+1} \operatorname{Tor}_1^R\big(H^p(P), H^q(Z)\big) \longrightarrow 0. \quad (\dagger)$$

This establishes the equivalence (1) $\Leftrightarrow$ (2). The sequence (†) considered for the complexes $H^q(Z)$ and $H^p(X)$ concentrated in degree zero gives the equivalence of conditions (2) and (3). □

Suppose now that we have a homomorphism $f\sigma^{-1}\colon R \to C$ in $\operatorname{Ho}\operatorname{Dga}(k)$, assuming that R is an algebra over a commutative base ring k. If $\sigma = 1_R$ (i.e. f is represented by a morphism of dg algebras rather than a fraction), the latter proposition says, using the notation of Remark 3.4, that

$$\mathcal{X} = \operatorname{Ker}(- \otimes_B^{\mathbf{L}} C) =$$
$$= \big\{X \in \mathbf{D}(R) \mid \operatorname{Tor}_i^R\big(H^p(X), H^q(C)\big) = 0 \text{ for all } p, q \in \mathbb{Z} \text{ and } i \geq 0\big\}.$$

This is very convenient as it is enough to consider modules rather than complexes.

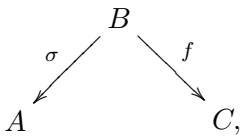
A complete analogy is true for general homomorphisms $f\sigma^{-1}$ starting at R, but more work is required. We first establish an auxiliary lemma, which is analogous to [Kel94a, Lemma 6.3].

Lemma 4.7. *Let k be a commutative ring and A be a dg algebra over k. Given a homomorphism $f\sigma^{-1}\colon A \to C$ in* $\operatorname{Ho}\operatorname{Dga}(k)$*, represented by homomorphisms of dg algebras $\sigma\colon B \to A$ and $f\colon B \to C$ as in Definition 3.1, such that C is a cofibrant dg algebra in the sense of Proposition 2.4, there exists a dg A-C-bimodule ${}_AZ_C$, homotopically projective as a left A-module, such that the functor*

$$- \otimes_A Z\colon \mathbf{D}(A) \longrightarrow \mathbf{D}(C)$$

is equivalent to $- \otimes_B^{\mathbf{L}} C\colon \mathbf{D}(A) \to \mathbf{D}(C)$ (compare to Remark 3.4).

Proof. Recall that $f\sigma^{-1}$ is a right fraction in $\operatorname{Dga}(k)$ of the form



where C is a cofibrant dg algebra and as such C is homotopically projective in $\mathbf{C}(k)$ by Proposition 2.4.

Let $v\colon {}_BV_C \to {}_BC_C$ be a homotopically projective resolution of C in $\mathbf{C}(B^{\mathrm{op}} \otimes_k C)$. Since C is homotopically projective in $\mathbf{C}(k)$, it follows from Lemma 2.3(2) that $B^{\mathrm{op}} \otimes_k C$ is homotopically projective as a left dg B-module. Applying Proposition 2.1 and Lemma 2.3(1), we deduce that any homotopically projective B-C-bimodule is a homotopically projective left dg B-module, an in particular so is ${}_BV_C$. Thus, if we put ${}_AZ_C = A \otimes_B V$,

then Z is homotopically projective in $\mathbf{C}(A)$ again by Lemma 2.3(2) and we have the following natural isomorphisms in $\mathbf{D}(C)$

$$X \otimes^{\mathbf{L}}_B C \cong X \otimes_B V \cong X \otimes_A (A \otimes_B V) = X \otimes_A Z$$

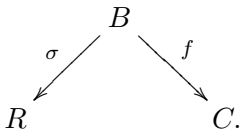
for each $X \in \mathbf{C}(A)$. □

Now we can compute the kernel of $- \otimes^{\mathbf{L}}_B C \colon \mathbf{D}(R) \to \mathbf{D}(C)$.

Lemma 4.8. *Let R be an algebra over k with* $\operatorname{w.gl.dim} R \leq 1$ *and let $f\sigma^{-1} \colon R \to C$ be a homomorphism in* $\operatorname{Ho}\operatorname{Dga}(k)$. *Using the notation of Remark 3.4, we put $\mathcal{X} = \operatorname{Ker}(- \otimes^{\mathbf{L}}_B C) \subseteq \mathbf{D}(R)$.*

Given $X \in \mathbf{D}(R)$, then $X \in \mathcal{X}$ if and only if $H^p(X) \in \mathcal{X}$ for every $p \in \mathbb{Z}$ if and only if $H^p(X) \otimes^{\mathbf{L}}_R H^q(C) = 0$ for every $p, q \in \mathbb{Z}$.

Proof. In order to give the expression $H^p(X) \otimes^{\mathbf{L}}_R H^q(C) = 0$ a good sense, we inspect the fraction



The quasi-isomorphism σ induces an isomorphism of k-algebras $R \cong H^0(B)$ and each cohomology $H^p(C)$ is naturally an $H^0(B)$-module via f.

In order to prove the proposition, we may without loss of generality assume that C is a cofibrant dg algebra over k. Indeed, otherwise we could take a trivial fibration $g \colon C' \to C$ in $\operatorname{Dga}(k)$ with C' cofibrant and, B being cofibrant, the map $f \colon B \to C$ would factor through g, keeping the class $\mathcal{X}$ unchanged.

After this reduction, we are in the situation of Lemma 4.7 and can interpret the functor $- \otimes^{\mathbf{L}}_B C \colon \mathbf{D}(R) \to \mathbf{D}(C)$ as $- \otimes_R Z$ for a suitable dg bimodule ${}_RZ_C$ which is homotopically projective as a complex of left R-modules. Proposition 4.6 now yields the equivalences

$$\begin{aligned} X \in \mathcal{X} \quad &\iff \quad H^p(X) \in \mathcal{X} \text{ for every } p \in \mathbb{Z} \\ &\iff \quad \operatorname{Tor}^R_i \big(H^p(X), H^q(Z)\big) = 0 \text{ for every } p, q \in \mathbb{Z} \text{ and } i \geq 0. \end{aligned}$$

For the last equivalence, notice that $H^q(Z) \cong H^q(C)$ as left R-modules for each $q \in \mathbb{Z}$. Indeed, consider the quasi-isomorphism $v \colon {}_BV_C \to {}_BC_C$ from the proof Lemma 4.7. Clearly $H^q(C) \cong H^q(V)$ as left R-modules via v. If we apply $- \otimes_B V$ to the quasi-isomorphisms of dg B-modules $\sigma \colon B_B \to R_B$, we get a quasi-isomorphisms $V \cong B \otimes_B V \to R \otimes_B V = Z$ since ${}_BV$ is homotopically projective in $\mathbf{C}(B^{\mathrm{op}})$, and the induced isomorphisms $H^q(V) \cong H^q(Z)$ are easily checked to be isomorphisms of left R-modules. □

We can make the statement of the last lemma even stronger, showing that only the zeroth cohomology is enough to determine the kernel of $- \otimes^{\mathbf{L}}_B C$.

Proposition 4.9. *Let R be a k-algebra such that* $\operatorname{w.gl.dim} R \leq 1$ *and $f\sigma^{-1} \colon R \to C$ be a homomorphism in* $\operatorname{Ho}\operatorname{Dga}(k)$. *Let $X \in \mathcal{D}(R)$; then the following are equivalent:*

(1) $X \otimes^{\mathbf{L}}_B C = 0$;
(2) $X \otimes^{\mathbf{L}}_R H^0(C) = 0$;

(3) $H^p(X) \otimes^{\mathbf{L}}_R H^0(C) = 0$ *for every* $p \in \mathbb{Z}$.

Proof. Consider the class $\mathcal{X} = \operatorname{Ker}(- \otimes^{\mathbf{L}}_B C) \subseteq \mathbf{D}(R)$ as in Lemma 4.8, and denote $S = H^0(C)$. Since σ is a quasi-isomorphism, $f\sigma^{-1}$ induces a homomorphism $R \to S$ of k-algebras. In view of Proposition 4.6 we ought to prove that

$$\mathcal{X} = \big\{X \in \mathbf{D}(R) \mid \operatorname{Tor}^R_i\big(H^p(X), S\big) = 0 \text{ for all } p, q \in \mathbb{Z} \text{ and } i = 0, 1\big\},$$

The inclusion $\subseteq$ is clear by Lemma 4.8. For the other inclusion, suppose that $\operatorname{Tor}^R_i\big(H^p(X), S\big) = 0$ for all $p \in \mathbb{Z}$ and $i = 0, 1$. If $q \in \mathbb{Z}$ is arbitrary, we have

$$H^p(X) \otimes_R H^q(C) = 0 \qquad \text{for all } q \in \mathbb{Z}$$

since $H^q(C)$ is a left S-module and $H^p(X) \otimes_R S = 0$ by the assumption. Since we also assume that $\operatorname{Tor}^R_i(H^p(X), S) = 0$ for all $i \geq 1$, we have isomorphisms

$$\operatorname{Tor}^R_n\big(H^p(X), H^q(C)\big) \cong \operatorname{Tor}^S_n\big(H^p(X) \otimes_R S, H^q(C)\big)$$

for all $p, q \in \mathbb{Z}$ and $n \geq 1$ by [CE56, Proposition 4.1.1] or [Mit73, The Mapping Theorem]. The latter term is zero thanks to our assumption that $H^p(X) \otimes_R S = 0$, showing that $X \in \mathcal{X}$ as required. □

Now we aim to state and prove the main result of the section. We remark that a result from [KŠ10] says the same as the theorem below, but under a much more restrictive condition that R is a one-sided hereditary ring.

Theorem 4.10. *Let R be a (possibly non-commutative) algebra of weak global dimension at most one over a commutative ring k. Then the assignment*

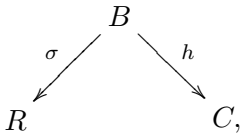
$$f \longmapsto \{X \in \mathbf{D}(R) \mid X \otimes^{\mathbf{L}}_R S = 0\}$$

is a bijection between

(1) *equivalence classes of homological epimorphisms $f\colon R \to S$ originating at R, and*
(2) *smashing localizing subcategories $\mathcal{X} \subseteq \mathbf{D}(R)$.*

Moreover, the class $\mathcal{X}$ corresponding to a given f consists precisely of the complexes $X \in \mathbf{D}(R)$ such that $H^n(X) \otimes_R S = 0 = \operatorname{Tor}^R_1\big(H^n(X), S\big)$ for all $n \in \mathbb{Z}$.

Proof. Suppose that $\mathcal{X} \subseteq \mathbf{D}(R)$ is a smashing localizing class. Then by Remark 3.4 and Proposition 3.5, viewing R as a dg algebra concentrated in degree 0, there is a homological epimorphism



in $\operatorname{Ho}\operatorname{Dga}(k)$ such that $\mathcal{X} = \{X \in \mathbf{D}(R) \mid X \otimes^{\mathbf{L}}_B C = 0\}$ in $\mathbf{D}(R)$. Let $S = H^0(C)$ and $f = H^0(h)H^0(\sigma)^{-1}\colon R \to S$ be the induced homomorphisms of ordinary k-algebras. By Proposition 4.9 we have

$$\mathcal{X} = \{X \in \mathbf{D}(R) \mid H^p(X) \otimes^{\mathbf{L}}_R S = 0 \text{ for all } p \in \mathbb{Z}\}. \qquad (\ddagger)$$

Now consider a morphism $\eta\colon R_R \to W_R$ in $\mathbf{C}(R)$ which represents an $\mathcal{X}^\perp$-reflection in $\mathbf{D}(R)$, assuming without loss of generality that W is homotopically projective. Note that, since W is up to isomorphism the image of C under the fully faithful functor $\sigma^* h_*\colon \mathbf{D}(C) \to \mathbf{D}(R)$, we have isomorphisms $\mathbf{D}(R)(W,W) \cong \mathbf{D}(C)(C,C) \cong H^0(C) = S$. Thus we have an isomorphism

$$S \cong \mathbf{D}(R)(W,W) \xrightarrow{\mathbf{D}(R)(\eta,W)} \mathbf{D}(R)(R,W) \cong H^0(W)$$

which means that $H^0(W) \cong S_R$ as R-modules. Since the mapping cone Q of η belongs to $\mathcal{X}$, equality (‡) implies that $Q \otimes_R S = 0$ and $\eta \otimes_R S$ is a quasi-isomorphism in $\mathbf{C}(S)$. In particular, η induces an isomorphism of S-modules $t\colon S \to H^0(W \otimes_R S)$ and $H^{-1}(W \otimes_R S) = 0$.

Applying Künneth's theorem to $W \otimes_R S$, we obtain a short exact sequence

$$0 \longrightarrow H^0(W) \otimes_R S \xrightarrow{i} H^0(W \otimes_R S) \longrightarrow \operatorname{Tor}_1^R(H^1(W), S) \longrightarrow 0$$

of S-modules and clearly t factors through i via the obvious morphism $S \to S \otimes_R S \cong H^0(W) \otimes_R S$ induced by f. The morphism i, being a monomorphism and a split epimorphism at the same time, is clearly an isomorphism, and so is the multiplication map $S \otimes_R S \to S$. Invoking Künneth's theorem once again, we also obtain a short exact sequence

$$0 \longrightarrow H^{-1}(W) \otimes_R S \longrightarrow H^{-1}(W \otimes_R S) \longrightarrow \operatorname{Tor}_1^R(H^0(W), S) \longrightarrow 0,$$

which tells us that $\operatorname{Tor}_1^R(S,S) = 0$. Hence $f\colon R \to S$ is a homological epimorphism.

Finally notice that if two homological epimorphisms $f\colon R \to S$ and $f'\colon R \to S'$ induce the same smashing localizing subcategory $\mathcal{X}$, then the essential images of $f_*\colon \mathbf{D}(S) \to \mathbf{D}(R)$ and $f'_*\colon \mathbf{D}(S') \to \mathbf{D}(R)$ also coincide by Remark 3.4. Clearly also $\operatorname{Im} f_* \cap \text{Mod-}R = \operatorname{Im} f'_* \cap \text{Mod-}R$, which implies that the essential images of the restriction functors

$$\text{Mod-}S \longrightarrow \text{Mod-}R \qquad \text{and} \qquad \text{Mod-}S' \longrightarrow \text{Mod-}R$$

are the same (see [AHKL11, Lemma 4.6]). Hence f and f' are equivalent homological epimorphisms by [GdlP87, Theorem 1.2]. □

5. A direct module theoretic approach

It is rather clear from the previous results that smashing localizations of $\mathbf{D}(R)$ for an algebra of weak global dimension at most one can be mostly described using module categories rather than invoking derived categories. We will show here how this approach can be worked out.

Suppose R is an algebra such that $\operatorname{w.gl.dim} R \leq 1$ and $\mathcal{X} \subseteq \mathbf{D}(R)$ is a smashing localizing class. Then Theorem 4.10 and [AHKL11, Lemma 4.6] imply that given the corresponding TTF triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ in $\mathbf{D}(R)$, there is a homological ring epimorphism $f\colon R \to S$ such that

$$\mathcal{X} = \big\{X \in \mathbf{D}(R) \mid H^n(X) \otimes_R S = 0 = \operatorname{Tor}_1^R\big(H^n(X), S\big) \text{ for all } n \in \mathbb{Z}\big\},$$
$$\mathcal{Y} = \big\{Y \in \mathbf{D}(R) \mid H^n(Y) \in \text{Mod-}S \text{ for all } n \in \mathbb{Z}\big\}.$$

Thus, both $\mathcal{X}$ and $\mathcal{Y}$ are determined by their intersections with Mod-R, which we denote $\mathcal{X}_0$ and $\mathcal{Y}_0$, respectively. Adjusting the results from [KŠ10, §2], we will see that $(\mathcal{X}_0, \mathcal{Y}_0)$ is a what is called an Ext-orthogonal pair there.

Theorem 5.1. *Let R be an algebra of* $\operatorname{w.gl.dim} R \leq 1$ *and let $f\colon R \to S$ be a homological epimorphism. Denote*

$$\mathcal{X}_0 = \big\{X' \in \text{Mod-}R \mid X' \otimes_R S = 0 = \operatorname{Tor}_1^R(X', S)\big\}.$$

Then, given any $M \in$ Mod-R, there is a 5-term exact sequence

$$\varepsilon_M\colon \quad 0 \to \operatorname{Tor}_1^R(M, S) \to X_M \to M \to M \otimes_R S \to X^M \to 0.$$

Moreover, the map $X_M \to M$ is an $\mathcal{X}_0$-coreflection and $M \to M \otimes_R S$ is an (Mod-S)*-reflection. Therefore, ε_M is unique up to a unique isomorphism and functorial in M, and the functor $M \mapsto \varepsilon_M$ commutes with direct limits.*

Proof. Let $\mathcal{X} = \{X \in \mathbf{D}(R) \mid H^p(X) \in \mathcal{X}_0 \text{ for all } p \in \mathbb{Z}\}$ be the smashing subcategory corresponding to f. In order to obtain ε_M, we simply consider the triangle

$$X \longrightarrow M \longrightarrow M \otimes^{\mathbf{L}}_R S \longrightarrow \Sigma X$$

as in §1.1 with $X \in \mathcal{X}$ and apply the cohomology functor. It follows from Theorem 4.10 that $X_M, X^M \in \mathcal{X}_0$. Further, $\operatorname{Ext}^i_R(X', N) = 0$ for each $i \geq 0$, $X' \in \mathcal{X}_0$ and $N \in$ Mod-S because of the TTF triple from Remark 3.4. The fact that we have an $\mathcal{X}_0$-coreflection and (Mod-S)-reflection in ε_M has been proved in [KŠ10, Lemma 2.9]. Finally, since both $\mathcal{X}_0$ and Mod-S are closed under direct limits in Mod-R, the assignment $M \mapsto \varepsilon_M$ commutes with direct limits by the same argument as for [KŠ10, Lemma 5.3(2)]. □

Furthermore, the essential images of Mod-$S \to$ Mod-R for homological epimorphisms $f\colon R \to S$ are simply determined by closure properties:

Proposition 5.2. *Let R be an algebra of weak global dimension at most one. Then the assignment $f \mapsto \mathcal{A} = \operatorname{Im} f_*$ yields a bijective correspondence between*

1. *equivalence classes of homological epimorphisms $f\colon R \to S$, and*
2. *subcategories $\mathcal{A} \subseteq$ Mod-R which are closed under limits, colimits and extensions.*

Proof. If we start with $\mathcal{A}$ as in (2), there is an algebra epimorphism $f\colon R \to S$ such that $\operatorname{Im} f_* = \mathcal{A}$ and f is unique up to equivalence; see [GdlP87, Theorem 1.2]. If we denote by DM the character module $\operatorname{Hom}_Z(M, \mathbb{Q}/\mathbb{Z})$, then

$$D \operatorname{Tor}_1^R(S, S) \cong \operatorname{Ext}^1_R(S, DS) \cong \operatorname{Ext}^1_S(S, DS) = 0,$$

so $f\colon R \to S$ is a homological epimorphism.

Suppose conversely that f is a homological epimorphism. Then $\operatorname{Im} f_*$ is closed under limits, colimits and extensions even without any restriction on R. Indeed, the closure under limits and colimits is clear, so suppose that we have a short exact sequence $0 \to X \to Y \to Z \to 0$ in Mod-R such that $X, Z \in \operatorname{Im} f_*$. Then we have a commutative diagram with isomorphisms in

the two marked columns

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & {\scriptstyle X\otimes_R f}\downarrow\cong & & {\scriptstyle Y\otimes_R f}\downarrow & & {\scriptstyle Z\otimes_R f}\downarrow\cong & & \\ \operatorname{Tor}_1^R(Z,S) & \longrightarrow & X\otimes_R S & \longrightarrow & Y\otimes_R S & \longrightarrow & Z\otimes_R S & \longrightarrow & 0 \end{array}$$

Since $\operatorname{Tor}_1^R(Z,S) \cong \operatorname{Tor}_1^S(Z,S) = 0$ by Proposition 4.3, $Y \otimes f$ is an isomorphism and $Y \in \operatorname{Im} f_*$. □

Finally, we mention a relation to universal localizations.

Definition 5.3. [Sch85, Ch. 4] A ring homomorphism $f\colon R \to S$ is called a *universal localization* if there exists a set $\mathfrak{S}$ of morphisms between finitely generated projective R-modules such that

(1) $\sigma \otimes_R S$ is an isomorphism of S-modules for all $\sigma \in \mathfrak{S}$, and
(2) every ring homomorphism $R \to S'$ such that $\sigma \otimes_R S'$ is an isomorphism of S'-modules for all $\sigma \in \mathfrak{S}$ factors uniquely through $f\colon R \to S$.

The following will be useful for future reference, especially in connection with the Telescope Conjecture studied in Section 8.

Proposition 5.4. *Let R be an algebra of weak global dimension at most one. If $f\colon R \to S$ is a universal localization, it is a homological epimorphism and it corresponds to a compactly generated localization in the correspondence of Theorem 4.10.*

If, moreover, R is right semihereditary (that is, every finitely submodule of a right projective module is again projective), then the bijection from Theorem 4.10 restricts to a bijection between

(1) *equivalence classes of universal localizations $f\colon R \to S$, and*
(2) *compactly generated localizing subcategories $\mathcal{X} \subseteq \mathbf{D}(R)$.*

Proof. By the construction in [Sch85, Theorem 4.1], a universal localization f of R with respect to $\mathfrak{S}$ is a ring epimorphism, and by [Sch85, Theorems 4.7 and 4.8] we have $\operatorname{Tor}_R^1(S,S) = 0$. Since all higher Tor groups vanish by our assumption, f is a homological epimorphism. The essential image of Mod-$S \to$ Mod-R consists precisely of the modules M such that $\operatorname{Hom}_R(\sigma, M)$ is an isomorphism for each $\sigma \in \mathfrak{S}$. Since a complex Y is in the essential image of $f_*\colon \mathbf{D}(S) \to \mathbf{D}(R)$ if and only if its cohomology is an S-module (see [AHKL11, Lemma 4.6]), we have

$$\operatorname{Im} f_* = \{Y \in \mathbf{D}(R) \mid \mathbf{R}\operatorname{Hom}_R(\sigma, Y) \text{ is an isomorphism for all } \sigma \in \mathfrak{S}\}.$$

Thus, the smashing localization of $\mathbf{D}(R)$ corresponding to f in the sense of Theorem 4.10 is generated by $\mathfrak{S}$, when we view its elements as 2-term perfect complexes.

Suppose now that R is right semihereditary. Then R is right coherent and the category mod-R of finitely presented right R-modules is hereditary abelian. Moreover, if C is a perfect complex in $\mathbf{D}(R)$, then $C = \bigoplus_{n\in\mathbb{Z}} H^n(C)[-n]$ and for every $n \in \mathbb{Z}$, $H^n(C)$ is a finitely presented module (see for instance [Kel07, Section 2.5] or [Kra07, Section 1.6]). Thus, any compactly generated localization of $\mathbf{D}(R)$ is generated by a set of finitely

presented R-modules and this yields precisely the same result as universally inverting (any choice of) projective resolutions of these modules. □

6. The classification for valuation domains

Now we are in a position to classify all smashing localizations (equivalently: homological epimorphisms) for valuation domains R, i.e. commutative domains whose ideals are totally ordered by inclusion. This will also reveal the amount of information which we need to know about R in order to reconstruct the lattice of smashing localizations (see [Kra05, BF11]): Knowing just the Zariski spectrum as a topological space is not enough (see Example 6.24 below), we also need to know which of the prime ideals are idempotent.

For properties of ideals of valuation domains we refer to [FS01, Chapter II]. In the sequel we will use without further mentioning it that the kernel of a homological ring epimorphism $\phi\colon R \to S$ is an idempotent ideal (see Lemma 4.5) and that an idempotent ideal of a valuation domain is a prime ideal. Note also that if $\mathfrak{p} \subseteq \mathfrak{q}$ are prime ideals, then $\mathfrak{p}$ is canonically an $R_\mathfrak{q}$-module, so that $\mathfrak{p}_\mathfrak{q} = \mathfrak{p}$ and $(R/\mathfrak{p})_\mathfrak{q} = R_\mathfrak{q}/\mathfrak{p}$.

Before starting our work on the classification, we state a useful lemma which also explains why valuation domains are a natural starting point.

Lemma 6.1. *Let R be a commutative ring. Then* $\operatorname{w.gl.dim} R \leq 1$ *if and only if $R_\mathfrak{p}$ is a valuation domain for every prime ideal $\mathfrak{p}$ of R.*

Proof. We refer to [Gla89, Corollary 4.2.6]. □

6.1. **From a homological epimorphism to a collection of intervals.** We will first show that a homological epimorphism $f\colon R \to S$ naturally induces a collection of disjoint intervals of $\operatorname{Spec} R$ satisfying certain conditions. Idempotent ideals will play an important role and, if S is semilocal, this will readily yield an explicit description of S.

Notation 6.2. We shall denote the collection of all idempotent ideals of R by $\operatorname{iSpec} R$ and view $(\operatorname{iSpec} R, \subseteq)$ as a totally ordered subset of the Zariski spectrum $(\operatorname{Spec} R, \subseteq)$.

The following are easy properties of the idempotent spectrum.

Lemma 6.3. *Let R be a valuation domain and $\mathcal{S} \subseteq \operatorname{Spec} R$ be a set. If $\mathcal{S}$ has no maximal element with respect to inclusion, then $\bigcup \mathcal{S}$ is an idempotent ideal. Any subset $\mathcal{S} \subseteq \operatorname{iSpec} R$ has a supremum and an infimum in* $\operatorname{iSpec} R$.

Proof. For the first part, it is easy to check that $\mathfrak{p} = \bigcup \mathcal{S}$ is a prime ideal. Note also that $\mathfrak{p}$ must be either idempotent or principal in $R_\mathfrak{p}$. Indeed $\mathfrak{p} \subseteq R_\mathfrak{p}$ is maximal and [FS01, Lemma 4.3(iv) and property (d), p. 69] apply. However, $\mathfrak{p}$ cannot be principal in $R_\mathfrak{p}$ since $\mathfrak{p} = \bigcup \mathcal{S}$. The second statement is a direct consequence of the first one. □

As an initial step in our classification we shall describe flat ring epimorphisms.

Proposition 6.4. *Let R be a valuation domain and let $f\colon R \to S$ be a flat ring epimorphism. Then f is injective and there is a prime ideal $\mathfrak{p}$ of R such that f is equivalent to the localization morphism $R \to R_\mathfrak{p}$.*

Proof. The kernel of f must vanish, since S is a flat R-module, hence torsion free. Localizing f at the zero ideal of R, we obtain the injective ring epimorphism $f \otimes_R Q \colon Q \to S \otimes_R Q$ where Q is the quotient field of R. Thus $f \otimes_R Q$ is an isomorphism (see [Laz69, Corollary IV.1.3] or apply Proposition 5.2 to Mod-Q) and, since S is flat, we have up to isomorphism a chain of ring embeddings $R \subseteq S \subseteq Q$. Now consider the set $\mathfrak{S} = \{r \in R \setminus \{0\} \mid r^{-1} \in S\}$. One easily checks that $\mathfrak{S}$ is a saturated multiplicative set in R, that $\mathfrak{p} = R \setminus \mathfrak{S}$ is a prime ideal, and that $S = R_{\mathfrak{p}}$. □

In order to understand general homological epimorphisms, we establish a connection between the maximal ideals of a homological factor of R and the promised collections of intervals in the poset $(\operatorname{Spec} R, \subseteq)$.

Proposition 6.5. *Let R be a valuation domain, $0 \neq f \colon R \to S$ be a homological epimorphism, and denote $\mathfrak{i} = \operatorname{Ker} f$. Then the following hold:*

(1) *There exists a prime ideal $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{i} \subseteq \mathfrak{p}$ and a surjective homological epimorphism $g \colon S \to R_{\mathfrak{p}}/\mathfrak{i}$ such that the composition $gf \colon R \to R_{\mathfrak{p}}/\mathfrak{i}$ is the canonical morphism. Moreover, there is a unique maximal ideal $\mathfrak{n}$ of S such that $g \colon S \to R_{\mathfrak{p}}/\mathfrak{i}$ is equivalent to the localization of S at $\mathfrak{n}$.*

(2) *If $\mathfrak{n}$ is a maximal ideal of S, then the localization morphism $S \to S_{\mathfrak{n}}$ is surjective and the composition*

$$R \xrightarrow{f} S \xrightarrow{\text{can}} S_{\mathfrak{n}}$$

is a homological epimorphism equivalent to $g \colon R \to R_{\mathfrak{q}}/\mathfrak{j}$, where $\mathfrak{j} \subseteq R$ is an idempotent ideal, $\mathfrak{q} = f^{-1}(\mathfrak{n})$ and $\mathfrak{j} \subseteq \mathfrak{q}$.

(3) *If $\mathfrak{n}' \neq \mathfrak{n}$ is another maximal ideal and $\mathfrak{j}' \subseteq \mathfrak{q}'$ are the corresponding primes in R with $\mathfrak{j}'$ idempotent, then the intervals*

$$[\mathfrak{j}, \mathfrak{q}] \qquad \text{and} \qquad [\mathfrak{j}', \mathfrak{q}']$$

in $(\operatorname{Spec} R, \subseteq)$ are disjoint. In particular we have

$$\operatorname{Tor}_n^R(R_{\mathfrak{q}}/\mathfrak{j}, R_{\mathfrak{q}'}/\mathfrak{j}') = 0$$

for $n \geq 0$.

Proof. Note that S is a commutative ring by Proposition 4.1(1) and that $\operatorname{w.gl.dim} S \leq 1$ by Lemma 4.5. In particular, every localization of S at a prime ideal is a valuation domain by Lemma 6.1.

By Lemma 4.5, f induces a homological ring epimorphism $R/\mathfrak{i} \to S$, where $R/\mathfrak{i}$ is a valuation domain, since $\mathfrak{i}$ is a prime ideal. Thus, without loss of generality we may assume that $\mathfrak{i} = 0$.

(1) Viewing S as an R-module via f, we shall consider its torsion submodule $t(S) = \{x \in S \mid \exists\, 0 \neq r \in R \text{ such that } rx = 0\}$. Clearly, $J = t(S)$ is an ideal of S and S/J is a torsion free, hence flat R-module. The composition $R \to S \to S/J$ is then a flat epimorphism and, by Proposition 6.4, there is a prime ideal $\mathfrak{p}$ of R such $R \to S/J$ is equivalent to the canonical morphism $R \to R_{\mathfrak{p}}$.

In particular, S/J is a local ring and we shall consider the unique maximal ideal $\mathfrak{n}$ of S which contains J. Clearly $\mathfrak{n}$ is mapped to $\mathfrak{p}$ under the surjection $g \colon S \to R_{\mathfrak{p}}$ and consequently $\mathfrak{p} = f^{-1}(\mathfrak{n})$. Note also that the composition

$h\colon R \xrightarrow{f} S \to S_{\mathfrak{n}}$ is injective. Indeed, $\operatorname{Ker} h$ consists of all elements $r \in R$ whose annihilator in S is not contained in $\mathfrak{n}$, but the R-torsion part $J = t(S)$ is contained in $\mathfrak{n}$, so $\operatorname{Ker} h = 0$. Therefore, $h\colon R \to S_{\mathfrak{n}}$ is a flat epimorphism since $S_{\mathfrak{n}}$ is a domain, and the combination of Proposition 6.4 with the equality $\mathfrak{p} = f^{-1}(\mathfrak{n})$ tells us that h and $R \to R_{\mathfrak{p}}$ are equivalent.

(2) Let now $\mathfrak{n}$ be an arbitrary maximal ideal of S and let $\psi_{\mathfrak{n}}\colon S \to S_{\mathfrak{n}}$ be the localization map. Consider the homological ring epimorphism $\psi_{\mathfrak{n}} f\colon R \to S_{\mathfrak{n}}$ and let $\mathfrak{j}$ be the kernel of $\psi_{\mathfrak{n}} f$. Then $\mathfrak{j}$ is an idempotent prime ideal of R, and $S_{\mathfrak{n}}$ is a flat $R/\mathfrak{j}$-module since $S_{\mathfrak{n}}$ is a domain. By Proposition 6.4 there is a prime ideal $\mathfrak{q}$ of R containing $\mathfrak{j}$ such that $S_{\mathfrak{n}} \cong R_{\mathfrak{q}}/\mathfrak{j}$ and we necessarily have $\mathfrak{q} = f^{-1}(\mathfrak{n})$.

Moreover, since $\mathfrak{j}S$ vanishes under the localization $\psi_{\mathfrak{n}}\colon S \to S_{\mathfrak{n}}$ by the very definition of $\mathfrak{j}$, $\psi_{\mathfrak{n}}$ canonically factors as $S \to S/\mathfrak{j}S \to (S/\mathfrak{j}S)_{\mathfrak{n}/\mathfrak{j}S} \cong S_{\mathfrak{n}}$. In particular, the $R/\mathfrak{j}$-torsion part $t'(S/\mathfrak{j}S)$ of $S/\mathfrak{j}S$ is contained in $\mathfrak{n}/\mathfrak{j}S$ and, by part (1), the epimorphism $S/\mathfrak{j}S \to (S/\mathfrak{j}S)_{\mathfrak{n}/\mathfrak{j}S}$ is equivalent to $S/\mathfrak{j}S \to (S/\mathfrak{j}S)/t'(S/\mathfrak{j}S)$. In particular, $S \to S_{\mathfrak{n}}$ is surjective.

(3) Suppose that we have two distinct maximal ideals $\mathfrak{n}, \mathfrak{n}'$ in S and the corresponding pairs $\mathfrak{j} \subseteq \mathfrak{q}$ and $\mathfrak{j}' \subseteq \mathfrak{q}'$ of ideals in R as in (2). Assume without loss of generality that $\mathfrak{j} \subseteq \mathfrak{j}'$. Since the localization map $\psi_{\mathfrak{n}}\colon S \to S_{\mathfrak{n}}$ is surjective by (2) and in particular $\mathfrak{n}$ is the unique maximal ideal containing $J = \operatorname{Ker} \psi_{\mathfrak{n}}$, we have $\operatorname{Tor}_n^S(S/J, S_{\mathfrak{n}'}) = 0$ for all $n \geq 0$. Indeed, there exists $x \in J \setminus \mathfrak{n}'$ and the multiplication by x must act on the Tor groups as zero and as an isomorphism at the same time. Thus,

$$\operatorname{Tor}_n^S(S/J, S_{\mathfrak{n}'}) = \operatorname{Tor}_n^S(S_{\mathfrak{n}}, S_{\mathfrak{n}'}) \cong \operatorname{Tor}_n^R(R_{\mathfrak{q}}/\mathfrak{j}, R_{\mathfrak{q}'}/\mathfrak{j}'),$$

since f is a homological ring epimorphism (see Proposition 4.3(2)), and $S_{\mathfrak{n}} \cong R_{\mathfrak{q}}/\mathfrak{j}$ and $S_{\mathfrak{n}'} \cong R_{\mathfrak{q}'}/\mathfrak{j}'$.

It remains to prove that the intervals $[\mathfrak{j}, \mathfrak{q}]$ and $[\mathfrak{j}', \mathfrak{q}']$ of $(\operatorname{Spec} R, \subseteq)$ are disjoint. This is easy now since if $\mathfrak{j}' \subseteq \mathfrak{q}$, then we would have $R_{\mathfrak{q}}/\mathfrak{j} \otimes_R R_{\mathfrak{q}'}/\mathfrak{j}' \cong R_{\mathfrak{q}\cap\mathfrak{q}'}/\mathfrak{j}' \neq 0$, a contradiction. □

Remark 6.6. It is rather clear from the latter proposition what the structure of homological epimorphisms is for $f\colon R \to S$ with S semilocal. In such a case the finitely many maximal ideals $\mathfrak{n}_i \subseteq S$ give us finitely many pairwise disjoint intervals $[\mathfrak{j}_i, \mathfrak{p}_i]$ in $(\operatorname{Spec} R, \subseteq)$ with all $\mathfrak{j}_i$ idempotent. Proposition 6.5 also provides us with a homological epimorphism

$$h\colon S \longrightarrow \prod_i R_{\mathfrak{p}_i}/\mathfrak{j}_i.$$

Moreover, $h_{\mathfrak{n}_i}$ is an isomorphism for every maximal ideal $\mathfrak{n}_i \subseteq S$, so that h itself is an isomorphism.

The non-semilocal case is more difficult. We shall focus on the problem which collection of intervals can occur in the conclusion of Proposition 6.5.

Definition 6.7. Let R be a valuation domain. An *admissible interval* $[\mathfrak{i}, \mathfrak{p}]$ is an interval in $(\operatorname{Spec} R, \subseteq)$ such that $\mathfrak{i}^2 = \mathfrak{i} \subseteq \mathfrak{p}$. The set of all admissible intervals will be denoted by $\operatorname{Inter} R$. We equip $\operatorname{Inter} R$ with a partial order: $[\mathfrak{i}, \mathfrak{p}] < [\mathfrak{i}', \mathfrak{p}']$ if $\mathfrak{p} \subsetneqq \mathfrak{i}'$ as ideals.

If $f\colon R \to S$ is a homological epimorphism, we denote by $\mathcal{I}(f)$ the collection of all admissible intervals $[\mathfrak{j}, \mathfrak{q}]$ which occur as in Proposition 6.5(2).

Thus, our task is to analyze the properties of $\mathcal{I}(f)$. First of all, it is easy to relate $\mathcal{I}(f)$ to the spectrum of S as a poset.

Lemma 6.8. *Let R be a valuation domain and $f\colon R \to S$ a homological epimorphism. Then the canonical map*

$$\operatorname{Spec} S \longrightarrow \operatorname{Spec} R, \qquad \mathfrak{n} \mapsto f^{-1}(\mathfrak{n})$$

restricts to a poset isomorphism between $(\operatorname{Spec} S, \subseteq)$ and the coproduct (= disjoint union)

$$\coprod_{[\mathfrak{j},\mathfrak{q}]\in\mathcal{I}(f)} [\mathfrak{j},\mathfrak{q}],$$

where $[\mathfrak{j},\mathfrak{q}]$ are viewed as subchains of $(\operatorname{Spec} R, \subseteq)$.

Proof. By Definition 6.7, there is a bijection between maximal ideals of S and elements of $\mathcal{I}(S)$. The rest follows from the fact that the primes below a maximal ideal $\mathfrak{n} \subseteq S$ correspond to the primes in the valuation domain $S_{\mathfrak{n}}$. □

Having this description at hand, the coming proposition encodes a crucial necessary condition on possible infinite collections of intervals coming from homological epimorphisms.

Proposition 6.9. *Let R be a valuation domain and $f\colon R \to S$ be a homological epimorphism.*

1. *If $\mathcal{S} = \{[\mathfrak{j}_\ell, \mathfrak{q}_\ell] \mid \ell \in \Lambda\} \subseteq \mathcal{I}(f)$ is a non-empty subset with no minimal element, then $\mathcal{I}(f)$ contains an element of the form $[\mathfrak{j}, \bigcap_{\ell\in\Lambda} \mathfrak{q}_\ell]$.*
2. *If $\mathcal{S} = \{[\mathfrak{j}_\ell, \mathfrak{q}_\ell] \mid \ell \in \Lambda\} \subseteq \mathcal{I}(f)$ is a non-empty subset with no maximal element, then $\mathcal{I}(f)$ contains an element of the form $[\bigcup_{\ell\in\Lambda} \mathfrak{j}_\ell, \mathfrak{q}]$.*

Proof. (1) Denote $\mathfrak{p} = \bigcap_{\ell\in\Lambda} \mathfrak{q}_\ell = \bigcap_{\ell\in\Lambda} \mathfrak{j}_\ell$ and, appealing to Lemma 6.3, let $\mathfrak{i} = \inf\{\mathfrak{j}_\ell \mid \ell \in \Lambda\}$ be the infimum taken in $(\operatorname{iSpec} R, \subseteq)$. One easily sees that $\mathfrak{p}$ is a prime ideal and, although it may happen that $\mathfrak{i} \subsetneqq \mathfrak{p}$ (see Example 6.21 below), we at least know that there is no idempotent ideal $\mathfrak{p}'$ such that $\mathfrak{i} \subsetneqq \mathfrak{p}' \subseteq \mathfrak{p}$.

We claim that $\mathcal{I}(f)$ must contain an element of the form $[\mathfrak{j}, \mathfrak{q}]$ such that $\mathfrak{j} \subseteq \mathfrak{i} \subseteq \mathfrak{q} \subseteq \mathfrak{p}$. To see that, let us denote for each $[\mathfrak{j}_\ell, \mathfrak{q}_\ell]$ the corresponding maximal ideal of S by $\mathfrak{n}_\ell$. Then $\mathfrak{j}_\ell = \operatorname{Ker}(R \to S_{\mathfrak{n}_\ell})$ contains $\mathfrak{i}$ for each $\ell \in \Lambda$. In particular, each $\psi_{\mathfrak{n}_\ell}\colon S \to S_{\mathfrak{n}_\ell}$ canonically factors through the projection $p\colon S \to S/\mathfrak{i}S$ and we infer that the kernel of the composition $pf\colon R \to S/\mathfrak{i}S$ is contained in $\mathfrak{p} = \bigcap_{\ell\in\Lambda} \mathfrak{j}_\ell$. Since pf is a homological epimorphism and $\operatorname{Ker} pf$ is idempotent by Lemma 4.5, we must have $\operatorname{Ker} pf \subseteq \mathfrak{i}$ and then clearly $\operatorname{Ker} pf = \mathfrak{i}$. Let $\mathfrak{m} \supseteq \mathfrak{i}S$ be the unique maximal ideal of S such that $\mathfrak{m}/\mathfrak{i}S$ fits Proposition 6.5(1) when applied to $pf\colon R \to S/\mathfrak{i}S$. If $[\mathfrak{j}, \mathfrak{q}] \in \mathcal{I}(f)$ is the interval corresponding to $\mathfrak{m}$, then clearly $\mathfrak{j} = \operatorname{Ker}(R \to S_{\mathfrak{m}}) \subseteq \mathfrak{i}$ and $\mathfrak{i} \subseteq f^{-1}(\mathfrak{m}) = \mathfrak{q}$. This proves the claim and reduces our task to showing that $\mathfrak{q} = \mathfrak{p}$.

Suppose by way of contradiction that $\mathfrak{p} \neq \mathfrak{q}$ and consider an element $x \in \mathfrak{p} \setminus \mathfrak{q}$. We shall denote $y = f(x)$ and by $u\colon S \to S[1/y]$ the corresponding localization. Suppose that $\mathfrak{n} \subseteq S$ is a maximal ideal and $[\mathfrak{j}', \mathfrak{q}'] \in \mathcal{I}(f)$ the corresponding interval in $\operatorname{Spec} R$. Then $S[1/y]_{\mathfrak{n}} \cong (R_{\mathfrak{q}'}/\mathfrak{j}')[1/x]$ canonically. Hence $S_{\mathfrak{n}} \to S[1/y]_{\mathfrak{n}}$ is either a zero map or an isomorphism depending on

whether $x \in \mathfrak{j}'$ or not, and in particular $S \to S[1/y]$ is surjective. Thus, $\operatorname{Spec} S[1/y]$ can be identified with a quasi-compact clopen set $V \subseteq \operatorname{Spec} S$ whose complement $\operatorname{Spec} R \setminus V$ is also quasi-compact. As in the noetherian case, we find an idempotent $e \in S$ such that $S \to S[1/y]$ is equivalent to $S \to S/(1-e)S$ as a ring epimorphism. Indeed, there exists $t \in S$ such that $1/y = t$ in $S[1/y]$, which forces the existence of $n \geq 0$ such that $y^n = ty^{n+1}$ in S. In particular $y^n = t^n y^{2n}$ and $e = t^n y^n$ is the idempotent which we are looking for.

Thus we have a homological epimorphism $g\colon R \to S/eS$ and $\operatorname{Ker} g = f^{-1}(eS)$ is an idempotent ideal in R by Lemma 4.5. As $y^n = ey^n \in eS$, we have $x^n \in \operatorname{Ker} g$. In particular $\mathfrak{q} \subsetneqq \operatorname{Ker} g$. On the other hand $\operatorname{Ker} g \subseteq \mathfrak{p}$ since $e = t^n y^n$ vanishes under $R \to S_{\mathfrak{n}_\ell} \cong R_{\mathfrak{q}_\ell}/\mathfrak{j}_\ell$ for each $\ell \in \Lambda$. To summarize, we have constructed an idempotent ideal $\operatorname{Ker} g \subseteq R$ such that $\mathfrak{q} \subsetneqq \operatorname{Ker} g \subseteq \mathfrak{p}$, in contradiction to our assumption that there are no idempotent ideals in that interval.

(2) Let $\mathfrak{p} = \bigcup_{\ell\in\Lambda} \mathfrak{j}_\ell$. It suffices to prove that $S \otimes_R k(\mathfrak{p}) \neq 0$ for the residue field $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}$. Indeed, then we get $S_\mathfrak{n} \otimes k(\mathfrak{p}) \neq 0$ for some maximal ideal $\mathfrak{n}$ of S and thus $R_\mathfrak{q}/\mathfrak{j} \otimes k(\mathfrak{p}) \neq 0$ for some $[\mathfrak{j}, \mathfrak{q}] \in \mathcal{I}(f)$, but since the intervals in $\mathcal{I}(f)$ are disjoint this implies that $\mathfrak{p} = \mathfrak{j}$.

Now note that $S \otimes_R R_\mathfrak{p}/\mathfrak{j}_\ell \neq 0$ for each $\ell \in \Lambda$ since we can always find $\ell' \in \Lambda$ and $\mathfrak{n}_{\ell'} \in \operatorname{Spec} S$ such that $[\mathfrak{j}_\ell, \mathfrak{q}_\ell] < [\mathfrak{j}_{\ell'}, \mathfrak{q}_{\ell'}]$, $\mathfrak{q}_{\ell'} = f^{-1}(\mathfrak{n}_{\ell'})$ and

$$S_{\mathfrak{n}_{\ell'}} \otimes_R R_\mathfrak{p}/\mathfrak{j}_\ell \cong R_{\mathfrak{q}_{\ell'}}/\mathfrak{j}_{\ell'} \otimes_R R_\mathfrak{p}/\mathfrak{j}_\ell \cong R_{\mathfrak{q}_{\ell'}}/\mathfrak{j}_{\ell'} \neq 0.$$

Observe further that $S \otimes_R k(\mathfrak{p})$ can be expressed as a direct limit of $(S \otimes_R R_\mathfrak{p}/\mathfrak{j}_\ell \mid \ell \in \Lambda)$, where the maps in the direct system are surjective. If $1 \otimes_R 1 \in S \otimes_R k(\mathfrak{p})$ were zero, standard properties of direct limits would imply that also $0 = 1 \otimes_R 1 \in S \otimes_R R_\mathfrak{p}/\mathfrak{j}_\ell$ for some $\ell \in \Lambda$, a contradiction. □

6.2. **Mazet presentations and abundance of idempotents.** So far we have mostly used the homological properties of $f\colon R \to S$. Now we are going to employ the fact that f is a ring epimorphism. The following concept will facilitate our discussion.

Definition 6.10. Let R, S be arbitrary (non-commutative) rings and $f\colon R \to S$ be a ring homomorphism. The *dominion* of f is the collection of all elements $s \in S$ such that for any pair $g_1, g_2\colon S \to T$ of ring homomorphisms with $g_1 f = g_2 f$ we have also $g_1(s) = g_2(s)$.

In connection to homotopy theory, dominions of ring homomorphisms $\mathbb{Z} \to S$ have been also studied in [BK72, BK73]. The following zig-zag criterion for the elements in the dominion was originally studied by Mazet [Maz68]. It was stated in the present form by Isbell [Isb69], who combined Mazet's results with those in [Sil67].

Proposition 6.11 (Isbell-Mazet-Silver)**.** *Let $f\colon R \to S$ be a ring homomorphism and $s \in S$. Then the following are equivalent:*

1. *s is in the dominion of f;*
2. *There exist natural numbers $m, n \geq 1$ and matrices $X \in M_{1\times m}(S)$, $Y \in M_{m\times n}(R)$ and $Z \in M_{n\times 1}(S)$ such that $s = X \cdot f(Y) \cdot Z$ (as 1×1-matrices) and $X \cdot f(Y)$, $f(Y) \cdot Z$ are (row and column, respectively) matrices over* $\operatorname{Im} f$.

Proof. The implication (2) $\Rightarrow$ (1) is easy. The argument for the converse is sketched in [Isb69, Theorem 1.1]. If $s \in S$ belongs to the dominion, the proof of [Sil67, Proposition 1.1] shows that $sm = ms$ for any S-S-bimodule M and any element $m \in M$. Applying this to $M = S \otimes_R S$ and $m = 1 \otimes_R 1$ implies that $s \otimes_R 1 = 1 \otimes_R s$. As explicitly computed in [Maz68] or [GdlP87, §1.4], the latter has as a consequence the existence of matrices X, Y, Z as in (2). □

Corollary 6.12. *Given $f\colon R \to S$, the dominion is the largest subring $S' \subseteq S$ such that $f\colon R \to S'$ is a ring epimorphism.*

This leads to the following definition.

Definition 6.13. Let $f\colon R \to S$ be a ring homomorphism and $s \in S$ be in the dominion. Then a triple $(P, Y, Q) \in M_{1\times n}(R) \times M_{m\times n}(R) \times M_{m\times 1}(R)$ is called a *Mazet presentation* of s over R if there exist matrices $X \in M_{1\times m}(S)$ and $Z \in M_{n\times 1}(S)$ such that

$$s = X \cdot f(Y) \cdot Z, \qquad f(P) = X \cdot f(Y), \quad \text{and} \quad f(Q) = f(Y) \cdot Z.$$

Note that the image of s under any ring homomorphism $g\colon S \to T$ (including $g = id_S$) is fully determined by (P, Y, Q). In fact, only P and Q suffice, but it will be more convenient for us to work with Y as well. For valuation domains, the situation simplifies as follows.

Lemma 6.14. *Let R be a valuation domain and $f\colon R \to S$ be a ring epimorphism. Then every $s \in S$ has a Mazet presentation (P, Y, Q) such that Y is a diagonal square matrix.*

Proof. Let (P, Y, Q) be an arbitrary Mazet presentation for $s \in S$. First we can turn $Y = (y_{ij})$ into a Smith normal form (that is, $y_{ij} = 0$ unless $i = j$ and $R \supseteq y_{11}R \supseteq y_{22}R \supseteq \cdots$) by applying equivalent row and column operations to Y and changing P and Q correspondingly. Indeed, the same proof as for discrete valuation domains applies and this is again closely related to the fact that, if we consider Y as a presentation matrix of an R-module N, then N is a direct sum of cyclically presented modules by [FS01, Theorem I.7.9].

Second, if Y is a diagonal $m \times n$ matrix, we can crop it to a square matrix of size $\min(m, n)$ and truncate P and Q correspondingly. As we have left out only zero entries, this will still be a presentation for s. □

Consider now a homological epimorphism $f\colon R \to S$. The following is an easy consequence of the results in §6.1.

Lemma 6.15. *Let R be a valuation domain and $f\colon R \to S$ be a homological epimorphism. Then we can identify S with a subring of $\prod_{[\mathfrak{j},\mathfrak{q}]\in\mathcal{I}(f)} R_\mathfrak{q}/\mathfrak{j}$ and f with the canonical map obtained from $R \to \prod_{[\mathfrak{j},\mathfrak{q}]\in\mathcal{I}(f)} R_\mathfrak{q}/\mathfrak{j}$ by restriction.*

Proof. Clearly the homomorphism $S \to \prod_{\mathfrak{n}\in\operatorname{Max} S} S_\mathfrak{n}$ is injective. The rest is easily deduced from Proposition 6.5 since we can canonically identify each $S_\mathfrak{n}$ with $R_\mathfrak{q}/\mathfrak{j}$ for some $[\mathfrak{j}, \mathfrak{q}] \in \mathcal{I}(f)$. □

Now we establish the key fact: The components of any fixed element $s \in S$ in $\prod_{[\mathfrak{j},\mathfrak{q}]\in\mathcal{I}(f)} R_\mathfrak{q}/\mathfrak{j}$ can come only from finitely many elements in R.

Proposition 6.16. *Let R be a valuation domain and $f\colon R \to S$ be a homological epimorphism with $S \subseteq \prod_{[\mathfrak{j},\mathfrak{q}]\in\mathcal{I}(f)} R_{\mathfrak{q}}/\mathfrak{j}$ as in the above lemma. Fix an element $s = (s_{[\mathfrak{j},\mathfrak{q}]})_{[\mathfrak{j},\mathfrak{q}]\in\mathcal{I}(f)}$. Then there exist an integer $k \geq 1$, and for each $1 \leq j \leq k$ an interval $[\mathfrak{p}_j, \mathfrak{p}'_j]$ in $\operatorname{Spec} R$ and an element $r_j \in R_{\mathfrak{p}'_j}/\mathfrak{p}_j$ such that:*

1. *Every $[\mathfrak{j},\mathfrak{q}] \in \mathcal{I}(f)$ is contained in $[\mathfrak{p}_j, \mathfrak{p}'_j]$ for some $1 \leq j \leq k$. That is, we have $\mathfrak{p}_j \subseteq \mathfrak{j} \subseteq \mathfrak{q} \subseteq \mathfrak{p}'_j$.*
2. *Whenever $1 \leq j \leq k$ and $[\mathfrak{j},\mathfrak{q}] \in \mathcal{I}(f)$ are such that $[\mathfrak{j},\mathfrak{q}]$ is contained in $[\mathfrak{p}_j, \mathfrak{p}'_j]$, then $s_{[\mathfrak{j},\mathfrak{q}]}$ is the image of r_j under the canonical map $R_{\mathfrak{p}'_j}/\mathfrak{p}_j \to R_{\mathfrak{q}}/\mathfrak{j}$.*

Proof. By Proposition 6.9(2) and Zorn's lemma, $\mathcal{I}(f)$ possesses a (unique) interval which is maximal with respect to the order on $\operatorname{Inter} R$. Let us denote this interval by $[\mathfrak{i}', \mathfrak{n}]$. Consider now a Mazet presentation (P, Y, Q) of $s \in S$ where Y is a square diagonal matrix, and consider all principal ideals of R generated by the entries in P, Y, Q which are contained in $\mathfrak{i}'$. Ordering these ideals by inclusion and removing duplicities results in a finite list

$$I_1 \subsetneqq I_2 \subsetneqq \cdots \subsetneqq I_\ell$$

of principal ideals of R. In order to facilitate the discussion, we also put $I_0 = 0$ and take for $I_{\ell+1} = \mathfrak{i}'$.

Now consider an integer j such that $0 \leq j \leq \ell$, let $\mathfrak{p}_j = \sqrt{I_j}$ and let $\mathfrak{p}'_j$ be the maximal prime ideal such that $\mathfrak{p}'_j \subseteq I_{j+1}$. Since a non-zero non-maximal prime ideal cannot be principal, we have $I_j \subsetneqq \mathfrak{p}_j$ unless $j = 0$ and $\mathfrak{p}'_j \subsetneqq I_{j+1}$ unless $j = \ell$. In particular, the images of P, Y, Q under the canonical map $g_j\colon R \to R_{\mathfrak{p}'_j}/\mathfrak{p}_j$ have either zeros or units in all entries. Now there are two possibilities: either there is an element $r_j \in R_{\mathfrak{p}'_j}/\mathfrak{p}_j$ with the Mazet presentation (P, Y, Q), or there is none. This depends only on the fact whether the systems of linear equations $X \cdot g_j(Y) = g_j(P)$ and $g_j(Y) \cdot Z = g_j(Q)$ have solutions X, Z over $R_{\mathfrak{p}'_j}/\mathfrak{p}_j$. In the first case such an element r_j is unique and whenever $[\mathfrak{j},\mathfrak{q}] \in \mathcal{I}(f)$ is contained in $[\mathfrak{p}_j, \mathfrak{p}'_j]$, then $s_{[\mathfrak{j},\mathfrak{q}]}$ must be the image of r_j. In the second case we claim that there cannot exist any $[\mathfrak{j},\mathfrak{q}] \in \mathcal{I}(f)$ which is contained in $[\mathfrak{p}_j, \mathfrak{p}'_j]$. Indeed, the systems of equations $X \cdot g_j(Y) = g_j(P)$ and $g_j(Y) \cdot Z = g_j(Q)$ have a very easy form since $g_j(Y)$ is diagonal and every entry of $g_j(Y), g_j(P), g_j(Q)$ is either a unit or zero. If there existed $[\mathfrak{j},\mathfrak{q}] \in \mathcal{I}(f)$ inside $[\mathfrak{p}_j, \mathfrak{p}'_j]$, units would stay units and zeros would stay zeros under the homomorphism $R_{\mathfrak{p}'_j}/\mathfrak{p}_j \to R_{\mathfrak{q}}/\mathfrak{j}$, so there could not be any element with Mazet's presentation (P, Y, Q) in $R_{\mathfrak{q}}/\mathfrak{j}$ either. But clearly $s_{[\mathfrak{j},\mathfrak{q}]}$ is such an element, a contradiction. This proves the claim.

Let us put all the intervals $[\mathfrak{p}_j, \mathfrak{p}'_j]$ for which there exists r_j as above on our list. It only can happen that some $[\mathfrak{j},\mathfrak{q}] \in \mathcal{I}(f)$ is not covered by any such $[\mathfrak{p}_j, \mathfrak{p}'_j]$ if either we have $\mathfrak{j} \subseteq I_i \subseteq \mathfrak{q}$ for some $0 \leq i \leq \ell$, or if $[\mathfrak{j},\mathfrak{q}] = [\mathfrak{i}', \mathfrak{n}]$. In either case, we simply add $[\mathfrak{j},\mathfrak{q}]$ and $s_{[\mathfrak{j},\mathfrak{q}]}$ to our list, resulting in at most finitely many additional intervals. It is easy to check that we have obtained a collection of intervals and elements as in the statement. □

Remark 6.17. By possibly removing finitely many intervals from the collection obtained by Proposition 6.16, we may without loss of generality assume that the collection is irredundant, i.e.

(1) no $[\mathfrak{p}_j, \mathfrak{p}'_j]$ is contained in $[\mathfrak{p}_i, \mathfrak{p}'_i]$ for any $i \neq j$, and
(2) each $[\mathfrak{p}_j, \mathfrak{p}'_j]$ contains an interval from $\mathcal{I}(f)$.

If we order the intervals such that $\mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq \cdots \subseteq \mathfrak{p}_k$, the irredundancy implies that $\mathfrak{p}'_j \subseteq \mathfrak{p}'_{j+1}$ for each $1 \leq i < k$.

Another important fact is that a much stronger reduction of the number of intervals is possible. As it turns out, we will be able to glue together any pair of overlapping intervals thanks to the following instance of the sheaf axiom (for a scheme-theoretic interpretation see [Sch05, Theorem 3.3]).

Lemma 6.18. *Let R be a valuation domain, let $k \geq 1$, and suppose that we are given for each $1 \leq j \leq k$ an interval $[\mathfrak{p}_j, \mathfrak{p}'_j]$ in* $\operatorname{Spec} R$ *and an element $r_j \in R_{\mathfrak{p}'_j}/\mathfrak{p}_j$. Suppose further that*

(1) $\mathfrak{p}_j \subseteq \mathfrak{p}_{j+1} \subseteq \mathfrak{p}'_j \subseteq \mathfrak{p}'_{j+1}$, *and*
(2) *the images of r_j and r_{j+1} under the canonical maps coincide in $R_{\mathfrak{p}'_j}/\mathfrak{p}_{j+1}$*

for each $1 \leq j < k$. Then there exists a unique element $r \in R_{\mathfrak{p}'_k}/\mathfrak{p}_1$ such that the image of r under $R_{\mathfrak{p}'_k}/\mathfrak{p}_1 \to R_{\mathfrak{p}'_j}/\mathfrak{p}_j$ equals r_j for each $1 \leq j \leq k$.

Proof. There is nothing to prove for $k = 1$ and the case $k = 2$ just amounts to the straightforward checking that the square with canonical maps

$$\begin{array}{ccc} R_{\mathfrak{p}'_2}/\mathfrak{p}_1 & \longrightarrow & R_{\mathfrak{p}'_1}/\mathfrak{p}_1 \\ \downarrow & & \downarrow \\ R_{\mathfrak{p}'_2}/\mathfrak{p}_2 & \longrightarrow & R_{\mathfrak{p}'_1}/\mathfrak{p}_2 \end{array}$$

is a pull-back of rings. Note that there we can without loss of generality assume that $\mathfrak{p}_1 = 0$ and $\mathfrak{p}'_2$ is the maximal ideal.

We proceed by induction for $k > 2$. By inductive hypothesis, there is a unique element $r' \in R_{\mathfrak{p}'_{k-1}}/\mathfrak{p}_1$ such that the image of r' under the canonical map $R_{\mathfrak{p}'_{k-1}}/\mathfrak{p}_1 \to (R_{\mathfrak{p}'_j}/\mathfrak{p}_j)$ equals r_j for all $1 \leq j < k$. Furthermore, the images of r' and r_k in $R_{\mathfrak{p}'_{k-1}}/\mathfrak{p}_k$ coincide by assumption (2) applied to $j = k - 1$. Thus, we can glue r' and r_k to a unique element $r \in R_{\mathfrak{p}'_k}/\mathfrak{p}_1$ using the argument for $k = 2$. □

As a consequence, we obtain the following dichotomy.

Proposition 6.19. *Let R be a valuation domain, let $f\colon R \to S$ be a homological epimorphism with $S \neq 0$, and let $\mathcal{I}(f)$ be the collection of intervals as in Definition 6.7. Then*

(1) *either $\mathcal{I}(f)$ contains a single element,*
(2) *or there are intervals $[\mathfrak{j}, \mathfrak{q}] < [\mathfrak{j}', \mathfrak{q}']$ in $\mathcal{I}(f)$ with no other interval of $\mathcal{I}(f)$ between them.*

In particular, either S is local or it has a non-trivial idempotent element.

Proof. Suppose that (2) does not hold, or equivalently that the order on $\mathcal{I}(f)$ is dense. Suppose further that $s = (s_{[\mathfrak{j},\mathfrak{q}]})_{[\mathfrak{j},\mathfrak{q}]\in\mathcal{I}(f)} \in S$ and $[\mathfrak{p}_j, \mathfrak{p}'_j]$ and $r_j \in R_{\mathfrak{p}'_j}/\mathfrak{p}_j$ is a corresponding collection of intervals and elements as in Proposition 6.16, which is irredundant and ordered as in Remark 6.17.

We first claim that then $\mathfrak{p}_{j+1} \subseteq \mathfrak{p}'_j$ for each $1 \leq j < k$. Indeed, suppose to the contrary that for instance $\mathfrak{p}'_1 \subsetneqq \mathfrak{p}_2$. Then using Proposition 6.9 and Zorn's lemma we can find

- a maximal element $[\mathfrak{j}', \mathfrak{q}']$ among those elements of $\mathcal{I}(f)$ which are contained in $[\mathfrak{p}_1, \mathfrak{p}'_1]$ and
- a minimal element $[\mathfrak{j}'', \mathfrak{q}'']$ among those elements of $\mathcal{I}(f)$ which are contained in $[\mathfrak{p}_2, \mathfrak{p}'_2]$.

Since each element of $\mathcal{I}(f)$ is contained in some interval $[\mathfrak{p}_j, \mathfrak{p}'_j]$, there cannot exist any element of $\mathcal{I}(f)$ between $[\mathfrak{j}', \mathfrak{q}']$ and $[\mathfrak{j}'', \mathfrak{q}'']$, contradicting that $\mathcal{I}(f)$ is densely ordered. This establishes the claim.

Since also Lemma 6.18(2) is satisfied for any collection of intervals and elements coming from the proof of Proposition 6.16 (all r_j have the same Mazet presentation over R), there is an element $r \in R_{\mathfrak{p}_k}/\mathfrak{p}_1$ such that the image of r under $R_{\mathfrak{p}_k}/\mathfrak{p}_1 \to R_{\mathfrak{q}}/\mathfrak{j}$ equals $s_{[\mathfrak{j},\mathfrak{q}]}$ for each $[\mathfrak{j}, \mathfrak{q}] \in \mathcal{I}(f)$.

Let us rephrase what we have just shown. Thanks to Proposition 6.9, $\mathcal{I}(f)$ has a unique minimal element $[\mathfrak{i}, \mathfrak{p}]$ and a unique maximal element $[\mathfrak{i}', \mathfrak{n}]$. If $\mathcal{I}(f)$ is densely ordered, we have shown that for every $s \in S$ there exists $r \in R_{\mathfrak{n}}/\mathfrak{i}$ such that s is the image of r under the morphism $R_{\mathfrak{n}}/\mathfrak{i} \to S$ induced by f. Put yet in other words, if $\mathcal{I}(f)$ is densely ordered, then $R_{\mathfrak{n}}/\mathfrak{i} \to S$ is surjective, S is necessarily local, and $\mathcal{I}(f)$ has a single element by the very definition. This proves the dichotomy between (1) and (2) in the statement.

For the second part, suppose that S is not local, fix some $[\mathfrak{j}, \mathfrak{q}] < [\mathfrak{j}', \mathfrak{q}']$ in $\mathcal{I}(f)$ with no other interval between them and fix $x \in \mathfrak{j}' \setminus \mathfrak{q}$. Then $S \to S[1/y]$ is surjective for $y = f(x)$ since $S_{\mathfrak{n}} \to S_{\mathfrak{n}}[1/y]$ is either zero or an isomorphism for every $\mathfrak{n} \in \operatorname{Max} S$. Now the same argument as for Proposition 6.9(1) provides us with a non-trivial idempotent $e \in S$. □

Corollary 6.20. *Given a homological epimorphism $f\colon R \to S$ where R is a valuation domain, and given any $[\mathfrak{j}_0, \mathfrak{q}_0] < [\mathfrak{j}_1, \mathfrak{q}_1]$ in $\mathcal{I}(f)$, there are intervals $[\mathfrak{j}, \mathfrak{q}], [\mathfrak{j}', \mathfrak{q}']$ in $\mathcal{I}(f)$ such that*

$$[\mathfrak{j}_0, \mathfrak{q}_0] \leq [\mathfrak{j}, \mathfrak{q}] < [\mathfrak{j}', \mathfrak{q}'] \leq [\mathfrak{j}_1, \mathfrak{q}_1]$$

and there is no other interval in $\mathcal{I}(f)$ between $[\mathfrak{j}, \mathfrak{q}]$ and $[\mathfrak{j}', \mathfrak{q}']$.

Proof. This follows by applying Proposition 6.19 to the composition $R \xrightarrow{f} S \to S_{\mathfrak{q}_1}/\mathfrak{j}_0 S$. □

In the non-local case, S is formally similar to a von Neumann regular ring in that the Zariski topology on $\operatorname{Max} S$ is totally disconnected. If S is semihereditary, this similarity can be formalized by noting that the localization of S at the set of all regular elements is von Neumann regular by [Gla89, Corollary 4.2.19]. Note also that in our situation, the regular elements of S are precisely those $s = (s_{[\mathfrak{j},\mathfrak{q}]})$ for which each component $s_{[\mathfrak{j},\mathfrak{q}]}$ is non-zero. Beware, however, that S might *not* be semihereditary:

Example 6.21. Suppose that R is a valuation domain with a countable descending chain $\mathfrak{i}_1 \supseteq \mathfrak{i}_2 \supseteq \cdots$ of idempotent ideals such that the intersection $\mathfrak{q} = \bigcap \mathfrak{i}_i$ is not idempotent. Such an example can be constructed by means of [FS01, Theorem II.3.8], where the value group $(G, \leq)$ is taken as follows: We put $H = \mathbb{Q}^{(\omega)}$ (a countable direct sum of copies of $\mathbb{Q}$) with the antilexicographic order and $G = \mathbb{Z} \times H$ with the components lexicographically ordered.

Denote now $R_j = R_{\mathfrak{i}_{j+1}} \times k(\mathfrak{i}_j) \times k(\mathfrak{i}_{j-1}) \times \cdots \times k(\mathfrak{i}_1)$, where $k(\mathfrak{i}_j)$ is the residue field of $\mathfrak{i}_j$, and consider the chain of obvious ring homomorphisms

$$R \to R_1 \to R_2 \to \cdots \to S = \varinjlim_j R_j.$$

One can check (see the results in §6.3) that $R \to S$ is a homological epimorphism and $\mathcal{I}(f) = \{[0, \mathfrak{q}]\} \cap \{[\mathfrak{i}_j, \mathfrak{i}_j] \mid j \geq 1\}$. But there are elements $s = (s_{[\mathfrak{j},\mathfrak{q}]}) \in S$ such that $s_{[0,q]} \neq 0$ and $s_{[\mathfrak{i}_j,\mathfrak{i}_j]} = 0$ for all $j \geq 1$. Then the ideal sS is not finitely presented, hence S is not semihereditary. One can also check that every regular element of S is a unit, so the localization at the set of regular elements is not von Neumann regular.

6.3. **From intervals to a homological epimorphism.** Now we finish the classification of homological epimorphisms starting at a valuation domain. In particular, given a suitable collection $\mathcal{I} \subseteq \operatorname{Inter} R$ (Definition 6.7) we construct the corresponding homological epimorphism $f_{(\mathcal{I})} \colon R \to R_{(\mathcal{I})}$.

Construction 6.22. Suppose that R is a valuation domain and $(\mathcal{I}, \leq)$ is a non-empty subchain of $(\operatorname{Inter} R, \leq)$ satisfying the conditions implied by Proposition 6.9 and Corollary 6.20. That is, we require:

(C1) If $\mathcal{S} = \{[\mathfrak{j}_\ell, \mathfrak{q}_\ell] \mid \ell \in \Lambda\}$ is a non-empty subset of $\mathcal{I}$ with no minimal element, then $\mathcal{I}$ contains an element of the form $[\mathfrak{j}, \bigcap_{\ell\in\Lambda} \mathfrak{q}_\ell]$.

(C2) If $\mathcal{S} = \{[\mathfrak{j}_\ell, \mathfrak{q}_\ell] \mid \ell \in \Lambda\}$ is a non-empty subset of $\mathcal{I}$ with no maximal element, then $\mathcal{I}$ contains an element of the form $[\bigcup_{\ell\in\Lambda} \mathfrak{j}_\ell, \mathfrak{q}]$.

(C3) Given any pair $[\mathfrak{j}_0, \mathfrak{q}_0] < [\mathfrak{j}_1, \mathfrak{q}_1]$ in $\mathcal{I}$, then there are elements $[\mathfrak{j}, \mathfrak{q}], [\mathfrak{j}', \mathfrak{q}']$ in $\mathcal{I}$ such that

$$[\mathfrak{j}_0, \mathfrak{q}_0] \leq [\mathfrak{j}, \mathfrak{q}] < [\mathfrak{j}', \mathfrak{q}'] \leq [\mathfrak{j}_1, \mathfrak{q}_1]$$

and there is no other interval in $\mathcal{I}$ between $[\mathfrak{j}, \mathfrak{q}]$ and $[\mathfrak{j}', \mathfrak{q}']$.

Denote by $[\mathfrak{i}, \mathfrak{p}]$ the unique minimal element of $\mathcal{I}$ and by $[\mathfrak{i}', \mathfrak{n}]$ the unique maximal element. Further denote by $R_{\mathcal{I}}$ the ring product $\prod_{[\mathfrak{j},\mathfrak{q}]\in\mathcal{I}} R_{\mathfrak{q}}/\mathfrak{j}$ and by $g_{\mathcal{I}} \colon R_{\mathfrak{n}}/\mathfrak{i} \to R_{\mathcal{I}}$ the canonical ring homomorphism. Clearly $g_{\mathcal{I}}$ is an embedding.

Consider now a partition of $\mathcal{I}$ into a finite disjoint union $\mathcal{I} = \mathcal{I}_0 \cup \cdots \cup \mathcal{I}_n$ of chains in $\operatorname{Inter} R$ which satisfies two simple conditions:

(a) Each $\mathcal{I}_\ell$, $0 \leq \ell \leq n$, is a subchain of $\mathcal{I}$ and has a minimal element $[\mathfrak{i}_\ell, \mathfrak{p}_\ell]$ and a maximal element $[\mathfrak{i}'_\ell, \mathfrak{m}_\ell]$.

(b) If $j < \ell$, then $[\mathfrak{j}, \mathfrak{q}] < [\mathfrak{j}', \mathfrak{q}']$ for each $[\mathfrak{j}, \mathfrak{q}] \in \mathcal{I}_j$ and $[\mathfrak{j}', \mathfrak{q}'] \in \mathcal{I}_\ell$.

In other words, we have subdivided $\mathcal{I}$ into finitely many intervals which enjoy properties (1)–(3) as $\mathcal{I}$ does itself.

Using this notation, we define a map

$$g_{(\mathcal{I}_0,\dots,\mathcal{I}_n)}\colon \prod_{\ell=0}^{n} R_{\mathfrak{n}_\ell}/\mathfrak{i}_\ell \longrightarrow R_{\mathcal{I}}$$

as the composition of the product of the maps

$$g_{\mathcal{I}_\ell}\colon R_{\mathfrak{n}_\ell}/\mathfrak{i}_\ell \longrightarrow R_{\mathcal{I}_\ell}$$

with the obvious isomorphism $\prod_{\ell=0}^{n} R_{\mathcal{I}_\ell} \cong R_{\mathcal{I}}$. Again $g_{(\mathcal{I}_0,\dots,\mathcal{I}_n)}$ is an embedding.

Another easy observation reveals that the images of $g_{(\mathcal{I}_0,\dots,\mathcal{I}_n)}$, where $(\mathcal{I}_0,\dots,\mathcal{I}_n)$ varies over all partitions of $\mathcal{I}$ satisfying conditions (a) and (b) above, form a direct system of subrings of $R_{\mathcal{I}}$. We denote by $R_{(\mathcal{I})}$ the direct union of all these images and by

$$f_{(\mathcal{I})}\colon R \longrightarrow R_{(\mathcal{I})}$$

the ring homomorphism induced by the composition $R \to R_{\mathfrak{n}}/\mathfrak{i} \xrightarrow{g_{\mathcal{I}}} R_{\mathcal{I}}$.

The highlight of the section is the following theorem, which together with Theorem 4.10 classifies smashing localizations of $\mathbf{D}(R)$.

Theorem 6.23. *Let R be a valuation domain. Then there is a bijection between:*

(i) *Subchains $\mathcal{I}$ of* $\operatorname{Inter} R$ *(cf. Definition 6.7) which satisfy conditions (C1)–(C3) from Construction 6.22.*
(ii) *Equivalence classes of homological epimorphisms $f\colon R \to S$.*

The bijection is given by assigning to a non-empty $\mathcal{I}$ from (i) the ring homomorphism $f_{(\mathcal{I})}\colon R \to R_{(\mathcal{I})}$ from Construction 6.22. We assign $R \to 0$ to $\mathcal{I} = \emptyset$. The converse is given by sending $f\colon R \to S$ to $\mathcal{I} = \mathcal{I}(f)$ (see Definition 6.7).

Proof. Suppose we have $\mathcal{I} \neq \emptyset$ as in (1) and consider $f_{(\mathcal{I})}\colon R \to R_{(\mathcal{I})}$. Since $f_{(\mathcal{I})}$ is a direct limit of homological epimorphisms of the form

$$R \longrightarrow \prod_{\ell=0}^{n} R_{\mathfrak{n}_\ell}/\mathfrak{i}_\ell, \qquad \mathfrak{i}_0 \subseteq \mathfrak{n}_0 \subsetneqq \mathfrak{i}_1 \subseteq \mathfrak{n}_1 \subsetneqq \cdots \subsetneqq \mathfrak{n}_n \subseteq \mathfrak{i}_n, \tag{§}$$

and since the Tor functors commute with direct limits, it follows that $f_{(\mathcal{I})}$ is a homological epimorphism.

Suppose now that $\mathcal{I}$ is a set of admissible intervals as in (i) and let $\mathcal{I}' = \mathcal{I}(f_{(\mathcal{I})})$. We claim that $\mathcal{I}' = \mathcal{I}$. To this end, let $[\mathfrak{i},\mathfrak{p}] \in \mathcal{I}'$. As then $R_{(\mathcal{I})} \otimes_R R_{\mathfrak{p}}/\mathfrak{i} \neq 0$ by the very definition of $\mathcal{I}'$, we deduce that there is an interval $[\mathfrak{j}_0,\mathfrak{q}_0] \in \mathcal{I}$ which overlaps $[\mathfrak{i},\mathfrak{p}]$. Indeed, otherwise we could take $\mathcal{I}_0 = \{[\mathfrak{j},\mathfrak{q}] \in \mathcal{I} \mid [\mathfrak{j},\mathfrak{q}] < [\mathfrak{i},\mathfrak{p}]\}$, $\mathcal{I}_1 = \{[\mathfrak{j},\mathfrak{q}] \in \mathcal{I} \mid [\mathfrak{j},\mathfrak{q}] > [\mathfrak{i},\mathfrak{p}]\}$ and $R_{\mathfrak{n}_0}/\mathfrak{i}_0 \times R_{\mathfrak{n}_1}/\mathfrak{i}_1$ as in Construction 6.22 (using conditions (C1) and (C2) on $\mathcal{I}$), but then $(R_{\mathfrak{n}_0}/\mathfrak{i}_0 \times R_{\mathfrak{n}_1}/\mathfrak{i}_1) \otimes_R R_{\mathfrak{p}}/\mathfrak{i} = 0$, so $R_{(\mathcal{I})} \otimes_R R_{\mathfrak{p}}/\mathfrak{i} = 0$, a contradiction. Note further that since $R_{\mathfrak{p}}/\mathfrak{i}$ is isomorphic to a localization of $R_{(\mathcal{I})}$ at a maximal ideal as an R-algebra, so that the obvious morphism $R_{\mathfrak{p}}/\mathfrak{i} \to R_{(\mathcal{I})} \otimes_R R_{\mathfrak{p}}/\mathfrak{i}$ is an isomorphism. This implies that $[\mathfrak{j}_0,\mathfrak{q}_0]$ contains $[\mathfrak{i},\mathfrak{p}]$. Indeed, otherwise we would encounter one of the following two cases:

(1) There is another interval in $\mathcal{I}$ overlapping $[\mathfrak{i},\mathfrak{p}]$. Then $R_{(\mathcal{I})}\otimes_R R_\mathfrak{p}/\mathfrak{i}$ would contain a nontrivial idempotent by Construction 6.22, using condition (C3). This is a contradiction to $R_{(\mathcal{I})}\otimes_R R_\mathfrak{p}/\mathfrak{i}\cong R_\mathfrak{p}/\mathfrak{i}$ being local.
(2) The interval $[\mathfrak{j}_0,\mathfrak{q}_0]$ is the only interval overlapping $[\mathfrak{i},\mathfrak{p}]$ and either $\mathfrak{j}_0\subseteq\mathfrak{i}\subseteq\mathfrak{q}_0\subsetneqq\mathfrak{p}$ or $\mathfrak{i}\subsetneqq\mathfrak{j}_0\subseteq\mathfrak{p}\subseteq\mathfrak{q}_0$ or $\mathfrak{i}\subsetneqq\mathfrak{j}_0\subseteq\mathfrak{q}_0\subsetneqq\mathfrak{p}$. In the first case we can take $\mathcal{I}_0=\{[\mathfrak{j},\mathfrak{q}]\in\mathcal{I}\mid[\mathfrak{j},\mathfrak{q}]\leq[\mathfrak{j}_0,\mathfrak{q}_0]\}$, $\mathcal{I}_1=\{[\mathfrak{j},\mathfrak{q}]\in\mathcal{I}\mid[\mathfrak{j},\mathfrak{q}]>[\mathfrak{i},\mathfrak{p}]\}$ and $R_{\mathfrak{q}_0}/\mathfrak{i}_0\times R_{\mathfrak{n}_1}/\mathfrak{i}_1$ as in Construction 6.22 (using condition (C1) to show that $\mathcal{I}_1$ has a minimum). Then $(R_{\mathfrak{q}_0}/\mathfrak{i}_0\times R_{\mathfrak{n}_1}/\mathfrak{i}_1)\otimes_R R_\mathfrak{p}/\mathfrak{i}\cong R_{\mathfrak{q}_0}/\mathfrak{i}$ and also $R_{(\mathcal{I})}\otimes_R R_\mathfrak{p}/\mathfrak{i}\cong R_{\mathfrak{q}_0}/\mathfrak{i}\not\cong R_\mathfrak{p}/\mathfrak{i}$, a contradiction. The other two cases lead to similar contradictions.

To summarize, we know so far that each $[\mathfrak{i},\mathfrak{p}]\in\mathcal{I}'$ is contained in a unique $[\mathfrak{j}_0,\mathfrak{q}_0]\in\mathcal{I}$.

Suppose conversely that we start with $[\mathfrak{j}_0,\mathfrak{q}_0]\in\mathcal{I}$. By the construction of $R_{(\mathcal{I})}$ we have $R_{(\mathcal{I})}\otimes_R R_{\mathfrak{q}_0}/\mathfrak{j}_0\cong R_{\mathfrak{q}_0}/\mathfrak{j}_0$ as R-algebras since all terms in the defining direct system have this property. Thus $(R_{(\mathcal{I})}/\mathfrak{j}_0R_{(\mathcal{I})})_{\mathfrak{q}_0}$ is local and there is a unique prime ideal $\mathfrak{n}\in\operatorname{Spec}R_{(\mathcal{I})}$ containing $\mathfrak{j}_0R_{(\mathcal{I})}$ such that the localization of $R_{(\mathcal{I})}/\mathfrak{j}_0R_{(\mathcal{I})}$ at $\mathfrak{n}$ is isomorphic to $R_{\mathfrak{q}_0}/\mathfrak{j}_0$ as R-algebra. Thus, invoking Lemma 6.8 for $f=f_{(\mathcal{I})}$, there is a unique interval $[\mathfrak{i},\mathfrak{p}]\in\mathcal{I}'=\mathcal{I}(f_{(\mathcal{I})})$ which contains $[\mathfrak{j}_0,\mathfrak{q}_0]$. This establishes the claim $\mathcal{I}'=\mathcal{I}$.

We have shown so far that the assignments $\mathcal{I}\mapsto f_{(\mathcal{I})}$ and $f\to\mathcal{I}(f)$ are well defined maps between the appropriate sets (recall Proposition 6.9 and Corollary 6.20), and that the composition $\mathcal{I}\mapsto f_{(\mathcal{I})}\mapsto\mathcal{I}(f_{(\mathcal{I})})$ is the identity on chains of admissible intervals. In order to prove the theorem, it suffices to show that $\mathcal{I}\mapsto f_{(\mathcal{I})}$ is a surjective assignment.

Thus suppose that we have $f\colon R\to S$ with $\mathcal{I}=\mathcal{I}(f)$. It is easy to check that f uniquely factors through any morphism $R\to\prod_{\ell=0}^n R_{\mathfrak{n}_\ell}/\mathfrak{i}_\ell$ in the direct system for $f_{(\mathcal{I})}\colon R\to R_{(\mathcal{I})}$ in Construction 6.22. One can see that for instance by Proposition 5.2, using the fact that the canonical homomorphism $S\cong S\otimes_R\prod_{\ell=0}^n R_{\mathfrak{n}_\ell}/\mathfrak{i}_\ell$ is bijective, which can be checked by localizing at maximal ideals of S. In particular we have a canonical morphism $g\colon R_{(\mathcal{I})}\to S$. Since $\mathcal{I}(f)=\mathcal{I}(f_{(\mathcal{I})})$, the map $(R_{(\mathcal{I})})_\mathfrak{n}\to S_\mathfrak{n}$ is an isomorphism for each $\mathfrak{n}\in\operatorname{Max}R_{(\mathcal{I})}$. Thus g is an isomorphism and we are done. □

Example 6.24. Let R be the ring which is called A in [Kel94b, §2]. The same ring can be obtained by invoking [FS01, Theorem II.3.8] for the totally ordered group $(\mathbb{Z}[1/\ell],+)$. Thus, R is a valuation domain, $\operatorname{Spec}R=\{0,\mathfrak{m}\}$ by [FS01, Proposition II.3.4], and $\mathfrak{m}^2=\mathfrak{m}$. Let Q be the quotient field of R and $k=R/\mathfrak{m}$ be the residue field. Our theorem says that we have precisely 5 distinct homological epimorphisms starting at R: $R\to0$, $R\to Q$, $R\to R$, $R\to k$ and $R\to Q\times k$, and only the first three are flat.

7. Flat epimorphisms

Now we will turn back to general commutative rings of weak global dimension ≤1. Our aim is to understand flat ring epimorphisms in this case. As it turns out, they precisely correspond to compactly generated Bousfield localizations, but the proof seems rather non-trivial.

We start with introducing some notations for future reference.

Notation 7.1. Let R be commutative, w. gl. dim $R \leq 1$ and $0 \neq f\colon R \to S$ be a flat ring epimorphism (so $S \neq 0$). Recall that by Lemma 6.1, $R_\mathfrak{p}$ is a valuation domain for every prime ideal $\mathfrak{p}$ of R. Thus, for every maximal ideal $\mathfrak{m}$ of R, $f \otimes_R R_\mathfrak{m}\colon R_\mathfrak{m} \to S_\mathfrak{m}$ is a flat epimorphism of the valuation domain $R_\mathfrak{m}$, hence by Proposition 6.4 there is a prime ideal $s(\mathfrak{m})$ of R such that $s(\mathfrak{m}) \subseteq \mathfrak{m}$ and $f \otimes_R R_\mathfrak{m}$ is equivalent to the localization $R_\mathfrak{m} \to R_{s(\mathfrak{m})}$.

Note also that in the situation of Notation 7.1, the map f is necessarily injective.

Lemma 7.2. *In the situation of Notation 7.1, we have:*

$$\{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q}S = S\} = \{\mathfrak{q} \in \operatorname{Spec} R \mid s(\mathfrak{m}) \subsetneqq \mathfrak{q} \subseteq \mathfrak{m}, \forall \mathfrak{m} \in \operatorname{Max} R, \mathfrak{m} \supseteq \mathfrak{q}\}.$$

Proof. For every prime ideal $\mathfrak{p} \in \operatorname{Spec} R$ we have an exact sequence

$$0 \longrightarrow \mathfrak{p} \otimes_R S \longrightarrow R \otimes_R S \longrightarrow R/\mathfrak{p} \otimes_R S \longrightarrow 0,$$

thus we may identify $\mathfrak{p} \otimes_R S$ with the S-ideal $\mathfrak{p}S$.

Let $\mathfrak{q} \in \operatorname{Spec} R$ be such that $\mathfrak{q}S = S$. Then, for every maximal ideal $\mathfrak{m}$ of R, $\mathfrak{q}S_\mathfrak{m} = S_\mathfrak{m} \cong R_{s(\mathfrak{m})}$, hence $\mathfrak{q} \not\subseteq s(\mathfrak{m})$. Thus, if $\mathfrak{m} \supseteq \mathfrak{q}$ we must have $s(\mathfrak{m}) \subsetneqq \mathfrak{q}$ since all primes below $\mathfrak{m}$ are totally ordered by inclusion.

Conversely, let $\mathfrak{q} \in \operatorname{Spec} R$ be such that $s(\mathfrak{m}) \subsetneqq \mathfrak{q}$ for every maximal ideal $\mathfrak{m}$ of R containing $\mathfrak{q}$. Assume, by way of contradiction that $\mathfrak{q}S \subsetneqq S$. Then there is a maximal ideal $\mathfrak{m}$ of R such that $\mathfrak{q}S_\mathfrak{m} \subsetneqq S_\mathfrak{m}$. Thus, $\mathfrak{q}R_{s(\mathfrak{m})} \subsetneqq R_{s(\mathfrak{m})}$, giving $\mathfrak{q} \subseteq s(\mathfrak{m}) \subseteq \mathfrak{m}$, a contradiction. □

We aim to prove that every flat epimorphism $f\colon R \to S$ as above is given by a compactly generated localization of $\mathbf{D}(R)$. The key role is played by Thomason's localization theory [Tho97] which classifies compactly generated localizations purely in terms of $\operatorname{Spec} R$ as a topological space. Let us recall the fundamentals.

Definition 7.3. Let R be a commutative ring. For $X \in \mathbf{D}(R)$ we define its *cohomological support* as

$$\operatorname{Supp} X = \{\mathfrak{p} \in \operatorname{Spec} R \mid X \otimes_R R_\mathfrak{p} \neq 0\}.$$

For a class of complexes $\mathcal{X}$, we define $\operatorname{Supp} \mathcal{X} = \bigcup_{X \in \mathcal{X}} \operatorname{Supp} X$.

A subset $Z \subseteq \operatorname{Spec} R$ is a *Thomason set* if it can be expressed as a union $Z = \bigcup Z_i$ with each Z_i Zariski closed and such that $\operatorname{Spec} R \setminus Z_i$ is quasi-compact. In other words, we have $Z_i = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \supseteq I_i\}$ for a finitely generated ideal $I_i \subseteq R$.

Proposition 7.4. *Let R be a commutative ring. Then there is a bijection between*

1. *Thomason sets $Z \subseteq \operatorname{Spec} R$, and*
2. *compactly generated localizing subcategories $\mathcal{X} \subseteq \mathbf{D}(R)$.*

given by the assignment $\mathcal{X} \mapsto Z = \operatorname{Supp} \mathcal{X}$.

Proof. We combine two results from the literature. Firstly [Tho97, Theorem 3.15] provides us with a similar bijection between Thomason sets and thick subcategories (i.e. triangulated and closed under summands) of the category of perfect complexes. In particular, if Z is a Thomason set and $\mathcal{C}$ is any set of

perfect complexes such that $Z = \operatorname{Supp}\mathcal{C}$, then the smallest thick subcategory of $\mathbf{D}(R)$ containing $\mathcal{C}$ is

$$\mathcal{C}' = \{X \in \mathbf{D}(R) \mid X \textit{ compact and } \operatorname{Supp} X \subseteq Z\}.$$

Secondly, [Nee92b, Theorem 2.1] establishes a bijection between thick subcategories of the category of perfect complexes and compactly generated Bousfield localizations of $\mathbf{D}(R)$ (cp. Definition 1.3). Starting with $\mathcal{C}$ as before, the localizing class $\mathcal{X}$ will be the smallest localizing class containing $\mathcal{C}$. Clearly $\operatorname{Supp}\mathcal{C} \subseteq \operatorname{Supp}\mathcal{X}$ and one easily sees also $\operatorname{Supp}\mathcal{C} \supseteq \operatorname{Supp}\mathcal{X}$ as closing $\mathcal{C}$ under coproducts, mapping cones and (de)suspensions cannot enlarge the support. □

Remark 7.5. Note that in the above correspondence, we only have proved $\mathcal{X} \subseteq \{X \in \mathbf{D}(R) \mid \operatorname{Supp} X \subseteq Z\}$, where $Z = \operatorname{Supp}\mathcal{X}$. We do not know whether these classes are equal in general, although they are in various cases. If R is commutative noetherian, the equality essentially follows from [Nee92a, Lemma 3.6]. If Z is Zariski closed with quasi-compact complement, the equality holds by [KP13, Theorem 2.2.4]. As we will show below, the equality also holds whenever $\operatorname{w.gl.dim} R \leq 1$.

To this end, we will need an auxiliary lemma which tells us how the support theory behaves with respect to localization.

Lemma 7.6. *Let R be a commutative ring such that* $\operatorname{w.gl.dim} R \leq 1$, *$\mathfrak{S} \subseteq R$ be a multiplicative subset, $\ell\colon R \to R_{\mathfrak{S}}$ be the localization morphism, and* $\operatorname{Spec}\ell\colon \operatorname{Spec} R_{\mathfrak{S}} \to \operatorname{Spec} R$ *be the induced morphism between the spectra. Suppose that $\mathcal{C} \subseteq \mathbf{D}(R)$ is a set of perfect complexes and $f\colon R \to S$ is a homological epimorphism corresponding to the Bousfield localization compactly generated by $\mathcal{C}$ (see Theorem 4.10 and §1.2). Then $f \otimes_R R_{\mathfrak{S}}\colon R_{\mathfrak{S}} \to S_{\mathfrak{S}}$ corresponds to the Bousfield localization of $\mathbf{D}(R_{\mathfrak{S}})$ compactly generated by $\mathcal{C}_{\mathfrak{S}} = \{C \otimes^{\mathbf{L}}_R R_{\mathfrak{S}} \mid C \in \mathcal{C}\}$.*

Proof. Clearly, $\operatorname{Supp}\mathcal{C}_{\mathfrak{S}} = (\operatorname{Spec}\ell)^{-1}(\operatorname{Supp}\mathcal{C})$ and the conclusion then follows from [Ste12, Proposition 2.6].

In more pedestrian terms, consider the set $\mathcal{C}'$ of perfect complexes of the form $R \xrightarrow{r} R$ which are concentrated in degrees -1 and 0 and with $r \in \mathfrak{S}$. Then it is straightforward to see that $\mathcal{C}' \otimes_R S = \{S \xrightarrow{f(r)} S\} \subseteq \mathbf{D}(S)$ compactly generates the localization of $\mathbf{D}(S)$ whose corresponding homological epimorphism is the ordinary localization $S \to S_{f(\mathfrak{S})}$ with respect to the multiplicative subset $f(\mathfrak{S}) \subseteq S$. Thus the composition $R \to S \to S_{f(\mathfrak{S})}$ corresponds to the localization of $\mathbf{D}(R)$ compactly generated by $\mathcal{C} \cup \mathcal{C}'$. The same composition can be also expressed as $R \to R_{\mathfrak{S}} \to S_{f(\mathfrak{S})}$, from which we see that $R_{\mathfrak{S}} \to S_{f(\mathfrak{S})}$ corresponds to the localization of $\mathbf{D}(R_{\mathfrak{S}})$ generated by $\mathcal{C}_{\mathfrak{S}} = \mathcal{C} \otimes_R R_{\mathfrak{S}}$. □

Now we can give the promised description of the class of acyclic objects for rings of weak global dimension at most 1.

Proposition 7.7. *Suppose that R is a commutative ring of* $\operatorname{w.gl.dim} R \leq 1$ *and $\mathcal{X}$ be a compactly generated localizing class in $\mathbf{D}(R)$. Let $Z \subseteq \operatorname{Spec} R$*

be the corresponding Thomason set and let $f\colon R \to S$ be the induced homological epimorphism. Then f is a flat epimorphism and we have

$$Z = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q}S = S\} \quad \textit{and} \quad \mathcal{X} = \{X \in \mathbf{D}(R) \mid \operatorname{Supp} X \subseteq Z\}.$$

Proof. Suppose first that R is a valuation domain, hence semihereditary. If $\mathcal{X}$ is generated as a localizing class by compact objects, it is by Proposition 5.4 and its proof generated by a set of finitely presented R-modules (viewed as complexes concentrated in degree 0). Since every finitely presented module over a valuation domain is a direct sum of modules of the form R/rR for some $r \in R$ (see [FS01, Theorem I.7.9]), it follows that every compactly generated Bousfield localization is generated by a set of 2-term perfect complexes of the form $R \xrightarrow{r} R$. As in the proof of Lemma 7.6, such a localization corresponds in terms of homological epimorphisms to an ordinary localization with respect to a multiplicative set. For a valuation domain, such a localization must be of the form $R \to R_{\mathfrak{p}}$ for $\mathfrak{p} \in \operatorname{Spec} R$; see the proof of Proposition 6.4. Hence we have $Z = \operatorname{Supp} \mathcal{X} = \{\mathfrak{q} \in \operatorname{Spec} R \mid r \in \mathfrak{q} \text{ for some } r \in R \setminus \mathfrak{p}\} = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \supsetneq \mathfrak{p}\} = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q}R_{\mathfrak{p}} = R_{\mathfrak{p}}\}$. Moreover, $\mathcal{X}$ is as required by Theorem 4.10.

Let now R be general and $\mathcal{C}$ be a set of perfect complexes generating $\mathcal{X}$. Then the morphism $f \otimes_R R_{\mathfrak{m}}\colon R_{\mathfrak{m}} \to S_{\mathfrak{m}}$ for each $\mathfrak{m} \in \operatorname{Max} R$ is by Lemma 7.6 equivalent to $R_{\mathfrak{m}} \to R_{s(\mathfrak{m})}$ as in Notation 7.1. In particular each $S_{\mathfrak{m}}$ is flat over $R_{\mathfrak{m}}$ and so S is flat over R. Further, $\mathfrak{q} \in Z$ if and only if $\mathfrak{q} \in \operatorname{Supp} \mathcal{C}$ if and only if $\mathfrak{q} \in \operatorname{Supp} \mathcal{C}_{\mathfrak{m}}$ for each $\mathfrak{m} \in \operatorname{Max} R$ such that $\mathfrak{m} \supseteq \mathfrak{q}$. Applying Lemma 7.6, we have $Z = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q}R_{s(\mathfrak{m})} = R_{s(\mathfrak{m})}\ \forall \mathfrak{m} \in \operatorname{Max} R, \mathfrak{m} \supseteq \mathfrak{q}\} = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q}S = S\}$.

Finally, by Theorem 4.10, Lemma 7.2 and the above discussion we have

$$\begin{aligned}\mathcal{X} &= \big\{X \in \mathbf{D}(R) \mid H^n(X) \otimes_R S = 0 \text{ for all } n \in \mathbb{Z}\big\} = \\ &= \big\{X \in \mathbf{D}(R) \mid H^n(X) \otimes_R R_{s(\mathfrak{m})} = 0 \text{ for all } n \in \mathbb{Z} \text{ and } \mathfrak{m} \in \operatorname{Max} R\big\} = \\ &= \{X \in \mathbf{D}(R) \mid \operatorname{Supp} X \subseteq Z\}. \quad \square\end{aligned}$$

As a consequence, we get the characterization of homological epimorphisms coming from compactly generated Bousfield localizations of $\mathbf{D}(R)$.

Theorem 7.8. *Let R be a commutative ring of weak global dimension at most 1. Then the correspondence from Theorem 4.10 restricts to the bijection between*

1. *equivalence classes of flat epimorphisms $f\colon R \to S$ originating at R, and*
2. *compactly generated localizing subcategories $\mathcal{X} \subseteq \mathbf{D}(R)$.*

If, moreover, R is semihereditary, then $f\colon R \to S$ is flat if and only if f is a universal localization.

Remark 7.9. Note that semiheredity is a strictly stronger assumption than $\operatorname{w.gl.dim} R \leq 1$, see Example 6.21 or [Gla05, Example 3.1.2].

Proof. If f corresponds to a compactly generated Bousfield localizations, it is flat by Proposition 7.7.

Suppose conversely that $f\colon R \to S$ is a flat ring epimorphism. We claim that $Z = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q}S = S\}$ is a Thomason set. Indeed, for any

$\mathfrak{q} \in Z$ write $1 = \sum_{i=1}^{n} a_i s_i$ with $a_i \in \mathfrak{q}$ and $s_i \in S$. Then the ideal $I = (a_1, \dots, a_n) \subseteq R$ is such that $IS = S$ and $I \subseteq \mathfrak{q}$. Thus, $\mathfrak{q} \in \operatorname{Supp} R/I \subseteq Z$ and $\operatorname{Supp} R/I$ is Zariski closed with quasi-compact complement. This proves the claim.

Let now $\mathcal{X} = \operatorname{Ker}(- \otimes_R S)$ be the localizing class corresponding to S (see Theorem 4.10). Then, using Notation 7.1 and Lemma 7.2, we have

$$\mathcal{X} = \big\{X \in \mathbf{D}(R) \mid H^n(X) \otimes_R R_{s(\mathfrak{m})} = 0 \text{ for all } n \in \mathbb{Z} \text{ and } \mathfrak{m} \in \operatorname{Max} R\big\} = \\ = \{X \in \mathbf{D}(R) \mid \operatorname{Supp} X \subseteq Z\}.$$

Thus, S describes the compactly generated localization corresponding to Z by Proposition 7.7.

The last part concerning universal localizations follows from Proposition 5.4. □

8. The Telescope Conjecture

Finally, we will discuss the Telescope Conjecture for rings of weak global dimension ≤ 1. Although we do not obtain a full classification of smashing localizations as in the case of valuation domains, we are still able to obtain an easy criterion characterizing when the Telescope Conjecture holds for $\mathbf{D}(R)$. In particular we will see that this is always the case when R is a commutative von Neumann regular (also known as absolutely flat) ring, generalizing [Ste12, Theorem 4.21].

Definition 8.1. Let $\mathcal{T}$ be a triangulated category with coproducts. We say that the *Telescope Conjecture* holds for $\mathcal{T}$ if every smashing localization of $\mathcal{T}$ is a compactly generated localization (see §1.2).

In fact, the Telescope Conjecture is a property of $\mathcal{T}$, it holds for some triangulated categories and fails for others. For $\mathbf{D}(R)$ with R commutative and $\operatorname{w.gl.dim} R \leq 1$, it asks for every homological epimorphism $f\colon R \to S$ to be flat. If R is even semihereditary, it equivalently requires that every homological epimorphism $f\colon R \to S$ is a universal localization (see also [KŠ10, §§6 and 7]). Now we can state the main result of the final section.

Theorem 8.2. *Let R be a commutative ring of weak global dimension ≤ 1. Then the following are equivalent:*

1. *The Telescope Conjecture holds for $\mathbf{D}(R)$;*
2. *Every homological epimorphism $f\colon R \to S$ is flat;*
3. *There is no $\mathfrak{p} \in \operatorname{Spec} R$ such that $\mathfrak{p}R_\mathfrak{p}$ is a non-zero idempotent ideal in $R_\mathfrak{p}$.*

Proof. (1) ⇔ (2) follows from Theorem 7.8. Assuming (2), let $\mathfrak{p} \in \operatorname{Spec} R$. If $0 \neq \mathfrak{p}R_\mathfrak{p}$ is idempotent in $R_\mathfrak{p}$, then $R \to R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is a non-flat homological epimorphism by Lemma 4.5, hence (2) ⇒ (3). Finally, assume (3) and let $f\colon R \to S$ be a homological epimorphism. Then $f \otimes_S R_\mathfrak{p}\colon R_\mathfrak{p} \to S_\mathfrak{p}$ must be flat for each $\mathfrak{p} \in \operatorname{Spec} R$ by Theorem 6.23. Hence f is flat and (2) follows. □

In particular, we have the following necessary condition.

Corollary 8.3. *If R is commutative, $\operatorname{w.gl.dim} R \leq 1$ and the Telescope Conjecture holds for $\mathbf{D}(R)$, then $(\operatorname{Spec} R, \subseteq)$ has the ascending chain condition on prime ideals.*

Proof. If there is an infinite chain $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq \cdots$ of primes of R, then $\mathfrak{p} = \bigcup_i \mathfrak{p}_i$ is also a prime ideal which is necessarily idempotent in its localization by Lemmas 6.1 and 6.3. □

We end by listing some classes of commutative semihereditary rings R studied in the literature such that $\mathbf{D}(R)$ satisfies the telescope conjecture.

(1) Recall that a commutative domain is a *Prüfer domain* if every localization at a maximal (or prime) ideal is a valuation domain. A Prüfer domain is *strongly discrete* [FS01, §III.7] if no non-zero prime ideal is idempotent. Then $\mathbf{D}(R)$ satisfies the Telescope Conjecture for a Prüfer domain R if and only if R is strongly discrete; see [FS01, Proposition III.7.4].
(2) If R a commutative Von Neumann regular ring, then $\mathbf{D}(R)$ satisfies the telescope conjecture. This generalizes [Ste12, Theorem 4.21]. In fact, R is semihereditary and every localization at a maximal ideal is a field (see e.g. [Gla89, Corollary 4.2.7]). Note that in this case every ideal of R is idempotent.

References

[AHKL11] Lidia Angeleri Hügel, Steffen Koenig, and Qunhua Liu. Recollements and tilting objects. *J. Pure Appl. Algebra*, 215(4):420–438, 2011.

[Bal05] Paul Balmer. The spectrum of prime ideals in tensor triangulated categories. *J. Reine Angew. Math.*, 588:149–168, 2005.

[BBD82] A. A. Beĭlinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.

[Bec14] Hanno Becker. Models for singularity categories. *Adv. Math.*, 254:187–232, 2014.

[BF11] Paul Balmer and Giordano Favi. Generalized tensor idempotents and the telescope conjecture. *Proc. Lond. Math. Soc. (3)*, 102(6):1161–1185, 2011.

[BIK08] Dave Benson, Srikanth B. Iyengar, and Henning Krause. Local cohomology and support for triangulated categories. *Ann. Sci. Éc. Norm. Supér. (4)*, 41(4):573–619, 2008.

[BIK11] Dave Benson, Srikanth B. Iyengar, and Henning Krause. Stratifying triangulated categories. *J. Topol.*, 4(3):641–666, 2011.

[BK72] A. K. Bousfield and D. M. Kan. The core of a ring. *J. Pure Appl. Algebra*, 2:73–81, 1972.

[BK73] A. K. Bousfield and D. M. Kan. Correction to "The core of a ring" (J. Pure Appl. Algebra **2** (1972), 73–81). *J. Pure Appl. Algebra*, 3:409, 1973.

[BO95] Alexey I. Bondal and Dmitri O. Orlov. Semiorthogonal decomposition for algebraic varieties. Preprint, arXiv:alg-geom/9506012v1, 1995.

[Bou79] A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.

[CE56] Henri Cartan and Samuel Eilenberg. *Homological algebra*. Princeton University Press, Princeton, N. J., 1956.

[DHS88] Ethan S. Devinatz, Michael J. Hopkins, and Jeffrey H. Smith. Nilpotence and stable homotopy theory. I. *Ann. of Math. (2)*, 128(2):207–241, 1988.

[DP08] W. G. Dwyer and J. H. Palmieri. The Bousfield lattice for truncated polynomial algebras. *Homology Homotopy Appl.*, 10(1):413–436, 2008.

[EJ00] Edgar E. Enochs and Overtoun M. G. Jenda. *Relative homological algebra*, volume 30 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter & Co., Berlin, 2000.

[FS01] László Fuchs and Luigi Salce. *Modules over non-Noetherian domains*, volume 84 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2001.

[GdlP87] P. Gabriel and J. A. de la Peña. Quotients of representation-finite algebras. *Comm. Algebra*, 15(1-2):279–307, 1987.

[GL91] Werner Geigle and Helmut Lenzing. Perpendicular categories with applications to representations and sheaves. *J. Algebra*, 144(2):273–343, 1991.

[Gla89] Sarah Glaz. *Commutative coherent rings*, volume 1371 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1989.

[Gla05] Sarah Glaz. Prüfer conditions in rings with zero-divisors. In *Arithmetical properties of commutative rings and monoids*, volume 241 of *Lect. Notes Pure Appl. Math.*, pages 272–281. Chapman & Hall/CRC, Boca Raton, FL, 2005.

[Hir03] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.

[Hov99] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[HPS97] Mark Hovey, John H. Palmieri, and Neil P. Strickland. Axiomatic stable homotopy theory. *Mem. Amer. Math. Soc.*, 128(610):x+114, 1997.

[Isb69] John R. Isbell. Epimorphisms and dominions. IV. *J. London Math. Soc. (2)*, 1:265–273, 1969.

[Jar97] J. F. Jardine. A closed model structure for differential graded algebras. In *Cyclic cohomology and noncommutative geometry (Waterloo, ON, 1995)*, volume 17 of *Fields Inst. Commun.*, pages 55–58. Amer. Math. Soc., Providence, RI, 1997.

[Kel94a] Bernhard Keller. Deriving DG categories. *Ann. Sci. École Norm. Sup. (4)*, 27(1):63–102, 1994.

[Kel94b] Bernhard Keller. A remark on the generalized smashing conjecture. *Manuscripta Math.*, 84(2):193–198, 1994.

[Kel98] Bernhard Keller. On the construction of triangle equivalences. In *Derived equivalences for group rings*, volume 1685 of *Lecture Notes in Math.*, pages 155–176. Springer, Berlin, 1998.

[Kel07] Bernhard Keller. Derived categories and tilting. In *Handbook of tilting theory*, volume 332 of *London Math. Soc. Lecture Note Ser.*, pages 49–104. Cambridge Univ. Press, Cambridge, 2007.

[KP13] Joachim Kock and Wolfgang Pitsch. Hochster duality in derived categories and point-free reconstruction of schemes. Preprint, arXiv:1305.1503v2, 2013.

[Kra00] Henning Krause. Smashing subcategories and the telescope conjecture—an algebraic approach. *Invent. Math.*, 139(1):99–133, 2000.

[Kra05] Henning Krause. Cohomological quotients and smashing localizations. *Amer. J. Math.*, 127(6):1191–1246, 2005.

[Kra07] Henning Krause. Derived categories, resolutions, and Brown representability. In *Interactions between homotopy theory and algebra*, volume 436 of *Contemp. Math.*, pages 101–139. Amer. Math. Soc., Providence, RI, 2007.

[Kra10] Henning Krause. Localization theory for triangulated categories. In *Triangulated categories*, volume 375 of *London Math. Soc. Lecture Note Ser.*, pages 161–235. Cambridge Univ. Press, Cambridge, 2010.

[KŠ10] Henning Krause and Jan Šťovíček. The telescope conjecture for hereditary rings via Ext-orthogonal pairs. *Adv. Math.*, 225(5):2341–2364, 2010.

[Laz69] Daniel Lazard. Autour de la platitude. *Bull. Soc. Math. France*, 97:81–128, 1969.

[Maz68] Pierre Mazet. Caractérisation des épimorphismes par relations et générateurs. In *Les épimorphismes d'anneaux*, Sem. P. Samuel (2), pages A2.01–A2.08. Paris, 1968.

[Mit73] Barry Mitchell. The cohomological dimension of a directed set. *Canad. J. Math.*, 25:233–238, 1973.

[ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[Nee92a] Amnon Neeman. The chromatic tower for $D(R)$. *Topology*, 31(3):519–532, 1992. With an appendix by Marcel Bökstedt.

[Nee92b] Amnon Neeman. The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel. *Ann. Sci. École Norm. Sup. (4)*, 25(5):547–566, 1992.

[Nee00] Amnon Neeman. Oddball Bousfield classes. *Topology*, 39(5):931–935, 2000.

[Nee01] Amnon Neeman. *Triangulated categories*, volume 148 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001.

[NS09] Pedro Nicolás and Manuel Saorín. Parametrizing recollement data for triangulated categories. *J. Algebra*, 322(4):1220–1250, 2009.

[NS13] Pedro Nicolás and Manuel Saorín. Pseudoepimorphisms and semiorthogonal decompositions. Private communication, 2013.

[Pau09] David Pauksztello. Homological epimorphisms of differential graded algebras. *Comm. Algebra*, 37(7):2337–2350, 2009.

[Rav84] Douglas C. Ravenel. Localization with respect to certain periodic homology theories. *Amer. J. Math.*, 106(2):351–414, 1984.

[Sch85] A. H. Schofield. *Representation of rings over skew fields*, volume 92 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1985.

[Sch05] Karl Schwede. Gluing schemes and a scheme without closed points. In *Recent progress in arithmetic and algebraic geometry*, volume 386 of *Contemp. Math.*, pages 157–172. Amer. Math. Soc., Providence, RI, 2005.

[Sil67] L. Silver. Noncommutative localizations and applications. *J. Algebra*, 7:44–76, 1967.

[ŠP12] Jan Šťovíček and David Pospíšil. On compactly generated torsion pairs and the classification of co-t-structures for commutative noetherian rings. Preprint, arXiv:1212.3122v1, 2012.

[Spa88] N. Spaltenstein. Resolutions of unbounded complexes. *Compositio Math.*, 65(2):121–154, 1988.

[SS00] Stefan Schwede and Brooke E. Shipley. Algebras and modules in monoidal model categories. *Proc. London Math. Soc. (3)*, 80(2):491–511, 2000.

[Ste75] Bo Stenström. *Rings of quotients*. Springer-Verlag, New York, 1975. Die Grundlehren der Mathematischen Wissenschaften, Band 217, An introduction to methods of ring theory.

[Ste12] Greg Stevenson. Derived categories of absolutely flat rings. Preprint, arXiv:1210.0399v1, 2012.

[Tho97] R. W. Thomason. The classification of triangulated subcategories. *Compositio Math.*, 105(1):1–27, 1997.

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