

A Family of 0-Simple Semihypergroups Related to Sequence A000070

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For any integer $n \geq 2$, let $\mathfrak{R}_0(n+1)$ be the class of 0-semihypergroups H of size $n+1$ such that $\{y\} \subseteq xy \subseteq \{0, y\}$ for all $x, y \in H - \{0\}$, all subsemihypergroups $K \subseteq H$ are 0-simple and, when $|K| \geq 3$, the fundamental relation β_K is not transitive. We determine a transversal of isomorphism classes of semihypergroups in $\mathfrak{R}_0(n+1)$ and we prove that its cardinality is the $(n+1)$ -th term of sequence A000070 in [21], namely, $\sum_{k=0}^n p(k)$, where $p(k)$ denotes the number of non-increasing partitions of integer k .

Keywords: Zero-semihypergroups, simple semihypergroups, graphs, integer sequences.

1 INTRODUCTION

According to [7, 9], a *fully simple semihypergroup* is a semihypergroup H such that all subsemihypergroup $K \subseteq H$ are simple and, when $|K| \geq 3$, the fundamental relation β_K is not transitive.

The class \mathfrak{F} of all fully simple semihypergroups has been characterized completely in [9] in terms of a small set of simple semihypergroups of size 3. In [7], the authors determine a transversal of isomorphism classes of finite semihypergroups in \mathfrak{F} . The structure of that transversal can be described by means of certain digraphs, which are transitive and acyclic, and allow

to derive an explicit formula to calculate the number of those isomorphism classes [4, 7]. Analogous results concerning enumeration of isomorphism classes of algebraic hyperstructures are found in [1, 6, 8].

Motivated by the abovementioned results, in the present paper we study the semihypergroups H such that all subsemihypergroups $K \subseteq H$ are 0-simple and, when $|K| \geq 3$, the fundamental relation β_K is not transitive. These semihypergroups are called *fully 0-simple* and we denote by \mathfrak{F}_0 their class.

The plan of this paper is the following. Hereafter, we introduce some basic definitions and notations to be used throughout the paper. In Section 2, we define fully 0-simple semihypergroups and provide some relevant properties of semihypergroups in \mathfrak{F}_0 . Notably we prove that, if $H \in \mathfrak{F}_0$ then $\prod_{i=1}^n z_i \neq \{0\}$ for every nonzero elements z_1, z_2, \dots, z_n in H .

Sections 3 and 4 are devoted to the introduction and a preliminary analysis of the class \mathfrak{R}_0 of R0-semihypergroups, that is, semihypergroups $H \in \mathfrak{F}_0$ such that $\{y\} \subseteq xy \subseteq \{0, y\}$ for all $x, y \in H - \{0\}$, which are the main subject of this paper. In particular, we find a characterization of R0-semihypergroups in terms of specific properties of the cardinality of hyperproducts xy with $x, y \in H - \{0\}$. Moreover, we introduce a special equivalence relation among the nonzero elements of $H \in \mathfrak{R}_0$. These preliminary results are exploited in Section 5 to associate a Boolean matrix to every finite R0-semihypergroup.

The main results of this paper are contained in Section 6, where we show a transversal of isomorphism classes of finite R0-semihypergroups and we prove that the number of isomorphism classes of semihypergroups in \mathfrak{R}_0 having size $n + 1$ is the $(n + 1)$ -th term of sequence A000070 in [21], namely, $\sum_{k=0}^n p(k)$.

1.1 Basic definitions and results

Throughout this paper we use just a few basic concepts and definitions which belong to common terminology in hyperstructure theory.

A *hypergroupoid* is a set H endowed with a hyperproduct, that is, a function $H \times H \mapsto \mathcal{P}^*(H)$ where $\mathcal{P}^*(H)$ denotes the family of all non-empty subsets of H . If the hyperproduct is associative, that is, $(xy)z = x(yz)$ for all $x, y, z \in H$, then H is a *semihypergroup*. If H is a semihypergroup then H is also a semihypergroup with respect to the hyperproduct \circ defined as $x \circ y = yx$ for all $x, y \in H$. The semihypergroup (H, \circ) is called *transposed* of H and is denoted by H^T . Clearly, the use of that term is motivated by the fact that, in the finite case, the multiplicative table of H^T is obtained by transposing the multiplicative table of H .

A non-empty subset K of a semihypergroup H is called a *subsemihypergroup* of H if it is closed with respect to multiplication, that is, $xy \subseteq K$ for all $x, y \in K$.

For any $x \in H$ we refer to $\widehat{x} = \bigcup_{n \geq 1} x^n$ as the *cyclic subsemihypergroup of H generated by the element x* . It is the smallest subsemihypergroup containing x .

If H is a semihypergroup, an element $0 \in H$ such that $x0 = \{0\}$ (resp., $0x = \{0\}$) for all $x \in H$ is called *right zero scalar element* or *right absorbing element* (resp., *left zero scalar element* or *left absorbing element*) of H [18]. If 0 is both right and left zero scalar element, then it is called *zero scalar*. A zero scalar element, if it exists, is unique.

A *simple semihypergroup* is a semihypergroup H such that $HxH = H$, for all $x \in H$. A semihypergroup H with a zero scalar element 0 is a *0-semihypergroup*. A 0-semihypergroup is called *0-simple* if $HxH = H$ for all $x \in H - \{0\}$.

Given a semihypergroup H , the relation β^* is the transitive closure of the relation $\beta = \bigcup_{n \geq 1} \beta_n$, where β_1 is the diagonal relation in H and, for every integer $n > 1$, β_n is defined recursively as follows:

$$x\beta_n y \iff \exists(z_1, \dots, z_n) \in H^n : \{x, y\} \subseteq \prod_{i=1}^n z_i.$$

The relations β and β^* are two of the best known *fundamental relations* on H [2, 12, 22]. Their relevance in semihypergroup theory stems from the following facts [19]: The quotient set H/β^* , equipped with the operation $\beta^*(x) \otimes \beta^*(y) = \beta^*(z)$ for all $x, y \in H$ and $z \in xy$, is a semigroup. Moreover, the relation β^* is the smallest strongly regular equivalence on H such that the quotient H/β^* is a semigroup. The interested reader can find all relevant definitions, many properties and applications of fundamental relations, even in more abstract contexts, also in [3, 5, 10, 11, 13–15, 17, 20].

Finally, we borrow from [12] a theorem which classifies the 0-simple semihypergroups of size 3 whose relation β is not transitive.

Theorem 1.1. *Up to isomorphisms, there exist fourteen 0-simple semihypergroups of size 3 where the relation β is not transitive. Their hyperproduct tables are the following:*

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We observe that $H_1, H_4, H_5,$ and H_6 are commutative semihypergroups. The semihypergroups $H_3, H_{11}, H_{12}, H_{13}, H_{14}$ are respectively the transposed semihypergroups of $H_2, H_7, H_8, H_9,$ and H_{10} . Moreover, $H_1, H_2, H_3, H_4, H_5,$ and H_6 coincide with their cyclic semihypergroups generated by the element 1.

2 FULLY 0-SIMPLE SEMIHYPERGROUPS

In this section we introduce semihypergroups H that fulfil the following conditions:

1. All subsemihypergroups of H (H itself included) are 0-simple; in particular, they have a zero scalar element.
2. The relation β in H and the relations β_K in all subsemihypergroups $K \subset H$ of size ≥ 3 are not transitive.

These semihypergroups are called *fully 0-simple semihypergroups*; their size is ≥ 3 since the relation β is transitive if $|H| \leq 2$. Actually, the semihypergroups in Theorem 1.1 are all the fully 0-simple semihypergroups of size 3, up to isomorphisms.

For notational simplicity, we denote by \mathfrak{F}_0 the class of fully 0-simple semihypergroups and we use 0 to indicate the zero scalar element of any $H \in \mathfrak{F}_0$. Furthermore, we use the notation H_+ to indicate the set of nonzero elements in H , that is, $H_+ = H - \{0\}$.

Hereafter, we prove some relevant properties of semihypergroups $H \in \mathfrak{F}_0$. We start by establishing preliminary results on subsemihypergroups and hyperproducts of elements in H .

Proposition 2.1. *If H is a 0-simple semihypergroup then $HH = H$.*

Proof. Obviously $0 \in HH$ because $\{0\} = 00$. Moreover, since H is 0-simple, for any fixed $x \in H_+$ we have $HxH = H$. Therefore, there exist $a, b \in H$ such that $x \in axb$. In consequence, there exists $d \in ax$ such that $x \in db \in HH$, whence $H \subseteq HH \subseteq H$. \square

Proposition 2.2. *If S is a subsemihypergroup of $H \in \mathfrak{F}_0$ such that $0 \notin S$, then $|S| = 1$.*

Proof. Let S be a subsemihypergroup of H such that $0 \notin S$. By definition, there exists in S a zero scalar element $0'$ and, by Proposition 2.1, $SS = S$. We have

$$(S \cup \{0\})(S \cup \{0\}) = SS \cup S0 \cup 0S \cup 00 = S \cup \{0\},$$

whence also $S \cup \{0\}$ is a 0-simple subsemihypergroup of H with 0 as zero scalar element. Now, since $0' \in S \cup \{0\}$ and $0' \neq 0$, we obtain:

$$S \cup \{0\} = (S \cup \{0\})0'(S \cup \{0\}) = S0'S \cup S0'0 \cup 00'S \cup 00'0 = \{0', 0\}.$$

Thus $S = \{0'\}$. \square

The following corollary is an immediate consequence of Proposition 2.2.

Corollary 2.3. *If $H \in \mathfrak{F}_0$ and S is a subsemihypergroup S of H such that $|S| \geq 2$, then the zero element of S is 0.*

Another consequence of Proposition 2.2 is the following result:

Proposition 2.4. *If $H \in \mathfrak{F}_0$ then there exist $x, y \in H_+$ such that $0 \in xy$.*

Proof. If $0 \notin xy$ for every $x, y \in H_+$, then $H_+H_+ \subseteq H_+$, hence H_+ is a subsemihypergroup of H . By Proposition 2.2, we have $|H_+| = 1$ and so $|H| = 2$. This fact leads to an absurdity because $H \in \mathfrak{F}_0$ and consequently $|H| \geq 3$. \square

In the forthcoming Theorem 2.7 we prove that if $H \in \mathfrak{F}_0$ then $xy \neq \{0\}$, for any pair x, y of non-zero elements of H . We premise the following lemma:

Lemma 2.5. *If $H \in \mathfrak{F}_0$ then we have $xx \neq \{0\}$, for every $x \in H_+$.*

Proof. If $xx = \{0\}$ then the set $S = \{0, x\}$ is a subsemihypergroup of H . Moreover we have $SxS = \{0\} \neq S$. This fact is an absurdity since $H \in \mathfrak{F}_0$ and S is not 0-simple. \square

Remark 2.6. If $H \in \mathfrak{F}_0$ then for every $x \in H_+$ such that $xx \neq \{x\}$ the cyclic semihypergroup $\widehat{x} = \bigcup_{n \geq 1} x^n$ generated by x has size ≥ 2 . Therefore it is a 0-simple subsemihypergroup of H , with 0 as zero scalar element.

Theorem 2.7. *If $H \in \mathfrak{F}_0$ then $ab \neq \{0\}$, for every $a, b \in H_+$.*

Proof. By absurd, suppose that there exist $a, b \in H_+$ such that $ab = \{0\}$. From Lemma 2.5 we get $a \neq b$. Then we have also $ba = \{0\}$. In fact, for any $c \in ba$ we have $cc \subseteq (ba)(ba) = b(ab)a = b0a = \{0\}$. Therefore $cc = \{0\}$, whence $c = 0$ by Lemma 2.5. So $ab = ba = \{0\}$ and $a \neq b$. Obviously, for every positive integers n, m we have

$$a^n b^m = b^m a^n = \{0\}. \tag{1}$$

Now, considering the cyclic subsemihypergroups $\widehat{a} = \bigcup_{n \geq 1} a^n$ and $\widehat{b} = \bigcup_{n \geq 1} b^n$, we obtain

$$(\widehat{a} \cup \widehat{b})(\widehat{a} \cup \widehat{b}) = \widehat{a}\widehat{a} \cup \widehat{a}\widehat{b} \cup \widehat{b}\widehat{a} \cup \widehat{b}\widehat{b} = \widehat{a} \cup \{0\} \cup \widehat{b},$$

and so we can distinguish the following two cases:

1. Case $aa = \{a\}$ and $bb = \{b\}$.
We have $\widehat{a} = \{a\}$, $\widehat{b} = \{b\}$ and the set $K = \{0, a, b\}$ is a subsemihypergroup of H with the hyperproduct table

	0	a	b
0	0	0	0
a	0	a	0
b	0	0	b

The relation β_K is transitive, thus contradicting the hypothesis that H is a fully 0-simple semihypergroup.

2. Case $bb \neq \{b\}$ or, equivalently, $aa \neq \{a\}$.
From Remark 2.6 we have that $0 \in \widehat{b}$ and

$$(\widehat{a} \cup \widehat{b})(\widehat{a} \cup \widehat{b}) = \widehat{a} \cup \widehat{b}.$$

Thus also $\widehat{a} \cup \widehat{b}$ is a 0-simple semihypergroup since $H \in \mathfrak{F}_0$. In consequence, from (1) we have

$$\widehat{a} \cup \widehat{b} = (\widehat{a} \cup \widehat{b})b(\widehat{a} \cup \widehat{b}) = \widehat{a}b\widehat{a} \cup \widehat{a}b\widehat{b} \cup \widehat{b}b\widehat{a} \cup \widehat{b}b\widehat{b} = \{0\} \cup \widehat{b} = \widehat{b}.$$

Now, from $a \in \widehat{a} \subseteq \widehat{b}$ and $a \neq b$, it ensues that there exists an integer $n > 1$ such that $a \in b^n$, whence $aa \subseteq ab^n = \{0\}$ and $aa = \{0\}$, which contradicts Lemma 2.5. □

Corollary 2.8. *Let $H \in \mathfrak{F}_0$. For every sequence z_1, z_2, \dots, z_n of elements in H_+ we have $\prod_{i=1}^n z_i \neq \{0\}$.*

Proof. By Theorem 2.7, the thesis is true when $n \in \{1, 2\}$. We suppose by induction it is true for $n - 1 \geq 2$ and we prove it for n . If $\prod_{i=1}^n z_i = \{0\}$ then $az_n = \{0\}$ for every $a \in \prod_{i=1}^{n-1} z_i$. Since $z_n \neq 0$, by Theorem 2.7 we obtain that $a = 0$ and so $\prod_{i=1}^{n-1} z_i = \{0\}$, which is absurd. □

Corollary 2.9. *Let $H \in \mathfrak{F}_0$. The set H_+ endowed by the hyperproduct*

$$a \star b = (ab) \cap H_+ \tag{2}$$

is a simple semihypergroup.

Proof. Firstly, we observe that $a \star b \neq \emptyset$ for every $a, b \in H_+$, so the operation \star is well defined, because of Theorem 2.7. Furthermore, the operation is trivially associative.

Finally, for any $x, y \in H_+$ there exist $a, b \in H_+$ such that $x \in (ayb) \cap H_+ = a \star y \star b \subseteq H_+ \star y \star H_+$, whence we have $H_+ \subseteq H_+ \star y \star H_+$ and we can conclude that (H_+, \star) is simple. □

The preceding result leads naturally to the following definition:

Definition 2.10. *Let $H \in \mathfrak{F}_0$. The semihypergroup (H_+, \star) defined by (2) is the nonzero part of H .*

We note in passing that the previous definition can be extended to semihypergroups with a zero element, as long as the hyperproduct (2) is well defined. The simple proof of the following fact, whose claim is included here for further reference, is omitted for brevity:

Corollary 2.11. *Let $H \in \mathfrak{F}_0$ and let (H_+, \star) be its nonzero part. If A is a subsemihypergroup of (H_+, \star) then $A \cup \{0\}$ is a subsemihypergroup of H .*

3 R0-SEMIHYPERGROUPS

In the previous section we proved that the semihypergroups $H \in \mathfrak{F}_0$ have no divisors of zero, in the sense that $xy \neq \{0\}$ for every $x, y \in H_+$. A similar property holds true also for all fully simple semihypergroups $H \in \mathfrak{F}$. In fact, as shown in [7, 9], these semihypergroups own a (right or left) zero scalar element 0 and for all $x, y \in H - \{0\}$ it holds either $\{y\} \subseteq xy \subseteq \{0, y\}$ or $\{x\} \subseteq xy \subseteq \{0, x\}$, depending on whether 0 is right or left zero scalar.

Although various similarities exist between fully simple semihypergroups and fully 0-simple semihypergroups, the specific role of the 0 element leads to very relevant differences between the two families. In fact, semihypergroups in \mathfrak{F}_0 show up a much larger diversity. For example, let R_n denote the *right zero semigroup of order n*, that is, the semigroup of n elements defined by $ab = b$ for all $a, b \in R_n$ [16, p. 3]; furthermore, let $L_n = R_n^T$ be the *left zero semigroup of order n*. It has been shown in [7] that the nonzero part of a semihypergroup $H \in \mathfrak{F}$ with $n + 1$ elements is either R_n or L_n , depending on whether the zero element of H is right or left zero scalar. On the other hand, in \mathfrak{F}_0 there are (finite) semihypergroups H whose nonzero part is neither R_n nor L_n . An example is the following:

	0	a	b	c
0	0	0	0	0
a	0	c	c	0, a, b
b	0	c	c	0, a, b
c	0	0, a, b	0, a, b	0, c

As a first step toward a deeper understanding of \mathfrak{F}_0 , hereafter we start considering those (finite) semihypergroups in \mathfrak{F}_0 whose nonzero part is R_n . In fact, the goal of this section is to prove that, with due modifications, various results shown in [7, 9] carry over semihypergroups $H \in \mathfrak{F}_0$ that satisfy the following condition:

$$\{y\} \subseteq xy \subseteq \{0, y\}, \text{ for all } x, y \in H_+. \tag{3}$$

These semihypergroups are called *R0-semihypergroups*. Analogously, we can define a *L0-semihypergroup* as a semihypergroup $H \in \mathfrak{F}_0$ such that $\{x\} \subseteq xy \subseteq \{0, x\}$ for all $x, y \in H_+$, that is, a semihypergroup whose nonzero part is L_n . These definitions can be extended naturally to the infinite case. It is rather apparent that an R0-semihypergroup is the transposed of an L0-semihypergroup, so that any property of the former can be translated into a property of the latter. For this reason, in what follows we concentrate on R0-semihypergroups. However, observe that no L0-semihypergroup is isomorphic to an R0-semihypergroup, because of the structure of their nonzero part.

Henceforth, we denote by \mathfrak{R}_0 the class of R0-semihypergroups. Theorem 1.1 allows us to list all R0-semihypergroups of size 3:

Theorem 3.1. *Up to isomorphisms, there exist four R0-semihypergroups of size 3. Their hyperproduct tables are indicated as H_{11} , H_{12} , H_{13} , and H_{14} in Theorem 1.1.*

In the forthcoming Theorem 3.3 we provide a characterization of all R0-semihypergroups. A preliminary lemma introduces a construction of semihypergroups whose nonzero part is a right zero semigroup.

Lemma 3.2. *Let H be a hypergroupoid such that*

- a) $\exists 0 \in H$ such that $x0 = 0x = \{0\}$, for every $x \in H$.
 Moreover, for all $x, y, z \in H - \{0\}$, we suppose
- b) $\{y\} \subseteq xy \subseteq \{0, y\}$;
- c) $|yz| = 1 \implies |xy| = |xz|$.

Then H is a 0-semihypergroup.

Proof. Let x, y, z be elements of H . We must prove that $(xy)z = x(yz)$. If $0 \in \{x, y, z\}$ the proof is trivial. Hence, we suppose that $x, y, z \in H - \{0\}$ and distinguish two cases, depending on whether $yz = \{z\}$ or $yz = \{0, z\}$.

- 1. Case $yz = \{z\}$.

By hypothesis c), we have also $|xy| = |xz|$.

If $|xy| = |xz| = 1$, by b), we have $xy = \{y\}$ and $xz = \{z\}$. Therefore $(xy)z = yz = \{z\} = xz = x(yz)$.

If $|xy| = |xz| = 2$, by b), we have $xy = \{0, y\}$ and $xz = \{0, z\}$. Thus we obtain $x(yz) = xz = \{0, z\} = \{0\} \cup yz = \{0, y\}z = (xy)z$.

- 2. Case $yz = \{0, z\}$.

In this case, $x(yz) = x\{0, z\} = \{0\} \cup xz = \{0, z\}$ because $z \in xz \subseteq \{0, z\}$. Finally, if $xy = \{y\}$ then $(xy)z = yz = \{0, z\}$, while, if $xy = \{0, y\}$ then $(xy)z = \{0, y\}z = \{0\} \cup yz = \{0, z\}$.

Thus, in any case, we have $(xy)z = x(yz)$ and H is a 0-semihypergroup. \square

Theorem 3.3. *Let H be a hypergroupoid such that $|H| \geq 3$. Then $H \in \mathfrak{R}_0$ if and only if H fulfills the following conditions:*

- a) $\exists 0 \in H$ such that $x0 = 0x = \{0\}$, for all $x \in H$;
- b) for all $x, y \in H - \{0\}$, $\{y\} \subseteq xy \subseteq \{0, y\}$;
- c) for all $x, y, z \in H - \{0\}$, $|yz| = 1 \implies |xy| = |xz|$;
- d) for every pair (x, y) of distinct elements in $H - \{0\}$ it holds $0 \in \{x, y\} \cdot \{x, y\}$.

Proof. Let $H \in \mathfrak{R}_0$. By definition, we have $H \in \mathfrak{F}_0$ and $\{y\} \subseteq xy \subseteq \{0, y\}$, for every $x, y \in H_+$. Therefore the conditions *a)* and *b)* are verified.

Moreover, for every $x, y \in H_+$ and $x \neq y$, the set $K = \{0, x, y\}$ is a sub-semihypergroup of H of size 3, whence $K \in \mathfrak{F}_0$. Then, by Proposition 2.4, also the condition *d)* is satisfied.

Finally, let $x, y, z \in H_+$ and $|yz| = 1$. If $|xy| = 1$, then $yz = \{z\}$ and $xy = \{y\}$. Therefore, $xz = x(yz) = (xy)z = yz = \{z\}$ and $|xy| = |xz|$. On the other hand, if $|xy| = 2$, then $xy = \{0, y\}$. Moreover, since $\{z\} \subseteq yz \subseteq \{0, z\}$, we obtain that $xz = x(yz) = (xy)z = \{0, y\}z = 0z \cup yz = \{0, z\}$ and so, also in this case, $|xy| = |xz|$.

On the contrary, suppose that H is a hypergroupoid of size ≥ 3 verifying conditions *a), b), c), d)*.

By Lemma 3.2, H is a 0-semihypergroup. Moreover, if $K \subseteq H$ is a sub-semihypergroup of H of size ≥ 3 , there exists a pair (x, y) of distinct elements in K , such that $0 \notin \{x, y\}$. By *d)*, we have

$$0 \in \{x, y\} \cdot \{x, y\} \subseteq K \cdot K \subseteq K,$$

and so K is a 0-semihypergroup. Moreover, for *b)*, if $x \in K - \{0\}$ and $y \in K$, we have $y \in yy \subseteq y(xy) = yxy \subseteq KxK$. Thus $K = KxK$, for every $x \in K - \{0\}$, that is K is 0-simple.

It remains to prove that the relation β_K is not transitive. In fact, we show that for every $x, y \in K - \{0\}$, we have $(x, 0) \in \beta_K, (0, y) \in \beta_K$ and $(x, y) \notin \beta_K$. By *b)*, we consider the following to cases:

1. $yy = \{0, y\}$ or $xx = \{0, x\}$;
2. $yy = \{y\}$ and $xx = \{x\}$.

In the first case, if $yy = \{0, y\}$ then we have $yx = \{0, x\}$, otherwise $yx = \{x\}$ and, by *c)*, it follows the contradiction $|yy| = |yx| = 1$ (clearly, if $xx = \{0, x\}$ we obtain $xy = \{0, y\}$).

In the second case, by *d)*, it is not restrictive to consider that $xy = \{0, y\}$. Therefore, we deduce again that $yx = \{0, x\}$, otherwise $yx = \{x\}$ and, in consequence, $\{x\} = xx = x(yx) = (xy)x = \{0, y\}x = 0x \cup yx = \{0, x\}$, that is an absurdity. Therefore, in both cases, we have that $(x, 0) \in \beta_K$ and $(0, y) \in \beta_K$.

Finally, we have $(x, y) \notin \beta_K$, because $\prod_{i=1}^n z_i = (\prod_{i=1}^{n-1} z_i)z_n \subseteq \{0, z_n\}$, for every integer $n \geq 2$ and for every $z_1, z_2, \dots, z_n \in K$. So H is a fully 0-simple semihypergroup such that $\{y\} \subseteq xy \subseteq \{0, y\}$, for every $x, y \in H_+$. This means that $H \in \mathfrak{R}_0$. □

4 PARTITIONING NONZERO ELEMENTS OF $H \in \mathfrak{R}_0$

In this section we give some results which will be used in the following to describe some properties of the Boolean matrices associated to finite semihypergroups in \mathfrak{R}_0 . For this purpose, we introduce the following notations:

If $H \in \mathfrak{R}_0$, for all $x \in H_+$ let

$$C(z) = \{y \in H_+ : |yz| = 1\}$$

$$R(z) = \{y \in H_+ : |zy| = 1\}.$$

Observe that $x \in C(y) \iff y \in R(x)$.

Proposition 4.1. *Let $H \in \mathfrak{R}_0$ and $z \in H_+$. Then:*

1. $y \in C(z) \Rightarrow |yy| = 1$;
2. $|C(z)| \leq 1$;
3. $C(z) = \emptyset \Rightarrow R(z) = \emptyset$.

Proof.

1. It follows at once from item c) of Theorem 3.3, by putting $x = y$.
2. Suppose that the set $C(z)$ contains two distinct elements x, y . Hence $|xz| = |yz| = 1$. From the preceding item 1) and item c) of Theorem 3.3, we obtain $|xx| = |xy| = |yx| = |yy| = 1$. This fact contradicts item d) of Theorem 3.3. We deduce $|C(z)| \leq 1$.
3. Arguing by contradiction, from item 1) we have $y \in R(z) \Rightarrow z \in C(y) \Rightarrow |zz| = 1 \Rightarrow z \in C(z)$. □

Item 2) of Proposition 4.1 leads us to organize nonzero elements in $H \in \mathfrak{R}_0$ into two types as follows:

Definition 4.2. *Let $H \in \mathfrak{R}_0$. An element $z \in H_+$ is said to be C0 if $|C(z)| = 0$ and C1 if $|C(z)| = 1$.*

Now, if $H \in \mathfrak{R}_0$, we introduce in H_+ the following relation:

$$x \sim y \iff C(x) = C(y) \neq \emptyset \text{ or } x = y. \tag{4}$$

The relation \sim is an equivalence relation on H_+ . In fact, it is obviously reflexive and symmetric. The following argument proves transitivity: Let

$x \sim y$ and $y \sim z$. If $x = y$ or $y = z$ then we have trivially $x \sim z$. Otherwise, $C(x) = C(y) \neq \emptyset$ and $C(y) = C(z) \neq \emptyset$. Therefore, $C(x) = C(z) \neq \emptyset$, whence $x \sim z$.

Equivalence classes of \sim form a partition of H_+ , whose properties are analyzed in the following proposition.

Proposition 4.3. *Let $H \in \mathfrak{R}_0$, and let \tilde{x} denote the \sim -class of $x \in H_+$. We have:*

1. *If $|xy| = 1$ then $C(x) = C(y) = \{x\}$, whence $x \sim y$.*
2. *If x is a C0 element then $\tilde{x} = \{x\}$.*
3. *If x is CI then $\tilde{x} = R(y)$ where $\{y\} = C(x)$.*

Proof.

1. If $|xy| = 1$ then $x \in C(y)$ and, by item 2 of Proposition 4.1, we have $C(y) = \{x\}$. From item 1 of Proposition 4.1 we also get $|xx| = 1$, whence $x \in C(x)$. Consequently, $C(x) = C(y) \neq \emptyset$ and $x \sim y$.
2. If x is C0 then $C(x) = \emptyset$, whence $x \sim y$ iff $x = y$.
3. If $C(x) = \{y\}$ and $z \sim x$ then $C(z) = \{y\}$, whence $z \in R(y)$. □

Since every non-empty subset of H_+ is a subsemihypergroup of the nonzero part of H , the following proposition is an immediate consequence of Corollary 2.11:

Proposition 4.4. *If $H \in \mathfrak{R}_0$ and $A \subseteq H_+$ then $A \cup \{0\}$ is a subsemihypergroup of H . In particular, $\tilde{x} \cup \{0\}$ is a subsemihypergroup of H , for every $x \in H_+$.*

5 THE BOOLEAN MATRIX OF A FINITE R0-SEMIHYPERGROUP

In this section we denote by $\mathfrak{R}_0(n+1)$ the class of semihypergroups in \mathfrak{R}_0 of size $n+1$, with $n \geq 2$. For ease of notation, hereafter we suppose that all semihypergroups in $\mathfrak{R}_0(n+1)$ have the same support $[n] = \{0, 1, 2, \dots, n\}$.

Definition 5.1. *Let $H = ([n], \circ)$ be a hypergroupoid fulfilling conditions a) and b) of Theorem 3.3 and let M_H be the $n \times n$ Boolean matrix M_H whose (i, j) -entry is defined in the following way, for every $i, j \in \{1, \dots, n\}$:*

$$[M_H]_{ij} = \begin{cases} 1 & \text{if } i \circ j = \{j\} \\ 0 & \text{if } i \circ j = \{0, j\}. \end{cases} \quad (5)$$

We say that M_H is the Boolean matrix associated to H .

By means of Definition 5.1, to any $H \in \mathfrak{R}_0(n + 1)$ we can associate a Boolean matrix which provides an alternative description of it. We can proceed also in the opposite way, and define a semihypergroup starting from a Boolean matrix; under appropriate conditions, that semihypergroup belongs to $\mathfrak{R}_0(n + 1)$. Indeed, let M be a given $n \times n$ Boolean matrix. On the set $[n]$ we define the hyperproduct \circ as follows:

$$i \circ j = \begin{cases} \{0\} & \text{if } i = 0 \text{ or } j = 0 \\ \{j\} & \text{if } i, j \in \{1, \dots, n\} \text{ and } [M]_{ij} = 1 \\ \{0, j\} & \text{if } i, j \in \{1, \dots, n\} \text{ and } [M]_{ij} = 0. \end{cases} \tag{6}$$

By construction, $H = ([n], \circ)$ is a hypergroupoid which fulfills hypotheses *a*) and *b*) of Theorem 3.3. Hence, we can restate that theorem as follows:

Theorem 5.2. *Let M be an $n \times n$ Boolean matrix. The hypergroupoid $H = ([n], \circ)$ defined by equation (6) is a semihypergroup in $\mathfrak{R}_0(n + 1)$ if and only if M fulfills the following conditions, for every $i, j, k \in \{1, \dots, n\}$:*

1. $[M]_{kj} = 1 \implies [M]_{ik} = [M]_{ij}$;
2. If $i \neq j$ then there exist $u, v \in \{i, j\}$ such that $[M]_{uv} = 0$.

Remark 5.3. *Two semihypergroups $H = ([n], \circ)$ and $H' = ([n], *)$ in $\mathfrak{R}_0(n + 1)$ are isomorphic if and only if there exists a permutation τ of $[n]$ such that $\tau(i \circ j) = \tau(i) * \tau(j)$. In that case,*

$$[M_H]_{ij} = [M_{H'}]_{\tau(i), \tau(j)}.$$

Consequently, if P denotes the $n \times n$ permutation matrix associated with τ ,

$$[P]_{ij} = \begin{cases} 1 & \text{if } \tau(i) = j \\ 0 & \text{if } \tau(i) \neq j, \end{cases}$$

then we have $P^T M_H P = M_{H'}$.

If $H \in \mathfrak{R}_0(n + 1)$ then $[M_H]_{ij} = 1$ if and only if $j \in R(i)$ or, equivalently, $i \in C(j)$. Actually, as a consequence of Proposition 4.1, we can observe in M_H the following properties:

1. if the i -th row of M_H has at least one entry equal to 1 then $[M_H]_{ii} = 1$;
2. in each column of M_H there is at most one entry equal to 1;

- 3. if the i -th column of M_H has all entries equal to 0 then all entries of i -th row are equal to 0.

As shown in the forthcoming Proposition 5.4, these properties can be used to bring M_H into a block diagonal form which reveals the inner structure of H and, ultimately, allows us to enumerate the isomorphism classes in $\mathfrak{R}_0(n + 1)$. In view of that result, we need to describe the structure of the Boolean matrices associated to semihypergroups introduced in Proposition 4.4. Hence, let $H \in \mathfrak{R}_0(n + 1)$ and $x \in H_+$. We have:

- If x is C0 then $\tilde{x} \cup \{0\}$ is a subsemihypergroup of H whose hyperproduct table is

	0	x
0	0	0
x	0	$0, x$

Its associated Boolean matrix is (0) .

- If x is C1 with $\tilde{x} = \{x_1, \dots, x_k\} = R(x_1)$ then $\tilde{x} \cup \{0\}$ is a subsemihypergroup of H whose hyperproduct table is

	0	x_1
0	0	0
x_1	0	x_1

when $k = 1$, and

	0	x_1	x_2	\dots	x_k
0	0	0	0	\dots	0
x_1	0	x_1	x_2	\dots	x_k
x_2	0	$0, x_1$	$0, x_2$	\dots	$0, x_k$
\vdots	\vdots	\vdots	\vdots		\vdots
x_m	0	$0, x_1$	$0, x_2$	\dots	$0, x_k$

when $k > 1$. The corresponding Boolean matrices are as follows:

$$D_1 = (1) \text{ and } D_k = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \text{ if } k \geq 2. \tag{7}$$

Proposition 5.4. *Let $H = ([n], \circ)$ be a hypergroupoid fulfilling conditions a) and b) of Theorem 3.3 and let M_H be its associated Boolean matrix. Then, $H \in \mathfrak{R}_0(n + 1)$ if and only if there exists a permutation matrix P such that $P^T M_H P$ has the block diagonal form*

$$\begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & B_m \end{pmatrix}, \tag{8}$$

where each diagonal block is either the 1×1 zero matrix or one of the matrices in (7).

Proof. Let $H \in \mathfrak{R}_0(n + 1)$ and let $\{x_1, x_2, \dots, x_m\}$ be a transversal of \sim -classes. Up to an isomorphism, we can suppose that elements belonging to the same \sim -class are consecutive and, if x_i is C1 then $\tilde{x}_i = R(x_i)$. Let B_1, \dots, B_m be the Boolean matrices associated to $\tilde{x}_i \cup \{0\}$ for $i = 1, \dots, m$. Hence, M_H has the form (8). Indeed, owing to item 1 of Proposition 4.3, if $i, j \in \{1, \dots, n\}$ belong to different \sim -classes then $[M_H]_{ij} = 0$.

To complete the proof, it is sufficient to assume that M_H has the form (8) and check the conditions in Theorem 5.2. In fact, it is not restrictive to consider the case where P is the identity matrix, owing to Remark 5.3.

1. Let $j, k \in \{1, \dots, n\}$ be such that $[M_H]_{kj} = 1$. Hence, if $i = k$ then $[M_H]_{ik} = [M_H]_{ij} = 1$ since by construction $[M_H]_{kk} = 1$. Otherwise we have $[M_H]_{ik} = [M_H]_{ij} = 0$ since every column of M_H contains at most one nonzero entry.
2. By construction M_H is upper triangular. Hence, for any two distinct integers $i, j \in \{1, \dots, n\}$ the 2×2 submatrix obtained by the i -th and the j -th rows and columns of M_H has at least one zero entry. □

5.1 The directed graph of $H \in \mathfrak{R}_0(n + 1)$

The Boolean matrix M_H associated to any 0-semihypergroup H fulfilling conditions a) and b) of Theorem 3.3 can be regarded as the adjacency matrix of a digraph \mathcal{G}_H whose vertex set is $\{1, \dots, n\}$. Figure 1 visualizes the content of Theorem 3.3 in terms of properties of \mathcal{G}_H ; the two pictures represent conditions c) and d) of that Theorem, respectively.

Moreover, if $H \in \mathfrak{R}_0(n + 1)$ then, owing to Proposition 5.4, it is immediate to recognize that \mathcal{G}_H is decomposed into $k_0 + k_1$ connected components,

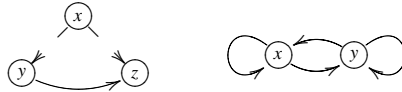


FIGURE 1

Graph-theoretic interpretation of conditions *c)* and *d)* in Theorem 3.3. Left: The solid edge implies that the two dashed edges are both present or both absent in \mathcal{G}_H . Right: The complete digraph with two nodes cannot occur as a subgraph in \mathcal{G}_H .

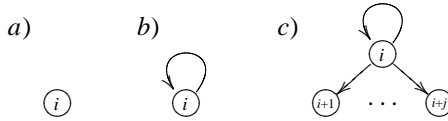


FIGURE 2

Graph-theoretic interpretation of Proposition 5.4. If $H \in \mathfrak{A}_0(n + 1)$ then each connected component of \mathcal{G}_H can assume one of the forms above. *a)* The element i is C0. *b)* The element i is C1 and $R(i) = \{i\}$. *c)* The element i is C1 and $R(i) = \{i, \dots, i + j\}$.

where k_0 is the number of C0 elements of H and k_1 is the number of \sim -classes in H made of C1 elements. Indeed, as illustrated in Figure 2, every C0 element corresponds to an isolated vertex in \mathcal{G}_H ; and every C1 element belongs to a connected component whose size equals that of its \sim -class.

6 A TRANSVERSAL OF ISOMORPHISM CLASSES OF SEMIHYPERGROUPS IN $\mathfrak{A}_0(N + 1)$

Hereafter we exhibit a transversal of isomorphism classes in $\mathfrak{A}_0(n + 1)$ and compute their number. For this purpose we recall that, if m is a positive integer then a *partition of m* or *m -partition* is a non-increasing sequence of positive integers p_1, p_2, \dots, p_r whose sum is m . Each p_i is called a *part* of the partition. It is conventional to write (p_1, p_2, \dots, p_r) to indicate a partition of n with r parts, and to denote by $p(m)$ the number of partitions of m ; moreover, $p(0) = 1$.

Let $H \in \mathfrak{A}_0(n + 1)$ and let H_0 and H_1 be the subsets of C0 and C1 elements in H , respectively. Clearly, we have

$$H_0 \cap H_1 = \emptyset \quad \text{and} \quad H = \{0\} \cup H_0 \cup H_1.$$

By Proposition 4.4, the sets $K_0 = H_0 \cup \{0\}$ and $K_1 = H_1 \cup \{0\}$ are subsemihypergroups of H . Note that K_0 and K_1 may not be fully 0-simple since the minimum size of a fully 0-simple semihypergroup is 3. In particular, we have

- a) If $H_0 = \{1, 2, \dots, n\}$ then $H_1 = \emptyset$ and the hyperproduct table of $H = K_0$ is the following:

	0	1	2	...	n
0	0	0	0	...	0
1	0	0, 1	0, 2	...	0, n
2	0	0, 1	0, 2	...	0, n
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
n	0	0, 1	0, 2	...	0, n

The associated Boolean matrix is the $n \times n$ zero matrix $0_{n,n}$.

- b) If $H_0 = \emptyset$ then $H = K_1$. By Proposition 5.4, there exists a n -partition (p_1, p_2, \dots, p_r) such that, up to a permutational similarity, the Boolean matrix M_H is the block diagonal matrix

$$D(p_1, p_2, \dots, p_r) = \begin{pmatrix} D_{p_1} & 0 & \cdots & 0 \\ 0 & D_{p_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_{p_r} \end{pmatrix}$$

where each block D_{p_i} has the form (7) for $i = 1, \dots, r$. Observe that there is no loss in generality if we suppose $p_1 \geq p_2 \geq \dots \geq p_r$.

- c) When both $H_0 \neq \emptyset$ and $H_1 \neq \emptyset$ then, up to a permutation which brings all C1 elements after all C0 elements, the Boolean matrix of $H \in \mathfrak{R}_0(n + 1)$ can be represented in a compact way by means of a block matrix,

$$M_H = \begin{pmatrix} M_0 & 0 \\ 0 & M_1 \end{pmatrix},$$

where M_0 and M_1 are the Boolean matrices associated to K_0 and K_1 , respectively.

We arrive at the following claim:

Theorem 6.1. *A transversal of the isomorphism classes in $\mathfrak{R}_0(n + 1)$ is the family of semihypergroups $H = ([n], \circ)$ defined by equation (6) from the following Boolean matrices:*

- a) $0_{n,n}$;
 b) $D(p_1, p_2, \dots, p_r)$, for any n -partition (p_1, p_2, \dots, p_r) ;

c) $\left(\begin{array}{c|c} 0_{k,k} & 0 \\ \hline 0 & D(p_1, p_2, \dots, p_r) \end{array} \right)$ for any integer $0 < k < n$ and for any $(n - k)$ -partition (p_1, p_2, \dots, p_r) .

Corollary 6.2. *The number of isomorphism classes of semihypergroups in $\mathfrak{R}_0(n + 1)$ is*

$$\phi_{n+1} = \sum_{k=0}^n p(k),$$

where $p(k)$ denote the number of non-increasing partitions of k .

Proof. By counting separately the number of Boolean matrices in the three cases listed in Theorem 6.1, the number of isomorphism classes of semihypergroups in $\mathfrak{R}_0(n + 1)$ is given by

$$\phi_{n+1} = 1 + p(n) + \sum_{k=1}^{n-1} p(k) = \sum_{k=0}^n p(k),$$

and we obtain the claim. □

The sequence ϕ_{n+1} is indexed as A000070 in the On-Line Encyclopedia of Integer Sequences [21]. The first two elements of that sequence, namely, $\phi_1 = 1$ and $\phi_2 = 2$, may be informally associated to the trivial 0-simple semihypergroups of order 1 and 2,

	0
0	0

	0	1
0	0	0
0	0	1

	0	1
0	0	0
0	0	0, 1

Clearly, the same sequence enumerates isomomorphism classes of L0-semihypergroups, discussed synthetically at the beginning of Section 3. Indeed, it is rather immediate to observe that these two classes are disjoint, and the transposition operator acts as an isomorphism between them.

7 CONCLUDING REMARKS

In this paper we introduced the class of semihypergroups $H \in \mathfrak{R}_0$ defined by the following axioms:

1. All subsemihypergroups of H (H itself included) are 0-simple; in particular, they have a zero scalar element.
2. The relation β in H and the relations β_K in all subsemihypergroups $K \subset H$ of size ≥ 3 are not transitive.
3. $\{y\} \subseteq xy \subseteq \{0, y\}$ for all $x, y \in H - \{0\}$.

In particular, we produced a transversal of isomorphism classes, in the finite case, and gave an explicit formula to enumerate them.

These semihypergroups have been obtained by a variant of fully simple semihypergroups [7, 9], consisting in a slightly different form of the first axiom. Actually, the class \mathfrak{R}_0 arises as a special subclass of the class \mathfrak{F}_0 of all 0-semihypergroups which are 0-simple and have a non-transitive relation β together with all subsemihypergroups $K \subseteq H$ with $|K| \geq 3$. As such, the present work is motivated by the extensive literature on fundamental relations in algebraic hyperstructures.

In order to outline possible directions for further research, we stress the fact that semihypergroups in \mathfrak{R}_0 can be regarded as extending a right zero semigroup by means of the introduction of a zero scalar element. That extension yields rather unexpected consequences on the behaviour of fundamental relations in the extended semihypergroups, while allowing a complete description of their isomorphism classes. Hence, it seems promising to consider extending in analogous ways other algebraic structures, such as groups and more general simple semigroups.

Moreover, it could be also of interest to examine properties of semihypergroups arising as nonzero parts of more general semihypergroups in \mathfrak{F}_0 .

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REFERENCES

- [1] H. Aghabozorgi, M. Jafarpour, B. Davvaz (2011). Enumeration of Varlet and Comer hypergroups. *Electron. J. Combin.* 18, Paper 31.
- [2] N. Antampoufis, S. Spartalis, T. Vougiouklis (2003). Fundamental relations in special extensions. *Algebraic hyperstructures and applications (Alexandroupoli-Orestiada, 2002)*, 81–89, Spanidis, Xanthi.
- [3] S. M. Anvariye, B. Davvaz (2009). Strongly transitive geometric spaces associated to hypermodules. *J. Algebra*, 322, 1340–1359.
- [4] W. E. Clark, X.-D. Hou (2013). Galkin quandles, pointed abelian groups, and sequence A000712. *Electron. J. Combin.*, 20, Paper 45.

- [5] B. Davvaz, V. Leoreanu-Fotea (2007). *Hyperring Theory and Applications*. International Academic Press, USA.
- [6] M. De Salvo, D. Fasino, D. Freni, G. Lo Faro (2013). Hypergroups with a strongly unilateral identity. *J. Mult.-Valued Logic Soft Comput.*, 21, 165–182.
- [7] M. De Salvo, D. Fasino, D. Freni, G. Lo Faro (2014). Fully simple semihypergroups, transitive digraphs, and sequence A000712. *Journal of Algebra*, 415, 65–87.
- [8] M. De Salvo, D. Freni, G. Lo Faro (2011). On the hypergroups of type U on the right of size five, with scalar identity. *J. of Mult.-Valued Logic Soft Comput.*, 17, 425–441.
- [9] M. De Salvo, D. Freni, G. Lo Faro (2014). Fully simple semihypergroups. *Journal of Algebra*, 399, 358–377.
- [10] M. De Salvo, G. Lo Faro (1999). On the n^* -complete hypergroups. *Discrete Mathematics*, 209, 177–188.
- [11] D. Fasino, D. Freni (2007). Existence of proper semi-hypergroups of type U on the right. *Discrete Math.*, 307, 2826–2836.
- [12] D. Fasino, D. Freni (2012). Fundamental relations in simple and 0-simple semi-hypergroups of small size. *Arab. J. Math.*, 1, 175–190.
- [13] D. Freni (2004). Strongly transitive geometric spaces: Applications to hypergroups and semigroups theory. *Comm. Algebra*, 32, 969–988.
- [14] D. Freni (2011). Minimal order semi-hypergroups of type U on the right. II. *Journal of Algebra*, 340, 77–89.
- [15] M. Gutan (1996). On the transitivity of the relation β in semi-hypergroups. *Rend. Circ. Mat. Palermo (2)*, 45, 189–200.
- [16] M. Howie (1995). *Fundamentals of semigroup theory*. Oxford University Press, New York.
- [17] H. M. Jafarabadi, N. H. Sarmin, M. R. Molaei (2012). Completely Simple and Regular Semi Hypergroups. *Bull. Malays. Math. Sci. Soc. (2)*, 35, 335–343.
- [18] J. Jantosciak, Ch. G. Massouros (2003). Strong identities and fortification in transposition hypergroups. *J. Discrete Math. Sci. Cryptogr.*, 6, 169–193.
- [19] H. Koskas (1970). Groupoïdes, demi-hypergroupes et hypergroupes. *J. Math. Pures et Appl.*, 49, 155–192.
- [20] S. Rasouli, D. Heidari, B. Davvaz (2009). η -relations and transitivity conditions of η on hyper-MV algebras. *J. Mult.-Valued Logic Soft Comput.*, 15, 517–524.
- [21] *The On-Line Encyclopedia of Integer Sequences*, <http://oeis.org/>
- [22] T. Vougiouklis (1999). Fundamental relations in hyperstructures. *Bull. Greek Math. Soc.*, 42, 113–118.