COMPUTING THE DIMENSION OF A MAJORANA REPRESENTATION OF THE HARADA-NORTON GROUP

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ABSTRACT. In this note we apply the theory of Association Schemes to compute the dimension of the subspace U of the 196,884-dimensional Conway-Norton-Griess algebra generated by the Majorana axes associated to the 2A-involutions of the Monster group that are contained in the Harada-Norton group.

1. INTRODUCTION

Let G be a finite group generated by a G-stable set T of involutions, V a real commutative (not necessarily associative) algebra equipped with a positive definite symmetric bilinear form and

$$\phi\colon G \to Aut(V)$$

a faithful representation of G on V (that is, for every $g \in G$, $g\phi$ is an isometry of V that preserves the algebra product). Let further

$$\psi\colon T \to V \setminus \{0\}$$

be an injective map such that for every $g \in G$ and $t \in T$,

$$(t\psi)(g\phi) = (g^{-1}tg)\psi. \tag{1}$$

The tuple

$$\mathcal{R} := (G, T, V, \phi, \psi)$$

is called a *Majorana representation* of G if $\mathcal{R}$ satisfies certain properties (axioms **M1-M7** in [7], which, for brevity, we don't state explicitly, as they won't be used in the sequel of this paper). The elements of $T\psi$ are called *Majorana axes* while the automorphisms in $T\phi$ are called *Majorana involutions*.

Given a Majorana representation $\mathcal{R} := (G, T, V, \phi, \psi)$ of a group G and a subset T_0 of T, it can be easily seen that

$$\mathcal{R} := (H, T_0, V, \phi|_H, \psi|_{T_0})$$

is a Majorana representation of $H := \langle T_0 \rangle$.

The fundamental example of a Majorana representation is given by the faithful representation of Monster group M on the Conway-Norton-Griess algebra V_M of dimension 196884, T being the set of $2A$-involutions of M. The concept of Majorana representation has been introduced by A. Ivanov in [9, Chapter 8] to provide an axiomatic framework for that representation.

By the above remark, any group H isomorphic to a subgroup H^* of the Monster group generated by a subset of $2A$-involutions, inherits a Majorana representation

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by restricting the one of M to H^*. In that case, the Majorana representation of H is said to be *based on the embedding* $H \to H^*$ in the Monster.

Majorana representations of many groups have alredy been investigated. The axioms of Majorana representation impliy that the Majorana representations of the dihedral group D_{2N}, with $2 \leq N \leq 6$, coincide with the Norton-Sakuma algebras of type NX, $X \in \{A, B, C\}$ (see [14] and [7]). Majorana representations of the groups $A_4, A_5, A_6, A_7, L_3(2), L_2(11)$ have been studied in [7], [10], [11, 12], [13] and [5] respectively. Moreover, in [2] further properties of a Majorana representation s of A_{12} are proved.

The purpose of this paper is to begin the investigation of Majorana representations of the Harada-Norton group HN. By the above remark, HN has a unique (up to equivalence) Majorana representation based on its embedding in M, since HN is uniquely (up to equivalence) embedded into M as the subgroup generated by the $2A$-involutions centralizing a $5A$-element [3]. Thus, for the remainder of this paper, let

$$(HN, T, V, \phi, \psi)$$

be the Majorana representation of HN based on its embedding in M, (in particular, using the notations of [3], T is the set of $2A$-involutions of HN). We prove the following.

Theorem 1. *Let* (HN, T, V, ϕ, ψ) *be the Majorana representation of* HN *based on its embedding in* M*. The* $\mathbb{R}$*-subspace* $\langle T\psi \rangle$ *of* V *is an* $\mathbb{R}[HN]$*-module of dimension* $18\,316$*, with irreducible submodules of dimensions* $1, 8910$ *and* 9405*.*

The proof relies on the following elementary result on Euclidean spaces:

Lemma 2. *Let* E *be a real Euclidean space with scalar product* $(\ ,\ )$ *and let* $v_1, \ldots, v_m$ *be vectors of* E*. Then the dimension of* $\langle v_1, \ldots, v_m \rangle$ *is equal to the rank of the Gram matrix* $((v_i, v_j))_{i,j}$

Let $T = \{t_1, \ldots, t_n\}$,

$$\gamma_{ij} := (t_i\psi, t_j\psi),$$

and let

$$\Gamma = (\gamma_{ij})_{ij}$$

be the Gram matrix of $(\ ,\ )$ associated to the n-tuple $(t_1, \ldots, t_n)$. By Lemma 2, we have that

$$\mathrm{rank}(\Gamma) = \dim_{\mathbb{R}}(\langle \mathrm{t}\psi \mid \mathrm{t} \in \mathrm{T} \rangle). \tag{2}$$

Note Γ is a $|T| \times |T|$ matrix and, in our case, $|T| = |HN : C_{HN}(t)| = 1539000$ (see [3]) making a direct computation of its rank quite hard to perform. On the other hand, the theory of Association Schemes allows us to reduce that computation to a much more manageable case.

Let

$$T_0, \ldots, T_d$$

be the orbitals of G on T, that is the orbits of G on $T \times T$. From equation (1) and the definition of γ_{ij} we have that

$$\gamma_{ij} = \gamma_{hk} \text{ if } (t_i, t_j) \text{ and } (t_h, t_k) \text{ belong to the same orbital of } G \text{ on } T.$$

Therefore, set, for $k \in \{0, \ldots, d\}$

$$\gamma_k := (t\psi, s\psi) \text{ for } (t, s) \in T_k \tag{3}$$

and for every $k \in \{0, \ldots, d\}$ let A_k be the matrix whose i, j-entry $a_{ij}(k)$ is as follows

$$a_{ij}(k) = \begin{cases} 1 & \text{if the pair } (t_i, t_j) \text{ is in } T_k \\ 0 & \text{otherwise} \end{cases}$$

Since $T_0, \ldots, T_d$ is a partition of $T \times T$, Equation (3) yields that

$$\Gamma = \sum_{k=0}^{d} \gamma_k A_k, \tag{4}$$

The pair $(T, \{T_0, \ldots, T_d\})$ is an association scheme (see [1]), and the matrices $A_0, \ldots, A_d$ are called the *adjacency matrices*. They are a basis for the *centralizer algebra* $\mathcal{C}$, that is the centralizer in $M_{n\times n}(\mathbb{R})$ of the subalgebra generated by the permutation matrices associated to the elements of HN in the action of HN on T. Moreover, for all $i, j \in \{0, \ldots, d\}$ there exist integers p_{ij}^k such that

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k. \tag{5}$$

The matrix B_i of size $d+1$ whose j, k entry is p_{ij}^k is called $i-th$ intersection matrix. Clearly B_i^t is the matrix associated to the endomorphism induced by A_i on $\mathcal{C}$ via left multiplication with respect to the basis $(A_0, \ldots, A_d)$, in particular B_i has the same eigenvalues as A_i.

Since the permutation character associated to the action of HN on T is multiplicity free, the centralizer algebra $\mathcal{C}$ is commutative and the matrices $A_0, \ldots, A_k$ are simultaneously diagonalizable by a real invertible matrix D. Thus from Equation (4) we get:

$$D^{-1}\Gamma D = \sum_{k=0}^{d} \gamma_k D^{-1} A_k D, \tag{6}$$

where all the matrices $D^{-1}\Gamma D$, and $D^{-1}A_k D$ for $k \in \{0, \ldots, 8\}$, are diagonal.

Now, clearly rank(Γ) (= rank$(\mathrm{D}^{-1}\Gamma\mathrm{D})$) is equal to the number of nonzero entries of $D^{-1}\Gamma D$. Since the coefficients γ_k are given by the Norton-Sakuma Theorem (see [9]), in order to compute the rank of Γ we are reduced to compute the eigenvalues of the matrices B_k and their multiplicities in the matrices A_k.

As usual, we shall choose the indexes of the orbitals so that T_0 is the diagonal and B_0 is the identity matrix. Moreover, by [8, Lemma 2.18.1*(ii)*] there is a unique orbital of size $|T| \cdot 1408$. We choose T_1 to be that orbital so that, again by [8, Lemma 2.18.1*(ii)*], the first intersection matrix B_1 is as follows:

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1408 & 53 & 32 & 18 & 4 & 2 & 0 & 0 & 0 \\ 0 & 50 & 0 & 2 & 12 & 0 & 2 & 0 & 0 \\ 0 & 450 & 32 & 100 & 32 & 50 & 32 & 0 & 0 \\ 0 & 350 & 672 & 112 & 160 & 100 & 92 & 160 & 0 \\ 0 & 504 & 0 & 504 & 288 & 356 & 312 & 320 & 0 \\ 0 & 0 & 672 & 672 & 552 & 650 & 720 & 640 & 1280 \\ 0 & 0 & 0 & 0 & 360 & 250 & 240 & 288 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 128 \end{pmatrix}.$$

Having B_1, the Theory of Association Schemes enables us to obtain the eigenvalues of all matrices A_i, for every $i \in \{0, \dots, d\}$, and their multiplicities, and, as a by-product, also all the other intersection matrices.

2. Computing eigenvalues

Let U be the permutation module associated to the action of HN on T and let

$$U = U_0 \dot{+} \dots \dot{+} U_r$$

be the decomposition of U into the direct sum of maximal common eigenspaces of the adjacency matrices A_i. Since, for each i, the rows of the matrices A_i have constant sum (say k_i), we have that $\langle {}^t(1,1,\dots,1)\rangle$ is a (common) k_i-eigenspace for A_i, for each i. As usual, we choose U_0 to be that eigenspace. By [1, Theorem 3.1], $r = d = 8$ and, for each $i \in \{0, \dots, 8\}$, U_i is an irreducible $\mathbb{R}[HN]$-module.

For $i, j \in \{0, \dots, 8\}$, let p_{ij} be the eigenvalue of A_j on U_i and let $P := (p_{ij})$ be the *first eigenmatrix* of the association scheme $(T, \mathcal{R})$.

Lemma 3. *The matrix P is the following*

$$P = \begin{pmatrix} 1 & 1408 & 2200 & 35200 & 123200 & 354816 & 739200 & 277200 & 5775 \\ 1 & 128 & 200 & 0 & 1600 & -2304 & 0 & 0 & 375 \\ 1 & 28 & -50 & -50 & -100 & 396 & -750 & 450 & 75 \\ 1 & 16 & 4 & -56 & -136 & -288 & 504 & 0 & -45 \\ 1 & -32 & 40 & -80 & 80 & 576 & -240 & -360 & 15 \\ 1 & -47 & -50 & 250 & 350 & -504 & 0 & 0 & 0 \\ 1 & -112 & 300 & 1000 & -2200 & -864 & -1800 & 3600 & 75 \\ 1 & 208 & -50 & 2200 & -2800 & 2016 & 4200 & -6300 & 525 \\ 1 & 208 & 100 & 1000 & 1400 & 2016 & -4200 & 0 & -525 \end{pmatrix}$$

Proof. Note that since A_0 is the identity matrix, $p_{i0} = 1$ for all i's. Moreover, $p_{0i} = k_i$ for all i's by the choice of U_0 and the numbers $k_i = |T_i|$ are known (see [8, Lemma 2.12.1]) .

By straightforward computation we get that the eigenvalues of B_1 are 1408, 128, 28, 16, -32, -47, -112, 208, 208.

Let us set

$$(\lambda_0, \dots, \lambda_8) = (1408, 128, 28, 16, -32, -47, -112, 208, 208).$$

For each $h \in \{0, \dots, 8\}$, let $\mathcal{S}_h$ be the linear system

$$(B_1 - \lambda_h Id)\, {}^t(1, \lambda_h, x_2, \dots x_8) = 0. \tag{7}$$

Let us consider Equation (5) with $i = 1$ and multiply each term by D on the right and D^{-1} on the left. We get

$$(D^{-1}A_1D)(D^{-1}A_jD) = \sum_{h=0}^{d} p_{ij}^h (D^{-1}A_hD). \tag{8}$$

Since matrices $D^{-1}A_hD$ are diagonal whith eigenvalues $p_{0h}, \dots, p_{8h}$ on the common eigenspaces $U_0, \dots, U_8$, from Equation (8) we get for every $h \in \{0, \dots, 8\}$ the relations

$$\lambda_k p_{kj} = \sum_{h=0}^{d} p_{ij}^h p_{kh}. \tag{9}$$

Thus, for each $h \in \{0, \dots, 8\}$, the system $\mathcal{S}_h$, admits the 7-tuple $(p_{h2}, \dots, p_{h8})$ as a solution.

Now, for $h \neq 7, 8$, by elementary theory of linear systems, we see that $\mathcal{S}_h$ have a unique solution. Thus it is $(p_{h2}, \dots, p_{h8})$ and, solving the systems $\mathcal{S}_0, \dots, \mathcal{S}_6$, we get the first seven rows of the matrix P.

In particular we get that $k_1 = 1408$, $k_2 = 2200$, $k_3 = 35200$, $k_4 = 123200$, $k_5 = 354816$, $k_6 = 739200$, $k_7 = 277200$, and $k_8 = 5775$.

We are now left with the last two rows of the matrix P, corresponding to the eigenvalue 208 of B_1.

The set of solutions of the system $\mathcal{S}_7$

$$(B_1 - 208Id)\,{}^t(1, 208, x_2, \dots, x_8) = 0$$

is

$$\{(25 - \frac{x}{7}, 1600 + \frac{8x}{7}, -700 - 4x, 2016, 8x, -3150 - 6x, x) \mid \text{ where } x \in \mathbb{R}\}.$$

Therefore, for suitable $x, y \in \mathbb{R}$, we can write the last two rows of the matrix P as follows

$$1, 208, 25 - \frac{x}{7}, 1600 + \frac{8x}{7}, -700 - 4x, 2016, 8x, -3150 - 6x, x$$

$$1, 208, 25 - \frac{y}{7}, 1600 + \frac{8y}{7}, -700 - 4y, 2016, 8y, -3150 - 6y, y.$$

Set $m_i = \dim_{\mathbb{R}}(U_i)$. The m_i's can be computed from the rows of P using the formula [BI, Theorem 4.1]:

$$m_i = \frac{|T|}{\sum_{j=0}^{d} k_i^{-1} p_{ij}^2}$$

from which we get $m_1 = 16929$, $m_2 = 267520$, $m_3 = 653125$, $m_4 = 365750$, $m_5 = 214016$, $m_6 = 8910$.

Comparing those values with the decomposition of the permutation module into irreducibles, we get that

$$m_1 + m_2 = 12749 = 3344 + 9405,$$

so that we may assume

$$m_7 = 3344 \text{ and } m_8 = 9405.$$

By the Column Orthogonality Relation of the first eigenmatrix [1, Theorem 3.5]

$$\sum_{k=0}^{d} m_k p_{ki} p_{kj} = |T| k_i \delta_{ij},$$

applied with $(i, j) = (0, 8)$ and $(i, j) = (8, 8)$, we get the quadratic system

$$\begin{cases} 3344x + 9405y = -3182025 \\ 3344x^2 + 9405y^2 = 3513943125 \end{cases}$$

whose solutions are

$$(x, y) = (525, -525) \text{ or } (x, y) = (1575/61, 62475/61).$$

Finally, to determine which of the two solutions is the right one, we use the formula in [BI, Theorem 3.6]

$$p_{ij}^h = \frac{1}{|T| k_h} tr(A_i A_j A_h) \tag{10}$$

(note that, by [4], the matrices A_i's are symmetric since the Frobenius-Schur indicator of the permutation character of HN on T is $+1$ [3]). In fact, we can compute $tr(A_iA_jA_h)$ by using the matrix P and we know that p_{ij}^h is an integer number.

Only in the case when $(x, y) = (525, -525)$ we get an integer value for the entries $p_{2,j}^k$ of the matrix B_2. □

Note that by Equation (10) we may now also compute all the intersection matrices B_i's.

Lemma 4. *The coefficients γ_k in the formula (6) are given in the following table*

k	$\lvert t^{C_G(s)}\rvert$	$(st)^G$	$\gamma_k = (s\psi, t\psi)$
0	1	1	1
1	1408	$5A$	$3/2^7$
2	2200	$2A$	$1/8$
3	35200	$3A$	$13/2^8$
4	123200	$4B$	$1/2^6$
5	354816	$5E(5A)$	$3/2^7$
6	739200	$6A$	$5/2^8$
7	277200	$4A$	$1/2^5$
8	5775	$2B$	0

Proof. For each k, the coefficient γ_k have been defined in (3) and by the Conway-Norton-Sakuma Theorem (see for example [9]) it depends only on the conjugacy class of the product ts, for (t, s) in the orbital T_k. The correspondence that associates to each orbital T_k of HN on T the conjugacy class of the products ts where $(t, s) \in T_k$ has been determined by Segev [14] and is given by the first two columns of the table. □

Now set $\overline{\Gamma} := D^{-1}\Gamma D$ and $\overline{A}_k := D^{-1}A_kD$ for each $k \in \{0, \ldots, 8\}$.
By Lemma 4, Equation (6) becomes

$$\overline{\Gamma} = \overline{A}_0 + \frac{3}{2^7}\overline{A}_1 + \frac{1}{8}\overline{A}_2 + \frac{13}{2^8}\overline{A}_3 + \frac{1}{2^6}\overline{A}_4 + \frac{3}{2^7}\overline{A}_5 + \frac{5}{2^8}\overline{A}_6 + \frac{1}{2^5}\overline{A}_7 + 0\overline{A}_8,$$

which gives the following eigenvalues of $\overline{\Gamma}$ on the subspaces $U_0, \ldots, U_8$ respectively

$$70875/2, 0, 0, 0, 0, 0, 875/8, 0, 225/4.$$

Hence

$$\dim_{\mathbb{R}}(\langle T\psi\rangle) = m_0 + m_6 + m_8 = 1 + 9405 + 8910 = 18\,316.$$

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