Global weak solutions for a model of two-phase flow with a single interface

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Abstract

We consider a simple nonlinear hyperbolic system modeling the flow of an inviscid fluid. The model includes as state variable the mass density fraction of the vapor in the fluid and then phase transitions can be taken into consideration; moreover, phase interfaces are contact discontinuities for the system. We focus on the special case of initial data consisting of two different phases separated by an interface. We find explicit bounds on the (possibly large) initial data in order that weak entropic solutions exist for all times. The proof exploits a carefully tailored version of the front tracking scheme.

1 Introduction

We consider the following nonlinear model for the one-dimensional flow of an inviscid fluid, where different phases can coexist:

$$\begin{cases} v_t - u_x & = 0\,, \\ u_t + p(v,\lambda)_x & = 0\,, \\ \lambda_t & = 0\,. \end{cases} \tag{1.1}$$

Here $t > 0$ and $x \in \mathbb{R}$; moreover, $v > 0$ is the specific volume, u the velocity and λ the mass-density fraction of vapor in the fluid. Then, we have $\lambda \in [0,1]$ and $\lambda = 0$ characterizes the liquid phase while $\lambda = 1$ the vapor phase. The pressure p is given by

$$p(v,\lambda) = \frac{a^2(\lambda)}{v}\,, \tag{1.2}$$

where a is a $\mathbf{C^1}$ function defined on $[0,1]$ and satisfying $a(\lambda) > 0$ for every $\lambda \in [0,1]$. We denote $U = (v,u,\lambda) \in \Omega \doteq (0,+\infty) \times \mathbb{R} \times [0,1]$.

System (1.1) is the homogeneous case of a more general model that was first introduced in [14]. If λ is constant, then (1.1) reduces to the isothermal p-system, where the global existence of weak solutions holds for initial data with arbitrary total variation [5, 18]. The global existence of weak solutions to the initial value problem for (1.1) was proved in [1] under a suitable condition on the total variation of the initial data and the assumption

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$a' > 0$; a different proof of an analogous result has been recently provided in [6]. The condition on the initial data was also stated in a slightly different way in [3] and requires, roughly speaking, that the total variation of both pressure and velocity is suitably bounded by the total variation of λ; then, it reminds of the famous condition introduced in [19] (see also [13]) for the system of isentropic gasdynamics.

A model analogous to (1.1) is also studied in [15, 16], where the pressure is $v^{-\gamma}$ and the state variable λ is replaced by the adiabatic exponent $\gamma > 1$; also in this case the global existence of solutions is proved under a condition that has the same flavor of that discussed above.

In this paper we focus on a particular class of initial data for (1.1): the state variable λ is constant both for $x < 0$ and for $x > 0$. More precisely, for $x \in \mathbb{R}$ we consider initial data

$$U(x,0) = U_o(x) = \big(v_o(x), u_o(x), \lambda_o(x)\big)\,, \tag{1.3}$$

where

$$\lambda_o(x) = \begin{cases} \lambda_\ell & \text{if } x < 0\,, \\ \lambda_r & \text{if } x > 0\,, \end{cases} \tag{1.4}$$

for two constant values $\lambda_\ell \neq \lambda_r \in [0,1]$. Phase interfaces are stationary in model (1.1); then, the assumption (1.4) reduces the study of the initial value problem for (1.1) to that of two initial value problems for two isothermal p-systems, which are coupled through the interface at $x = 0$. In other words, the flow remains in the two phases characterized by the values λ_ℓ and λ_r as long as a solution exists.

The problem we are dealing with can be understood in a different way as follows. Phase interfaces are contact discontinuities for system (1.1); then, in a sense, we fall into the general framework of the perturbation of a Riemann solution. For this subject we refer to [9, 10, 11, 12, 21], where however the perturbation is small in the **BV** norm. In our case the perturbation leaves unchanged the initial datum for λ but it is not necessarily small in the other state variables. The problem of a *small* perturbation of a Riemann solution and the related existence of globally defined solutions was thoroughly studied in [17].

The main result of this paper concerns the global existence of weak solutions to the initial value problem (1.1), (1.3), (1.4), provided that (1.2) holds and the initial data satisfy suitable conditions. The focus is precisely on weakening as much as possible such conditions, allowing for large initial data: the result in [1] mentioned above clearly applies to the present situation, but it is here greatly improved. The proof follows the same steps as that of Theorem 2.2 there. However, several novelties have been introduced here:

- a Glimm functional that better accounts for nonlinear interactions with the phase wave;
- refined interaction estimates on the amplitude of the reflected waves;
- an original treatment of *non-physical* waves in the front tracking algorithm;
- a simpler proof of the decay of the reflected waves at a geometric rate, as the number of reflections increases.

In particular, as a consequence of this new approach, we require no conditions on the maximal amplitude of the phase wave, differently from [1, (2.8)] and the equivalent formulation in [3, (3.6)].

In spite of the fact that initial data (1.4) seem to reduce system (1.1) to two systems of two conservation laws, we cannot avoid the introduction of non-physical waves [8] in the scheme, as a formal example in [2] shows. Nevertheless, we can obtain an immediate bound on the number of non-physical fronts in the (x,t)-plane by attaching them to the front $x = 0$ (which carries the contact discontinuity) in a sense that will be specified later: this represents a remarkable algorithmic advantage and the main feature of the front tracking used here. On the other hand, we recall that if λ is constant then non-physical waves need not to be introduced, see [5, 7].

The plan of the paper is the following. In Section 2 we state our main result, while in Section 3 we first provide some information on the Riemann problem and then show how to treat non-physical waves by introducing a composite wave together with the phase wave. Consequently, we introduce two solvers to be used in the front-tracking scheme, which shows up in Section 4. Section 5 deals with interactions while in the last Section 6 we prove the convergence and consistency of the algorithm and make a comparison with the result in [1]. In a final short appendix we show how the damping coefficient c introduced in (5.15), which plays a key role in the paper, is also fundamental in the stability analysis of Riemann problems in the sense of [21].

2 Main Result

In this section we state our existence theorem. First, we define $a_r = a(\lambda_r)$, $a_\ell = a(\lambda_\ell)$ and

$$\delta_2 = 2\,\frac{a_r - a_\ell}{a_r + a_\ell}\,. \tag{2.1}$$

Notice that δ_2 ranges over $(-2, 2)$ as soon as a_r, a_ℓ range over $\mathbb{R}_+$. The quantity δ_2 measures the strength of the contact discontinuity located at $x = 0$ and it does not change by interactions with waves of the other families.

We denote $p_o(x) \doteq p\left(v_o(x), \lambda_o(x)\right)$.

Theorem 2.1. *Assume* (1.2) *and consider initial data* (1.3), (1.4) *with* $v_o(x) \geq \underline{v} > 0$, *for some constant* $\underline{v}$. *Let* δ_2 *be as in* (2.1).

There exists a strictly decreasing function $\mathcal{K}$ *defined for* $r \in (0, 2)$ *and satisfying*

$$\lim_{r\to 0+} \mathcal{K}(r) = +\infty\,, \qquad \lim_{r\to 2-} \mathcal{K}(r) = \frac{2}{3}\log\left(2+\sqrt{3}\right), \tag{2.2}$$

such that, if $\delta_2 \neq 0$ *and the initial data satisfy*

$$\mathrm{TV}\left(\log(p_o)\right) + \frac{1}{\min\{a_r, a_\ell\}}\mathrm{TV}(u_o) < \mathcal{K}(|\delta_2|)\,, \tag{2.3}$$

then the Cauchy problem (1.1), (1.3) *has a weak entropic solution* (v, u, λ) *defined for* $t \in [0, +\infty)$. *If* $\delta_2 = 0$ *the same conclusion holds with* $\mathcal{K}(|\delta_2|)$ *replaced by* $+\infty$ *in* (2.3).

Moreover, the solution is valued in a compact set of Ω *and there is a constant* $C = C(\delta_2)$ *such that for every* $t \in [0, +\infty)$ *we have*

$$\mathrm{TV}\left(v(t,\cdot), u(t,\cdot)\right) \leq C\,. \tag{2.4}$$

The properties listed in (2.2) can be directly deduced from the analytical expression of $\mathcal{K}$, that is

$$\mathcal{K}(r) = \frac{2}{1+r}\log\left(\frac{2}{r} + 1 + \frac{2}{r}\sqrt{1+r}\right). \tag{2.5}$$

Therefore, condition (2.3) is *explicit.* We recall that related results of global existence of solutions with large data [13, 15, 16, 19] do not precise the threshold of smallness of the initial data.

Moreover, we observe that condition (2.3) is trivially satisfied if

$$\mathrm{TV}\left(\log(p_o)\right) + \frac{1}{\min\{a_r, a_\ell\}}\mathrm{TV}(u_o) \leq \frac{2}{3}\log\left(2+\sqrt{3}\right), \tag{2.6}$$

because of (2.2). Then, problem (1.1), (1.3) has a global solution if (2.6) is satisfied and $v_o(x) \geq \underline{v} > 0$ holds. This is a striking difference with respect to the results in [1, 3], where the corresponding bound in the right-hand side vanishes at a critical threshold. Moreover, Theorem 2.1 improves the main result in [1], when restricted to the case of a single contact discontinuity. At last, we point out that if $\delta_2 = 0$ we recover the result of [18].

It is left open the question of whether the global existence of solutions to (1.1), (1.3) for any **BV** initial data v_o, u_o occurs, opposite to the possibility of the blow-up in finite time for certain **BV** data.

3 The Riemann problem and the composite wave

In this section we first briefly recall some basic facts about system (1.1), its wave curves and the solution to the Riemann problem; we refer to [1] and the literature cited therein for more details. Next, we introduce a composite wave which sums up the effects of the contact discontinuity and the non-physical waves. We then show two Riemann solvers that make use of the composite wave.

Under assumption (1.2) system (1.1) is strictly hyperbolic in Ω with eigenvalues $e_1 = -\sqrt{-p_v(v,\lambda)}$, $e_2 = 0$, $e_3 = \sqrt{-p_v(v,\lambda)}$; the eigenvalues e_1 and e_3 are genuinely nonlinear while e_2 is linearly degenerate.

For $i = 1, 3$, the right shock-rarefaction curves through the point $U_o = (v_o, u_o, \lambda_o)$ for (1.1) are

$$v \mapsto \left(v, u_o + 2a(\lambda_o)h(\varepsilon_i), \lambda_o\right), \qquad v > 0\,,\ i = 1, 3\,, \tag{3.1}$$

where the strength ε_i of an i-wave is defined as

$$\varepsilon_1 = \frac{1}{2}\log\left(\frac{v}{v_o}\right), \quad \varepsilon_3 = \frac{1}{2}\log\left(\frac{v_o}{v}\right) \tag{3.2}$$

and the function h is defined by

$$h(\varepsilon) = \begin{cases} \varepsilon & \text{if } \varepsilon \geq 0\,, \\ \sinh\varepsilon & \text{if } \varepsilon < 0\,. \end{cases} \tag{3.3}$$

Then, rarefaction waves have positive strengths and shock waves have negative strengths. The wave curve through U_o for $i = 2$ is defined by

$$\lambda \mapsto \left(v_o \frac{a^2(\lambda)}{a^2(\lambda_o)}, u_o, \lambda\right), \qquad \lambda \in [0, 1]\,.$$

Then, the pressure is constant along a 2-curve; the strength of a 2-wave is defined by

$$\varepsilon_2 = 2\,\frac{a(\lambda) - a(\lambda_o)}{a(\lambda) + a(\lambda_o)}\,.$$

Now, we consider the Riemann problem for (1.1) with initial condition

$$(v, u, \lambda)(0, x) = \begin{cases} (v_\ell, u_\ell, \lambda_\ell) = U_\ell & \text{if } x < 0\,, \\ (v_r, u_r, \lambda_r) = U_r & \text{if } x > 0\,, \end{cases} \tag{3.4}$$

for U_ℓ and U_r in Ω. We write $p_r = a_r^2/v_r$, $p_\ell = a_\ell^2/v_\ell$.

Proposition 3.1 ([1]). *The Riemann problem* (1.1), (3.4) *has a unique* Ω*-valued solution in the class of solutions consisting of simple Lax waves, for any pair of states* U_ℓ, U_r *in* Ω.

Moreover, if ε_i *is the strength of the i-wave,* $i = 1, 2, 3$, *then*

$$\varepsilon_3 - \varepsilon_1 = \frac{1}{2}\log\left(\frac{p_r}{p_\ell}\right), \qquad 2\left(a_\ell h(\varepsilon_1) + a_r h(\varepsilon_3)\right) = u_r - u_\ell\,, \tag{3.5}$$

$$\varepsilon_2 = 2\,\frac{a_r - a_\ell}{a_r + a_\ell}\,.$$

The proof of Theorem 2.1 relies on a wave-front tracking algorithm that introduces non-physical waves [8], which, however, are only needed to solve some Riemann problems involving interactions with the 2-wave. Following [1], two states U_ℓ and U_r as in (3.4) can be connected by a non-physical wave if $v_\ell = v_r$ and $\lambda_\ell = \lambda_r$; the strength of a non-physical wave is defined as

$$\delta_0 = u_r - u_\ell\,. \tag{3.6}$$

Then, a non-physical wave changes neither the side values of v nor those of λ, while a 2-wave does not change the side values of u. This suggests to define a new wave by composing the 2-wave with a non-physical wave, with the condition that we assign *zero speed* to non-physical waves and we locate them at $x = 0$. The order of composition does not matter, because a 2-wave and a non-physical wave act on different state variables. This procedure differs from the one used in [1]. Then, we define the *composite* $(2, 0)$*-wave curve* through a point $U_o = (v_o, u_o, \lambda_\ell)$ by

$$u \mapsto \left((a_r^2/a_\ell^2)v_o, u, \lambda_r\right) \tag{3.7}$$

and its strength by

$$\delta_{2,0} = u - u_o\,.$$

The above definition of strength is motivated by the fact that the quantity δ_2 remains constant at any interaction with 1- or 3-waves [1]. Clearly, a $(2, 0)$-wave reduces to the 2-wave as long as non-physical waves are missing. At last, we notice that the pressure does not change across a $(2, 0)$-wave.

In this way, we are left to deal with waves of family 1, 3 and a single composite $(2, 0)$-wave, which is no more entropic. A Riemann solver analogous to that provided in Proposition 3.1 is needed; however, since we have a single contact discontinuity δ_2 and we are going to use the Riemann solver only to solve interactions, we state the following result into such a form.

Proposition 3.2 (Pseudo Accurate Solver). *Consider the interaction at time* t *of a* $\delta_{2,0}$*-wave with an i-wave of strength* δ_i, $i = 1, 3$. *Then the Riemann problem at time* t *has a unique* Ω*-valued solution, which is formed by waves* ε_1, $\delta_{2,0}$, ε_3, *where* ε_1, ε_3 *belong to the first and the third family, respectively. Moreover, we have*

$$\varepsilon_3 - \varepsilon_1 = \frac{1}{2}\log\left(\frac{p_r}{p_\ell}\right), \qquad 2\left(a_\ell h(\varepsilon_1) + a_r h(\varepsilon_3)\right) = u_r - u_\ell - \delta_{2,0}\,. \tag{3.8}$$

Proof. We only consider the case $i = 1$ and refer to Figure 1; the other case is analogous. Consider the auxiliary problem in Figure 1(b), where $V'_\ell = U_\ell + (0, \delta_{2,0}, 0)$. We simply shifted the left state in order to be able to solve the interaction as if it was with an actual 2-wave. Indeed, by Proposition 3.1 we uniquely find ε_1, ε_3 and states V'_p, V'_q such that (3.8) holds. Then, the interaction in Figure 1(a) is solved by the same waves ε_1, ε_3 and by states $U'_p = V'_p - (0, \delta_{2,0}, 0)$, $U'_q = V'_q$. Finally, (3.8) holds by construction.

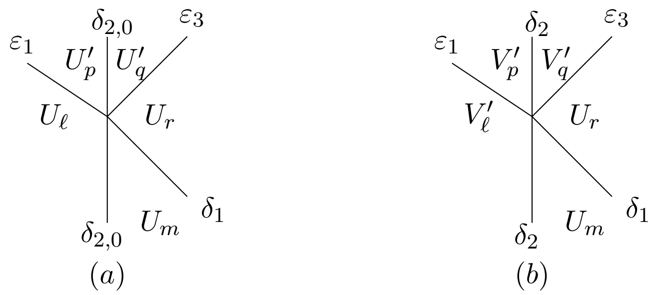

Figure 1: (a): interaction with the $(2,0)$-wave solved with the Pseudo Accurate solver, case $i = 1$; (b): the auxiliary problem. Here and in the following interaction diagrams, the horizontal and vertical axes are x and t, respectively.

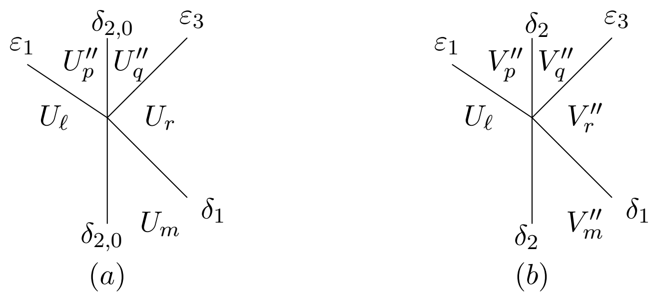

Figure 2: (a): interaction with the $(2,0)$-wave solved with the Pseudo Accurate solver, case $i = 1$; (b): the auxiliary problem.

Notice that we get the same result by shifting the other two states at the right. Indeed, consider the auxiliary problem in Figure 2(b), where $V''_r = U_r - (0, \delta_{2,0}, 0)$ and $V''_m = U_m - (0, \delta_{2,0}, 0)$.

By Proposition 3.1 we uniquely find ε_1, ε_3 (the same as before, since $u''_r - u_l = u_r - u'_l = u_r - u_l - \delta_{2,0}$) and states V''_p, V''_q. Then, the interaction in Figure 2(a) is solved by the same waves ε_1, ε_3 and by states $U''_p = V''_p$ and $U''_q = V''_q + (0, \delta_{2,0}, 0)$. It is then straightforward to check that $U'_p = U''_p$ and $U'_q = U''_q$. □

Another solver is used below. We introduce it in the same framework of Proposition 3.1.

Proposition 3.3 (Pseudo Simplified Solver)**.** *Consider the interaction at time t of a $\delta_{2,0}$-wave with an i-wave of strength δ_i, $i = 1, 3$. Then the Riemann problem at time t can be solved by an i-wave of the same strength δ_i and a unique wave $\varepsilon_{2,0}$, where*

$$\varepsilon_{2,0} = \begin{cases} \delta_{2,0} + 2(a_r - a_\ell)h(\delta_1) & \text{if } i = 1\,, \\ \delta_{2,0} - 2(a_r - a_\ell)h(\delta_3) & \text{if } i = 3\,. \end{cases} \tag{3.9}$$

Proof. We refer to Figure 3. We recall that the commutation of a 1-wave (or a 3-wave) with the 2-wave δ_2 only modifies the u component.

In the case when a 1-wave interacts, it is easy to check that $u_q = u_\ell + 2a_\ell h(\delta_1)$ and $u_m = u_\ell + \delta_{2,0}$; then, we compute $\varepsilon_{2,0}$ by $u_\ell + 2a_\ell h(\delta_1) + \varepsilon_{2,0} = u_\ell + \delta_{2,0} + 2a_r h(\delta_1)$. The other case is analogous. □

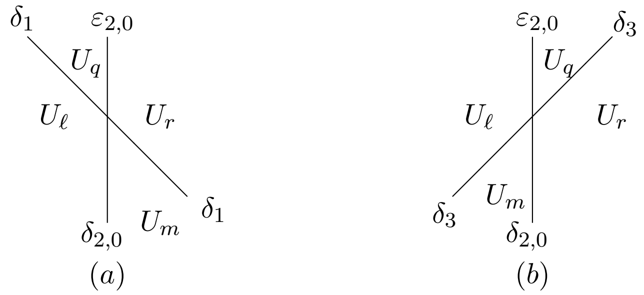

Figure 3: Interactions with the $(2,0)$-wave solved with the Pseudo Simplified solver. (a): from the right $(i = 1)$; (b): from the left $(i = 3)$.

4 Approximate solutions

We use Propositions 3.2 and 3.3 to build up the piecewise-constant approximate solutions to (1.1) that are needed for the wave-front tracking scheme. We first approximate the initial data (1.3): for any $\nu \in \mathbb{N}$ we take a sequence (v_o^ν, u_o^ν) of piecewise constant functions with a finite number of jumps such that, denoting $p_o^\nu = a^2(\lambda_o)/v_o^\nu$,

1. $\mathrm{TV}\log(p_o^\nu) \le \mathrm{TV}\log(p_o)$, $\mathrm{TV}u_o^\nu \le \mathrm{TV}u_o$;
2. $\lim_{x\to-\infty}(v_o^\nu, u_o^\nu)(x) = \lim_{x\to-\infty}(v_o, u_o)(x)$;
3. $\|(v_o^\nu, u_o^\nu) - (v_o, u_o)\|_{\mathbf{L}^1} \le 1/\nu$.

We introduce two strictly positive parameters: $\eta = \eta_\nu$, that controls the size of rarefactions, and a threshold $\rho = \rho_\nu$, that determines which of the two Pseudo Riemann solver is to be used. Here follows a description of the scheme that improves the algorithm of [1] and adapts it to the current situation.

(i) At time $t = 0$ we solve the Riemann problems at each point of jump of $(v_o^\nu, u_o^\nu, \lambda_o)(\cdot, 0+)$ as follows: shocks are not modified while rarefactions are approximated by fans of waves, each of them having size less than η. More precisely, a rarefaction of size ε is approximated by $N = [\varepsilon/\eta] + 1$ waves whose size is $\varepsilon/N < \eta$; we set their speeds to be equal to the characteristic speed of the state at the right. Then $(v, u, \lambda)(\cdot, t)$ is defined until some wave fronts interact; by slightly changing the speed of some waves we can assume that only *two* fronts interact at a time.

(ii) When two wave fronts of the families 1 or 3 interact, we solve the Riemann problem at the interaction point. If one of the incoming waves is a rarefaction, after the interaction it is prolonged (if it still exists) as a single discontinuity with speed equal to the characteristic speed of the state at the right. If a new rarefaction is generated, we employ the Riemann solver described in step *(i)* and split the rarefaction into a fan of waves having size less than η.

(iii) When a wave front of family 1 or 3 with strength δ interacts with the composite wave at a time $t > 0$, we proceed as follows:

- if $|\delta| \ge \rho$, we use the *Pseudo Accurate solver* introduced in Proposition 3.2, partitioning the possibly new rarefaction according to *(i)*;
- if $|\delta| < \rho$, we use the *Pseudo Simplified solver* of Proposition 3.3.

5 Interactions

In this section we analyze the interactions of waves. If $\delta_2 = 0$, i.e. if $a(\lambda_\ell) = a(\lambda_r)$, then the initial data (1.3) reduce (1.1) to a p-system where the pressure p only depends on v. The results of [1, 5] apply and we recover the famous result of [18]. Then, we assume from now on that $\delta_2 \neq 0$. For simplicity, we focus on the case

$$a(\lambda_\ell) < a(\lambda_r)\,.$$

As a consequence we have $\delta_2 > 0$; the other case is entirely similar.

For $t > 0$ at which no interactions occur, and for $\xi \geq 1$, $K_{np} > 0$, $K \geq 1$ to be determined, we introduce the functionals

$$L = \sum_{\substack{i=1,3\\ \gamma_i>0}} |\gamma_i| + \xi \sum_{\substack{i=1,3\\ \gamma_i<0}} |\gamma_i| + K_{np}|\gamma_{2,0}|\,, \tag{5.1}$$

$$V = \sum_{\substack{i=1,3\\ \gamma_i>0,\,\mathcal{A}}} |\gamma_i| + \xi \sum_{\substack{i=1,3\\ \gamma_i<0,\,\mathcal{A}}} |\gamma_i|\,, \qquad Q = \delta_2 V,$$

$$F = L + K\,Q\,. \tag{5.2}$$

By γ_i we mean the strength of a generic i-wave ($i = 1, 3$) located at some point x and by $\gamma_{2,0}$ the strength of the composite wave. The summation in V is performed only over the set $\mathcal{A}$ of waves *approaching* the front carrying the composite wave, namely the waves of the family 1 (and 3) located at the right (left, respectively) of $x = 0$. The term Q is then the "usual" quadratic interaction potential due to the contact discontinuity at $x = 0$. We also introduce

$$\bar{L} = \sum_{i=1,3} |\gamma_i| = \frac{1}{2}\mathrm{TV}\left(\log p(t,\cdot)\right)\,.$$

Remark 5.1. The functional defined in (5.2) differs from [1, (5.1)] because of the presence of the parameter ξ in V and, consequently, in the interaction potential Q, leading to better estimates and a more general result.

Under the notation of Figure 4, we shall make use of the identities [20]

$$\varepsilon_3 - \varepsilon_1 = \alpha_3 + \beta_3 - \alpha_1 - \beta_1\,, \tag{5.3}$$

$$a_\ell h(\varepsilon_1) + a_r h(\varepsilon_3) = a_\ell h(\alpha_1) + a_m h(\alpha_3) + a_m h(\beta_1) + a_r h(\beta_3)\,. \tag{5.4}$$

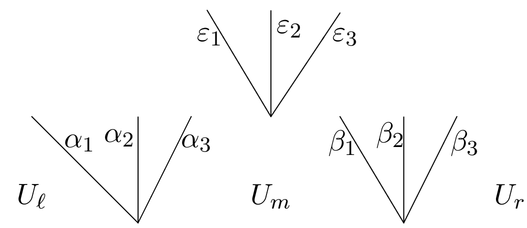

Figure 4: A general interaction pattern.

5.1 Interactions with the composite wave

We first consider the interactions of a 1- or 3-wave with a $(2, 0)$-wave. We notice that they give rise to the following pattern of solutions:

$$\begin{array}{llllll} (2,0)\times 1R & \to & 1R + (2,0) + 3R\,, & (2,0)\times 1S & \to & 1S + (2,0) + 3S\,,\\ 3R\times(2,0) & \to & 1S + (2,0) + 3R\,, & 3S\times(2,0) & \to & 1R + (2,0) + 3S\,. \end{array} \tag{5.5}$$

In the following we often assume that, for some fixed $m > 0$, any interacting i-wave, $i = 1, 3$, with strength δ_i satisfies

$$|\delta_i| \le m\,. \tag{5.6}$$

We usually denote with δ_k (and ε_k) the interacting waves (respectively, the waves produced by the interaction).

Lemma 5.2. *Assume that a wave δ_i, $i = 1, 3$, interacts with a $\delta_{2,0}$-wave.*

If the Riemann problem is solved by the Pseudo Accurate solver, then the strengths ε_i of the outgoing waves satisfy $\varepsilon_{2,0} = \delta_{2,0}$ and

$$|\varepsilon_i - \delta_i| = |\varepsilon_j| \le \frac{1}{2}\,\delta_2|\delta_i|, \quad i, j = 1, 3,\ i \ne j\,, \tag{5.7}$$

$$|\varepsilon_1| + |\varepsilon_3| \le \begin{cases} |\delta_1| + \delta_2|\delta_1| & \text{if } i = 1\,, \\ |\delta_3| & \text{if } i = 3\,. \end{cases} \tag{5.8}$$

If the Riemann problem is solved by the Pseudo Simplified procedure and we assume (5.6)*, then there exists $C_o = C_o(m)$ such that*

$$|\varepsilon_{2,0} - \delta_{2,0}| \le C_o\,\delta_2|\delta_i|\,. \tag{5.9}$$

Proof. The estimates (5.7) and (5.8) easily follow from Proposition 3.2. The proof of the second part relies on the estimates of [1, Proposition 5.12]; we have

$$|\varepsilon_{2,0} - \delta_{2,0}| = 2|a_r - a_\ell|\,|h(\delta_i)| \le 2a_r\frac{\sinh m}{m}\delta_2\,|\delta_i|\,,$$

whence (5.9) immediately follows once we set $C_o(m) \doteq 2a_r \sinh m/m$. □

Proposition 5.3. *Assume that a wave δ_i, $i = 1, 3$, interacts with a $\delta_{2,0}$-wave at time t. In the cases where the Pseudo Accurate procedure is used, then $\Delta F(t) = F(t^+) - F(t^-) < 0$ if it holds*

$$K > \max\left\{\frac{\xi - 1}{2}\,,\,1\right\}\,. \tag{5.10}$$

In the cases where the Pseudo Simplified procedure is used, then $\Delta F(t) < 0$ if it holds

$$K_{np} < \frac{K}{C_o}\,. \tag{5.11}$$

Proof. We first consider the case where the Pseudo Accurate solver is used and use the notation of Figure 1. By (5.3) and Lemma 5.2, we have

$$\begin{cases} \varepsilon_1 - \delta_1 = \varepsilon_3, & |\varepsilon_1| - |\delta_1| = |\varepsilon_3|\,, & \text{if } i = 1, \\ \varepsilon_1 + \delta_3 = \varepsilon_3, & |\delta_3| - |\varepsilon_1| = |\varepsilon_3|\,, & \text{if } i = 3\,. \end{cases}$$

$\boxed{i = 1.}$ If the interacting wave is a rarefaction, then $\Delta L = 2|\varepsilon_3| \le \delta_2|\delta_1|$ and $\Delta V = -|\delta_1|$. Therefore, by (5.10) we deduce

$$\Delta F = \Delta L + K\,\delta_2\Delta V \le \{1 - K\}\,\delta_2|\delta_1| < 0\,. \tag{5.12}$$

If the interacting wave is a shock, we have the same estimates with ξ as a factor.

$\boxed{i = 3.}$ If the interacting wave is a shock, then $\Delta L = |\varepsilon_1| + \xi|\varepsilon_3| - \xi|\delta_3| = -(\xi - 1)|\varepsilon_1| \le 0$, $\Delta V = -\xi|\delta_3| < 0$ and

$$\Delta F = -(\xi - 1)|\varepsilon_1| - K\delta_2\xi|\delta_3| \le -K\delta_2\xi|\delta_3| < 0\,. \tag{5.13}$$

If the wave is a rarefaction, then $\Delta L = \xi|\varepsilon_1| + |\varepsilon_3| - |\delta_3| = (\xi - 1)|\varepsilon_1| \leq (\xi - 1)\,\delta_2|\delta_3|/2$ and $\Delta V = -|\delta_3|$. By (5.10) we obtain again

$$\Delta F = \Delta L + K\,\delta_2\,\Delta V \leq \delta_2|\delta_3|\left\{\frac{\xi - 1}{2} - K\right\} < 0\,. \tag{5.14}$$

If the Pseudo Simplified solver is used, then $\Delta V \leq -|\delta_i|$ $(i = 1, 3)$ and $\Delta L = K_{np}|\varepsilon_{2,0}| - K_{np}|\delta_{2,0}| \leq K_{np}C_o\delta_2|\delta_i|$ by (5.9). Hence, by (5.11) we get

$$\Delta F \leq \delta_2|\delta_i|(K_{np}C_o - K) < 0\,.$$

□

5.2 Interactions between 1- and 3-waves

In this subsection we analyze the interactions between 1- and 3-waves, see Figure 5.

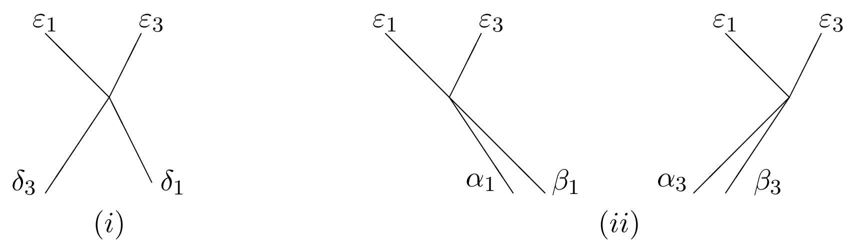

Figure 5: Interactions of 1- and 3-waves.

Lemma 5.4. *For the interaction patterns in Figure 5, the following holds.*

(i) Two interacting waves of different families cross each other without changing strengths.

(ii) Let α_i, β_i be two interacting waves of the same family and ε_1, ε_3 the outgoing waves.

(ii.a) If both incoming waves are shocks, then the outgoing wave of the same family is a shock and satisfies $|\varepsilon_i| > \max\{|\alpha_i|, |\beta_i|\}$; the reflected wave is a rarefaction.

(ii.b) If the incoming waves have different signs, then the reflected wave is a shock; both the amounts of shocks and rarefactions of the i-family decrease across the interaction. Moreover for $j \neq i$ and $\alpha_i < 0 < \beta_i$ one has

$$|\varepsilon_j| \leq c(\alpha_i) \cdot \min\{|\alpha_i|, |\beta_i|\}\,, \qquad c(z) \doteq \frac{\cosh z - 1}{\cosh z + 1}\,. \tag{5.15}$$

Remark 5.5. The inequality (5.15) generalizes the one stated in [1, Lemma B.1] for the case SR, $RS \to SS$; moreover, in that case we provide below a simpler proof.

Proof of Lemma 5.4. We only need to prove (5.15), the rest being already proved in [1, Lemmas 5.4–5.6]. For simplicity we assume $i = 3$ and distinguish between two cases according to the outgoing wave ε_3. Indeed, we remark that there exists a function $x_o(\cdot)$ such that ε_3 is a rarefaction iff $\beta_3 \geq x_o(|\alpha_3|)$; see [1, Lemma B.1]. In the limiting case $\beta_3 = x_o(|\alpha_3|)$ the shock and the rarefaction cancel each other and $\varepsilon_3 = 0$; the interaction gives only rise to the reflected wave ε_1. By setting $x = |\beta_3|$ and $z = |\alpha_3|$, from (5.3) and (5.4) we find the equation valid for $\varepsilon_3 = 0$, namely

$$\sinh(x - z) - \sinh z + x = 0\,,$$

which implicitly defines the function $x = x_o(z)$.

$\boxed{SR,\, RS \to SR}$ The starting point is to specialize (5.3) and (5.4) to the present case:

$$|\varepsilon_1| + |\varepsilon_3| = -|\alpha_3| + |\beta_3|\,, \tag{5.16}$$
$$\sinh(|\varepsilon_1|) - |\varepsilon_3| = \sinh(|\alpha_3|) - |\beta_3|\,. \tag{5.17}$$

By summing up (5.16) and (5.17) we find that

$$\sinh(|\varepsilon_1|) + |\varepsilon_1| = \sinh(|\alpha_3|) - |\alpha_3|\,. \tag{5.18}$$

To prove (5.15) it is enough to prove that

$$|\varepsilon_1| \le c(\alpha_3)|\alpha_3|\,. \tag{5.19}$$

Indeed, from (5.16) we infer that $|\alpha_3| < |\beta_3|$ and therefore (5.19) implies (5.15).

To prove (5.19), we introduce the notation $|\varepsilon_1| = y$ and $|\alpha_3| = z$, so that (5.18) rewrites as

$$G(y,z) \doteq \sinh y + y - \sinh z + z = 0\,.$$

By a simple application of the Implicit Function Theorem, there exists a function $y = y(z) \ge 0$, defined for all $z \ge 0$, such that $G\big(y(z), z\big) = 0$.

Since $G_y(y,z) = \cosh y + 1 > 0$, in order to prove that $y(z) \le c(z)z$ it is enough to prove that $g(z) \doteq G(c(z)z, z) > 0$, that is

$$g(z) = (c(z)+1)z + \sinh(c(z)z) - \sinh z > 0\,. \tag{5.20}$$

Using the fact that $c(z)z < z$, the Mean Value Theorem and the simple identity

$$1 + c(z) = \big(1 - c(z)\big)\cosh z\,,$$

we find that

$$g(z) = \big(c(z)+1\big)\,z + \big(c(z)z - z\big)\cosh\zeta > z\left[c(z) + 1 + \big(c(z)-1\big)\cosh z\right] = 0\,,$$

for $c(z)z < \zeta < z$. Hence, we have proved (5.20).

$\boxed{SR,\, RS \to SS}$ Again, we start from (5.3) and (5.4) that can now be rewritten as

$$|\varepsilon_1| - |\varepsilon_3| = -|\alpha_3| + |\beta_3|\,, \tag{5.21}$$
$$\sinh(|\varepsilon_1|) + \sinh(|\varepsilon_3|) = \sinh(|\alpha_3|) - |\beta_3|\,.$$

Set $x = |\beta_3|$, $y = |\varepsilon_1|$, $z = |\alpha_3|$ and define the function

$$F(x,y;z) = \sinh y + \sinh(y - x + z) - \sinh z + x\,,$$

which is subject to the constraints

$$z \ge 0, \quad 0 \le x < x_o(z), \quad \max\{0, x - z\} < y < \min\{x, z\}\,.$$

By the Implicit Function Theorem, there exists a function $y = y(x;z)$ such that $F\big(x, y(x;z); z\big) \equiv 0$. Moreover, by denoting with y' the derivative of y with respect to x and so on, we have

$$y' = -\frac{F_x}{F_y}\,, \qquad y'' = -\frac{F_{xx} + 2F_{xy}y' + F_{yy}(y')^2}{F_y}\,,$$

where

$$F_x = 1 - \cosh(y - x + z) < 0, \quad F_y = \cosh(y - x + z) + \cosh y > 0\,,$$
$$F_{xx} = -F_{xy} = \sinh(y - x + z) > 0\,, \quad F_{yy} = \sinh(y - x + z) + \sinh y > 0\,.$$

Therefore $y' > 0$ and

$$y''(x; z) = -\frac{\sinh(y - x + z)\,(1 - y')^2 + \sinh(y)\,(y')^2}{F_y} < 0\,.$$

Hence $x \mapsto y(x; z)$ is concave down and thus

$$y(x; z) \leq y'(0; z)x = c(z)x\,.$$

To complete the proof of (5.15), it remains to prove that $y(x; z) \leq c(z)z$. To do this, simply recall that $y' > 0$ and then

$$y(x; z) \leq y\left(x_o(z); z\right) \leq c(z)z\,,$$

where the last inequality holds because it coincides with (5.19) in the limiting case $\beta_3 = x_o(z)$, $z = |\alpha_3|$. □

Remark 5.6. Under the notation of the proof of case *(ii.b)* in Lemma 5.4, i.e., $x = \beta_i$, $z = |\alpha_i|$, we see that the size of the reflected shock is

$$|\varepsilon_j| = \begin{cases} y(x; z) & \text{if } x \leq x_o(z)\,, \\ y(z) & \text{if } x > x_o(z)\,. \end{cases} \tag{5.22}$$

The strength ε_j is a continuous function of x since $y\left(x_o(z); z\right) = y(z)$ for every z. In particular, assume that $\beta_i > x_o(|\alpha_i|)$, so that ε_i is a rarefaction. For β_i in this range, the size of ε_j does not change by (5.22) and the part of β_i exceeding $x_o(|\alpha_i|)$ is entirely propagated along ε_i. This holds since the interaction only affects that part of β_i whose amplitude is exactly $x_o(|\alpha_i|)$. We refer to Figure 6 for a graph of $|\varepsilon_j|$ as a function of β_i.

We notice that this behavior of ε_j is mimicked by the damping coefficient c in (5.15), which only depends on the size of α_i.

Remark 5.7. In case *(ii.a)* of Lemma 5.4, one can prove for the reflected rarefaction that

$$|\varepsilon_j| \leq d\left(\max\{|\alpha_i|, |\beta_i|\}\right) \min\left\{|\alpha_i|, |\beta_i|\right\}\,, \tag{5.23}$$

for a suitable function $d(z) > c(z)$; see [1, Lemma 5.6]. Estimate (5.23) is analogous to (5.15) but the damping coefficient $d\left(\max\{|\alpha_i|, |\beta_i|\}\right)$ cannot be replaced by $c\left(\max\{|\alpha_i|, |\beta_i|\}\right)$. This easily follows by a second order expansion of the function $\tau(a, b)$ in [1, Lemma 5.6] or simply by arguing as in the proof of case *(ii.b)*. However, we shall see in the following proposition that the decreasing of the functional F only depends on the coefficient c and not on d.

Proposition 5.8. *Consider the interactions of two wave fronts of the same family* 1 *or* 3*, and assume* (5.6)*. Then* $\Delta F \leq 0$ *if*

$$1 < \xi \leq \frac{1}{c(m)} \quad \textit{and} \quad K \leq \frac{\xi - 1}{\delta_2}\,. \tag{5.24}$$

Proof. The proof takes into account the possible wave configurations. We use the notation of Lemma 5.4 and assume $i = 3$.

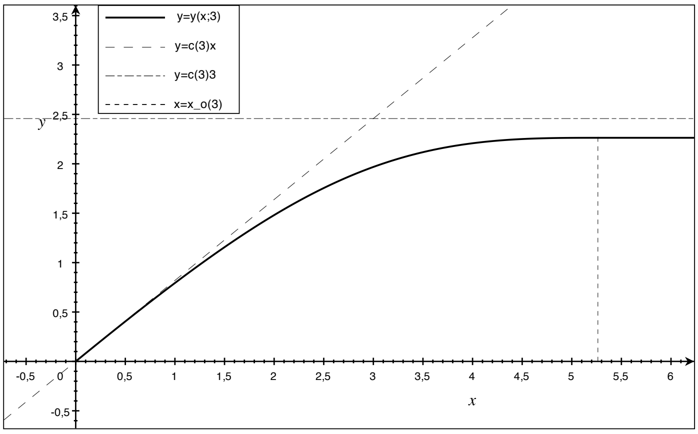

Figure 6: The reflected shock in case *(ii.b)* of Lemma 5.4. The solid curve is the graph of $|\varepsilon_j| = y$ as a function of $\beta_i = x$, for $|\alpha_i| = z = 3$; see (5.22). The vertical line marks the passage of ε_i from shock to rarefaction; on its right, $|\varepsilon_j|$ assume the constant value $y = (x_o(z); z)$. The two remaining dashed lines refer to the bounds in (5.15); in particular, since $\lim_{z\to+\infty}\big(c(z)z - y(x_o(z); z)\big) = 0$, the horizontal bound becomes asymptotically accurate.

$\boxed{SS \to RS}$ We start by proving that

$$\Delta L + |\varepsilon_1|(\xi - 1) = 0\,, \tag{5.25}$$

that holds for all $\xi \geq 1$. Indeed, in this case one has $\Delta\bar{L} = 0$ by (5.3) and then

$$\Delta L + (\xi - 1)|\varepsilon_1| = \xi(|\varepsilon_1| + |\varepsilon_3| - |\alpha_3| - |\beta_3|) = 0\,.$$

If $\Delta V > 0$ then $\Delta V = |\varepsilon_1|$; hence, by (5.24) and (5.25) we obtain

$$\Delta F \leq |\varepsilon_1|\left\{-(\xi - 1) + K\delta_2\right\} \leq 0\,.$$

$\boxed{SR,\, RS \to SR,\, SS}$ Assume $\alpha_3 < 0 < \beta_3$. We now prove the stronger inequality

$$\Delta L + |\varepsilon_1|\xi(\xi - 1) \leq 0\,. \tag{5.26}$$

If ε_3 is a shock, then we use (5.21), (5.15) and $(5.24)_1$ to obtain

$$\begin{aligned}
\Delta L + |\varepsilon_1|\xi(\xi - 1) &= \xi^2|\varepsilon_1| + \xi(|\varepsilon_3| - |\alpha_3|) - |\beta_3| \\
&= \xi^2|\varepsilon_1| + \xi(|\varepsilon_1| - |\beta_3|) - |\beta_3| \\
&= (\xi + 1)(\xi|\varepsilon_1| - |\beta_3|) \leq 0\,.
\end{aligned}$$

Therefore (5.26) holds in this case.

On the other hand, if ε_3 is a rarefaction, then the left hand side of (5.26) turns out to be

$$\xi^2|\varepsilon_1| + |\varepsilon_3| - \xi|\alpha_3| - |\beta_3|\,.$$

From (5.16) we have $|\varepsilon_3| < |\beta_3|$, while (5.15) and $(5.24)_1$ imply $\xi|\varepsilon_1| \leq |\alpha_3|$. This completely proves (5.26).

If $\Delta V > 0$, then $\Delta V = \xi|\varepsilon_1|$ and hence

$$\Delta F \leq \xi|\varepsilon_1|\left\{-(\xi - 1) + K\delta_2\right\} \leq 0 \tag{5.27}$$

by $(5.24)_2$. This concludes the proof of the lemma. □

5.3 Decreasing of the functional F and control of the variations

In order that $\Delta F \le 0$ at any interaction, we need K to satisfy both (5.10) and $(5.24)_2$:

$$\max\left\{\frac{\xi-1}{2}\,,\,1\right\} < K \le \frac{\xi-1}{\delta_2}\,. \tag{5.28}$$

This is possible if $1+\delta_2<\xi$; hence, by $(5.24)_1$ we require that ξ satisfies

$$1+\delta_2 < \xi \le \frac{1}{c(m)}\,. \tag{5.29}$$

In turn, this is possible if

$$c(m) < \frac{1}{1+\delta_2}\,. \tag{5.30}$$

We notice that inequality (5.30) is certainly satisfied if $c(m)\le 1/3$ because $\delta_2<2$. Therefore, we choose the parameters m, ξ and K as follows:

1. We determine the maximum size m of the waves in the approximate solution by assuming (5.30); we recall that c is a strictly increasing function of m and then it is invertible.
2. We choose ξ in the non-empty interval defined by (5.29) and then choose K to satisfy (5.28) with strict inequalities:

   $$\max\left\{\frac{\xi-1}{2}\,,\,1\right\} < K < \frac{\xi-1}{\delta_2}\,. \tag{5.31}$$

   The strict inequality on the right of (5.31) is needed both for the control on the number of interactions [1, Lemma 6.2] and for the decay of the reflected waves as the number of interactions increases, see (6.3) and Proposition 6.4.
3. We choose K_{np} so that (5.11) holds.

We collect the results of the previous subsection into a single proposition.

Proposition 5.9 (Local decreasing)**.** *Consider the interaction of any two waves at time t. Let $m>0$ be such that* (5.30) *holds and $C_o = C_o(m)$ as in Lemma 5.2. If ξ, K, K_{np} satisfy* (5.29)*,* (5.31) *and* (5.11)*, respectively, then*

$$\Delta F(t) \le 0\,. \tag{5.32}$$

6 The convergence and consistency of the algorithm

In this section we finally conclude the proof of Theorem 2.1, focusing on the convergence and consistency of the front tracking algorithm.

For the algorithm to be well-defined, one has to verify that the total number of wave fronts and interactions is finite, besides the fact that the size of rarefaction waves remains small. We already anticipated in the introduction that the algorithm used here to construct the approximate solutions offers the advantage of getting quickly a bound on the total number of wave fronts. As a matter of fact, at every interaction producing more than two outgoing waves the interaction potential F decreases by a fixed positive amount; hence, as in [1, Lemma 6.2] one can prove that for large times any interaction involves only two incoming and two outgoing fronts. The other two requirements are accomplished as in [1, Proposition 6.3] and [1, Lemma 6.1], respectively. In particular, the size ε of any rarefaction wave is bounded by

$$0 < \varepsilon < \eta\left(1+\frac{\delta_2}{2}\right) < 2\eta\,. \tag{6.1}$$

The convergence follows from a standard application of Helly's Theorem, while for the consistency we need refined estimates to control the total size of the composite wave.

6.1 Control of the total size of the composite wave

The wave-front tracking scheme exploits the notion of generation order of a wave to prove that the strength of the composite wave tends to zero as the approximation parameter ν tends to infinity: this means that the $(2,0)$-wave becomes an entropic 2-wave in the limit. More specifically, for a physical wave γ of family 1 or 3 we define its generation order k_γ as in [1, §6.2]; on the other hand, for the $(2,0)$-wave we proceed as follows. We assign order 1 to the $(2,0)$-wave generated at $t = 0+$; then, we keep its order unchanged in the cases where the Pseudo Accurate solver is used, while we set it to be equal to $k_\gamma + 1$ when the Pseudo Simplified solver is used with a physical wave γ.

For any $k = 1, 2, \ldots$, we define

$$\begin{aligned} L_k &= \sum_{\substack{\gamma>0\\ k_\gamma=k}} |\gamma| + \xi \sum_{\substack{\gamma<0\\ k_\gamma=k}} |\gamma| + K_{np}\, L_k^0\,, \\ V_k &= \sum_{\substack{\gamma>0,\,\mathcal{A}\\ k_\gamma=k}} |\gamma| + \xi \sum_{\substack{\gamma<0,\,\mathcal{A}\\ k_\gamma=k}} |\gamma|\,, \qquad Q_k \;=\; \delta_2 V_k\,, \\ F_k &= L_k + K\, Q_k\,, \end{aligned}$$

where γ ranges over the set of 1- and 3-waves, as for (5.1). Above we denoted

$$L_k^0 = \sum_{\tau_k<t} |\varepsilon_{2,0} - \delta_{2,0}|(\tau_k)\,, \tag{6.2}$$

with τ_k denoting the interaction times where the outgoing composite wave has order of generation k. As a consequence, only the times τ_k where the Pseudo Simplified solver is used give positive summands in (6.2): when the Pseudo Accurate solver is used we have $\varepsilon_{2,0} = \delta_{2,0}$.

For $k \in \mathbb{N}$, we introduce:

- I_k = set of times when two waves α, β of same family interact, with $\max\{k_\alpha, k_\beta\} = k$;
- J_k = set of times when a 1- or a 3-wave of order k interacts with the $(2,0)$-wave.

We set $\mathcal{T}_k = I_k \cup J_k$ and define

$$\mu \doteq \max\left\{\frac{1}{2K-1}, \frac{\xi}{2K+1}, \frac{K\delta_2+1}{\xi}, \frac{K_{np}C_o}{K}\right\}\,. \tag{6.3}$$

We notice that $0 < \mu < 1$ by (5.31) and (5.11).

Proposition 6.1. *Let m, ξ, K and K_{np} satisfy the assumptions of Proposition 5.9 and assume that* (5.6) *for every wave. Then the following holds, for $\tau \in \mathcal{T}_h$, $h \geq 1$:*

$$\Delta F_h < 0\,, \qquad \Delta F_{h+1} > 0\,, \tag{6.4}$$

$$\Delta F_k = 0 \qquad \textit{if} \;\; k \geq h+2\,. \tag{6.5}$$

Moreover,

$$[\Delta F_{h+1}]_+ \leq \mu\left([\Delta F_h]_- - \sum_{\ell=1}^{h-1} \Delta F_\ell\right). \tag{6.6}$$

Remark 6.2. Notice that Proposition 6.1 let us improve Proposition 5.9. Indeed, recalling that $\mathcal{T}_h = I_h \cup J_h$, Proposition 6.1 implies, for $\tau \in I_h$,

$$\Delta F = \sum_{\ell=1}^{h-1} \Delta F_\ell \, - \, [\Delta F_h]_- \, + \, [\Delta F_{h+1}]_+ \le -(1-\mu)[\Delta F_h]_- < 0\,,$$

while for $\tau \in J_h$, being $\sum_{\ell=1}^{h-1}[\Delta F_\ell]_+ = 0$, it gives

$$\Delta F \;=\; -[\Delta F_h]_- \, + \, [\Delta F_{h+1}]_+ \le -(1-\mu)[\Delta F_h]_- < 0\,.$$

Then, estimate (6.6) quantifies the decrease in the functional F and thus improves (5.32).

Proof of Proposition 6.1. If $k \ge h+2$, no wave of order k is involved and then (6.5) holds. To prove (6.4) and (6.6), we distinguish between two cases.

$\boxed{\tau \in I_h}$ (Interactions between waves of 1-, 3-family).

Clearly the F_k's do not vary when a 1-wave interacts with a 3-wave. Then we consider interactions of waves of the same family, see Figure 7*(a)*.

Since $\tau \in I_h$, then $\Delta L_{h+1} > 0$ and $0 \le \Delta Q_{h+1} \le \delta_2 \Delta L_{h+1}$. Also, $\Delta F_h = \Delta L_h + K\Delta Q_h < 0$, since both terms in the sum are negative or zero. This proves (6.4).

By (5.25) and (5.26) (see also [1, (6.10)]), we have that

$$[\Delta L_{h+1}]_+ \;\le\; \frac{1}{\xi}\left([\Delta L_h]_- - \sum_{\ell=1}^{h-1} \Delta L_\ell\right). \tag{6.7}$$

By (6.7), the estimate $0 \le \Delta Q_{h+1} \le \delta_2 \Delta L_{h+1}$ and (6.3) we deduce that

$$0 < \Delta F_{h+1} \le (1+K\delta_2)[\Delta L_{h+1}]_+ \le \mu\left([\Delta L_h]_- - \sum_{\ell=1}^{h-1} \Delta L_\ell\right). \tag{6.8}$$

We now prove that

$$[\Delta Q_h]_- - \sum_{\ell=1}^{h-1} \Delta Q_\ell \ge 0\,, \tag{6.9}$$

for which we only have to consider the case when $\Delta Q_\ell > 0$ for an $\ell \le h-1$. In this case, $[\Delta Q_h]_- - \sum_{\ell=1}^{h-1} \Delta Q_\ell = -\delta_2\, \Delta V \ge \delta_2(-\Delta L + |\varepsilon_1|) \ge 0$ because of (5.25), (5.26); this proves (6.9). Therefore, for $\tau \in I_h$, estimate (6.6) follows from (6.8) and (6.9).

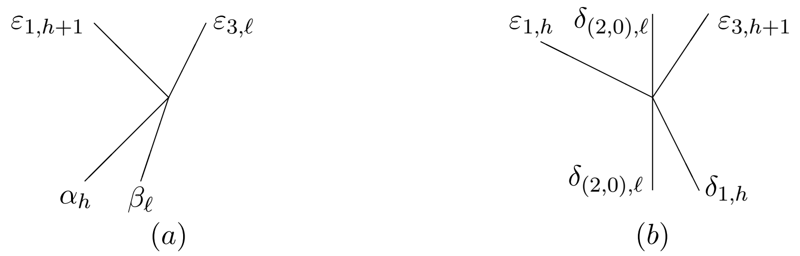

Figure 7: Interactions of waves; h and ℓ denote generation orders. *(a)*: interaction of 3-waves with $h \ge \ell$; *(b)*: interaction between a 1-wave and the $(2,0)$-wave solved by the Pseudo Accurate solver.

$\boxed{\tau \in J_h}$ (Interactions with the $(2,0)$-wave).

Since no wave of order $\leq h-1$ interact, then (6.6) reduces to

$$[\Delta F_{h+1}]_+ \leq \mu [\Delta F_h]_- \,. \tag{6.10}$$

To prove (6.10), we first consider the case where the Pseudo Accurate solver is used, see Figure 7*(b)*. Assume that a 1-wave δ_1 of order h interacts with the $(2,0)$-wave. By (5.5), the reflected wave ε_3 is of the same type of the interacting wave and the transmitted one ε_1. If $\delta_1 > 0$, then $\varepsilon_1 > 0$ and $\varepsilon_3 > 0$; by Lemma 5.2 this leads to

$$\Delta F_h = \Delta L_h + K\Delta Q_h \leq \frac{\delta_2|\delta_1|}{2} - K\delta_2|\delta_1| = -(2K-1)\frac{\delta_2|\delta_1|}{2} < 0$$

by (5.28) and then, because of (6.3), to

$$[\Delta F_{h+1}]_+ = \Delta L_{h+1} = |\varepsilon_3| \leq \frac{\delta_2|\delta_1|}{2} \leq \frac{1}{2K-1}[\Delta F_h]_- \leq \mu[\Delta F_h]_- \,.$$

The last estimate is also valid when $\delta_1 < 0$ (the only difference is that in the previous computations there is a factor ξ both in ΔF_h and in ΔF_{h+1}).

On the other hand, if we consider the interaction with a wave δ_3 of order h belonging to the third family, then the reflected wave ε_1 will be of a type different from that of δ_3 and ε_3. In this case, we first suppose $\delta_3, \varepsilon_3 > 0$; then, $\varepsilon_1 < 0$. As a consequence we have

$$\Delta F_h = -|\varepsilon_1| - K\delta_2|\delta_3| \leq -(1+2K)|\varepsilon_1|$$

and, therefore,

$$[\Delta F_{h+1}]_+ = \xi|\varepsilon_1| = \frac{\xi}{1+2K}\left[(1+2K)|\varepsilon_1|\right] \leq \frac{\xi}{1+2K}[\Delta F_h]_- \leq \mu[\Delta F_h]_- \,,$$

because of (6.3). In the other case, i.e. when $\delta_3, \varepsilon_3 < 0$ and $\varepsilon_1 > 0$, we have

$$\Delta F_h = -\xi|\varepsilon_1| - K\xi\delta_2|\delta_3| \leq -\xi(1+2K)|\varepsilon_1|$$

and

$$[\Delta F_{h+1}]_+ = |\varepsilon_1| \leq \frac{1}{\xi(1+2K)}[\Delta F_h]_- \leq \mu[\Delta F_h]_- \,.$$

Now, we consider the case when the interacting wave has strength $|\delta| < \rho$ and then the Pseudo Simplified solver is used. In this case a non-physical error of size $|\varepsilon_{2,0} - \delta_{2,0}|$ and order $h+1$ appears. Thus, again by Lemma 5.2,

$$0 < \Delta F_{h+1} = K_{np}\Delta L^0_{h+1} \leq K_{np}C_o\delta_2|\delta|, \qquad \Delta L_h = 0\,, \qquad \Delta Q_h \leq -\delta_2|\delta|\,.$$

Consequently, $[\Delta F_h]_- \geq K\delta_2|\delta|$ and

$$[\Delta F_{h+1}]_+ \leq \frac{K_{np}C_o}{K}[\Delta F_{k-1}]_- \leq \mu[\Delta F_{k-1}]_- \,.$$

Then (6.10) is proved. Finally we notice that, in all the above cases for $\tau \in J_h$, (6.4) holds. □

Now, we proceed similarly as in [1, Proposition 6.7] to obtain a recursive estimate for F_k. Indeed, the functional F_k increases at times $\tau \in \mathcal{T}_{k-1}$, it decreases at $\tau \in \mathcal{T}_k$, while it has not a definite sign for times $\tau \in \mathcal{T}_h$ with $h \geq k+1$. For F_1 we have:

$$F_1(t) = F_1(0) - \sum_{\mathcal{T}_1} [\Delta F_1]_- + \sum_{h>1} \sum_{\mathcal{T}_h} \Delta F_1 \,, \tag{6.11}$$

while for F_k with $k \geq 2$ we use that $F_k(0) = 0$ to obtain

$$F_k(t) = \sum_{\mathcal{T}_{k-1}} [\Delta F_k]_+ - \sum_{\mathcal{T}_k} [\Delta F_k]_- + \sum_{h>k} \sum_{\mathcal{T}_h} \Delta F_k \,. \tag{6.12}$$

Here above we assumed that summations are done over interaction times $\tau < t$; the same notation is used in the following. We consider now the last terms in (6.11), (6.12):

$$\sum_{h>k} \sum_{\mathcal{T}_h} \Delta F_k \,, \qquad k \geq 1 \,.$$

The above contribution is different from zero (and then possibly positive) only if the interaction involves two waves of the same family, one of order k and the other of order h, with $h > k$. We denote by $\mathcal{T}_{h,k}$ the set of times at which an interaction of this type occurs. Clearly $\mathcal{T}_{h,k} \subset \mathcal{T}_h$.

Moreover, we define the quantity

$$\alpha_k(t) = \sum_{\tau \in \mathcal{T}_{k-1}, \tau < t} [\Delta F_k(\tau)]_+ \,, \qquad k \geq 2 \,, \tag{6.13}$$

that is, the first term on the right hand side of (6.12). Hence we rewrite (6.11), (6.12) as

$$0 \leq F_1(t) = F_1(0) - \sum_{\mathcal{T}_1} [\Delta F_1]_- + \sum_{h>1} \sum_{\mathcal{T}_{h,1}} \Delta F_1 \,, \tag{6.14}$$

$$0 \leq F_k(t) = \alpha_k - \sum_{\mathcal{T}_k} [\Delta F_k]_- + \sum_{h>k} \sum_{\mathcal{T}_{h,k}} \Delta F_k \,, \qquad k \geq 2 \,. \tag{6.15}$$

Proposition 6.3. *For $k \geq 2$ one has*

$$\alpha_k \leq \mu^{k-1} F_1(0) + \sum_{h \geq k} \sum_{\ell=1}^{k-1} \sum_{\mathcal{T}_{h,\ell}} \Delta F_\ell \,. \tag{6.16}$$

Proof. For $k = 2$, we use (6.6) and the positivity of F_1 to get

$$\begin{aligned}
\alpha_2 = \sum_{\mathcal{T}_1} [\Delta F_2]_+ \;\leq\; \mu \sum_{\mathcal{T}_1} [\Delta F_1]_- \;\leq\; \mu \left\{ F_1(0) + \sum_{h>1} \sum_{\mathcal{T}_{h,1}} \Delta F_1 \right\} \\
\leq \mu F_1(0) + \sum_{h \geq 2} \sum_{\mathcal{T}_{h,1}} \Delta F_1 \,,
\end{aligned}$$

which is (6.16) for $k = 2$.

By induction, assume that (6.16) holds for some $k \geq 2$. Since $F_k \geq 0$, from (6.15) we get

$$\sum_{\mathcal{T}_k} [\Delta F_k]_- \leq \alpha_k \, + \sum_{h>k} \sum_{\mathcal{T}_{h,k}} \Delta F_k \,.$$

Now, by definition (6.13), by estimate (6.6) and the previous inequality we find

$$\begin{aligned}
\alpha_{k+1} = \sum_{\mathcal{T}_k} [\Delta F_{k+1}]_+ &\leq \mu \sum_{\mathcal{T}_k} [\Delta F_k]_- \;-\; \mu \sum_{\ell<k} \sum_{\mathcal{T}_{k,\ell}} \Delta F_\ell \\
&\leq \mu \alpha_k \;+\; \mu \sum_{h>k} \sum_{\mathcal{T}_{h,k}} \Delta F_k \;-\; \mu \sum_{\ell<k} \sum_{\mathcal{T}_{k,\ell}} \Delta F_\ell \,.
\end{aligned}$$

By using the induction hypothesis (6.16), we get

$$\alpha_{k+1} \le \mu^k F_1(0) + \mu \underbrace{\sum_{\substack{h,\ell \\ h\ge k>\ell}} \sum_{\mathcal{T}_{h,\ell}} \Delta F_\ell}_{(I)} + \mu \sum_{h>k} \sum_{\mathcal{T}_{h,k}} \Delta F_k \; - \; \mu \underbrace{\sum_{\ell<k} \sum_{\mathcal{T}_{k,\ell}} \Delta F_\ell}_{(II)} \,.$$

Notice that

$$(I) = (II) + \sum_{\substack{h,\ell \\ h>k>\ell}} \sum_{\mathcal{T}_{h,\ell}} \Delta F_\ell \,,$$

so that

$$\begin{aligned} \alpha_{k+1} &\le \mu^k F_1(0) + \mu \sum_{\substack{h,\ell \\ h>k>\ell}} \sum_{\mathcal{T}_{h,\ell}} \Delta F_\ell \; + \; \mu \sum_{h>k} \sum_{\mathcal{T}_{h,k}} \Delta F_k \\ &= \mu^k F_1(0) + \mu \sum_{\substack{h,\ell \\ h>k\ge\ell}} \sum_{\mathcal{T}_{h,\ell}} \Delta F_\ell \end{aligned}$$

from which we deduce (6.16) for $k+1$, since $\mu < 1$. □

Proposition 6.4. *For $k \ge 2$ one has*

$$\tilde{F}_k(t) \doteq \sum_{j\ge k} F_j(t) \le \mu^{k-1} F_1(0) \,. \tag{6.17}$$

Proof. For $k \ge 2$ we have $\tilde{F}_k(0) = 0$. Moreover, we also deduce:

- $\Delta\tilde{F}_k(\tau) = 0$ for $\tau \in \mathcal{T}_h$, $h \le k-2$, by (6.5);
- $\Delta\tilde{F}_k(\tau) = \Delta F_k(\tau) > 0$ for $\tau \in \mathcal{T}_{k-1}$, by (6.4);
- at last, for all $\tau \in \mathcal{T}_h$, $h \ge k$,

  $$\Delta\tilde{F}_k(\tau) \le -\sum_{\ell=1}^{k-1} \Delta F_\ell(\tau) \,,$$

  by the property $\Delta F(\tau) < 0$, see Remark 6.2.

As a consequence of the above properties, using also (6.13) and (6.16), we find

$$\begin{aligned} \tilde{F}_k(t) &= \alpha_k + \sum_{h\ge k} \sum_{\mathcal{T}_h} \Delta\tilde{F}_k \\ &\le \mu^{k-1} F_1(0) + \sum_{h\ge k} \sum_{\ell=1}^{k-1} \sum_{\mathcal{T}_{h,\ell}} \Delta F_\ell - \sum_{h\ge k} \sum_{\ell=1}^{k-1} \sum_{\mathcal{T}_{h,\ell}} \Delta F_\ell \; = \; \mu^{k-1} F_1(0) \,. \end{aligned}$$

□

We can now proceed to determine parameters ρ and η as in [1]. Fix $\eta > 0$ such that $\eta = \eta_\nu \to 0$ as $\nu \to \infty$ and estimate the total number of waves of order $< k$. Then, for the strength of the composite wave it holds

$$\begin{aligned} |\gamma_{2,0}|(t) &\le \tilde{L}_k(t) + \sum_{\substack{h<k \\ \tau_h<t}} |\varepsilon_{2,0} - \delta_{2,0}|(\tau_h) \le \\ &\le \mu^{k-1} \cdot L(0) \cdot (1 + K\delta_2) + C_o \rho \, \delta_2 \, [\text{number of fronts of order} < k] < \frac{1}{\nu} \,, \end{aligned}$$

by choosing k sufficiently large to have the first term $\le 1/(2\nu)$ and, then, $\rho = \rho_\nu$ small enough to have the second term also $\le 1/(2\nu)$.

Remark 6.5. Proposition 6.3 improves Lemma 6.6 in [1], because of ΔF_ℓ on the right hand side of (6.16) in place of $[\Delta F_\ell]_+$. This is obtained under the same local interaction estimates (6.4)–(6.6). Moreover, Proposition 6.4 is only based on Proposition 6.3 and on $\Delta F < 0$. Hence the same argument could be applied to the general case treated in [1], and improve the related result by avoiding some technical assumptions due to the presence of non-physical waves.

Finally, we prove the global decreasing of F.

Proposition 6.6 (Global decreasing)**.** *We choose parameters m, ξ, K, K_{np} as in Proposition 5.9, that is, that satisfy* (5.30), (5.29), (5.31) *and* (5.11), *respectively.*

Moreover, we assume that

$$\bar{L}(0+) \leq m\,c(m) \tag{6.18}$$

and that the approximate solution is defined in $[0,T]$. *Then we have that* (5.6) *is satisfied and therefore that $\Delta F(t) \leq 0$ for every $t \in (0,T]$. In particular, every shock wave of size δ_i and generation order $k \geq 1$ satisfies*

$$|\delta_i| \leq \mu^{k-1} m\,. \tag{6.19}$$

Proof. By Propositions 5.3 and 5.8 we know that $\Delta F \leq 0$ if (5.6) holds.

Recalling that the maximum size of a rarefaction is smaller than 2η, see (6.1), we need to check (5.6) only for shocks. Hence, once that m is chosen, it is enough to assume $\eta < m/2$.

By (6.18) we deduce that $L(0+) \leq m$ and by a recursion argument we find that for every $t \leq T$

$$F(t) \leq F(0+) \leq L(0+)(1 + K\delta_2) \leq \xi^2 \bar{L}(0+)\,.$$

This implies that the size δ_i of a shock, at time t, satisfies

$$|\delta_i| \leq \frac{1}{\xi} F(t) \leq \frac{1}{\xi} F(0) \leq \xi \bar{L}(0+) \leq \frac{1}{c(m)} \bar{L}(0+) \leq m$$

and in particular (5.6), which is (6.19) for $k = 1$.

Moreover, recalling (6.17) in Proposition 6.4, for $k \geq 2$ we have

$$|\delta_i| \leq \frac{1}{\xi} F_k(t) \leq \frac{\mu^{k-1}}{\xi} F_1(0+) = \frac{\mu^{k-1}}{\xi} F(0+) \leq \mu^{k-1} m$$

which is (6.19). □

Remark 6.7. As an example of choice of the parameters we can take $\xi = 3$, so that the left hand side of (5.29) is satisfied for every δ_2 (remind that $0 < \delta_2 < 2$). Then K, m will satisfy

$$1 < K < \frac{2}{\delta_2}\,, \qquad c(m) \leq 1/3 \tag{6.20}$$

which gives $\cosh m \geq 2$, that is

$$m \geq \bar{m} = \cosh^{-1}(2) = \log\left(2 + \sqrt{3}\right)\,. \tag{6.21}$$

Therefore, if $\bar{L}(0+) \leq \frac{1}{3} \log\left(2 + \sqrt{3}\right)$ and for any δ_2, the functional F with $\xi = 3$ and K as in (6.20) is decreasing on $\mathbb{R}_+$.

6.2 Proof of Theorem 2.1 and a comparison

In this last section we accomplish the proof of Theorem 2.1 and compare the result we obtain with that proved in [1, 3].

Proof of Theorem 2.1. It only remains to reinterpret the choice of the parameter m in terms of the assumption (2.3) on the initial data. Recalling Proposition 6.6, (5.30) and since

$$\bar{L}(0+) \leq \frac{1}{2}\mathrm{TV}\left(\log(p_o)\right) + \frac{1}{2\inf a_o}\mathrm{TV}(u_o)\,,$$

we look for m satisfying

$$|\delta_2| < \frac{1}{c(m)} - 1 \;=\; \frac{2}{\cosh m - 1} \doteq w(m)\,, \tag{6.22}$$

$$\mathrm{TV}\left(\log(p_o)\right) \;+\; \frac{1}{\min\{a_r, a_\ell\}}\mathrm{TV}(u_o) < 2m\,c(m) \doteq z(m)\,. \tag{6.23}$$

Notice that $w(m)$ is strictly decreasing from $\mathbb{R}_+$ to $\mathbb{R}_+$, while $z(m)$ is strictly increasing on the same sets. Since $|\delta_2| < 2$, we restrict the choice of the parameter to have $w(m) \in (0, 2)$, that is $\cosh m > 2$ and then $m > \bar{m}$, where $\bar{m}$ is given in (6.21).

We can now define

$$\mathcal{K}(r) \doteq z\left(w^{-1}(r)\right) = \frac{2}{1+r}\,c^{-1}\left(\frac{1}{1+r}\right)\,, \qquad r \in (0, 2)\,, \tag{6.24}$$

which is explicitly given in (2.5). Hence, if the assumption (2.3) holds, namely

$$\mathrm{TV}\left(\log(p_o)\right) \;+\; \frac{1}{\min\{a_r, a_\ell\}}\mathrm{TV}(u_o) < \mathcal{K}(|\delta_2|)\,,$$

it is easy to prove that one can choose $m > \bar{m}$ such that (6.22), (6.23) hold. Finally, in order to pass to the limit and prove the convergence to a weak solution, one can proceed as in [8]. Theorem 2.1 is, therefore, completely proved. □

Now, we make a comparison between Theorem 2.1 and the main result in [1], which was proved to be equivalent to Theorem 3.1 of [3]. Condition (3.7) of the latter theorem, when applied to the current problem, can be written as

$$\mathrm{TV}\left(\log(p_o)\right) + \frac{1}{\min\{a_r, a_\ell\}}\mathrm{TV}(u_o) < H(|\delta_2|)\,, \tag{6.25}$$

where the function $H(r)$ is only defined for $r < 1/2$ by

$$H(r) \doteq 2(1-2r)k^{-1}(r)\,, \qquad k(m) = \frac{1-\sqrt{d(m)}}{2-\sqrt{d(m)}}\,. \tag{6.26}$$

Here above, $d(m)$ is the damping coefficient introduced in [1, Lemma 5.6], see Remark 5.7.

Hence, the result of Theorem 2.1 is new for $1/2 \leq |\delta_2| < 2$, including the case where the 2-wave may be arbitrarily large, i.e. $|\delta_2|$ close to 2. In order to compare (6.25) with (2.3) in the common range $|\delta_2| < 1/2$, we set $r = |\delta_2| \in (0, 1/2)$ and rewrite H as

$$H(r) = 2(1-2r)\,d^{-1}\left(\left(\frac{1-2r}{1-r}\right)^2\right)\,.$$

Comparing this expression with (2.5), we notice that $1/(1+r) > (1-2r)$. Moreover, we have

$$\frac{1}{1+r} > \left(\frac{1-2r}{1-r}\right)^2;$$

since $c < d$ and c is strictly increasing, we have also that $c^{-1}\left(1/(1+r)\right) > k^{-1}(r)$. We deduce that $\mathcal{K}(r) > H(r)$ for $0 \le r < 1/2$; see Figure 8. Then, the conditions on the initial data obtained here considerably improve the ones required in the previous works [1, 3], albeit the latter were given for a more general case.

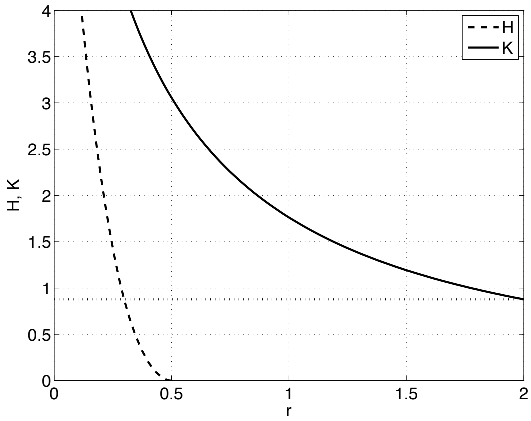

Figure 8: The functions H (dashed line) and $\mathcal{K}$ (solid line). The horizontal dotted line gives the asymptotic value $\frac{2}{3}\log(2+\sqrt{3})$ of $\mathcal{K}$ for $r \to 2-$.

A Another interpretation of the damping coefficient c

The function c introduced in (5.15) plays a fundamental role in controlling the size of the weight ξ assigned to shock waves in the front-tracking scheme, see Proposition 5.8. In this appendix we show that the same coefficient c also appears in the stability analysis of the Riemann problems of system (1.1), see [21, 4].

In [21] Schochet proves that if the solution of a Riemann problem satisfies some *finiteness conditions* (also called BV-stability conditions), then *small* perturbations of bounded variation of its initial data give rise to a solution defined globally in time. The analysis for system (1.1) was done in [4], where it was proved that there are solutions to suitable Riemann problems that do not satisfy such conditions.

Proposition A.1. *Let us consider the pattern formed by a* 1*-shock* ε_1*, a contact discontinuity* ε_2 *and a* 3*-shock* ε_3 *as in Figure 9. Then, the finiteness condition of [21] for this pattern can be written as*

$$c(\varepsilon_1)c(\varepsilon_3)\varepsilon_2^2 - \left(c(\varepsilon_1)+c(\varepsilon_3)\right)|\varepsilon_2| + 2\left(1-c(\varepsilon_1)c(\varepsilon_3)\right) > 0\,. \tag{A.1}$$

This condition makes explicit the analogous one provided in [4, (14)]. We remark that condition (A.1) is satisfied for *every* shock ε_3 (for example) if it holds in the degenerate case $c(\varepsilon_3) = 1$, [4]; in such a case, it simply reduces to $1 + |\varepsilon_2| \le 1/c(\varepsilon_1)$, which reminds of (5.29).

Proof of Proposition A.1. Maintaining the notation of [4, Lemma 1.2], we denote the states lying between the waves by U_0, U_1, U_2, U_3, from left to right; see Figure 9. We use $c_1 = a_1/v_1$, $c_2 = a_2/v_2$ to indicate the characteristic speeds and $s_- = -a_1/\sqrt{v_1 v_0}$, $s_+ = a_2/\sqrt{v_2 v_3}$ to indicate the speeds of the shocks of the first and third family, respectively. Finally, we write $L_\pm$, $R_\pm$ for the left and right eigenvectors of the first and third family, while we let $[U]_\pm$ be the variation of U along the 1- and 3-shock.

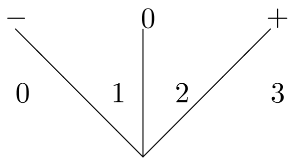

Figure 9: States for the Riemann problem.

Let us introduce the following quantities

$$A = |R^{(-)}| = \left| \frac{c_1 + s_-}{c_1 - s_-} \cdot \frac{L_+(U_1) \cdot [U]_-}{L_-(U_1) \cdot [U]_-} \right| , \quad B = |R^{(+)}| = \left| \frac{c_2 - s_+}{c_2 + s_+} \cdot \frac{L_-(U_2) \cdot [U]_+}{L_+(U_2) \cdot [U]_+} \right| ,$$

which represent some coefficients of the reflection matrices $R^{(-)}_{>,\leq}$ and $R^{(+)}_{<,\geq}$ appearing in [4]. By performing simple computations, we find that $A = c(\varepsilon_1)$ and $B = c(\varepsilon_3)$. Indeed, we have

$$\frac{L_+(U_1) \cdot [U]_-}{L_-(U_1) \cdot [U]_-} = \frac{-c_1(v_1 - v_0) + (u_1 - u_0)}{c_1(v_1 - v_0) + (u_1 - u_0)}$$

and, recalling that along a shock of the first family it holds $u_1 - u_0 = -s_-(v_1 - v_0)$, the previous quantity becomes $(-c_1 - s_-)/(c_1 - s_-)$. Therefore,

$$A = \Big(\frac{c_1 + s_-}{c_1 - s_-}\Big)^2 = \left(\frac{v_0/v_1 - \sqrt{v_0/v_1}}{v_0/v_1 + \sqrt{v_0/v_1}}\right)^2 .$$

By definition (3.2) we get $v_0/v_1 = \exp(-2\varepsilon_1)$ and, consequently,

$$A = \Big(\frac{\exp(-\varepsilon_1/2) - \exp(\varepsilon_1/2)}{\exp(-\varepsilon_1/2) + \exp(\varepsilon_1/2)}\Big)^2 = \tanh^2(\varepsilon_1/2) = \frac{\cosh(\varepsilon_1) - 1}{\cosh(\varepsilon_1) + 1} = c(\varepsilon_1) .$$

Analogous calculations hold for $B = (\cosh(\varepsilon_3) - 1)/(\cosh(\varepsilon_3) + 1) = c(\varepsilon_3)$. By substituting such A, B in [4, (14)] we obtain (A.1) and the proposition is proved. □

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