COEXISTENCE AND OPTIMAL CONTROL PROBLEMS FOR A DEGENERATE PREDATOR-PREY MODEL

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Abstract. In this paper we present a predator-prey mathematical model for two biological populations which dislike crowding. The model consists of a system of two degenerate parabolic equations with nonlocal terms and drifts. We provide conditions on the system ensuring the periodic coexistence, namely the existence of two non-trivial non-negative periodic solutions representing the densities of the two populations. We assume that the predator species is harvested if its density exceeds a given threshold. A minimization problem for a cost functional associated with this process and with some other significant parameters of the model is also considered.

Key words: Degenerate parabolic equations, positive periodic solutions, optimal control problems.
2009 MSC: Primary: 35K65, 49J20, Secondary: 47H11, 47H04.

1. Introduction

There is a vast literature on mathematical models in biology and population dynamics and the last decades witnessed a growing interest toward models that take into account the effects of spatial inhomogeneity and diffusion on the evolution of a population. Mathematically, this translates into studying mainly systems of partial differential equations of parabolic type with particular attention to the existence of positive steady states, positive periodic solutions, their stability properties, permanence and extinction issues as the following far-from-exhaustive list of references is meant to illustrate.

In [1] an autonomous and competitive Lotka-Volterra system of parabolic type is considered and conditions are given on the size of diffusivity coefficients in order to guarantee that the model has spatially non homogeneous steady states (see also [2, 3, 4] for results on stability of coexistence steady states). The existence of periodic solutions of an autonomous reaction diffusion systems is proved in [5]. In particular, in [6] nonlocal delays are added to a predator-prey parabolic system and the global asymptotical stability of its steady states is discussed together with the occurrence of Hopf bifurcations.

One of the first papers to consider the predator-prey non-autonomous situation is [7] where the coefficients are T-periodic in time and the existence of componentwise positive periodic solutions is proved with different boundary conditions. Existence and stability of positive periodic solutions to reaction-diffusion systems are also considered in the papers [8, 9, 10, 11, 12, 13, 14] when the right hand side is in the classical predator-prey or competitive form [12], ratio-dependent of Holling type-III [9], monotone in some suitable sense [8, 10, 11, 14]. The techniques employed involve monotone iteration schemes, upper and lower solutions, comparison results. The question of permanence and extinction is also considered in [9, 12] while the presence of delayed terms is allowed in [10, 11, 13].

In order to model a more general way of spreading behavior in space the use of degenerate parabolic operators has been proposed. For instance, replacing the usual $-\Delta u$ term by a degenerate elliptic operator as $-\Delta u^m$, as proposed in [15, 16, 17, 18], is a way of modeling the diffusion of species that dislike crowding. The periodic problem for single degenerate parabolic equations has been already considered in a series of papers including the significant results in [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30] under different kinds of degeneracy for the elliptic operator.

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Of more importance to our situation are the problems studied in [31, 32, 33, 34, 35]. A porous media type model (see the monographs [36, 37] for detailed references on this equation and on degenerate parabolic equations) is studied in [31, 32] were a single degenerate equation featuring a nonlocal term has been considered and the existence of a nonnegative nontrivial periodic solution has been proved. Nonlocal terms are a way to express that the evolution of a population in a point of space does not depend only on nearby density but also on the total amount of population (see [38]). Analogous results are shown in [33, 34] for p-Laplacean and in [35] for a doubly degenerate parabolic equation. In [33, 38, 39, 40, 41] an optimization problem is also considered in which some of the coefficient functions are taken as control parameters and the existence of an optimal control minimizing (or maximizing) a suitable cost functional is proved. Concerning systems of degenerate parabolic equations, there are few papers dealing with the subject. In [42, 43] the existence, continuability and asymptotic behavior of the positive solutions of a system of possibly degenerate semilinear parabolic equations are considered under different boundary conditions. On the other hand, [41] deals with the positive periodic problem for a system of degenerate and delayed reaction-diffusion equations with non-local terms of the following form:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u^m = \left(f(x,t) + \displaystyle\int_\Omega [-K_1(\xi,t)u^2(\xi,t-\tau_1) + K_2(\xi,t-\tau_2)v^2] d\xi \right) u, \\ \dfrac{\partial v}{\partial t} - \Delta v^m = \left(g(x,t) + \displaystyle\int_\Omega [K_3(\xi,t)u^2(\xi,t-\tau_3) - K_4(\xi,t-\tau_4)v^2] d\xi \right) v. \end{cases} \tag{1}$$

In this paper we consider a predator-prey model that is described by the following system of degenerate parabolic equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nabla \cdot \left[\nabla u^m + 2\vec{b}u^\delta \right] = \left(f(x,t) - N_1(x,t)u - \dfrac{N_2(x,t)v}{K_0+u} - \displaystyle\int_\Omega K_1(\xi,t)u^2(\xi,t)d\xi \right) u, \\ \dfrac{\partial v}{\partial t} - \nabla \cdot \left[\nabla v^n + 2\vec{\beta}v^\gamma \right] = \left(\dfrac{K_3(x,t)u}{K_0+u} - K_4(x,t)v - \displaystyle\int_\Omega \left[K_2(\xi,t)v^2(\xi,t) + h\big(\xi, v_0(t)\big)v(\xi,t) \right] d\xi \right) v. \end{cases} \tag{2}$$

Here u and v denote respectively the densities of the prey and the predator, sharing the same territory Ω, which is a bounded domain of $\mathbb{R}^N$, $1 \leq N \leq 3$ (even if the dimension is mathematically irrelevant in the arguments we develop), with smooth boundary $\partial\Omega$. Beside the degeneracy of the elliptic operators, the model features the terms $2\vec{b}u^\delta, 2\vec{\beta}v^\gamma$ which, in a biological setting, simulate the effects of drifts within the system (caused, for example, by currents due either to natural fluid movements or by mechanical stirring of the system medium, when u, v represent populations living in water). We shall only consider the situation $1 \leq \delta \leq m$ and $1 \leq \gamma \leq n$. In particular, the cases $\delta = \gamma = 1$ with $m > 1, n > 1$ are allowed, corresponding to the Fokker-Plank equation situation with degenerate diffusion. The terms $N_1(x,t)u^2(x,t), K_4(x,t)v^2(x,t)$ represent the natural mortality based on the logistic equation (see [44]). The term $N_2(x,t)v(x,t)u(x,t)/[K_0 + u(x,t)]$ accounts for losses of preys due to the grazing by predators, while the symmetric term $K_3(x,t)u(x,t)v(x,t)/[K_0 + u(x,t)]$ measures the effect of predation on the growth rate of predators. The nonlocal terms $u(x,t)\int_\Omega K_1(\xi,t)u^2(\xi,t)d\xi$ and $v(x,t)\int_\Omega K_2(\xi,t)v^2(\xi,t)d\xi$ describe the effect of the competition for food among the members of each species through the weighted fraction of individuals that actually interact at time $t > 0$.

Moreover, the term $-[\int_\Omega h(\xi, v_0)v(\xi,t)d\xi]v(x,t)$ is added in the second equation to include a continuous harvesting effect on the predators in a region $\Omega' \subset \Omega$. This situation occurs, for instance, in the management of fisheries with v standing for the concentration of fish that are caught and u standing for the concentration of their (live) food. The harvesting is modeled as an increase in the death rate of the prey that depends non-locally on v in Ω' as we explain hereafter. First of all, a sample of the prey concentration $v_0(t) = \int_{\Omega''} v(x,t)dx/|\Omega''|$ is measured for $x_0 \in \Omega''$, where $\Omega'' \subset \Omega'$ is a small region. For simplicity, v can be considered constant in Ω'' so that $v_0(t) \triangleq v(x_0,t)$ for $x_0 \in \Omega''$. If $v_0(t)$ exceeds a suitable threshold $a > 0$ the harvesting of v in Ω' takes place at an intensity that increases linearly with $v_0(t)$ and reaches its maximum intensity, say $\omega > 0$, if $v_0(t)$ is larger than or equal to another suitable threshold $b > a$; if $v_0(t)$ is below a, no harvesting occurs. Therefore we consider the following expression for the function h:

$$h\big(\xi, v_0(t)\big) \triangleq \frac{\omega}{|\Omega'|}\chi_{\Omega'}(\xi)H\big(v_0(t)\big) \quad \text{with} \quad H(z) \triangleq \chi_{(a,b)}(z)\frac{z-a}{b-a} + \chi_{[b,+\infty)}(z),$$

where χ_A is the characteristic function of the set A.

Assuming $K_0 > 0$ and the T-periodicity of the functions f, N_j, K_i ($j = 1,2$ and $i = 1,2,3,4$), we provide conditions on system (2) ensuring the existence of a pair of non-trivial non-negative Hölder continuous functions (u,v) solving (2) in the weak sense, cf. ([36], Definition 5.4), and satisfying the following boundary conditions

$$\begin{cases} u(x,t) = v(x,t) = 0 & \text{for } (x,t) \in \partial\Omega \times (0,T], \\ u(x,0) = u(x,T) & \text{for } x \in \Omega, \\ v(x,0) = v(x,T) & \text{for } x \in \Omega. \end{cases} \tag{3}$$

One of our results, contained in Theorem 2.1, is as follows:

Theorem 1.1. *If $m,n > 1$, $m \geq \delta > (m+1)/2$, $n \geq \gamma > (n+1)/2$, f is non-negative non-trivial and N_1, K_3, K_4 are strictly positive, then system* (2) *has a pair of non-negative non-trivial periodic weak solutions (u,v) belonging to $C^{\alpha,\alpha/2}(\overline{Q}_T)$, for some $\alpha > 0$.*

Observe that unlike the $m = 1$ or $n = 1$ cases, we do not require a size estimate on $f, K_3, \vec{b}, \vec{\beta}$ for this result. Note also that we do not require the strict positivity of f, unlike earlier work, but we need the strict positivity of N_1, K_3, K_4. If δ, γ are "smaller", stricter conditions will be needed, as we show below. In particular, in some cases we shall require that $\vec{b}, \vec{\beta}$ vanish in some part of Ω for all t. Intuitively, in the case of species living in a lagoon, this corresponds to a small part of the lagoon being always stagnant. We also remark that our proofs will immediately lead, in some cases, to the existence of positive periodic solutions as a consequence of known results [45],[36].

We also point out that the presence of drift terms and the fact that the growth factor $K_3u/(K_0+u)$ of v depends on u make the analysis of system (2) significatively different and more difficult than that of (1) in [41]. In fact, looking at the conditions (b)–(d) in Theorem 2.1, when $2\delta \leq m+1$ in the prey equation we have to impose that the drift vector field $\vec{b}$ is small in some suitable sense with respect to the preys growth rate f in order to guarantee the non-triviality of the obtained solutions. On the other hand, when $2\gamma \leq n+1$ in the predators equation in (2) (see conditions (α)–(δ) in Theorem 2.1), the predators growth rate depends on the preys concentration u and, thus, we have to require that the drift vector field $\vec{\beta}$ is similarly smaller than $K_3u_*/(K_0+u_*)$ where u_* is any non-trivial and non-negative periodic solution of the uncoupled preys equation (i.e. with $v \equiv 0$). Part of the paper is then devoted to obtain some estimates on $K_3u_*/(K_0+u_*)$, since u_* is not known. The fact that our conditions involve integrals plays a critical role in these arguments. This differentiate our results from the preceding ones.

Our results are based on degree theory arguments, along the general lines of those used in [38]. We recall that the presence of nonlocal terms seems to render upper/lower solutions arguments difficult to apply as illustrated in [38]. Nevertheless we find cut-off arguments useful in our presentation.

In the last part of the paper we consider a cost functional associated to (2)–(3), which evaluates both the cost of controlling the growth rate of the prey u to the values $f(x,t)$, the grazing of v on u to the values $K_3(x,t)$, the cost and the benefit due to the harvesting of the predator. Thus Section 3 of the paper is devoted to the minimization of the considered cost functional. In fact, assuming as a control parameters the threshold b of the piecewise linear function H representing the characteristic of the harvesting, its maximum intensity per unit of time ω, the intrinsic growth rate $f(x,t)$ of the prey and the intensity of grazing $K_3(x,t)$ of v on u, we prove that the cost functional attains its minimum on the set of solutions to (2)–(3) corresponding to the control parameters.

2. Preliminaries and coexistence result

We assume that the domain $\Omega \subset \mathbb{R}^N$ is bounded, open and has smooth boundary and we set $Q_T = \Omega \times (0,T)$ for a fixed $T > 0$. The gradient ∇ and Laplacean Δ differential operators are always meant with respect to the space variable $x \in \mathbb{R}^N$ unless otherwise stated. All functions on the right hand sides of system (2) are assumed to be bounded and non-negative. In particular we will require N_1, K_4 to be strictly positive. Moreover, $\vec{b}, \vec{\beta} \in W^{1,\infty}(Q_T)$. The constants m, n are assumed bigger or equal to one. We observe that the left hand sides of system (2) are of the classical "porous media" type. There is a vast literature on this subject. We refer the interested reader to the book by Vazquez, [36], where many references can be found. We begin by collecting in Lemma 2.1 results that are either simple consequences or special cases of known results. As mentioned earlier, in Lemma 2.1 — and throughout the paper — by a solution we mean a weak generalized solution defined in the usual way ([36], Definition 5.4). For the reader's convenience, we sketch a brief proof.

Lemma 2.1.

(a) *Let* $0 \leq g \in L^\infty(Q_T)$ *and consider the Dirichlet periodic problem*

$$
\begin{cases}
\ell_1(w) \triangleq w_t - \nabla \cdot \left[A\nabla w + 2\vec{B}w\right] + cw = g & \text{in } Q_T \\
w = 0 & \text{on } \partial\Omega \times [0,T] \\
w(x,0) = w(x,T) & \text{in } \Omega
\end{cases}
\tag{4}
$$

with $0 < a \leq A(x,t)$ *for some constant* a, *and* $A, \vec{B}, c$ *in* $L^\infty(Q_T)$. *If* $\operatorname{ess\,inf} c > \|\vec{B}\|_\infty^2/a$, *then* (4) *has a unique solution* $w \geq 0$, *of class* $C^{\alpha,\alpha/2}(\overline{Q}_T)$ *for some* $\alpha > 0$.

(b) *Let* $\epsilon \geq 0$, $c \geq 0, 1 \leq q \leq p$ *with* $\vec{B}, g$ *smooth. There exists an upper solution* $z > 0$, *independent of* ϵ, c, *such that*

$$
\ell(z) \triangleq z_t - \nabla \cdot \left[\nabla(z^p + \epsilon z) + 2\vec{B}z^q\right] + cz \geq \|g\|_\infty \qquad \text{in } \overline{Q}_T,
\tag{5}
$$

and $z(x,0) = z(x,T)$.

Proof. (a) Existence is immediate from the usual Poincaré map argument, given that the condition on c ensures that the elliptic part of ℓ_1 is uniformly definite. These conditions also yield the uniqueness and the positivity of w. Indeed, let us consider the bilinear form

$$
\mathcal{B}(u,v) = \int_\Omega v\left\{cu - \nabla \cdot \left[A\nabla u + 2u\vec{B}\right]\right\}dx
$$

for functions satisfying Dirichlet boundary conditions on $\partial\Omega$. Straightforward computations show that

$$
\begin{aligned}
\mathcal{B}(u,u) &= \int_\Omega \left(cu^2 + A|\nabla u|^2 + 2u\vec{B}\nabla u\right) \\
&\geq (\operatorname{ess\,inf} c)\|u\|_2^2 - 2\|\vec{B}\|_\infty\|u\|_2\|\nabla u\|_2 + a\|\nabla u\|_2^2 \\
&\geq \left(a - \frac{\|\vec{B}\|_\infty^2}{\operatorname{ess\,inf} c}\right)\|\nabla u\|_2^2
\end{aligned}
$$

and, in particular, $\mathcal{B}(u,u) = 0$ if and only if $u \equiv 0$. Now, if w solves $\ell_1(w) = 0$, then by T-periodicity we get

$$
0 = \iint_{Q_T} w\ell_1(w) = \int_0^T \mathcal{B}(w,w),
$$

and, thus, $w \equiv 0$ and the uniqueness of the solution of $\ell_1(w) = g$ follows. Moreover, if $\ell_1(w) = g \geq 0$, then we obtain again by T-periodicity

$$
0 \leq \iint_{Q_T} gw^- = \iint_{Q_T} \ell_1(w)w^- = \int_0^T \mathcal{B}(w,w^-) = \int_0^T \mathcal{B}(-w^-,w^-) = -\int_0^T \mathcal{B}(w^-,w^-) \leq 0
$$

and, hence, $w \geq 0$. Here $w^- = \max\{0,-w\}$. Finally the regularity of w is a consequence of classical results: [46], [47].

(b) We consider explicitly the case $p > 1$, since the case $p = 1$ is well known. It is useful to first introduce a comparison function $v(x)$ given by $v(x) = C(1 - x_1^{-D})$, where without loss of generality, we may assume that if $x = (x_1, \ldots, x_N) \in \overline{\Omega}$ then $1 < \lambda \leq x_1 \leq \Lambda$ for some constants λ, Λ. We choose C, D to ensure:

$$
\begin{cases}
-\nabla \cdot \left[\nabla v + \epsilon\nabla v^{1/p} + 2\vec{B}v^{q/p}\right] \geq \|g\|_\infty \\
v > 0
\end{cases}
\tag{6}
$$

since $c \geq 0$. Observing that $-\nabla \cdot [\nabla v^{1/p}] \geq 0$ by direct calculation, it suffices that

$$
C\left\{D(D+1) + \left[\left(-\nabla \cdot 2\vec{B}\right)C^{q/p-1}\left(1 - x_1^{-D}\right)^{q/p}\right]x_1^{D+2} - \frac{2q}{p}|\vec{B}|v^{q/p-1}Dx_1\right\} \geq \|g\|_\infty x_1^{D+2}.
$$

We first choose D such that

$$
D > \frac{2q}{p}\|\vec{B}\|_\infty\Lambda\left(1 - \lambda^{-D}\right)^{q/p-1} + 1.
$$

Next, taking into account that $q/p - 1 < 0$ and $p > 1$, we choose $C_0 \geq 1$ sufficiently large so that:

$$
\left[-2\|\nabla \cdot \vec{B}\|_\infty C_0^{q/p-1}\left(1 - \Lambda^{-D}\right)^{q/p}\right]x_1^{D+2} > -1.
$$

Finally, we choose $C \geq C_0$ such that:

$$C > \frac{\|g\|_\infty x_1^{D+2}}{2D-1}$$

whence estimate (6) follows.

Putting $v = z^p$ then yields (5). $\square$

We now employ Lemma 2.1 to obtain the coexistence result. We have:

Theorem 2.1. *Assume $m, n \geq 1$ and that N_1, K_4 are strictly positive. Then system (2) has a solution (u, v) in Q_T, with non-trivial, non-negative functions u, v belonging to a space $C^{\alpha,\alpha/2}(\overline{Q}_T)$ determined by the problem data, if two of the following conditions are satisfied: one for the first (prey) equation of system (2), the other for the second (predator) equation of system (2). These are as follows:*

- *Conditions on the first equation:*
  - (a) *If $2m \geq 2\delta > m+1$ then $f \geq 0$ non-trivial in $\overline{Q}_T$.*
  - (b) *If $2\delta < m+1$ then there exists $\Omega' \subset\subset \Omega$ such that $\vec{b} = \vec{0}$ and f non-trivial in $\Omega' \times (0, T)$.*
  - (c) *If $2\delta = m+1$ with $m > 1$, then there exists $\Omega' \subset\subset \Omega$ in which $\int_0^T \left[f - \delta^2 \frac{|\vec{b}|^2}{m}\right] > 0$.*
  - (d) *If $m = \delta = 1$, then the least eigenvalue μ_1 of the elliptic Dirichlet problem:*

$$\begin{cases} -\Delta\tau + \left[\dfrac{1}{T}\displaystyle\int_0^T \left(|\vec{b}|^2 - \nabla \cdot \vec{b} - f\right)\right]\tau = \mu_1 \tau & \text{in } \Omega \\ \tau = 0 & \text{on } \partial\Omega \end{cases}$$

    is negative.
- *Conditions on the second equation:*
  - (α) *If $2n \geq 2\gamma > n+1$, then $K_3 u_* \geq 0$ is non-trivial.*
  - (β) *If $2\gamma < n+1$, then there exists an open set $\Omega'' \subset \Omega$ such that $\vec{\beta} = \vec{0}$ and $K_3 u_*$ non-trivial in $\Omega'' \times (0, T)$.*
  - (γ) *If $2\gamma = n+1$ with $n > 1$, then there exists an open set $\Omega'' \subset\subset \Omega$ in which*

$$\int_0^T \left[\frac{K_3 u_*}{K_0 + u_*} - \gamma^2 \frac{|\vec{\beta}|^2}{n}\right] > 0.$$

  - (δ) *If $n = \gamma = 1$, then the least eigenvalue λ_1 of the elliptic Dirichlet problem:*

$$\begin{cases} -\Delta\tau + \left[\dfrac{1}{T}\displaystyle\int_0^T \left(|\vec{\beta}|^2 - \nabla \cdot \vec{\beta} - \dfrac{K_3 u_*}{K_0 + u_*}\right)\right]\tau = \lambda_1 \tau & \text{in } \Omega \\ \tau = 0 & \text{on } \partial\Omega \end{cases}$$

    is negative.

Here, by u_* we denote any non-trivial solution of the first equation of the decoupled system (2), i.e. with $v \equiv 0$.

Before passing to the proof, we comment that conditions (a)–(d) are sufficient for a non-trivial solution u_* to exist, as can be seen by applying the arguments of the proof of Theorem 2.1 only to the first equation of (2). Furthermore, condition (d) holds if, for example,

$$\int_\Omega \left[\frac{1}{T}\int_0^T \left(|\vec{b}|^2 - \nabla \cdot \vec{b} - f\right) + \nu_1\right] \phi_1^2 < 0$$

where ν_1, ϕ_1 denote the least eigenvalue, eigenvector of

$$\begin{cases} -\Delta\phi = \nu\phi & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

An identical remark applies to condition (δ). Observe, as well, that conditions $(c), (d), (\gamma), (\delta)$ involve the time averages of $\vec{b}, f, \vec{\beta}, K_3 u_*$. This allows for considerable variations in time of those coefficients, as long as their averages are suitable.

Proof. We modify and regularize system (2) in the following way

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nabla \cdot \left[\nabla (u^m + \epsilon u) + 2\vec{b} u^\delta \right] + cu = \left(f - N_1 u - \dfrac{N_2 v}{K_0 + u} - \displaystyle\int_\Omega K_1 u^2 d\xi + c \right) u + \epsilon \triangleq F_1(u, v, \epsilon), \\ \dfrac{\partial v}{\partial t} - \nabla \cdot \left[\nabla (v^n + \epsilon v) + 2\vec{\beta} v^\gamma \right] + cv = \left(\dfrac{K_3 u}{K_0 + u} - K_4 v - \displaystyle\int_\Omega (K_2 v^2 + hv) d\xi + c \right) v + \epsilon \triangleq F_2(u, v, \epsilon). \end{cases} \tag{7}$$

with $0 < \epsilon \leq 1$ constant, and $c \geq 0$ to be chosen. Next, we apply Lemma 2.1(b) to the first equation in (7) — with the choices $p = m$, $q = \delta$, $\vec{B} = \vec{b}$ and $g = \|f^2/(4N_1)\|_\infty + 1$ — and construct the upper solution z_1 to the first equation in (7). In the same way, we apply lemma 2.1(b) to the second equation of (7) — with $p = n$, $q = \gamma$, $\vec{B} = \vec{b}$ and $g = \|K_3^2/(4K_4)\|_\infty + 1$ — to obtain z_2.

For any function $\xi : Q_T \to \mathbb{R}$ put:

$$\tilde{\xi} \triangleq \begin{cases} 0 & \text{if } \xi \leq 0 \\ \xi & \text{if } 0 < \xi \leq z_1 \\ z_1 & \text{if } \xi > z_1 \end{cases} \quad \text{and} \quad \hat{\xi} \triangleq \begin{cases} 0 & \text{if } \xi \leq 0 \\ \xi & \text{if } 0 < \xi \leq z_2 \\ z_2 & \text{if } \xi > z_2 \end{cases}$$

and choose

$$c \geq \sup_{0 \leq \mu \leq \|z_1\|_\infty} \left[\frac{\|\vec{b}\|_\infty^2 \mu^{2\delta - 2}}{\mu^{m-1} + \epsilon} \right] + \sup_{0 \leq \mu \leq \|z_2\|_\infty} \left[\frac{\|\vec{\beta}\|_\infty^2 \mu^{2\gamma - 2}}{\mu^{n-1} + \epsilon} \right]. \tag{8}$$

We next consider the linear system

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \nabla \cdot \left[\left(m(\tilde{\xi}_1)^{m-1} + \epsilon \right) \nabla u_1 + 2\vec{b} (\tilde{\xi}_1)^{\delta - 1} u_1 \right] + cu_1 = F_1(\tilde{\xi}_1, \hat{\xi}_2, \epsilon), \\ \dfrac{\partial v_1}{\partial t} - \nabla \cdot \left[\left(n(\hat{\xi}_2)^{n-1} + \epsilon \right) \nabla v_1 + 2\vec{\beta} (\hat{\xi}_2)^{\gamma - 1} v_1 \right] + cv_1 = F_2(\tilde{\xi}_1, \hat{\xi}_2, \epsilon) \end{cases} \tag{9}$$

subject to Dirichlet and periodic conditions. Observe that given (ξ_1, ξ_2) we can find solutions (u_1, v_1) as a consequence of the choice of c by Lemma 2.1(a). We thus establish a map $(\xi_1, \xi_2) \mapsto (u_1, v_1)$ between $C^{\alpha, \alpha/2}$ spaces — for some $\alpha > 0$ — which is continuous and completely continuous by suitable choices of $\alpha > 0$ (again by Lemma 2.1(a)). It follows that system (9) has a fixed point (u_1, v_1) by the usual Degree Theory Arguments.

Next, we claim that $\tilde{u}_1 = u_1$ and $\hat{v}_1 = v_1$. Indeed, let us consider the difference between the first equation in (9) and the equation satisfied by z_1 written as:

$$(z_1)_t - \nabla \cdot \left[\nabla (z_1^m + \epsilon z_1) + 2\vec{b} z_1^\delta \right] + cz_1 \geq \|f^2/(4N_1)\|_\infty + 1 + cz_1 \,.$$

Multiplying the obtained inequality by $(u_1 - z_1)^+$, integrating over Q_T and recalling that $\tilde{u}_1 = z_1$ wherever $(u_1 - z_1)^+ \neq 0$, we have

$$\iint_{Q_T} \{ (m z_1^{m-1} + \epsilon) |\nabla (u_1 - z_1)^+|^2 + 2 z_1^{\delta - 1} (u_1 - z_1)^+ \vec{b} \cdot \nabla (u_1 - z_1)^+ + c[(u_1 - z_1)^+]^2 \}$$
$$\leq - \iint_{Q_T} \left(N_1 z_1^2 - f z_1 - \epsilon + \frac{f^2}{4N_1} + 1 \right) (u_1 - z_1)^+$$

by T-periodicity and boundary conditions. Now,

$$- \iint_{Q_T} \left(N_1 z_1^2 - f z_1 - \epsilon + \frac{f^2}{4N_1} + 1 \right) (u_1 - z_1)^+ \leq - \iint_{Q_T} N_1 \left(z_1 - \frac{f}{2N_1} \right)^2 (u_1 - z_1)^+ \leq 0$$

and

$$\begin{aligned} 0 &\geq \iint_{Q_T} \{ (m z_1^{m-1} + \epsilon) |\nabla (u_1 - z_1)^+|^2 + 2 z_1^{\delta - 1} (u_1 - z_1)^+ \vec{b} \cdot \nabla (u_1 - z_1)^+ + c[(u_1 - z_1)^+]^2 \} \\ &\geq \iint_{Q_T} \{ (m z_1^{m-1} + \epsilon) |\nabla (u_1 - z_1)^+|^2 - 2 \|\vec{b}\|_\infty z_1^{\delta - 1} (u_1 - z_1)^+ |\nabla (u_1 - z_1)^+| + c[(u_1 - z_1)^+]^2 \} \\ &\geq \iint_{Q_T} \left(c - \frac{\|\vec{b}\|_\infty z_1^{2\delta - 2}}{m z_1^{m-1} + \epsilon} \right) [(u_1 - z_1)^+]^2. \end{aligned}$$

Therefore we obtain that $(u_1 - z_1)^+ \equiv 0$ thanks to the choice of c in (8). In an analogous way it is possible to deduce that $(v_1 - z_2)^+ \equiv 0$.

We thus have found solutions (u_ϵ, v_ϵ) of (7) that are nonnegative nontrivial, due to: $F_1 \geq \epsilon > 0$, $F_2 \geq \epsilon > 0$, and are uniformly bounded by z_1, z_2 respectively, that is: independently of ϵ for $0 < \epsilon \leq 1$. By a standard argument (see also the proof of [41, Theorem 2.1] for a similar situation) it is possible to show that (u_ϵ, v_ϵ) converges to a T-periodic non-negative solution (u, v) of (2) as ϵ goes to zero along a suitable sequence. We just mention how to deal with the presence of drift terms in (7). If we multiply by $\phi \in C_c^\infty(\Omega)$ the first equation of (7), integrating over Q_T and using the Banach-Steinhaus theorem we get

$$\iint_{Q_T} |\nabla u_\epsilon^m|^2 dxdt \leq M$$

for any ϵ and some $M > 0$. Therefore $\nabla u_\epsilon^m + 2\vec{b}u_\epsilon^\delta \rightharpoonup \nabla u^m + 2\vec{b}u^\delta$ weakly in $L^2(Q_T)$ as $\epsilon \to 0$ along a suitable sequence. The same procedure and conclusion apply to the second equation of (7).

We now show that both components of (u, v) are non-trivial. For convenience we note that the first equation of (7) can be written as

$$\frac{\partial u_\epsilon}{\partial t} - \nabla \cdot \Big[\nabla(u_\epsilon^m + \epsilon u_\epsilon) + 2\vec{b}u_\epsilon^\delta\Big] + cu_\epsilon = \big(f - O(u_\epsilon, v_\epsilon) + c\big)u_\epsilon + \epsilon$$

where $O(u_\epsilon, v_\epsilon) \to 0$ as $(u_\epsilon, v_\epsilon) \to 0$. Next we let $\epsilon \to 0$ and observe that we need only show that $u_\epsilon, v_\epsilon \not\to 0$ pointwise as $\epsilon \to 0$. We claim that $u_\epsilon \not\to 0$, for otherwise we first note that $v_\epsilon \not\to 0$. Indeed, if we assume that both $u_\epsilon \to 0$ and $v_\epsilon \to 0$, we choose a nontrivial $C_0^\infty(\Omega)$ function $\phi(x)$, further specified below. We then observe: Let $w = u_\epsilon + \epsilon^2 > 0$. Completing the square shows:

$$0 \leq (mu_\epsilon^{m-1} + \epsilon)w^2 \left|\nabla\left(\frac{\phi}{w}\right)\right|^2 - 2\delta\phi w u_\epsilon^{\delta-1}\vec{b} \cdot \nabla\left(\frac{\phi}{w}\right) + \frac{\delta^2|\vec{b}|^2 u_\epsilon^{2(\delta-1)}\phi^2}{mu_\epsilon^{m-1} + \epsilon}\,.$$

Integrating over Q_T and expanding gives:

$$\begin{aligned}
0 \leq& \iint_{Q_T} \left[(mu_\epsilon^{m-1} + \epsilon)|\nabla\phi|^2 - 2\delta\phi u_\epsilon^{\delta-1}\vec{b} \cdot \nabla\phi + \delta^2|\vec{b}|^2 \frac{u_\epsilon^{2(\delta-1)}}{mu_\epsilon^{m-1} + \epsilon}\phi^2\right] \\
&- \iint_{Q_T} \frac{\phi^2}{w}\Big[-\nabla \cdot \big[(mu_\epsilon^{m-1} + \epsilon)\nabla w\big] - 2\delta u_\epsilon^{\delta-1}\vec{b} \cdot \nabla w\Big] \\
=& \, I_1 - I_2
\end{aligned}$$

Now we observe:

$$I_2 = \iint_{Q_T} \frac{\phi^2}{w}\Big[-\nabla \cdot \big[(mu_\epsilon^{m-1} + \epsilon)\nabla u_\epsilon\big] - 2\delta u_\epsilon^{\delta-1}\vec{b} \cdot \nabla u_\epsilon\Big]$$

and thus, formally:

$$I_2 = \iint_{Q_T} \frac{\phi^2}{u_\epsilon + \epsilon^2}\left[-\frac{\partial u_\epsilon}{\partial t} + \big(f - O(u_\epsilon, v_\epsilon)\big)u_\epsilon + \epsilon + 2u_\epsilon^\delta \nabla \cdot \vec{b}\right].$$

We note that by periodicity and Steklov averages:

$$\iint_{Q_T} \frac{\phi^2}{u_\epsilon + \epsilon^2}\frac{\partial u_\epsilon}{\partial t} = \int_\Omega \phi^2 \left\{\ln[u_\epsilon + \epsilon^2]\Big|_0^T\right\} = 0.$$

Thus

$$I_2 = \iint_{Q_T} \frac{\phi^2}{u_\epsilon + \epsilon^2}\left\{\big(f - O(u_\epsilon, v_\epsilon)\big)(u_\epsilon + \epsilon^2) + \epsilon\left[1 - \epsilon\big(f - O(u_\epsilon, v_\epsilon)\big)\right] + 2u_\epsilon^\delta \nabla \cdot \vec{b}\right\}.$$

Passing to the limit as $\epsilon \to 0$ we obtain, by obvious choice of ϕ, an immediate contradiction to $I_1 - I_2 \geq 0$ due to condition (a) or (b) accordingly to the case $2\delta > m + 1$ or $2\delta < m + 1$. If $2\delta = m + 1$ and $m > 1$, then observe that $\delta > 1$ implies $u_\epsilon^\delta/(u_\epsilon + \epsilon^2) \leq u_\epsilon^{\delta-1} \to 0$, and we have a contradiction to $I_1 - I_2 \geq 0$ due to condition (c). Finally, if $m=\delta=1$ then

$$
\begin{aligned}
0 \leq & I_1 - I_2 \\
= & \iint_{Q_T} \left[(1+\epsilon)|\nabla\phi|^2 - 2\phi\vec{b}\cdot\nabla\phi + \frac{|\vec{b}|^2\phi^2}{1+\epsilon}\right] - \iint_{Q_T} \phi^2\left[f + 2\nabla\cdot\vec{b} - O(u_\epsilon, v_\epsilon)\right] \\
& - \iint_{Q_T} \frac{\epsilon\phi^2}{u_\epsilon+\epsilon^2}\left[1-\epsilon\big(f+2\nabla\cdot\vec{b} - O(u_\epsilon,v_\epsilon)\big)\right] \\
= & \iint_{Q_T} \left[(1+\epsilon)|\nabla\phi|^2 + \phi^2\left(\frac{|\vec{b}|^2}{1+\epsilon} - \nabla\cdot\vec{b} - f + O(u_\epsilon, v_\epsilon)\right)\right] \\
& - \iint_{Q_T} \frac{\epsilon\phi^2}{u_\epsilon+\epsilon^2}\left[1-\epsilon\big(f+2\nabla\cdot\vec{b} - O(u_\epsilon,v_\epsilon)\big)\right].
\end{aligned}
$$

Since the last integral is positive for ϵ small enough, condition (d) gives the needed contradiction to the non-negativity of $I_1 - I_2$ as $\epsilon \to 0$ if ϕ is chosen to be an eigenfunction relative to μ_1.

Next we observe that if $v_\epsilon \not\to 0$ then $u_\epsilon \not\to 0$, since otherwise we would have $v_\epsilon \to v_* > 0$ and $u_\epsilon \to 0$ along a subsequence and direct integration on Q_T of the second equation formally gives as $\epsilon \to 0$:

$$
\iint_{Q_T} \frac{\partial v_*}{\partial t} - \int_0^T \int_{\partial\Omega} \frac{\partial v_*^n}{\partial n} + \iint_{Q_T} K_4 v_*^2 \leq 0.
$$

The first integral is zero by periodicity, the second non-negative by the boundary condition and the non-negativity of v_*. We then obtain the contradiction $\iint_{Q_T} K_4 v_*^2 \leq 0$.

Finally suppose $v_\epsilon \to 0$. Then $u_\epsilon \to u_* \geq 0$ a non-trivial periodic solution of the first (decoupled) equation, and we need only repeat the earlier arguments with $f, \vec{b}, m, \delta$ replaced by $K_3 u_*/(K_0+u_*), \vec{\beta}, n, \gamma$ respectively.

We thus obtained the existence of solutions (u,v). That these belong in some $C^{\alpha,\alpha/2}$ is a consequence of the results of Porzio and Vespri, [48], due to the boundedness of (u,v) (by z_1, z_2). □

We observe that some of the conditions in Theorem 2.1 — namely, those on the second equation — involve the unknown function u_*. A first obvious way to eliminate the dependence on u_* in conditions (α) and (β) is to assume that $K_3 > 0$ in Q_T. If we knew that each non-trivial and non-negative T-periodic solution u_* of the first uncoupled equation is positive everywhere inside Q_T, conditions (α) and (β) could be further simplified by directly dropping u_*. However, the strong maximum principle does not hold in general here due to the degeneracy of the elliptic part of the differential operator and it cannot be used to show that a non-trivial non-negative solution is actually positive [36, §1.2]. On the other hand, the support of a non-negative solution of the porous medium equation without lower order terms expands in time and includes in finite time every point of the domain Ω [36, Theorem 14.3]. This fact and the T-periodicity of a solution are now enough to guarantee that a T-periodic non-negative non-trivial solution of the porous medium equation without lower order terms must be positive in Q_T. As for analogous properties in presence of lower order terms, we mention that there are recent papers dealing with the subject which seems to be still under investigation. In particular, in [49] an intrinsic Harnack estimate is proved for degenerate parabolic equations with degeneracy of two types: p-Laplacean and porous media (and it has been extended to doubly degenerate equations in [50]). The results in those papers, on the one hand, imply that, if $u(x_0,t_0) > 0$, then $u(x_0,t) > 0$ for all $t > t_0$ and, on the other, have been used in [51, 52] to show that the expansion of the positivity set does occur also with lower order terms in the case of p-Laplace type of degenerate equations. The validity of the analogous result for the porous medium equations still seems to be an open problem.

However, it is sometimes possible to eliminate the dependence on u_* in (α)–(γ) and obtain explicit — albeit somewhat crude — sufficient conditions for coexistence. We illustrate this remark by considering condition (γ) in Theorem 2.1, and recall that $0 \leq u_* \leq z_1$ by construction, with z_1 a super-solution that can be explicitly estimated in terms of the problem data. We then have:

Corollary 2.1. *Let $\vec{b} \equiv \vec{0}$ and assume that $n > 1, 2\gamma = n+1$.*

1. *Assume $m > 1$ and let ϕ_1, ν_1 be the eigenvector/eigenvalue of $-\Delta$ as given after the statement of Theorem 2.1. Put:*

$$V(\eta) \triangleq (\eta + K_0) \int_0^\eta \frac{m\xi^{m-2}}{\xi + K_0} d\xi$$
$$R(x,t) \triangleq f(x,t) - \nu_1 V\big(z_1(x)\big) - \int_\Omega K_1(\xi,t) z_1^2(\xi) d\xi.$$

If:

$$\operatorname*{ess\,inf}_{Q_T} \frac{K_3}{N_1} \iint_{Q_T} \frac{R\phi_1}{z_1 + K_0} > \iint_{Q_T} \frac{\phi_1 \gamma^2 |\vec{\beta}|^2}{n},$$

then condition (γ) of Theorem 2.1 holds.

2. *Assume $m = 1$ and condition (d) of Theorem 2.1. If*

$$\frac{\gamma^2}{n} \iint_{Q_T} |\vec{\beta}|^2 \frac{(K_0 + z_1)(N_1 + |\Omega| K_1 z_1)}{K_3} < -\mu_1 \iint_{Q_T} \tau_1^2, \tag{10}$$

where μ_1, τ_1 are the principal eigenvalue/eigenvector in condition (d) with $\|\tau_1\|_\infty = 1$, then condition (γ) of Theorem 2.1 holds.

Proof. Observe first that z_1 depends on $\|f^2/N_1\|_\infty$, whence z_1 will be small if $N_1 \gg f^2$. Consequently the conditions of the Corollary are not void. Put $u_* = u$ in this proof for notational simplicity.

Case 1: $m > 1$. Choosing $\phi_1/[(u + K_0)(u + \epsilon)]$ as a test function in the equation for u yields

$$\iint_{Q_T} \nabla u^m \cdot \nabla \left[\frac{\phi_1}{(u + K_0)(u + \epsilon)} \right] = \iint_{Q_T} \left(f - N_1 u - \int_\Omega K_1 u^2 \right) \frac{u\phi_1}{(u + K_0)(u + \epsilon)}.$$

Now,

$$\begin{aligned}
\iint_{Q_T} \nabla u^m \cdot \nabla \left[\frac{\phi_1}{(u + K_0)(u + \epsilon)} \right] &= \iint_{Q_T} \frac{\nabla u^m \cdot \nabla \phi_1}{(u + K_0)(u + \epsilon)} - \iint_{Q_T} \frac{m\phi_1 u^{m-1}(2u + K_0 + \epsilon)}{(u + K_0)^2 (u + \epsilon)^2} |\nabla u|^2 \\
&\le \iint_{Q_T} \nabla \left[\int_0^u \frac{m\xi^{m-1}}{(\xi + K_0)(\xi + \epsilon)} d\xi \right] \cdot \nabla \phi_1 \\
&= \nu_1 \iint_{Q_T} \left[\int_0^u \frac{m\xi^{m-1}}{(\xi + K_0)(\xi + \epsilon)} d\xi \right] \phi_1.
\end{aligned}$$

Recalling $m > 1$ and letting $\epsilon \to 0$ gives

$$\nu_1 \iint_{Q_T} \left[\int_0^u \frac{m\xi^{m-2}}{\xi + K_0} d\xi \right] \phi_1 \ge \iint_{Q_T} \left(f - N_1 u - \int_\Omega K_1 u^2 \right) \frac{\phi_1}{u + K_0}.$$

Rearranging we obtain

$$\iint_{Q_T} \frac{N_1 u \phi_1}{u + K_0} \ge \iint_{Q_T} \frac{R\phi_1}{z_1 + K_0},$$

and, finally,

$$\iint_{Q_T} \frac{\phi_1 K_3 u}{K_0 + u} \ge \operatorname*{ess\,inf}_{Q_T} \frac{K_3}{N_1} \iint_{Q_T} \frac{R\phi_1}{z_1 + K_0} > \iint_{Q_T} \frac{\phi_1 \gamma^2 |\vec{\beta}|^2}{n}$$

and the first result follows.

Case 2: $m = 1$. Let $\phi \in C_c^\infty(\Omega)$ a non-negative function to be specified later. Using ϕ^2/u as a test function in the equation for u we obtain

$$\iint_{Q_T} \left(2\frac{\phi}{u} \nabla u \cdot \nabla \phi - \frac{\phi^2}{u^2} |\nabla u|^2 \right) = \iint_{Q_T} \left[f - N_1 u - \int_\Omega K_1 u^2 \right] \phi^2$$

and, therefore,

$$0 \le \iint_{Q_T} u^2 \left| \nabla \frac{\phi}{u} \right|^2 = \iint_{Q_T} |\nabla \phi|^2 - \iint_{Q_T} \left[f - N_1 u - \int_\Omega K_1 u^2 \right] \phi^2.$$

If we take any sequence $\phi_k \in C_c^\infty(\Omega)$ such that $\phi_k \to \tau_1$, then we have as $k \to +\infty$

$$\begin{aligned}
-\mu_1 \iint_{Q_T} \tau_1^2 &\leq \iint_{Q_T} \left[N_1 u + \int_\Omega K_1 u^2\right] \tau_1^2 \\
&\leq \iint_{Q_T} (N_1 u + |\Omega| K_1 u^2) \\
&\leq \iint_{Q_T} \frac{K_3 u}{K_0 + u} \frac{(K_0 + z_1)(N_1 + |\Omega| K_1 z_1)}{K_3}
\end{aligned}$$

and condition (γ) follows from (10). □

Remark 1. It is clear from the argument in case 2 that condition (10) can be replaced by the requirement that

$$\frac{\gamma^2}{n} \iint_{Q_T} |\vec{\beta}|^2 \frac{(K_0 + z_1)(N_1 + |\Omega| K_1 z_1)}{K_3} < \iint_{Q_T} (f\phi^2 - |\nabla \phi|^2)$$

for some $\phi \in C_0^1(\Omega)$ such that $\|\phi\|_\infty = 1$.

3. An optimization problem

In this section we aim at minimizing a cost function associated with (2)–(3). Similar problems were considered in [33, 38, 39, 40, 41]. Here, specifically, assuming as control parameters b and ω in the harvesting term $\int_\Omega h\big(\xi, v_0(t)\big) v(\xi, t) d\xi$ (where we recall that $v_0(t) \triangleq v(x_0, t) = v(x, t)$ with $x_0 \in \Omega''$ for all $x \in \Omega''$), the growth rate $f(x,t)$ of u and the intensity $K_3(x,t)$ of the grazing of v on u, we consider the cost functional $C : \big(C(\overline{Q}_T)\big)^2 \times \big(L^\infty(Q_T)\big)^2 \times (\mathbb{R}_+)^2 \to \mathbb{R}$ defined by

$$\begin{aligned}
C(u, v, f, K_3, b, \omega) \triangleq &\iint_{Q_T} \big[c_1\big(x, t, f(x,t)\big) + c_2\big(x, t, K_3(x,t)\big)\big] dxdt - \iint_{Q_T} g\big(x, t, v(x,t)\big) dxdt \\
&+ c_3(b, \omega) \iint_{Q_T} h\big(x, v_0(t)\big) v(x,t) dxdt,
\end{aligned}$$

where $c_1\big(x, t, f(x,t)\big)$ is the cost of controlling the intrinsic growth rate f of the prey to the values $f(x,t), c_2\big(x, t, K_3(x,t)\big)$ is the cost of controlling the intensity of the grazing of v on u to the values $K_3(x,t), c_3(b, \omega)$ is the unitary cost of the harvesting of the predator and g is the gain relative to the harvesting. We assume the continuity of the functions $c_i, i = 1, 2, 3,$ and g with respect to their arguments.

For given positive constants $r_1, r_2, b_0, b_1, \omega_0$ and ω_1 consider the following sets.

$$\begin{aligned}
U \triangleq &\big\{(f, K_3, b, \omega) \in \big(L^\infty(Q_T)\big)^2 \times (\mathbb{R}_+)^2 : 0 \leq f(x,t) \leq r_1, 0 \leq K_3(x,t) \leq r_2, \text{ for a.a. } (x,t) \in Q_T, \\
&\text{satisfying one of the conditions } (a)\text{-}(d) \text{ and } (\alpha)\text{-}(\delta) \text{ respectively, } a + b_0 \leq b \leq b_1,\ \omega_0 \leq \omega \leq \omega_1\big\}
\end{aligned}$$

and

$$\begin{aligned}
S \triangleq \big\{(u, v) : (u, v) \in \big(C^{\alpha, \alpha/2}(\overline{Q}_T)\big)^2 &\text{ is a solution of (2)–(3) corresponding to} \\
&(f, K_3, b, \omega) \in U \text{ with } u, v \geq 0 \text{ in } \overline{Q}_T,\ u, v \neq 0\big\}.
\end{aligned}$$

The set S is nonempty. In fact, for all $(f, K_3, b, \omega) \in U$, Theorem 2.1 ensures the existence of a T-periodic solution $(u, v) \in \big(C^{\alpha, \alpha/2}(\overline{Q}_T)\big)^2$ of (2)–(3) with $u \geq 0$ and $v \geq 0$ in $\overline{Q}_T$, $u, v \neq 0$ and α depending on $r_i, i = 1, 2,$ radius of the ball $B(0, r_i) \subset L^\infty(Q_T)$.

Fix $(f, K_3, b, \omega) \in U$ and consider the map $\psi_{(f, K_3, b, \omega)} : Q_T \times \mathbb{R}^6 \to \mathbb{R}^3$ given by

$$\begin{aligned}
&\psi_{(f, K_3, b, \omega)}(x, t, \alpha, \beta, \alpha_1, \beta_1, \alpha_2, \beta_2) \triangleq (y_1, y_2, y_3) \quad \text{with:} \\
&\qquad y_1 = \left(f(x,t) - N_1(x,t)\alpha - \frac{N_2(x,t)\beta}{K_0 + \alpha} - \alpha_1\right) \alpha, \\
&\qquad y_2 = \left(\frac{K_3(x,t)\alpha}{K_0 + \alpha} - K_4(x,t)\beta - \beta_1 - \alpha_2\right) \beta, \\
&\qquad y_3 = c_1\big(x, t, f(x,t)\big) + c_2\big(x, t, K_3(x,t)\big) + c_3(b, \omega)\Gamma(x, \beta_2, b, \omega)\beta - g(x, t, \beta),
\end{aligned}$$

where

$$\Gamma(x, \beta_2, b, \omega) \triangleq \chi_{\Omega'}(x) \frac{H(\beta_2, b)\omega}{|\Omega'|} \quad \text{and} \quad H(\beta_2, b) \triangleq \chi_{(a,b)}(\beta_2) \frac{\beta_2 - a}{b - a} + \chi_{[b, +\infty)}(\beta_2). \tag{11}$$

Consider now the multivalued map $\Psi : Q_T \times \mathbb{R}^8 \multimap \mathbb{R}^3$ defined as follows:

$\Psi(x,t,\alpha,\beta,\alpha_1,\beta_1,\alpha_2,\beta_2,b,\omega) \triangleq \{\psi_{(f,K_3,b,\omega)}(x,t,\alpha,\beta,\alpha_1,\beta_1,\alpha_2,\beta_2) : f \equiv \gamma_1 \in [0,r_1], K_3 \equiv \gamma_2 \in [0,r_2]\}$.

For $(u,v) \in \big(C(\overline{Q}_T)\big)^2$ and $(b,\omega) \in [a+b_0,b_1] \times [\omega_0,\omega_1]$ we put

$$\hat{\Psi}(u,v,b,\omega)(x,t) \triangleq \Psi\big(x,t,u(x,t),v(x,t),\phi_1(u)(t),\phi_2(v)(t),\phi_3(v)(t),\phi_4(v)(t),b,\omega\big)$$

for a.a. $(x,t) \in Q_T$, where $\phi_i(w)(t) = \int_\Omega K_i(x,t)w^2(x,t)dx$, $i=1,2$, $\phi_3(v)(t) = \int_\Omega h\big(x,v_0(t)\big)v(x,t)dx$, and $\phi_4(v)(t) = v_0(t)$. Let $\Sigma := \overline{S}$ where the closure is in the $C(\overline{Q}_T) \times C(\overline{Q}_T)$-topology. We are now in the position to prove the following result.

Theorem 3.1. *The cost functional C attains the minimum in a point $(u,v,f,K_3,b,\omega) \in \Sigma \times B(0,r_1) \times B(0,r_2) \times [a+b_0,b_1] \times [\omega_0,\omega_1]$, where (u,v) is a solution of (2)–(3) corresponding to (f,K_3,b,ω).*

Proof. First observe that $\inf_S C$ is finite. Let $\{(u_n,v_n,f_n,K_{3,n},b_n,\omega_n)\}$ be a minimizing sequence for the cost functional C, where $\{(u_n,v_n)\} \subset S$ and $\{(f_n,K_{3,n},b_n,\omega_n)\} \subset U$ is the corresponding sequence of control parameters. Define

$$\psi_n(x,t) \triangleq \psi_{(f_n,K_{3,n},b_n,\omega_n)}\big(x,t,u_n(x,t),v_n(x,t),\phi_1(u_n)(t),\phi_2(v_n)(t),\phi_3(v_n)(t),\phi_4(v_n)(t)\big).$$

Clearly, $\psi_n(x,t) \in \hat{\Psi}(u_n,v_n,b_n,\omega_n)(x,t)$ for a.a. $(x,t) \in Q_T$ and $\hat{\Psi}(u_n,v_n,b_n,\omega_n)(x,t)$ is a nonempty, compact, convex set in $\mathbb{R}^3$ for a.a. $(x,t) \in Q_T$ and any $n \in \mathbb{N}$.

Let $\mathcal{G} : \big(L^\infty(Q_T)\big)^3 \to \big(C(\overline{Q}_T)\big)^2 \times C([0,T])$ be the solution map defined as

$$\mathcal{G}(\hat{y}_1,\hat{y}_2,\hat{y}_3) = (u,v,z), \qquad \text{with } \psi = (\hat{y}_1,\hat{y}_2,\hat{y}_3),$$

if and only if

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nabla \cdot \Big[\nabla u^m + 2\vec{b}u^\delta\Big] = \hat{y}_1 & \text{in } Q_T \\ \dfrac{\partial v}{\partial t} - \nabla \cdot \Big[\nabla v^n + 2\vec{\beta}v^\gamma\Big] = \hat{y}_2 & \text{in } Q_T \\ u(\cdot,t)|_{\partial\Omega} = v(\cdot,t)|_{\partial\Omega} = 0, & \text{for a.a. } t \in (0,T), \\ u(x,0) = u(x,T), & \text{in } \Omega \\ v(x,0) = v(x,T), & \text{in } \Omega \end{cases} \tag{12}$$

and

$$\begin{cases} \dot{z}(t) = \int_\Omega \hat{y}_3(x,t)dx, & \text{a.e. in } [0,T], \\ z(0) = 0. \end{cases} \tag{13}$$

Since $\mathcal{G}$ is a compact map and the sequence $\{\psi_n\} \subset \big(L^\infty(Q_T)\big)^3$ is bounded, we have that $(u_n,v_n) \to (u,v)$, with $u,v \geq 0$ in $\overline{Q}_T$, and $z_n \to z$ in $\big(C(\overline{Q}_T)\big)^2$ and $C([0,T])$ respectively. Moreover $(b_n,\omega_n) \to (b,\omega) \in [a+b_0,b_1] \times [\omega_0,\omega_1]$. On the other hand, $\psi_n \rightharpoonup \psi$ weakly in $\big(L^2(Q_T)\big)^2 \times L^2\big((0,T)\big)$.

We now show that

$$\psi(x,t) \in \hat{\Psi}(u,v,b,\omega)(x,t)$$

for a.a $(x,t) \in Q_T$. By the weak convergence of ψ_n to ψ we have that

$$\limsup_{n\to\infty} \langle \eta, \psi_n(x,t)\rangle \geq \langle \eta, \psi(x,t)\rangle \geq \liminf_{n\to\infty} \langle \eta, \psi_n(x,t)\rangle$$

for all $\eta \in \mathbb{R}^3$ and for a.a. $(x,t) \in Q_T$. Therefore

$$\limsup_{n\to\infty} \Big[\sup\langle \eta, \hat{\Psi}(u_n,v_n,b_n,\omega_n)(x,t)\rangle\Big] \geq \langle \eta, \psi(x,t)\rangle \geq \liminf_{n\to\infty} \Big[\inf\langle \eta, \hat{\Psi}(u_n,v_n,b_n,\omega_n)(x,t)\rangle\Big].$$

By the continuity of $\hat{\Psi}(u,v,b,\omega)(x,t)$ with respect to $u(x,t),v(x,t),b,\omega$ we obtain

$$\sup\langle \eta, \hat{\Psi}(u,v,b,\omega)(x,t)\rangle \geq \langle \eta, \psi(x,t)\rangle \geq \inf\langle \eta, \hat{\Psi}(u,v,b,\omega)(x,t)\rangle,$$

for any $\eta \in \mathbb{R}^3$ and for a.a. $(x,t) \in Q_T$. By the convexity of the set $\hat{\Psi}(u,v,b,\omega)(x,t)$ it follows that

$$\psi(x,t) \in \hat{\Psi}(u,v,b,\omega)(x,t)$$

for a.a. $(x,t) \in Q_T$. By the measurable selection Theorem, see e.g. [53], there exist $(f,K_3) \in B(0,r_1) \times B(0,r_2) \subset \big(L^\infty(Q_T)\big)^2$ such that

$$\psi(x,t) = \psi_{(f,K_3,b,\omega)}\big(x,t,u(x,t),v(x,t),\phi_1(u)(t),\phi_2(v)(t),\phi_3(v)(t),\phi_4(v)(t)\big). \tag{14}$$

Arguing as in the proof of Theorem 2.1 we get that $\nabla u_n^m + 2\vec{b}u_n^\delta \rightharpoonup \nabla u^m + 2\vec{b}u^\delta$ and $\nabla v_n^m + 2\vec{b}v_n^\delta \rightharpoonup \nabla v^m + 2\vec{b}v^\delta$ weakly in $L^2(Q_T)$. Moreover, the inverse operator of the left hand side of (13) is weakly continuous in $L^2\big((0,T)\big)$. In conclusion, $(u,v) \in \Sigma$ is the weak solution of (12) and z is the solution of (13), where $(\hat{y}_1, \hat{y}_2, \hat{y}_3)$ are the components of the function ψ given in (14). Hence, (u,v) is a solution of (2)–(3), with $u, v \geq 0$ in $\overline{Q}_T$ and $(u,v) \in \big(C^{\alpha,\alpha/2}(\overline{Q}_T)\big)^2$, corresponding to the control parameters (f, K_3, b, ω) and

$$z(T) = C(u, v, f, K_3, b, \omega) = \inf_S C.$$

□

Observe that the arguments used in the proof of Theorem 3.1 when applied to the two first components of the vector multivalued function $\hat{\Psi}(u, v, b, \omega)$ also show that each pair $(u,v) \in \Sigma \setminus S$ is a solution of (2)–(3) corresponding to some control parameters $(f, K_3, b, \omega) \in B(0, r_1) \times B(0, r_2) \times [a + b_0, b_1] \times [\omega_0, \omega_1]$. Furthermore in Theorem 3.1 the minimum of the cost functional C could be attained at a point $(u,v) \in \Sigma$ for which at least one element of the pair (u,v) is zero. This is due to the fact that the control set U is not closed with respect to the multivalued approach used in the proof of Theorem 3.1. If it is of interest to avoid such a situation, namely to have the minimum of C at a point $(u, v, f, K_3, b, \omega)$ with u and v different from zero, then we must strengthen the conditions of Theorem 2.1. For instance, we can replace conditions (a) and (α) by

- (a') If $2m \geq 2\delta > m + 1$ then there exists $Q' \subset Q_T$ such that $f \geq f_0$ in Q' for some positive constant f_0.
- (α') If $2n \geq 2\gamma > n + 1$ then there exists $Q'' \subset Q_T$ such that $K_3 \geq K_{3,0}$ in Q'' for some positive constant $K_{3,0}$.

Defining the control set $\tilde{U}$ as

$$\tilde{U} \triangleq \{(f, K_3, b, \omega) \in \big(L^\infty(Q_T)\big)^2 \times (\mathbb{R}_+)^2 : 0 \leq f(x,t) \leq r_1, 0 \leq K_3(x,t) \leq r_2, \text{for a.a. } (x,t) \in Q_T \\ \text{satisfying } (a') \text{ and } (\alpha') \text{ respectively}, a + b_0 < b < b_1,\ \omega_0 \leq \omega \leq \omega_1\},$$

we can proceed as in the proof of Theorem 3.1 to obtain that the measurable selections f and K_3 of the limit point of the minimizing sequence belong to $\tilde{U}$, thus the cost functional C assumes the minimum in the corresponding solution set $\tilde{S}$.

The same considerations apply to the other possible cases of Theorem 2.1 by replacing the conditions therein with the more restrictive corresponding pointwise conditions. In particular, conditions like those of Corollary 2.1 become essential in order to define explicitly the analogue of the control set $\tilde{U}$ in case of $2\gamma = n + 1$.

Acknowledgements

G. Fragnelli, P. Nistri and D. Papini were supported by PRIN 2008 "Controllo Nonlineare: metodi geometrici e applicazioni".

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