Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

NON-TRIVIAL NON-NEGATIVE PERIODIC SOLUTIONS OF A SYSTEM OF DOUBLY DEGENERATE PARABOLIC EQUATIONS WITH NONLOCAL TERMS

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(Communicated by the associate editor name)

Abstract. The aim of the paper is to provide conditions ensuring the existence of non-trivial non-negative periodic solutions to a system of doubly degenerate parabolic equations containing delayed nonlocal terms and satisfying Dirichlet boundary conditions. The employed approach is based on the theory of the Leray-Schauder topological degree theory, thus a crucial purpose of the paper is to obtain a priori bounds in a convenient functional space, here $L^2(Q_T)$, on the solutions of certain homotopies. This is achieved under different assumptions on the sign of the kernels of the nonlocal terms. The considered system is a possible model of the interactions between two biological species sharing the same territory where such interactions are modeled by the kernels of the nonlocal terms. To this regard the obtained results can be viewed as coexistence results of the two biological populations under different intra and inter specific interferences on their natural growth rates.

1. **Introduction.** In this paper we consider a system of doubly degenerate parabolic equations with delayed nonlocal terms and Dirichlet boundary conditions of the form

$$\begin{cases} \ell^{m,p}[u] = \left[a - \int_\Omega K_1(\xi,t)u^2(\xi,t-\tau_1)d\xi + \int_\Omega K_2(\xi,t)v^2(\xi,t-\tau_2)d\xi\right] u & \text{in } Q_T, \\ \ell^{n,q}[v] = \left[b + \int_\Omega K_3(\xi,t)u^2(\xi,t-\tau_3)d\xi - \int_\Omega K_4(\xi,t)v^2(\xi,t-\tau_4)d\xi\right] v & \text{in } Q_T, \\ u(x,t) = v(x,t) = 0, \quad \text{for } (x,t) \in \partial\Omega \times (0,T), \\ u(\cdot,0) = u(\cdot,T) \text{ and } v(\cdot,0) = v(\cdot,T), \end{cases} \tag{1}$$

and we look for continuous weak solutions. Here $\ell^{m,p}[u] := u_t - \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m)$ ($\ell^{n,q}[v]$ is similarly defined), Ω is an open bounded domain of $\mathbb{R}^N$ with smooth

2000 *Mathematics Subject Classification.* Primary: 35K65, 35B10; Secondary: 47H11.

Key words and phrases. Doubly degenerate parabolic equations, non-negative periodic solutions, topological degree.

Research supported by the grant P.R.I.N. 2008 "Controllo Nonlineare: metodi geometrici e applicazioni".

boundary $\partial\Omega$, $Q_T := \Omega \times (0,T), T > 0$, $\tau_i \in (0,+\infty)$, $m,n > 1$, $s^m = |s|^{m-1}s$, $p,q > 2$, and $K_i, a, b \in L^\infty(Q_T), i = 1,2,3,4$, are extended to $\Omega\times\mathbb{R}$ by T-periodicity.

Let $A[u] := \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m)$ and observe that

1. if $m = 1$ then $A[u] = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$,
2. if $m > 1$ and, if we set $l := (m-1)(p-1)$, the operator $A[u]$ becomes $m^{p-1}\operatorname{div}(|u|^l|\nabla u|^{p-2}\nabla u)$, which is the operator considered by Ivanov in [10] and in [11].

Following [10] and [11] we say that each of the equations of 1 is of

1. *slow diffusion* type if $m > \frac{1}{p-1}$,
2. *normal diffusion* type if $m = \frac{1}{p-1}$,
3. *fast diffusion* type if $m < \frac{1}{p-1}$.

Since we assume $p > 2$ and $m > 1$, only the slow diffusion occurs here. As pointed out in [22] for a single equation, with $m = 1$ and $p > 2$, the slow diffusion in the biological models is more realistic, since the speed of propagation of perturbations in the degenerate case is finite while in the non-degenerate case is infinite. Specifically, [22] deals with a periodic optimal control problem governed by a parabolic Volterra-Lotka type equation, where the Laplacian is replaced by the p-Laplacian with $p > 2$. Furthermore, in the case when $m = 1$, several authors, see [1] and [18], studied the existence of positive steady state solutions $u(x)$, $x \in \Omega \subset \mathbb{R}^N, N \geq 3$, for an equation involving the p-Laplacian and governing the population density of one biological species whose reproduction follows the logistic growth in presence of an harvesting term (see [6] for a system of two evolution equations). Parabolic systems with the p-Laplacian, $p > 1$, have been extensively studied in [4].

For problem 1 when $n = m > 1$ and $p = q = 2$ the authors in [5] proved the existence of non-negative periodic solutions (u,v), with $u \neq 0$ and $v \neq 0$, representing the population densities of two interacting biological species sharing the same territory $\Omega \subset \mathbb{R}^N$. This coexistence problem was studied under different conditions on the sign of the functions K_i, $i = 1,2,3,4$, which model the different kinds of interaction between the species, in particular if the species are cooperative, i.e. $K_2, K_3 \geq 0$, or competitive, i.e. $K_2, K_3 \leq 0$. In [5] the system is formed by two equations of the porous media type and some relevant results of the related theory, see [21], were employed. In the case of a single equation, with $m > 1$ and $p = 2$, Huang, Wang and Ke in [9] provided conditions to ensure the existence of non-trivial non-negative periodic solutions when the right hand side contains a coercive nonlocal term of general type. The same problem has been solved in [26] when $m = 1$ and $p > 2$. Wang and Gao in [23] extended these existence results to the case of a doubly degenerate equation, namely when $m > 1$ and $p > 2$. In turn, all these results extend those of Allegretto and Nistri obtained in [2] for $m = 1$ and $p = 2$. Many other papers deal with the problem of the existence of periodic solutions for degenerate parabolic equations, we cite here among others, [15], [16], [24] and [25]. In all the above cited papers the approach is based on the same topological tools employed to solve 1. Finally, we mention a very recent paper [14] which deals with a biological aggregation model of some biological species such as insect swarms and bacterial colonies. The model is represented by an evolution equation in $\mathbb{R}$ with nonlinear diffusion, which takes into account both of the tendency of the species to aggregate, when the gradient of the density of population increases, and of the anti-crowding effect when the density increases. Due to this second effect the resulting equation is of the porous medium type; while, following the existing literature on the subject, the aggregation effect is modeled by means of a nonlocal term depending on the density through a suitable kernel. The paper shows that, for compactly supported non-negative smooth initial data, the gradient of the density blows up in finite time.

In this paper we deal with the general problem 1 which is a possible model of the interactions of two biological species, with density u and v respectively, disliking crowding, i.e. $m, n > 1$, see [7], [8] and [17], and whose diffusion involves as in [1], [18] and [22] the p-Laplacian, in our case $p, q > 2$. As pointed out before, this situation corresponds to the slow diffusion, see e.g. [10] and [11]. In 1 a and b are the natural growth rates of the populations. The nonlocal terms $\int_\Omega K_i(\xi,t)u^2(\xi,t-\tau_i)d\xi$ and $\int_\Omega K_i(\xi,t)v^2(\xi,t-\tau_i)d\xi$ evaluate a weighted fraction of individuals that actually interact at time $t > 0$. The functions K_1, K_4 are supposed to be non-negative and they measure the competition for food among each species. While K_2, K_3 model the influence of a population on the other one. The delayed densities u, v at time $t - \tau_i$, that appear in the nonlocal terms, take into account the time needed to an individual to become adult, and, thus to interact and to compete. Therefore, the term on the right hand side of each equation in 1 denotes the actual increasing rate of the population at $(x,t) \in Q_T$.

Due to the double degeneracy of the equations of the system 1 we follow the standard technique of the parabolic regularization to obtain a family of regularized non-degenerate systems. These systems will depend on two parameters $\epsilon, \eta > 0$ (see 2 in the sequel), and by the Leray-Schauder topological degree theory we will establish the existence of a family of non-negative, periodic solutions $(u_{\epsilon\eta}, v_{\epsilon\eta})$ with non-trivial components. Our approach, based on the topological degree, requires a priori bounds in some functional space, in our case $L^2(Q_T)$, on all the possible solutions of a suitable defined homotopy joining the original problem with a simpler problem for which the topological degree is different from zero in an open set defined by means of the a priori bounds and not containing zero. To this goal is devoted the first part of Section 2, which ends with the main existence result of the paper: Theorem 2.4. Precisely, by means of a condition involving the a priori bounds and the first eigenvalue of the Laplacian with Dirichlet boundary conditions, we prove that the solution $(u_{\epsilon\eta}, v_{\epsilon\eta})$ of the regularized system converges as $\epsilon, \eta \to 0$ to a non-negative periodic solution pair (u, v) of 1 such that $u \neq 0$ and $v \neq 0$. In other words, Theorem 2.4 provides a general coexistence result for the two biological species whose dynamics is modeled by 1. As pointed out before, the crucial assumption for all the results of Section 2 are represented by the a priori bounds in $L^2(Q_T)$ on the solutions of the considered homotopies, Lemma 2.3 provides a condition ensuring that such a priori bounds are indeed in $L^\infty(Q_T)$. In Section 3, under different assumptions on the sign of the functions K_i, $i = 1, 2, 3, 4$, we present several results establishing the sought-after a priori bounds for the coexistence result. Specifically, assuming that $K_i(x,t) \geq \underline{k}_i > 0$, $i = 1, 4$, for a.a. $(x,t) \in Q_T$: the so-called coercive case, Theorem 3.1 provides a precise evaluation of the constants of the a priori bounds in $L^2(Q_T)$ for $(u_{\epsilon\eta}, v_{\epsilon\eta})$ whatever the sign of K_i, $i = 2, 3$, in Q_T. In the non-coercive case, namely in the case when we allow the functions K_i, $i = 1, 4$, to vanish on a subset of Q_T of positive measure, under the assumption that $K_i \leq 0$ (weak competitive case), or $K_i \leq -\overline{k}_i < 0$ (strong competitive case), $i = 2, 3$, Theorems 3.2 and 3.3 respectively guarantee the needed a priori bounds for Theorem 2.4. Finally, in Theorem 3.4, under more restrictive assumptions on m, n, p and q, also in the non-coercive case we obtain a priori bounds for Theorem 2.4 independently on the sign of the functions K_i, $i = 2, 3$.

We point out that the results in this paper concern only the strict doubly degenerate case: $m, n > 1$ and $p, q > 2$, this is due to the arguments employed to show that we obtain neither trivial nor semi-trivial periodic solutions. In fact, the constants that bounds from below the norms of our solutions (i.e. r_0 of Proposition 2.2 and λ_ν in 25) are not well defined in the limit cases, that is when any of the two equations of 1 fails to be doubly degenerate (i.e. when either one of m, n is 1 or one of p, q is 2). However, one can handle all the possible limit cases by examining which kind of degeneracy (if any) appears in each equation of 1 and by employing the corresponding estimation strategy as it is exploited in [2, 5, 9, 23, 26] or in the present paper. Therefore in what follows, unless otherwise stated, we assume that $m, n > 1$ and $p, q > 2$.

2. **The regularized problem.**

Throughout the paper we assume that $a, b, K_i \in L^\infty(Q_T), i = 1, 2, 3, 4$, in 1. We now recall the definition of a weak solution to 1.

Definition 2.1. A pair of functions (u, v) is said to be a weak solution of 1 if $u, v \in C(\overline{Q}_T)$, $u^m \in L^p\big(0, T; W_0^{1,p}(\Omega)\big)$, $v^n \in L^q\big(0, T; W_0^{1,q}(\Omega)\big)$ and (u, v) satisfies

$$
\begin{aligned}
0 = \iint_{Q_T} \Big\{ & -u\frac{\partial \varphi}{\partial t} + |\nabla u^m|^{p-2}\nabla u^m \nabla\varphi - au\varphi \\
& + u\varphi \int_\Omega [K_1(\xi, t)u^2(\xi, t-\tau_1) - K_2(\xi, t)v^2(\xi, t-\tau_2)]d\xi \Big\} \, dxdt
\end{aligned}
$$

and

$$
\begin{aligned}
0 = \iint_{Q_T} \Big\{ & -v\frac{\partial \varphi}{\partial t} + |\nabla v^n|^{q-2}\nabla v^n \nabla\varphi - bv\varphi \\
& + v\varphi \int_\Omega [-K_3(\xi, t)u^2(\xi, t-\tau_3) + K_4(\xi, t)v^2(\xi, t-\tau_4)]d\xi \Big\} \, dxdt,
\end{aligned}
$$

for any $\varphi \in C^1(\overline{Q}_T), \varphi(x, T) = \varphi(x, 0)$ for any $x \in \Omega$, and $\varphi(x, t) = 0$ for any $(x, t) \in \partial\Omega \times [0, T]$.

Here and in the following we assume that the functions $t \to u(\cdot, t)$ and $t \to v(\cdot, t)$ are extended from $[0, T]$ to $\mathbb{R}$ by T-periodicity so that (u, v) is a solution for all t. Moreover, since $u^m = |u|^{m-1}u, m > 1$, its derivative is $m|u|^{m-1}$ and is positive for $u \neq 0$, namely u^m is invertible on $\mathbb{R}$. In what follows we omit the absolute value in the derivative.

Due to the double degeneracy of the equation we consider, as in [23], the following regularized (non-degenerate) problem:

$$
\begin{cases}
\ell^{m,p}_{\epsilon,\eta,1}[u] = \left[a - \displaystyle\int_\Omega K_1(\xi, t)u^2(\xi, t-\tau_1)d\xi + \int_\Omega K_2(\xi, t)v^2(\xi, t-\tau_2)d\xi\right] u & \text{in } Q_T \\
\ell^{n,q}_{\epsilon,\eta,1}[v] = \left[b + \displaystyle\int_\Omega K_3(\xi, t)u^2(\xi, t-\tau_3)d\xi - \int_\Omega K_4(\xi, t)v^2(\xi, t-\tau_4)d\xi\right] v & \text{in } Q_T \\
u(\cdot, t)|_{\partial\Omega} = v(\cdot, t)|_{\partial\Omega} = 0, \quad \text{for a.a. } t \in (0, T), \\
u(\cdot, 0) = u(\cdot, T) \text{ and } v(\cdot, 0) = v(\cdot, T),
\end{cases}
\tag{2}
$$

where $\ell^{m,p}_{\epsilon,\eta,\sigma}[u] := u_t - \operatorname{div}\{[|(\sigma m u^{m-1} + \epsilon)\nabla u|^2 + \eta]^{\frac{p-2}{2}}(\sigma m u^{m-1} + \epsilon)\nabla u\}$ ($\ell^{n,q}_{\epsilon,\eta,\sigma}[v]$ is similarly defined) with $\epsilon, \eta \in (0, 1/2)$ and $\sigma \in [0, 1]$. A solution (u, v) of 1 will be then obtained as the limit, for $\epsilon, \eta \to 0$, of the solutions $(u_{\epsilon\eta}, v_{\epsilon\eta})$ of 2, which will be functions in $L^p\big(0,T; W_0^{1,p}(\Omega)\big) \cap C(\overline{Q}_T)$ and $L^q\big(0,T; W_0^{1,q}(\Omega)\big) \cap C(\overline{Q}_T)$, respectively, satisfying 2 in the usual weak sense.

To deal with the existence of weak T-periodic solutions $(u_{\epsilon\eta}, v_{\epsilon\eta})$ of system 2, with $u_{\epsilon\eta}, v_{\epsilon\eta} \geq 0$ in Q_T, we introduce, for any $\epsilon, \eta \in (0, 1/2)$, the map $G_{\epsilon\eta} : [0,1] \times L^\infty(Q_T) \times L^\infty(Q_T) \to L^\infty(Q_T) \times L^\infty(Q_T)$ as follows:

$$(\sigma, f, g) \mapsto (u_{\epsilon\eta}, v_{\epsilon\eta}) = G_{\epsilon\eta}(\sigma, f, g)$$

if and only if $(u_{\epsilon\eta}, v_{\epsilon\eta})$ solves the following uncoupled problem

$$\begin{cases} \ell^{m,p}_{\epsilon,\eta,\sigma}[u] = f, & \text{in } Q_T, \\ \ell^{n,q}_{\epsilon,\eta,\sigma}[v] = g, & \text{in } Q_T, \\ u(\cdot,t)|_{\partial\Omega} = v(\cdot,t)|_{\partial\Omega} = 0, & \text{for a.a. } t \in (0,T), \\ u(\cdot,0) = u(\cdot,T) \text{ and } v(\cdot,0) = v(\cdot,T). & \end{cases} \tag{3}$$

The map $G_{\epsilon\eta}$ is well defined since the elliptic part of the parabolic operators $\ell^{m,p}_{\epsilon,\eta,\sigma}$ and $\ell^{n,q}_{\epsilon,\eta,\sigma}$ with Dirichlet boundary condition are m-accretive in $L^1(\Omega)$ for any $\sigma \in [0,1]$ and $\epsilon, \eta > 0$ sufficiently small. In fact, they satisfy the structure conditions of [3] and so [3, Proposition 2.4] applies. Finally, [20, Proposition IV.4.1] ensures that the solution (u, v) of 3 is unique. Consider now

$$f(\alpha, \beta) := \left(a - \int_\Omega K_1(\xi, \cdot)\, \alpha^2(\xi, \cdot - \tau_1) d\xi + \int_\Omega K_2(\xi, \cdot)\, \beta^2(\xi, \cdot - \tau_2) d\xi \right) \alpha$$

and

$$g(\alpha, \beta) := \left(b + \int_\Omega K_3(\xi, \cdot)\, \alpha^2(\xi, \cdot - \tau_3) d\xi - \int_\Omega K_4(\xi, \cdot)\, \beta^2(\xi, \cdot - \tau_4) d\xi \right) \beta,$$

where α and β belong to $L^\infty(Q_T)$. Clearly, if the non-negative functions $u_{\epsilon\eta}, v_{\epsilon\eta} \in L^\infty(Q_T)$ are such that $(u_{\epsilon\eta}, v_{\epsilon\eta}) = G_{\epsilon\eta}\big(1, f(u_{\epsilon\eta}, v_{\epsilon\eta}), g(u_{\epsilon\eta}, v_{\epsilon\eta})\big)$, then $(u_{\epsilon\eta}, v_{\epsilon\eta})$ is also a solution of 2 (with $u_{\epsilon\eta} \geq 0$ and $v_{\epsilon\eta} \geq 0$) in Q_T. Hence, the existence of a non-negative solution of 2 is equivalent to the existence of a fixed point (α, β) of the map $(\alpha, \beta) \to G_{\epsilon\eta}\big(1, f(\alpha, \beta), g(\alpha, \beta)\big)$ with $\alpha \geq 0$ and $\beta \geq 0$.

Let $T_{\epsilon\eta}(\sigma, \alpha, \beta) := G_{\epsilon\eta}\big(\sigma, f(\alpha, \beta), g(\alpha, \beta)\big)$. Analogously to [5], by using classical regularity results of [13], one can prove the next result for the regularized problem 3.

Lemma 2.2. *Let $(\alpha, \beta) \in L^\infty(Q_T) \times L^\infty(Q_T)$ and let $\epsilon, \eta \in (0, 1/2)$. Then $(u_{\epsilon\eta}, v_{\epsilon\eta}) = T_{\epsilon\eta}(\sigma, \alpha, \beta)$ is a compact continuous map from $[0,1] \times L^\infty(Q_T) \times L^\infty(Q_T) \to L^\infty(Q_T) \times L^\infty(Q_T)$. Moreover $u_{\epsilon\eta}, v_{\epsilon\eta} \in C(\overline{Q}_T)$.*

Our aim is to prove the existence of T-periodic solutions $u_{\epsilon\eta}, v_{\epsilon\eta} \in C(\overline{Q}_T)$, $u_{\epsilon\eta}$, $v_{\epsilon\eta} > 0$ in Q_T, of the regularized problem 2 for all $\epsilon > 0$ and $\eta > 0$ small enough as positive fixed points of the map $(\alpha, \beta) \to T_{\epsilon\eta}(1, \alpha, \beta)$. As a first step we prove the following result.

Proposition 2.1. *Assume that $a, b, K_i \in L^\infty(Q_T)$ for $i = 1, 2, 3, 4$. If the non-trivial pair $(u_{\epsilon\eta}, v_{\epsilon\eta})$ solves*

$$(u, v) = G_{\epsilon\eta}\big(\sigma, \rho f(u^+, v^+) + (1-\sigma), \rho g(u^+, v^+) + (1-\sigma)\big), \tag{4}$$

for some $\sigma \in [0,1]$ and $\rho \in [0,1]$, then

$$u_{\epsilon\eta}(x,t) \geq 0 \text{ and } v_{\epsilon\eta}(x,t) \geq 0 \quad \text{for any } (x,t) \in Q_T.$$

Moreover, if $u_{\epsilon\eta} \neq 0$ or $v_{\epsilon\eta} \neq 0$ then $u_{\epsilon\eta} > 0$ or $v_{\epsilon\eta} > 0$ in Q_T, respectively.

Proof. Assume that $(u_{\epsilon\eta}, v_{\epsilon\eta})$ solves 4 with $u_{\epsilon\eta} \neq 0$ for some $\sigma \in [0,1]$ and $\rho \in [0,1]$. We first prove that $u_{\epsilon\eta} \geq 0$. Multiplying the first equation of 3, where $f(\alpha,\beta)$ is replaced by $\rho f(u^+_{\epsilon\eta}, v^+_{\epsilon\eta}) + (1-\sigma)$, by $u^-_{\epsilon\eta} := \min\{0, u_{\epsilon\eta}\}$, integrating on Q_T and passing to the limit in the Steklov averages $(u_{\epsilon\eta})_h \in H^1(Q_{T-\delta})$, $\delta, h > 0$, in the standard way [13, p. 85], we obtain

$$\iint_{Q_T} [|\nabla(\sigma u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} \nabla(\sigma u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta}) \nabla u^-_{\epsilon\eta} = \iint_{Q_T} (1-\sigma) u^-_{\epsilon\eta}$$

by the T-periodicity of $u_{\epsilon\eta}$ and taking into account that $u^+_{\epsilon\eta} u^-_{\epsilon\eta} = 0$. Hence we obtain

$$\iint_{Q_T} [|\nabla(\sigma u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} (\sigma m u^{m-1}_{\epsilon\eta} + \epsilon) |\nabla u^-_{\epsilon\eta}|^2 \leq 0,$$

that is

$$\begin{aligned} 0 \geq & \frac{4\sigma m}{(m+1)^2} \iint_{Q_T} [|\nabla(\sigma u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} |\nabla (u^-_{\epsilon\eta})^{\frac{m+1}{2}}|^2 \\ & + \epsilon \iint_{Q_T} [|\nabla(\sigma u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} |\nabla u^-_{\epsilon\eta}|^2 \end{aligned}$$

and so, in particular,

$$\iint_{Q_T} [|\nabla(\sigma u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} |\nabla u^-_{\epsilon\eta}|^2 \leq 0. \tag{5}$$

Since $[|\nabla(\sigma u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} > 0$, 5 implies that

$$|\nabla u^-_{\epsilon\eta}|^2 = 0$$

a.e. in Q_T. Thus

$$\iint_{Q_T} |\nabla u^-_{\epsilon\eta}|^2 = 0.$$

The Poincaré inequality gives

$$0 \leq \int_\Omega |u^-_{\epsilon\eta}|^2 \leq c \int_\Omega |\nabla u^-_{\epsilon\eta}|^2,$$

for some $c > 0$. Integrating over $(0,T)$, we have

$$0 \leq \iint_{Q_T} |u^-_{\epsilon\eta}|^2 \leq c \iint_{Q_T} |\nabla u^-_{\epsilon\eta}|^2 = 0,$$

which, together with the boundary conditions and the fact that $u^-_{\epsilon\eta} \in C(\overline{Q}_T)$, implies $u^-_{\epsilon\eta}(x,t) = 0$ for all $(x,t) \in Q_T$. Thus $u_{\epsilon\eta}(x,t) = u^+_{\epsilon\eta}(x,t) \geq 0$ for all $(x,t) \in Q_T$. Now we prove that $u_{\epsilon\eta} > 0$ in Q_T. Since $u_{\epsilon\eta}$ is non-trivial, there exists $(x_0,t_0) \in \Omega \times (0,T]$ such that $u_{\epsilon\eta}(x_0,t_0) > 0$. Let $\psi \in C^\infty_0(\Omega)$ be a non-negative function such that $0 < \psi(x_0) < u_{\epsilon\eta}(x_0,t_0)$ and, for $M > 0$, let z be a solution of

$$\begin{cases} z_t - \Delta(\sigma z^m + \epsilon z) + Mz = 0, & (x,t) \in \Omega \times (t_0, t_0+T], \\ z(\cdot,t)|_{\partial\Omega} = 0, & \text{for } t \in [t_0, t_0+T], \\ z(\cdot,t_0) = \psi(\cdot). \end{cases}$$

Since $a - \int_\Omega K_1(\xi,\cdot) u^2_{\epsilon\eta}(\xi, \cdot - \tau_1) d\xi + \int_\Omega K_2(\xi,\cdot) v^2_{\epsilon\eta}(\xi, \cdot - \tau_2) d\xi \in L^\infty(Q_T)$, we can choose M large enough so that, by the comparison theorem,

$$u_{\epsilon\eta}(x,t) \geq z(x,t) \quad \text{for any } (x,t) \in \Omega \times [t_0, t_0+T].$$

By the maximum principle, $z(x,t) > 0$ for any $(x,t) \in \Omega \times [t_0, t_0 + T]$. Therefore, by T-periodicity, $u_{\epsilon\eta}(x,t) > 0$ for all $(x,t) \in Q_T$. In the same way, one can prove that $v_{\epsilon\eta} \neq 0$ implies $v_{\epsilon\eta}(x,t) > 0$ for all $(x,t) \in Q_T$. □

Observe that if $\rho = 0$, by using the arguments of the proof of Proposition 2.1 to show that $u^-_{\epsilon\eta} = 0$, it can be shown that $(u,v) = G_{\epsilon\eta}(1,0,0)$ if and only if $(u,v) = (0,0)$.

The following result guarantees that the solutions $(u_{\epsilon\eta}, v_{\epsilon\eta})$ of 2 we are going to find are not bifurcating from the trivial solution $(0,0)$ as ϵ and η range in $(0,1/2)$. To this aim assume that

$$\frac{1}{T}\iint_{Q_T} e_1^2 a > \mu_1 \qquad \text{and} \qquad \frac{1}{T}\iint_{Q_T} e_1^2 b > \mu_1 \tag{6}$$

and let r_0 be the following positive quantity

$$\min\left\{\left(\frac{1}{2m}\right)^{\frac{1}{m-1}}, \left(\frac{1}{2n}\right)^{\frac{1}{n-1}}, \left(\frac{\iint_{Q_T} e_1^2 a - T\mu_1}{M_1}\right)^{\frac{p}{p-2}}, \left(\frac{\iint_{Q_T} e_1^2 b - T\mu_1}{M_2}\right)^{\frac{q}{q-2}}\right\}.$$

Here μ_1 is the first eigenvalue of the problem

$$\begin{cases} -\Delta z = \mu z, & x \in \Omega, \\ z = 0, & x \in \partial\Omega, \end{cases}$$

e_1 is the associated positive eigenfunction such that $\|e_1\|_{L^2(\Omega)} = 1$,

$$\begin{aligned} M_1 :=& \|K_1\|_{L^1} + \|K_2\|_{L^1} \\ &+ 2^{\frac{p^2-4}{2p}} \max_{\overline{\Omega}} |\nabla e_1|^2 (|\Omega|T)^{\frac{2}{p}} (\|a\|_{L^1} + \|K_2\|_{L^1}|\Omega| + T|\Omega|)^{\frac{p-2}{p}} \end{aligned}$$

and

$$\begin{aligned} M_2 :=& \|K_3\|_{L^1} + \|K_4\|_{L^1} \\ &+ 2^{\frac{q^2-4}{2q}} \max_{\overline{\Omega}} |\nabla e_1|^2 (|\Omega|T)^{\frac{2}{q}} (\|b\|_{L^1} + \|K_3\|_{L^1}|\Omega| + T|\Omega|)^{\frac{q-2}{q}}. \end{aligned}$$

Proposition 2.2. *Assume that 6 is satisfied. If the non-trivial pair $(u_{\epsilon\eta}, v_{\epsilon\eta})$ solves $(u,v) = G_{\epsilon\eta}\big(\sigma, f(u^+,v^+) + (1-\sigma), g(u^+,v^+) + (1-\sigma)\big)$, for some $\sigma \in [0,1]$, then*

$$\max\{\|u_{\epsilon\eta}\|_{L^\infty}, \|v_{\epsilon\eta}\|_{L^\infty}\} \geq r_0.$$

Moreover $\deg\big((u,v) - T_{\epsilon\eta}(1,u^+,v^+), B_r, 0\big) = 0$ *for all $r \in (0, r_0)$.*

Proof. By contradiction, assume that for some $\sigma \in [0,1]$ and $r \in (0,r_0)$ there exists a pair $(u_{\epsilon\eta}, v_{\epsilon\eta}) \neq (0,0)$ such that $(u_{\epsilon\eta}, v_{\epsilon\eta}) = G_{\epsilon\eta}\big(\sigma, f(u^+_{\epsilon\eta}, v^+_{\epsilon\eta}) + (1-\sigma), g(u^+_{\epsilon\eta}, v^+_{\epsilon\eta}) + (1-\sigma)\big)$ with $\|u_{\epsilon\eta}\|_{L^\infty} \leq r$ and $\|v_{\epsilon\eta}\|_{L^\infty} \leq r$. Assume that $u_{\epsilon\eta} \neq 0$ and take $\phi \in C_0^\infty(\Omega)$. Since by Proposition 2.1 we have $u_{\epsilon\eta} > 0$ in Q_T, we can multiply the equation

$$\begin{aligned} \ell^{m,p}_{\epsilon,\eta,\sigma}[u_{\epsilon\eta}] = \Big[a - &\int_\Omega K_1(\xi,t) u^2_{\epsilon\eta}(\xi, t-\tau_1) d\xi \\ &+ \int_\Omega K_2(\xi,t) v^2_{\epsilon\eta}(\xi, t-\tau_2) d\xi\Big] u_{\epsilon\eta} + (1-\sigma) \end{aligned} \tag{7}$$

by $\phi^2/u_{\epsilon\eta}$, integrate over Q_T and pass to the limit in the Steklov averages in order to obtain

$$\begin{aligned}
&-\iint_{Q_T} \frac{\phi^2}{u_{\epsilon\eta}} \operatorname{div}\{[|\nabla(\sigma u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} \nabla(\sigma u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})\} \\
&= \iint_{Q_T} \left[a\phi^2 + (1-\sigma)\frac{\phi^2}{u_{\epsilon\eta}}\right] - \iint_{Q_T} \phi^2 \left[\int_\Omega K_1(\xi,t) u_{\epsilon\eta}^2(\xi, t-\tau_1) d\xi\right] dxdt \qquad (8) \\
&+ \iint_{Q_T} \phi^2 \left[\int_\Omega K_2(\xi,t) v_{\epsilon\eta}^2(\xi, t-\tau_2) d\xi\right] dxdt
\end{aligned}$$

by the T-periodicity of $u_{\epsilon\eta}$. Moreover, a straightforward computation shows that

$$\begin{aligned}
&-\iint_{Q_T} \frac{\phi^2}{u_{\epsilon\eta}} \operatorname{div}\{[|\nabla(\sigma u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} \nabla(\sigma u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})\} \\
&= \iint_{Q_T} \nabla\left(\frac{\phi^2}{u_{\epsilon\eta}}\right) [|\nabla(\sigma u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} \nabla(\sigma u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta}) \\
&= \iint_{Q_T} [|\nabla(\sigma u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} (m\sigma u_{\epsilon\eta}^{m-1} + \epsilon) \left[|\nabla\phi|^2 - u_{\epsilon\eta}^2 \left|\nabla\left(\frac{\phi}{u_{\epsilon\eta}}\right)\right|^2\right] \\
&\le \iint_{Q_T} [|\nabla(\sigma u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} (m\sigma u_{\epsilon\eta}^{m-1} + \epsilon)|\nabla\phi|^2.
\end{aligned}$$

Since $r < (1/2m)^{1/(m-1)}$ and $\epsilon < 1/2$, then $m\sigma u_{\epsilon\eta}^{m-1} + \epsilon \le m u_{\epsilon\eta}^{m-1} + \epsilon < 1/2 + \epsilon < 1$ and

$$\begin{aligned}
&-\iint_{Q_T} \frac{\phi^2}{u_{\epsilon\eta}} \operatorname{div}\{[|\nabla(\sigma u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} \nabla(\sigma u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})\} \\
&< \iint_{Q_T} [|\nabla(\sigma u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} |\nabla\phi|^2 \\
&\le 2^{\frac{p-2}{2}} \iint_{Q_T} |\nabla(\sigma u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^{p-2} |\nabla\phi|^2 + (2\eta)^{\frac{p-2}{2}} \iint_{Q_T} |\nabla\phi|^2.
\end{aligned}$$

Since $\eta < 1/2$ and applying the Hölder inequality with $s := p/(p-2)$, it follows

$$\begin{aligned}
&-\iint_{Q_T} \frac{\phi^2}{u_{\epsilon\eta}} \operatorname{div}\{[|\nabla(\sigma u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} \nabla(\sigma u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})\} \\
&< 2^{\frac{p-2}{2}} \iint_{Q_T} |\nabla(\sigma u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^{p-2} |\nabla\phi|^2 + \iint_{Q_T} |\nabla\phi|^2 \qquad (9) \\
&\le 2^{\frac{p-2}{2}} \|\nabla\phi\|_{L^p(\Omega)}^2 T^{\frac{2}{p}} \left[\iint_{Q_T} |\nabla(\sigma u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^p\right]^{\frac{p-2}{p}} + \iint_{Q_T} |\nabla\phi|^2.
\end{aligned}$$

Hence, combining 8 and 9, we obtain

$$\begin{aligned}
&\iint_{Q_T} \phi^2 \left[a - \int_\Omega K_1(\xi,t) u_{\epsilon\eta}^2(\xi, t-\tau_1) d\xi + \int_\Omega K_2(\xi,t) v_{\epsilon\eta}^2(\xi, t-\tau_2) d\xi\right] dxdt \\
&< 2^{\frac{p-2}{2}} \|\nabla\phi\|_{L^p(\Omega)}^2 T^{\frac{2}{p}} \left[\iint_{Q_T} |\nabla(\sigma u_{\epsilon\eta}^m + u_{\epsilon\eta})|^p\right]^{\frac{p-2}{p}} + \iint_{Q_T} |\nabla\phi|^2.
\end{aligned}$$

Taking $\phi(x) \to e_1(x)$ in $H_0^1(\Omega)$ one has

$$
\begin{aligned}
0 <& 2^{\frac{p-2}{2}} \|\nabla e_1\|^2_{L^p(\Omega)} T^{\frac{2}{p}} \left[\iint_{Q_T} |\nabla(\sigma u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})|^p \right]^{\frac{p-2}{p}} + \iint_{Q_T} |\nabla e_1|^2 \\
&- \iint_{Q_T} e_1^2 \left[a - \int_\Omega K_1(\xi,t) u^2_{\epsilon\eta}(\xi, t-\tau_1) d\xi + \int_\Omega K_2(\xi,t) v^2_{\epsilon\eta}(\xi, t-\tau_2) d\xi \right] dxdt \\
=& 2^{\frac{p-2}{2}} \|\nabla e_1\|^2_{L^p(\Omega)} T^{\frac{2}{p}} \left[\iint_{Q_T} |\nabla(\sigma u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})|^p \right]^{\frac{p-2}{p}} + \mu_1 T - \iint_{Q_T} a e_1^2 \\
&+ \iint_{Q_T} [K_1(\xi,t) u^2_{\epsilon\eta}(\xi, t-\tau_1) - K_2(\xi,t) v^2_{\epsilon\eta}(\xi, t-\tau_2)] d\xi dt.
\end{aligned}
\tag{10}
$$

Now we estimate the term $[\iint_{Q_T} |\nabla(\sigma u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})|^p]^{(p-2)/p}$. Multiplying 7 by $(\sigma u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})$, integrating over Q_T, using the T-periodicity and passing to the limit in the Steklov averages, one has

$$
\begin{aligned}
&\iint_{Q_T} [|\nabla(\sigma u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} |\nabla(\sigma u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})|^2 \\
\le& \iint_{Q_T} \left[a + \int_\Omega K_2(\xi,t) v^2_{\epsilon\eta}(\xi, t-\tau_2) d\xi \right] u_{\epsilon\eta}(u^m_{\epsilon\eta} + u_{\epsilon\eta}) dxdt \\
&+ (1-\sigma) \iint_{Q_T} (u^m_{\epsilon\eta} + u_{\epsilon\eta}) \\
\le& \iint_{Q_T} a(u^{m+1}_{\epsilon\eta} + u^2_{\epsilon\eta}) + \int_0^T \left[\int_\Omega u^{m+1}_{\epsilon\eta} \right] \left[\int_\Omega K_2(\xi,t) v^2_{\epsilon\eta}(\xi, t-\tau_2) d\xi \right] dt \\
&+ \int_0^T \left[\int_\Omega u^2_{\epsilon\eta} \right] \left[\int_\Omega K_2(\xi,t) v^2_{\epsilon\eta}(\xi, t-\tau_2) d\xi \right] dt + \iint_{Q_T} (u^m_{\epsilon\eta} + u_{\epsilon\eta}) \\
\le& \|a\|_{L^1}(r^{m+1} + r^2) + \int_0^T \left[\int_\Omega u^{m+1}_{\epsilon\eta} \right] \left[\int_\Omega K_2(\xi,t) v^2_{\epsilon\eta}(\xi, t-\tau_2) d\xi \right] dt \\
&+ \int_0^T \left[\int_\Omega u^2_{\epsilon\eta} \right] \left[\int_\Omega K_2(\xi,t) v^2_{\epsilon\eta}(\xi, t-\tau_2) d\xi \right] dt + |\Omega| T (r^m + r).
\end{aligned}
$$

By Hölder's inequality:

$$
\begin{aligned}
&\iint_{Q_T} [|\nabla(\sigma u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} |\nabla(\sigma u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})|^2 \\
\le& \|a\|_{L^1}(r^{m+1} + r^2) + \int_0^T \left[\int_\Omega u^{m+1}_{\epsilon\eta} \right] \left[\int_\Omega K_2(\xi,t) v^2_{\epsilon\eta}(\xi, t-\tau_2) d\xi \right] dt \\
&+ \int_0^T \left[\int_\Omega u^2_{\epsilon\eta} \right] \left[\int_\Omega K_2(\xi,t) v^2_{\epsilon\eta}(\xi, t-\tau_2) d\xi \right] dt + |\Omega| T (r^m + r) \\
\le& \|a\|_{L^1}(r^{m+1} + r^2) + |\Omega| \|K_2\|_{L^1} r^{m+3} + |\Omega| \|K_2\|_{L^1} r^4 + |\Omega| T (r^m + r).
\end{aligned}
$$

Since $r < 1$, we obtain

$$
\begin{aligned}
&\iint_{Q_T} [|\nabla(\sigma u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} |\nabla(\sigma u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})|^2 \\
\le& 2r(\|a\|_{L^1} + |\Omega| \|K_2\|_{L^1} + T|\Omega|).
\end{aligned}
$$

Thus

$$\iint_{Q_T} |\nabla(\sigma u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^p \leq \iint_{Q_T} [|\nabla(\sigma u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} |\nabla(\sigma u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^2$$
$$\leq 2r(\|a\|_{L^1} + \|K_2\|_{L^1}|\Omega| + T|\Omega|)$$

and

$$\left[\iint_{Q_T} |\nabla(\sigma u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^p\right]^{\frac{p-2}{p}} \leq [2r(\|a\|_{L^1} + \|K_2\|_{L^1}|\Omega| + T|\Omega|)]^{\frac{p-2}{p}}.$$

The previous inequality, 10 and the Hölder inequality imply

$$\begin{aligned}
\iint_{Q_T} ae_1^2 - \mu_1 T <& 2^{\frac{p-2}{2}} \|\nabla e_1\|^2_{L^p(\Omega)} T^{\frac{2}{p}} \left[\iint_{Q_T} |\nabla(\sigma u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^p\right]^{\frac{p-2}{p}} \\
&+ \iint_{Q_T} [K_1(\xi,t) u_{\epsilon\eta}^2(\xi, t-\tau_1) - K_2(\xi,t) v_{\epsilon\eta}^2(\xi, t-\tau_2)] d\xi dt \\
\leq& 2^{\frac{p^2-4}{2p}} \|\nabla e_1\|^2_{L^p(\Omega)} T^{\frac{2}{p}} r^{\frac{p-2}{p}} (\|a\|_{L^1} + \|K_2\|_{L^1}|\Omega| + T|\Omega|)^{\frac{p-2}{p}} \\
&+ (\|K_1\|_{L^1} + \|K_2\|_{L^1}) r^2 \\
\leq& r^{\frac{p-2}{p}} \big[2^{\frac{p^2-4}{2p}} \max_{\overline{\Omega}} |\nabla e_1|^2 (|\Omega| T)^{\frac{2}{p}} (\|a\|_{L^1} + \|K_2\|_{L^1}|\Omega| + T|\Omega|)^{\frac{p-2}{p}} \\
&+ \|K_1\|_{L^1} + \|K_2\|_{L^1}\big].
\end{aligned} \tag{11}$$

Thus

$$r_0^{\frac{p-2}{p}} \leq \frac{\iint_{Q_T} ae_1^2 - \mu_1 T}{M_1} \leq r^{\frac{p-2}{p}}$$

that is a contradiction. The same argument applies if $v_{\epsilon\eta} \neq 0$.
Let us now fix any $r \in (0, r_0)$. We just proved that

$$(u,v) \neq G_{\epsilon\eta}\big(\sigma, f(u^+, v^+) + (1-\sigma), g(u^+, v^+) + (1-\sigma)\big),$$

for all $(u,v) \in \partial B_r$ and for all $\sigma \in [0,1]$. Hence the topological degree of $(u,v) - G_{\epsilon\eta}\big(\sigma, f(u^+,v^+) + (1-\sigma), g(u^+,v^+) + (1-\sigma)\big)$ is well defined in B_r for all $\sigma \in [0,1]$. From the homotopy invariance of the Leray-Schauder degree, we have

$$\begin{aligned}
&\deg\big((u,v) - T_{\epsilon\eta}(1, u^+, v^+), B_r, 0\big) \\
=& \deg\big((u,v) - G_{\epsilon\eta}(0, f(u^+,v^+) + 1, g(u^+,v^+) + 1), B_r, 0\big)
\end{aligned}$$

and the last degree is zero since the equation

$$(u,v) = G_{\epsilon\eta}\big(0, f(u^+,v^+) + 1, g(u^+,v^+) + 1\big)$$

admits neither trivial nor non-trivial solutions in B_r. □

The next lemma is crucial to prove Proposition 2.3.

Lemma 2.3. *Let $K > 0$ and assume that u is a non-negative periodic continuous function such that*

$$u_t - \operatorname{div}\{[|\nabla(u^m + \epsilon u)|^2 + \eta]^{\frac{p-2}{2}} \nabla(u^m + \epsilon u)\} \leq Ku, \quad \textit{for a.e. } (x,t) \in Q_T$$

and $u(\cdot,t)|_{\partial\Omega} = 0$, for $t \in [0,T]$. Then there exists $R > 0$ and independent of ϵ and η such that

$$\|u\|_{L^\infty} \leq R.$$

Proof. We follow Moser's technique to show the stated a priori bounds. Multiplying

$$u_t - \mathrm{div}\{[|\nabla(u^m + \epsilon u)|^2 + \eta]^{\frac{p-2}{2}} \nabla(u^m + \epsilon u)\} \le Ku$$

by u^{s+1}, with $s \ge 0$, and integrating over Ω, we have

$$\begin{aligned} K\|u(t)\|^{s+2}_{L^{s+2}(\Omega)} \ge& \frac{1}{s+2}\frac{d}{dt}\|u(t)\|^{s+2}_{L^{s+2}(\Omega)} \\ &+ \int_\Omega [|\nabla(u^m + \epsilon u)|^2 + \eta]^{\frac{p-2}{2}} \nabla(u^m + \epsilon u)\nabla u^{s+1}, \end{aligned}$$

namely

$$\begin{aligned} K\|u(t)\|^{s+2}_{L^{s+2}(\Omega)} \ge& \frac{1}{s+2}\frac{d}{dt}\|u(t)\|^{s+2}_{L^{s+2}(\Omega)} \\ &+ (s+1)\int_\Omega [|\nabla(u^m + \epsilon u)|^2 + \eta]^{\frac{p-2}{2}} (mu^{m-1} + \epsilon)u^s|\nabla u|^2. \end{aligned}$$

Since $p > 2$, $m > 1$ and

$$u^{(m-1)(p-2)}|\nabla u|^{p-2} \le (mu^{m-1} + \epsilon)^{p-2}|\nabla u|^{p-2} \le [|\nabla(u^m + \epsilon u)|^2 + \eta]^{\frac{p-2}{2}},$$

we have

$$\frac{1}{s+2}\frac{d}{dt}\|u(t)\|^{s+2}_{L^{s+2}(\Omega)} + \int_\Omega u^{(p-1)(m-1)+s}|\nabla u|^p \le K\|u(t)\|^{s+2}_{L^{s+2}(\Omega)}.$$

This implies

$$\begin{aligned} K(s+2)\|u(t)\|^{s+2}_{L^{s+2}(\Omega)} \ge& \frac{d}{dt}\|u(t)\|^{s+2}_{L^{s+2}(\Omega)} \\ &+ \frac{s+2}{[m(p-1)+s+1]^p}\int_\Omega \left|\nabla u^{\frac{m(p-1)+s+1}{p}}\right|^p. \end{aligned} \tag{12}$$

For ϵ and η fixed and $k = 1, 2, \ldots$, setting

$$s_k := 2p^k + \frac{p^k - p}{p-1} + m - 1, \quad \alpha_k := \frac{p(s_k+2)}{m(p-1) + s_k + 1}, \quad w_k := u^{\frac{m(p-1)+s_k+1}{p}},$$

we obtain by 12

$$\frac{d}{dt}\|w_k(t)\|^{\alpha_k}_{L^{\alpha_k}(\Omega)} + \frac{s_k+2}{[m(p-1)+s_k+1]^p}\|\nabla w_k(t)\|^p_{L^p(\Omega)} \le K(s_k+2)\|w_k(t)\|^{\alpha_k}_{L^{\alpha_k}(\Omega)}. \tag{13}$$

Observe that since $s_k \to +\infty$, as $k \to +\infty$, there exists k_0 such that $\alpha_k \in (1, p)$ for all $k \ge k_0$. By the interpolation and the Sobolev inequalities, it results

$$\|w_k(t)\|_{L^{\alpha_k}(\Omega)} \le \|w_k(t)\|^{\theta_k}_{L^1(\Omega)}\|w_k(t)\|^{1-\theta_k}_{L^s(\Omega)} \le C\|w_k(t)\|^{\theta_k}_{L^1(\Omega)}\|\nabla w_k(t)\|^{1-\theta_k}_{L^p(\Omega)}$$

for all $k \ge k_0$. Here $\theta_k = (s - \alpha_k)/[\alpha_k(s-1)]$, $s > p$ is fixed (say $s = p^*$ if $p < n$, where $p^* := np/(n-p)$) and C is a positive constant. Using the fact that $\|w_k(t)\|_{L^1(\Omega)} = \|w_{k-1}(t)\|^{\alpha_{k-1}}_{L^{\alpha_{k-1}}(\Omega)}$ and defining $x_{k-1} := \sup_{t\in\mathbb{R}}\|w_{k-1}(t)\|_{L^{\alpha_{k-1}}(\Omega)}$, one has

$$\begin{aligned} \|w_k(t)\|^{\frac{p}{1-\theta_k}}_{L^{\alpha_k}(\Omega)} \le& C\|w_{k-1}(t)\|^{p\alpha_{k-1}\frac{\theta_k}{1-\theta_k}}_{L^{\alpha_{k-1}}(\Omega)}\|\nabla w_k(t)\|^p_{L^p(\Omega)} \\ \le& Cx_{k-1}^{p\alpha_{k-1}\frac{\theta_k}{1-\theta_k}}\|\nabla w_k(t)\|^p_{L^p(\Omega)}, \end{aligned}$$

for all $k \geq k_0$. Thus, by 13,

$$
\begin{aligned}
\frac{d}{dt}\|w_k(t)\|^{\alpha_k}_{L^{\alpha_k}(\Omega)} \leq & K(s_k+2)\|w_k(t)\|^{\alpha_k}_{L^{\alpha_k}(\Omega)} \\
& - C\frac{s_k+2}{[m(p-1)+s_k+1]^p}\|w_k(t)\|^{\frac{p}{1-\theta_k}}_{L^{\alpha_k}(\Omega)} x_{k-1}^{p\alpha_{k-1}\frac{\theta_k}{\theta_k-1}} \\
= & \left(K - \frac{C}{(m(p-1)+s_k+1)^p}\|w_k(t)\|^{\frac{p}{1-\theta_k}-\alpha_k}_{L^{\alpha_k}(\Omega)} x_{k-1}^{p\alpha_{k-1}\frac{\theta_k}{\theta_k-1}}\right) \\
& \times (s_k+2)\|w_k(t)\|^{\alpha_k}_{L^{\alpha_k}(\Omega)},
\end{aligned} \tag{14}
$$

for all $k \geq k_0$. By the positiveness of w_k it results that the map $t \mapsto \|w_k(t)\|^{\alpha_k}_{L^{\alpha_k}(\Omega)}$ is increasing, hence 14 implies

$$
\|w_k(t)\|_{L^{\alpha_k}(\Omega)} \leq \left(\frac{K}{M_k} x_{k-1}^{p\alpha_{k-1}\frac{\theta_k}{1-\theta_k}}\right)^{\eta_k}, \tag{15}
$$

for all $k \geq k_0$, where $\eta_k := (1-\theta_k)[p-\alpha_k(1-\theta_k)]$ and $M_k := C/[m(p-1)+s_k+1]^p$. By definition of x_k and 15 we get

$$
x_k \leq \left(\frac{K}{M_k}\right)^{\eta_k} x_{k-1}^{\nu_k}
$$

for all $k \geq k_0$, with $\nu_k := p\alpha_{k-1}\theta_k/[p-\alpha_k(1-\theta_k)]$.

If $x_{k-1} \leq 1$, using the fact that $x_{k-1} = \sup_{t\in\mathbb{R}} \|u(t)\|_{s_{k-1}+2}^{\frac{m(p-1)+s_{k-1}+1}{p}}$, one has $\|u\|_{L^\infty} \leq 1$. Now, assume $x_{k-1} > 1$ and observe that there exists $\overline{k}_0$ such that, for all $k \geq \overline{k}_0$, $\eta_k := (1-\theta_k)/[p-\alpha_k(1-\theta_k)] \leq 1/(p\theta)$ and $\nu_k \leq p$. Here $\theta := (s-p)/[p(s-1)]$. Without loss of generality, assume $k_0 = \max\{\overline{k}_0, k_0\}$. Then, there exists a positive constant A such that

$$
\begin{aligned}
x_k \leq & \left(\frac{K}{C}\right)^{\eta_k} [m(p-1)+s_k+1]^{p\eta_k} x_{k-1}^{\nu_k} \\
\leq & \left(\frac{K}{C}\right)^{\eta_k} \left(mp + \frac{2p^{k+1}}{p-1}\right)^{p\eta_k} x_{k-1}^{\nu_k} \\
\leq & Ap^{\frac{k+1}{\theta}} x_{k-1}^{p}
\end{aligned}
$$

for all $k \geq k_0$. Thus

$$
\begin{aligned}
\log x_k \leq & \log A + \frac{k+1}{\theta}\log p + p\log x_{k-1} \\
\leq & \log A \sum_{i=0}^{k-k_0-1} p^i + \frac{\log p}{\theta}\sum_{i=k_0+2}^{k+1} ip^{k+1-i} + p^{k-k_0}\log x_{k_0} \\
\leq & \frac{\log p}{\theta} p^{k+1-(k-k_0)(k_0+2)}\left[\frac{(k+1)(k+2)}{2} - \frac{(k_0+1)(k_0+2)}{2}\right] \\
& + \log A\frac{1-p^{k-k_0}}{1-p} + p^{k-k_0}\log x_{k_0}.
\end{aligned}
$$

It follows

$$
x_k \leq A^{\frac{1-p^{k-k_0}}{1-p}} p^{\frac{p^{k+1-(k-k_0)(k_0+2)}}{\theta}\left[\frac{(k+1)(k+2)}{2} - \frac{(k_0+1)(k_0+2)}{2}\right]} x_{k_0}^{p^{k-k_0}}.
$$

Since $x_k = \sup_{t\in\mathbb{R}} \|u(t)\|_{s_k+2}^{\frac{m(p-1)+s_k+1}{p}}$, we obtain

$$\begin{aligned}\sup_{t\in\mathbb{R}} \|u(t)\|_{L^\infty(\Omega)} &\leq \limsup_{k\to\infty} \|u(t)\|_{s_k+2} \\ &\leq \limsup_{k\to\infty} \Big\{ A^{\frac{p}{m(p-1)+s_k+1}\frac{1-p^{k-k_0}}{1-p}} x_{k_0}^{\frac{p^{k-k_0+1}}{m(p-1)+s_k+1}} \\ &\quad \times p^{\frac{p[k+2-(k-k_0)(k_0+2)]}{\theta(m(p-1)+s_k+1)}\left[\frac{(k+1)(k+2)}{2}-\frac{(k_0+1)(k_0+2)}{2}\right]} \Big\} =: R,\end{aligned}$$

where R is a positive constant independent of ϵ and η as claimed. □

Next, we show that the map $I - G_{\epsilon\eta} : \{1\} \times L^\infty(Q_T) \times L^\infty(Q_T) \to L^\infty(Q_T) \times L^\infty(Q_T)$ has the Leray - Schauder topological degree different from zero in the cone of non-negative functions.

Proposition 2.3. *Assume that $K_1(x,t), K_4(x,t) \geq 0$ for a.e. $(x,t) \in Q_T$ and that there are $C_1, C_2 > 0$ such that*

$$\|u_{\epsilon\eta}\|_{L^2}^2 \leq C_1 \text{ and } \|v_{\epsilon\eta}\|_{L^2}^2 \leq C_2 \tag{16}$$

for all solution pairs $(u_{\epsilon\eta}, v_{\epsilon\eta})$ of

$$(u,v) = G_{\epsilon\eta}\big(1, \rho f(u^+,v^+), \rho g(u^+,v^+)\big) \tag{17}$$

and all $\epsilon, \eta \in (0,1/2)$ and $\rho \in (0,1]$. Then there is a constant $R > 0$ such that

$$\|u_{\epsilon\eta}\|_{L^\infty}, \|v_{\epsilon\eta}\|_{L^\infty} < R$$

for all solution pairs $(u_{\epsilon\eta}, v_{\epsilon\eta})$ of 17 and all $\epsilon, \eta \in (0,1/2)$ and $\rho \in (0,1]$. Moreover, one has that

$$\deg\big((u,v) - G_{\epsilon\eta}\big(1, \rho f(u^+,v^+), \rho g(u^+,v^+)\big), B_R, 0\big) = 1.$$

Proof. Assume $u_{\epsilon\eta} \neq 0$, thus $u_{\epsilon\eta} > 0$ and $v_{\epsilon\eta} \geq 0$ in Q_T by Proposition 2.1. Multiplying by $u_{\epsilon\eta}$ the first equation of 2, where $f(u,v)$ is replaced by $\rho f(u,v)$, integrating over Ω and using the Steklov averages $(u_{\epsilon\eta})_h$, $\delta, h > 0$ we obtain

$$\begin{aligned}&\rho\left[\|a\|_{L^\infty(Q_T)} + \|K_2\|_{L^\infty(Q_T)} \int_\Omega (v_{\epsilon\eta})_h^2(\xi, t-\tau_2)d\xi\right] \\ &\geq \frac{\int_\Omega \{|[m(u_{\epsilon\eta})_h^{m-1}+\epsilon]\nabla(u_{\epsilon\eta})_h|^2+\eta\}^{\frac{p-2}{2}}[m(u_{\epsilon\eta})_h^{m-1}+\epsilon]|\nabla(u_{\epsilon\eta})_h|^2}{\int_\Omega (u_{\epsilon\eta})_h^2} \\ &\quad + \frac{1}{2}\frac{d}{dt}\log\int_\Omega (u_{\epsilon\eta})_h^2.\end{aligned} \tag{18}$$

Since $t \mapsto \|u(t)\|_{L^2(\Omega)}$ is continuous in $[0,T]$, there exist t_1 and t_2 in $[0,T]$ such that

$$\int_\Omega u_{\epsilon\eta}^2(x,t_1)dx = \min_{t\in[0,T]} \int_\Omega u_{\epsilon\eta}^2(x,t)dx$$

and

$$\int_\Omega u_{\epsilon\eta}^2(x,t_2)dx = \max_{t\in[0,T]} \int_\Omega u_{\epsilon\eta}^2(x,t)dx.$$

Integrating 18 between t_1 and t_2 and passing to the limit as $h \to 0$ we obtain

$$\frac{1}{2}\log\frac{\max_{t\in[0,T]}\int_\Omega u_{\epsilon\eta}^2(x,t)dx}{\min_{t\in[0,T]}\int_\Omega u_{\epsilon\eta}^2(x,t)dx} \leq T\|a\|_{L^\infty} + \|K_2\|_{L^\infty} C_2,$$

or, equivalently,

$$\max_{t\in[0,T]}\int_\Omega u_{\epsilon\eta}^2(x,t)dx \le C\min_{t\in[0,T]}\int_\Omega u_{\epsilon\eta}^2(x,t)dx, \tag{19}$$

where C is independent of ϵ,η and ρ. Hence, there is a constant $\gamma>0$, independent of ϵ,η and ρ, such that

$$\max_{t\in[0,T]}\int_\Omega u_{\epsilon\eta}^2(x,t)dx \le \gamma.$$

Otherwise, inequality 19 would imply that the solutions $u_{\epsilon\eta}$ are unbounded in $L^2(Q_T)$ as ϵ,η range in $(0,1/2)$ and ρ in $(0,1]$, against our assumption 16. Of course, an analogous inequality holds for $v_{\epsilon\eta}$.

Now, we have

$$\begin{aligned}\ell^{m,p}_{\epsilon,\eta,1}[u_{\epsilon\eta}] \le &\left(\|a\|_{L^\infty}+\|K_2\|_{L^\infty}\max_{t\in[0,T]}\int_\Omega v_{\epsilon\eta}^2(x,t)dx\right)u_{\epsilon\eta}\\ \le&(\|a\|_{L^\infty}+\gamma\|K_2\|_{L^\infty})u_{\epsilon\eta},\end{aligned} \tag{20}$$

that is

$$\frac{\partial u_{\epsilon\eta}}{\partial t}-\operatorname{div}\{[|(mu_{\epsilon\eta}^{m-1}+\epsilon)\nabla u_{\epsilon\eta}|^2+\eta]^{\frac{p-2}{2}}(mu_{\epsilon\eta}^{m-1}+\epsilon)\nabla u_{\epsilon\eta}\}\le Ku_{\epsilon\eta},$$

where $K:=\|a\|_{L^\infty}+\gamma\|K_2\|_{L^\infty}$ and $\ell^{m,p}_{\epsilon,\eta,1}$ is given in 2. By Lemma 2.3 we conclude that $\|u_{\epsilon\eta}\|_{L^\infty}\le R_1$ for some $R_1>0$ independent of ρ, η and ϵ. Analogously, $\|v_{\epsilon\eta}\|_{L^\infty}\le R_2$ for some constant $R_2>0$. Therefore it is enough to choose $R>\max\{R_1,R_2\}$.

The homotopy invariance property of the Leray-Schauder degree implies that

$$\begin{aligned}&\deg\big((u,v)-T_{\epsilon\eta}(1,u^+,v^+),B_R,0\big)\\ =&\deg\big((u,v)-G_{\epsilon\eta}(1,\rho f(u^+,v^+),\rho g(u^+,v^+)),B_R,0\big),\end{aligned}$$

for any $\rho\in[0,1]$. If we take $\rho=0$, using the fact that $G_{\epsilon\eta}$ at $\rho=0$ is the zero map, it results

$$\deg\big((u,v)-T_{\epsilon\eta}(1,u^+,v^+),B_R,0\big)=\deg\big((u,v),B_R,0\big)=1.$$

□

The next result is our main tool to obtain coexistence results for 1.

Theorem 2.4. *Assume that $K_1(x,t),K_4(x,t)\ge 0$ for a.e. $(x,t)\in Q_T$ and that there are $C_1,C_2>0$ such that*

$$\|u_{\epsilon\eta}\|_{L^2}^2\le C_1 \text{ and } \|v_{\epsilon\eta}\|_{L^2}^2\le C_2 \tag{21}$$

for all solution pairs $(u_{\epsilon\eta},v_{\epsilon\eta})$ of

$$(u,v)=G_{\epsilon\eta}\big(1,\rho f(u^+,v^+),\rho g(u^+,v^+)\big) \tag{22}$$

and all $\epsilon,\eta\in(0,1/2)$ and $\rho\in(0,1]$. If

$$\theta(C_1,C_2):=\min\left\{\frac{1}{T}\iint_{Q_T}e_1^2a-\mu_1-\frac{\underline{k}_2C_2}{T},\frac{1}{T}\iint_{Q_T}e_1^2b-\mu_1-\frac{\underline{k}_3C_1}{T}\right\}>0, \tag{23}$$

where the non-negative constants $\underline{k}_2,\underline{k}_3$ are such that $-\underline{k}_2\le K_2(x,t)$ and $-\underline{k}_3\le K_3(x,t)$ for a.e. $(x,t)\in Q_T$, then problem 1 has a T-periodic non-negative solution (u,v) with non-trivial u,v.

Proof. By assumption $\theta(C_1, C_2) > 0$ and so 6 holds. Moreover, there is $R > r > 0$, independent of ϵ and η, such that

$$\deg\big((u,v) - G_{\epsilon\eta}(1, f(u^+, v^+), g(u^+, v^+)), B_R \setminus \overline{B}_r, 0\big) = 1,$$

for any $\epsilon, \eta \in (0, 1/2)$, by Proposition 2.2 and the excision property of the topological degree.

Let us fix any $\epsilon, \eta \in (0, 1/2)$. There is $\sigma_0 = \sigma_0(\epsilon, \eta) \in (0, 1)$ such that still

$$\deg\big((u,v) - G_{\epsilon\eta}(\sigma, f(u^+, v^+) + (1-\sigma), g(u^+, v^+) + (1-\sigma)), B_R \setminus \overline{B}_r, 0\big) = 1$$

for all $\sigma \in [\sigma_0, 1]$, by the continuity of Leray-Schauder degree. This implies that the set of solution triples $(\sigma, u, v) \in [0,1] \times (B_R \setminus \overline{B}_r)$ such that

$$(u,v) = G_{\epsilon\eta}\big(\sigma, f(u^+, v^+) + (1-\sigma), g(u^+, v^+) + (1-\sigma)\big) \tag{24}$$

contains a continuum $\mathcal{S}_{\epsilon\eta}$ with the property that

$$\mathcal{S}_{\epsilon\eta} \cap \big[\{\sigma\} \times \big(B_R \setminus \overline{B}_r\big)\big] \neq \emptyset \qquad \text{for all } \sigma \in [\sigma_0, 1].$$

Now, all the pairs (u,v) such that $(1, u, v) \in \mathcal{S}_{\epsilon\eta}$ are T-periodic solutions of 2 with $(u,v) \neq (0,0)$ and, hence, satisfy 21. Since the L^2-norm is continuous with respect to the L^∞-norm and $\mathcal{S}_{\epsilon\eta}$ is a continuum, for every $\nu > 0$ there is $\sigma_\nu \in [\sigma_0, 1)$ such that

$$\|u\|_{L^2}^2 \leq C_1 + \nu \qquad \text{and} \qquad \|v\|_{L^2}^2 \leq C_2 + \nu$$

for all (u,v) with $(\sigma, u, v) \in \mathcal{S}_{\epsilon\eta}$ and $\sigma \in [\sigma_\nu, 1]$. Observe that, if $(\sigma, u, v) \in \mathcal{S}_{\epsilon\eta}$ for $\sigma < 1$, then u and v are *positive* solutions of 24. Moreover, if ν is sufficiently small, then we still have $\theta(C_1 + \nu, C_2 + \nu) > 0$.

Now, setting

$$K_p := \|K_1\|_{L^1} + 2^{\frac{p^2-4}{2p}} \max_{\overline{\Omega}} |\nabla e_1|^2 (|\Omega| T)^{\frac{2}{p}} (\|a\|_{L^1} + \|K_2\|_{L^1}|\Omega| + T|\Omega|)^{\frac{p-2}{p}}$$

$$K_q := \|K_4\|_{L^1} + 2^{\frac{q^2-4}{2q}} \max_{\overline{\Omega}} |\nabla e_1|^2 (|\Omega| T)^{\frac{2}{q}} (\|b\|_{L^1} + \|K_3\|_{L^1}|\Omega| + T|\Omega|)^{\frac{q-2}{q}},$$

we can prove that, if ν is sufficiently small, then

$$\begin{aligned} \|u\|_{L^\infty}, \|v\|_{L^\infty} \geq \min\Bigg\{ &\left[\frac{1}{2m}\right]^{\frac{1}{m-1}}, \left[\frac{1}{2n}\right]^{\frac{1}{n-1}}, \left[\frac{T\theta(C_1+\nu, C_2+\nu)}{K_p}\right]^{\frac{p}{p-2}}, \\ &\left[\frac{T\theta(C_1+\nu, C_2+\nu)}{K_q}\right]^{\frac{q}{q-2}} \Bigg\} =: \lambda_\nu \end{aligned} \tag{25}$$

for all u, v such that $(\sigma, u, v) \in \mathcal{S}_{\epsilon\eta}$ and $\sigma \in [\sigma_\nu, 1)$. Indeed, let (u,v) be a solution of 24. Arguing by contradiction, assume that $\|u\|_{L^\infty} < \lambda_\nu$ and proceeding as in the proof of Proposition 2.2 (see 11 and recall that $u > 0$ since (u,v) solves 24 with $\sigma < 1$) we obtain the inequality

$$\iint_{Q_T} e_1^2 a - \mu_1 T < \lambda_\nu^{\frac{p-2}{p}} K_p + \underline{k}_2 (C_2 + \nu).$$

Thus, the definition of θ implies that

$$T\theta(C_1+\nu, C_2+\nu) \leq \iint_{Q_T} e_1^2 a - \mu_1 T - \underline{k}_2(C_2+\nu) < \lambda_\nu^{\frac{p-2}{p}} K_p,$$

which is a contradiction with the definition of λ_ν. The same argument shows that $\|v\|_{L^\infty} \geq \lambda_\nu$.

Now, if we let $\sigma \to 1$ and $\nu \to 0$, then we obtain that 2 has at least a solution $(u_{\epsilon\eta}, v_{\epsilon\eta})$ such that $\|u_{\epsilon\eta}\|_{L^\infty}, \|v_{\epsilon\eta}\|_{L^\infty} \geq \lambda_0$, since $\mathcal{S}_{\epsilon\eta}$ is a continuum and $\lambda_\nu \to \lambda_0$ as $\nu \to 0$.

Finally, we show that a solution (u, v) of 1 with non-trivial $u, v \geq 0$ is obtained as a limit of $(u_{\epsilon\eta}, v_{\epsilon\eta})$ as $\epsilon, \eta \to 0$ since λ_0 is independent of ϵ and η.

Since $u_{\epsilon\eta}, v_{\epsilon\eta}$ are Hölder continuous in $\overline{Q}_T$, bounded in $C(\overline{Q}_T)$ uniformly in $\epsilon, \eta > 0$ and the structure conditions of [19] (see also [11]) are satisfied for the equations of system 1, whenever $\epsilon, \eta \in \left(0, \frac{1}{2}\right)$, [19, Theorem 1.2] applies to conclude that the inequality

$$|u_{\epsilon\eta}(x_1, t_1) - u_{\epsilon\eta}(x_2, t_2))| \leq \Gamma(|x_1 - x_2|^\beta + |t_1 - t_2|^{\frac{\beta}{2}})$$

holds for any $(x_1, t_1), (x_2, t_2) \in \overline{Q}_T$, where the constants $\Gamma > 0, \beta \in (0, 1)$ are independent of $\|u_{\epsilon\eta}\|_{L^\infty}$. The same inequality holds for $v_{\epsilon\eta}$. Therefore, by the Ascoli-Arzelà Theorem, a subsequence of $(u_{\epsilon\eta}, v_{\epsilon\eta})$ converges uniformly in $\overline{Q}_T$ to a pair (u, v) satisfying

$$\lambda_0 \leq \|u\|_{L^\infty}, \|v\|_{L^\infty} \leq R.$$

Moreover, from 20 we have

$$\frac{\partial u_{\epsilon\eta}}{\partial t} - \operatorname{div}\{[|\nabla(u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} \nabla(u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})\} \leq C u_{\epsilon\eta}, \tag{26}$$

where C is a positive constant independent of ϵ and η. Multiplying 26 by $u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta}$, integrating over Q_T and passing to the limit in the Steklov averages $(u_{\epsilon\eta})_h$, one has

$$\begin{aligned} \iint_{Q_T} |\nabla u_{\epsilon\eta}^m|^p \leq & \iint_{Q_T} (|\nabla u_{\epsilon\eta}^m|^2 + \eta)^{\frac{p-2}{2}} |\nabla u_{\epsilon\eta}^m|^2 \\ \leq & \iint_{Q_T} [|\nabla(u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} |\nabla(u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^2 \\ \leq & C \iint_{Q_T} (u_{\epsilon\eta}^{m+1} + \epsilon u_{\epsilon\eta}^2) \\ \leq & M, \end{aligned} \tag{27}$$

and

$$\begin{aligned} \iint_{Q_T} |\nabla(u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^p \leq & \iint_{Q_T} [|\nabla(u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} |\nabla(u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^2 \\ \leq & C \iint_{Q_T} (u_{\epsilon\eta}^{m+1} + \epsilon u_{\epsilon\eta}^2) \\ \leq & M, \end{aligned} \tag{28}$$

by the T-periodicity of $u_{\epsilon\eta}$, its non-negativity and its boundedness in $L^\infty(Q_T)$. Here M is positive and independent of ϵ and η. An analogous estimates hold for $v_{\epsilon\eta}$.

By 27, the sequences $u_{\epsilon\eta}^m, v_{\epsilon\eta}^n$ are uniformly bounded in $L^p\big(0, T; W_0^{1,p}(\Omega)\big)$ and in $L^q\big(0, T; W_0^{1,q}(\Omega)\big)$, respectively. Thus, up to subsequence if necessary, $(u_{\epsilon\eta}^m, v_{\epsilon\eta}^n)$ weakly converges in $L^p\big(0, T; W_0^{1,p}(\Omega)\big) \times L^q\big(0, T; W_0^{1,q}(\Omega)\big)$ and in $C(\overline{Q}_T) \times C(\overline{Q}_T)$ to $(h, k) = (u^m, v^n)$. In particular $(u^m, v^n) \in L^p\big(0, T; W_0^{1,p}(\Omega)\big) \times L^q\big(0, T; W_0^{1,q}(\Omega)\big)$.

We finally claim that the pair (u, v) satisfies the identities

$$0 = \iint_{Q_T} \Big\{ -u\frac{\partial \varphi}{\partial t} + |\nabla u^m|^{p-2}\nabla u^m \cdot \nabla\varphi - au\varphi + u\varphi \int_\Omega [K_1(\xi,t)u^2(\xi,t-\tau_1) - K_2(\xi,t)v^2(\xi,t-\tau_2)]d\xi \Big\} dxdt$$

and

$$0 = \iint_{Q_T} \Big\{ -v\frac{\partial \varphi}{\partial t} + |\nabla v^n|^{q-2}\nabla v^n \cdot \nabla\varphi - bv\varphi + v\varphi \int_\Omega [-K_3(\xi,t)u^2(\xi,t-\tau_3) + K_4(\xi,t)v^2(\xi,t-\tau_4)]d\xi \Big\} dxdt,$$

for any $\varphi \in C^1(\overline{Q}_T)$, $\varphi(x,T) = \varphi(x,0)$ for any $x \in \Omega$ and $\varphi(x,t) = 0$ for any $(x,t) \in \partial\Omega \times [0,T]$, that is (u,v) is a generalized solution of 1. The approach for doing this is standard, in the sequel we report it for the reader's convenience. By 28, there exists a positive constant M such that

$$\iint_{Q_T} |\nabla(u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})|^p \le M \quad \text{and} \quad \iint_{Q_T} |\nabla(u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})|^2 \le M.$$

Thus, by the Hölder inequality with $r := \frac{2(p-1)}{p}$, one has

$$\begin{aligned} &\iint_{Q_T} |[|\nabla(u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} \nabla(u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})|^{\frac{p}{p-1}} \\ \le & M_p \left[\iint_{Q_T} |\nabla(u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})|^p + \iint_{Q_T} |\nabla(u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})|^{\frac{p}{p-1}} \right] \\ \le & M, \end{aligned}$$

for some other positive constants M and M_p. This implies that there exists $H \in \left(L^{\frac{p}{p-1}}(Q_T)\right)^N$ such that $[|\nabla(u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} \nabla(u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})$ weakly converges to H in $\left(L^{\frac{p}{p-1}}(Q_T)\right)^N$ as $\epsilon, \eta \to 0$. Now it is easy to prove that

$$0 = \iint_{Q_T} \Big\{ -u\frac{\partial \varphi}{\partial t} + H \cdot \nabla\varphi - au\varphi + u\varphi \int_\Omega [K_1(\xi,t)u^2(\xi,t-\tau_1) - K_2(\xi,t)v^2(\xi,t-\tau_2)]d\xi \Big\} dxdt \tag{29}$$

for any $\varphi \in C^1(\overline{Q}_T)$, $\varphi(x,T) = \varphi(x,0)$ for any $x \in \Omega$ and $\varphi(x,t) = 0$ for any $(x,t) \in \partial\Omega \times [0,T]$ (and, by density, for any T-periodic $\varphi \in L^p\big(0,T; W^{1,p}_0(\Omega)\big)$). It remains to prove that for every $\varphi \in C^1(\overline{Q}_T)$

$$\iint_{Q_T} |\nabla u^m|^{p-2}\nabla u^m \cdot \nabla\varphi = \iint_{Q_T} H \cdot \nabla\varphi. \tag{30}$$

To this aim consider the matrix function $H(Y) := (|Y|^2 + \eta)^{\frac{p-2}{2}} Y$. Then

$$H'(Y) = (|Y|^2 + \eta)^{\frac{p-2}{2}} I + (p-2)(|Y|^2 + \eta)^{\frac{p-4}{2}} YY^T$$

is a positive definite matrix and, taken $v \in L^p\big(0,T; W^{1,p}_0(\Omega)\big)$, there exists a matrix Y such that

$$\begin{aligned} 0 \le & \langle H'(Y)(\nabla(u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta}) - \nabla v), \nabla(u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta}) - \nabla v\rangle \\ = & \langle H(\nabla(u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta})) - H(\nabla v), \nabla(u^m_{\epsilon\eta} + \epsilon u_{\epsilon\eta}) - \nabla v\rangle \end{aligned}$$

The previous inequality is equivalent to

$$0 \leq \iint_{Q_T} \Big\{ [|\nabla(u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} \nabla(u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta}) \\ -(|\nabla v|^2 + \eta)^{\frac{p-2}{2}} \nabla v \Big\} \cdot \nabla[(u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta}) - v],$$

for all $v \in L^p\big(0,T; W_0^{1,p}(\Omega)\big)$. Multiplying by $u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta}$ the first equation of 2, integrating over Q_T and using the periodicity of $u_{\epsilon\eta}$, one has

$$\iint_{Q_T} [|\nabla(u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} |\nabla(u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^2 \\ = \iint_{Q_T} \Big[a - \int_\Omega K_1(\xi,t) u_{\epsilon\eta}^2(\xi, t-\tau_1) d\xi \\ + \int_\Omega K_2(\xi,t) v_{\epsilon\eta}^2(\xi, t-\tau_2) d\xi \Big] (u_{\epsilon\eta}^{m+1} + \epsilon u_{\epsilon\eta}^2) dxdt.$$

Thus

$$\iint_{Q_T} [|\nabla(u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} \nabla(u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta}) \cdot \nabla v \\ + \iint_{Q_T} (|\nabla v|^2 + \eta)^{\frac{p-2}{2}} \nabla v \cdot \nabla[(u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta}) - v] \\ \leq \iint_{Q_T} \Big[a - \int_\Omega K_1(\xi,t) u_{\epsilon\eta}^2(\xi, t-\tau_1) d\xi \\ + \int_\Omega K_2(\xi,t) v_{\epsilon\eta}^2(\xi, t-\tau_2) d\xi \Big] (u_{\epsilon\eta}^{m+1} + \epsilon u_{\epsilon\eta}^2) dxdt.$$

Letting $\epsilon, \eta \to 0$ and using 28, we have

$$\iint_{Q_T} \big[H \cdot \nabla v + |\nabla v|^{p-2} \nabla v \cdot \nabla(u^m - v) \big] \\ \leq \iint_{Q_T} \Big[a - \int_\Omega K_1(\xi,t) u^2(\xi, t-\tau_1) d\xi + \int_\Omega K_2(\xi,t) v^2(\xi, t-\tau_2) d\xi \Big] u^{m+1} dxdt$$

On the other hand, take $u^m = \varphi$ in 29 and obtain

$$\iint_{Q_T} H \cdot \nabla u^m \\ = \iint_{Q_T} \Big[a - \int_\Omega K_1(\xi,t) u^2(\xi, t-\tau_1) d\xi + \int_\Omega K_2(\xi,t) v^2(\xi, t-\tau_2) d\xi \Big] u^{m+1} dxdt.$$

This implies

$$0 \leq \iint_{Q_T} (H - |\nabla v|^{p-2} \nabla v) \cdot \nabla(u^m - v). \tag{31}$$

Taking $v := u^m - \lambda\varphi$, with $\lambda > 0$ and $\varphi \in C^1(\overline{Q}_T)$, we get

$$0 \leq \iint_{Q_T} (H - |\nabla(u^m - \lambda\varphi)|^{p-2} \nabla(u^m - \lambda\varphi)) \cdot \nabla\varphi.$$

Letting $\lambda \to 0$ yields

$$0 \leq \iint_{Q_T} (H - |\nabla u^m|^{p-2} \nabla u^m) \cdot \nabla\varphi.$$

If in 31 we take $v := u^m + \lambda\varphi$, with $\lambda > 0$, $\varphi \in C^1(\overline{Q}_T)$ and letting again $\lambda \to 0$, then

$$\iint_{Q_T} (H - |\nabla u^m|^{p-2}\nabla u^m) \cdot \nabla\varphi \leq 0.$$

Thus 30 holds. □

Remark 2.1. Proposition 2.2, with assumption 6, does not guarantee that both components of a non-trivial solution $(u_{\epsilon\eta}, v_{\epsilon\eta})$ of the regularized problem 2 are positive and, in fact, the proof of such a positivity is one of the main issues we had to handle in the proof of the previous theorem with the help of the stronger assumption $\theta(C_1, C_2) > 0$. However, if the cooperative case $K_2(x,t), K_3(x,t) \geq 0$, i.e. $\underline{k}_2 = \underline{k}_3 = 0$, is considered, then we have that

$$\theta(C_1, C_2) = \min\left\{\frac{1}{T}\iint_{Q_T} e_1^2 a - \mu_1, \frac{1}{T}\iint_{Q_T} e_1^2 b - \mu_1\right\} = \theta_0$$

and the condition $\theta_0 > 0$ is equivalent to 6. In particular, coexistence in the cooperative case follows from $\theta_0 > 0$ and a priori bounds on $(u_{\epsilon\eta}, v_{\epsilon\eta})$, even if the constants C_1, C_2 are not explicitly known.

Remark 2.2. The assumption $\theta(C_1, C_2) > 0$ is used to show 25 and, therefore, grants the non-triviality of both the components of the non-negative T-periodic solution (u, v) that is given by Theorem 2.4. From a biological point of view, this hypothesis requires that the growth rates a, b of the species are sufficiently large with respect to the terms that model the competition between them. In other words, it reasonably states that the competitive interaction between the two species should not prevail the growth capacity of the species if extinction has to be avoided.

However, when we proved the lower bounds 25, we used the estimate 11 for the first equation of the system in order to show that $\|u\|_{L^\infty(Q_T)}$ was not smaller than λ_ν and, implicitly, we used for $\|v\|_{L^\infty}$ the analogous estimate that holds for the second equation. On the other hand, if we use the second equation for $\|u\|_{L^\infty}$ (and the first equation for $\|v\|_{L^\infty}$) we obtain a different choice for λ_ν and, in particular, for η. In fact, we can prove a version of Theorem 2.4 with

$$\zeta(C_1, C_2) := \min\left\{\frac{1}{T}\iint_{Q_T} e_1^2 a - \mu_1 - \frac{\overline{k}_1 C_1}{T}, \frac{1}{T}\iint_{Q_T} e_1^2 b - \mu_1 - \frac{\overline{k}_4 C_2}{T}\right\} > 0 \tag{32}$$

in place of $\theta(C_1, C_2) > 0$, where $0 \leq K_i(x,t) \leq \overline{k}_i$ a.e. in Q_T for $i = 1, 4$. Specifically, it turns out that

$$\lambda_\nu := \min\left\{\left[\frac{1}{2m}\right]^{\frac{1}{m-1}}, \left[\frac{1}{2n}\right]^{\frac{1}{n-1}}, \left[\frac{T\zeta(C_1+\nu, C_2+\nu)}{K_p'}\right]^{\frac{p}{p-2}}, \left[\frac{T\zeta(C_1+\nu, C_2+\nu)}{K_q'}\right]^{\frac{q}{q-2}}\right\}$$

in 25, where

$$K_p' := \|K_2\|_{L^1} + 2^{\frac{p^2-4}{2p}} \max_{\overline{\Omega}} |\nabla e_1|^2 (|\Omega|T)^{\frac{2}{p}} (\|a\|_{L^1} + \|K_2\|_{L^1}|\Omega| + T|\Omega|)^{\frac{p-2}{p}}$$

$$K_q' := \|K_3\|_{L^1} + 2^{\frac{q^2-4}{2q}} \max_{\overline{\Omega}} |\nabla e_1|^2 (|\Omega|T)^{\frac{2}{q}} (\|b\|_{L^1} + \|K_3\|_{L^1}|\Omega| + T|\Omega|)^{\frac{q-2}{q}}.$$

We observe that also the assumption $\zeta(C_1, C_2) > 0$ has a biological meaning: it requires that the competition inside each species does not prevail on the growth rate of the species itself. The feasibility of the two conditions 23 and 32 depends on the constants C_1, C_2. As we will see in Section 3 there are cases, namely Theorems 3.1 and 3.3, in which one of them is never satisfied (see the next Remark 3.1).

3. A priori bounds in $L^2(Q_T)$.

We apply Theorem 2.4 by looking for explicit a priori bounds in $L^2(Q_T)$ for the solutions of the approximating problems 2 in different situations. We consider two main different cases. In the first one, which we call the "coercive case", we assume that $K_i(x,t) \geq \underline{k}_i > 0$ a.e. in Q_T for $i = 1, 4$. In the second one, the "non-coercive case", we allow the non-negative functions K_1, K_4 to vanish on sets with positive measure. We distinguish also between cooperative and competitive situations by imposing sign conditions on K_2, K_3 and having in mind the biological interpretation of model 1.

3.1. The coercive case.

Theorem 3.1. *Assume that*

1. *there are constants $\underline{k}_i > 0$, $i = 1, 4$, and $\underline{k}_i, \overline{k}_i \geq 0$, $i = 2, 3$, such that $\underline{k}_1 \underline{k}_4 > \overline{k}_2 \overline{k}_3$ and*
$$K_i(x,t) \geq \underline{k}_i \text{ for } i = 1, 4 \qquad \text{and} \qquad -\underline{k}_i \leq K_i(x,t) \leq \overline{k}_i \text{ for } i = 2, 3,$$
for a.e. $(x,t) \in Q_T$;
2. *condition 23 of Theorem 2.4, that is $\theta(C_1, C_2) > 0$, is satisfied with*
$$C_1 = \frac{T\underline{k}_4}{\underline{k}_1 \underline{k}_4 - \overline{k}_2 \overline{k}_3} \left(\|a\|_{L^\infty} + \frac{\overline{k}_2}{\underline{k}_4} \|b\|_{L^\infty} \right)$$
$$C_2 = \frac{T\underline{k}_1}{\underline{k}_1 \underline{k}_4 - \overline{k}_2 \overline{k}_3} \left(\|b\|_{L^\infty} + \frac{\overline{k}_3}{\underline{k}_1} \|a\|_{L^\infty} \right).$$

Then problem 1 has a non-negative T-periodic solution (u, v) with non-trivial u, v.

Proof. We just need to show that $\|u_{\epsilon\eta}\|_{L^2}^2 \leq C_1$ and $\|v_{\epsilon\eta}\|_{L^2}^2 \leq C_2$ for any solution $(u_{\epsilon\eta}, v_{\epsilon\eta})$ of 22. Then, assume $u_{\epsilon\eta} \neq 0$, thus $u_{\epsilon\eta} > 0$ and $v_{\epsilon\eta} \geq 0$ in Q_T by Proposition 2.1. Multiplying the first equation of 2

$$\ell_{\epsilon,\eta,1}^{m,p}[u_{\epsilon\eta}] = \left[a - \int_\Omega K_1(\xi,t) u_{\epsilon\eta}^2(\xi, t-\tau_1) d\xi + \int_\Omega K_2(\xi,t) v_{\epsilon\eta}^2(\xi, t-\tau_2) d\xi \right] u_{\epsilon\eta}$$

by $u_{\epsilon\eta}^{p-1}$, integrating over Ω and using the Steklov averages $(u_{\epsilon\eta})_h \in H^1(Q_{T-\delta})$, $\delta, h > 0$, it results

$$\begin{aligned} &\|a\|_{L^\infty} - \int_\Omega K_1(\xi,t)(u_{\epsilon\eta})_h^2(\xi, t-\tau_1) d\xi + \int_\Omega K_2(\xi,t)(v_{\epsilon\eta})_h^2(\xi, t-\tau_2) d\xi \\ \geq &\frac{\int_\Omega \{|\nabla[(u_{\epsilon\eta})_h^m + \epsilon(u_{\epsilon\eta})_h]|^2 + \eta\}^{\frac{p-2}{2}} \nabla[(u_{\epsilon\eta})_h^m + \epsilon(u_{\epsilon\eta})_h] \nabla(u_{\epsilon\eta})_h^{p-1}}{\int_\Omega (u_{\epsilon\eta})_h^p} \\ &+ \frac{1}{p} \frac{d}{dt} \log \int_\Omega (u_{\epsilon\eta})_h^p. \end{aligned}$$

Integrating the equation 18 over $[0,T]$, and passing to the limit as $h\to 0$, by the T-periodicity of $u_{\epsilon\eta}$, we have that

$$\begin{aligned}&\int_0^T \frac{\int_\Omega [|\nabla(u_{\epsilon\eta}^m+\epsilon u_{\epsilon\eta})|^2+\eta]^{\frac{p-2}{2}}\nabla(u_{\epsilon\eta}^m+\epsilon u_{\epsilon\eta})\nabla u_{\epsilon\eta}^{p-1}}{\int_\Omega u_{\epsilon\eta}^p}\\ \leq & T\|a\|_{L^\infty}-\underline{k}_1\|u_{\epsilon\eta}\|_{L^2}^2+\overline{k}_2\|v_{\epsilon\eta}\|_{L^2}^2.\end{aligned}$$

Now

$$\begin{aligned}&\int_\Omega [|\nabla(u_{\epsilon\eta}^m+\epsilon u_{\epsilon\eta})|^2+\eta]^{\frac{p-2}{2}}\nabla(u_{\epsilon\eta}^m+\epsilon u_{\epsilon\eta})\nabla u_{\epsilon\eta}^{p-1}\\ =&\int_\Omega [|\nabla(u_{\epsilon\eta}^m+\epsilon u_{\epsilon\eta})|^2+\eta]^{\frac{p-2}{2}}(mu_{\epsilon\eta}^{m-1}+\epsilon)(p-1)u_{\epsilon\eta}^{p-2}|\nabla u_{\epsilon\eta}|^2\\ \geq&\int_\Omega (mu_{\epsilon\eta}^{m-1}+\epsilon)^{p-1}u_{\epsilon\eta}^{p-2}|\nabla u_{\epsilon\eta}|^p\\ \geq&\epsilon^{p-1}\int_\Omega u_{\epsilon\eta}^{p-2}|\nabla u_{\epsilon\eta}|^p\\ =&\epsilon^{p-1}\left[\frac{p}{2(p-1)}\right]^p\int_\Omega\left|\nabla u_{\epsilon\eta}^{\frac{2(p-1)}{p}}\right|^p.\end{aligned}$$

Setting μ_p the first positive eigenvalue of the problem

$$\begin{cases}-\operatorname{div}(|\nabla z|^{p-2}\nabla z)=\mu|z|^{p-2}z, & x\in\Omega\\ z=0, & x\in\partial\Omega,\end{cases}\tag{33}$$

(see, for example, [12]), using the Hölder and the Poincaré inequalities and the fact that $p>2$, one has

$$\begin{aligned}\|u_{\epsilon\eta}\|_{L^p(\Omega)}^{2(p-1)}\leq&|\Omega|^{\frac{p-2}{p}}\|u_{\epsilon\eta}\|_{L^{2(p-1)}(\Omega)}^{2(p-1)}\\ =&|\Omega|^{\frac{p-2}{p}}\left\|u_{\epsilon\eta}^{\frac{2(p-1)}{p}}\right\|_{L^p(\Omega)}^p\\ \leq&\frac{|\Omega|^{\frac{p-2}{p}}}{\mu_p}\left\|\nabla u_{\epsilon\eta}^{\frac{2(p-1)}{p}}\right\|_{L^p(\Omega)}^p,\end{aligned}$$

thus

$$\begin{aligned}&\int_\Omega [|\nabla(u_{\epsilon\eta}^m+\epsilon u_{\epsilon\eta})|^2+\eta]^{\frac{p-2}{2}}\nabla(u_{\epsilon\eta}^m+\epsilon u_{\epsilon\eta})|\nabla u_{\epsilon\eta}|^{p-1}\\ \geq&\epsilon^{p-1}\left[\frac{p}{2(p-1)}\right]^p\int_\Omega\left|\nabla u_{\epsilon\eta}^{\frac{2(p-1)}{p}}\right|^p\\ \geq&\frac{\mu_p\epsilon^{p-1}}{|\Omega|^{\frac{p-2}{p}}}\left[\frac{p}{2(p-1)}\right]^p\left(\int_\Omega u_{\epsilon\eta}^p\right)^{\frac{2(p-1)}{p}}.\end{aligned}$$

Since by the Jensen inequality,

$$\int_\Omega u_{\epsilon\eta}^{p-2}\leq|\Omega|^{\frac{2}{p}}\left(\int_\Omega u_{\epsilon\eta}^p\right)^{\frac{p-2}{p}}=\left(\int_\Omega u_{\epsilon\eta}^p\right)^{\frac{2(p-1)}{p}-1},$$

then

$$\begin{aligned}&\int_0^T \frac{\int_\Omega [|\nabla(u_{\epsilon\eta}^m+\epsilon u_{\epsilon\eta})|^2+\eta]^{\frac{p-2}{2}}\nabla(u_{\epsilon\eta}^m+\epsilon u_{\epsilon\eta})\nabla u_{\epsilon\eta}^{p-1}}{\int_\Omega u_{\epsilon\eta}^p}\\ \geq&\frac{\mu_p\epsilon^{p-1}}{|\Omega|^{\frac{p-2}{p}}}\left[\frac{p}{2(p-1)}\right]^p\int_0^T\left(\int_\Omega u_{\epsilon\eta}^p\right)^{\frac{2(p-1)}{p}-1}\\ \geq&\frac{\mu_p\epsilon^{p-1}}{|\Omega|}\left[\frac{p}{2(p-1)}\right]^p\iint_{Q_T}u_{\epsilon\eta}^{p-2}.\end{aligned}$$

Thus

$$\frac{\mu_p\epsilon^{p-1}}{|\Omega|}\left[\frac{p}{2(p-1)}\right]^p\|u_{\epsilon\eta}\|_{L^{p-2}}^{p-2}\leq T\|a\|_{L^\infty}-\underline{k}_1\|u_{\epsilon\eta}\|_{L^2}^2+\overline{k}_2\|v_{\epsilon\eta}\|_{L^2}^2. \tag{34}$$

The same procedure, when it is applied to the second equation of 2, leads to

$$\frac{\mu_q\epsilon^{q-1}}{|\Omega|}\left[\frac{q}{2(q-1)}\right]^q\|v_{\epsilon\eta}\|_{L^{q-2}}^{q-2}\leq T\|b\|_{L^\infty}-\underline{k}_4\|v_{\epsilon\eta}\|_{L^2}^2+\overline{k}_3\|u_{\epsilon\eta}\|_{L^2}^2. \tag{35}$$

Hence from 34 and 35 we have

$$\begin{aligned}\|u_{\epsilon\eta}\|_{L^2}^2\leq&\frac{T\|a\|_{L^\infty}+\overline{k}_2\|v_{\epsilon\eta}\|_{L^2}^2-\frac{\mu_p\epsilon^{p-1}}{|\Omega|}\left[\frac{p}{2(p-1)}\right]^p\|u_{\epsilon\eta}\|_{L^{p-2}}^{p-2}}{\underline{k}_1},\\ \|v_{\epsilon\eta}\|_{L^2}^2\leq&\frac{T\|b\|_{L^\infty}+\overline{k}_3\|u_{\epsilon\eta}\|_{L^2}^2-\frac{\mu_q\epsilon^{q-1}}{|\Omega|}\left[\frac{q}{2(q-1)}\right]^q\|v_{\epsilon\eta}\|_{L^{q-2}}^{q-2}}{\underline{k}_4}.\end{aligned}$$

These two inequalities imply that

$$\begin{aligned}\left(1-\frac{\overline{k}_2\overline{k}_3}{\underline{k}_1\underline{k}_4}\right)\|u_{\epsilon\eta}\|_{L^2}^2<&\frac{T}{\underline{k}_1}\left(\|a\|_{L^\infty}+\frac{\overline{k}_2}{\underline{k}_4}\|b\|_{L^\infty}\right)\\ \left(1-\frac{\overline{k}_2\overline{k}_3}{\underline{k}_1\underline{k}_4}\right)\|v_{\epsilon\eta}\|_{L^2}^2<&\frac{T}{\underline{k}_4}\left(\|b\|_{L^\infty}+\frac{\overline{k}_3}{\underline{k}_1}\|a\|_{L^\infty}\right)\end{aligned}$$

for any $\epsilon,\eta\in(0,1/2)$ and the desired bounds follow since $\overline{k}_2\overline{k}_3<\underline{k}_1\underline{k}_4$. □

As immediate consequences of the previous results we obtain the following corollaries for the cooperative and the competitive cases.

Corollary 3.1. *Assume that*

1. *there are constants* $\underline{k}_i>0$, $i=1,4$, *and* $\overline{k}_i\geq 0$, $i=2,3$, *such that* $\underline{k}_1\underline{k}_4>\overline{k}_2\overline{k}_3$ *and*

$$K_i(x,t)\geq\underline{k}_i \text{ for } i=1,4 \qquad\text{and}\qquad 0\leq K_i(x,t)\leq\overline{k}_i \text{ for } i=2,3,$$

   for a.e. $(x,t)\in Q_T$;
2. *condition 6 holds,*

then problem 1 has a non-negative T*-periodic solution* (u,v) *with non-trivial* u,v.

Corollary 3.2. *Assume that*

1. *there are constants* $\underline{k}_i>0$, $i=1,4$, *and* $\underline{k}_i\geq 0$, $i=2,3$, *such that*

$$K_i(x,t)\geq\underline{k}_i \text{ for } i=1,4 \qquad\text{and}\qquad -\underline{k}_i\leq K_i(x,t)\leq 0 \text{ for } i=2,3,$$

   for a.e. $(x,t)\in Q_T$;

2. *condition 23 of Theorem 2.4, that is $\theta(C_1, C_2) > 0$, is satisfied with*

$$C_1 = \frac{T}{\underline{k}_1}\|a\|_{L^\infty} \quad \text{and} \quad C_2 = \frac{T}{\underline{k}_4}\|b\|_{L^\infty},$$

then problem 1 has a non-negative T-periodic solution (u, v).

We observe that the condition $\overline{k}_2\overline{k}_3 < \underline{k}_1\underline{k}_4$ of Theorem 3.1 is crucial to establish the a priori L^2-bounds on the solution pairs $(u_{\epsilon\eta}, v_{\epsilon\eta})$ of 2. Roughly speaking this condition guarantees that the terms in the equations that contribute to the growth of the respective species do not prevail on the whole on those limiting the growth.

On the other hand, when the strict positivity of the functions K_1 and K_4 is relaxed, obtaining the needed a priori bounds becomes more difficult (at least with our approach). In fact, we are able to obtain simple a priori bounds in the non-coercive case when some sign condition is imposed on the functions K_2 and K_3 (weak and strong competition, see Subsections 3.2 and 3.3), but we have to impose the technical restriction $\min\{n(q-1), m(p-1)\} > 3$ to obtain a result like Theorem 3.1 with no sign condition on the functions K_2 and K_3 (and with rather complicated constants C_1, C_2, see Subsection 3.4).

3.2. **The non-coercive case: weak competition.**

Theorem 3.2. *Assume that*

1. *$K_i(x,t) \geq 0$, $i = 1,4$ and $-\underline{k}_i \leq K_i(x,t) \leq 0$, $i = 2,3$ for a.e. $(x,t) \in Q_T$ and for some non-negative constants $\underline{k}_i$, $i = 2,3$;*
2. *condition 23 of Theorem 2.4, that is $\theta(C_1, C_2) > 0$, is satisfied with*

$$C_1 = |\Omega| T \left\{ \frac{1}{\mu_p}\left[\frac{m(p-1)+1}{p}\right]^p \|a\|_{L^\infty}\right\}^{\frac{2}{m(p-1)-1}}$$

and

$$C_2 = |\Omega| T \left\{ \frac{1}{\mu_q}\left[\frac{n(q-1)+1}{q}\right]^q \|b\|_{L^\infty}\right\}^{\frac{2}{n(q-1)-1}},$$

where μ_p and μ_q are defined in 33. Then problem 1 has a T-periodic non-negative solution (u, v) with non-trivial u, v.

Proof. We begin by finding the bound for the non-negative solutions $u_{\epsilon\eta}$ of the first equation of 22. Since, by the Hölder inequality with $r := \frac{m(p-1)+1}{2}$,

$$\int_\Omega u_{\epsilon\eta}^2 dx \leq |\Omega|^{\frac{m(p-1)-1}{m(p-1)+1}} \left(\int_\Omega u_{\epsilon\eta}^{m(p-1)+1} dx\right)^{\frac{2}{m(p-1)+1}},$$

and, by the Poincaré inequality,

$$\begin{aligned}
\int_\Omega \left(u_{\epsilon\eta}^{\frac{m(p-1)+1}{p}}\right)^p \leq& \frac{1}{\mu_p}\int_\Omega \left|\nabla u_{\epsilon\eta}^{\frac{m(p-1)+1}{p}}\right|^p \\
=& \frac{1}{\mu_p}\left\{\frac{[m(p-1)+1]}{p}\right\}^p \int_\Omega u_{\epsilon\eta}^{(m-1)(p-1)}|\nabla u_{\epsilon\eta}|^p \\
\leq& C_p \int_\Omega (m u_{\epsilon\eta}^{m-1} + \epsilon)^{p-1}|\nabla u_{\epsilon\eta}|^p \\
\leq& C_p \int_\Omega [|\nabla(u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}}(m u_{\epsilon\eta}^{m-1} + \epsilon)|\nabla u_{\epsilon\eta}|^2,
\end{aligned}$$

where $C_p := \{[m(p-1)+1]/p\}^p/\mu_p$, we obtain

$$|\Omega|^{\frac{m(p-1)-1}{m(p-1)+1}} C_p^{\frac{2}{m(p-1)+1}} \left\{ \int_\Omega [|\nabla(u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} (m u_{\epsilon\eta}^{m-1} + \epsilon)|\nabla u_{\epsilon\eta}|^2 \right\}^{\frac{2}{m(p-1)+1}}$$
$$\geq \int_\Omega u_{\epsilon\eta}^2.$$

Integrating over $[0,T]$ it results

$$|Q_T|^{\frac{m(p-1)-1}{m(p-1)+1}} \left\{ C_p \iint_{Q_T} [|\nabla(u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} (m u_{\epsilon\eta}^{m-1} + \epsilon)|\nabla u_{\epsilon\eta}|^2 \right\}^{\frac{2}{m(p-1)+1}}$$
$$\geq \|u_{\epsilon\eta}\|_{L^2}^2,$$

by Hölder's inequality with $r = \frac{m(p-1)+1}{2}$. Multiplying the first equation of 22 by $u_{\epsilon\eta}$, integrating in Q_T and passing to the limit in the Steklov averages $(u_{\epsilon\eta})_h$, we obtain

$$\iint_{Q_T} [|\nabla(u_{\epsilon\eta}^m + \epsilon u_{\epsilon\eta})|^2 + \eta]^{\frac{p-2}{2}} (m u_{\epsilon\eta}^{m-1} + \epsilon)|\nabla u_{\epsilon\eta}|^2 \leq \|a\|_{L^\infty} \|u_{\epsilon\eta}\|_{L^2}^2$$

by the T-periodicity of $u_{\epsilon\eta}$ and the non-positivity of the function K_2. Thus

$$\|u_{\epsilon\eta}\|_{L^2}^2 \leq T|\Omega| \left(C_p \|a\|_{L^\infty}\right)^{\frac{2}{m(p-1)-1}}.$$

In an analogous way we obtain that

$$\|v_{\epsilon\eta}\|_{L^2}^2 \leq T|\Omega| \left\{ \frac{1}{\mu_q} \left[\frac{(n(q-1)+1)}{q} \right]^q \|b\|_{L^\infty} \right\}^{\frac{2}{n(q-1)-1}},$$

if $v_{\epsilon\eta}$ is a solution of the second equation of 22. □

The arguments of the proof of Theorem 3.2 can be easily adapted to show the following result for the case of a single equation with a non-coercive local term.

Corollary 3.3. *Assume $m > 1$, $p > 2$ and that the function K is non-negative. If*

$$\frac{1}{T} \iint_{Q_T} e_1^2 a - \mu_1 > 0,$$

then the problem

$$\begin{cases} u_t - \operatorname{div}(|\nabla u^m|^{p-2} \nabla u^m) = \left[a - \int_\Omega K(\xi,t) u^2(\xi, t-\tau) d\xi \right] u, \\ u(\cdot,t)|_{\partial\Omega} = 0, \quad \textit{for a.e. } t \in (0,T), \\ u(\cdot,0) = u(\cdot,T), \end{cases}$$

has a non-trivial T-periodic solution $u \geq 0$.

3.3. **The non-coercive case: strong competition.**

Theorem 3.3. *Assume that*

1. *$K_i(x,t) \geq 0$, $i = 1,4$ and $-\underline{k}_i \leq K_i(x,t) \leq -\overline{k}_i < 0$, $i = 2,3$ for a.e. $(x,t) \in Q_T$ and for some positive constants $\underline{k}_i, \overline{k}_i$, $i = 2,3$;*

2. *condition 32, i.e. $\zeta(C_1, C_2) > 0$, is satisfied with*

$$C_1 = T\max\left\{|\Omega|T\left[\frac{1}{\mu_p}\left(\frac{m(p-1)+1}{p}\right)^p\|a\|_{L^\infty}\right]^{\frac{2}{m(p-1)-1}}, \frac{\|b\|_{L^\infty}}{\overline{k}_3}\right\}$$

$$C_2 = T\max\left\{|\Omega|T\left[\frac{1}{\mu_q}\left(\frac{n(q-1)+1}{q}\right)^q\|b\|_{L^\infty}\right]^{\frac{2}{n(q-1)-1}}, \frac{\|a\|_{L^\infty}}{\overline{k}_2}\right\}.$$

Then problem 1 has a non-negative T-periodic solution (u, v) with non-trivial u, v.

Proof. If $(u_{\epsilon\eta}, v_{\epsilon\eta})$ is a solution of 22 with $u_{\epsilon\eta} \equiv 0$, or respectively $v_{\epsilon\eta} \equiv 0$, one can argue as in the proof of Theorem 3.2 to obtain that

$$\|v_{\epsilon\eta}\|_{L^2}^2 \le |\Omega|T\left[\frac{1}{\mu_q}\left(\frac{n(q-1)+1}{q}\right)^q\|b\|_{L^\infty}\right]^{\frac{2}{n(q-1)-1}},$$

or respectively,

$$\|u_{\epsilon\eta}\|_{L^2}^2 \le |\Omega|T\left[\frac{1}{\mu_p}\left(\frac{m(p-1)+1}{p}\right)^p\|a\|_{L^\infty}\right]^{\frac{2}{m(p-1)-1}}.$$

If $u_{\epsilon\eta} \neq 0$, then $u_{\epsilon\eta} > 0$ and $v_{\epsilon\eta} \ge 0$ in Q_T by Proposition 2.1. Moreover $u_{\epsilon\eta} \in C(\overline{Q}_T)$ and, hence, there exists $t_1 \in [0, T]$ such that

$$\int_\Omega u_{\epsilon\eta}^2(x, t_1)dx = \min_{t\in[0,T]}\int_\Omega u_{\epsilon\eta}^2(x, t)dx.$$

Multiplying the first equation of 22 by $u_{\epsilon\eta}$, integrating over Ω and using the Steklov averages $(u_{\epsilon\eta})_h$ we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u_{\epsilon\eta})_h^2 \le \left[\|a\|_{L^\infty} - \overline{k}_2\int_\Omega (v_{\epsilon\eta})_h^2(\xi, t-\tau_2)d\xi\right]\int_\Omega (u_{\epsilon\eta})_h^2.$$

Hence, we have

$$\frac{d}{dt}\left[\exp\left\{2\int_{t_1}^t\left(\overline{k}_2\int_\Omega (v_{\epsilon\eta})_h^2(\xi, s-\tau_2)d\xi - \|a\|_{L^\infty}\right)ds\right\}\int_\Omega (u_{\epsilon\eta})_h^2(x,t)dx\right] \le 0,$$

for $t \ge t_1$, which implies that

$$\exp\left\{2\int_{t_1}^t\left[\overline{k}_2\int_\Omega (v_{\epsilon\eta})_h^2(\xi, s-\tau_2)d\xi - \|a\|_{L^\infty}\right]ds\right\}\int_\Omega (u_{\epsilon\eta})_h^2(x,t)dx$$
$$\le \int_\Omega (u_{\epsilon\eta})_h^2(x,t_1)dx$$

for $t \ge t_1$, and, passing to the limit as $h \to 0$ and taking $t = t_1 + T$,

$$\exp\left\{2\overline{k}_2\int_{t_1}^{t_1+T}\int_\Omega v_{\epsilon\eta}^2(\xi, t-\tau_2)d\xi dt - 2T\|a\|_{L^\infty}\right\}\int_\Omega u_{\epsilon\eta}^2(x, t_1+T)dx$$
$$\le \int_\Omega u_{\epsilon\eta}^2(x, t_1)dx$$
$$= \int_\Omega u_{\epsilon\eta}^2(x, t_1+T)dx.$$

Therefore we have that

$$\iint_{Q_T} v_{\epsilon\eta}^2 = \int_{t_1}^{t_1+T}\int_\Omega v_{\epsilon\eta}^2(\xi, t-\tau_2)d\xi dt \le \frac{T\|a\|_{L^\infty}}{\overline{k}_2}$$

by the T-periodicity of $v_{\epsilon\eta}$.

If $v_{\epsilon\eta} \neq 0$, then we can prove that $\|u_{\epsilon\eta}\|_{L^2}^2 \leq T\|b\|_{L^\infty}/\overline{k}_3$ in a similar way. Finally, Remark 2.2 allows to apply Theorem 2.4 with 23 replaced by 32. □

Observe that the conditions $\overline{k}_2 > 0$ and $\overline{k}_3 > 0$ are essential to establish the a priori bounds of the last theorem.

3.4. **The non-coercive case: $\min\{n(q-1), m(p-1)\} > 3$.** In the case that $\min\{n(q-1), m(p-1)\} > 3$, we are able to find explicit bounds (although complicated) without any assumption on the sign of the functions K_2, K_3, as shown in the next result.

Theorem 3.4. *Assume $\min\{n(q-1), m(p-1)\} > 3$ and that*

1. *$K_i(x,t) \geq 0$, $i = 1,4$ and $K_i(x,t) \leq \overline{k}_i$, $i = 2,3$ for a.e. $(x,t) \in Q_T$ and for some positive constants $\overline{k}_i$, $i = 2,3$;*
2. *condition 23 of Theorem 2.4, that is $\theta(C_1, C_2) > 0$, is satisfied with*

$$
\begin{aligned}
C_1 =& T\left\{\frac{[n(q-1)-1][m(p-1)-1]}{[n(q-1)-1][m(p-1)-1]-4}\left[(2M_p^2\|a\|_{L^\infty}^2)^{\frac{2}{m(p-1)-1}}\right.\right.\\
&\left.\left.+\left(2M_p^2\overline{k}_2^{-2}(2M_q^2\|b\|_{L^\infty}^2)^{\frac{2}{n(q-1)-1}}\right)^{\frac{2}{m(p-1)-1}}\right]\right\}^{1/2},\\
C_2 =& T\left\{\frac{[n(q-1)-1][m(p-1)-1]}{[n(q-1)-1][m(p-1)-1]-4}\left[(2M_q^2\|b\|_{L^\infty}^2)^{\frac{2}{n(q-1)-1}}\right.\right.\\
&\left.\left.+\left(2M_q^2\overline{k}_3^{-2}(2M_p^2\|a\|_{L^\infty}^2)^{\frac{2}{m(p-1)-1}}\right)^{\frac{2}{n(q-1)-1}}\right]\right\}^{1/2},
\end{aligned}
\tag{36}
$$

where

$$
M_p = \frac{|\Omega|^{\frac{m(p-1)-1}{2}}}{\mu_p}\left[\frac{m(p-1)+1}{p}\right]^p \quad \text{and} \quad M_q = \frac{|\Omega|^{\frac{n(q-1)-1}{2}}}{\mu_q}\left[\frac{n(q-1)+1}{q}\right]^q. \tag{37}
$$

Then problem 1 has a non-negative T-periodic solution (u,v) with non-trivial u, v.

Proof. Let $(u_{\epsilon\eta}, v_{\epsilon\eta})$ be a solution of 22. We have

$$
\begin{aligned}
\mu_p\left[\frac{p}{m(p-1)+1}\right]^p\int_\Omega u_{\epsilon\eta}^{m(p-1)+1} &\leq \left[\frac{p}{m(p-1)+1}\right]^p\int_\Omega\left|\nabla u_{\epsilon\eta}^{\frac{m(p-1)+1}{p}}\right|^p\\
&\leq \int_\Omega (mu_{\epsilon\eta}^{m-1}+\epsilon)^{p-1}|\nabla u_{\epsilon\eta}|^p\\
&\leq \int_\Omega [|\nabla(u_{\epsilon\eta}^m+\epsilon u_{\epsilon\eta})|^2+\eta]^{\frac{p-2}{2}}\nabla(u_{\epsilon\eta}^m+\epsilon u_{\epsilon\eta})\nabla u_{\epsilon\eta}
\end{aligned}
\tag{38}
$$

by the Poincaré inequality. Since $m > 1$ and $p > 2$ it results

$$
\left(\int_\Omega u_{\epsilon\eta}^2\right)^{\frac{m(p-1)+1}{2}} \leq |\Omega|^{\frac{m(p-1)-1}{2}}\int_\Omega u_{\epsilon\eta}^{m(p-1)+1}.
$$

The previous inequality and 38 imply

$$\begin{aligned}
&\frac{\mu_p}{|\Omega|^{\frac{m(p-1)-1}{2}}}\left[\frac{p}{m(p-1)+1}\right]^p\left(\int_\Omega u_{\epsilon\eta}^2\right)^{\frac{m(p-1)+1}{2}} \\
&\le \int_\Omega [|\nabla(u_{\epsilon\eta}^m+\epsilon u_{\epsilon\eta})|^2+\eta]^{\frac{p-2}{2}}\nabla(u_{\epsilon\eta}^m+\epsilon u_{\epsilon\eta})\nabla u_{\epsilon\eta}.
\end{aligned} \tag{39}$$

Multiplying the first equation of 22 by $u_{\epsilon\eta}$, integrating in Q_T and passing to the limit in the Steklov averages $(u_{\epsilon\eta})_h$, we obtain by the T-periodicity of $u_{\epsilon\eta}$

$$\begin{aligned}
&\iint_{Q_T} [|\nabla(u_{\epsilon\eta}^m+\epsilon u_{\epsilon\eta})|^2+\eta]^{\frac{p-2}{2}}\nabla(u_{\epsilon\eta}^m+\epsilon u_{\epsilon\eta})\nabla u_{\epsilon\eta} \\
&\le \int_0^T\left[\|a\|_{L^\infty}+\overline{k}_2\int_\Omega v_{\epsilon\eta}^2(\xi,t-\tau_2)d\xi\right]\left(\int_\Omega u_{\epsilon\eta}^2\right)dt \\
&\le \left[\int_0^T\left(\|a\|_{L^\infty}+\overline{k}_2\int_\Omega v_{\epsilon\eta}^2(\xi,t-\tau_2)d\xi\right)^2dt\right]^{\frac{1}{2}}\left[\int_0^T\left(\int_\Omega u_{\epsilon\eta}^2\right)^2\right]^{\frac{1}{2}}.
\end{aligned} \tag{40}$$

By 39 and 40 it follows

$$\begin{aligned}
&\int_0^T\left(\int_\Omega u_{\epsilon\eta}^2\right)^{\frac{m(p-1)+1}{2}} \\
\le& M_p\left[\int_0^T\left(\|a\|_{L^\infty}+\overline{k}_2\int_\Omega v_{\epsilon\eta}^2(\xi,t-\tau_2)d\xi\right)^2dt\right]^{\frac{1}{2}}\left[\int_0^T\left(\int_\Omega u_{\epsilon\eta}^2\right)^2\right]^{\frac{1}{2}},
\end{aligned}$$

where M_p is defined in 37. On the other hand,

$$\int_0^T\left(\int_\Omega u_{\epsilon\eta}^2\right)^2 \le T^{\frac{m(p-1)-3}{m(p-1)+1}}\left[\int_0^T\left(\int_\Omega u_{\epsilon\eta}^2\right)^{\frac{m(p-1)+1}{2}}\right]^{\frac{4}{m(p-1)+1}}$$

by Hölder's inequality with $r=\frac{m(p-1)+1}{4}$. Therefore, setting $U=\int_0^T(\int_\Omega u_{\epsilon\eta}^2)^2$ and $V=\int_0^T(\int_\Omega v_{\epsilon\eta}^2)^2$, these last two inequalities imply

$$\begin{aligned}
U \le& T^{\frac{m(p-1)-3}{m(p-1)-1}}M_p^{\frac{4}{m(p-1)-1}}\left[\int_0^T\left(\|a\|_{L^\infty}+\overline{k}_2\int_\Omega v_{\epsilon\eta}^2\right)^2\right]^{\frac{2}{m(p-1)-1}} \\
\le& T^{\frac{m(p-1)-3}{m(p-1)-1}}M_p^{\frac{4}{m(p-1)-1}}\left[2T\|a\|_{L^\infty}^2+2\overline{k}_2^2V\right]^{\frac{2}{m(p-1)-1}} \\
\le& T\left(2M_p^2\|a\|_{L^\infty}^2\right)^{\frac{2}{m(p-1)-1}}+T^{\frac{m(p-1)-3}{m(p-1)-1}}\left(2M_p^2\overline{k}_2^2\right)^{\frac{2}{m(p-1)-1}}V^{\frac{2}{m(p-1)-1}}
\end{aligned}$$

thanks to the fact that $\min\{n(q-1),m(p-1)\}>3$. In an analogous way we can show that

$$V\le T\left(2M_q^2\|b\|_{L^\infty}^2\right)^{\frac{2}{n(q-1)-1}}+T^{\frac{n(q-1)-3}{n(q-1)-1}}\left(2M_q^2\overline{k}_3^2\right)^{\frac{2}{n(q-1)-1}}U^{\frac{2}{n(q-1)-1}},$$

where M_q is defined in 36. Hence,

$$\begin{aligned}U \leq & T^{\frac{m(p-1)-3}{m(p-1)-1}} \left(2M_p^2\overline{k}_2^2\right)^{\frac{2}{m(p-1)-1}} \Bigg[T\left(2M_q^2\|b\|_{L^\infty}^2\right)^{\frac{2}{n(q-1)-1}} \\ & + T^{\frac{n(q-1)-3}{n(q-1)-1}} \left(2M_q^2\overline{k}_3^2 U\right)^{\frac{2}{n(q-1)-1}}\Bigg]^{\frac{2}{m(p-1)-1}} + T\left(2M_p^2\|a\|_{L^\infty}^2\right)^{\frac{2}{m(p-1)-1}} \\ \leq & T\left(2M_p^2\|a\|_{L^\infty}^2\right)^{\frac{2}{m(p-1)-1}} + T\left[2M_p^2\overline{k}_2^2\left(2M_q^2\|b\|_{L^\infty}^2\right)^{\frac{2}{n(q-1)-1}}\right]^{\frac{2}{m(p-1)-1}} \\ & + \left(T^{\frac{[m(p-1)-2][n(q-1)-1]-2}{[n(q-1)-1][m(p-1)-1]}} 2M_p^2\overline{k}_2^2\left(2M_q^2\overline{k}_3^2 U\right)^{\frac{2}{n(q-1)-1}}\right)^{\frac{2}{m(p-1)-1}}.\end{aligned}$$

The last inequality has the form:

$$U \leq \alpha + \beta U^{\frac{4}{[n(q-1)-1][m(p-1)-1]}},$$

with $\alpha, \beta > 0$. Since $m(p-1) > 3$ the function $f(U) := \alpha + \beta U^{\frac{4}{[n(q-1)-1][m(p-1)-1]}}$ is concave, and then

$$U \leq f(U) \leq f(U_0) + f'(U_0)(U - U_0), \tag{41}$$

where $U_0 := \beta^{\frac{[n(q-1)-1][m(p-1)-1]}{[n(q-1)-1][m(p-1)-1]-4}}$. Using the fact that $f(U_0) = \alpha + U_0$ and 41, one has

$$U \leq \frac{[n(q-1)-1][m(p-1)-1]}{[n(q-1)-1][m(p-1)-1]-4}\alpha + \beta^{\frac{[n(q-1)-1][m(p-1)-1]}{[n(q-1)-1][m(p-1)-1]-4}}.$$

A final application of Hölder's inequality shows that $\|u_{\epsilon\eta}\|_{L^2}^2 \leq T^{1/2}U^{1/2} = C_1$. The argument for $v_{\epsilon\eta}$ proceeds in a similar way. □

In the cooperative case we can ignore the explicit value of the constants in 36 and obtain the following cleaner looking corollary thanks to Remark 2.1.

Corollary 3.4. *Assume that*

1. $\min\{n(q-1), m(p-1)\} > 3$;
2. $K_i(x,t) \geq 0$ *for* $i = 1, 4$, *for a.e.* $(x,t) \in Q_T$, *and there are positive constants* $\overline{k}_2, \overline{k}_3$ *such that*
$$0 \leq K_i(x,t) \leq \overline{k}_i \text{ for } i = 2, 3,$$
for a.e. $(x,t) \in Q_T$;
3. *condition 6 holds,*

then problem 1 has a non-negative T*-periodic solution* (u, v) *with non-trivial* u, v.

Another consequence of Theorem 3.4 is the following corollary for the competitive case.

Corollary 3.5. *Assume that*

1. $\min\{n(q-1), m(p-1)\} > 3$;
2. $K_i(x,t) \geq 0$ *for* $i = 1, 4$, *for a.e.* $(x,t) \in Q_T$, *and there are non-negative constants* $\underline{k}_2, \underline{k}_3$ *such that*
$$-\underline{k}_i \leq K_i(x,t) \leq 0 \text{ for } i = 2, 3,$$
for a.e. $(x,t) \in Q_T$;

3. condition 23 of Theorem 2.4, that is $\theta(C_1, C_2) > 0$, is satisfied with

$$C_1 = T\left\{\frac{[n(q-1)-1][m(p-1)-1]}{[n(q-1)-1][m(p-1)-1]-4}\left(2M_p^2\|a\|_{L^\infty}^2\right)^{\frac{2}{m(p-1)-1}}\right\}^{1/2},$$

$$C_2 = T\left\{\frac{[n(q-1)-1][m(p-1)-1]}{[n(q-1)-1][m(p-1)-1]-4}\left(2M_q^2\|b\|_{L^\infty}\right)^{\frac{2}{n(q-1)-1}}\right\}^{1/2},$$

where

$$M_p = \frac{|\Omega|^{\frac{m(p-1)-1}{2}}}{\mu_p}\left[\frac{m(p-1)+1}{p}\right]^p \text{ and } M_q = \frac{|\Omega|^{\frac{n(q-1)-1}{2}}}{\mu_q}\left[\frac{n(q-1)+1}{q}\right]^q.$$

Then problem 1 has a non-negative T-periodic solution (u, v).

Remark 3.1. Observe that the a priori bounds C_1, C_2 in Theorem 3.1 do not allow to replace the condition 23 with 32, since

$$\min\left\{\frac{\overline{k}_1\underline{k}_4}{\underline{k}_1\underline{k}_4 - \overline{k}_2\overline{k}_3}, \frac{\underline{k}_1\overline{k}_4}{\underline{k}_1\underline{k}_4 - \overline{k}_2\overline{k}_3}\right\} \geq 1,$$

which implies that $\zeta(C_1, C_2) \leq -\mu_1$. Analogously, in Theorem 3.3 the condition 32 cannot be replaced by 23 since in this case we have that

$$\min\left\{\frac{\underline{k}_2}{\overline{k}_2}, \frac{\underline{k}_3}{\overline{k}_3}\right\} \geq 1,$$

which implies that $\theta(C_1, C_2) \leq -\mu_1$. Finally, in the other cases, namely Theorems 3.2 and 3.4, the constants C_1, C_2 allow to employ indifferently any of the two conditions.

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Received xxxx 20xx; revised xxxx 20xx.

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