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## Identification of two cracks in a rod by minimal resonant and antiresonant frequency data

Lourdes Rubio<sup>a</sup>, José Fernández-Sáez<sup>b</sup>, Antonino Morassi<sup>c,\*</sup><sup>a</sup> Department of Mechanical Engineering, Carlos III University of Madrid, Avda. de la Universidad 30, 28911 Leganés, Madrid, Spain<sup>b</sup> Department of Continuum Mechanics and Structural Analysis, Carlos III University of Madrid, Avda. de la Universidad 30, 28911 Leganés, Madrid, Spain<sup>c</sup> Università degli Studi di Udine, Dipartimento di Ingegneria Civile e Architettura, via Cotonificio 114, 33100, Udine, Italy

### ARTICLE INFO

#### Article history:

Received 24 December 2013

Accepted 14 January 2015

Available online 16 February 2015

#### Keywords:

Multiple cracks

Damage identification

Natural frequencies

Antiresonant frequencies

Rods

### ABSTRACT

In this paper we consider the identification of two cracks of equal severity in a uniform free–free rod under longitudinal vibration. Each crack is simulated by a translational spring connecting the two adjacent segments of the rod and the cracks are considered to be small. We show that the inverse problem can be formulated and solved in terms of three frequency data only, corresponding to a suitable set of low resonant and antiresonant frequencies. Closed-form expressions of the damage parameters in terms of the measured frequency shifts are obtained. The paper improves existing results available in the literature, since the use of antiresonant frequencies allows us to exclude all the symmetrical crack locations occurring when only natural frequency are used as data. The analysis also explains why the use of high frequency data introduces spurious damage locations in the inverse problem solution. Numerical simulations show that if accurate input data are available then damage identification leads to satisfactory results.

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## 1. Introduction

Dynamic methods are a useful diagnostic tool for several applications in mechanical and civil engineering. Diagnostic techniques based on natural frequency data, in particular, have the advantage of being simple to carry out in practice, since they require a limited experimental commitment and can be easily repeatable during the structure's lifetime. In addition, frequencies can be measured more easily than can mode shapes, and are less affected by experimental errors. This class of diagnostic methods operates on a global scale and does not require a priori information on the damaged area. Their global character, however, has the disadvantage of introducing synthetic information on the formulation of the inverse problem. Therefore, to be effective, these techniques often need additional information, such as a knowledge of the undamaged configuration and of the characteristics of the defect to be identified (localized or diffuse damage, for example). Another aspect worth of noticing is the relatively low sensitivity of the natural frequencies to damage, see, among other contributions, [1,2]. This is a peculiarity also of other damage indicators [3], and negative effects can be controlled by reducing the experimental errors and using accurate mechanical models for the interpretation of the measurements [4].

\* Corresponding author. Tel.: +39 0432 558739; fax: +39 0432 558700.

E-mail addresses: [lrubio@ing.uc3m.es](mailto:lrubio@ing.uc3m.es) (L. Rubio), [ppfer@ing.uc3m.es](mailto:ppfer@ing.uc3m.es) (J. Fernández-Sáez), [antonino.morassi@uniud.it](mailto:antonino.morassi@uniud.it) (A. Morassi).

As a confirmation of the growing interest on this class of diagnostic methods, we recall that an extensive line of research on damage identification by frequency data has been developed over the past three decades since the pioneering paper by Adams et al. [5]. The damages commonly considered have localized nature, are notches or cracks, as this class of defects frequently occurs in engineering applications. Most of the results concern damage identification in one-dimensional elements, such as beams or bars. Indeed, in addition to the high number of applications, cracked beams have the advantage of being described by consolidated mechanical models and being relatively manageable from the mathematical point of view, at least in the case of cracks that remain open during the vibration of the system. Of course, there are contexts in which it is necessary to take into account the phenomena of opening/closing of cracks, and more sophisticated nonlinear models of crack must be implemented, see, among other contributions, [6] and [7] for an analysis of the direct problem in beams and rotors, respectively, and [8] for the identification of breathing cracks in a vibrating beam. Without claiming of completeness, the reader may refer to [9–13] for an overview of some recent contributions on damage detection based on frequency data.

Despite the extensive literature on identification of cracks in rods and beams by frequency data, some basic problems are still open. Among the problems for which a satisfactory knowledge is not available yet, there is the identification of multiple cracks.

One of the first contributions to the treatment of the direct problem is due to Ostachowicz and Krawczuk [14], who considered the effect of two open cracks on the lower natural frequencies and vibration modes of a cantilever beam in bending. Each crack was modelled as a massless linear elastic rotational spring located at the cracked cross-section, according to arguments of Fracture Mechanics [15]. Later on, Ruotolo and Shifrin [16] presented an efficient technique for solving the eigenvalue problem of the free bending vibration of a multicroaked beam. Main advantage of their approach was in the reduction of the differential equations between cracks to a single differential equation on the whole beam axis interval. The method was later applied to longitudinal vibration of a multi-cracked rod by Ruotolo and Surace [17]. In this case, each crack is included in the one-dimensional rod model as a massless linear elastic translational spring located at the damaged cross-section. Among the recent contributions, worth of mention is the study developed by Caddemi and Calió [18], who derived exact closed-form solutions for the free-vibration of a uniform Euler–Bernoulli beam in the presence of multiple open cracks mathematically modelled as Dirac's delta functions in the bending stiffness coefficient. Other interesting contributions to the direct problem appeared in the last few years, but, since our main goal is the analysis of the inverse problem, we refer the reader to the introduction of the paper [18] for an updated overview on this topic.

Results on the inverse problem of identifying multiple (open) cracks in rods and beams from frequency data are less numerous, see the updated state-of-the-art on identification and conditioning monitoring for multi-cracked structures by Sekhar [19] (Section 5), and [20] for an application to model-based identification of two cracks in a rotor system.

Assuming as above the linear concentrated flexibility model to describe cracks in rods and beams, one approach consists in considering as many natural frequencies as the unknowns of the problem (two unknowns for each crack, the position and the severity), and then solve the system formed by the characteristic equation written for all the natural frequencies in terms of the damage parameters, see, for example, [21,22]. Inverse transcendental eigenvalue problems for the identification of multiple open cracks in a longitudinally vibrating rod were considered by Singh [23]. An iterative procedure based on a suitable Taylor series expansion of the system of characteristic equations for the damaged rod was used to estimate the damage parameters. Singh noticed that the possible presence of spurious solutions can be avoided by carefully selecting the data and using simultaneously natural frequency and antiresonant frequency measurements. This is a powerful class of methods, but has the drawback of requiring a strong support on numerical simulation, with the consequence of making difficult to find out general properties, such as, for example, the indication of optimal data to be used in order to reduce the non-uniqueness effects in the inverse problem solution.

Another approach to multi-cracked identification transforms the inverse problem to an optimization issue. It consists in determining the damage parameters such that the natural frequencies of the mechanical model are closest (in some least square sense) to those found experimentally, see [24,25] and, for a linearized version suitable in the case of small cracks, [26,12]. An error function which measures the distance between experimental and analytical frequency values is minimized via gradient-type methods. This class of techniques allows us to dealing with a large number of cracks and system of high complexity (beams of variable profile under general set of end conditions), but the approach has several drawbacks mainly connected with the non-convexity of the error function and, as a consequence, with the appearance of several local and global minima. Basic questions such as how accurate the description of the reference configuration has to be or how many data are necessary to ensure uniqueness of the solution (at least in local sense) are rarely discussed in the literature and are mainly still open.

In the case of multi-cracked rods under longitudinal vibration, an attempt to use the classical results of the spectral theory for Sturm–Liouville operators has been done in [27]. The authors proved that knowledge of the highest part of a single spectrum of a rod with multiple cracks suffices to determine uniquely the (unordered) set of lengths of the segments of bar separating the cracks. Unfortunately, no reconstruction algorithm for the position of the cracks was provided in that study. The most general result along this direction has been obtained by Shifrin [28]. Shifrin proposed a constructive procedure for identifying the position and severity of multiple cracks in a longitudinally vibrating rod by the knowledge of two full spectra corresponding to different boundary conditions. It is worth noticing that, different from most of the methods available in the literature, Shifrin's technique does not assume that the number of cracks is known. However, the reconstruction procedure needs a large number of eigenvalues (independently on the number of cracks) to obtain stable and accurate estimates: two infinite spectra, in principle, or at least a number of eigenvalues of the order of 10 for each spectrum in the applications presented in the above-mentioned paper [28]. This seems to be a possible limitation of the method because, first, it is difficult to obtain accurate measurement of high frequencies in practice and, second, because the analytical model of the rod under longitudinal vibration loses accuracy as the eigenvalue order increases.

Taking into account the above results, a different line of research has been developed in the last years by Narkis [29] and Morassi and co-workers [30,31]. It relies on the use of minimal number of low frequency data to identify localized damages in rods and beams. The reduction in the request of input information is somehow compensated by the introduction of suitable a priori information on the problem. Precisely, it is assumed that the damage is small, that is the damaged configuration is a perturbation of the initial one, and that the number of cracks to be identified is known. The first assumption is not a severe limitation because in most practical applications it is crucial to detect the damage right as it arises. Concerning the second hypothesis, in many engineering applications few cracks usually are expected to occur and their number could be estimated on the basis of the engineering experience on the problem. Alternatively, it is well known that other methods, such as the variational ones (see, for example, the analysis by [10]) or the Generalized Fourier Coefficient Method [32,33], can be used to foresee the number of localized damages.

Along this line of research, the only known result for multiple cracks is due to Morassi and Rollo [34]. It concerns the identification of two cracks having equal severity in a uniform beam by frequency data. Focussing the attention on a free–free longitudinally vibrating rod (discussed in the last paragraph of Section 3 of the above-mentioned paper), the authors proved that the inverse problem of identifying the positions and the common severity of the two cracks can be formulated and solved in terms of the first three natural frequencies of the rod. The inverse problem turns out to be ill-posed, namely cracks with different severity in two sets of different locations can produce identical changes in first three natural frequencies. One source of ill-posedness is the symmetry of the problem, which prevents to distinguish symmetrical positions of the cracks when only natural frequencies are used as data.

In this paper we improve the result by Morassi and Rollo [34] showing that all the symmetrical damage configurations can be excluded by using suitable resonant frequency and antiresonant frequency data.

A typical result of our analysis, shown in Section 3, is as follows. Let  $H = H(\omega)$  be the frequency response function determined by exciting and measuring the response of the rod at the same end, and consider as spectral data the first and the second antiresonant frequency of  $H(\omega)$  and the first (elastic) resonance of the rod. We show that there exist only two damage configurations, corresponding to different values of the damage severity and different positions of the cracks, which correspond to the given spectral data. One configuration is the exact one, the other one is a spurious solution coming from the mathematical nature of the inverse problem, see also Singh [23] for a detailed numerical treatment of this aspect. The spurious solution could be detected and excluded from the analysis by crossing the above results with those obtained using information on high resonant and antiresonant frequency data. This part is discussed in Section 4.

It is worth noticing that the damage parameters are obtained as solutions of second degree polynomial equations, whose coefficients depend on the measured data. Therefore, they can be determined by simple closed form expressions in terms of the eigenfrequency shifts induced by the damage. This fact is not trivial at all, since the resolving system of equations is highly non-linear with respect to the crack positions, in spite of the fact that the function expressing the dependence of the eigenvalue on the crack severity is linearized in a neighborhood of the undamaged rod. It should be also recalled that the proposed procedure does not introduce additional experimental burden, since antiresonant data can be extracted together with natural frequencies by the same frequency response function  $H(\omega)$  measured at one end of the rod. In Section 5 it is shown that the above results can be extended to the torsional vibration of shafts with two circumferential cracks, a model problem that has several applications in rotor dynamics.

A series of numerical simulations performed on free–free cracked rods supported the theoretical result. Numerical results, presented in Section 6, show that if the frequency data used in identification are affected by errors relatively small with respect to the variations of the frequency data induced by the damage, then damage identification leads to satisfactory indications.

## 2. Formulation of the problem and perturbation analysis

In the present section we introduce the sensitivity of natural frequencies and antiresonant frequencies to cracks and we formulate the inverse problem.

Let us consider a straight thin rod under longitudinal vibration and with free–free end conditions (F–F). We assume that the rod is uniform and has two cracks of equal severity at two different cross-sections of abscissa  $s_1$  and  $s_2$ , with  $0 < s_i < L$  for  $i = 1, 2$ , where  $L$  is the length of the rod. Each crack is assumed to remain open during vibration and is modelled as a massless longitudinal linearly elastic spring with stiffness  $K$ , see, for example, Freund and Herrmann [15] or Cabib et al. [35]. The value of  $K$  can be determined in terms of the geometry of the cracked cross-section and of the material properties of the beam, see Section 6 for a specific expression in the case of rectangular cross-section. The free undamped longitudinal vibrations of the rod with radian frequency  $\omega$  and spatial amplitude  $w = w(x)$  are governed by the following eigenvalue problem:

$$\begin{cases} EAw'' + \lambda\gamma Aw = 0, & x \in (0, s_1) \cup (s_1, s_2) \cup (s_2, L), & \text{(a)} \\ w'(s_i^-) = w'(s_i^+), & i = 1, 2, & \text{(b)} \\ K(w(s_i^+) - w(s_i^-)) = EAw(s_i), & i = 1, 2, & \text{(c)} \\ w'(0) = w'(L) = 0, & & \text{(d)} \end{cases} \quad (1)$$

where  $\lambda = \omega^2$ ,  $E$  is the (constant) Young's modulus of the material,  $A$  is the area of the transversal cross-section and  $\gamma$  is the (uniform) volume mass density of the material. Under our assumptions, there exists an infinite sequence of real numbers  $\{\lambda_n^{F-F}\}_{n=0}^{\infty}$  (eigenvalues), with  $0 = \lambda_0^{F-F} < \lambda_1^{F-F} < \lambda_2^{F-F} < \dots$  and  $\lim_{n \rightarrow \infty} \lambda_n^{F-F} = \infty$ , such that the problem (1a)–(1d) has no trivial

solution  $w_n^{F-F} = w_n^{F-F}(x)$  (eigenfunction associated to the eigenvalue  $\lambda_n^{F-F}$ ). The undamaged system corresponds to taking in (1a)–(1d) the limit as  $K \rightarrow \infty$  or, equivalently, as  $\epsilon \equiv 1/K \rightarrow 0^+$ . The mass-normalized eigenpairs of the undamaged rod  $\{\lambda_n^{F-F(U)}, w_n^{F-F(U)}(x)\}_{n=0}^\infty$  can be evaluated explicitly and are equal to

$$\lambda_n^{F-F(U)} = \frac{E}{\gamma} \left( \frac{n\pi}{L} \right)^2, \quad w_n^{F-F(U)}(x) = \sqrt{\frac{2}{\gamma AL}} \cos \left( \frac{n\pi x}{L} \right), \quad n = 0, 1, 2, \dots \quad (2)$$

As the cracks are small, namely  $\epsilon$  is small enough, we can use a standard approach to find the first order perturbation on the eigenvalues with  $\epsilon$ , see, for example, the analysis by Morassi [36] and Loya et al. [37]. By taking

$$\lambda_n^{F-F} = \lambda_n^{F-F(U)} + \epsilon \Delta \lambda_n^{F-F}, \quad (3)$$

we find

$$\delta \lambda_n^{F-F} \equiv \epsilon \Delta \lambda_n^{F-F} = - \frac{(N_n^{(U)}(s_1))^2 + (N_n^{(U)}(s_2))^2}{K} \quad (4)$$

for every  $n \geq 0$ , where the axial force associated to the  $n$ th undamaged vibration mode is given by

$$N_n^{(U)}(x) = EA \frac{d}{dx} (w_n^{F-F(U)}(x)). \quad (5)$$

The first eigenvalue is insensitive to damage, since the corresponding vibration mode is a longitudinal rigid motion. Inserting the expression (2) of the eigenpairs of the undamaged rod into Eq. (4) and elaborating, we find

$$C_n^{F-F} = \frac{1}{K} \left( \sin^2 \frac{2n\pi s_1}{L} + \sin^2 \frac{2n\pi s_2}{L} \right), \quad n \geq 1, \quad (6)$$

where the non-negative quantity

$$C_n^{F-F} = - \frac{\delta \lambda_n^{F-F}}{2EA \lambda_n^{F-F(U)}}, \quad n \geq 1, \quad (7)$$

only depends on the undamaged rod properties and on the eigenvalue shift induced by the damage on the  $n$ th eigenvalue.

The effect of the two cracks on antiresonances of frequency response functions (FRF) of the rod will be now investigated. It is well known that antiresonances are the zeros of the FRF  $H^{F-F}(\omega, x_i, x_o)$ , where  $x_i$  and  $x_o$  are the abscissas of the excitation point and of the measurement point, respectively, and  $\omega$  is the frequency variable, see the paper by Wahl et al. [38]. When  $x_i = x_o$ , the zeros of the FRF  $H^{F-F}(\omega, x_i, x_i)$  are the frequencies of a rod in which the longitudinal displacement at the cross-section of abscissa  $x_i$  is hindered. Therefore, under the assumption of small cracks, on proceeding as above and with the same notation, the first order variation of the (square of the)  $m$ th antiresonance of the point FRF  $H^{F-F}(\omega, x_i, x_i)$  with respect to  $\epsilon = 1/K$  can be evaluated by Eqs. (3) and (4).

Let us consider, in particular, the FRF  $H^{F-F}(\omega, 0, 0)$  of the F–F rod for  $x_i = x_o = 0$ . The antiresonances of  $H^{F-F}(\omega, 0, 0)$  are the (square root of the) eigenvalues of the rod with left end, at  $x=0$ , fixed, namely the eigenvalues  $\lambda_m^{S-F}$  of the supported–free rod (denoted as S–F in the sequel). It follows that their first order variation with respect to damage coincides with the first order variation  $\delta \lambda_m^{S-F} = \epsilon \Delta \lambda_m^{S-F}$  of the eigenvalues  $\lambda_m^{S-F}$  of the S–F rod. The eigenpairs of the undamaged S–F rod are given by

$$\lambda_m^{S-F(U)} = \frac{E}{\gamma} \left( \frac{(1+2m)\pi}{2L} \right)^2, \quad w_m^{S-F(U)}(x) = \sqrt{\frac{2}{\gamma AL}} \sin \left( \frac{(1+2m)\pi x}{2L} \right), \quad (8)$$

$m = 0, 1, 2, \dots$

Let us introduce the quantities

$$C_m^{S-F} = - \frac{\delta \lambda_m^{S-F}}{2EA \lambda_m^{S-F}}, \quad m \geq 0, \quad (9)$$

analogous to those appearing in (7). By proceeding as exemplified above and under the same notation, we can write the following additional relationships between the measured quantities  $C_m^{S-F}$  and the unknowns  $\{s_1, s_2, K\}$ :

$$C_m^{S-F} = \frac{1}{K} \left( \cos^2 \frac{(2m+1)\pi s_1}{2L} + \cos^2 \frac{(2m+1)\pi s_2}{2L} \right), \quad m \geq 0. \quad (10)$$

Our diagnostic problem consists in finding the severity of the two cracks  $K$ ,  $K > 0$ , and their positions  $s_1$  and  $s_2$ , with  $0 < s_i < L$ ,  $i = 1, 2$ , from natural frequency data taken on the F–F rod and from antiresonant frequency data extracted from the FRF  $H^{F-F}(\omega, 0, 0)$ . We note that the two damage configurations  $\{(s_1, K), (s_2, K), s_1 < s_2\}$  and  $\{(s_2, K), (s_1, K), s_2 < s_1\}$  are indistinguishable (since the two cracks have the same severity). Then, without loss of generality we can assume  $s_1 < s_2$ .

In the next section we shall show how to select minimal data in order to properly formulate the inverse diagnostic problem and to find closed-form expressions of the damage parameters.

### 3. Identification of two equal cracks in a F-F rod

In this section we assume that frequency data consist of the first and second antiresonant frequencies of the FRF  $H^{F-F}(\omega, 0, 0)$ , namely  $\lambda_0^{S-F}$  and  $\lambda_1^{S-F}$  (note that the antiresonances  $\lambda_m^{S-F}$  are enumerated from  $m=0$ ), and the first resonant frequency  $\lambda_1^{F-F}$  of the F-F rod.

By inserting the expressions of the corresponding eigenmodes into Eqs. (7) and (9), we obtain the following system of three nonlinear equations in the unknowns  $\{K, s_1, s_2\}$ :

$$\begin{cases} \frac{1}{K} \left( \cos^2 \frac{2\pi s_1}{2L} + \cos^2 \frac{2\pi s_2}{2L} \right) = C_1, & (a) \\ \frac{1}{K} \left( \sin^2 \frac{2\pi s_1}{L} + \sin^2 \frac{2\pi s_2}{L} \right) = C_2, & (b) \\ \frac{1}{K} \left( \cos^2 \frac{3\pi s_1}{2L} + \cos^2 \frac{3\pi s_2}{2L} \right) = C_3, & (c) \end{cases} \quad (11)$$

where we have defined

$$C_1 = C_0^{S-F}, \quad C_2 = C_1^{F-F}, \quad C_3 = C_1^{S-F}. \quad (12)$$

Note that  $C_i > 0$  for every  $i = 1, 2, 3$ . No general analytical technique seems to be available for solving the above nonlinear system in terms of the damage parameters. We shall prove that, by introducing a suitable change of the damage position variables, the system (11a)–(11c) can be rewritten in a form such that the explicit resolution method developed by Morassi and Rollo [34] can be applied.

As a first step, we use the trigonometric identity  $\sin^2 \alpha + \cos^2 \alpha = 1$  and we rewrite (11a)–(11c) as

$$\begin{cases} \sin^2 \frac{2\pi s_1}{2L} + \sin^2 \frac{2\pi s_2}{2L} = 2 - KC_1, & (a) \\ \sin^2 \frac{2\pi s_1}{L} + \sin^2 \frac{2\pi s_2}{L} = KC_2, & (b) \\ \sin^2 \frac{3\pi s_1}{2L} + \sin^2 \frac{3\pi s_2}{2L} = 2 - KC_3. & (c) \end{cases} \quad (13)$$

The key idea is to introduce two crack location variables different from those used by Morassi and Rollo [34], namely

$$x = \cos \frac{\pi s_1}{L} \in (-1, 1), \quad y = \cos \frac{\pi s_2}{L} \in (-1, 1). \quad (14)$$

Note that the function  $f(s) = \cos(\pi s/L)$  is a one-to-one correspondence between the interval  $(0, L)$  and the interval  $(-1, 1)$ . Therefore, if we were able to find the unknowns  $\{x, y\}$ , then we could determine uniquely the associated damage locations  $\{s_1, s_2\}$ .

By using the trigonometrical identity  $\sin^2 \alpha/2 = (1 - \cos \alpha)/2$  in (13a), we obtain

$$x + y = 2(KC_1 - 1). \quad (15)$$

In Eq. (13b) we use again the identity  $\sin^2 \alpha + \cos^2 \alpha = 1$ , and we find

$$x^2 + y^2 = 2 - KC_2. \quad (16)$$

Additional tricks are needed in Eq. (13c). For reader convenience, and also because this part is useful to understand the selection criterion for high frequency data suggested in the next section, we provide some more details on this step. The underlying idea is to use the trigonometric identity  $\sin(\alpha + \beta) = (\sin \alpha \cos \beta + \sin \beta \cos \alpha)$  for  $\alpha$  and  $\beta$  equal to the arguments of the trigonometric functions appearing on the left-hand side of the first two equations (13a) and (13b) e.g.,  $\alpha = \pi s_i/2L$  and  $\beta = \pi s_i/L$ ,  $i=1,2$ . Next, the peculiar structure of the system (13a)–(13c) is exploited with the aim of expressing  $(x^3 + y^3)$  in terms of the measured data only. More precisely, by using the identity

$$\sin^2 \frac{3\pi s_i}{2L} = -2 \cos^3 \frac{\pi s_i}{L} + \frac{3}{2} \cos \frac{\pi s_i}{L} + \frac{1}{2}, \quad i = 1, 2, \quad (17)$$

Eq. (13c) becomes

$$\cos^3 \frac{\pi s_1}{L} + \cos^3 \frac{\pi s_2}{L} = \frac{KC_3 - 1}{2} + \frac{3}{4} \left( \cos \frac{\pi s_1}{L} + \cos \frac{\pi s_2}{L} \right). \quad (18)$$

Inserting (15) in the right-hand side of (18), Eq. (13c) can be written as

$$x^3 + y^3 = -2 + \frac{K}{2} (3C_1 + C_3). \quad (19)$$

Taking Eqs. (15), (16), and (19) into account, the original system (13a)–(13c) rewritten in terms of the new variables  $x$  and  $y$  shows a structure similar to that of the system governing the determination of two open cracks in a F-F rod from the first three natural frequencies considered by Morassi and Rollo (see Eqs. (12a)–(12c) of [34] specialized to the axial vibration case).



At this stage we take advantage of the method illustrated in [34] for finding the explicit solution of the nonlinear system governing the inverse damage detection problem. Therefore, instead of presenting all the details of the analysis, in the sequel we shall point out how the procedure presented in the above-mentioned paper can be adapted to the present case.

Observing that

$$x^3 + y^3 = (x+y) \left[ \frac{3}{2}(x^2 + y^2) - \frac{1}{2}(x+y)^2 \right] \quad (20)$$

and using Eqs. (15), (16), and (19) for representing the quantities  $(x+y)$ ,  $(x^2 + y^2)$ , and  $(x^3 + y^3)$  in terms of the data, respectively, the parameter  $K$  turns out to be a root of the second degree polynomial equation:

$$4K^2 C_1^3 + 3C_1(C_2 - 4C_1)K + \left( \frac{15}{2}C_1 - 3C_2 + \frac{1}{2}C_3 \right) = 0. \quad (21)$$

Following the lines of the proof shown in [34], it can be proved that Eq. (21) always has two real positive roots for every set of data  $\{C_1, C_2, C_3\}$ , that is, there exist two values of the stiffness  $K$  of the spring simulating the damage:

$$K_{1,2} = \frac{-3(C_2 - 4C_1) \mp [9(C_2 - 4C_1)^2 - 16C_1(\frac{15}{2}C_1 - 3C_2 + \frac{1}{2}C_3)]^{1/2}}{8C_1^2}, \quad (22)$$

where the indexes 1 and 2 correspond to  $-$  and  $+$  sign on the right-hand side, respectively.

Once  $K$  is found, we can localize the cracks. We note that the role of the two spatial variables  $x$  and  $y$  in the system formed by Eqs. (15), (16), and (19) is perfectly interchangeable. Therefore, it is enough to determine the position variable  $x$ . By using Eq. (15) to express  $y$  in terms of  $x$  and substituting in (16), the variable  $x$  turns out to be a root of the second degree polynomial equation:

$$2x^2 - 4x(KC_1 - 1) + 4(KC_1 - 1)^2 - (2 - KC_2) = 0. \quad (23)$$

The above equation always has two distinct real solutions in  $(-1, 1)$ , namely

$$x_{\mp}(K) = \frac{2(KC_1 - 1) \mp [8KC_1 - 4K^2C_1^2 - 2KC_2]^{1/2}}{2}, \quad (24)$$

where symbols  $x_-(K)$  and  $x_+(K)$  correspond to  $-$  sign and  $+$  sign on the right-hand side, respectively, and where the dependence of  $x_{\mp}$  on  $K$  is explicitly indicated. Note that, by (15), the solution  $y_-(K)$  corresponding to  $x_-(K)$  is such that

$$y_-(K) = 2(KC_1 - 1) - x_-(K) = \frac{2(KC_1 - 1) + [8KC_1 - 4K^2C_1^2 - 2KC_2]^{1/2}}{2} = x_+(K) \quad (25)$$

and, similarly, corresponding to  $x_+(K)$  we have  $y_+(K) = x_-(K)$ . Clearly, the two damage configurations  $\{K, x_-(K), y_-(K) = x_+(K)\}$  and  $\{K, x_+(K), y_+(K) = x_-(K)\}$  coincide and then, for a fixed value of the stiffness  $K$ , there exists only one solution, say  $\{K, x_-(K), x_+(K)\}$ , of the system of Eqs. (15), (16), and (19). As a consequence, since the function  $f(s) = \cos \pi s/L$  is one-to-one on  $(0, L)$ , for any fixed value of  $\bar{K}$  there exists only one solution  $\{\bar{K}, s_-(\bar{K}) = (L/\pi)\arccos(x_-(\bar{K})), s_+(\bar{K}) = (L/\pi)\arccos(x_+(\bar{K}))\}$  of our inverse problem.

In conclusion, we have shown that two cracks of the same severity  $K_1$  (evaluated via expression (22) with minus sign) located at the cross-sections of abscissa  $s_-(K_1)$  and  $s_+(K_1)$  (evaluated via expression (24), with  $K = K_1$ , and using the inverse of (14)) produce changes in the considered set of spectral data identical to those induced by two cracks having the common severity  $K_2$  (evaluated via expression (22) with plus sign) located at the cross-sections of abscissa  $s_-(K_2)$  and  $s_+(K_2)$  (evaluated via expression (24), with  $K = K_2$ , and using the inverse of (14)). One of these two solutions corresponds to the actual damaged configuration of the rod. The other solution is a spurious one, and it is a consequence of the mathematical nature of the diagnostic problem. In other words, the damage detection problem formulated with as many unknowns as data turns out to be ill-posed, since it has no unique solution even if the number of unknowns is exactly equal to the number of frequency data. However, it should be emphasized that the effect of the non-uniqueness is not particularly dramatic, especially if we consider that closed-form expressions for the damage parameters are available. These closed-form expressions allow us to determine the damage parameters by minimal computational effort and also permit a direct assessment of the effects that possible perturbations of the data (due to experimental or modelling errors, for example) have on the solution of the diagnostic problem. We will see in the next section that the intrinsic non-uniqueness of the inverse problem can be removed by merging the above results with those obtained using additional high frequency data.

#### 4. High frequency input data

We analyze how the procedure proposed in the previous section can be adapted to include high frequency information as input data. The underlying idea will be illustrated in the case in which frequency data consist of the second and the fifth antiresonant frequency of the FRF  $H^{F-F}(\omega, 0, 0)$ , namely  $\lambda_1^{S-F}$  and  $\lambda_4^{S-F}$ , and the third resonant frequency  $\lambda_3^{F-F}$  of the F-F rod.

The system analogous to (13a)–(13c) now reads as

$$\begin{cases} \sin \frac{23\pi s_1}{2L} + \sin \frac{23\pi s_2}{2L} = 2 - KC_1, & \text{(a)} \\ \sin \frac{23\pi s_1}{L} + \sin \frac{23\pi s_2}{L} = KC_2, & \text{(b)} \\ \sin \frac{9\pi s_1}{2L} + \sin \frac{9\pi s_2}{2L} = 2 - KC_3, & \text{(c)} \end{cases} \quad (26)$$

where

$$C_1 = C_1^{S-F}, \quad C_2 = C_3^{F-F}, \quad C_3 = C_4^{S-F}. \quad (27)$$

By proceeding as exemplified in the previous section, Eqs. (26a) and (26b) can be rewritten in terms of  $\{\cos 3\pi s_1/L, \cos 3\pi s_2/L\}$  as

$$\cos \frac{3\pi s_1}{L} + \cos \frac{3\pi s_2}{L} = 2(KC_1 - 1), \quad (28)$$

$$\cos^2 \frac{3\pi s_1}{L} + \cos^2 \frac{3\pi s_2}{L} = 2 - KC_2. \quad (29)$$

Noticing that  $9\pi s_i/2L = 3\pi s_i/L + 3\pi s_i/2L$ ,  $i=1,2$ , it is easy to see that also Eq. (13c) can be written in terms of  $\{\cos 3\pi s_i/L, \cos^3 3\pi s_i/L\}$  only,  $i=1,2$ . More precisely, by introducing the new variable positions

$$x = \cos \frac{3\pi s_1}{L}, \quad y = \cos \frac{3\pi s_2}{L}, \quad (30)$$

system (26a)–(26c) takes exactly the same structure of equations (15), (16), and (19), with, of course, the new meaning of the coefficients  $C_i$ ,  $i=1,2,3$ , and of the damage position variables  $x$  and  $y$ .

Observing that  $C_1 > 0$ , the solution of (26a)–(26c) can be carried out along the same procedure illustrated in the previous section, that is, the stiffness parameter  $K$  and the associated spatial variable position  $x(K)$  turn out to be the solutions of second order polynomial equations analogous to (22) and (24), respectively. Therefore, using the notation introduced in the previous section, the complete set of solutions of the system (26a)–(26c) is given by  $\{K_1, x_-(K_1), x_+(K_1)\}$  and  $\{K_2, x_-(K_2), x_+(K_2)\}$ .

The main difference with respect to the “low” frequency case emerges in the inversion of the variable position functions  $x = x(s_1)$  and  $y = y(s_2)$  (here the dependence on the stiffness parameter is omitted to simplify the notation). The function  $f(s) = \cos 3\pi s/L$  is no longer one-to-one in  $(0, L)$ , and then, by inversion, more crack locations (generally three distinct abscissa values) may correspond to a given value of the position variables  $x$  and  $y$ . The set of solutions includes also the correct position of the two cracks, but more spurious solutions occur. Nevertheless, the additional information coming from high frequency data can be usefully included in the inverse problem solution derived in previous section, since, crossing the results with those determined using “low” frequency data, the spurious solution appeared therein can be eliminated and, finally, the position of the two cracks can be uniquely identified, see Section 6 for an application.

The above analysis can be extended to include other triplets of high frequency data. With reference to frequency values of the undamaged rod, the key points in selecting the spectral input data are the following:

- (i) the order of the natural frequency and the order of the lower antiresonant frequency must be chosen such that the square root of the resonant frequency value is twice the square root of the antiresonant frequency value;
- (ii) the order of the higher antiresonant frequency must be chosen such that the corresponding square root of the frequency value is equal to the sum of the square root of the lower antiresonance frequency and the square root of the resonant frequency.

For instance, for the system of equations (26a)–(26c) we can easily check that  $\sqrt{\lambda_3^{F-F(U)}} = 2\sqrt{\lambda_1^{S-F(U)}}$  and  $\sqrt{\lambda_4^{S-F(U)}} = \sqrt{\lambda_1^{S-F(U)}} + \sqrt{\lambda_3^{F-F(U)}}$ . Other sets of data that fulfill conditions (i) and (ii) are, for example,  $\{\lambda_2^{S-F}, \lambda_5^{F-F}, \lambda_7^{S-F}\}$  and  $\{\lambda_3^{S-F}, \lambda_7^{F-F}, \lambda_{11}^{S-F}\}$ . For all these choices, the system of equations can be solved explicitly and closed form expressions  $\{K, x(K), y(K)\}$  of the damage parameters can be provided. Choices of data that do not satisfy the conditions (i) and (ii) are of course possible, but, in those cases, finding exact closed form solutions is an open question and recourse to numerical analysis is in order.

## 5. An extension to multi-cracked torsionally rotating shafts

In this section we present an extension of the above methodology to the identification of two circumferential cracks in a torsionally vibrating shaft.

Circumferential cracks often appear in a large variety of machinery. The free torsional vibration of a cylindrical shaft was found to be considerably influenced by the presence of circumferential cracks by Dimarogonas and Massouros [39] in their experimental/analytical research, see also [40] for an updated review on this topic. According to Dimarogonas and Massouros [39], a circumferential crack in a torsional vibrating shaft having uniform circular cross-section can be described by means of a torsional linearly elastic spring located at the cracked cross-section. The value  $K_T$  of the spring stiffness can be expressed in

terms of the crack geometry as suggested in the above-mentioned paper [39]. Therefore, the free undamped vibrations of a shaft with free ends and two cracks of equal severity located at different cross-sections  $s_i \in (0, L)$ ,  $i=1,2$ , are described by the following eigenvalue problem

$$\begin{cases} GI_0 \varphi'' + \mu \gamma I_0 w = 0, & x \in (0, s_1) \cup (s_1, s_2) \cup (s_2, L), & (a) \\ \varphi'(s_i^-) = \varphi'(s_i^+), & i = 1, 2, & (b) \\ K_T(\varphi(s_i^+) - \varphi(s_i^-)) = GI_0 \varphi'(s_i), & i = 1, 2, & (c) \\ \varphi'(0) = \varphi'(L) = 0, & & (d) \end{cases} \quad (31)$$

where  $\mu = \omega^2$ , with  $\omega$  being the radian frequency of the motion,  $G$  is the (constant) shear modulus of the material,  $I_0$  is the polar moment of inertia of the transversal cross-section and  $\gamma$  is the (uniform) volume mass density of the material.

Comparison between (1a)–(1d) and (31a)–(31d) shows that there is clearly a one-to-one correspondence  $\{E, A, w, \lambda\} \leftrightarrow \{G, I_0, \varphi, \mu\}$  between the longitudinal and torsional systems. Then, the  $n$ th eigenpair  $\{\mu_n^{F-F(U)}, \varphi_n^{F-F(U)}(x)\}$  is equal to

$$\mu_n^{F-F(U)} = \frac{G}{\gamma} \left( \frac{n\pi}{L} \right)^2, \quad \varphi_n^{F-F(U)}(x) = \sqrt{\frac{2}{\gamma I_0 L}} \cos \left( \frac{n\pi x}{L} \right), \quad (32)$$

where  $n = 0, 1, 2, \dots$ . If the cracks are small, namely  $K_T$  is large enough, the first order perturbation  $\delta\mu_n^{F-F} \equiv (\mu_n^{F-F} - \mu_n^{F-F(U)})$  on the  $n$ th eigenvalue with respect to  $1/K_T$  is given by

$$\delta\mu_n^{F-F} = -2\mu_n^{F-F(U)} \frac{GI_0}{LK_T} \left( \sin^2 \frac{2n\pi s_1}{L} + \sin^2 \frac{2n\pi s_2}{L} \right), \quad (33)$$

where  $n \geq 1$ . The 0th eigenvalue is insensitive to damage.

Let us denote by  $H(\omega, 0, 0)$  the FRF obtained by applying a torsional couple at  $x=0$  and measuring the rotation at the same end. It turns out that the antiresonant frequencies of  $H(\omega, 0, 0)$  are the eigenvalues  $\{\mu_m^{S-F}\}_{m=0}^\infty$  of the shaft with the end at  $x=0$  fixed. The corresponding eigenpairs of the undamaged S–F shaft are

$$\mu_m^{S-F(U)} = \frac{G}{\gamma} \left( \frac{(1+2m)\pi}{2L} \right)^2, \quad \varphi_m^{S-F(U)}(x) = \sqrt{\frac{2}{\gamma I_0 L}} \sin \frac{(1+2m)\pi x}{2L}, \quad (34)$$

$m = 0, 1, 2, \dots$ , and the first order changes of antiresonant frequencies  $\delta\mu_m^{S-F} \equiv (\mu_m^{S-F} - \mu_m^{S-F(U)})$  have the expression

$$\delta\mu_m^{S-F} = -2\mu_m^{S-F(U)} \frac{GI_0}{LK_T} \left( \cos^2 \frac{(2m+1)\pi s_1}{2L} + \cos^2 \frac{(2m+1)\pi s_2}{2L} \right), \quad (35)$$

$m \geq 0$ . Then, by comparing the expressions (6), (10) and (33), (35) we can conclude that the procedure shown in previous sections for identifying two equal cracks in a longitudinally vibrating rod can be transferred step-by-step to the identification of two circumferential cracks in a torsional vibrating shaft with free ends.

## 6. Applications

In Sections 3 and 4 it was shown how to use changes in resonant and antiresonant frequencies of a free-free longitudinally vibrating rod with two small open cracks so as to assess the location as well the magnitude of the damage. The present section is devoted to outlining some applications of numerical character.

The inverse problem of damage detection is formulated by using pseudo-experimental data, that is, the frequencies are obtained from the direct problem in undamaged conditions and in simulated damage conditions defined by the damage parameters  $\{(s_1, K), (s_2, K)\}$ .

The specimen is the straight thin uniform rod of length  $L$ , shown in Fig. 1. The cross-section is rectangular with width  $h$ ,  $h/L = 0.1$ , and depth  $b$ . The material is steel, with Young's modulus being  $E = 2.1 \times 10^2$  GPa, Poisson ratio  $\nu = 0.3$  and volume mass density equal to  $\gamma = 7850$  kg/m<sup>3</sup>. In this section, we express the stiffness  $K$  of the elastic spring simulating the crack in terms of the dimensionless flexibility parameter  $\delta$  as

$$K = \frac{EA}{L\delta}, \quad (36)$$

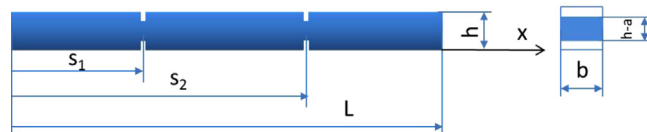


Fig. 1. Rod with two cracks.



where  $\delta$  can be determined as a function of the crack ratio  $\alpha = a/h$  as (see, for instance, [17])

$$\delta = \frac{2h(1-\nu)^2}{L}g(\alpha), \tag{37}$$

with

$$g(\alpha) = 0.7314\alpha^8 - 1.0368\alpha^7 + 0.5803\alpha^6 + 1.2055\alpha^5 - 1.0368\alpha^4 + 0.2381\alpha^3 + 0.9852\alpha^2. \tag{38}$$

An exhaustive series of numerical simulations has been carried out for different locations of the cracks and various levels of damage. Two main different damage scenarios among several studied are presented and commented in this section: they are illustrative of the main features of the inverse problem and of the identification technique. The first case,  $\delta=0.00200$ , is characterized by “small” damage, that is, the value of the flexibility  $\delta$  is chosen such that the variations of the lower frequencies and antiresonances are about 0.1–0.3% of the referential undamaged values. The other case involves “moderate” damage,  $\delta=0.01797$ , and it corresponds to variations of the same spectral data about 1–3%. In both cases, identification results are presented for two sets of damage locations, namely  $s_1 = L/2$ ,  $s_2 = 3L/5$  (close cracks, denoted by C in the sequel) and  $s_1 = L/4$  and  $s_2 = 3L/5$  (distant cracks, case D).

We start by considering the results obtained in the absence of errors on the data.

A first series of tests has been carried out using low frequency data in identification. The eigenvalues of the undamaged rod and their values associated with the cases of damage are shown in Table 1. The latter are obtained by solving in exact way the eigenvalue problem (1a)–(1d) for resonant frequencies and the analogous one for antiresonant frequencies. The results of identification are summarized in Table 2. It is possible to observe that the pair of solutions predicted by the theory contains a satisfactory estimate of the actual solution of the problem. The discrepancies between identified and actual damage parameters connected with this solution are exclusively due to the perturbation assumption of small damage and turn out to be negligible, both for damage location and severity. Deviations are typically smaller for less severe damage, as it is expected because the inverse problem is linearized in a neighborhood of the undamaged rod.

Table 3 shows the effects of damage on high frequency data, namely on the resonant/antiresonant frequencies used in Section 4. This set of data leads to the identification results shown in Table 4. As it was mentioned in Section 4, and under the same notation, the complete set of solutions of the system (27a)–(27c) is given by  $\{K_1, x_-(K_1), x_+(K_1)\}$  and  $\{K_2, x_-(K_2), x_+(K_2)\}$ , where the variable positions  $x_- = x_-(K_i)$  and  $x_+ = x_+(K_i)$  are uniquely determined in terms of  $K_i$  (or, equivalently,  $\delta_i$ ),  $i=1,2$ . The function  $f(s) = \cos 3\pi s/L$  expressing the variable positions  $x_-$  and  $x_+$  in terms of the

**Table 1**

Low resonant and antiresonant frequencies  $f = L\sqrt{\gamma/E}\sqrt{\lambda}$  for the undamaged rod ( $f^U$ ) and their values associated to the cases of damage (free of error data). C=close cracks; D=distant cracks. Note:  $\Delta = 100 \times (f^U - f)/f^U$ .

$f$	Undam.	Small damage				Moderate damage			
		Case C	$\Delta$	Case D	$\Delta$	Case C	$\Delta$	Case D	$\Delta$
$f_1^{F-F}$	3.14159	3.12968	0.38	3.13280	0.28	3.03820	3.29	3.06474	2.45
$f_0^{S-F}$	1.57080	1.56814	0.17	1.56704	0.24	1.54727	1.50	1.53794	2.09
$f_1^{S-F}$	4.71239	4.69922	0.28	4.70250	0.21	4.59908	2.40	4.62427	1.87

**Table 2**

Results of damage identification for low resonant/antiresonant frequency data (cases free of error). Determination of the damage severity  $\delta = EA/LK$  (where  $K$  is given by (22)) and the corresponding damage locations  $s_1$  and  $s_2$  (Eq. (24) and the inverse of (14)). C=close cracks; D=distant cracks. Percentage errors are indicated in brackets for the correct solution only.

Damage parameters	Small damage $\delta=0.00200$		Moderate damage $\delta=0.01797$	
	Case C $s_1/L = 0.50$ $s_2/L = 0.60$	Case D $s_1/L = 0.25$ $s_2/L = 0.60$	Case C $s_1/L = 0.50$ $s_2/L = 0.60$	Case D $s_1/L = 0.25$ $s_2/L = 0.60$
$\delta_1$	0.0036	0.0026	0.0284	0.0222
$s_-(\delta_1)$	0.5275	0.2964	0.5082	0.2937
$s_+(\delta_1)$	0.9135	0.7702	0.8773	0.7652
$\delta_2$	0.0020	0.0020	0.0170	0.0174
	(0.0)	(0.0)	(5.4)	(3.2)
$s_-(\delta_2)$	0.4945	0.2497	0.4736	0.2491
	(1.1)	(0.1)	(5.3)	(0.4)
$s_+(\delta_2)$	0.6063	0.6002	0.6299	0.6050
	(-1.0)	(0.0)	(-5.0)	(-0.8)

**Table 3**

High resonant and antiresonant frequencies  $f = L\sqrt{\gamma/E}\sqrt{\lambda}$  for the undamaged rod ( $f^U$ ) and their values associated to the cases of damage (free of error data). C=close cracks; D=distant cracks. Note:  $\Delta = 100 \times (f^U - f)/f^U$ .

$f$	Undam.	Small damage				Moderate damage			
		Case C	$\Delta$	Case D	$\Delta$	Case C	$\Delta$	Case D	$\Delta$
$f_3^{F-F}$	9.42478	9.39961	0.27	9.40894	0.17	9.21197	2.26	9.28942	1.44
$f_1^{S-F}$	4.71239	4.69922	0.28	4.70250	0.21	4.59908	2.40	4.62427	1.87
$f_4^{S-F}$	14.13717	14.11300	0.17	14.10320	0.24	13.90420	1.65	13.82800	2.19

**Table 4**

Results of damage identification for high resonant/antiresonant frequency data. Determination of the damage severity  $\delta = EA/LK$  (where  $K$  is given by (22)) and the corresponding damage locations  $\{s_-^k\}_{k=1}^3$  and  $\{s_+^j\}_{j=1}^3$  for  $x_-$  and  $x_+$ , respectively (Eq. (24) and the inverse of (30)). Actual values of damage severity:  $\delta = 0.00200$  for small damage and  $\delta = 0.01797$  for moderate damage. C=close cracks; D=distant cracks.

Damage level	Severity	Damage positions		Severity	Damage positions	
	$\delta_1$	$\frac{s_-^k(\delta_1)}{L}$	$\frac{s_+^j(\delta_1)}{L}$	$\delta_2$	$\frac{s_-^k(\delta_2)}{L}$	$\frac{s_+^j(\delta_2)}{L}$
D-Small	0.0020	0.2500	0.0663	0.0016	0.1919	0.0399
		0.4167	0.6004		0.4747	0.6268
		0.9166	0.7330		0.8586	0.7065
D-Moderate	0.0175	0.2505	0.0637	0.0136	0.1900	0.0339
		0.4161	0.6030		0.4766	0.6327
		0.9172	0.7303		0.8567	0.7006
C-Small	0.0020	0.1672	0.0662	0.0032	0.2791	0.0941
		0.4995	0.6005		0.3876	0.5725
		0.8338	0.7329		0.9457	0.7608
C-Moderate	0.0170	0.1710	0.0629	0.0264	0.1639	0.0910
		0.4956	0.6038		0.5028	0.5756
		0.8377	0.7295		0.8305	0.7577

physical damage coordinates  $s_-$  and  $s_+$  (with a slight abuse of notation) is not one-to-one. Then, the inversion of the function  $f$  with respect to  $s$  gives rise to (typically) three distinct possible crack locations, namely  $\{s_-^k\}_{k=1}^3$  and  $\{s_+^j\}_{j=1}^3$  for  $x_-$  and  $x_+$ , respectively. These results are shown in Table 4. It can be seen that the set of possible damage locations always includes a good estimate of the actual position of the two cracks.

By comparing the estimates of the damage severity shown in Tables 2 and 4, it is possible to remove the spurious solution occurring when low frequency data only are used in identification, thus leading to a unique solution for the double-crack identification problem. For instance, in the case of distant cracks and moderate level of damage, the identified values of the dimensionless flexibility coefficient  $\delta$  are (0.0174, 0.0222) and (0.0136, 0.0175) in Tables 2 and 4, respectively. Clearly, the pair of closer flexibility values is given by 0.0174 and 0.0175. Therefore, the spurious solution in Table 2 corresponds to  $\delta = 0.0222$ .

For the sake of completeness, we note that numerical simulations using high-frequency data have not led to positive results in a certain number of cases. These cases refer to close cracks having intermediate position near a cross-section of zero sensitivity for all three resonant/antiresonant frequencies considered as data. In a free-free uniform rod, the only cross-section with this property is located at  $x = L/3$ . A detailed analysis of the numerical simulations shows that Eq. (22) leads to imaginary values of the stiffness of the elastic spring simulating the crack when the damage positions  $s_1$  and  $s_2$  belong to the interval  $(0.25L, 0.45L)$  and  $(s_1 + s_2)/2 \geq L/3$ . This result is not in contradiction with the general property that Eq. (22) always has two real roots. In fact, this general property has been deduced from Eqs. (15), (16), and (19), which are the linearized version of the full non-linear inverse problem when  $1/K$  is small enough. The stiffness parameter  $K$  is expressed in terms of the coefficients  $C_i$ ,  $i = 1, 2, 3$ , which, in turn, depend on resonant/antiresonant frequency shifts induced by the damage. In real measurements, the frequency shifts include both the term linear in  $1/K$  and a higher order term that, of course, the formulation of the linearized inverse problem cannot treat successfully. This non-linear term attains its maximum effect just when the linear part vanishes, i.e., when the positions of the two cracks are close to a point of zero sensitivity. The numerical simulations also show that, in previous cases, the failure of identification is more frequent when the damage is more severe. This behavior is rather expected, because our formulation of the inverse problem is based on the assumption that the damaged system is a small perturbation of the virgin system.

Finally, in order to test the robustness of identification to possible errors, numerical simulation has been repeated in the presence of random noise on the frequency data. In practical applications, one of main sources of error is due to the

**Table 5**

Results of damage identification for low resonant/antiresonant frequency data (cases with random error). Determination of the damage severity  $\delta = EA/LK$  (where  $K$  is given by (22)) and the corresponding damage locations  $s_1$  and  $s_2$  (Eq. (24) and the inverse of (14)). C=close cracks; D=distant cracks.

Case	Statistical property	Damage severity $\delta$		Damage positions			
				$s_- (\delta)/L$		$s_+ (\delta)/L$	
		Err. Lev. 1	Err. Lev. 2	Err. Lev. 1	Err. Lev. 2	Err. Lev. 1	Err. Lev. 2
D-Small	Mean	0.0020	0.0022	0.2364	0.2142	0.5403	0.5044
	Std. Dev.	0.0003	0.0005	0.0804	0.0975	0.0693	0.0885
	Mean/Exact	1.0001	1.1001	0.9456	0.8568	0.9005	0.8407
D-Moderate	Mean	0.0174	0.0174	0.2497	0.2479	0.6062	0.6027
	Std. Dev.	0.0003	0.0005	0.0113	0.0204	0.0183	0.0295
	Mean/Exact	0.9680	0.9680	0.9988	0.9916	1.0103	1.0045
C-Small	Mean	0.0023	0.0025	0.3933	0.2905	0.6174	0.5892
	Std. Dev.	0.0003	0.0005	0.0830	0.1091	0.0769	0.0527
	Mean/Exact	1.1501	1.2501	0.7866	0.5810	1.0290	0.9820
C-Moderate	Mean	0.0179	0.0181	0.4731	0.4681	0.6347	0.6381
	Std. Dev.	0.0009	0.0011	0.0229	0.0280	0.0340	0.0384
	Mean/Exact	0.9959	1.0070	0.9462	0.9362	1.0578	1.0635

inaccuracy of the analytical model that is used to interpret the experiments. The analytical model (1a)–(1d) that we have used to describe the small longitudinal vibrations of a cracked rod turns out to be very accurate, both for low natural frequencies [30] and antiresonant frequencies [31]. However, damage-induced changes are typically small and, therefore, it may happen that even small errors can affect the outcome of the identification. In order to evaluate the effect of errors on the data, resonant and antiresonant frequencies were perturbed as follows:

$$\sqrt{\lambda}^{\text{pert}} = \sqrt{\lambda}(1 + \tau), \quad (39)$$

where  $\tau$  is a real random Gaussian variable with zero mean.

As an example, Table 5 shows the statistical properties of the results of identification in the case of low frequency data, corresponding to a Monte Carlo simulation on a population of 10 000 samples for each damage configuration. Note that the results shown in Table 5 refer only to the solution of the system (11a)–(11c) corresponding to the actual value of the damage parameters (see last paragraph of Section 3).

Different error magnitudes have been taken into account depending on the type of measure done (natural frequency or antiresonant). For the case of natural frequencies, two normal distributions with standard deviations  $\sigma_1^R = 0.00033$  (Error Level 1) and  $\sigma_2^R = 0.00067$  (Error Level 2) have been considered. Therefore, the maximum error magnitude is approximately equal to  $3\sigma$ , corresponding to 0.1% and 0.2% of the nominal resonant frequency value for Error Level 1 and Error Level 2, respectively. The measurement errors in the antiresonant frequencies are higher than those corresponding to natural frequencies. Thus, the standard deviations of the random distributions affecting antiresonant measurements were taken as  $\sigma_1^{AR} = 0.00067$  and  $\sigma_2^{AR} = 0.00133$  corresponding to maximum errors of 0.2% and 0.4% of the nominal antiresonant frequency value for Error Level 1 and Error Level 2, respectively. These values are close to average modelling errors found on low natural frequencies and antiresonances of real cracked rods [31].

The analyzed cases correspond to small ( $\delta=0.00200$ ) and moderate ( $\delta=0.01797$ ) damage, and to distant ( $s_1/L = 0.25, s_2/L = 0.6$ ) and close ( $s_1/L = 0.5, s_2/L = 0.6$ ) cracks. The results of Table 5 show that significant differences appear in the identification of small damage. These discrepancies are due to the fact that maximum errors on the data are of the same order of frequency-shifts induced by the damage, both for close and distant cracks. The method provides much better results in the case of moderate damage, with error on the average value not exceeding the 5–6% of the nominal value, both for the position and the intensity of the damage. For completeness, we notice that complex values have been obtained in a certain number of simulations. The percentage of such cases is negligible or small for distant cracks and moderate damage (about 0.5 and 10%, for Error Levels 1 and 2, respectively), whereas takes large values in the case of close and small cracks (about 75%, both for Error Levels 1 and 2). The results collected in Table 5 correspond to real values of the damage parameter estimates.

## 7. Conclusions

Few general results are available for the inverse problem of identifying multiple cracks in vibrating beams by frequency measurements. The present paper is a contribution to this issue. More precisely, we have focussed on detecting two open cracks of equal severity in a free–free longitudinally vibrating beam. It was shown how appropriate use of natural frequency and antiresonant frequency data can be useful to reduce the intrinsic non-uniqueness of the damage location problem, which occurs when only frequency data are used.

The diagnostic method is based on an explicit expression of the first-order change in natural frequencies and antiresonances induced by open cracks in a rod, under the assumption that the damaged system is a perturbation of the virgin system. Closed-form expressions of the damage parameters in terms of the data have been obtained. Extensions to multiple crack detection in torsionally vibrating shafts are also provided.

The analysis allowed us to better clarify the role of the frequency/antiresonant data in the inverse problem solution and to find optimal choices of the input data in order to reduce the non-uniqueness effects. In particular, it was explained why the use of high frequency/antiresonant data introduces spurious solutions in the localization of the damage. In this regard, the choice of using the first and second antiresonant frequencies of the frequency response function evaluated at one end of the rod and the first resonant frequency turns out to be the optimal one. The optimality of this choice is also supported by the fact that lower frequencies/antiresonances are easy to measure and are less affected by modelling errors.

Numerical results are generally in good agreement with the theory when exact analytical data are employed in identification. An extensive series of numerical simulations in the presence of random errors suggested that if accurate frequency/antiresonant estimates are available, then the location and the severity of the cracks can be accurately identified. It turns out that, in the inverse problem solution, random errors are usually amplified strongly in the case of close and small cracks. Numerical simulations have also shown that the identification fails when high frequencies are used and cracks are close to a cross-section of zero sensitivity for all the input spectral data. It is likely that this indeterminacy can be removed by introducing a proper treatment of the full non-linear diagnostic problem for not necessarily small cracks. General results along this direction are not available yet – even for a single crack – and this problem requires further investigation.

The present study leaves an important question open, namely the identification of two small cracks having *different* severities. The formulation of this more general inverse problem requires, at least, the introduction of one additional spectral information. A preliminary analysis carried out by the authors shows that the corresponding set of nonlinear equations takes a structure significantly different from the system of Eqs. (15), (16), and (19) of the present paper. One can also verify that the technique used in Section 3 to find the damage parameters breaks down in the case of different severities. It is likely that new ideas and methods must be developed to deal with this challenging inverse problem.

## Acknowledgements

The work of A. Morassi is supported by the Carlos III University of Madrid-Banco de Santander Chairs of Excellence Programme for the 2013–2014 Academic Year.

A. Morassi wishes to thank the colleagues of the Universidad Carlos III de Madrid, especially Professors L. Rubio and J. Fernández-Sáez, for the warm hospitality at the Department of Engineering Mechanics.

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