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Mathematical modeling of fall armyworm spodoptera frugiperda infestations in maize crops and its impact on final maize biomass

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MATHEMATICAL MODELING OF FALL ARMYWORM *SPODOPTERA FRUGIPERDA* INFESTATIONS IN MAIZE CROPS AND ITS IMPACT ON FINAL MAIZE BIOMASS

Salamida Daudi

A Dissertation Submitted in Partial Fulfilment of the Requirements for the Degree of Doctor of Philosophy in Mathematical and Computer Sciences and Engineering of the Nelson Mandela African Institution of Science and Technology

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ABSTRACT

Fall armyworm (FAW-*Spodoptera frugiperda*), a highly destructive and fast spreading agricultural pest native to North and South America, poses a real threat to global food security. It is estimated that intermittent FAW outbreaks could cause up to $US 13 billion per annum in crop losses throughout sub-Saharan Africa. Considering this projected loss it is imperative that various tools and techniques be utilized to infer on the various factors that affect FAW maize interaction and in-turn affect the final maize biomass. Mathematical modeling has proved to be an important tool that is capable of shedding light on the FAW-maize interaction dynamics. In this study, three mathematical models were proposed to evaluate the impact of memory effects and controls, seasonality and Integrated Pest Management strategy (farming awareness and larvae predation) on FAW infestations in maize crops and on final maize biomass. Firstly, to evaluate the impact of memory effects and control, a new dynamical system for FAW-maize biomass interaction via Caputo fractional-order operator was proposed and analyzed. In the proposed model, four equilibrium points which revealed the existence of a threshold parameter defined by $\mathscr{R}_0$ were computed and analyzed. Further, it was observed that, $\mathscr{R}_0$, the average number of newborns produced by one individual female moth during its life span was an integral component for stability of the aforementioned model equilibria. Secondly, to evaluate the implications of seasonality on FAW maize interaction and on the final maize biomass, a non-autonomous mathematical model was proposed and analyzed. The analysis revealed that the model solution was non-negative, unique, permanent and bounded admitting global asymptotic and continuous periodic function. Further, the model was extended into an optimal control problem with the aim of determining optimal pesticides and traditional methods that are capable of minimizing FAW egg and larvae populations at minimum cost. Results from the study demonstrated that a combination of pesticides use at low intensity with traditional methods at higher intensity could eradicate FAW in a maize field in a period less than half the life span of the crop in the field. Thirdly, to evaluate the impact of farming awareness campaigns and larvae predation, a fractional-order model that incorporated farming awareness campaigns and larvae predation was proposed and analysed. Overall, the study highlighted that, non-time dependent farming awareness campaigns should be close to 100% all the time to eradicate the FAW. However, when time-dependent farming awareness was implemented, it was observed that even less than 50% intensity level could lead to eradication of FAW. In all the proposed models, comprehensive numerical simulations were carried out in MATLAB programming language to support the analytical findings. In a nutshell, the results of this study showed that mathematical models can be important tools to evaluate FAW and maize interaction dynamics.

AUTHOR'S DECLARATION

I, Salamida Daudi, do hereby declare to the Senate of Nelson Mandela African Institution of Science and Technology that this dissertation titled: Mathematical Modeling of Fall Armyworm (*Spodoptera frugiperda*) Infestations in Maize Crops and its Impact on final Maize Biomass is my own original work and that it has neither been submitted nor presented for degree award in any other institution.

Salamida Daudi (Candidate)	1/08/2022 Date

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Prof. Dimitry Kuznetsov (Supervisor 3)	1/08/2022 Date

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CERTIFICATION

The undersigned certify that they have read and hereby recommend for acceptance by the Nelson Mandela African Institution of Science and Technology the dissertation entitled: Mathematical Modeling of Fall armyworm *Spodoptera frugiperda* Infestations in Maize Crops and its Impact on final Maize Biomass, in fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematical and Computer Sciences and Engineering of the Nelson Mandela African Institution of Science and Technology.

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DEDICATION

This dissertation is dedicated to my late father and mother, Daudi Ngulyati and Mwalu Dame. May their souls rest in peace, Amen. Special thanks are also dedicated to my wife Catherine M. Baisha, my wonderful children; Alfa, Daudi, and Daniel.

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LIST OF ABBREVIATIONS

Bt	Bacillus thuringiensis
CABI	Centre for Agriculture and Biosciences International
CIMMYT	International Maize and Wheat Improvement Center
Cry	Crystal protein gene
FAO	Food and Agriculture Organization of the United Nations
FAW	Fall Armyworm
GEM	Germplasm Enhancement of Maize
GM	Genetically Modified
IITA	International Institute of Tropical Agriculture
IPM	Integrated Pest Management
SSA	Sub-Saharan Africa
TARI	Tanzania Agricultural Research Institute
USAID	United States Agency for International Development
WEMA	Water Efficient Maize for Africa
WHO	World Health Organization

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CHAPTER ONE

INTRODUCTION

1.1 Background of the Problem

Maize (*Zea mays L.*) is ranked the third most important cereal grain after wheat and rice globally and is also referred to as the "Queen of Cereals" due to its high genetic yield potential (Jeyaraman, 2017). The demand for maize is increasing, not only because of its higher nutritional value but also its ability to feed the growing global population and contribution to food security (Kandel & Poudel, 2020). According to FAO (2018*b*), food security is a situation that exists when all people, at all times, have physical, social, and economic access to sufficient, safe, and nutritious food that meets their dietary needs and food preferences for an active and healthy life.

Maize is believed to have originated from central Mexico about 7000 years ago from a wild grass, and was transformed by Native Americans into a better source of food (Bariw *et al.*, 2020). It is one of the three most widely cultivated crops in the world (Sharon *et al.*, 2020), with USA, China, and Brazil contributing about 63% of the global maize production, followed by Mexico, Argentina, India, Ukraine, Indonesia, France, Canada, Nigeria, South Africa, and other Sub-Sahara African countries including East Africa (Sharon *et al.*, 2020).

Goergen *et al.* (2016) estimated the total maize harvest in Africa at 40 million hectares, with Nigeria being the top producer (16%), followed by Ethiopia, and Tanzania. Globally, maize is a staple food for more than 1.2 billion people in Sub-Saharan Africa (SSA) (Jeyaraman, 2017; FAO, 2018*a*). Worldwide maize consumption is estimated to be more than 116 million tons with, 30% and 21% of the consumption occurring globally and in SSA, respectively.

In Africa, around 14 countries in SSA consume 85–95% of white maize as their staple food (De Groote *et al.*, 2020). However, sustainable production of maize crops in these countries, and consequently, the livelihood of maize growers has recently been constrained by the significant widespread infestation, damage, and destruction caused by the interaction between maize biomass and exotic (invasive) pest called the fall armyworm (FAW) (*Spodoptera frugiperda*) (Lepidoptera: Noctuidae). In this interaction, FAW extremely affects maize plants by eating the plant materials which are collectively called "biomass" (Kandel & Poudel, 2020).

FAW is a highly destructive and fast spreading agricultural pest native to North and South America (Kandel & Poudel, 2020; Day *et al.*, 2017). It is an infecting insects of members from a family of *Poaceace* which include major food crops such as corn, sorghum, rice, wheat, maize, and diverse pasture (Rukundo *et al.*, 2020). Prior studies suggest that FAW prefers maize than other crops and pastures (De Groote *et al.*, 2020). In maize, FAW attacks all crop stages from the emergence of seedlings through to ear development. They defoliate and destroy young plants; whorl damage can result in yield losses, and ear feeding can result in reduced yields and grain quality (Sisay *et al.*, 2019).

According to Battude *et al.* (2016) and Chowdhury *et al.* (2019), biomass comprises plant materials such as tissues of greenish parts of the plant, corn or kernel, leaves and stems. In particular, the FAW feeds on maize biomass in all stages from seedling emergence to ear development with much preference on maize biomass (foliage or whorls) and then to ear, cob, and kernels. Cognizant of this, the effect becomes severe when the biomass of the growing points and photosynthetic areas of maize plants is attacked. This reduces the ability of maize plants manufacture their own food which subsequently reduce the final grain yield which is highly needed to feed the growing population (Bhusal & Chapagain, 2020; Prasanna *et al.*, 2018).

The current estimates from 12 African countries suggest an annual loss of 4.1 to a massive 17.7 million tons of maize due to FAW infestations (De Groote *et al.*, 2020). In particular, farm-level estimates from Ghana and Zambia suggest yield losses of 22–67% (Day *et al.*, 2017), 47% in Kenya (Rukundo *et al.*, 2020) and 9.4% in Zimbabwe (Baudron *et al.*, 2019) due to FAW. As outlined by the United-Nations (2014) and Shiferaw *et al.* (2013), the world's population is expected to reach 9.3 billion by the end of 2050, with an approximated yearly increase of more than 80% of the global increase, and a quarter of this increase is expected to occur in developing countries (De Groote *et al.*, 2020). This unprecedented global population increase poses a serious challenge for maize producers and policymakers, especially regarding the minimization of food losses due to the effect of interaction between maize biomass and FAW (Faithpraise *et al.*, 2015).

1.1.1 Life Cycle of the FAW

The FAW as an holometabolous insect undergoes complete metamorphosis with four stages of a life cycle: (a) eggs, (b) six growth stages of larval development (instars), (c) pupa, and (d) adult moth. The life cycle of FAW is completed in approximately 30 days during the summer, but 60 days in the spring and autumn, and between 80-90 days during the winter (FAO, 2018*a*). The number of generations occurring in an area depends on the appearance of the dispersing adults. For instances, studies in Kansas have reported generations between one to two, while those in Carolina and Louisiana, have reported three and four generations, respectively (FAO, 2018*b*).

(i) Egg stage

The egg is dome-shaped; the base is flattened and the egg curves upward to a broadly rounded point at the apex. The egg measures about 0.4 mm and 0.3 mm in diameter and height respectively. The number of eggs per egg-mass varies considerably but is often 100 to 200, and total egg production per female averages about 1500 with a maximum of over 2000 (Bista *et al.*, 2020; Sharanabasappa *et al.*, 2019). The eggs are sometimes deposited in layers, but most are spread over a single layer attached to foliage. The female also deposits a layer of grayish scales between the eggs and over the egg mass that imparts a furry or moldy appearance (Prasanna *et al.*, 2018). The duration of the egg stage is only 2-3 days during the summer months (FAO, 2018*a*). Figure 1 illustrates egg mass of FAW.

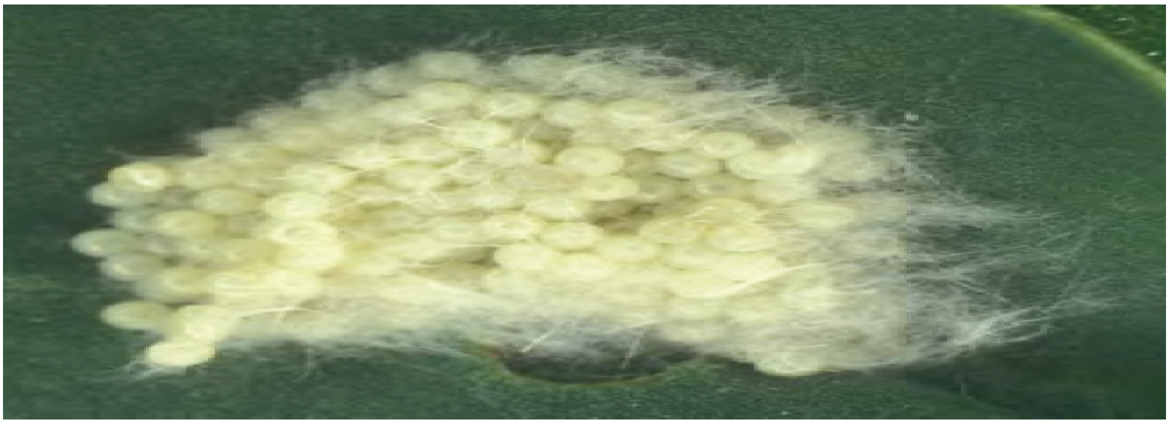


Figure 1: Egg mass of FAW (FAO, 2018*b*)

(ii) Larval stage

There are usually six instars in FAW. The head capsule widths are about 0.35, 0.45, 0.75, 1.3, 2.0, and 2.6 mm for instars 1 to 6 respectively. The larvae attain lengths of about 1.7, 3.5, 6.4, 10.0, 17.2, and 34.2 mm during these instars respectively. The young larvae are greenish with a black head which turns orangish in the 2nd instar. In the 2nd, but particularly the 3rd instar, the dorsal surface of the body becomes brownish, and lateral white lines begin to form (Abrahams *et al.*, 2017).

In the 4th to the 6th instars, the head is reddish brown, mottled with white, and the brownish body bears white subdorsal and lateral lines. Elevated spots occur dorsally on the body; these are usually dark in color, and bear spines. The face of the mature larva is also marked with a white inverted "Y" and the epidermis of the larva is rough or granular in texture when examined closely (Bista *et al.*, 2020). However, this larva does not feel rough to touch, like the corn earworm, (*Helicoverpa zea*) (Boddie), because it lacks the microspines found on the corn earworm (FAO, 2018*a*).

In addition to the typical brownish form, the FAW larva may mostly be green dorsally. In the green form, the dorsal elevated spots are pale rather than dark. The larvae tend to conceal themselves during the brightest time of the day. The duration of the larval stage tends to be about 14 days during the summer, and 30 days during cool weather. The mean development time was determined to be 3.3, 1.7, 1.5, 1.5, 2.0, and 3.7 days for instars 1 to 6, respectively, when larvae were reared at 25° C (FAO, 2018*a*). Figure 2 depicts the larvae instar stages of FAW.

Figure 2: (a) FAW Larvae from the 1st instar to the 6th instar (left to right) (b) FAW larvae at 6th stage (FAO, 2018*b*)

(iii) Pupa stage

Pupation normally takes place in the soil, at a depth 2-8 cm. The larva constructs a loose oval cocoon of 20-30 mm in length, by tying together particles of soil with silk. If the soil is too hard, the larvae may web together leaf debris and other material to form a cocoon on the soil surface. The pupa is reddish-brown in color, and measures 14-18 mm in length and about 4.5 mm in width. The duration of the pupal stage is about 8-9 days during the summer, but reaches 20 to 30 days during the winter in Florida (FAO, 2018*a*). Figure 3 shows the pupae stage of the FAW.

The pupal stage of fall armyworm cannot withstand protracted periods of cold weather. Recent study of the winter survival of the pupal stage in Florida suggest that, 51% of the survival in southern Florida only 27.5% survival in central Florida, and 11.6% survival in northern Florida (Pitre & Hogg, 1983).

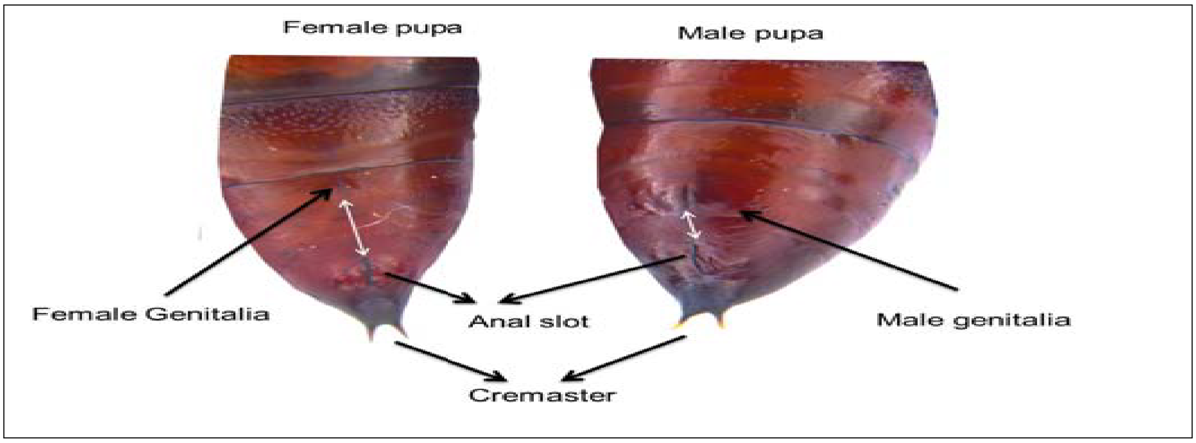


Figure 3: Pupae stage of the FAW (FAO, 2018*b*)

(iv) Adult stage

Adult moths have a wingspan of 32-40 mm. In the male moth, the forewing is generally shaded gray and brown, with triangular white spots at the tip and near the center. The forewings of females are less distinctly marked, ranging from a uniform grayish brown to a fine mottling of gray and brown. The hindwing is iridescent silver-white with a narrow dark border in both sexes (Abrahams *et al.*, 2017). Figure 4 shows the male and female adult moth of the FAW.

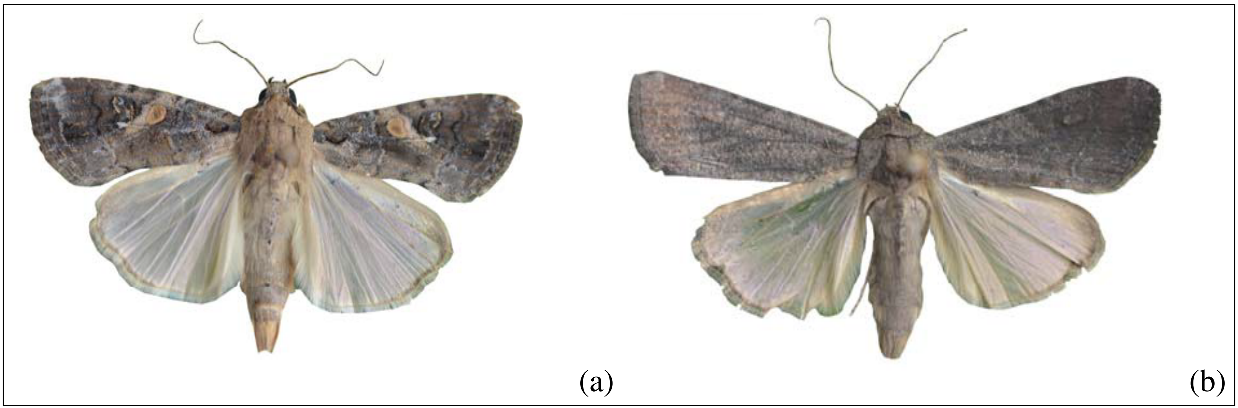


Figure 4: (a) Male adult of the FAW (b) Female adult of the FAW (FAO, 2018*b*)

Adult moth are nocturnal, and most active during warm, humid evenings. After a pre-oviposition period of 3-4 days, the female normally deposits most of her eggs during the first 4-5 days of life, but some oviposition occurs for up to three weeks. The duration of the adult moth's life is estimated to average about 10 days, with a range of about 7-21 days (Prasanna *et al.*, 2018). Figure 5 shows the complete life cycle of the FAW.

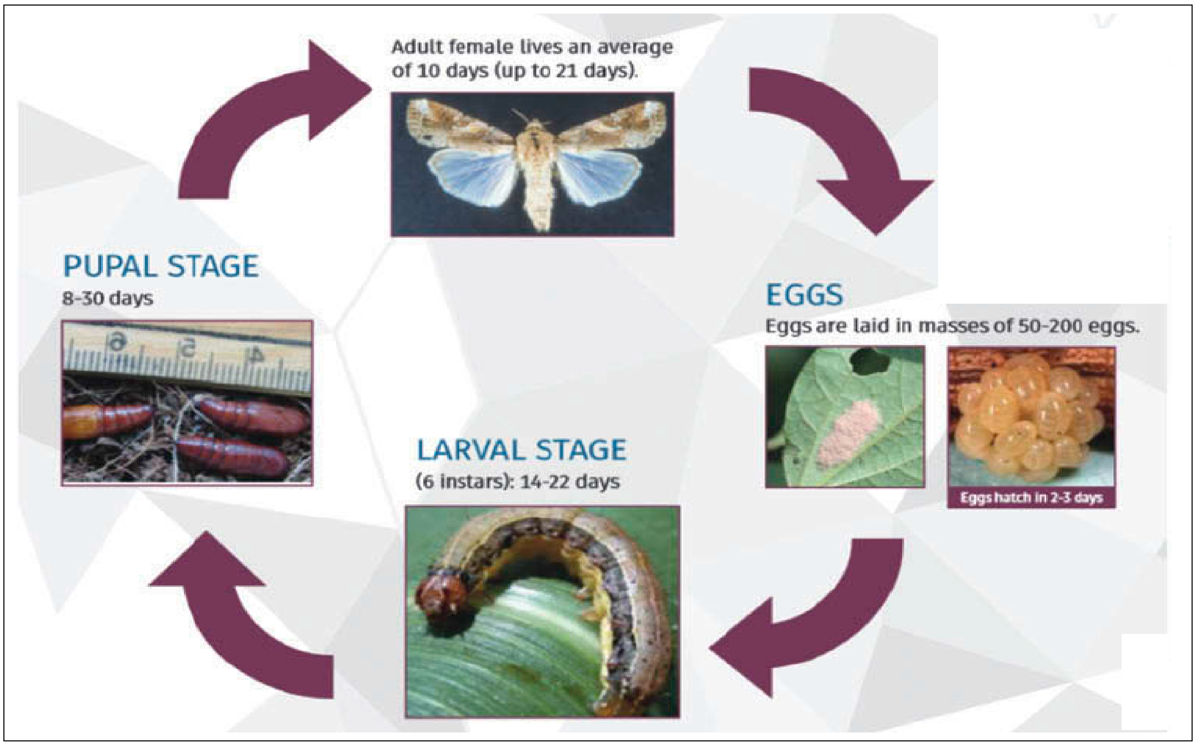


Figure 5: The life cycle of the FAW (Assefa & Ayalew, 2019)

1.1.2 Host Plants

The FAW, is a destructive insect pest that feeds on 353 host plants in 76 families with the most agriculturally important found in Poaceace (106), Asteraceae (31), and Fabaceae (31) (Day *et al.*, 2017). The most frequently destroyed plants are field corn and sweet corn, sorghum, Bermudagrass, and grass weeds such as crabgrass, Digitaria spp. When the larvae are very numerous they defoliate the preferred plants, acquire an “armyworm” habit and disperse in large numbers, consuming nearly all vegetation in their path (Assefa & Ayalew, 2019).

Many host records reflect such periods of abundance, and are not truly indicative of oviposition and feeding behavior under normal conditions (Prasanna *et al.*, 2018). Field crops that are frequently attacked by FAW include; alfalfa, barley, Bermudagrass, buckwheat, cotton, clover, corn, oat, millet, peanut, rice, ryegrass, sorghum, sugarbeet, Sudangrass, soybean, sugarcane, timothy, tobacco, and wheat. Among vegetable crops, only sweet corn is regularly damaged, but others are attacked occasionally (Sisay *et al.*, 2019).

Other crops that are sometimes attacked by FAW are apples, grapes, oranges, papayas, peachs, strawberry and a number of flowers. Weeds known to serve as hosts include *bentgrass, Agrostis sp.; crabgrass, Digitaria spp.; Johnson grass, Sorghum halepense; morning glory, Ipomoea spp.; nutsedge, Cyperus spp.; pigweed, Amaranthus spp.;* and *sandspur, Cenchrus tribuloides* (Fotso Kuate *et al.*, 2019).

1.1.3 Damage to Plants

The larvae of FAW cause damage by consuming foliage. Young larvae initially consume leaf tissue from one side, leaving the opposite epidermal layer intact (Shiferaw *et al.*, 2013). By the 2$^{\text{nd}}$ or 3$^{\text{rd}}$ instar, larvae begin to make holes in leaves, and eat from the edges inwards. Feeding on the whorl of corn often produces a characteristic row of perforations on the leaves. Due to cannibalistic behavior, larval densities are usually reduced to one or two per plant when feed in close proximity to one another (Assefa & Ayalew, 2019).

Older larvae cause extensive defoliation, often leaving only the ribs and stalks of corn plants, or a ragged, torn appearance. Marenco *et al.* (1992) studied the effects of FAW injury on the early vegetative growth of sweet corn in Florida. Their study showed that, the early whorl stage was least sensitive to injury, the mid whorl stage was intermediately sensitive, and the late whorl stage the most sensitive to injury. They further noted that, mean densities of 0.2-0.8 larvae per plant during the late whorl stage could reduce crop yield by 5 to 20%.

Larvae can also burrow into the growing points of plants (bud, whorl, etc.), destroying the growth potential of plants, or clipping the leaves. In corn, the larvae sometimes burrow into the ear, feeding on kernels in the same manner as the corn earworm, *Helicoverpa zea*. Unlike corn earworm, which tends to feed down through the silk before attacking the kernels at the tip of the ear, the FAW feeds by burrowing through the husk on the side of the ear (FAO, 2018*a*).

In a study by Pannuti *et al.* (2016) on larval feeding behavior, it was shown that although young (vegetative stage) leaf tissue is suitable for larval growth and survival on more mature plants, the leaf tissue is unsuitable, and the larvae tend to settle and feed on the ear zone, and particularly on the silk tissues. However, silk was not very suitable for growth and larvae attaining the corn kernels display the fastest rate of development. Similarly, although the closed tassel was suitable with respect to survival, it resulted in poor growth. Thus, tassel tissue may be suitable for initial feeding, perhaps until the larvae locate the silk and ears, but feeding on tassel tissue only is suboptimal.

Understanding how often the effects of the FAW on maize biomass occurs and spread in the field not only helps to plan a good strategy to prevent the effect but is also a guiding tool for its management. As reported by Chapman *et al.* (2000), the effect caused by FAW on maize starts immediately after hatching when the young larvae feeds on maize biomass (foliage or whorls) and then move to the growing points of the maize plants, leaving semi-transparent patches called windows and developing kernels inside the cobs of older plants protected by leaf bracts (Figure 6).

Figure 6: The damage of maize at different stages of plant caused by larvae of the FAW

The small larvae hide in the joints between the biomass and stem of the maize plants and move out during the night to feed on biomass. The FAW larvae are voracious feeders causing huge damages to older plants by feeding on both vegetative and reproductive structures where it defoliate the host when foliage or whorls is attacked and damage the young plant when the stem and growing point are attacked. When the number of larvae increases in a field, they begin to defoliate every plant that comes on their way while spreading in the maize field. However, when food is limited the older FAW larvae exhibit a cannibalistic behavior on the smaller larvae (Assefa & Ayalew, 2019; Bhusal & Chapagain, 2020).

The severity and extent of FAW infestations are enhanced following the onset of the wet season when the wind-borne immigrations of adult moths are attracted to lay eggs which transform into larvae within 2 to 5 days (Bhusal & Chapagain, 2020). The newly hatched larvae benefit from the flush of green maize biomass vegetation resulting from the rain and develop rapidly over three weeks and outbreaks can have a very high density. The larvae can severely devastate maize plantations over several thousand square kilometers with a very high population density (De Groote *et al.*, 2020; Faithpraise *et al.*, 2015).

These severe effects of FAW outbreaks particularly occur when rainstorms follow droughts (Bhusal & Chapagain, 2020). Since The pest does not have the ability to diapause (a biological resting period), infestations by FAW occur continuously throughout the year where the pest is endemic. In non-endemic areas, migratory FAW arrive when environmental conditions allow and may have as few as one generation before they become locally extinct.

1.1.4 Natural Enemies

The predators of FAW are general predators that attack many other larvaes, among them are various ground beetles (Coleoptera: Carabidae); the striped earwig, *Labidura riparia* (Pallas) (Dermaptera: Labiduridae); the spined soldier bug, *Podisus maculiventris* (Say) (Hemiptera:

Pentatomidae); and the insidious flower bug, *Orius insidiosus* (Say) (Hemiptera: Anthocoridae). Vertebrates such as birds, skunks, and rodents also consume larvae and pupae readily. Predation may be quite important, as Pair & Gross Jr (1984) demonstrated 60 to 90% loss of pupae to predators in Georgia.

Cool, wet springs followed by warm, humid weather in the overwintering areas favor survival and reproduction of FAW, allowing it to escape suppression by natural enemies. Once dispersal northward begins, the natural enemies are left behind. Therefore, although FAW has many natural enemies, few act effectively enough to prevent crop injury (Assefa & Ayalew, 2019; Chapman *et al.*, 2000) .

1.1.5 Introduction and Spread of FAW in Africa

The FAW is native to the western hemisphere, particularly North and South America (Pannuti *et al.*, 2016), where the insect has been a problem pest in crops for several decades (FAO, 2018*a*). Several reports confirmed that FAW was initially established in São Tomé and Príncipe, Benin, Nigeria, and Togo in 2016 (Prasanna *et al.*, 2018). It is believed that FAW came into Africa through stowaways on commercial aircrafts, in cargo, or airplanes (Day *et al.*, 2017).

As of December 2018, the trans-boundary pest was reported to be present in almost all SSA countries and in August 2018 the pest was detected in Yemen and India Rukundo *et al.* (2020). As was predicted by modelling (Day *et al.*, 2017), FAW has spread to all SSA countries, parts of the Middle East and Asia (Rwomushana *et al.*, 2018), and there are chances that the pest will spread to Europe. The spread is suspected to be attributable to natural migration and trade (Day *et al.*, 2017). The map in Figure 7 shows the distribution of the FAW in Africa as at August 2017.

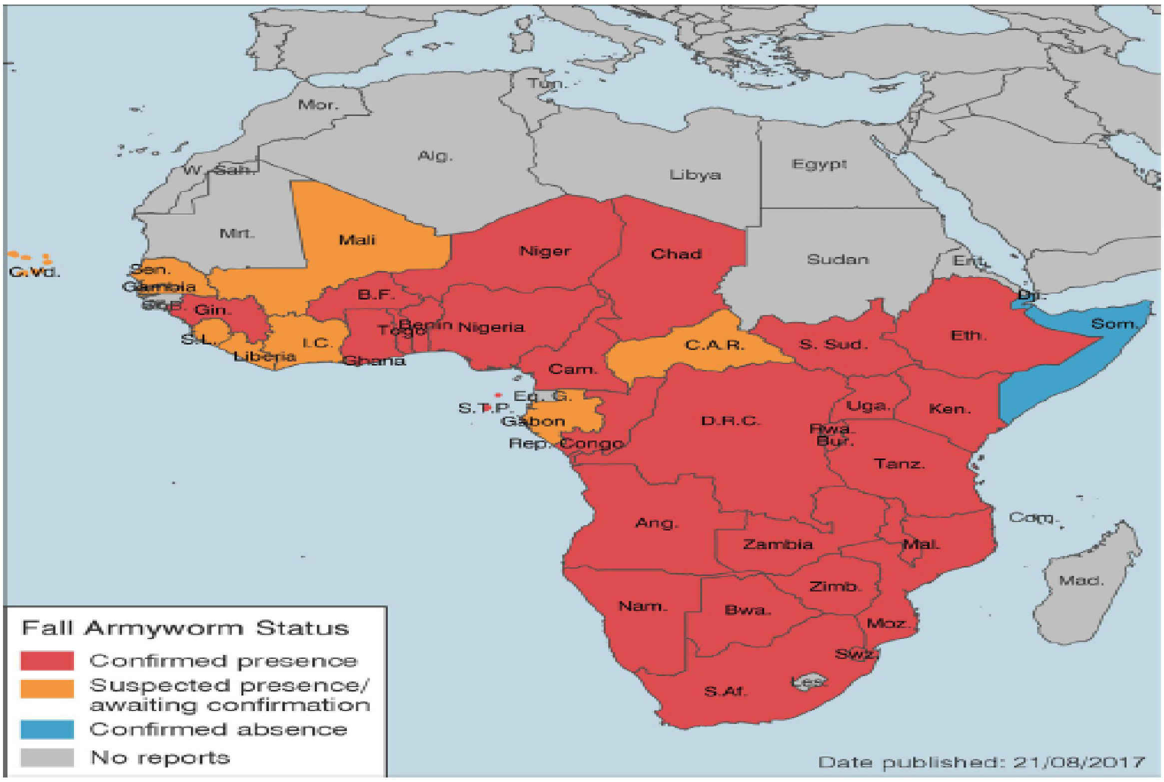
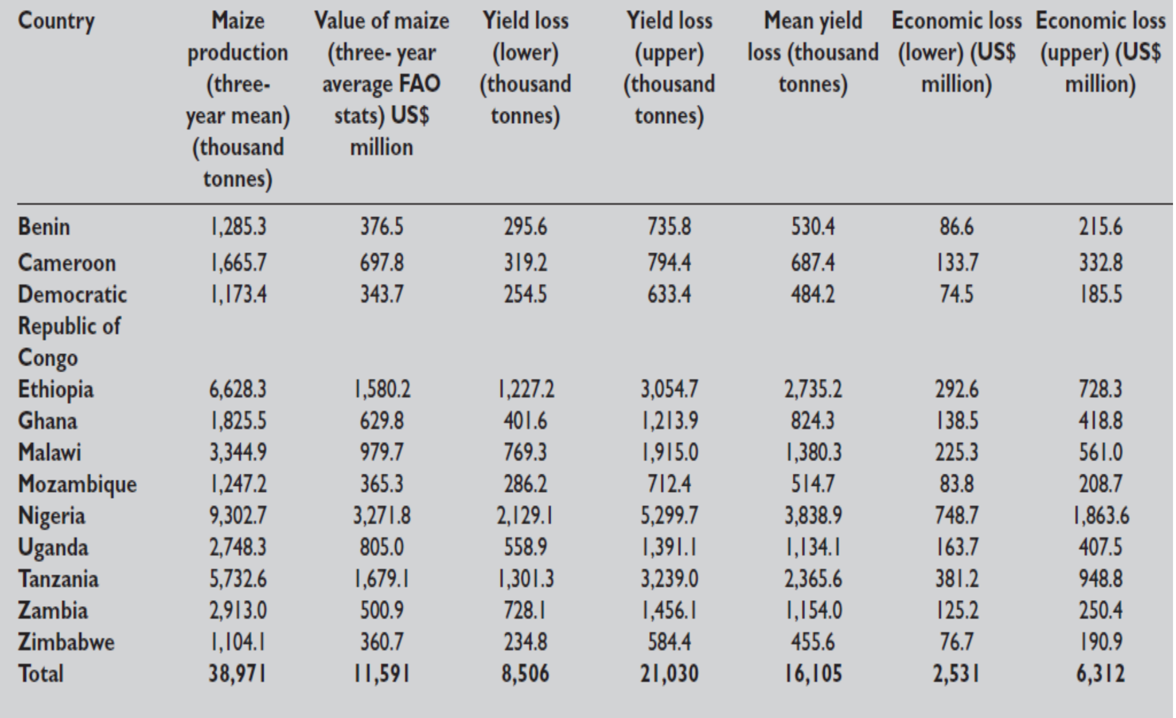


Figure 7: The distribution of the fall armyworm, *Spodoptera frugiperda* **(J.E. Smith) in Africa as at August 2017 (Day** ***et al.*****, 2017)**

1.1.6 Impact of FAW on Crop Production in Africa

Maize is the most widely grown crop in Africa and a staple food for around half of the continent's population. The crop is grown across diverse agro-ecological zones (AEZs) where over 200 million people depend on it for food security (Day *et al.*, 2017). Since its invasion in Africa the FAW has had a major economic and environmental implications. The costs of management, losses of grain yield, hunger or food insufficiency, losses of quality and quantity of crops, and risks of chemical pesticides on health and environments are the major ones (Figure 8). The estimated percentage yield losses of maize across African countries as reported by various authors indicated variations, for instance, 22% in Ghana, 67% in Zambia, 32% in Ethiopia, 47% in Kenya, 11.57% in Zimbabwe, and 57% in Namibia (Yigezu & Wakgari, 2020; Kandel & Poudel, 2020).



Country	Maize production (three-year mean) (thousand tonnes)	Value of maize (three- year average FAO stats) US$ million	Yield loss (lower) (thousand tonnes)	Yield loss (upper) (thousand tonnes)	Mean yield loss (thousand tonnes)	Economic loss (lower) (US$ million)	Economic loss (upper) (US$ million)
Benin	1,285.3	376.5	295.6	735.8	530.4	86.6	215.6
Cameroon	1,665.7	697.8	319.2	794.4	687.4	133.7	332.8
Democratic Republic of Congo	1,173.4	343.7	254.5	633.4	484.2	74.5	185.5
Ethiopia	6,628.3	1,580.2	1,227.2	3,054.7	2,735.2	292.6	728.3
Ghana	1,825.5	629.8	401.6	1,213.9	824.3	138.5	418.8
Malawi	3,344.9	979.7	769.3	1,915.0	1,380.3	225.3	561.0
Mozambique	1,247.2	365.3	286.2	712.4	514.7	83.8	208.7
Nigeria	9,302.7	3,271.8	2,129.1	5,299.7	3,838.9	748.7	1,863.6
Uganda	2,748.3	805.0	558.9	1,391.1	1,134.1	163.7	407.5
Tanzania	5,732.6	1,679.1	1,301.3	3,239.0	2,365.6	381.2	948.8
Zambia	2,913.0	500.9	728.1	1,456.1	1,154.0	125.2	250.4
Zimbabwe	1,104.1	360.7	234.8	584.4	455.6	76.7	190.9
Total	**38,971**	**11,591**	**8,506**	**21,030**	**16,105**	**2,531**	**6,312**

Figure 8: Estimated lower and upper yield and economic losses in the 12 maize-producing countries included in the study (Day *et al.*, 2017)

The overall estimated potential impacts of FAW on Africa's maize yield losses is assumed to range from 4.1 to 20.6 million tons annually, out of the total expected production of 39.3 million tons (Day *et al.*, 2017). The capital losses in terms of money are also estimated to be 1088 to 4661 USD million annually, out of the total expected value of USD 10343 million per year (Yigezu & Wakgari, 2020; Kandel & Poudel, 2020). In SSA countries alone, the yield losses of maize was estimated to be USD 13 billion per annum after 2018, thereby threatening the livelihoods of millions of poor farmers (Yigezu & Wakgari, 2020; Kandel & Poudel, 2020).

As of mid-March 2017, damages to hundreds of hectares of maize planted earlier in the year in Tanzania in Kagera, Geita, Kongwa (Tanga), Simiyu, Mwanza, Morogoro, Kilimanjaro, Njombe, western region of Rukwa and 3000 hectares of maize farms in Chalinze ward (Pwani) in the coastal region were reported (Makirita *et al.*, 2019). Nkasi district, in Rukwa, was particularly affected, as many households in the region depend on maize for access to income and food. These findings of the two authors indicated that yield losses caused by FAW increased from year to year. Using the data from Ghana and Zambia as reported by Rwomushana *et al.* (2018), the estimates of the potential impacts on national yield and revenue in 10 other major maize-producing countries in maize producing seasons are shown in Table 8.

1.1.7 Effect of Climate change on FAW

Being originally a tropical insect, FAW performs better in hot climates where the lower and upper limits of tolerance of temperature are $10°C$ and $42°C$, respectively (Caniço *et al.*, 2020). The optimal range of temperature for its development is between $30°C$ and $35°C$, and its survival and development rates seems to be affected by humidity (Caniço *et al.*, 2020). Depending on the weather conditions, the development cycle of FAW can be significantly affected. Like any other insect pests, weather conditions of different seasons have an effect on FAW dynamics.

Prior studies on pest biology have shown that, the distribution and abundance of FAW pests dynamics is largely influenced by relationship between their developmental rates and fluctuation of weather condition (Du Plessis *et al.*, 2020). In particular, different development stages of insects are favored by fluctuation of weather condition, hence, fluctuation of weather condition influence the development rates, duration of life-cycles, and, ultimately, the survival of insects (Du Plessis *et al.*, 2020). Moreover, an increase in the ambient temperature to the near thermal optimum for insects causes increases their metabolism, and, consequently, their activities (Du Plessis *et al.*, 2020).

Since weather condition fluctuates in the natural environment, it follows that the development rates of insects varies seasonally. For FAW in particular, prior studies suggest that their populations in a given area directly depends on the time of the year, host plants availability, and weather conditions (Caniço *et al.*, 2020). Under unfavorable weather condition for the development and reproduction of the FAW, it is forced to migrate to other suitable locations for survival (Westbrook *et al.*, 2016; Caniço *et al.*, 2020). For example, According to Prasanna *et al.* (2018), duration of the eqqs stage is time-dependent mostly is 2 to 3 days during the warm summer months while the larval stage tends to be about 14 days during the warm summer months and 30 days during cooler weather with a mean development time of 3.3, 1.7, 1.5, 1.5, 2.0, and 3.7 days for instars 1 to 6, respectively, when reared at $25°C$.

1.1.8 FAW Control Strategies and Associated Challenges

A number of strategies can be used to control the poised by FAW whenever they infest in a field (Matova *et al.*, 2020). The control strategies are classified in the following categories: (a) Synthetic and botanical pesticides control practices (b) Cultural agronomic practices (c) Biological control practices.

(i) Synthetic and botanical pesticides control practices

This entails the use of synthetic and botanical pesticides to control the spread of FAW. This approach is considered to be the most expensive relative to other control strategies (Matova *et al.*, 2020).

(ii) Cultural agronomic practices

Farming practices such as weeding, inter-cropping maize with pumpkin and crop rotation are classified under cultural agronomic practices. Prior studies have shown that intercropping of two or more crops, or inclusion of non-host crop plants in the field, can reduce FAW oviposition on the maize plant (Matova *et al.*, 2020).

(iii) Biological control practices

The FAW has several natural enemies, such as predators, parasitoids, and pathogens that regulate its population levels. In some cases, intercropping creates an environment that favors development and growth of a population of natural enemies, large enough to control FAW (FAO, 2018*b*). This phenomenon has resulted in popularization of the "push–pull technology" (PPT) which is currently being recommended for FAW control (Matova *et al.*, 2020).

The PPT is based on intercropping maize with greenleaf desmodium [*Desmodium intortum* (Mill.) Urb.] and bordering the intercrop with *Brachiaria* 'Mulato II' (Midega *et al.*, 2018). The *Desmodium* protects the maize by emitting semiochemicals that repel (push) the moths that are concurrently attracted (pulled) by semiochemicals released by the border crop. Midega *et al.* (2018) and Hailu *et al.* (2018) reported that FAW infestations can be reduced by at least 80% in a field where the technology is being implemented.

Following the invasion of SSA by FAW, a combination of the aforementioned strategies have been used to control FAW and the process of implementing more than one control strategy is known as Integrate Pest management (IPM). The IPM strategy is based on the principle of controlling a pest using a combination of methods while causing the minimum possible damage to the environment, animals, and people. In particular, IPM combines cultural, biological, host-plant resistance, and safe pesticide control methods (Hurley & Mitchell, 2014; Onstad, 2014; Matova *et al.*, 2020).

The FAW IPM strategies are targeted at preventing or avoiding pest infestations, and management of established infestations. This involves routine scouting to identify and respond to infestations, to suppress the pest using the IPM triangle strategies, that is, minimum application of safe pesticides, provision of safe, scientifically proven or evidence-based options to farmers, and managing insect resistance to pesticides (Onstad, 2014; Prasanna *et al.*, 2018; Matova *et al.*, 2020).

1.1.9 Challenge on Control Strategy of FAW

Farmers' choice of FAW control strategy is affected by various factors, which include availability of a control strategy (including its effectiveness and ease of use) and resources, gender, and age among other issues (Matova *et al.*, 2020). Recently, control of FAW and its associated damages on maize is a challenge in Africa particulary to small holder farmers due to its feeding and sheltering behavior on the host plant. Different common management strategy for the FAW which include pesticide sprays, bilogical control and genetically modified crops (Bt maize) were applied in weatern countries and later in Africa. However, the FAW young larvae hide in the maize funnel during the day but emerges at night to feed on the maize biomass (Day *et al.*, 2017).

Therefore, spraying application of pesticides are not effective due to the tendency of the younger and older FAW larvae stay inside the maize funnel and so are protected by the foliage from spray application. Furthermore, Use of pesticides to control FAW predisposes farmers to harmful insecticide contamination and also destabilizes the ecosystem by killing non-target organisms (Matova *et al.*, 2020). Hence, development of sustainable and environment-friendly control strategies for FAW is paramount. Additionally, its short life cycle, ability to travel across large geographical areas, and its wide host range promote its rapid multiplication and spread, making it difficult to control (Onstad, 2014; Prasanna *et al.*, 2018).

In addition, most small holder farmers in African countries cannot afford repeated spray of pesticide and according to De Groote *et al.* (2020), approximately 10 000 people die per day each year in developing countries from pesticide when applied in huge amount and 400 000 people suffer acutely as pesticide travel through food chain. FAW being a recent invader in the continent, genetically modified crops (Bt maize) not available to majority of the small holder farmers and information on natural enemies (Biological Control) associated with this pest is not well documented for Africa (Onstad, 2014).

1.2 Statement of the Problem

FAW and other diseases have been threatening our food security at all levels of life, causing considerable economic losses for decades. Within the last decade, several scientists have utilized mathematical models to describe the plant-pest interaction dynamics (Yigezu & Wakgari, 2020; Kandel & Poudel, 2020) and to assess effectiveness of the control measures (Day *et al.*, 2017). Despite all these efforts, however, mathematical models for understanding the implications of FAW-maize interaction are still lacking, regardless of serious threats to food security caused by FAW infestations in maize fields. In particular, to the best of the author's knowledge, there are no mathematical models in the literature that have been developed to qualitatively and quantitatively assess the effects of rolling out either single or multiple FAW intervention strategies on pest population density over time and the final maize biomass during an infestation in a maize field. To fill this gap, this study presents three new mathematical models for FAW and maize interaction. Mathematical models developed in this study were quantitatively and qualitatively used to describe multiple scenarios, simulate the impact of interventions and provide estimates for the final maize biomass. In particular, the first mathematical model aimed at determining the effects of intervention and memory effects on FAW-maize interaction dynamics over time and the resultant final maize biomass. There is no doubt that seasonal variations and daily changes in weather patterns affect the existence of both the pest and plant population dynamics as highlighted in the introduction. Based on these facts, the second mathematical model focused on qualitatively and quantitatively analysing the effects of seasonal variations and intervention on FAW-maize interaction over time and the resultant final maize biomass. Unlike insecticides, bio-control methods (larvae predatio) are environmentally friendly. Based on this and other several advantages their use in managing the pests has been increasing within the last decade. Motivated by the aforementioned facts, the final mathematical model, sought to qualitatively and quantitatively estimate changes in pest population density over time and resultant maize biomass due to the implementation of bio-control methods during an outbreak.

1.3 Research Objectives

1.3.1 Main Objective

The main objective of this study was to construct, calibrate (using data in literature) and analyse mathematical models for FAW-maize interaction, that incorporate either single or multiple intervention strategies, with a goal to estimate changes in pest population density over time and the resultant maize biomass.

1.3.2 Specific Objectives

The specific objectives of the study were:

(i) To construct, calibrate and analyse a new mathematical model for FAW-maize interaction that incorporates intervention strategies, with a goal to estimate the changes in pest population density over time and the resultant maize biomass.

(ii) To construct, calibrate and analyse a new mathematical model for FAW-maize interaction that incorporates effects of seasonal variations and intervention strategy, with a goal to estimate changes in pest population density over time and the resultant maize biomass.

(iii) To construct, calibrate and analyze a new mathematical model for FAW-maize interaction that incorporates farming awareness campaigns and bio-control methods, with a goal to estimate changes in pest population density over time and the resultant maize biomass

1.4 Research Questions

This study was influenced by the following research questions:

(i) What are the effects of intervention strategies on FAW population density over time and the resultant maize biomass, during an outbreak?

(ii) What are the effects of seasonal variations and intervention strategies on FAW population density over time and the resultant maize biomass, during an outbreak?

(iii) What are the effects of farming awareness campaigns and bio-control (larvae predation) methods on FAW population density over time and the resultant maize biomass, during an outbreak?

1.5 Rationale of the Study

Maize is the second most important cereal crop in the world after wheat, contributing substantially to the total cereal grain production in the world economy as a trade, food, feed, and industrial grain crop. The demand for maize is increasing, not only because of its higher nutritional benefits but also its ability to feed the growing global population and contribution to food security. However, the production of this crop and consequently the livelihood of the growers is threatened by the invasion and widespread infestation of the fall army worm (FAW), Spodoptera frugiperda (J. E. Smith) (Lepidoptera: Noctuidae) which has led to substantial maize yield losses.

Therefore, understanding the spread, effects and symptoms of the FAW on maize biomass will help small holders farmers in Sub-Sahara African counties be aware and plan a good strategy to prevent and manage the effect of FAW in the field. The study will also help agricultural officers, policy makers and other stake holders to select the best control strategies and be able to implement them to minimize the migration of the pest to other location. Besides, knowing the effects of seasonal (specific temperature) variations on the dynamics of FAW will help the stake holders to allocate and implement resources at the right place and time. In addition to that, the selected topic of study will add knowledge in the literature review for further studies.

1.6 Significance of the Study

This study will study will make several immeasurable contributions, among the following:

(i) Enhance the existing knowledge on the role of memory effects on plant-pest interaction, particularly FAW-maize interaction.

(ii) Enhance the existing knowledge on influence of seasonality on FAW-maize interaction and the associated effects on the final maize biomass.

(iii) Enhance the existing knowledge on the impact of IPM strategy on FAW management and maize productivity following FAW infestations in a maize field.

1.7 Delineation of the Study

In recent years, the FAW has spread globally and emerged in countries where it had rarely or never before been present, posing a real threat to global food security (Assefa & Ayalew, 2019). Current estimates from 12 African countries suggest an annual loss of 4.1 to a massive 17.7 million tons of maize due to FAW (De Groote *et al.*, 2020). It is imperative that mathematical models be proposed and utilized to assess the implications of FAW infestations in a maize field and its impact on maize productivity (maize biomass). Mathematical modeling has become a tool used to explore many real world phenomena. Mathematical modeling, analysis and simulation for plant-pest interaction have proved to be an essential guiding tool that could give a sound direction to policy makers and farmers on how to increase plant productivity (Bokil *et al.*, 2019). The models and results presented in this dissertation are new and make a significant contribution to the existing body of knowledge on FAW infestation in a maize field and its impact on the final maize biomass.

1.8 Mathematical Preliminaries

1.8.1 Preliminaries on the Caputo fractional calculus

We begin by introducing the definition of Caputo fractional derivative and state related theorems (see, (Caputo, 1967; Diethelm, 2010; Podlubny, 1999)) that will be utilized to derive important results in this work.

Definition 1.1

Suppose that $q > 0, t > a, q, a, t \in \mathbb{R}$. The Caputo fractional derivative is given by

$$ {}_{a}^{c}D_{t}^{q} f(t) = \frac{1}{\Gamma(n-q)} \int_{a}^{t} \frac{f^{n}(\xi)}{(t-\xi)^{q+1-n}} d\xi, \quad n-1 < q, n \in \mathbb{N}. \tag{1.1} $$

Where the Gamma function $\Gamma(n-q)$ in equation (1.1) generalizes the factorial and allows n, to take non-integer values.

Definition 1.2

Let $q > 0$. The fractional integral of order q for a function $f(t) \in C_a (a \geq -1)$, initially defined by Riemann-Liouville which is presented in (Podlubny, 1999) is defined by

$$ J^{q} f(t) = \frac{1}{\Gamma(q)} \int_{0}^{t} (t-\xi)^{q-1} f(\xi) d\xi, \tag{1.2} $$

$J^{0} f(t) = f(t)$.

Definition 1.3

As seen in (Liang *et al.*, 2012). Let $q > 0$, $n-1 < q < n - \mathbb{N}$. Suppose that $f(t), f'(t), ..., f^{(n-1)}(t)$ are continuous on $[t_0, \infty)$ and the exponential order and that ${}_{t_0}^{c}D_{t}^{q} f(t)$ is piece-wise continuous on $[t_0, \infty)$. Then

$$ \mathscr{L}\{{}_{t_0}^{c}D_{t}^{q} f(t)\} = s^{q} \mathscr{F}(s) - \sum_{k=0}^{n-1} s^{q-k-1} f^{(k)}(t_0) \tag{1.3} $$

where $\mathscr{F}(s) = \mathscr{L}\{f(t)\}$.

Lemma 1.1

As observed in Vargas-De-León (2015). Let $x(\cdot)$ be a continuous and differentiable function with $x(t) \in \mathbb{R}_+$. Then, for any time instant $t \geq t_0$, one has

$$ {}_{t_0}^{c}D_t^q \left(x(t) - x^* - x^* \ln \frac{x(t)}{x^*} \right) \leq \left(1 - \frac{x^*}{x(t)} \right) {}_{t_0}^{c}D_t^q x(t), \qquad x^* \in \mathbb{R}^+, \qquad \forall q \in (0,1). \quad (1.4) $$

Lemma 1.2

Ahmed *et al.* (2007) considered the following fractional order system:

$$ \left. \begin{array}{lcl} {}_{t_0}^{c}D^q x(t) & = & f(t,x), \\ x(0) & = & x_0 \end{array} \right\} \quad (1.5) $$

where $f(t,x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$. The equilibrium points (1.5) are locally asymptotically stable if all eigenvalues $\lambda_i (i = 1,2,3,4...)$ of the Jacobian matrix $\frac{\partial f(t,x)}{\partial x}$ evaluated at the equilibrium points satisfy the following condition:

$$ |\arg(\lambda_i)| > \frac{q\pi}{2}. \quad (1.6) $$

1.8.2 Existence and Uniqueness Theorem

Definition 1.4

Suppose that $D = \{(t,y) | a \leq t \leq b \text{and} - \infty < y < \infty\}$ and that $f(t,y)$is continuous on D.If f satisfy a Lipschitz condition on D in the variable y,then the solution $y(t)$ for the initial value problem

$$ \dot{y} = f(t,y), \; a \leq t \leq b, \; y(a) = \alpha \quad (1.7) $$

exists and unique (Richard, 2005).

1.8.3 Adam-Bashforth-Moulton Method

Adam-Bashforth-Moulton method is a numerical technique developed by John Couch Adam for approximating solution of a dynamical system (Richard, 2005). Given a differential equation

$$ \frac{d^\alpha x(t)}{dt^\alpha} = f(t,x(t)), \quad (1.8) $$

for fractional order, Adam-Bashforth-Moulton method as in (Diethelm, 2010) is a variant of the one step method given by

$$ x_{n+1} = \sum_{i=0}^{[\alpha]-1} \frac{t_{n+1}^i}{i!} x_0^i + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} a_{i,n+1} f(t_i, x_i) + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1}, x_{n+1}^p), \quad (1.9) $$

where $t_i = ih$ with some fixed h and :

$$a_{i,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & i = 0, \\ (n-i+2)^{\alpha+1} + (n-i)^{\alpha+1} - 2(n-i+1)^{\alpha+1}, & 1 \leq i \leq n. \end{cases} \tag{1.10}$$

To determine the error in this method, by assuming that $t_i = ih = \frac{i\tau}{N}$ where $\tau = b - a$, with some $N \in \mathbb{N}$, and we have:)

$$\max_{0 \leq i \leq N} |x(t_i) - x_i| = \begin{cases} \mathscr{O}(h^2), & \alpha \leq 1, \\ \mathscr{O}(h^{1+\alpha}), & \alpha < 1. \end{cases}$$

1.8.4 Logistic Equation

According to Iannelli & Pugliese (2015), an equation

$$\dot{N} = rNF(N), \text{ and } F(N) = 1 - \frac{N}{K} \tag{1.11}$$

with an intrinsic carrying capacity K is said to be a logistic equation if:

(i) $F(0) = 1$ (the population grows exponentially with growth rate r when N is small).

(ii) $F(K) = 0$ (the population stops growing at the carrying capacity K).

(iii) $F(N) < 0$ when $N > K$ (the population decays when it is larger then the carrying).

1.8.5 Reproduction Number

The reproduction number $\mathscr{R}_0$ which also is a threshold quantity is the average number of offspring generated by an adult female during its entire lifespan (Iannelli & Pugliese, 2015). The $\mathscr{R}_0$ provides the necessary condition for the persistence or extinction of an individual in the population as well as the stability of the equilibrium points. In particular, we have that:

(i) When $\mathscr{R}_0 < 1$ the pest extinct in the population and the equilibrium point is asymptotically stable.

(ii) When $\mathscr{R}_0 > 1$ the pest persist in the population and the equilibrium point is asymptotically unstable.

1.8.6 Population Dynamics

Population dynamics involves two processes: reproduction and death of an individual (Iannelli & Pugliese, 2015). The equation which describes the population dynamics of a population can be written in the general form:

$$\dot{x} = F(x), \text{ or } \dot{x} = xf(x) \text{ with,} \tag{1.12}$$

$$F(x) = B(x) - D(x) \text{ or } f(x) = b(x) - d(x) \tag{1.13}$$

where $B(x)$ is the absolute reproduction, $D(x)$ is the absolute mortality rates of individuals, $b(x)$ is the corresponding per capita reproduction, and $d(x)$ is the death rates, that is fecundity and mortality.

1.8.7 Routh-Hurwitz Criteria

Lemma 1.3
Given the polynomial

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + a_3\lambda^{n-3} + a_4\lambda^{n-4} + \cdots + a_{n-1}\lambda + a_n, \tag{1.14}$$

where the coefficients a_i for $(i = 1,...,n)$ are real constants. According to Allen (2007), the n Hurwitz matrices using the coefficients a_i of the characteristic polynomial defined as follows:

$$H_1 = \begin{bmatrix} a_1 \end{bmatrix}, \quad H_2 = \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix}, \quad H_3 = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix}, \tag{1.15}$$

and

$$H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix}, \tag{1.16}$$

where $a_j = 0$ if $j > n$. All of the roots of the polynomial $P(\lambda)$ are negative or have negative real part if and only if the determinants of all Hurwitz matrices are positive:

$$det(H_j) > 0, \qquad j = 1,2,...,n. \tag{1.17}$$

Routh-Hurwitz criteria for $n = 2, 3$, and 4 are as follows

$$
\begin{aligned}
&(C1) \qquad n=2: \qquad a_1 > 0, \quad \text{and} \quad a_2 > 0, \\
&(C2) \qquad n=3: \qquad a_1 > 0, \quad a_3 > 0, \quad \text{and} \quad a_1 a_2 > a_3 \\
&(C3) \qquad n=4: \qquad a_1 > 0, \quad a_3 > 0, \quad a_4 > 0, \quad \text{and} \quad a_1 a_2 a_3 > a_3^2 + a_1^2 a_4.
\end{aligned} \tag{1.18}
$$

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

Mathematical modelling is commonly regarded as the art of applying mathematics to a real world problem to better understand the problem (Cheng, 2009). Yanagimoto (2005) argues that mathematical modelling is not just a process of solving a real-life problem using mathematics, but the process that involves applying mathematics in a situations where the results are useful in society. In recent years, Mathematical models have proved to be an important tool to explore and guide policy formulation in many real world phenomena. Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs) with and without memory effects are some of the tools that have commonly been used to formulate equation(s) that mirror real world problem(s) (Tang *et al.*, 2010; Faithpraise *et al.*, 2015; Bokil *et al.*, 2019; Helikumi *et al.*, 2020). Models enable researchers to create frameworks that can be accurately utilized to conceptualize and communicate ideas regarding the behavior of a particular system (Keeling, 2005). Through these frameworks, solutions to phenomena that are difficult to measure in the field can be found.

This chapter provides a review of mathematical models for plant-pest interaction. According to Páezchávez *et al.* (2017), such mathematical models are essential for understanding and providing useful abstractions of the underlying biological phenomena and ecological interactions taking place in pest control applications .

2.2 Mathematical Models of Plant-Pest Interactions

Recently, a number of mathematical models have been developed to explore plant-pest interactions (Jiang *et al.*, 2005; Hui & Zhu, 2006; Pearce *et al.*, 2006; Rafikov *et al.*, 2008; Tang *et al.*, 2010; Kang *et al.*, 2013; Faithpraise *et al.*, 2015; Gao *et al.*, 2016; Anguelov *et al.*, 2017; Páezchávez *et al.*, 2017; Bokil *et al.*, 2019; Chowdhury *et al.*, 2019; Li *et al.*, 2019; Abraha *et al.*, 2021) to mention a few.

Liu *et al.* (2005) utilized a mathematical model to asses the impact of spraying pesticides at a fixed time on the pest reproductive cycles. Among several outcomes, their study showed that there exists optimal time of pest control if the pesticides were to be applied just before each birth pulse of the cycle.

Jiang *et al.* (2005) developed an autonomous piecewise linear system with impulses governed by state feedback control to study the dynamics of a stage-structured pest management system.

The authors determined the sufficient conditions of existence and stability of periodic solutions through the method of the sequence convergence rule and the analogue of the Poincare criterion. The authors also illustrated the attractive region of periodic solutions. Utilizing the Poincare map, bifurcation diagrams of the periodic solutions were obtained. In addition, the superiority of the state feedback control strategy was also discussed.

Pearce *et al.* (2006) utilized a two Lotka-Voltera predator-prey model to investigate the role of pest predators on the dynamics of pest-plant interactions. In their study, the pest such as mites,insects, spiders and snails were considered as predators while Egg, Larval, and pupal parasites were considered as prey. Their aim was to investigate the dynamics and compare the results for the ordinary differential model corresponding to continuous biological control with those for the impulsive differential equation model corresponding to impulsive biological control. The results indicated that under sufficient conditions, with impulsive differential equation model, the pest population were reduced below economic threshold relative to ordinary differential model.

Tang *et al.* (2010) proposed impulsive differential equation models or hybrid dynamical system to model the introduction of a periodic IPM strategy which includes periodic spraying of pesticide and release of natural enemies at critical time. From their study, optimal periods for spraying were determined.

Kang *et al.* (2013) proposed a stage-structured pest control model with impulse effects by state feedback control to determine the optimal control strategies for agricultural pests. The sufficient conditions for existence and attractiveness of order one periodic solution were determined using the method of successor functions. It was established that the superiority of the state feedback control strategy occurred due to the need to monitor the sum of immature and mature pest populations.

Faithpraise *et al.* (2015) developed a mathematical framework to determine the role of naturally beneficial insects on controlling African armyworm (AAW) pest infestations in cereal crop fields. The model incorporated egg and larvae predators. The results from the study demonstrated that the proposed model was valuable in offering possible solutions for the control of AAW outbreaks.

Gao *et al.* (2016) developed a compartmental model for plant-pest interaction to represent the dynamics of plant disease in a periodic environment. The model incorporated impulsive rouging as a control strategy. The results from the study showed that when the infection rate of plants is high, it may be impossible to eradicate the disease by simply rouging the infectious plants, hence, there is a need to also identify latently infected plants and rouge them. This study also noted that, increased replanting is not a good strategy for disease control.

Páezchávez *et al.* (2017) proposed and analyzed mathematical models ranging from classical smooth differential equations to differential equations with reformulation of impulses with a goal to investigate the impact of combining chemical (pesticides), biological (natural enemy predation, bio-pesticides), and cultural (rouging, replanting) methods to control pests. In a nutshell, the proposed model investigated the implications of the IPM strategy on minimizing pest population in a crop field. Important results that include fold and flip bifurcation of limit cycles, periodic doubling cascade leading to chaotic behavior, and hysteric effects were observed.

Chowdhury *et al.* (2019) formulated and extensively investigated a continuous and discrete predator-prey models concerning the IPM strategy. From their study, optimal threshold levels for the IPM strategy that are essential to efficiently minimize plant attack were determined. Discrete host parasitoid models have also been proposed for circumstances when the timing of pesticide application leads to the death of parasitoid, and four different cases involving the timing of pesticide applications were also investigate by Anguelov *et al.* (2017).

Li *et al.* (2019) proposed pest control models that incorporated birth pulse and were based on the assumption that pesticides killed adult pests or larvaes or both. Using numerical simulations, , the author demonstrated that the optimal time for pesticide applications was different for different elimination rates of larvae and adults moth.

Abraha *et al.* (2021) developed a mathematical model for plant-pest interaction with an aim to investigate the impact of time delays and optimal use of bio-pesticides by considering plant biomass and the effect of farming awareness. The time delay factor incorporated by the authors in this study accounted for the time needed by farmers to become aware of the control strategies or taking the necessary steps following pest attack. The results from their study highlighted that optimal control can be an essential tool for designing the control strategies for plant-pest interactions.

The aforementioned studies and those cited therein certainly improved our qualitative and quantitative knowledge on plant-pest interactions. However, despite all these studies, several challenges remain in the mathematical modeling of plant-pest interaction. First, a majority of the aforementioned studies were general and not pest-specific which implies that their results were also general. However, in those studies which were not general, plants such as *Jatropha curcas* and the associated pests were considered. Practically, pests are not general, they follow different biological development cycles, hence more informative plant-pest interaction models need to be pest specific and closely follow the life cycle of the pest involved. For example, some pest diapause (for example Tarnished plant bugs) while others such as FAW, the ability to diapause is not present. In short, diapause is a state of arrested growth or reproduction of many hibernating or estivating arthropods. There is no doubt that these difference alter plant-pest interaction dynamics.

CHAPTER THREE

MATERIALS AND METHODS

3.1 Introduction

In this Chapter, three mathematical models for FAW-maize interactions have been developed, with the aim of evaluating the implications of this interactions on the final maize biomass (biomass at the end of the season). The models presented in this chapter are original (which extend many of the published models in the literature) and incorporate all the relevant biological details that characterize FAW-maize interactions.

The first model was developed to evaluate the impact of memory effects and control on the final maize biomass due to FAW-maize interactions. There is no doubt that mathematical models of plant-pest interactions reviewed in Chapter 2 and several others cited therein have certainly produced many useful results and improved the existing knowledge on plant-pest interactions. However, one of the limitations of these studies is that their models were based on integer-order ordinary differential equations. Recent studies suggest that models that use integer-order differential equations do not adequately capture memory effects as well as hereditary properties, which are inherent in many real world problems (Helikumi *et al.*, 2020). As such, in recent years, fractional calculus has become an intriguing field. Several researchers have shown that models that utilize fractional calculus are more likely to replicate real world problems compared to those that use integer-order differential equations since fractional-order differential equations are able to capture memory effects (Helikumi *et al.*, 2020; Mouaouine *et al.*, 2018).

Like any other insect pest, the weather conditions in maize growing seasons have an effect on FAW dynamics. Prior studies on pest biology have shown that the distribution and abundance of pests is largely influenced by relationship between their developmental rates and fluctuation of weather conditions (Du Plessis *et al.*, 2020). In particular, different development stages of insects are favored by different ranges of weather conditions, hence, variations of weather conditions influence the development rates, duration of life cycles, and, ultimately, the survival of insects (Du Plessis *et al.*, 2020). Moreso, an increase in the ambient temperature to the near thermal optimum for insects causes increases their metabolism, and, consequently, their activities (Du Plessis *et al.*, 2020).

Since weather conditions fluctuates in the natural environment, it follows that the development rates of insects vary seasonally. For FAW in particular, prior studies suggest that their populations in a given area directly depends on the time of the year, host plant availability, and weather conditions (Caniço *et al.*, 2020). Under unfavourable weather conditions for the development and reproduction, the FAW is forced to migrate to other suitable locations for survival (Caniço *et al.*, 2020; Westbrook *et al.*, 2016). In the present study, a second mathematical model was proposed and analyzed to evaluate the effect of seasonality on FAW-maize interaction in a maize growing season.

Host plant resistance and chemical insecticide use are ranked among the best ways of effectively managing FAW during an outbreak (Sisay *et al.*, 2019). However, due to the limited financial capacity of the smallholder farmers and governments of most African countries, it follows that either massive spraying of chemical insecticides or use of genetically modified crops remain elusive. Thus, effective management of the pest in the African continent remains a challenge (Sisay *et al.*, 2019). Besides, the excessive use of chemical insecticides is associated with negative environmental effects and can lead to the development of pesticide resistance (Matova *et al.*, 2020).

At the backdrop of this, integrated pest management (IPM) has attracted more attention among researchers and its application is also increasing the crop yield in the field (Abraha *et al.*, 2021). In a nutshell, IPM entails the use of several of low-cost cultural practices to manage FAW during an outbreak. Thus the IPM approach seeks to minimize the reliance of pesticide use by emphasizing the contribution of biological control agents. In the present study, a third mathematical frame work to evaluate the impact of IPM on effective management of FAW during an outbreak was proposed and analyzed. The model incorporated mass media campaigns and biological control (larvae predation). To gain insights into the qualitative features of FAW-maize interactions and its impact on the final maize biomass, the following mathematical models were developed and analyzed.

3.2 FAW-Maize Interaction Model with Memory Effects and Control

This section describes a fractional order model that was developed and analyzed in the present study to gain insights into the impact of memory effects on the final maize biomass due to FAW-maize interactions. To analyze the impact of memory effects and control on final maize biomass, a fractional-order model with control strategies was formulated consisting of two populations: maize biomass and the FAW population where one of the populations, the FAW was stage-structured to give a total of five populations. Meanwhile, the FAW population was divided into four classes which represented the FAW life cycle: egg stage $E(t)$, Larval stage $L(t)$, pupal stage $P(t)$, and adult stage (Moth) $A(t)$. Although the FAW typically has six larval instars, however, to reduce complexity of the model in a biological sensible way, all larval instars were represented by class $L(t)$. The life cycle of the FAW starts when eggs are laid in masses on maize biomass, mostly underside of these biomass. For maize biomass in particular, when plant biomass (plant seeds) are planted at a time $t = 0$ in a field whose maize biomass carrying capacity is K_M, the maize biomass emerge in a period of 0 to 7 days. It was assumed that the planting of maize seed per hectare at the beginning of the season is done in one day. Further, it was also assumed that weather condition, environment condition and the planting system of maize seeds favour seed germination and the corresponding growth of maize biomass with no natural death rate before harvest. In this regard, $M(t)$ represented the population density of maize biomass per hectare. Therefore, formulation of a fractional order calculus was utilized and a new mathematical framework was developed based on the assumptions listed in (i)-(v):

(i) Maize biomass was modeled via a logistic growth in the absence of FAW. The biomass decreases due to consumption by FAW larvae at rate β. Thus, the dynamics of maize biomass over time were modeled as shown in equation (3.1):

$$ {}_{a}^{c}D_{t}^{q}M(t) = r^{q}M\left(1-\frac{M}{K_{M}^{q}}\right)-\beta^{q}LM, \tag{3.1}$$

where r is the growth rate of maize biomass

(ii) The dynamics of FAW egg population in relation to time grows logistically and were modeled by the equation (3.2):

$$ {}_{a}^{c}D_{t}^{q}E(t) = b^{q}\left(1-\frac{E}{K_{E}^{q}}\right)wA-(\alpha_{E}^{q}+u_{E}^{q}+\mu_{E}^{q})E, \tag{3.2}$$

In the term $b\left(1-\frac{E}{K_E}\right)wA$ of equation (3.2), b represented egg laying rate for an adult female FAW, that is, an average number of eggs each adult female FAW will lay per day, K_E represented the egg carrying capacity, that is, the availability of space to lay eggs, w was the proportion of female adult FAW, α_E was the egg hatching rate and u_E accounted for the effects of intervention strategies a farmer would implement once they observe eggs laid on the maize biomass, μ_E was the egg mortality rate. FAW larvae generally emerge simultaneously 3-5 days following oviposition (FAO, 2018*b*). The efficiency with which FAW larvae converted consumed maize biomass into larvae's biomass was modelled by parameter $e(0 < e < 1)$.

(iii) The emergence of the FAW larvae following oviposition and their entire growth in the presence of maize biomass is a logistic growth. Therefore, equation (3.3) summarizes the dynamics of the FAW larvae population over time:

$$ {}_{a}^{c}D_{t}^{q}L(t) = \alpha_{E}^{q}\left(1-\frac{L}{K_{L}^{q}}\right)E+\theta^{q}LM-(\alpha_{L}^{q}+u_{L}^{q}+\mu_{L}^{q})L. \tag{3.3}$$

In equation (3.3), the transition rate from the egg stage to larvae was α_E. The older larvae of FAW exhibit a cannibalistic behavior on the smaller larvae when food is limited (Chapman *et al.*, 2000). Hence, the model assumed that the death rate due to lack of food was proportional to the smaller larvae $\alpha_E E$ and to the coefficient L/K_L that represented the availability of food for each larvae. Therefore K_L represented the availability of food and space for the larvae population, μ_L natural mortality rate of the larvae, and $1/\alpha_L$ the average duration of the larval stage which was estimated to range from 14-30 days (Assefa & Ayalew, 2019; Chapman *et al.*, 2000; FAO, 2018*b*). In particular, it estimated that this duration is shorter, around 14 days during warm summer months and longer, around 30 days during cooler weather (FAO, 2018*b*; Chapman *et al.*, 2000). Hence, parameter u_L modelled the role of intervention strategies implemented by the farmer, which may be use of pesticides or handpicking of the larvae. The term $\theta^q LM$ represents the interaction of the larvae and maize biomass which results in conversion of maize biomass into larvae biomass. Hence we can write $\theta^q = e^q \beta^q$, where e^q is the leaf-impact factor or efficiency of biomass conversion.

(iv) Pupation of the FAW normally occurs in the soil at a depth of 2-8 cm (Chapman *et al.*, 2000). Here, the larva constructs a loose cocoon which is oval in shape and 20-30 mm in length, through tying soil particles together with silk (Assefa & Ayalew, 2019). In areas where soil is too hard, larvae web together leaf debris and other materials to form a cocoon on the soil surface (Assefa & Ayalew, 2019). Therefore, equation (3.4) represents the dynamics of pupal stage:

$$ {}_a^c D_t^q P(t) = \alpha_L^q L - (\mu_P^q + \alpha_p^q + u_P^q) P, \tag{3.4} $$

where μ_P was the natural mortality rate, $1/\alpha_P$ the duration of the pupal stage which is approximately 8-9 days during the summer, however, during winter it may reach 20-30 days (Assefa & Ayalew, 2019; FAO, 2018*b*). The parameter u_P accounted for the effects of artificial intervention strategies aimed at reducing FAW pupae population. It was worth noted that the pupal stage of FAW does not enter a diapause period to withstand protracted periods of winter or summer seasons in the absence of host plants (Assefa & Ayalew, 2019).

(v) Adult female FAW are responsible for laying eggs on the surface of biomass, a process which usually starts after a pre-oviposition period of 3-4 days, and continue until they become 3 week old. Therefore, equation (3.5) summarizes the dynamics of the adult FAW:

$$ {}_a^c D_t^q A(t) = \alpha_P^q P - (\mu_A^q + u_A^q) A, \tag{3.5} $$

where μ_P accounted for the proportion of FAW pupa population that successfully progressed to the adult stage, u_A denoted the effects of intervention strategies, and $1/\mu_A$ the life span of the adult FAW, which is estimated to average about 10 days, with a range of about 7-21 days (Assefa & Ayalew, 2019). It was worth noted that the duration of FAW life cycle lasts for about 30 days at 28°C but may take longer, 60-90 days when the weather is cooler (FAO, 2018*b*). In addition, under favorable conditions the FAW has a potential to feed and breed on maize leaves year-round (FAO, 2018*b*).

The summary of the definitions of model state variables and parameters are given in Tables 1 and 2.

Table 1: Description of state variables of FAW-maize interaction model used in this study

Symbol	Definition
$M(t)$	Maize biomass per plant
$E(t)$	FAW egg population
$L(t)$	FAW larvae population
$P(t)$	FAW pupae population
$A(t)$	FAW adult moth population
$Z(t)$	FAW Larvae predator population

Table 2: Model parameters and their biological definitions

Symbol	Definition
b	Number of eggs laid per day per female moth
b_L	Growth rate of larvae
w	Proportion of adult female moth
α_E^{-1}	Average duration of egg stage
α_L^{-1}	Development time of the larva
α_P^{-1}	Development time of pupae
μ_A^{-1}	Moth life span
K_M	Maximum carrying capacity of the maize biomass
K_E	Egg environmental carrying capacity
K_L	Larvae environmental carrying capacity
μ_E	Natural mortality rate of eggs
μ_L	Natural mortality rate of larvae
μ_P	Natural mortality rate of pupae
β	Plant attack rate by the larvae
r	Growth rate of maize biomass
e	Efficiency of biomass conversion
u_E	Intervention strategies on egg population
u_L	Intervention strategies on larvae population
u_P	Intervention strategies on pupae population
η^{-1}	Average life span of larvae predator
σ	Attack rate of the larvae by predators
$.\ \rho$	Efficiency of conversion
η	Average life span of predators

Based on the assumptions stated above on the dynamics of the FAW in a maize biomass population density and mathematical preliminaries on the Caputo fractional calculus presented in subsection (1.8.1), the following mathematical model was presented in system (3.6) and the model flow diagram is in Figure (9):

$$\left.\begin{array}{rcl} {}_{a}^{c}D_{t}^{q}M(t) & = & r^{q}M\left(1-\dfrac{M}{K_{M}^{q}}\right)-\beta^{q}LM, \\ {}_{a}^{c}D_{t}^{q}E(t) & = & b^{q}\left(1-\dfrac{E}{K_{E}^{q}}\right)wA-(\alpha_{E}^{q}+u_{E}^{q}+\mu_{E}^{q})E, \\ {}_{a}^{c}D_{t}^{q}L(t) & = & \alpha_{E}^{q}\left(1-\dfrac{L}{K_{L}^{q}}\right)E+\theta^{q}LM-(\alpha_{L}^{q}+u_{L}^{q}+\mu_{L}^{q})L, \\ {}_{a}^{c}D_{t}^{q}P(t) & = & \alpha_{L}^{q}L-(\mu_{P}^{q}+\alpha_{p}^{q}+u_{P}^{q})P, \\ {}_{a}^{c}D_{t}^{q}A(t) & = & \alpha_{P}^{q}P-(\mu_{A}^{q}+u_{A}^{q})A. \end{array}\right\} \tag{3.6}$$

where the symbol ${}_{a}^{c}D_{t}^{q}$ in (3.6) represented the Caputo fractional derivative of order q $(0 < q < 1)$. The Caputo fractional derivative of order q is defined in Podlubny (1999) as shown in equation (3.7):

$$ {}_{a}^{c}D_{t}^{q}f(t)=\frac{1}{\Gamma(n-q)}\int_{0}^{t}\frac{f^{n}(\xi)}{(t-\xi)^{q+1-n}}d\xi, \quad n-1<q<n\in\mathbb{N}, \tag{3.7}$$

where Γ represented the gamma function.

Remark 3.1

Note that, to avoid flaws regarding the time dimension, q was introduced in the model parameters (right-hand side) of system (3.6) so that the dimensions of these parameters became $(time)^{-q}$ which was in agreement with the left-hand side of the model.

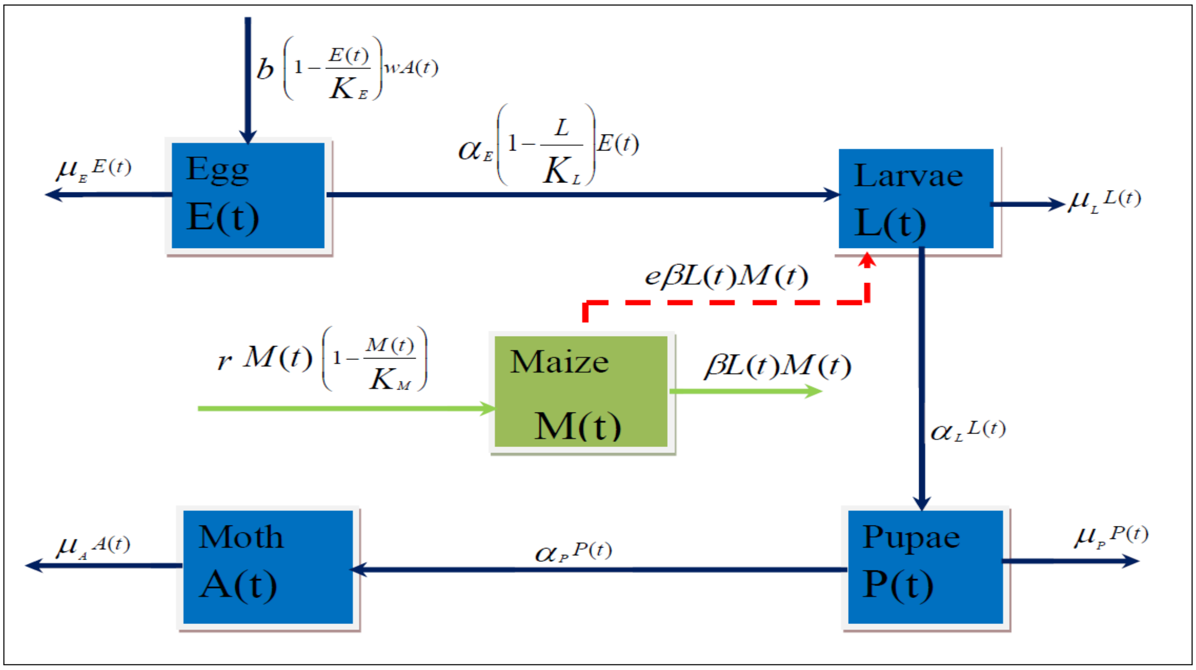


Figure 9: Model flow diagram for system (3.6) illustrating the dynamics of FAW in a field of maize biomass

3.2.1 Positivity and Boundedness of Model Solutions

Since model (3.6)) was formulated to evaluate the impact of memory effects and control on FAW-maize interaction and on final maize biomass, it was essential to investigate its biological and mathematical feasibility as follows:

Theorem 3.1

There exists a unique solution for the fractional-order model (3.6) in $(0,\infty)$. Moreover, the solution is non-negative for all $t>0$ and it remains in $\mathbb{R}^5_+$.

Proof. In proving this theorem, Firstly, it was demonstrated that $\mathbb{R}^5_+ = \{(M,E,L,P,A) \in \mathbb{R}^5_+ : M \geq 0, E \geq 0, L \geq 0, P \geq 0, A \geq 0\}$ is positively invariant. For that, it was demonstrated that on each hyper-plane bounding the non-negative orthant, the vector field pointed to $\mathbb{R}^5_+$. Therefore, the following cases 1-3 presented below were considered as follows:

Case 1. It was assumed that there exists a $t_* > t_0$ such that $M(t_*) = 0$, and $M(t) < 0$ for $t \in (t_*, t_1]$, where t_1 is sufficiently close to t_*. If $M(t_*) = 0$, then it follows that:

$$ {}_{t_0}^{c}D^q M(t_*) = 0. \tag{3.8}$$

Therefore ${}_{t_0}^{c}D^q M(t) \geq 0$ for all $t \in [t_*, t_1]$.

Case 2. It was assumed that there exists a $t_* > t_0$ such that $E(t_*) = 0$, and $E(t) < 0$ for $t \in (t_*, t_1]$, where t_1 is sufficiently close to t_*. If $E(t_*) = 0$, then it follows that:

$$ {}_{t_0}^{c}D^q E(t_*) = b^q WA > 0, \tag{3.9}$$

Therefore it follows that ${}_{t_0}^{c}D^{q}E(t) > 0$ for all $t \in [t_*, t_1]$.

Case 3. It was assume that there exists a $t_* > t_0$ such that $L(t_*) = 0$, and $L(t) < 0$ for $t \in (t_*, t_1]$, where t_1 is sufficiently close to t_*. If $L(t_*) = 0$, then it follows that:

$$
{}_{t_0}^{c}D^{q}L(t_*) = \alpha_E^q E > 0, \tag{3.10}
$$

From the last two equations of system (3.6), it can easily be verified that:

$$
\left.\begin{array}{l}
{}_{t_0}^{c}D^{q}P(t) = \alpha_L^q L > 0, \\
{}_{t_0}^{c}D^{q}A(t) = \alpha_P^q P > 0.
\end{array}\right\} \tag{3.11}
$$

From the above discussion, it was observed that each hyper-plane bounding the non-negative orthant, the vector field points to $\mathbb{R}_+^5$, that is, all the solutions of system (3.6) remains non-negative for all $t \geq 0$. □

Theorem 3.2

Let $\mathscr{X}(t) = (M(t), E(t), L(t), P(t), A(t))$ be the unique of the model (3.6) for $t \geq 0$. Then, the solution $\mathscr{X}(t)$ is bounded above, that is, $\mathscr{X}(t) \in \Omega$ where Ω denotes the feasible region and is given by:

$$
\Omega = \left\{ \Big(E(t), L(t), P(t), A(t)\Big) \left| \begin{array}{l} 0 \leq E(t) \leq K_E \\ 0 \leq L(t) \leq C_L \\ 0 \leq P(t) \leq C_P \\ 0 \leq A(t) \leq C_A \end{array} \right. \right\}, \tag{3.12}
$$

which its interior, denoted by $int(\Omega)$, is given by:

$$
int(\Omega) = \left\{ \Big(E(t), L(t), P(t), A(t)\Big) \left| \begin{array}{l} 0 < E(t) < K_E \\ 0 < L(t) < C_L \\ 0 < P(t) < C_P \\ 0 < A(t) < C_A \end{array} \right. \right\}. \tag{3.13}
$$

Proof. Secondly, it was demonstrated that the solutions of system (3.6) are bounded $\forall t \geq 0$. For biological relevance the least possible lower bound for each of the variables in system (3.6) is zero. Based on this, discussion was on determining the upper-bound for these variables. Moreover, it was easily established that for biological relevance, the following conditions should hold,$0 \leq M(t) \leq K_M$ and $0 \leq E(t) \leq K_E$. For instance,

$$
\begin{aligned}
{}_{a}^{c}D_t^{q}M(t) &= r^q M\left(1 - \frac{M}{K_M^q}\right) - \beta^q L M \\
&\leq r^q M\left(1 - \frac{M}{K_M^q}\right).
\end{aligned} \tag{3.14}
$$

Therefore, it follows that $\limsup_{t\infty} M(t) \leq K_M$. Based on these bounds, it was found that:

$$\begin{aligned}
{}^{c}_{a}D^{q}_{t}L(t) &= \alpha_E^q\left(1-\frac{L}{K_L^q}\right)E+\theta^q LM-(\alpha_L^q+u_L^q+\mu_L^q)L. \\
&\leq \alpha_E^q K_E^q-\left[\left(\alpha_L^q+u_L^q+\mu_L^q+\frac{\alpha_E^q K_E^q}{K_L^q}\right)-\theta^q K_M^q\right]L. \qquad (3.15)
\end{aligned}$$

Applying the Laplace transform leads to:

$$s^q\mathscr{L}[L(t)]-s^{q-1}L(0)\leq\frac{\alpha_E^q K_E^q}{s}-\left[\left(\alpha_L^q+u_L^q+\mu_L^q+\frac{\alpha_E^q K_E^q}{K_L^q}\right)-\theta^q K_M^q\right]\mathscr{L}[L(t)]. \qquad (3.16)$$

Grouping like terms one gets:

$$\begin{aligned}
\mathscr{L}(L(t)) &\leq \frac{\alpha_E^q K_E^q s^{-1}}{s^q+\left[\left(\alpha_L^q+u_L^q+\mu_L^q+\frac{\alpha_E^q K_E^q}{K_L^q}\right)-\theta^q K_M^q\right]} \\
&\quad+\frac{s^{q-1}L(0)}{s^q+\left[\left(\alpha_L^q+u_L^q+\mu_L^q+\frac{\alpha_E^q K_E^q}{K_L^q}\right)-\theta^q K_M^q\right]} \\
&= \frac{\alpha_E^q K_E^q s^{q-(1+q)}}{s^q+\left[\left(\alpha_L^q+u_L^q+\mu_L^q+\frac{\alpha_E^q K_E^q}{K_L^q}\right)-\theta^q K_M^q\right]} \\
&\quad+\frac{s^{q-1}L(0}{s^q+\left[\left(\alpha_L^q+u_L^q+\mu_L^q+\frac{\alpha_E^q K_E^q}{K_L^q}\right)-\theta^q K_M^q\right]}. \qquad (3.17)
\end{aligned}$$

Applying the inverse Laplace transform leads to:

$$\begin{aligned}
L(t) &\leq \mathscr{L}^{-1}\left\{\frac{\alpha_E^q K_E^q s^{q-(1+q)}}{s^q+\left[\left(\alpha_L^q+u_L^q+\mu_L^q+\frac{\alpha_E^q K_E^q}{K_L^q}\right)-\theta^q K_M^q\right]}\right\} \\
&\quad+\mathscr{L}^{-1}\left\{\frac{s^{q-1}L(0}{s^q+\left[\left(\alpha_L^q+u_L^q+\mu_L^q+\frac{\alpha_E^q K_E^q}{K_L^q}\right)-\theta^q K_M^q\right]}\right\} \\
&\leq \alpha_E^q K_E^q t^q E_{q,q+1}\left(-\left[\left(\alpha_L^q+u_L^q+\mu_L^q+\frac{\alpha_E^q K_E^q}{K_L^q}\right)-\theta^q K_M^q\right]t^q\right) \\
&\quad+L(0)E_{q,1}\left(-\left[\left(\alpha_L^q+u_L^q+\mu_L^q+\frac{\alpha_E^q K_E^q}{K_L^q}\right)-\theta^q K_M^q\right]t^q\right) \\
&\leq \max\left\{\frac{\alpha_E^q K_E^q}{\left[\left(\alpha_L^q+u_L^q+\mu_L^q+\frac{\alpha_E^q K_E^q}{K_L^q}\right)-\theta^q K_M^q\right]},L(0)\right\} \\
&\quad\times\left(\left[\left(\alpha_L^q+u_L^q+\mu_L^q+\frac{\alpha_E^q K_E^q}{K_L^q}\right)-\theta^q K_M^q\right]t^q E_{q,q+1}\right.
\end{aligned}$$

$$
\begin{aligned}
& \times\left(-\left[\left(\alpha_L^q + u_L^q + \mu_L^q + \frac{\alpha_E^q K_E^q}{K_L^q}\right) - \theta^q K_M^q\right] t^q\right) \\
& +E_{q,1}\left(-\left[\left(\alpha_L^q + u_L^q + \mu_L^q + \frac{\alpha_E^q K_E^q}{K_L^q}\right) - \theta^q K_M^q\right] t^q)\right) \\
= \; & \frac{C}{\Gamma(1)} := C_L,
\end{aligned} \tag{3.18}
$$

where $C_L = \max\left\{\left[\left(\alpha_L^q + u_L^q + \mu_L^q + \frac{\alpha_E^q K_E^q}{K_L^q}\right) - \theta^q K_M^q\right], L(0)\right\}$. Thus, $L(t)$ is bounded from above. From the equation for the pupa population, it follows that:

$$
{}_a^c D_t^q P(t) = \alpha_L^q L - (\mu_P^q + \alpha_p^q + u_P^q)P \le \alpha_L^q C_L - (\mu_P^q + \alpha_p^q + u_P^q)P. \tag{3.19}
$$

Applying the Laplace transform leads to:

$$
s^q \mathscr{L}[P(t)] - s^{q-1}P(0) \le \frac{\alpha_L^q C_L}{s} - (\mu_P^q + \alpha_p^q + u_P^q)\mathscr{L}[P(t)]. \tag{3.20}
$$

Combining the like terms one gets:

$$
\begin{aligned}
\mathscr{L}(P(t)) & \le \alpha_L^q C_L \frac{s^{-1}}{s^q + (\mu_P^q + \alpha_p^q + u_P^q)} + P(0)\frac{s^{q-1}}{s^q + (\mu_P^q + \alpha_p^q + u_P^q)} \\
& = \alpha_L^q C_L \frac{s^{q-(1+q)}}{s^q + (\mu_P^q + \alpha_p^q + u_P^q)} + P(0)\frac{s^{q-1}}{s^q + (\mu_P^q + \alpha_p^q + u_P^q)}.
\end{aligned} \tag{3.21}
$$

Applying the inverse Laplace transform leads to:

$$
\begin{aligned}
P(t) & \le \mathscr{L}^{-1}\left\{\alpha_L^q C_L \frac{s^{q-(1+q)}}{s^q + (\mu_P^q + \alpha_p^q + u_P^q)}\right\} + P(0)\mathscr{L}^{-1}\left\{\frac{s^{q-1}}{s^q + (\mu_P^q + \alpha_p^q + u_P^q)}\right\} \\
& \le \alpha_L^q C_L t^q E_{q,q+1}(-(\mu_P^q + \alpha_p^q + u_P^q)t^q) + P(0)E_{q,1}(-(\mu_P^q + \alpha_p^q + u_P^q)t^q) \\
& \le \frac{\alpha_L^q C_L}{(\alpha_P^q + \mu_p^q + u_P^q)}(\mu_P^q + \alpha_p^q + u_P^q)t^q E_{q,q+1}(-(\alpha_P^q + \mu_p^q + u_P^q)t^q) \\
& \quad +P(0)E_{q,1}(-(\alpha_P^q + \mu_p^q + u_P^q)t^q) \\
& \le \max\left\{\frac{\alpha_L^q C_L}{(\alpha_P^q + \mu_p^q + u_P^q)}, P(0)\right\}((\alpha_P^q + \mu_p^q + u_P^q)t^q E_{q,q+1}(-(\alpha_P^q + \mu_p^q + u_P^q)t^q) \\
& \quad +E_{\alpha,1}(-(\alpha_P^q + \mu_p^q + u_P^q)t^q)) \\
& = \frac{C}{\Gamma(1)} := C_P,
\end{aligned} \tag{3.22}
$$

where $C_P = \max\left\{\dfrac{\alpha_L^q C_L}{(\alpha_P^q+\mu_p^q)}, P(0)\right\}$. Thus, $P(t)$ is bounded from above. From the last equation of system (3.6) one gets

$$ {}_a^cD_t^q A(t) = \alpha_P^q P - (\mu_A^q + u_A^q)A \le \alpha_P^q C_P - (\mu_A^q + u_A^q)A. \tag{3.23}$$

By applying the Laplace transform it follows that:

$$ s^q \mathscr{L}[A(t)] - s^{q-1}A(0) \le \frac{\alpha_P^q C_P}{s} - (\mu_A^q + u_A^q)\mathscr{L}[A(t)]. \tag{3.24}$$

Grouping similar terms leads to:

$$\begin{aligned} \mathscr{L}(A(t)) &\le \alpha_P^q C_P \frac{s^{-1}}{s^q + (\mu_A^q + u_A^q)} + A(0)\frac{s^{q-1}}{s^q + (\mu_A^q + u_A^q)} \\ &= \alpha_P^q C_P \frac{s^{q-(1+q)}}{s^q + \mu_A^q} + A(0)\frac{s^{q-1}}{s^q + \mu_A^q}. \end{aligned} \tag{3.25}$$

Utilizing inverse Laplace transform one gets:

$$\begin{aligned} A(t) &\le \mathscr{L}^{-1}\left\{\alpha_P^q C_P \frac{s^{q-(1+q)}}{s^q + (\mu_A^q + u_A^q)}\right\} + A(0)\mathscr{L}^{-1}\left\{\frac{s^{q-1}}{s^q + (\mu_A^q + u_A^q)}\right\} \\ &\le \alpha_P^q C_P t^q E_{q,q+1}(-(\mu_A^q + u_A^q)t^q) + A(0)E_{q,1}(-(\mu_A^q + u_A^q)t^q) \\ &\le \frac{\alpha_P^q C_P}{(\alpha_A^q + u_A^q)}(\mu_A^q + u_A^q)t^q E_{q,q+1}(-(\mu_A^q + u_A^q)t^q) + A(0)E_{q,1}(-(\mu_A^q + u_A^q)t^q) \\ &\le \max\left\{\frac{\alpha_P^q C_P}{(\mu_A^q + u_A^q)}, A(0)\right\}((\mu_A^q + u_A^q)t^q \\ &\quad E_{q,q+1}(-(\mu_A^q + u_A^q)t^q) + E_{q,1}(-(\mu_A^q + u_A^q)t^q)) \\ &= \frac{C}{\Gamma(1)} = C_A, \end{aligned} \tag{3.26}$$

where $C_P = \max\left\{\dfrac{\alpha_P^q C_P}{(\mu_A^q + u_A^q)}, A(0)\right\}$. Thus, $A(t)$ is bounded from above. This completes the proof. □

3.2.2 Model Equilibria

By direct calculations it was observed that, system (3.6) has four equilibrium points and these are:

(i) Trivial equilibrium:

$$\mathscr{E}^1 = \{E^1 = 0, L^1 = 0, P^1 = 0, A^1 = 0, M^1 = 0\}. \tag{3.27}$$

(ii) First axial equilibrium point:

$$\mathscr{E}^2 = \{M^2 = K_M^q, E^2 = 0, L^2 = 0, P^2 = 0, A^2 = 0\}. \tag{3.28}$$

(iii) Second axial equilibrium point: $\mathscr{E}^3 = \{M^3, E^3, L^3, P^3, A^3\}$, where:

$$\mathscr{E}^3 : \left\{ \begin{array}{l} E^3 = \dfrac{K_E^q K_L^q m_1 m_2 m_3 m_4}{\alpha_E^q (b^q W K_L^q \alpha_L^q \alpha_P^q + K_E^q m_1 m_3 m_4)} \left(\dfrac{b^q W \alpha_E^q \alpha_L^q \alpha_P^q}{m_1 m_2 m_3 m_4} - 1 \right), \\[2ex] L^3 = \dfrac{K_E^q K_L^q m_1 m_2 m_3 m_4}{b^q W (K_E \alpha_E + K_L^q m_2) \alpha_L^q \alpha_P^q} \left(\dfrac{b^q W \alpha_E^q \alpha_P^q \alpha_L^q}{m_1 m_2 m_3 m_4} - 1 \right), \\[2ex] P^3 = \dfrac{K_E^q K_L^q m_1 m_2 m_3 m_4}{b^q W (K_E \alpha_E + K_L^q m_2) \alpha_P^q m_3} \left(\dfrac{b^q W \alpha_E^q \alpha_L^q \alpha_P^q}{m_1 m_2 m_3 m_4} - 1 \right), \\[2ex] A^3 = \dfrac{K_E^q K_L^q m_1 m_2 m_3 m_4}{b^q W (K_E \alpha_E + K_L^q m_2) m_3 m_4} \left(\dfrac{b^q W \alpha_E^q \alpha_L^q \alpha_P^q}{m_1 m_2 m_3 m_4} - 1 \right), \\[2ex] M^3 = 0, \end{array} \right\} \tag{3.29}$$

with

$$\begin{aligned} m_1 &= (\mu_E^q + \alpha_E^q + u_E^q), \qquad m_2 = (\mu_L^q + \alpha_L^q + u_L^q), \\ m_3 &= (\mu_P^q + \alpha_p^q + u_P^q), \qquad m_4 = (\mu_A^q + u_A^q). \end{aligned} \tag{3.30}$$

It was observed that this equilibrium point makes biological sense whenever

$$\frac{b^q W \alpha_E^q \alpha_L^q \alpha_P^q}{m_1 m_2 m_3 m_4} > 1. \tag{3.31}$$

Let

$$\begin{aligned} \mathscr{R}_0 &= b^q W \left(\frac{\alpha_E^q}{\mu_E^q + \alpha_E^q + u_E^q} \right) \left(\frac{\alpha_L^q}{\mu_L^q + \alpha_L^q + u_L^q} \right) \left(\frac{\alpha_P^q}{\mu_P^q + \alpha_P^q + u_P^q} \right) \left(\frac{1}{\mu_A^q + u_A^q} \right) \\ &= \frac{b^q W \alpha_E^q \alpha_L^q \alpha_P^q}{m_1 m_2 m_3 m_4}. \end{aligned} \tag{3.32}$$

Biologically, $\mathscr{R}_0$ is a threshold quantity that accounts for the persistence of the FAW population, thus when $\mathscr{R}_0 > 1$ the FAW population persists and they will be an attack on maize biomass and finally the population of maize plants extinct. Hence, $\mathscr{R}_0$ can precisely be defined as the average number of off-springs generated by an adult female FAW during its entire life span. Precisely, it was noted that, a proportion W of moth will each lay b eggs per day for an average duration of $\dfrac{1}{\mu_A^q + u_A^q}$, laid egg has a probability $\dfrac{\alpha_E^q}{\mu_E^q + \alpha_E^q + u_E^q}$ of surviving to become larva, larvae that emerge following oviposition has a probability of $\dfrac{\alpha_L^q}{\mu_L^q + \alpha_L^q + u_L^q}$ of surviving to become pupa which also has a probability $\dfrac{\alpha_P^q}{\mu_P^q + \alpha_P^q + u_P^q}$ of surviving to become moth.

(iv) Interior equilibrium point:

$$
\mathscr{E}^4 : \left\{
\begin{aligned}
E^4 &= \frac{-b^q K_E^q W \alpha_L^q \alpha_P^q h_2 + b^q K_E^q W \alpha_P^q \sqrt{h_2^2 - 4h_1 h_3}}{-b^q W \alpha_L^q \alpha_P^q h_2 + b^q W \alpha_P^q \sqrt{h_2^2 - 4h_1 h_3} - 2K_E^q h_1 m_1 m_4}, \\
L^4 &= \frac{-h_2 + \sqrt{h_2^2 - 4h_1 h_3}}{2h_1}, \\
P^4 &= \frac{-\alpha_L^q h_2 + \alpha_L^q \sqrt{h_2^2 - 4h_1 h_3}}{2h_1 m_3}, \\
A^4 &= \frac{-\alpha_P^q \alpha_L^q h_2 + \alpha_P^q \alpha_L^q \sqrt{h_2^2 - 4h_1 h_3}}{2h_1 m_3 m_4}, \\
M^* &= \frac{2h_1 r^q K_M^q - \beta_q K_M^q h_2 + \beta^q K_M^q \sqrt{h_2^2 - 4h_1 h_3}}{2h_1 r},
\end{aligned}
\right\} \tag{3.33}
$$

where

$$
\left.
\begin{aligned}
h_1 &= b^q K_L^q \theta^q K_M^q W \alpha_P^q \alpha_L^q, \\
h_2 &= -(b^q \theta^q r^q W K_L^q K_M^q \alpha_L^q \alpha_P^q + \theta^q \beta^q e^q K_E^q K_L^q K_M^q m_1 m_2 m_3 + \\
&\quad b^q r^q W K_E^q K_L^q \alpha_E^q \alpha_L^q \alpha_P^q - b^q r^q W K_L^q \alpha_L^q \alpha_P^q m_2), \\
h_3 &= -(\theta^q K_E K_L^q K_M^q m_1 m_2 m_3 + r^q K_E^q K_L^q m_1 m_2^2 m_3),
\end{aligned}
\right\} \tag{3.34}
$$

Basing on (3.33), h_1, h_2 and h_3, the $\Delta = (h_2^2 - 4h_1 h_3) > 0$ implying that the equilibrium point $\mathscr{E}^4$ has a unique feasible equilibrium.

3.2.3 Local Stability of Equilibrium Points

In this section, local stability behavior of the four equilibrium points computed earlier using the Jacobian matrix and the Routh-Hurwitz Criteria (1.3) were examined and discussed. The Jacobian matrix for model (3.6) is given by:

$$J(M,E,L,P,A) = \begin{bmatrix} 0 & -\beta^q M & 0 & 0 & \Theta_1 \\ \Theta_2 & 0 & 0 & \Theta_3 & 0 \\ \Theta_4 & -m_2 - \dfrac{\alpha_E^q}{K_L} & 0 & 0 & \theta^q L \\ 0 & \alpha_L^q & -m_3 & 0 & 0 \\ 0 & 0 & \alpha_P^q & -m_4 & 0 \end{bmatrix}, \tag{3.35}$$

with:

$$\begin{aligned} \Theta_1 &= r^q - \frac{2Mr^q}{K_M} - \beta^q L, \qquad \Theta_2 = -m_1 - \frac{b^q W A}{K_E}, \\ \Theta_3 &= b^q W\left(1 - \frac{E}{K_E}\right), \qquad \Theta_4 = \left(1 - \frac{L}{K_L^q}\right)\alpha_E^q. \end{aligned} \tag{3.36}$$

(i) Trivial equilibrium point:

Evaluating the Jacobian matrix (3.35) about $\mathscr{E}^1$ it follows that:

$$J(\mathscr{E}^1) = \begin{bmatrix} r^q & 0 & 0 & 0 & 0 \\ 0 & -m_1 & 0 & 0 & b^q W \\ 0 & \alpha_E^q & -m_2 & 0 & 0 \\ 0 & 0 & \alpha_L^q & -m_3 & 0 \\ 0 & 0 & 0 & \alpha_P^q & -m_4 \end{bmatrix}. \tag{3.37}$$

From the Jacobian matrix (3.37), it was observed that the trivial equilibrium point is locally stable if all eigenvalues $\lambda_i (i = 1,2,3,4)$ of the $J(\mathscr{E}^1)$ satisfy the following condition: $|\arg(\lambda_i)| > \frac{q\pi}{2}$ (Ahmed *et al.*, 2007). It was evident that one of the eigenvalues of (3.37) is $r^q > 0$. Thus, the other equilibrium points were obtained from the following characteristic equation:

$$\lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda + c_4 = 0, \tag{3.38}$$

with

$$\left.\begin{array}{rcl} c_1 & = & m_1+m_2+m_4, \\ c_2 & = & m_1m_2+(m_1+m_2)(m_3+m_4)+m_3m_4, \\ c_3 & = & m_1m_2(m_3+m_4)+m_3m_4(m_1+m_2), \\ c_4 & = & m_1m_2m_3m_4-b^qW\alpha_E^q\alpha_L^q\alpha_P^q \\ & = & m_1m_2m_3m_4(1-\mathscr{R}_0). \end{array}\right\} \qquad (3.39)$$

The Routh-Hurwitz criteria for local asymptotic stability of the equilibrium point $\mathscr{E}^1$ were as shown in equation (3.40)

$$\left.\begin{array}{rcl} \mathscr{H}_1 & = & c_1>0, \quad c_3>0, \qquad c_4>0, \\ \mathscr{H}_2 & = & c_1c_2c_3-c_3^2-c_1^2c_4>0. \end{array}\right\} \qquad (3.40)$$

Based on equation (3.38), it was observed that all the coefficients of the characteristic polynomial were positive whenever $\mathscr{R}_0<1$, implying that condition $\mathscr{H}_1$ holds for $\mathscr{R}_0<1$. Since it was also established that the trivial equilibrium point $\mathscr{E}^1$ had another eigenvalue r^q which is always positive, the positivity of condition $\mathscr{H}_2$ was not investigated, hence it was concluded that $\mathscr{E}^1$ is an unstable equilibrium point.

(ii) First axial equilibrium point $\mathscr{E}^2$:

Evaluating the Jacobian matrix (3.35) about $\mathscr{E}^2$ leads to:

$$J(\mathscr{E}^2)=\begin{bmatrix} -r^q & 0 & -\beta^qK_M^q & 0 & 0 \\ 0 & -m_1 & 0 & 0 & b^qW \\ 0 & \alpha_E^q & \theta^qK_M^q-m_2 & 0 & 0 \\ 0 & 0 & \alpha_L^q & -m_3 & 0 \\ 0 & 0 & 0 & \alpha_P^q & -m_4 \end{bmatrix}. \qquad (3.41)$$

From (3.41), it was observed that one of the eigenvalues is $-r^q<0$ and the other eigenvalues are roots of the characteristic equation (3.42) :

$$\lambda^4+b_1\lambda^3+b_2\lambda^2+b_3\lambda+b_4=0, \qquad (3.42)$$

with:

$$\left.\begin{array}{rcl} b_1 & = & m_1+m_2+m_4-\theta^qK_M^q, \\ b_2 & = & (m_1+m_2)(m_3+m_4)+m_1m_2+m_3m_4 \\ & & -\theta^qK_M^q(m_1+m_3+m_4), \\ b_3 & = & m_1(m_3m_4+m_2(m_3+m_4))+m_2m_3m_4 \\ & & -\theta^qK_M^q(m_1(m_3+m_4)+m_3m_4), \\ b_4 & = & m_1m_2m_3m_4((1-\mathscr{R}_0)-\theta^qK_M^q). \end{array}\right\} \qquad (3.43)$$

The Routh-Hurwitz criteria for local asymptotic stability of the equilibrium point $\mathscr{E}^2$ are as follows:

$$\left.\begin{aligned}\widehat{\mathscr{H}_1} &= b_1 > 0, \quad b_3 > 0, \qquad b_4 > 0, \\ \widehat{\mathscr{H}_2} &= b_1 b_2 b_3 - b_3^2 - b_1^2 b_4 > 0.\end{aligned}\right\} \tag{3.44}$$

If conditions specified in (3.44) hold then the equilibrium point $\mathscr{E}^2$ is locally asymptotically stable.

(iii) Second axial equilibrium point $\mathscr{E}^3$:

Evaluating the Jacobian matrix (3.35) about $\mathscr{E}^3$ one gets:

$$J(\mathscr{E}^3) = \begin{bmatrix} r^q - \beta^q L^3 & 0 & 0 & 0 & 0 \\ 0 & -\widehat{n_1} & 0 & 0 & \widehat{n_2} \\ \widehat{n_3} & \widehat{n_4} & -\widehat{n_5} & 0 & 0 \\ 0 & 0 & \alpha_L^q & -m_3 & 0 \\ 0 & 0 & 0 & \alpha_P^q & -m_4 \end{bmatrix}, \tag{3.45}$$

with

$$\begin{aligned}\widehat{n_1} &= -m_1 - \frac{b^q W A^3}{K_E}, \qquad \widehat{n_2} = b^q W\left(1 - \frac{E^3}{K_E}\right), \qquad \widehat{n_3} = \theta^q L \\ \widehat{n_4} &= \left(1 - \frac{L^3}{K_L^q}\right), \qquad \widehat{n_5} = m_2 + \frac{\alpha_E^q E}{K_L}.\end{aligned} \tag{3.46}$$

From (3.45) we can observe that $-r^q\left(\frac{\beta^q L^3}{r^q} - 1\right)$ is an eigenvalue and other eigenvalues can be determined from the following characteristic polynomial

$$\lambda^4 + d_1\lambda^3 + d_2\lambda^2 + d_3\lambda + d_4 = 0,$$

with

$$\begin{aligned} d_1 &= m_1 + m_3 + m_4 + \widehat{n_5}, \\ d_2 &= \widehat{n_1}\widehat{n_5} + m_3 m_4 + (m_3 + m_4)(\widehat{n_1} + \widehat{n_5}), \\ d_3 &= \widehat{n_1}\widehat{n_5}(m_3 + m_4) + m_3 m_4(\widehat{n_1} + \widehat{n_5}), \\ d_4 &= m_3 m_4 \widehat{n_1}\widehat{n_5} - \widehat{n_2}\widehat{n_4}\alpha_L^q \alpha_P^q. \end{aligned} \tag{3.47}$$

Ahmed *et al.* (2007) presented some Routh-Hurwitz stability conditions for fractional order system. One well known Routh-Hurwitz conditions is that an equilibrium point is locally stable if all eigenvalues of the community matrix satisfy the following condition $|\arg(\lambda_i)| > \frac{q\pi}{2}$. The Routh-Hurwitz criteria for local asymptotic stability of the equilibrium point $\mathscr{E}^3$ are

$$\left.\begin{aligned} \xi_1 &= d_1 > 0, \quad d_3 > 0, \qquad d_4 > 0, \\ \xi_2 &= d_1 d_2 d_3 - d_3^2 - d_1^2 d_4 > 0. \end{aligned}\right\} \tag{3.48}$$

Since the existence of the equilibrium point $\mathscr{E}^3$ is based on $\mathscr{R}_0 > 1$, (3.29), one can conclude that the equilibrium point $\mathscr{E}^3$ is locally asymptotically stable provided the conditions in (3.48) hold and (i) $r^q < \beta^q L^3$ (ii) $\mathscr{R}_0 > 1$.

(iv) Interior equilibrium point $\mathscr{E}^4$:

Evaluating the Jacobian matrix (3.35) about $\mathscr{E}^4$ one gets:

$$J(\mathscr{E}^4) = \begin{bmatrix} n_1 & 0 & -n_2 & 0 & 0 \\ 0 & -n_3 & 0 & 0 & n_4 \\ n_5 & n_6 & n_7 & 0 & 0 \\ 0 & 0 & \alpha_L^q & -m_3 & 0 \\ 0 & 0 & 0 & \alpha_P^q & -m_4 \end{bmatrix}, \tag{3.49}$$

with

$$\begin{aligned} n_1 &= r^q - \frac{2Mr^q}{K_M} - \beta^q L, \qquad n_2 = -\beta^q M, \qquad n_3 = -m_1 - \frac{b^q WA}{K_E} \\ n_4 &= b^q W \left(1 - \frac{E}{K_E}\right), \qquad n_5 = \theta^q L, \qquad n_6 = \left(1 - \frac{L}{K_L^q}\right) \alpha_E^q, \\ n_7 &= \theta^q M - m_2 - \frac{\alpha_E^q E}{K_L}. \end{aligned} \tag{3.50}$$

The characteristic equation of (3.49) is

$$\lambda^5 + z_1 \lambda^4 + z_2 \lambda^3 + z_3 \lambda^2 + z_4 \lambda + z_5 = 0, \tag{3.51}$$

where

$$\begin{aligned} z_1 &= m_3 + m_4 + n_3 - n_1 - n_7, \\ z_2 &= n_2 n_5 - n_1 n_3 + m_3(m_4 + n_3 - n_1 - n_7) + n_1 n_7 - n_3 n_7 \\ &\quad -m_4(n_1 - n_3 + n_7), \\ z_3 &= n_3(n_1 n_7 + n_2 n_5) + m_4(n_2 n_5 - n_3 n_7 + n_1(n_7 - n_3)) \\ &\quad -m_3(n_1(n_3 - n_7) + n_3 n_7 + m_4(n_1 - n_3 + n_7) - n_2 n_5), \\ z_4 &= n_3 m_4(n_2 n_5 + n_1 n_7) + m_3(n_3(n_2 n_5 + n_1 n_7) \\ &\quad +m_4(n_2 n_5 - n_3 n_7 + n_1(n_7 - n_3))) \\ &\quad -\alpha_L^q \alpha_P^q n_4 n_6, \\ z_5 &= \alpha_L^q \alpha_P^q n_1 n_4 n_6 + n_3 m_3 m_4(n_2 n_5 + n_1 n_7). \end{aligned} \tag{3.52}$$

The Routh-Hurwitz criteria that are necessary and sufficient for local asymptotic stability of the equilibrium point $\mathscr{E}^4$ are that the Hurwitz determinant H_i are all positive (1.3). For a fifth degree polynomial these criteria are

$$\left.\begin{aligned} H_1 &= z_1 > 0, \\ H_2 &= z_1 z_2 - z_3 > 0, \\ H_3 &= z_1 z_2 z_3 + z_1 z_5 - z_1^2 z_4 - z_3^2 > 0, \\ H_4 &= (z_3 z_4 - z_2 z_5)(z_1 z_2 - z_3) - (z_1 z_4 - z_5)^2 > 0, \\ H_5 &= c_5 H_4 > 0. \end{aligned}\right\} \tag{3.53}$$

This leads to the following result.

Theorem 3.3

The interior equilibrium point $\mathscr{E}^4$ is locally asymptotically stable if conditions in (3.53) hold, otherwise it is unstable.

3.2.4 Global Stability of Equilibrium Points

In this section, global stability of the equilibrium points determined earlier that is $\mathscr{E}^1$, $\mathscr{E}^2$, $\mathscr{E}^3$, and $\mathscr{E}^4$ were studied.

(i) Trivial equilibrium point $\mathscr{E}^1$

Let us consider the following Lyapunov function:

$$\begin{aligned} \mathscr{U}_1(M,E,L,P,A) &= M(t) + \left(\frac{m_4}{b^q W}\right) E(t) + \left(\frac{m_1 m_4}{b^q W \alpha_E^q}\right) L(t) \\ &\quad + \left(\frac{m_1 m_2 m_4}{b^q W \alpha_E^q \alpha_L^q}\right) P(t) + \left(\frac{m_1 m_2 m_3 m_4}{b^q W \alpha_E^q \alpha_L^q \alpha_P^q}\right) A(t). \end{aligned} \tag{3.54}$$

As one can observe, Lyapunov functional $\mathscr{U}_1(M,E,L,P,A)$ is defined, continuous and positive definite for all $M(t)$, $E(t)$, $L(t)$, $P(t)$ and $A(t)$. It is evident that $\mathscr{U}_1$ vanishes at $\mathscr{E}^1$. The fractional derivative of $\mathscr{U}(t)$ along the solutions of system (3.6) lead to:

$$\begin{aligned} {}^{c}_{t_0}D^q_t \mathscr{U}_1(t) &= {}^{c}_{t_0}D^q_t M(t) + \left(\frac{m_4}{b^q W}\right) {}^{c}_{t_0}D^q_t E(t) + \left(\frac{m_1 m_4}{b^q W \alpha_E^q}\right) {}^{c}_{t_0}D^q_t L(t) \\ &\quad + \left(\frac{m_1 m_2 m_4}{b^q W \alpha_E^q \alpha_L^q}\right) {}^{c}_{t_0}D^q_t P(t) + \left(\frac{m_1 m_2 m_3 m_4}{b^q W \alpha_E^q \alpha_L^q \alpha_P^q}\right) {}^{c}_{t_0}D^\alpha_t A(t) \\ &= r^q M(t)\left(1 - \frac{M(t)}{K_M^q}\right) - \beta^q L(t) M(t) \\ &\quad + \left(\frac{m_4}{b^q W}\right)\left(b^q\left(1 - \frac{E(t)}{K_E^q}\right) W A(t) - m_1 E(t)\right) \\ &\quad + \left(\frac{m_1 m_4}{b^q W \alpha_E^q}\right)\left(\alpha_E^q\left(1 - \frac{L(t)}{K_L^q}\right) E(t) + \theta^q L(t) M(t) - m_2 L(t)\right) \end{aligned}$$

$$\begin{aligned}
& +\left(\frac{m_1 m_2 m_4}{b^q W \alpha_E^q \alpha_L^q}\right)\left(\alpha_L^q L(t) - m_3 P(t)\right) \\
& +\left(\frac{m_1 m_2 m_3 m_4}{b^q W \alpha_E^q \alpha_L^q \alpha_P^q}\right)\left(\alpha_P^q P(t) - m_4 AV\right) \\
= \; & -m_4 \frac{E(t)A(t)}{K_E} - m_1 m_2 \frac{E(t)L(t)}{K_L} - \frac{m_1 m_2 m_3 m_4^2}{b^q W \alpha_E^q \alpha_L^q \alpha_P^q} \\
& \times \left(1 - \frac{b^q W \alpha_E^q \alpha_L^q \alpha_P^q}{m_1 m_2 m_3 m_4}\right) A(t) - \theta^q \left(\frac{b^q W \alpha_E^q \beta^q}{m_1 m_4 \theta^q} - 1\right) L(t)M(t) \\
& + r^q M(t)\left(1 - \frac{M(t)}{K_M^q}\right) \\
= \; & -m_4 \frac{E(t)A(t)}{K_E} - m_1 m_2 \frac{E(t)L(t)}{K_L} - \frac{m_4}{\mathscr{R}_0}(1-\mathscr{R}_0)A(t) \\
& -\theta^q \left(\frac{b^q W \alpha_E^q \beta^q}{m_1 m_4 \theta^q} - 1\right) L(t)M(t) + r^q M(t)\left(1 - \frac{M(t)}{K_M^q}\right). \qquad (3.55)
\end{aligned}$$

One can note that ${}_{t_0}^{c}D_t^q \mathscr{U}_1(t) = 0$ if $M(t) = K_M^q$, $\mathscr{R}_0 = 1$ and $m_1 m_4 \theta^q \leq b^q W \alpha_E^q \beta^q$. Thus ${}_{t_0}^{c}D_t^q \mathscr{U}_1(t)$ is negative definite if $M(t) = K_M^q$, $\mathscr{R}_0 \leq 1$ and $m_1 m_4 \theta^q \leq b^q W \alpha_E^q \beta^q$. The above discussion leads to the following theorem.

Theorem 3.4

The trivial equilibrium point $\mathscr{E}^1$ is globally asymptotically stable if $M(t) = K_M^q$, $\mathscr{R}_0 \leq 1$ and $m_1 m_4 \theta^q \leq b^q W \alpha_E^q \beta^q$ otherwise it is unstable.

(ii) First axial equilibrium point $\mathscr{E}^2$:

Define the function function:

$$\begin{aligned}
\mathscr{U}_2(M,E,L,P,A) \; = \; & M(t) - M^* - M^* \ln \frac{M(t)}{M^*} + \left(\frac{m_4}{b^q W}\right) E(t) \\
& + \left(\frac{m_1 m_4}{b^q W \alpha_E^q}\right) L(t) + \left(\frac{m_1 m_2 m_4}{b^q W \alpha_E^q \alpha_L^q}\right) P(t) \\
& + \left(\frac{m_1 m_2 m_3 m_4}{b^q W \alpha_E^q \alpha_L^q \alpha_P^q}\right) A(t). \qquad (3.56)
\end{aligned}$$

Evidently the function $\mathscr{U}_2(M,E,L,P,A)$ is defined, continuous and positive definite for all $M(t)$, $E(t)$, $L(t)$, $P(t)$ and $A(t)$. Furthermore, it $\mathscr{U}_2$ vanishes at $\mathscr{E}^2$. Hence, the fractional derivative of $\mathscr{U}_2(t)$ along the solutions of system

$${}_{t_0}^{c}D_t^q \mathscr{U}_2(t) \; \leq \; -r^q M^* \left(1 - \frac{M(t)}{K_M^q}\right)\left(1 - \frac{M(t)}{M^*}\right) - m_4 \frac{E(t)A(t)}{K_E}$$

$$
\begin{aligned}
& -m_1m_2\frac{E(t)L(t)}{K_L} - \frac{m_1m_2m_3m_4^2}{b^qW\alpha_E^q\alpha_L^q\alpha_P^q}\left(1-\frac{b^qW\alpha_E^q\alpha_L^q\alpha_P^q}{m_1m_2m_3m_4}\right)A(t) \\
& -\frac{\beta^q b^q W\alpha_E^q + m_1m_4\theta^q}{b^qW\alpha_E^q}\left(1-\frac{\beta^q b^q W\alpha_E^q}{m_1m_4\theta^q+\beta^q b^q W\alpha_E^q}\frac{M^*}{M(t)}\right) \\
= \; & -r^qM^*\left(1-\frac{M(t)}{K_M^q}\right)\left(1-\frac{M(t)}{M^*}\right) - m_4\frac{E(t)A(t)}{K_E} - m_1m_2\frac{E(t)L(t)}{K_L} \\
& -\frac{\beta^q b^q W\alpha_E^q + m_1m_4\theta^q}{b^qW\alpha_E^q}\left(1-\frac{\beta^q b^q W\alpha_E^q M^*}{(m_1m_4\theta^q+\beta^q b^q W\alpha_E^q)M(t)}\right) \\
& -\frac{m_1m_2m_3m_4^2}{b^qW\alpha_E^q\alpha_L^q\alpha_P^q}(1-\mathscr{R}_0)A(t). \qquad (3.57)
\end{aligned}
$$

Therefore ${}_{t_0}^{c}D_t^q\mathscr{U}_2(t)$ is negative definite if the following conditions hold (i) $\mathscr{R}_0 \leq 1$, (ii) $M < M^*$ (iii) $\beta^q b^q W\alpha_E^q M^* \leq (m_1m_4\theta^q + \beta^q b^q W\alpha_E^q)M(t)$. This leads to the following theorem.

Theorem 3.5

The trivial equilibrium point $\mathscr{E}^2$ is globally asymptotically stable if the following conditions hold (i) $\mathscr{R}_0 \leq 1$, (ii) $M < M^*$ (iii) $\beta^q b^q W\alpha_E^q M^* \leq (m_1m_4\theta^q + \beta^q b^q W\alpha_E^q)M(t)$, otherwise it is unstable.

(iii) Global stability of equilibrium points $\mathscr{E}^3$ and $\mathscr{E}^4$:

The following Lyapunov function was used to investigate the global stability of the equilibrium points $\mathscr{E}^3$ and $\mathscr{E}^4$:

$$
\begin{aligned}
\mathscr{U}_3(t) = \; & a_0\left[M(t)-M^*-M^*\ln\left(\frac{M(t)}{M^*}\right)\right] + a_1\left[E(t)-E^*-E^*\ln\left(\frac{E(t)}{E^*}\right)\right] \\
& + a_2\left[L(t)-L^*-L^*\ln\left(\frac{L(t)}{L^*}\right)\right] + a_3\left[P(t)-P^*-P^*\ln\left(\frac{P(t)}{P^*}\right)\right] \\
& + a_4\left[A(t)-A^*-A^*\ln\left(\frac{A(t)}{A^*}\right)\right], \qquad (3.58)
\end{aligned}
$$

where a_1, a_2, a_3 and a_4 are positive constants to be determined. Let $g_0(M) = r^q\left(1-\frac{M}{K_M^q}\right)$, $g_1(E,A) = b^q\left(1-\frac{E}{K_E^q}\right)WA$ and $g_2(E,L) = \alpha_E^q\left(1-\frac{L}{K_L^q}\right)E$. Recall that at this equilibrium the following identities hold:

$$
\left.
\begin{aligned}
& g_0(M) = \beta^q L^* M^*, \qquad g_1(E^*,A^*) = m_1E^*, \\
& g_2(E^*,L^*) + \theta^q L^* M^* = m_2L^*, \\
& \alpha_L^q L^* - m_3P^*, \qquad \alpha_P^q P^* = m_4A^*.
\end{aligned}
\right\} \qquad (3.59)
$$

Setting

$$\left.\begin{array}{ll} a_1 = g_2(E^*,L^*), & a_3 = \dfrac{g_1(E^*,A^*)g_2(E^*,L^*)}{\alpha_L^q L^*}, \\[2ex] a_2 = g_1(E^*,A^*), & a_4 = \dfrac{g_1(E^*,A^*)g_2(E^*,L^*)}{\alpha_P^q P^*}. \end{array}\right\} \tag{3.60}$$

It follows from Lemma 1.1 that

$$\begin{aligned} {}_{t_0}^{c}D_t^{\alpha}\mathscr{U}_2(t) \;\leq\;& g_0(M^*)\left(1-\frac{M^*}{M}\right)\left(\frac{g(M)}{g(M^*)}-\frac{LM}{L^*M^*}\right) \\ & +\theta^q g_1(E^*,A^*)\left(1-\frac{L}{L^*}\right)\left(1-\frac{M}{M^*}\right) \\ & +g_1(E^*,A^*)g_2(E^*,L^*)\left(1-\frac{E}{E^*}-\frac{E^*g_1(E,A)}{Eg_1(E^*,A^*)}+\frac{g_1(E,A)}{g_1(E^*,A^*)}\right) \\ & +g_1(E^*,A^*)g_2(E^*,L^*)\left(3-\frac{A}{A^*}-\frac{A^*P}{AP^*}-\frac{P^*L}{PL^*}\right. \\ & \left.-\frac{L^*g_2(E,L)}{Lg_2(E^*,L^*)}+\frac{g_2(E,L)}{g_2(E^*,L^*)}\right). \end{aligned} \tag{3.61}$$

Let $\Phi(x) = 1-x+\ln x$, for $x > 0$. It follows that $\Phi(x) \leq 0$, with the equality satisfied if and only if $x = 1$. Using this relation leads to

$$\begin{aligned} & 1-\frac{E}{E^*}-\frac{E^*g_1(E,A)}{Eg_1(E^*,A^*)}+\frac{g_1(E,A)}{g_1(E^*,A^*)} \\ & \quad = \Phi\left(\frac{E^*g_1(E,A)}{Eg_1(E^*,A^*)}\right)-\frac{E}{E^*}+\frac{g_1(E,A)}{g_1(E^*,A^*)}-\ln\left(\frac{E^*g_1(E,A)}{Eg_1(E^*,A^*)}\right) \\ & \quad \leq \frac{g_1(E,A)}{g_1(E^*,A^*)}-\ln\left(\frac{g_1(E,A)}{g_1(E^*,A^*)}\right)-\frac{E}{E^*}+\ln\left(\frac{E}{E^*}\right). \end{aligned} \tag{3.62}$$

Similarly, one can write

$$\begin{aligned} 3 \;-\;& \frac{A}{A^*}-\frac{A^*P}{AP^*}-\frac{P^*L}{PL^*}-\frac{L^*g_2(E,L)}{Lg_2(E^*,L^*)}+\frac{g_2(E,L)}{g_2(E^*,L^*)} \\ =\;& \Phi\left(\frac{A^*P}{AP^*}\right)+\Phi\left(\frac{A^*P}{AP^*}\right)+\Phi\left(\frac{L^*g_2(E,L)}{Lg_2(E^*,L^*)}\right)-\frac{A}{A^*} \\ & +\frac{g_2(E,L)}{g_2(E^*,L^*)}-\ln\left(\frac{A^*g_2(E,L)}{Ag_2(E^*,L^*)}\right) \\ \leq\;& \frac{g_2(E,L)}{g_2(E^*,L^*)}-\ln\left(\frac{g_2(E,L)}{g_2(E^*,L^*)}\right)-\frac{A}{A^*}+\ln\left(\frac{A}{A^*}\right). \end{aligned} \tag{3.63}$$

Therefore ${}_{t_0}^{c}D_t^q\mathscr{U}_3(t)$ is negative definite if the following conditions hold

(i) $\left(1-\frac{M^*}{M}\right)\left(\frac{g(M)}{g(M^*)}-\frac{LM}{L^*M^*}\right)\leq 0$

(ii) $\left(1-\frac{L}{L^*}\right)\left(1-\frac{M}{M^*}\right)\leq 0.$

Therefore the following theorem holds.

Theorem 3.6

The equilibrium point(s) $\mathscr{E}^3$ and $\mathscr{E}^4$ is globally asymptotically stable if the following conditions hold

(i) $\left(1-\frac{M^*}{M}\right)\left(\frac{g(M)}{g(M^*)}-\frac{LM}{L^*M^*}\right)\leq 0$

(ii) $\left(1-\frac{L}{L^*}\right)\left(1-\frac{M}{M^*}\right)\leq 0.$

otherwise it is unstable.

3.3 FAW-Maize Interaction Model with Seasonality

As highlighted earlier in this chapter, the influence of seasonal variations on FAW dynamics is well documented. However, despite all the documentation, mathematical models meant to evaluate the effects of seasonal variations on FAW-maize interaction are still lacking. To that end, the second model proposed in this study was meant to investigate the effects of seasonality on maize biomass dynamics and on final maize biomass. Biologically, maize seed planted at the beginning of the season at time $t=0$ germinates in $0-7$ days (Du Plessis *et al.*, 2020).

Depending on the variety of maize seed planted, harvest of this crop which occurs at the end of the season (90 - 164) days is influenced by the weather variations within a season. Since maize growth is affected by weather condition fluctuation, the growth rates of their parts such as leaves, cobs, kernel, and stems which in fact called biomass according to Chowdhury *et al.* (2019) and Battude *et al.* (2016) are weather dependent. Motivated by recent mathematical models for plant-pest interactions (Anguelov *et al.*, 2017; Faithpraise *et al.*, 2015; Páezchávez *et al.*, 2017; Hui & Zhu, 2006; Liang *et al.*, 2012; Kang *et al.*, 2013; Tang *et al.*, 2010; Chowdhury *et al.*, 2019; Rafikov *et al.*, 2008), in this study a non-autonomous model for FAW infestations in a field of maize biomass under assumption that weather condition fluctuation in a season have an impact on maize biomass and FAW dynamics.

The developed model subdivides the FAW population of life cycle into subclasses as: eggs population $E(t)$, larvae population $L(t)$, pupae population $P(t)$ and adult population which are also known as moth $A(t)$. On the other hand, we let the variable $M(t)$ denote the population density of maize biomass which grow logistically in the absence of the larvae with carrying capacity K_M and a net seasonal growth rate $r(t)$. It was assumed that larvae with a mortality rate $\mu_L(t)$ is the only threat to maize biomass throughout its growth period and the adult moth takes over in the reproduction process. The life cycle of FAW starts when eggs are laid in masses on maize biomass, mostly underside of these biomass (Matova *et al.*, 2020). It was also assumed that production of FAW egg is a logistic growth $b(t)\left[1-\frac{E(t)}{K_E(t)}\right]wA(t)$ with $b(t)$ representing the average number of eggs laid by a proportion w of moth, $A(t)$ which are females.

Since the growth of maize plants depends on seasonal variations, it suffices to assume that the egg carrying capacity $K_E(t)$, egg hatching rate $\alpha_E(t)$ and egg mortality rate $\mu_E(t)$ are season-dependent. Furthermore, FAW larvae generally emerge simultaneously three to five days following oviposition. Although, the FAW has six larval instar stages, we have considered this as single group called larvae in order to reduce complexity of the model. Since the population size of maize biomass is finite and independent of weather fluctuation and because the rate at which FAW larvae consumes food decreases, a Holling type II functional response also known as the saturating functional response is included in the equation capturing the dynamics of maize density and larvae population with half saturation constant $a(t)$.

In particular, when FAW larvae feed on maize biomass, the FAW larvae with an average duration of $\frac{1}{\alpha_L(t)}$ in the larval stage convert maize biomass into larvae's biomass at the rate $e(t)$. Finally, pupation of the FAW normally occurs in the soil, at a depth of 2-8cm (Capinera, 2002). It was assume that, duration of the pupal stage with natural mortality rate $\mu_P(t)$ is denoted by $\frac{1}{\alpha_P(t)}$ which after 8 days in the soil escapes as adult moth and start the cycle again.

The formulation of this model (3.64) was also supported by the following assumptions:

(i) Once the FAW moths migrate into the field, no migration out of the field occurs before the harvest

(ii) In a natural environment there are weather condition fluctuations which have an impact on maize biomass and FAW dynamics.

(iii) Population dynamics of FAW egg in the absence of any predators grow logistically.

(iv) The only source of food for the larvae is maize biomass so that in its absence, larvae becomes extinct.

(v) Maize biomass carrying capacity K_M is not affected by weather condition fluctuation.

A summary of the definitions of model state variables and parameters are given in Tables 1 and 2. Therefore, the proposed model was summarized by the system of non-linear ordinary differential equations provided in (3.64) and the model flow diagram is in Figure 10:

$$\left.\begin{aligned}
\frac{dM(t)}{dt} &= r(t)M(t)\left[1-\frac{M(t)}{K_M}\right]-\frac{\beta(t)M(t)}{a(t)+M(t)}L(t),\\
\frac{dE(t)}{dt} &= b(t)wA(t)\left[1-\frac{E(t)}{K_E(t)}\right]-[\alpha_E(t)+\mu_E(t)]E(t),\\
\frac{dL(t)}{dt} &= \alpha_E(t)E(t)+\frac{e\beta(t)M(t)}{a(t)+M(t)}L(t)-[\alpha_L(t)+\mu_L(t)]L(t)\\
&\quad -\theta(t)L^2(t),\\
\frac{dP(t)}{dt} &= \alpha_L(t)L(t)-[\alpha_p(t)+\mu_P(t)]P(t),\\
\frac{dA(t)}{dt} &= \alpha_P(t)P(t)-\mu_A(t)A(t).
\end{aligned}\right\} \qquad (3.64)$$

where $\mu_A(t)$ represents natural mortality rate of the moth (adult moth) and $\theta(t)$ is the density-dependent death rate of larvae population. Thus, $-\theta(t)L^2(t)$ reflects of predation, intra-specific and interspecific competition that is known to exist on FAW larvae population. Precisely, prior studies suggest that when food is limited, the older FAW larvae exhibit a cannibalistic behavior on the smaller larvae (Assefa & Ayalew, 2019; Chapman *et al.*, 2000). All model parameters that are functions of time depend on seasonal variations. For biological significance, it was assumed that all these parameters are continuous and bounded functions defined on $\mathbb{R}_+$.

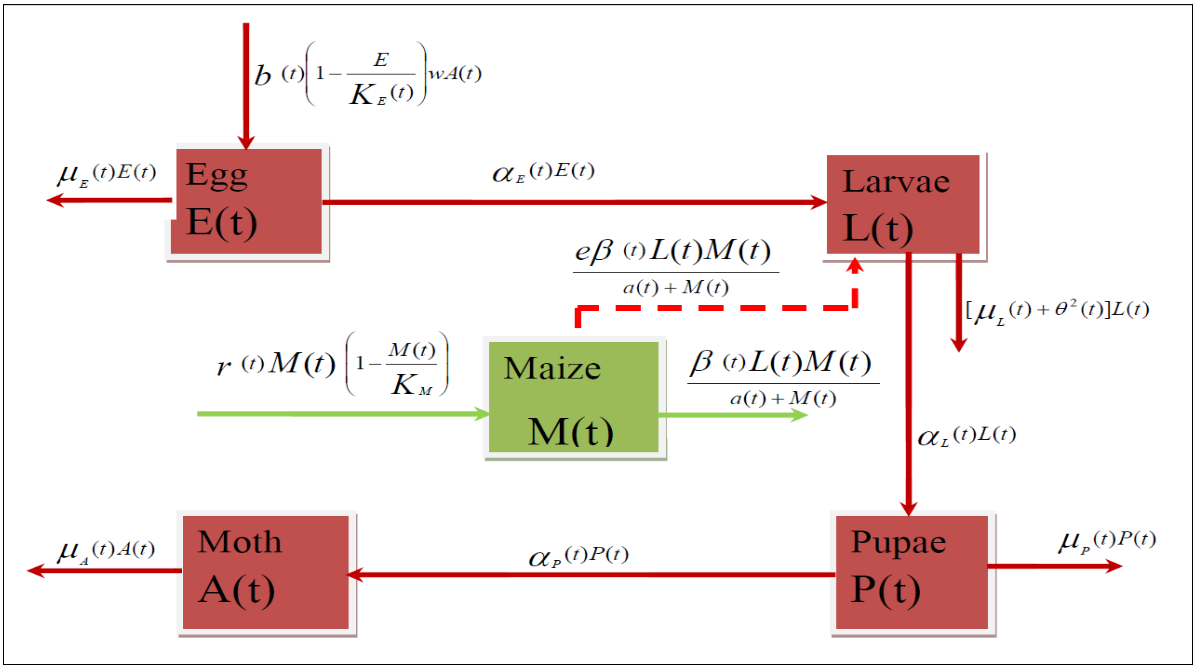


Figure 10: Schematic representation of the model (3.64); continuous arrows indicate either inflow or outflow transition between compartments

Comprehensive mathematical expressions to model parameters that are meant to capture seasonal fluctuations are given in (3.65):

$$\left.\begin{array}{lcll}
r(t) & = & r_0[1+r_1\cos(2\pi t\omega^{-1})], & b(t)=b_0[1+b_1\cos(2\pi t\omega^{-1})],\\
\beta(t) & = & \beta_0[1+\beta_1\cos(2\pi t\omega^{-1})], & K_E(t)=K_{E_0}[1+K_{E_1}\cos(2\pi t\omega^{-1})],\\
a(t) & = & a_0[1+a_1\cos(2\pi t\omega^{-1})], & \alpha_E(t)=\alpha_{E_0}[1+\alpha_{E_1}\cos(2\pi t\omega^{-1})],\\
\mu_E(t) & = & \mu_{E_0}[1+\mu_{E_1}\cos(2\pi t\omega^{-1})], & \alpha_L(t)=\alpha_{L_0}[1+\alpha_{L_1}\cos(2\pi t\omega^{-1})],\\
\mu_L(t) & = & \mu_{L_0}[1+\mu_{L_1}\cos(2\pi t\omega^{-1})], & \theta(t)=\theta_0[1+\theta_1\cos(2\pi t\omega^{-1})],\\
\alpha_P(t) & = & \alpha_{P_0}[1+\alpha_{P_1}\cos(2\pi t\omega^{-1})], & \mu_A(t)=\mu_{A_0}[1+\mu_{A_1}\cos(2\pi t\omega^{-1})],\\
\mu_P(t) & = & \mu_{P_0}[1+\mu_{P_1}\cos(2\pi t\omega^{-1})], & e(t)=e_0[1+e_1\cos(2\pi t\omega^{-1})],
\end{array}\right\} \quad (3.65)$$

where $\omega > 0$ denote the period. Further, r_0, β_0, K_{E0}, a_0, b_0, α_{E_0}, μ_{E_0}, α_{L_0}, μ_{L0}, θ_0, α_{P_0} and μ_{A_0} are the baseline values or the time averages of $r(t)$, $\beta(t)$, $K_E(t)$, $a(t)$, $b(t)$, $\alpha_E(t)$, $\mu_E(t)$, $\alpha_L(t)$, $\mu_L(t)$, $\theta(t)$, $\alpha_P(t)$, $\mu_A(t)$, respectively, and r_1, K_{M_1}, β_1, K_{E_1}, a_1, b_1, α_{E_1}, μ_{E_1}, e_1, α_{L_1}, μ_{L_1}, θ_1, α_{P_1} and μ_{A_1} denote the magnitude of seasonal fluctuations. Note that $0 < r_1 < 1$, $0 < \beta_1 < 1$, $0 < K_{E_1} < 1$, $0 < a_1 < 1$, $0 < b_1 < 1$, $0 < \alpha_{E_1} < 1$, $0 < \mu_{E_1} < 1$, $0 < \alpha_{L_1} < 1$, $0 < \mu_{L_1} < 1$, $0 < \theta_1 < 1$, $0 < \alpha_{P_1} < 1$ and $0 < \mu_{A_1} < 1$. From (3.65) one can observe that all model parameters that account for seasonal fluctuations are periodic (with period $\omega > 0$ days), continuous and bounded below and above. Since $\left|\cos(2\pi t\omega^{-1})\right| \leq 1$ and $0 < r_1 < 1$, it follows that:

$$r_0(1-r_1) \leq r(t) \leq r_0(1+r_1). \tag{3.66}$$

From (3.66), it follows that $r(t)$ is bounded below and above. By following the same approach one can easily verify that all the other periodic model parameters are bounded below and above.

3.3.1 Positivity, Boundedness and Permanence of Model Solutions

In this section, dynamical behavior of the FAW-maize interaction model (3.64) with seasonality was studied; that is, investigation of positive invariance, boundedness, permanence, non-persistence and global stability. This was done with the aid of the definition (3.5) and (3.6):

Definition 3.5

The solution set of system (3.64) is said to be ultimately bounded if there exists $\mathscr{M} > 0$, such that for every solution $(M(t), E(t), L(t), P(t), A(t))$ of (3.64), there exists $T > 0$, such that $\|(M(t), E(t), L(t), P(t), A(t))\| \leq \mathscr{M}$ for all $t > T$, where $\mathscr{M}$ is independent of a particular solution while T may depend on the solution.

Definition 3.6

System (3.64) is said to be permanent if there exist positive constants δ, Δ with $0 < \delta < \Delta$ such that:

$$\begin{aligned}
&\min\left\{\liminf_{t\to+\infty} M(t),\quad \liminf_{t\to+\infty} E(t),\quad \liminf_{t\to+\infty} L(t),\quad \liminf_{t\to+\infty} P(t),\quad \liminf_{t\to+\infty} A(t)\right\} \geq \delta,\\
&\max\left\{\limsup_{t\to+\infty} M(t),\quad \limsup_{t\to+\infty} E(t),\quad \limsup_{t\to+\infty} L(t),\quad \limsup_{t\to+\infty} P(t),\quad \limsup_{t\to+\infty} A(t)\right\} \leq \Delta,
\end{aligned}$$

hold for all solutions of (3.64) with positive initial values. System (3.64) is said to be non-persistent if there is a positive solution $(M(t), E(t), L(t), P(t), A(t))$ of system (3.64) satisfying:

$$\min\left\{\liminf_{t\to+\infty} M(t),\quad \liminf_{t\to+\infty} E(t),\quad \liminf_{t\to+\infty} L(t),\quad \liminf_{t\to+\infty} P(t),\quad \liminf_{t\to+\infty} A(t)\right\} = 0. \tag{3.67}$$

For $u \in C(\mathbb{R})$, define $\Phi(u(s)) = \dfrac{u(s)}{a(s)+u(s)}$. As one can observe, $\Phi(u)$ is a monotonic increasing function. Define:

$$\left.\begin{aligned}
&\mathscr{M}_1 = \sup_{t\in\mathbb{R}}\{K_M\}, \qquad m_1 = \inf_{t\in\mathbb{R}}\left\{\frac{K_M}{r(t)}[r(t)-\beta(t)\mathscr{M}_3]\right\},\\
&\mathscr{M}_2 = \sup_{t\in\mathbb{R}}\left\{\frac{b(t)w\mathscr{M}_5}{b(t)w\mathscr{M}_5+\alpha_E(t)+\mu_E(t)}\right\},\\
&m_2 = \sup_{t\in\mathbb{R}}\left\{\frac{b(t)wm_5}{b(t)wm_5+\alpha_E(t)+\mu_E(t)}\right\},\\
&\mathscr{M}_3 = \sup_{t\in\mathbb{R}}\left\{\alpha_E(t)\mathscr{M}_2+\frac{e\beta(t)\Phi(\mathscr{M}_1)(t)-(\mu_L(t)+\alpha_L(t))}{\theta(t)}\right\},\\
&\mathscr{M}_4 = \sup_{t\in\mathbb{R}}\left\{\frac{\alpha_L(t)\mathscr{M}_3}{\mu_P(t)+\alpha_P(t)}\right\},\\
&\mathscr{M}_5 = \sup_{t\in\mathbb{R}}\left\{\frac{\alpha_P(t)\mathscr{M}_4}{\mu_A(t)}\right\},\\
&m_3 = \inf_{t\in\mathbb{R}}\left\{\alpha_E(t)m_2+\frac{e\beta(t)\Phi(m_1)(t)-(\mu_L(t)+\alpha_L(t))}{\theta(t)}\right\},\\
&m_4 = \inf_{t\in\mathbb{R}}\left\{\frac{\alpha_L(t)m_3}{\mu_P(t)}\right\}, \qquad m_5 = \inf_{t\in\mathbb{R}}\left\{\frac{\alpha_P(t)m_4}{\mu_A(t)}\right\}.
\end{aligned}\right\} \tag{3.68}$$

One can now define Condition (H1) as follows:

$$(H1):\left\{\begin{aligned}
&\inf_{t\in\mathbb{R}}\{[r(t)-\beta(t)\mathscr{M}_3]\}>0,\\
&\inf_{t\in\mathbb{R}}\left\{\alpha_E(t)\mathscr{M}_2+\frac{e\beta(t)\Phi(\mathscr{M}_1)(t)-(\mu_L(t)+\alpha_L(t))}{\theta(t)}\right\}>0,\\
&\inf_{t\in\mathbb{R}}\left\{\alpha_E(t)m_2+\frac{e\beta(t)\Phi(m_1)(t)-(\mu_L(t)+\alpha_L(t))}{\theta(t)}\right\}>0.
\end{aligned}\right\} \tag{3.69}$$

Under Condition (H1), the set:

$$\Omega = \left\{ \Big(M(t),E(t),L(t),P(t),A(t)\Big) \;\middle|\; \begin{aligned}
&m_1 \le M(t) \le \mathscr{M}_1,\\
&m_2 \le E(t) \le \mathscr{M}_2,\\
&m_3 \le L(t) \le \mathscr{M}_3,\\
&m_4 \le P(t) \le \mathscr{M}_4,\\
&m_5 \le A(t) \le \mathscr{M}_5
\end{aligned} \right\}. \tag{3.70}$$

was defined which lead us to theorem (3.7):

Theorem 3.7

(a) If condition (H1) holds, then the set Ω is positively-invariant and permanent for system (3.64), and the set Ω_δ defined by:

$$\Omega = \left\{ \Big(M(t), E(t), L(t), P(t), A(t) \Big) \left| \begin{array}{l} m_1^\delta \leq M(t) \leq \mathscr{M}_1^\delta, \\ m_2^\delta \leq E(t) \leq \mathscr{M}_2^\delta, \\ m_3^\delta \leq L(t) \leq \mathscr{M}_3^\delta, \\ m_4^\delta \leq P(t) \leq \mathscr{M}_4^\delta, \\ m_5^\delta \leq A(t) \leq \mathscr{M}_5^\delta, \end{array} \right. \right\}. \tag{3.71}$$

is ultimately bounded region of (3.64), where $\delta > 0$ is sufficiently small such that $m_i^\delta > 0 (i = 1, 2, 3, 4, 5)$ and (H1) holds when $\mathscr{M}_i$ and m_i are replaced by $\mathscr{M}_i^\delta$ and m_i^δ, respectively. Further, it was defined that $\mathscr{M}_i^\delta = \mathscr{M}_i + \delta$, $m_i^\delta = m_i - \delta$. **(b)** If Condition (H1) holds, then system (3.64) has at least one positive $\omega-$ periodic solution $(M^*(t), E^*(t), L^*(t), P^*(t), A^*(t)) \in \Omega$.

Biological implications of Theorem 3.7: Theorem 3.7 implies that model system (3.64) is biologically well-poised, that is., the population of species under consideration are non-negative and bounded.

Proof. To prove Theorem 3.7 **(a)**, the following lemma given by Bai, Yu, Fan and Kang (2020) was used to demonstrate the positivity, boundedness and permanence of a non-autonomous predator-prey system with a generalist predator.

Lemma 3.4

Let $p(t)$ and $q(t)$ be bounded continuous functions defined on $\mathbb{R}$ with $\inf_{t\in\mathbb{R}} p(t) > 0$ and $\inf_{t\in\mathbb{R}} q(t) > 0$. If the positive function $u(t)$ satisfies:

$$u'(t) \leq p(t)u(t) - q(t)u^2(t), \qquad t \in [t_0, +\infty), \tag{3.72}$$

then $\limsup_{t\to+\infty} u(t) \leq \sup_{t\in\mathbb{R}} \frac{p(t)}{q(t)}$. Moreover, $u(t) \leq \sup_{t\in\mathbb{R}} \frac{p(t)}{q(t)}$ for all $t \in [t_0, +\infty)$ if $0 < u(t_0) \leq \sup_{t\in\mathbb{R}} \frac{p(t)}{q(t)}$. While if the positive function $u(t)$ satisfies:

$$u'(t) \geq p(t)u(t) - q(t)u^2(t), \qquad t \in [t_0, +\infty), \tag{3.73}$$

then $\liminf_{t\to+\infty} u(t) \geq \inf_{t\in\mathbb{R}} \frac{p(t)}{q(t)}$. Moreover, $u(t) \geq \inf_{t\in\mathbb{R}} \frac{p(t)}{q(t)}$ for all $t \in [t_0, +\infty)$ if $0 < u(t_0) \geq \inf_{t\in\mathbb{R}} \frac{p(t)}{q(t)}$.

In what follows, the proof of Theorem 3.7 was presented as follows.

From system (3.64), one can obtain the following expressions:

$$\left.\begin{aligned}
M(t) &= M(t_0)\exp\left\{\int_{t_0}^{t}\left[r(s)M(s)\left[1-\frac{M(s)}{K_M}\right]-\frac{\beta(s)M(s)}{a(s)+M(s)}L(s)\right]ds\right\},\\
E(t) &= E(t_0)\exp\left\{\int_{t_0}^{t}\left[b(s)wA(s)\left[1-\frac{E(s)}{K_E(s)}\right]-[\alpha_E(s)+\mu_E(s)]E(s)\right]ds\right\},\\
L(t) &= L(t_0)\exp\left\{\int_{t_0}^{t}\left[\alpha_E(s)E(s)+\frac{e\beta(s)M(s)}{a(s)+M(s)}L(s)-[\alpha_L(s)+\mu_L(s)]L(s)\right.\right.\\
&\qquad \left.\left.-\theta(s)L^2(s)\right]ds\right\},\\
P(t) &= P(t_0)\exp\left\{\int_{t_0}^{t}\left[\alpha_L(s)L(s)-[\alpha_p(s)+\mu_P(s)]P(s)\right]ds\right\},\\
A(t) &= A(t_0)\exp\left\{\int_{t_0}^{t}\left[\alpha_P(s)P(s)-\mu_A(s)A(s)\right]ds\right\}.
\end{aligned}\right\} \qquad (3.74)$$

From (3.74), one can observe that all the solutions of model (3.64) are non-negative.

Next, Theorem 3.7 **(b)** was proved under assumption that Ω is positively-invariant for system (3.64). It was assumed that $(M(t), E(t), L(t), P(t), A(t))$ be a unique solution of system (3.64) with $(M(t_0), E(t_0), L(t_0), P(t_0), A(t_0)) \in \Omega$.

From the first equation of (3.64) and the positivity solutions of (3.64), one has:

$$M'(t) \;\leq\; r(t)M(t)\left[1-\frac{M(t)}{K_M}\right], \qquad t \geq t_0, \qquad (3.75)$$

and by Lemma 3.4 and $0 < M(t_0) \leq \mathscr{M}_1$, $M(t) \leq \mathscr{M}_1$, $t \geq t_0$.

From the second equation of system (3.64), one has:

$$\begin{aligned}
E(t) &\leq b(t)wA(t)-[b(t)wA(t)+\alpha_E(t)+\mu_E(t)]E(t)\\
&\leq b(t)w\mathscr{M}_5-[b(t)w\mathscr{M}_5+\alpha_E(t)+\mu_E(t)]E(t),, \qquad t \geq t_0, \qquad (3.76)
\end{aligned}$$

by Lemma 3.4 and $0 < E(t_0) \leq \mathscr{M}_2$, $E(t) \leq \mathscr{M}_2$, $t \geq t_0$.

From the third equation of model (3.64), one has:

$$L'(t) \;\leq\; \alpha_E(t)+e\beta(t)\Phi(\mathscr{M}_1)(t)L(t)-[\alpha_L(t)+\mu_L(t)]L(t)-\theta(t)L^2(t), \qquad t \geq t_0 (3.77)$$

by Lemma 3.4 and $0 < L(t_0) \leq \mathscr{M}_3$, $L(t) \leq \mathscr{M}_3$, $t \geq t_0$.

From the fourth equation of system (3.64), one has:

$$P'(t) \leq \ [\mu_P(t)+\alpha_P(t)]\left[\frac{\alpha_L(t)\mathscr{M}_3}{[\mu_P(t)+\alpha_P(t)]}-P(t)\right], \qquad t \geq t_0, \tag{3.78}$$

by Lemma 3.4 and $0 < P(t_0) \leq \mathscr{M}_4$, $P(t) \leq \mathscr{M}_4, t \geq t_0$.
From the last equation of system (3.64), one has:

$$A'(t) \leq \ \mu_A(t)\left[\frac{\alpha_P(t)\mathscr{M}_4}{\mu_A(t)}-A(t)\right], \qquad t \geq t_0. \tag{3.79}$$

Again from the first equation of system (3.64), one gets:

$$\begin{aligned} M'(t) &\geq r(t)M(t)-\frac{r(t)}{K_M}M^2(t)-\beta(t)L(t)M(t), \\ &\geq (r(t)-\beta(t)\mathscr{M}_3)-\frac{r(t)}{K_M}M^2(t), \qquad t \geq t_0, \end{aligned} \tag{3.80}$$

and by Lemma 3.4 and $M(t_0) \geq m_1 > 0$, one obtain $M(t) \geq m_1, t \geq t_0$. From the second equation of system (3.64), the following result is obtained:

$$\begin{aligned} E(t) &\geq b(t)wA(t)-[b(t)wA(t)+\alpha_E(t)+\mu_E(t)]E(t) \\ &\geq b(t)wm_5-[b(t)wm_5+\alpha_E(t)+\mu_E(t)]E(t), \qquad t \geq t_0. \end{aligned} \tag{3.81}$$

By Lemma 3.4 and $E(t_0) \geq m_2 > 0$, it follows that $E(t) \geq m_2$, holds for $t \geq t_0$. From the third equation of system (3.64), I got:

$$L'(t) \ \geq \ \alpha_E(t)+e\beta(t)\Phi(m_1)(t)L(t)-[\alpha_L(t)+\mu_L(t)]L(t)-\theta(t)L^2(t), \qquad t \geq t_0 \tag{3.82}$$

It follows from Lemma 3.4 and $L(t_0) \geq m_3 > 0$ that$L(t) \geq m_3$, holds for $t \geq t_0$. From the fourth equation of system (3.64), I got:

$$P'(t) \geq \ [\mu_P(t)+\alpha_P(t)]\left[\frac{\alpha_L(t)m_3}{[\mu_P(t)+\alpha_P(t)]}-P(t)\right], \qquad t \geq t_0, \tag{3.83}$$

By Lemma 3.4 and $P(t_0) \geq m_4 > 0$ it follows that $P(t) \geq m_3$, holds for $t \geq t_0$. Furthermore, from the last equation of system (3.64), I have:

$$A'(t) \geq \ \mu_A(t)\left[\frac{\alpha_P(t)m_4}{\mu_A(t)}-A(t)\right], \qquad t \geq t_0, \tag{3.84}$$

Based on the above proofs, it implies Lemma 3.4 and $A(t_0) \geq m_5$ that $A(t) \geq m_5$ holds for all

$t \geq t_0$. Therefore, Ω is positively-invariant with respect to system (3.64).

Next we prove that if Theorem 3.7 **(a)** and Condition (H1) holds, then system (3.64) is permanent. This started by assuming that $(M(t),\ E(t),\ L(t),\ P(t),\ A(t))$ is a unique solution of system (3.64) with positive initial value $(M(t_0),\ E(t_0),\ L(t_0),\ P(t_0),\ A(t_0))$. If a sufficiently small $\delta > 0$ chosen such that $m_i^\delta (i = 1,2,3,4,5)$, and each inequality of (H1) holds when $\mathscr{M}_i$ and m_i are replaced by $\mathscr{M}_i^\delta > 0$ and $m_i^\delta > 0$, respectively, then by Lemma 3.4, it follows that $\limsup_{t\to+\infty} M(t) \leq \mathscr{M}_1$, implying that there exists $T_0 > t_0$ such that for $t > T_0$, $M(t) \leq \mathscr{M}_1^\delta$. Then from the first equation of system (3.64), we obtain:

$$M'(t) \leq r(t)M(t) - \frac{r(t)}{K_M}M^2(t), \qquad t > T_0, \tag{3.85}$$

which yields by Lemma 3.4 that:

$$\limsup_{t\to+\infty} M(t) \leq \sup_{t\in\mathbb{R}}\{K_M\}. \tag{3.86}$$

Thus, by the arbitrariness of δ, it follows that $\limsup_{t\to+\infty} M(t) \leq \mathscr{M}_1$. Then, there exists $T_1 > T_0$ such that for $t > T_1$, $M(t) \leq \mathscr{M}_1^\delta$, and:

$$E(t) \quad \leq \quad b(t)w\mathscr{M}_5^\delta - [b(t)w\mathscr{M}_5^\delta + \alpha_E(t) + \mu_E(t)]E(t), \qquad t > T_1. \tag{3.87}$$

It follows from Lemma 3.4 and $\inf_{t\in\mathbb{R}}\{b(t)w\mathscr{M}_5^\delta - [b(t)w\mathscr{M}_5^\delta + \alpha_E(t) + \mu_E(t)]E(t)\} > 0$ that

$$\limsup_{t\to+\infty} E(t) \leq \sup_{t\in\mathbb{R}}\left\{\frac{b(t)w\mathscr{M}_5^\delta}{[b(t)w\mathscr{M}_5^\delta + \alpha_E(t) + \mu_E(t)]E(t)}\right\}. \tag{3.88}$$

Thus, by the arbitrariness of δ, I have $\limsup_{t\to+\infty} E(t) \leq \mathscr{M}_2$, and there exists $T_2 > T_1$ such that for $t > T_2$, $E(t) \leq \mathscr{M}_2^\delta$, and

$$L'(t) \quad \leq \quad \alpha_E(t) + e\beta(t)\Phi(\mathscr{M}_1^\delta)(t)L(t) - [\alpha_L(t) + \mu_L(t)]L(t) - \theta(t)L^2(t), \qquad t > T_2 \tag{3.89}$$

It follows from Lemma 3.4 and $\inf_{t\in\mathbb{R}}\{e\beta(t)\Phi(\mathscr{M}_1^\delta)(t)L(t) - [\alpha_L(t) + \mu_L(t)]\} > 0$ that:

$$\limsup_{t\to+\infty} L(t) \leq \sup_{t\in\mathbb{R}}\left\{\alpha_E(t)\mathscr{M}_2^\delta + \frac{e\beta(t)\Phi(\mathscr{M}_1^\delta)(t) - (\mu_L(t) + \alpha_L(t))}{\theta(t)}\right\}. \tag{3.90}$$

Thus, by the arbitrariness of δ, I have $\limsup_{t\to+\infty} L(t) \leq \mathscr{M}_3$, and there exists $T_3 > T_2$ such that for $t > T_3$, $L(t) \leq \mathscr{M}_3^\delta$, and:

$$P'(t) \leq \alpha_L(t)\mathscr{M}_3^\delta - (\mu_P(t) + \alpha_P(t))P(t), \qquad t > T_3, \tag{3.91}$$

which yields by Lemma 3.4 that:

$$\limsup_{t\to+\infty} P(t) \le \sup_{t\in\mathbb{R}} \left\{ \frac{\alpha_L(t)\mathscr{M}_3^\delta}{[\mu_P(t)+\alpha_P(t)]} \right\}. \tag{3.92}$$

Thus, by the arbitrariness of δ, one has $\limsup_{t\to+\infty} P(t) \le \mathscr{M}_4$. Then, there exists $T_4 > T_3$ such that for $t > T_4$, $P(t) \le \mathscr{M}_4^\delta$, and:

$$A'(t) \le \alpha_P(t)\mathscr{M}_4^\delta - \mu_A(t)P(t), \qquad t > T_4,$$

which yields by Lemma 3.4 that:

$$\limsup_{t\to+\infty} A(t) \le \sup_{t\in\mathbb{R}} \left\{ \frac{\alpha_P(t)\mathscr{M}_4^\delta}{\mu_A(t)} \right\}. \tag{3.93}$$

Thus, by the arbitrariness of δ, one can obtain $\limsup_{t\to+\infty} A(t) \le \mathscr{M}_5$. Then, there exists $T_5 > T_4$ such that for $t > T_4$, $A(t) \le \mathscr{M}_5^\delta$. This completes the proof of Theorem 3.7(i).

3.3.2 Global Asymptotic Stability of the Boundary Solution

In this section, global asymptotic stability of the boundary solution were investigated. It was assumed that maize biomass is the only food source for the FAW in this case; such that in the absence of the maize plant biomass, the FAW population becomes extinct. Hence the only boundary solution of system (3.64) is $(M(t),0,0,0,0)$. Substituting this boundary solution into system (3.64) one gets:

$$\frac{dM(t)}{dt} = r(t)M(t)\left(1-\frac{M(t)}{K_M}\right). \tag{3.94}$$

Equation (3.94) is well known non-autonomous logistic equation. As illustrated in equation (3.66), $r(t)$ is continuous $\omega-$periodic, bounded below and above by positive constants. According to Fan and Wang (1998), one can obtain the following results:

Lemma 3.5

If $r(t)$ is a continuous periodic function $\omega-$, and bounded below and above by strictly positive reals for all $t \in \mathbb{R}$, the logistic equation (3.94) has exactly a solution $M^g(t)$ bounded below and above by positive constants (Fan & Wang, 1998). Precisely, this solution is given by:

$$M^g(t) = \left[\exp\left(\int_0^\omega r(s)ds\right)-1\right]\cdot\left[\int_t^{t+\omega}\frac{r(s)}{K_M}\cdot\exp\left(-\int_s^t r(\tau)d\tau)ds\right)\right]^{-1}. \tag{3.95}$$

In addition, $M^g(t)$ is globally asymptotically-stable for $M(t)$ with positive initial value $M(t_0) =$

$M_0 > 0$ in the sense $\lim_{t\to+\infty} |M(t) - M^g(t)| = 0$.

By Lemma 3.5, one can obtain the following result:
Lemma 3.6
System (3.64) admits a unique positive $\omega-$periodic solution $M^g(t)$, 0, 0, 0, 0) which is globally asymptotically-stable for $M(t)$ with positive initial value $M(t_0) = M_0 > 0$ in the sense $\lim_{t\to+\infty} |M(t) - M^g(t)| = 0$.

For a continuous and periodic function $g(t)$ with periodic ω, we denote:

$$\mathscr{A}(g) := \frac{1}{\omega}\int_0^{\omega} g(t)dt. \tag{3.96}$$

Lemma 3.7
If $r(t)$ is a continuous $\omega-$periodic function, then the null solution of (3.94) is globally asymptotically-stable provided that one of the following two conditions is met (Bai *et al.*, 2020):

(i) $\mathscr{A}(r) < 0$;

(ii) $\mathscr{A}(r) = 0$, and $\mathscr{A}(r/K_M) < 0$.

Note: If $\mathscr{A}(r) > 0$ and $\mathscr{A}(r/K_M) > 0$, then (3.94) has a unique positive $\omega-$ periodic solution $M^g(t)$ which is globally asymptotically-stable (Tineo, 1995). Thus, when r/K_M is non-negative with $\mathscr{A}(r/K_M) > 0$, the null solution of (3.94) is globally stable if and only if $\mathscr{A}(r) \leq 0$.

Finally, I provide the proof of Theorem 3.7**(b)**: Define a Poincare mapping $\mathscr{F} : \mathbb{R}^5 \to \mathbb{R}^5$ as follows:

$$\begin{aligned} \mathscr{F}(\xi) &= (M(t_0+\omega,t_0,\xi),E(t_0+\omega,t_0,\xi),L(t_0+\omega,t_0,\xi),P(t_0+\omega,t_0,\xi), \\ &\quad A(t_0+\omega,t_0,\xi)) \\ \xi &= (M_0,E_0,L_0,P_0,A_0) \in \mathbb{R}^5, \end{aligned} \tag{3.97}$$

where $(M(t_0+\omega,t_0,\xi),E(t_0+\omega,t_0,\xi),L(t_0+\omega,t_0,\xi),P(t_0+\omega,t_0,\xi),A(t_0+\omega,t_0,\xi))$ represents the solution of (3.64) through (t_0,ξ), $\xi = (M_0,E_0,L_0,P_0,A_0) \in \mathbb{R}^5$. Then $\mathscr{F}(\Omega) \cap \Omega$ by the positive invariant property of Ω. The continuity of $\mathscr{F}$ can be guaranteed by the continuity of solution of (3.64) with respect to initial value. Note that Ω is a bounded, closed, convex set in $\mathbb{R}^5$. Therefore, it follows from Brouwer's fixed point theorem that the operator $\mathscr{F}$ has at least one fixed point $\xi^* = (M^*(t),E^*(t),L^*(t),P^*(t),A^*(t))$ in Ω, which is a positive $\omega-$ periodic solution of system (3.64). The proof is complete. □

3.3.3 The Optimal Control Problem

In this section, the basic model (3.64) was extended to incorporate time dependent intervention strategies with the main goal of reducing FAW egg and larvae populations. Considering the extent of damage FAW can cause in a short period of time, it is imperative that once this pest has been identified in a maize field, necessary control approaches should be implemented timeously.

Prior studies suggest that the effective management of FAW depends on the integration of several control strategies including the biological, host plant resistance, and use of chemical insecticides (Rukundo *et al.*, 2020). Here, model (3.64) was reformulated to incorporate new parameters $u_1(t)$ and $u_2(t)$, known as controls. Control $u_1(t)$ models the efforts of traditional control methods like handpicking and destruction of FAW egg masses and larvae on FAW dynamics. Control $u_2(t)$ accounts for the efforts of chemical pesticide use on FAW dynamics. Without loss in generality, herein the term traditional methods was used to describe handpicking and destruction of FAW egg masses and larvae. Utilizing the same variable and parameter names as in system (3.64), the new system of nonlinear differential equations incorporating time dependent controls was summarized by the following equations:

$$\left.\begin{aligned}
\frac{dM(t)}{dt} &= r(t)M(t)\left[1-\frac{M(t)}{K_M}\right]-\frac{\beta(t)M(t)}{a(t)+M(t)}L(t),\\
\frac{dE(t)}{dt} &= b(t)wA(t)\left[1-\frac{E(t)}{K_E(t)}\right]-[\alpha_E(t)+\mu_E(t)+u_1(t)]E(t),\\
\frac{dL(t)}{dt} &= \alpha_E(t)E(t)+\frac{e\beta(t)M(t)}{a(t)+M(t)}L(t)-[\alpha_L(t)+\mu_L(t)+u_1(t)+u_2(t)]L(t)\\
&\quad -\theta(t)L^2(t),\\
\frac{dP(t)}{dt} &= \alpha_L(t)L(t)-[\alpha_p(t)+\mu_P(t)]P(t),\\
\frac{dA(t)}{dt} &= \alpha_P(t)P(t)-\mu_A(t)A(t).
\end{aligned}\right\} \tag{3.98}$$

As one can note, in (3.98) control efforts $u_1(t)$ aims to reduce the egg and larvae population while the use of chemical insecticide $u_2(t)$ mainly reduces the density of larvae population only. For model (3.98) to be biologically meaningful we set:

$$M(0)\geq 0, \qquad E(0)\geq 0, \qquad L(0)\geq 0, \qquad P(0)\geq 0, \qquad A(0)\geq 0. \tag{3.99}$$

The control set for the controls was considered as follows:

$$\Gamma = \{(u_1(t), u_2(t))) \,\big|\, 0 \leq u_1(t) \leq u_{1\max}, 0 \leq u_2(t) \leq u_{2\max}\}, \tag{3.100}$$

where $u_{1\max}$ and $u_{2\max}$ represents the upper bounds for the efforts of traditional methods and use of chemical insecticide respectively. If $u_i = 0$, $(i = 1,2)$, it implies absence of time dependent control measures.

A control strategy is said to be successful control strategy if it can reduce the egg and larvae population thereby reducing or eradicating FAW population in the field. As such, the main goal was to identify a pair of control (u_1^*, u_2^*) that minimize the population FAW egg and larvae at minimal cost. Therefore the objective functional was proposed as follows:

$$J(u_1(t), u_2(t)) = \int_0^T \left[C_1 E(t) + C_2 L(t) + \frac{W_1}{2} u_1^2(t) + \frac{W_2}{2} u_2^2(t)\right] dt. \tag{3.101}$$

subject to the constraints (3.98), where C_1, C_2, W_1 and W_2 are balancing coefficients (non-negative) converting the integrals into monetary quantity over a finite period of time, T days. In (3.101), control efforts $(u_1^*(t), u_2^*(t))$ with coefficients W_1 and W_2 are assumed to be nonlinear-quadratic, since a quadratic structure in the control has mathematical advantages such as: (i) controls will not disappear after differentiation and this implies that the solution is unique (ii) all control strategies (u_1^*, u_2^*) are bounded implying that efforts for implementing these controls are also bounded and characterized (iii) if the control set is compact and convex, it follows that the Hamiltonian attains its minimum over the control set at a unique point. Further, the goal was to find a control pair $(u_1^*(t), u_2^*(t)) \in U$ such that condition (3.102) is attained:

$$J(u_1^*(t), u_2^*(t)) = \min_U J(u_1(t), u_2(t)) \tag{3.102}$$

subject to the state equations in system (3.98) with initial conditions.

In order to study the existence of an optimal control pair (u_1^*, u_2^*), techniques in Fleming and Rishel's work (Fleming & Rishel, 1975) were utilized. Theorem 3.8 (i) and (ii) state the existence of the optimal controls and the their characterization. The proofs of these controls are presented in Appendix 3.

Theorem 3.8

(a) There exists an optimal control pair (u_1^*, u_2^*) to the problem (3.98). **(b)** Given an optimal control $u = (u_1, u_2) \in U$ and corresponding state solutions M, E, L, P and P, there exists adjoint functions λ_i, $i = 1,2,3,4,5$ satisfying:

$$\frac{d\lambda_1}{dt} = -\left[r(t) - \frac{2r(t)M(t)}{K_M} - \frac{\beta(t)L(t)}{a(t)+M(t)} + \frac{\beta(t)L(t)M(t)}{(a(t)+M(t))^2}\right]\lambda_1(t)$$

$$
\begin{aligned}
& & & -\left[\frac{e\beta(t)L(t)M(t)}{a(t)+M(t)}-\frac{e\beta(t)L(t)M(t)}{(a(t)+M(t))^2}\right]\lambda_3(t),\\
\frac{d\lambda_2}{dt} &=& & -C_1+\left[\frac{b(t)wA(t)}{K_E(t)}+\alpha_E(t)+\mu_E(t)+u_1(t)\right]\lambda_2(t)-\alpha_E(t)\lambda_3(t),\\
\frac{d\lambda_3}{dt} &=& & -C_2+\frac{\beta(t)M(t)}{a(t)+M(t)}\lambda_1(t)-\left[\frac{e\beta(t)M(t)}{a(t)+M(t)}-2\theta(t)L(t)-\alpha_L(t)-\mu_L(t)\right]\lambda_3(t)\\
& & & +[u_1(t)+u_2(t)]\lambda_3(t)-\alpha_L(t)\lambda_4(t),\\
\frac{d\lambda_4}{dt} &=& & (\alpha_P(t)+\mu_P(t))\lambda_4(t)-\alpha_P(t)\lambda_5(t),\\
\frac{d\lambda_5}{dt} &=& & -b(t)w\left(1-\frac{E(t)}{K_E(t)}\right)\lambda_2(t)+\mu_A(t)\lambda_5(t), \qquad (3.103)
\end{aligned}
$$

with transversality condition $\lambda_i(T)=0$, for $i=1,2,3,4,5$. Furthermore, these optimal controls are characterized by:

$$
\left.
\begin{aligned}
u_1 &= \min\left[u_{1\max},\max\left(\frac{E(t)\lambda_2(t)+L(t)\lambda_3(t)}{W_1},0\right)\right],\\
u_2 &= \min\left[u_{2\max},\max\left(\frac{L(t)\lambda_3(t}{W_2},0\right)\right]
\end{aligned}
\right\} \qquad (3.104)
$$

Theorem 3.8 **(a)** There exists an optimal control pair (u_1^*,u_2^*) to the problem (3.98).

Proof. Suppose that $\mathbf{f}(t,\mathbf{x},\mathbf{u})$ be the right-hand side of the (3.98) whereby $\mathbf{x}=(M,E,L,P,A)$ and $\mathbf{u}=(u_1(t),u_2(t))$ represent the vector of state variables and control functions respectively. The requirements for the existence of optimal control as presented in Fleming & Rishel (1975) were listed as follows:

1. The function $\mathbf{f}$ is of class C^1 and there exists a constant C such that $|\mathbf{f}(t,0,0)|\le C, |\mathbf{f}_{\mathbf{x}}(t,\mathbf{x},\mathbf{u})|\le C(1+|\mathbf{u}|)$, $|\mathbf{f}_{\mathbf{u}}(t,\mathbf{x},\mathbf{u})|\le C$;

2. the admissible set of all solutions to system (3.98) with corresponding control in Ω is non-empty;

3. $\mathbf{f}(t,\mathbf{x},\mathbf{u})=\mathbf{a}(t,\mathbf{x})+\mathbf{b}(t,\mathbf{x})\mathbf{u}$;

4. the control set $U=[0,u_{1\max}]\times[0,u_{2\max}]$ is compact, closed, and convex;

5. the integrand of the objective functional is convex in U.

To verify these conditions, it was given that:

$$\mathbf{f}(t,\mathbf{x},\mathbf{u}) = \begin{bmatrix} \bar{m}_1 \\ \bar{m}_2 \\ \bar{m}_3 \\ \bar{m}_4 \\ \bar{m}_5 \end{bmatrix}, \tag{3.105}$$

with

$$\left.\begin{aligned}
\bar{m}_1 &= r(t)M(t)\left[1-\frac{M(t)}{K_M}\right] - \frac{\beta(t)M(t)}{a(t)+M(t)}L(t), \\
\bar{m}_2 &= b(t)wA(t)\left[1-\frac{E(t)}{K_E(t)}\right] - [\alpha_E(t)+\mu_E(t)+u_1(t)]E(t), \\
\bar{m}_3 &= \alpha_E(t)E(t) + \frac{e\beta(t)M(t)}{a(t)+M(t)}L(t) - [\alpha_L(t)+\mu_L(t)+u_1(t)+u_2(t)]L(t) \\
&\quad -\theta(t)L^2(t), \\
\bar{m}_4 &= \alpha_L(t)L(t) - [\alpha_p(t)+\mu_P(t)]P(t), \\
\bar{m}_5 &= \alpha_P(t)P(t) - \mu_A(t)A(t).
\end{aligned}\right\} \tag{3.106}$$

From (3.105), it is evident that $\mathbf{f}(t,\mathbf{x},\mathbf{u})$ is of class C^1 and $|\mathbf{f}(t,\mathbf{0},\mathbf{0})| = 0$. Furthermore, one can easily compute $|\mathbf{f}_\mathbf{x}(t,\mathbf{x},\mathbf{u})|$ and $|\mathbf{f}_\mathbf{u}(t,\mathbf{x},\mathbf{u})|$, and demonstrate that:

$$|\mathbf{f}(t,\mathbf{0},\mathbf{0})| \le C, \quad |\mathbf{f}_\mathbf{x}(t,\mathbf{x},\mathbf{u})| \le C(1+|\mathbf{u}|) \quad \text{and} \quad |\mathbf{f}_\mathbf{u}(t,\mathbf{x},\mathbf{u})| \le C. \tag{3.107}$$

Due to the condition 1, the existence of the unique solution for condition 2 for bounded control is satisfied. On the other hand, the quantity $\mathbf{f}(t,\mathbf{x},\mathbf{u})$ is expressed as a linear function of the control variables which satisfy the condition 3. ☐

Theorem 3.8 **(b)** Given an optimal control $u = (u_1,u_2) \in U$ and corresponding state solutions M, E, L, P and P, there exists adjoint functions λ_i, $i = 1,2,3,4,5$ satisfying equations (3.103) and (3.104).

Proof. To characterize the optimal control problem, Pontryagin's Maximum Principle (Pontryagin, 2018; Shell, 1969) was used to formulate the following Hamiltonian function:

$$\begin{aligned}
H(t) &= C_1E(t) + C_2L(t) + \frac{W_1}{2}u_1^2(t) + \frac{W_2}{2}u_2^2(t) \\
&\quad +\lambda_1(t)\left[r(t)M(t)\left[1-\frac{M(t)}{K_M}\right] - \frac{\beta(t)M(t)}{a(t)+M(t)}L(t)\right] \\
&\quad +\lambda_2(t)\left[b(t)wA(t)\left[1-\frac{E(t)}{K_E(t)}\right] - [\alpha_E(t)+\mu_E(t)+u_1(t)]E(t)\right]
\end{aligned}$$

$$
\begin{aligned}
&+\lambda_3(t)\left[\alpha_E(t)E(t)+\frac{e\beta(t)M(t)}{a(t)+M(t)}L(t)-[\alpha_L(t)+\mu_L(t)+u_1(t)+u_2(t)]L(t)\right.\\
&\left.-\theta(t)L^2(t)\right]+\lambda_4(t)\left[\alpha_L(t)L(t)-[\alpha_p(t)+\mu_P(t)]P(t)\right]\\
&+\lambda_5(t)\left[\alpha_P(t)P(t)-\mu_A(t)A(t)\right]. \qquad (3.108)
\end{aligned}
$$

Next, adjoint equations was determined as follows; $\frac{\partial\lambda_i}{dt}=-\frac{\partial H}{\partial \mathbf{x}}$, where $\mathbf{x}=(M(t), E(t), L(t), P(t), A(t))$, with transversality condition $\lambda_i(T)=0$ for $i=1,2,3,4,5$, and obtained the results in equation (3.103). Now, we minimize the Hamiltonian with respect to the controls. Note that we have required the convexity for minimization, $\frac{\partial^2 H}{\partial u_i^2}=W_i>0$, $i=1,2$. On the interior of the control set, we have:

$$
\left.\frac{\partial H}{\partial u_1}=0\Rightarrow u_1=\frac{E(t)\lambda_2(t)+L(t)\lambda_3(t)}{W_1}, \quad \text{and,} \quad \frac{\partial H}{\partial u_2}=0\Rightarrow u_2=\frac{L(t)\lambda_3(t}{W_2}.\right\} \qquad (3.109)
$$

Using the standard arguments and the bounds for the controls, one gets the characterization of this optimal pair (3.104). ☐

3.4 FAW-Maize Interaction Model with Farming Awareness and Larvae Predation

To evaluate the role of farming awareness campaigns and FAW larvae predation (IPM) on the dynamics of FAW-maize interaction and in-turn on final maize biomass, a mathematical model based on Caputo fractional derivative (Podlubny, 1999) was developed under the following assumption:

(i) The FAW population was subdivided into two essential classes namely; larvae population $L(t)$ and the adult FAW population $A(t)$. FAW larvae predator population was modeled by $Z(t)$. Meanwhile, the variable $M(t)$ was used to account for the dynamics of maize biomass.

(ii) In the absence of FAW, maize biomass was assumed to follow logistic growth with net growth rate r and carrying capacity K_M. It was laso assumed that FAW larvae consumes maize biomass at the rate β which in-turn the biomass are converted into larvae's biomass at the rate e. Farming awareness campaigns during an outbreak were assumed to reduce the attack rate of the maize crop by FAW larvae by a factor $1-u$, with $0\leq u\leq 1$. Thus $u=0$ imply that awareness has no impact on reducing the attack rate of the maize plant by FAW whereas $u=1$ imply that farming awareness is 100% efficient to protect the maize crop from FAW attack during an outbreak.

(iii) The dynamics of the FAW larvae were assumed to follow a logistic growth model, with net growth rate b_L and the carrying capacity K_L. It was assumed that larvae progress to the adult stage after approximately $1/\alpha_L$ days. Further, FAW larvae and adult populations were assumed to decrease due to natural mortality at rates μ_L and μ_A, respectively. Apart from natural mortality, both populations (FAW larvae and adult) were assumed to further diminish due to mortality attributed to mitigation strategies carried out by farmers as a result of awareness, at rate ud, where d is the mortality rate of FAW larvae. Note that if awareness does not have an impact ($u = 0$) on FAW populations then these populations suffer natural mortality only.

(iv) Even though biological control may not replace conventional insecticides, a number of parasitoids, predators and pathogens (birds, rodents, beetles, earwigs) readily attack larvae (Assefa & Ayalew, 2019). To account for the effect of larvae predation parameters σ and ρ were used to account for the attack rate of the larvae by predators and the efficiency of prey conversion, respectively. The average life span of predators was assumed to be $1/\eta$ days.

The summary of the definitions of model state variables and parameters are given in Tables 1 and 2. Based on the assumptions above the proposed model was summarized by the following system equations (3.110) and the model flow diagram is in Figure 11:

$$\left.\begin{array}{rcl}
{}_a^cD_t^qM(t) &=& r^qM\left(1-\dfrac{M}{K_M^q}\right)-\beta^q(1-u)LM,\\
{}_a^cD_t^qL(t) &=& b_L^qA\left(1-\dfrac{L}{K_L^q}\right)+e\beta^q(1-u)LM-\sigma^qZL\\
&& -(\mu_L+\alpha_L+ud)L,\\
{}_a^cD_t^qA(t) &=& \alpha_L^qL-(\mu_A^q+ud)A,\\
{}_a^cD_t^qZ(t) &=& \rho\sigma^qLZ-\eta^qZ,
\end{array}\right\} \tag{3.110}$$

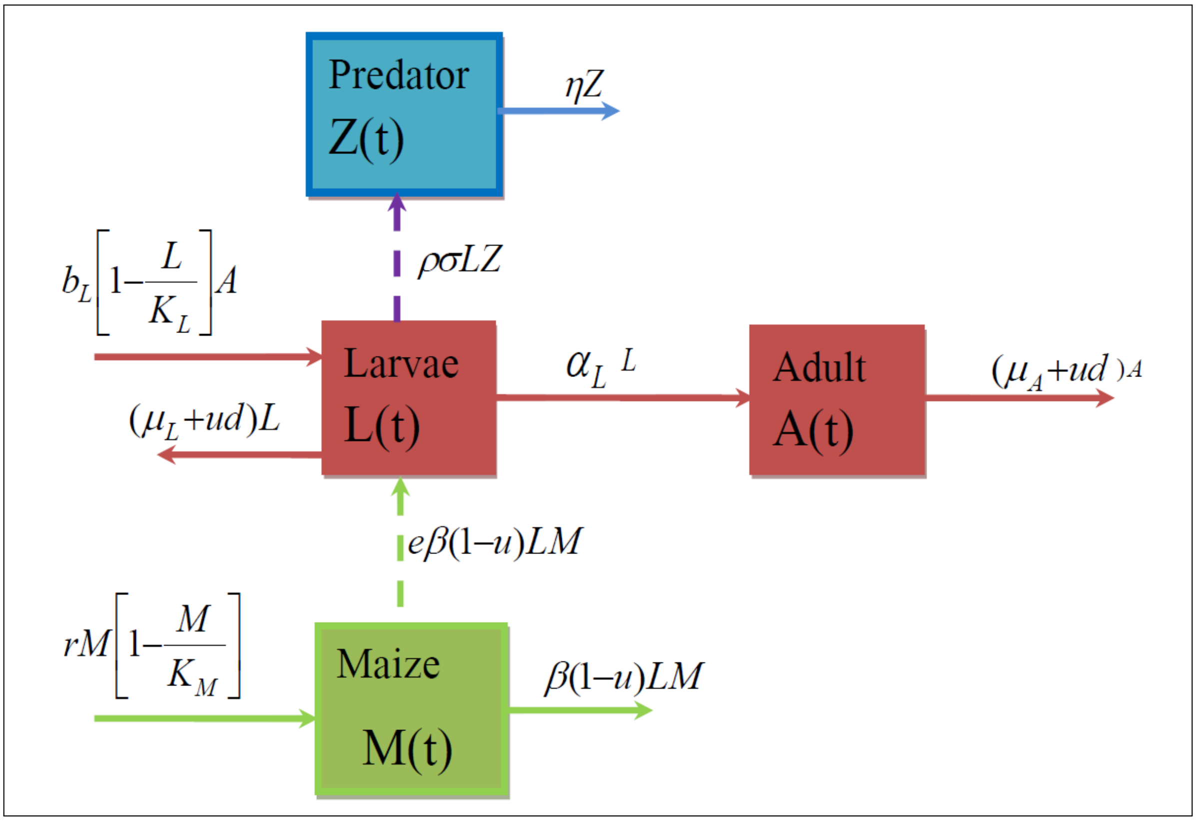


Figure 11: Model flow diagram for system (3.110)

The analytical results of the FAW-maize interaction model (3.110) with farming awareness and larvae predation were comprehensively discussed. The discussion commenced by proving on positivity and boundedness of solutions.

3.4.1 Positivity and Boundedness of Solutions

In this section, results on the positivity and boundednes of solutions of the proposed fractional order model (3.110) were established. In particular, the model was investigated if it is mathematically and biological well-poised. It follows from (3.110) that:

Theorem 3.9

Model (3.110) is positively invariant and bounded in $\mathbb{R}^4_+$.

Proof. For $M(0) \geq 0$, $L(0) \geq 0$, $A(0) \geq 0$, $Z(0) \geq 0$, we have:

$$\left.\begin{array}{rcl} {}_a^cD_t^q M(t)\,|_{M=0} & = & 0, \\ {}_a^cD_t^q L(t)\,|_{L=0} & = & b_L^q A \geq 0, \\ {}_a^cD_t^q A(t)\,|_{A=0} & = & \alpha_L^q L \geq 0, \\ {}_a^cD_t^q Z(t)\,|_{Z=0} & = & 0. \end{array}\right\} \tag{3.111}$$

Based on the results in (3.111), it follows that model (3.110) is positively invariant in $\mathbb{R}^4_+$.

Further, from the first equation of model (3.110) we demonstrate that $M(t) \leq K_M$, $\forall t \geq 0$. If there exists t_0 such that $M(t_0) > K_M$, then due to the continuity of $M(t)$:

$$\exists B_\varepsilon(t_0) : \forall t \in B_\varepsilon(t_0) : M(t) > K_M, \tag{3.112}$$

so,

$$rM\left(1 - \frac{M}{K_M}\right) < 0. \tag{3.113}$$

Thus ${}^{c}_{t_0}D^q M(t) < 0$. From the continuity of $M(t)$ and $\dfrac{dM}{dt} = \lim\limits_{q \to 1^-} {}^{c}_{t_0}D^q M(t) < 0$, hence we conclude that $M(t)$ is a decreasing function for all $t \geq 0$ and it follows that $0 \leq M(t) \leq M(0) \leq K_M$, $\forall t \geq 0$, and this is a contradiction to (3.112). Thus $M(t) \leq K_M$, for all $t \geq 0$. Using a similar approach it can easily be verified that $0 \leq L(t) \leq K_L$. Now, from the third equation of system (3.110) we have:

$$\begin{aligned} {}^{c}_{a}D^q_t A(t) &= \alpha^q_L L - (\mu^q_A + ud^q)A \\ &\leq \alpha^q K^q_L - (\mu^q_A + ud^q)A. \end{aligned} \tag{3.114}$$

Applying the Laplace transform one gets:

$$s^q \mathscr{L}[A(t)] - s^{q-1}A(0) \leq \frac{\alpha^q_L K^q_L}{s} - (\mu^q_a + ud^q)\mathscr{L}[A(t)]. \tag{3.115}$$

After combining like terms one gets:

$$\begin{aligned} \mathscr{L}[A(t)] &\leq \alpha^q_L K^q_L \frac{s^{-1}}{s^q + (\mu^q_a + ud^q)} + A(0)\frac{s^{q-1}}{s^q + (\mu^q_A + ud)} \\ &= \alpha^q_L K^q_L \frac{s^{q-(1+q)}}{s^q + (\mu^q_a + ud^q)} + A(0)\frac{s^{q-1}}{s^q + (\mu^q_a + ud^q)}. \end{aligned} \tag{3.116}$$

Applying the inverse Laplace transform leads to:

$$\begin{aligned} A(t) &\leq \mathscr{L}^{-1}\left\{\alpha^q_L K^q_L \frac{s^{q-(1+q)}}{s^q + (\mu^q_a + ud^q)}\right\} + A(0)\mathscr{L}^{-1}\left\{\frac{s^{q-1}}{s^q + (\mu^q_a + ud^q)}\right\} \\ &\leq \alpha^q_L K^q_L t^q E_{q,q+1}(-(\mu^q_a + ud^q)t^q) + A(0)E_{q,1}(-(\mu^q_a + ud^q)t^q) \\ &\leq \frac{\alpha^q_L K^q_L}{(\mu^q_A + ud)}(\mu^q_A + ud)t^q E_{q,q+1}(-(\mu^q_a + ud^q)t^q) \\ &\quad + A(0)E_{q,1}(-(\mu^q_a + ud^q)t^q) \end{aligned}$$

$$\begin{aligned}
&\leq \max\left\{\frac{\alpha^q K_L^q}{(\mu_a^q+ud^q)}, A(0)\right\}((\mu_a^q+ud^q)t^q E_{q,q+1}(-(\mu_a^q+ud^q)t^q) \\
&\quad +E_{q,1}(-(\mu_a^q+ud^q)t^q)) \\
&= \frac{C}{\Gamma(1)} = C_A, \qquad (3.117)
\end{aligned}$$

where E_q is the Mittag-Leffler function and $C_A = \max\left\{\frac{\alpha_L^q K_L^q}{(\mu_a^q+ud^q)}, P(0)\right\}$. Thus, $A(t)$ is bounded from above. From the last equation of system (3.168) we have:

$$\begin{aligned}
{}_a^cD_t^q Z(t) &= \rho\sigma^q LZ - \eta^q Z \\
&\leq -(\eta^q - \rho\sigma^q K_L^q)Z. \qquad (3.118)
\end{aligned}$$

Applying the Laplace transform in the previous inequality, we get:

$$s^q \mathscr{L}[Z(t)] - s^{q-1}Z(0) \leq -(\eta^q - \rho\sigma^q K_L^q)\mathscr{L}[Z(t)], \qquad (3.119)$$

that can be written as

$$\mathscr{L}[Z(t)] \leq Z(0)\frac{s^{q-1}}{s^q + (\eta^q - \rho\sigma^q K_L^q)}. \qquad (3.120)$$

Applying the inverse Laplace transforms leads to:

$$Z(t) \leq Z(0)E_q[-(\eta^q - \rho\sigma^q K_L^q)t^q]. \qquad (3.121)$$

Hence, we conclude that $Z(t)$ is bounded. □

Biological implications: Theorem 3.9 implies that our proposed model (3.110) is mathematically and biologically well-defined, i.e., the population of species under consideration is non-negative and bounded.

3.4.2 Equilibrium Points and their Existence

In this section, equilibrium points of model (3.110) and their existence were computed and analyzed. Therefore, solving the model system (3.110), the fractional-order model has the following equilibrium points:

(**a**) The trivial equilibrium point $\mathscr{E}_0 : (M_0, L_0, A_0, Z_0) = (0,0,0,0)$ always exists. Biologically, this equilibrium point $\mathscr{E}_0$ reflects the total absence of the species defined in (3.110) either through extinction or intervention strategies.

(**b**) The pest-extinction equilibrium point

$$\mathscr{E}_1 : (M_1, L_1, A_1, Z_1) \quad = \quad (K_M, 0, 0, 0). \tag{3.122}$$

This equilibrium point $\mathscr{E}_1$ reflects the growth of maize biomass in the field when there is no dynamics of FAW and its predator. Thus, in the absence of FAW larvae, the expected maize biomass at the end of season will be of high yield.

(**c**) The plant-extinction equilibrium point $\mathscr{E}_2 : (M_2, L_2, A_2, Z_2)$ where

$$\left.\begin{aligned} M_2 &= 0, \qquad L_2 = \frac{\eta^q}{\rho}, \qquad A_2 = \frac{\eta^q \alpha_L^q}{(\mu_A^q + ud^q)\rho}, \\ Z_2 &= \frac{b_L^q \eta^q + \rho K_L^q (\mu_L^q + \alpha_L^q + ud^q)}{\rho \sigma^q K_L^q} \left(\frac{b_L^q \rho K_L^q}{b_L^q \eta^q + \rho K_L^q (\mu_L^q + ud^q + \alpha_L^q)} - 1 \right). \end{aligned}\right\} \tag{3.123}$$

Let $\mathscr{R}_1 = \dfrac{b^q \rho K_L^q}{b^q \eta^q + \rho K_L^q (\mu_L^q + ud^q + \alpha_L^q)}$. Thus, $Z_2 > 0$ if $\mathscr{R}_1 > 1$. The term $\mathscr{R}_1$ is a threshold condition for the existence of the FAW predator in the ecosystem. Biologically this term accounts for the ability of the FAW predator population to grow. If $\mathscr{R}_1 \leq 1$, the FAW predator will extinct otherwise it will persists. Moreover, from the terms that define $\mathscr{R}_1$, one can observe that the growth of FAW predator depends on the availability of the FAW larvae. Thus, when the FAW predator exists, growth of the FAW predator entirely depends on the availability of FAW larvae.

(**d**) The plant and predator-extinction equilibrium point $\mathscr{E}_3 : (M_3, L_3, A_3, Z_3)$ where

$$\left.\begin{aligned} M_3 &= 0, \qquad L_3 = \frac{\eta^q K_L^q}{b_L^q} \left(\frac{b_L^q}{(\mu_L^q + \alpha_L^q + ud^q)} - 1 \right), \\ A_3 &= \frac{\alpha_L^q K_L^q (\mu_L^q + \alpha_L^q + ud^q)}{b^q (\mu_A^q + ud^q)} \left(\frac{b_L^q}{(\mu_L^q + \alpha_L^q + ud^q)} - 1 \right), \\ Z_3 &= 0. \end{aligned}\right\} \tag{3.124}$$

Let $\mathscr{R}_2 = \dfrac{b^q}{(\mu_L^q + \alpha_L^q + ud^q)}$. Thus, $L_3 > 0$ and $A_3 > 0$ if $\mathscr{R}_2 > 1$. The equilibrium point $\mathscr{E}_3$ implies that, the FAW larvae population exist if the threshold quantity $\mathscr{R}_2$ is greater than unity. Biologically $\mathscr{R}_2$ represents the reproductive rate of FAW larvae in the absence of FAW predators. Precisely, the produced larvae b_L^q will remain in the environment for a period $\dfrac{1}{(\mu_L^q + \alpha_L^q + ud^q)}$ days.

(**e**) The predator-extinction equilibrium point is $\mathscr{E}_4 : (M_4, L_4, A_4, Z_4)$ where

$$\left.\begin{array}{rcl} M_4 &=& \dfrac{K_M^q\left[r^q b_L^q + \beta^q K_L^q(\mu_L^q + \alpha_L^q + ud^q)\left(1 - \dfrac{b_L^q}{(\mu_L^q + \alpha_L^q + ud^q)}\right)\right]}{r^q b_L^q + e(\beta^q(1-u))^2 K_M^q K_L^q}, \\[2ex] L_4 &=& \dfrac{r^q K_L^q}{r^q b_L^q + e(\beta^q(1-u))^2 K_M^q K_L^q}\left(\dfrac{b_L^q + e\beta^q(1-u)K_M^q}{(\mu_L^q + \alpha^q + ud^q)} - 1\right), \\[2ex] A_4 &=& \dfrac{r^q \alpha^q k_L^q}{(\mu_A^q + ud^q)(\mu_L^q + \alpha_L + ud^q)\tilde{n}}\left(\dfrac{b^q + e\beta^q(1-u)K_M^q}{(\mu_L^q + \alpha_L^q + ud^q)} - 1\right), \\ Z_4 &=& 0. \end{array}\right\} \qquad (3.125)$$

Let $\mathscr{R}_3 = \dfrac{b^q + e\beta^q(1-u)K_M^q}{(\mu_L^q + \alpha_L^q + ud^q)}$. One can observe that the FAW populations persist if $\mathscr{R}_3 > 1$.

The equilibrium point $\mathscr{E}_4$ revealed that, in the absence of FAW predators, growth of the FAW population significantly increases due to availability of the food source (maize biomass). In addition, it can also be noted that, growth of FAW larvae population $\mathscr{R}_2$ have a significant impact on maize biomass, since an increase in $\mathscr{R}_2$ will decrease M_4.

(**f**) The coexistence equilibrium point $\mathscr{E}_5 : (M_5, L_5, A_5, Z_5)$ where

$$\left.\begin{array}{rcl} M_5 &=& \dfrac{K_M^q \beta^q \eta^q}{r^q \rho^q}\left(\dfrac{r^q \rho}{\beta^q \eta^q} - 1\right) \qquad L_5 = \dfrac{\eta^q}{\eta^q}, \qquad A_5 = \dfrac{\eta^q \alpha_L^q}{(\mu_A^q + ud^q)\rho}, \\[2ex] Z_5 &=& \dfrac{K_M^q e(\beta^q(1-u))^2 \eta^q}{r^q \rho \sigma^q}\left(\dfrac{r^q \rho^q}{\beta^q \eta^q} - 1\right) \\ && + \dfrac{\sigma^q(K_L^q r^q \rho(\mu_L^q + \alpha_L^q + ud^q) + b^q}{r^q \rho}\left(\dfrac{b^q K_L^q r^q \rho}{b^q r^q \eta^q(\mu_L^q + \alpha_L^q + ud^q)K_L^q r^q \rho)} - 1\right). \end{array}\right\} \qquad (3.126)$$

Let $\mathscr{R}_4 = \dfrac{r^q \rho}{\beta^q \eta^q}$ and $\mathscr{R}_5 = \dfrac{b^q K_L^q r^q \rho}{b^q r^q \eta^q(\mu_L^q + \alpha_L^q + ud^q)K_L^q r^q \rho)}$. We can observe that $M_5 > 0$ if $\mathscr{R}_4 > 1$ and $Z_5 > 0$ if $\mathscr{R}_4 > 1$ and $\mathscr{R}_5 > 1$. The threshold quantities $\mathscr{R}_4$ and $\mathscr{R}_5$, represents the persistence of the FAW predator in the presence of FAW and its food source (maize biomass). Biologically, all the populations defined by system (3.110) co-exist if $\mathscr{R}_4 > 1$ and $\mathscr{R}_5 > 1$ where $\mathscr{R}_4$ reflects the impact of FAW larvae predator on the persistence of maize biomass. From, both $\mathscr{R}_4$ and $\mathscr{R}_5$, one can deduce that if FAW predators have a long life span, that is η^q, then final maize biomass will be high.

3.4.3 Local Stability Analysis of the Equilibrium Points

The local stability analysis for the fractional order model (3.110) around the equilibrium points were obtained by computing the Jacobian matrix corresponding to equilibrium points. The Jacobian matrix of system (3.110) is given as follows:

$$J(M,L,A,Z) = \begin{bmatrix} r^q - \beta^q L - \dfrac{2r^q M}{K_M^q} & -\beta^q M & 0 & 0 \\ e\beta^q(1-u)L & n & 0 & -\sigma^q L \\ 0 & \alpha_L^q & -(\mu_A^q + ud^q) & 0 \\ 0 & \rho Z & 0 & -\eta^q + \rho L \end{bmatrix}. \tag{3.127}$$

with $n = b^q + e\beta^q(1-u)M - \sigma^q Z - (\mu_L^q + \alpha_L^q + ud^q) - \dfrac{2b^q L}{K_L^q}$. Ahmed *et al.* (2007) studied some Routh-Hurwitz stability conditions for fractional order systems. In this case, well-known Routh-Hurwitz conditions are necessary and sufficient for $|\arg(\lambda_i)| > \dfrac{q\pi}{2}$ to be satisfied. To investigate the local stability of the equilibrium points for model (3.110), Jacobian matrix (3.127), Lemma 1.2, Lemma 1.3 and Theorem 3.10 were used:

Theorem 3.10 (i) The trivial equilibrium point $\mathscr{E}_0$ is locally asymptotically unstable.

(ii) If $b^q < (\mu_L^q + \alpha_L^q + ud^q)$, then the pest-extinction equilibrium point $\mathscr{E}_1$ is locally asymptotically stable.

(iii) If $r^q\rho < \beta^q\eta^q$ and condition $(C1)$ of Lemma 1.3 holds, then the equilibrium point $\mathscr{E}_2$ is locally asymptotically stable, otherwise it is unstable.

(iv) If $b^q + e\beta^q(1-u)K_M^q < (\mu_L^q + \alpha^q + ud^q)$ and condition $(C1)$ of Lemma 1.3 holds, then the equilibrium point $\mathscr{E}_4$ is locally asymptotically stable, otherwise it is unstable.

(vi) If condition $(C2)$ of Lemma 1.3 holds, then the equilibrium point $\mathscr{E}_5$ is locally asymptotically stable, otherwise it is unstable.

Proof. **(i)** The Jacobian matrix of system (3.110) evaluated at $\mathscr{E}_0$ is

$$J(\mathscr{E}_0) = \begin{bmatrix} r^q & 0 & 0 & 0 & \\ 0 & b^q - (\mu_L^q + \alpha_L^q + ud^q) & 0 & 0 & \\ 0 & \alpha_L^q & -(\mu^q + ud^q) & 0 & \\ 0 & 0 & 0 & 0 & -\eta^q \end{bmatrix} \tag{3.128}$$

The eigenvalues of matrix $J(\mathscr{E}_0)$ are $\lambda_1 = r^q > 0$, $\lambda_2 = b^q - (\mu_L^q + \alpha_L^q + ud^q)$, $\lambda_3 = -(\mu_A + ud^q)$ and $\lambda_4 = -\eta^q$. Since $\lambda_1 > 0$ it follows that the trivial equilibrium point $\mathscr{E}_0$ is locally asymptotically unstable.

(ii) The Jacobian matrix of system (3.110) evaluated at $\mathscr{E}_1$ is

$$J(\mathscr{E}_1) = \begin{bmatrix} -r^q & 0 & 0 & 0 & \\ 0 & b^q - (\mu_L^q + \alpha_L^q + ud^q) & 0 & 0 & \\ 0 & \alpha_L^q & -(\mu^q + ud^q) & 0 & \\ 0 & 0 & 0 & 0 & -\eta^q \end{bmatrix} \tag{3.129}$$

The eigenvalues of matrix $J(\mathscr{E}_1)$ are $\lambda_1 = -r^q$, $\lambda_2 = b^q - (\mu_L^q + \alpha_L^q + ud^q)$, $\lambda_3 = -(\mu_A + ud^q)$ and $\lambda_4 = -\eta^q$. Following Theorem 3.10 (ii), it can be observed that the equilibrium point $\mathscr{E}_1$ is locally asymptotically stable if $b^q < (\mu_L^q + \alpha_L^q + ud^q)$

(iii) The Jacobian matrix of system (3.110) evaluated at $\mathscr{E}_2$ is

$$J(\mathscr{E}_2) = \begin{bmatrix} r^q - \beta^q L_2 & 0 & 0 & 0 \\ e\beta^q(1-u)L_2 & \widetilde{m} & 0 & -\sigma^q L_2 \\ 0 & \alpha_L^q & -(\mu_A^q + ud^q) & 0 \\ 0 & \rho Z_2 & 0 & -\eta^q + \rho L_2 \end{bmatrix}. \tag{3.130}$$

with $\widetilde{m} = b^q - \sigma^q Z_2 - (\mu_L^q + \alpha_L^q + ud^q) - \dfrac{2b^q L_2}{K_L^q}$. The eigenvalues of matrix (3.130) are $\lambda_1 = r^q - \dfrac{\beta^q \eta^q}{\rho}$, $\lambda_2 = -(\mu_A^q + ud^q)$ and the remaining eigenvalues can be obtained from the reduced matrix

$$\widetilde{J}(\mathscr{E}_2) = \begin{bmatrix} b^q - \sigma^q Z_2 - (\mu_L^q + \alpha_L^q + ud^q) - \dfrac{2b^q L_2}{K_L^q} & -\sigma^q L_2 \\ \rho Z_2 & -\eta^q + \rho L_2 \end{bmatrix}, \tag{3.131}$$

whose characteristic equation is as follows

$$\lambda^2 + a_1 \lambda + a_2 = 0, \tag{3.132}$$

with

$$\begin{aligned} a_1 &= \eta^q + \sigma^q Z_2 + (\mu_L^q + \alpha_L^q + ud^q) - b^q, \\ &= \eta^q + \sigma^q \frac{b^q \eta^q + \rho K_L^q(\mu_L^q + \alpha^q + ud^q)}{\rho \sigma^q K_L^q} \left(\frac{b^q \rho K_L^q}{b^q \eta^q + \rho K_L^q(\mu_L^q + ud^q + \alpha^q)} - 1 \right) \\ &\quad + b^q \left(\frac{(\mu_L^q + \alpha_L^q + ud^q)}{b^q} - 1 \right) \end{aligned}$$

$$\begin{aligned} a_2 &= \sigma^q \eta^q Z_2 + (\eta^q - \rho L_2)\left((\mu_L^q + \alpha_L^q + ud^q) + \frac{2bL_2}{K_L} - b^q \right) \\ &= \sigma^q \eta^q \frac{b^q \eta^q + \rho K_L^q(\mu_L^q + \alpha^q + ud^q)}{\rho \sigma^q K_L^q} \left(\frac{b^q \rho K_L^q}{b^q \eta^q + \rho K_L^q(\mu_L^q + ud^q + \alpha^q)} - 1 \right) \end{aligned} \tag{3.133}$$

Therefore if $r^q\rho < \beta^q\eta^q$ and condition $(C1)$ of Lemma 1.3 holds, then the equilibrium point $\mathscr{E}_2$ is locally asymptotically stable, otherwise it is unstable.

(iv) The Jacobian matrix of system (3.110) evaluated at $\mathscr{E}_3$ is

$$J(\mathscr{E}_3) = \begin{bmatrix} r^q - \beta^q L_3 & 0 & 0 & 0 \\ e\beta^q L_3 & \Phi & 0 & -\sigma^q L_3 \\ 0 & \alpha_L^q & -(\mu_A^q + ud^q) & 0 \\ 0 & \rho Z_3 & 0 & -\eta^q + \rho L_3 \end{bmatrix}. \tag{3.134}$$

with

$$\Phi = b^q - (\mu_L^q + \alpha_L^q + ud^q) - \frac{2b^q L_3}{K_L^q}.$$

One can observe that $\lambda_1 = r^q - \frac{\beta^q\eta^q}{\rho}$, $\lambda_2 = -(\mu_A^q + ud^q)$ are some of the eigenvalues of the Jacobian matrix (3.134), hence matrix (3.134) reduces to

$$\widetilde{J}(\mathscr{E}_3) = \begin{bmatrix} b^q - (\mu_L^q + \alpha_L^q + ud^q) - \dfrac{2b^q L_3}{K_L^q} & -\sigma^q L_3 \\ \rho Z_3 & -\eta^q + \rho L_3 \end{bmatrix}. \tag{3.135}$$

From (3.135) we have the characteristic equation

$$\lambda^2 + \bar{a}_1 \lambda + \bar{a}_2 = 0, \tag{3.136}$$

with

$$\begin{aligned} \bar{a}_1 &= b^q \left(\frac{(\eta^q + \mu_L^q + \alpha_L^q + ud^q)}{b^q} - 1 \right) + 2\eta^q \left(\frac{b^q}{(\mu_L^q + \alpha^q + ud^q)} - 1 \right), \\ \bar{a}_2 &= \eta^q(\mu_L^q + \alpha^q + ud^q)\left(1 - \frac{b^q}{(\mu_L^q + \alpha^q + ud^q)} \right)\left(1 - \frac{\rho K_L^q}{b^q}\left(\frac{b^q}{(\mu_L^q + \alpha^q + ud^q)} - 1 \right) \right) \\ &\quad + 2(\eta^q)^2 \left(1 - \rho \frac{\eta^q K_L^q}{b^q}\left(\frac{b^q}{(\mu_L^q + \alpha^q + ud^q)} - 1 \right) \right)\left(\frac{b^q}{(\mu_L^q + \alpha^q + ud^q)} - 1 \right). \end{aligned} \tag{3.137}$$

Thus, if $r^q\rho < \beta^q\eta^q$ and condition $(C1)$ of Lemma 1.3 holds, then the equilibrium point $\mathscr{E}_3$ is locally asymptotically stable, otherwise it is unstable.

(**v**) The Jacobian matrix of system (3.110) evaluated at $\mathscr{E}_4$ is

$$J(\mathscr{E}_4)=\begin{bmatrix} r^q-\beta^q L_4-\dfrac{2r^q M_4}{K_M^q} & -\beta^q M_4 & 0 & 0 \\ e\beta^q(1-u)L_4 & \bar{n} & 0 & -\sigma^q L_4 \\ 0 & \alpha_L^q & -(\mu_A^q+ud^q) & 0 \\ 0 & 0 & 0 & -\eta^q+\rho L_4 \end{bmatrix}, \tag{3.138}$$

with $\bar{n}=b^q+e\beta^q(1-u)M_4-(\mu_L^q+\alpha_L^q+ud^q)-\dfrac{2b^q L_4}{K_L^q}$. The eigenvalues of $J(\mathscr{E}_4)$ are $\lambda_1=-(\mu_A^q+ud^q)$ and

$$\begin{aligned}\lambda_2 &= -\eta^q+\rho L_4 \\ &= -\eta^q-\frac{\rho r^q K_L^q}{r^q b^q+e(\beta^q(1-u))^2 K_M^q K_L^q}\left(1-\frac{b^q+e\beta^q(1-u)K_M^q}{(\mu_L^q+\alpha^q+ud^q)}\right).\end{aligned} \tag{3.139}$$

Hence, matrix (3.138) reduces to

$$\widetilde{J}(\mathscr{E}_4)=\begin{bmatrix} w_1 & -\beta^q M_4 \\ e\beta^q(1-u)L_4 & w_2 \end{bmatrix}, \tag{3.140}$$

with

$$\begin{aligned} w_1 &= r^q-\beta^q L_4-\frac{2r^q M_4}{K_M^q} \\ &= r^q-\frac{\beta^q r^q K_L^q}{r^q b^q+e(\beta^q(1-u))^2 K_M^q K_L^q}\left(\frac{b^q+e\beta^q(1-u)K_M^q}{(\mu_L^q+\alpha^q+ud^q)}-1\right) \\ &\quad -\frac{2r^q\left[r^q b^q+\beta^q K_L^q(\mu_L^q+\alpha_L^q+ud^q)\left(1-\dfrac{b^q}{(\mu_L^q+\alpha_L^q+ud^q)}\right)\right]}{r^q b^q+e(\beta^q(1-u))^2 K_M^q K_L^q}, \\ w_2 &= b^q+e\beta^q(1-u)M_4-(\mu_L^q+\alpha_L^q+ud^q)-\frac{2b^q L_4}{K_L^q} \\ &= b^q+e\beta^q(1-u)\frac{K_M^q\left[r^q b^q+\beta^q K_L^q(\mu_L^q+\alpha_L^q+ud^q)\left(1-\dfrac{b^q}{(\mu_L^q+\alpha_L^q+ud^q)}\right)\right]}{r^q b^q+e(\beta^q(1-u))^2 K_M^q K_L^q} \\ &\quad -(\mu_L^q+\alpha_L^q+ud^q)-\frac{2b^q r^q}{r^q b^q+e(\beta^q(1-u))^2 K_M^q K_L^q}\left(\frac{b^q+e\beta^q(1-u)K_M^q}{(\mu_L^q+\alpha^q+ud^q)}-1\right). \end{aligned}$$

From (3.140), the corresponding characteristic equation is

$$\lambda^2 + \tilde{a}_1\lambda + \tilde{a}_2 = 0, \tag{3.141}$$

with

$$\begin{aligned}
\tilde{a}_1 &= -(w_1 + w_2), \\
\tilde{a}_2 &= w_1 w_2 + \frac{e\beta^{2q}(1-u)^2 r^q K_L^q}{r^q b^q + e(\beta^q(1-u))^2 K_M^q K_L^q}\left(\frac{b^q + e\beta^q(1-u)K_M^q}{(\mu_L^q + \alpha_L^q + ud^q)} - 1\right).
\end{aligned}$$

Therefore, if $b^q + e\beta^q(1-u)K_M^q < (\mu_L^q + \alpha_L^q + ud^q)$ and condition $(C1)$ of Lemma 1.3 holds, then the equilibrium point $\mathscr{E}_4$ is locally asymptotically stable, otherwise it is unstable. Since all the variables are non-zero at the coexistence equilibrium point, it follows that matrix J (3.127) is the Jacobian matrix of system (3.110) at this equilibrium point. From (3.127) one can observe, that $\lambda_1 = -(\mu_A^q + ud^q)$ and the remainder can be obtained from the following reduced matrix:

$$\bar{J}(\mathscr{E}_5) = \begin{bmatrix} \bar{w}_1 & -\beta^q M_5 & 0 \\ e\beta^q L_5 & \bar{w}_2 & -\sigma^q L_5 \\ 0 & \rho Z_5 & \bar{w}_3 \end{bmatrix}. \tag{3.142}$$

where

$$\begin{aligned}
\bar{w}_1 &= r^q - \beta^q L_5 - \frac{2r^q M_5}{K_M^q}, \\
\bar{w}_2 &= b^q + e\beta^q(1-u)M_5 - \sigma^q Z_5 - (\mu_L^q + \alpha_L^q + ud^q) - \frac{2b^q L_5}{K_L^q}, \\
\bar{w}_3 &= -\eta^q + \rho L_5.
\end{aligned} \tag{3.143}$$

The corresponding characteristic equation at $\mathscr{E}_5$ becomes

$$\lambda^3 + a_1^*\lambda^2 + a_2^*\lambda + a_3^* = 0,$$

with

$$\begin{aligned}
a_1^* &= -(\bar{w}_1 + \bar{w}_2 + \bar{w}_3), \\
a_2^* &= \bar{w}_1(\bar{w}_2 + \bar{w}_2) + \bar{w}_2\bar{w}_3 + \sigma\rho L_5 Z_5 + e(\beta^q)^2 L_5 M_5, \\
a_3^* &= -\bar{w}_1(\sigma^q\rho L_5 Z_5 + \bar{w}_2\bar{w}_3) - e(\beta^q)^2 L_5 M_5 \bar{w}_3.
\end{aligned} \tag{3.144}$$

Since $\lambda_1 < 0$, it follows that condition $(C2)$ of Lemma 1.3 holds, then the equilibrium point $\mathscr{E}_5$ is locally asymptotically stable, otherwise it is unstable. This completes the proof. □

3.4.4 Global Stability Analysis of the Equilibrium Points

In this section, global stability of the equilibrium points of the model were investigated by constructing a Lyapunov functions. To simplify the analysis, it was assumed that $g_0(M) = r^q M(1-M/K_M)$ and $g_1(L,A) = b_L(1-L/K_L)A$.

Theorem 3.11

The trivial equilibrium point $\mathscr{E}_0$ is globally asymptotically stable whenever

$$eg_0(M)+g_1(L,A) \leq \frac{(\mu_L^q+\alpha_L^q+ud^q)(\mu_A^q+ud^q)A}{\alpha_L^q}+\frac{\sigma^q\eta^q}{\rho}Z.$$

Proof. Let us consider the following Lyapunov function

$$U_0(t) = eM(t)+L(t)+\frac{(\mu_L^q+\alpha_L^q+ud^q)}{\alpha_L^q}A(t)+\frac{1}{\rho}Z(t). \tag{3.145}$$

The fractional derivative of (3.145) along the solutions of system (3.110) leads to

$$\begin{aligned}
{}_a^cD_t^qU_0(t) &\leq {}_a^cD_t^q[eM(t)] + {}_a^cD_t^qL(t) + {}_a^cD_t^q\left[\frac{(\mu_L^q+\alpha_L^q+ud^q)}{\alpha_L^q}A(t)\right] + {}_a^cD_t^q\left[\frac{1}{\rho}Z(t)\right] \\
&= e[g_0(M)-\beta^q(1-u)LM]+g_1(L,A)+e\beta^q(1-u)LM-\sigma^qZL \\
&\quad -(\mu_L^q+\alpha_L^q+ud^q)L+\frac{(\mu_L^q+\alpha_L^q+ud^q)}{\alpha_L^q}\left[\alpha_L^qL-(\mu_A^q+ud^q)A\right] \\
&\quad +\frac{1}{\rho}\left[\rho\sigma^qLZ-\eta^qZ\right] \\
&= eg_0(M)+g_1(L,A)-\frac{(\mu_L^q+\alpha_L^q+ud^q)(\mu_A^q+ud^q)A}{\alpha_L^q}-\frac{\eta^q}{\rho}Z.
\end{aligned} \tag{3.146}$$

It follows that if $M(t)=M_0$, $L(t)=L_0$, $A(t)=A_0$ and $Z(t)=Z_0$, then ${}_a^cD_t^qU_1(t)=0$. However, if

$$eg_0(M)+g_1(L,A) \leq \frac{(\mu_L^q+\alpha_L^q+ud^q)(\mu_A^q+ud^q)A}{\alpha_L^q}+\frac{\sigma^q\eta^q}{\rho}Z<0,$$

then ${}_a^cD_t^qU_1(t)<0$ and the trivial equilibrium point $\mathscr{E}_0$ is globally asymptotically stable, otherwise it is unstable. This completes the proof. □

Theorem 3.12
The equilibrium point $\mathscr{E}_1$ is globally asymptotically stable whenever

$$\begin{aligned}&eg_0(M)\left(1-\frac{M^*}{M}+\beta^q(1-u)\frac{LM^*}{g_0(M)}\right)+g_1(L,A)-\frac{(\mu_L^q+\alpha_L^q+ud^q)(\mu_A^q+ud^q)A}{\alpha_L^q}\\&-\frac{\sigma^q\eta^q}{\rho}Z\le 0.\end{aligned} \tag{3.147}$$

Proof. Let us consider the following Lyapunov function:

$$\begin{aligned}U_1(t) &= e\left[M(t)-M_1-M_1\ln\left(\frac{M(t)}{M_1}\right)\right]+L(t)+\frac{(\mu_L^q+\alpha_L^q+ud^q)}{\alpha_L^q}A(t)\\&\quad+\frac{1}{\rho^q}Z(t).\end{aligned} \tag{3.148}$$

The fractional derivative of (3.148) along the solutions of system (3.110) leads to

$$\begin{aligned}{}_a^cD_t^qU_1(t) &\le e\left(1-\frac{M^*}{M(t)}\right){}_a^cD_t^qM(t)+{}_a^cD_t^qL(t)+{}_a^cD_t^q\left[\frac{(\mu_L+\alpha_L+ud)}{\alpha_L}A(t)\right]\\&\quad+{}_a^cD_t^q\left[\frac{1}{\rho}Z(t)\right]\\&= e\left(1-\frac{M_1}{M(t)}\right)(g_0(M)-\beta^q(1-u)LM)+g_1(L,A)\\&\quad+e\beta^q(1-u)LM-\sigma^qZL\\&\quad-(\mu_L+\alpha_L+ud)L+\frac{(\mu_L^q+\alpha_L^q+ud^q)}{\alpha_L^q}\left[\alpha_L^qL-(\mu_A^q+ud^q)A\right]\\&\quad+\frac{1}{\rho}\left[\rho\sigma^qLZ-\eta^qZ\right]\\&= eg_0(M)\left(1-\frac{M_1}{M}+\beta(1-u)\frac{LM_1}{g_0(M)}\right)+g_1(L,A)\\&\quad-\frac{(\mu_L+\alpha_L+ud)(\mu_A^q+ud^q)A}{\alpha_L^q}-\frac{\sigma^q\eta^q}{\rho}Z.\end{aligned} \tag{3.149}$$

It follows that if $M(t)=M_1$, $L(t)=L_1$, $A(t)=A_1$ and $Z(t)=Z_1$, then ${}_a^cD_t^qU_1(t)=0$. However, if

$$\begin{aligned}&eg_0(M)\left(1-\frac{M_1}{M}+\beta^q(1-u)\frac{LM_1}{g_0(M)}\right)+g_1(L,A)-\frac{(\mu_L^q+\alpha_L^q+ud^q)(\mu_A^q+ud)A}{\alpha_L^q}\\&-\frac{\sigma^q\eta^q}{\rho}Z<0,\end{aligned} \tag{3.150}$$

then ${}_{a}^{c}D_{t}^{q}U_{1}(t) < 0$ and the trivial equilibrium point $\mathscr{E}_1$ is globally asymptotically stable, otherwise it is unstable. This completes the proof. □

Theorem 3.13

The equilibrium point $\mathscr{E}_2$ is globally asymptotically stable whenever

$$\begin{aligned} & g_1(L_2,A_2)\left(1-\frac{L}{L_2}-\frac{L_2g_1(L,A)}{L_2g(L_2,A_2)}+\frac{g_1(L,A)}{g_1(L_2,A_2)}\right)+L_2\left(1+\frac{L}{L_2}-\frac{A}{A_2}-\frac{LA_2}{L_2A}\right) \\ & +eg_0(M)-e\beta^q(1-u)L_2M \leq 0. \end{aligned} \tag{3.151}$$

Proof. Consider the Lyapunov functional

$$\begin{aligned} U_1(t) \;=\; & eM(t)+\left[L(t)-L_2-L_2\ln\left(\frac{L(t)}{L_2}\right)\right]+\frac{1}{\alpha_L^q}\left[A(t)-A_2-A_2\ln\left(\frac{A(t)}{A_2}\right)\right] \\ & +\frac{1}{\rho}\left[Z(t)-Z_2-Z_2\ln\left(\frac{Z(t)}{Z_2}\right)\right]. \end{aligned} \tag{3.152}$$

The fractional derivative of (3.152) along the solutions of system (3.110) leads to

$$\begin{aligned} {}_{a}^{c}D_{t}^{q}U_2(t) \;\leq\; & e\,{}_{a}^{c}D_{t}^{q}M(t)+\left(1-\frac{L^3}{L(t)}\right){}_{a}^{c}D_{t}^{q}L(t)+\frac{1}{\alpha_L^q}\left(1-\frac{M^3}{M(t)}\right){}_{a}^{c}D_{t}^{q}A(t) \\ & +\frac{1}{\rho}\left(1-\frac{Z^3}{Z(t)}\right){}_{a}^{c}D_{t}^{q}Z(t). \end{aligned} \tag{3.153}$$

At the equilibrium point $\mathscr{E}_2$ we have the following identities:

$$(\mu_L^q+\alpha_L^q+ud^q)L_2=g_1(L_2,A_2)-\sigma^qZ_2L_2, \qquad (\mu_A^q+ud^q)A_2=\alpha_L^qL_2, \qquad \eta^q=\sigma^q\rho L_2. \tag{3.154}$$

Making use of these identities leads to

$$\begin{aligned} {}_{a}^{c}D_{t}^{q}U_2(t) \;\leq\; & g_1(L_2,A_2)\left(1-\frac{L}{L_2}-\frac{L_2g_1(L,A)}{L_2g_1(L_2,A_2)}+\frac{g_1(L,A)}{g_1(L_2,A_2)}\right) \\ & +L_2\left(1+\frac{L}{L_2}-\frac{A}{A_2}-\frac{LA_2}{L_2A}\right) \\ & +eg_0(M)-e\beta^q(1-u)L_2M. \end{aligned} \tag{3.155}$$

One can easily verify that at the equilibrium point $\mathscr{E}_3$, ${}_{a}^{c}D_{t}^{q}U_2(t)=0$ and ${}_{a}^{c}D_{t}^{q}U_2(t)<0$ if and only if:

$$g_1(L_2,A_2)\left(1-\frac{L}{L_2}-\frac{L_2g_1(L,A)}{L_2g_1(L_2,A_2)}+\frac{g_1(L,A)}{g_1(L_2,A_2)}\right)+L_2\left(1+\frac{L}{L_2}-\frac{A}{A_2}-\frac{LA_2}{L_2A}\right)$$

$$+eg_0(M) - e\beta^q(1-u)L_2M < 0. \tag{3.156}$$

Hence if the above condition holds then $\mathscr{E}_2$ is globally asymptotically stable. This completes the proof. □

Theorem 3.14

The equilibrium point $\mathscr{E}_3$ is globally asymptotically stable whenever

$$\begin{aligned} & g_1(L_3,A_3)\left(1-\frac{L}{L_3}-\frac{L_3g_1(L,A)}{L_3g_1(L_3,A_3)}+\frac{g_1(L,A)}{g_1(L_3,A_3)}\right)+L_3\left(1+\frac{L}{L_3}-\frac{A}{A_3}-\frac{LA_3}{L_3A}\right) \\ & +eg_0(M)-e\beta^q(1-u)L_3M-\eta^qZ\left(1-\frac{\sigma^q}{\eta^q}L_3\right)\leq 0. \end{aligned} \tag{3.157}$$

Proof. Consider the Lyapunov functional

$$\begin{aligned} U_3(t) &= eM(t)+\left[L(t)-L_3-L_3\ln\left(\frac{L(t)}{L_3}\right)\right]+\frac{1}{\alpha_L^q}\left[A(t)-A_3-A_3\ln\left(\frac{A(t)}{A_3}\right)\right] \\ &\quad +\frac{1}{\rho}Z(t). \end{aligned} \tag{3.158}$$

At the equilibrium point $\mathscr{E}_3$ we have the identities:

$$(\mu_L^q+\alpha_L^q+ud^q)L_3 = g_1(L_3,A_3), \qquad (\mu_A^q+ud^q)A_3 = \alpha_L^qL_3. \tag{3.159}$$

Utilizing these identities leads to the following result:

$$\begin{aligned} {}_a^cD_t^qU_3(t) &\leq g_1(L_3,A_3)\left(1-\frac{L}{L_3}-\frac{L_3g_1(L,A)}{L_3g_1(L_3,A_3)}+\frac{g_1(L,A)}{g_1(L_3,A_3)}\right) \\ &\quad +L_3\left(1+\frac{L}{L_3}-\frac{A}{A_3}-\frac{LA_3}{L_3A}\right) \\ &\quad +eg_0(M)-e\beta^q(1-u)L_3M-\eta^q\left(1-\frac{\sigma^q}{\eta^q}L_3\right). \end{aligned} \tag{3.160}$$

It follows that if $M(t)=M_3$, $L(t)=L_3$, $A(t)=A_3$ and $Z(t)=Z_3$, then ${}_a^cD_t^qU_3(t)=0$. However, if

$$\begin{aligned} & g_1(L_3,A_3)\left(1-\frac{L}{L_3}-\frac{L_3g_1(L,A)}{L_3g_1(L_3,A_3)}+\frac{g_1(L,A)}{g_1(L_3,A_3)}\right) \\ & +L_3\left(1+\frac{L}{L_3}-\frac{A}{A_3}-\frac{LA_3}{L_3A}\right) \end{aligned}$$

$$+eg_0(M)-e\beta^q(1-u)L_3M-\eta^q\left(1-\frac{\sigma^q}{\eta^q}L_3\right))<0, \tag{3.161}$$

then ${}_a^cD_t^qU_3(t)<0$ and it follows that equilibrium point $\mathscr{E}_3$ is globally asymptotically stable, otherwise it is unstable. This completes the proof. $\square$

Theorem 3.15

The equilibrium point $\mathscr{E}_4$ is globally asymptotically stable whenever

$$\begin{aligned}&g_0(M_4)\left(\frac{L}{L_4}+\frac{g_0(M)}{g_0(M_4)}-\frac{M_4}{M}\frac{g_0(M)}{g_0(M_4)}-\frac{LM}{L_4M_4}\frac{g_0(M)}{g_0(M_4)}\right)\\&+L_4\left(1+\frac{L}{L_4}-\frac{A}{A_4}-\frac{LA_4}{L_4A}\right)\\&+g_1(L_4,A_4)\left(1+\frac{g_1(L,A)}{g_1(L_4,A_4)}-\frac{L}{L_4}-\frac{L_4}{L}\frac{g_1(L,A)}{g_1(L_4,A_4)}\right)\\&+e\beta^q(1-u)L_4M_4\left(1+\frac{LM}{L_4M_4}-\frac{L}{L_4}-\frac{L_4g_1(L,A)}{Lg(L_4,A_4)}\right)\leq 0.\end{aligned} \tag{3.162}$$

Proof. Consider the Lyapunov functional

$$\begin{aligned}U_4(t) \;=\; &+\left[M(t)-M_4-M_4\ln\left(\frac{M(t)}{M_4}\right)\right]+\left[L(t)-L_4-L_4\ln\left(\frac{L(t)}{L_4}\right)\right]\\&+\frac{1}{\alpha_L^q}\left[A(t)-A_4-A_4\ln\left(\frac{A(t)}{A_4}\right)\right]\\&+\frac{1}{\rho}\left[Z(t)-Z_4-Z_4\ln\left(\frac{Z(t)}{Z_4}\right)\right].\end{aligned} \tag{3.163}$$

At the equilibrium point $\mathscr{E}_4$ we have the following identities:

$$\begin{aligned}&g_0(M_4)=\beta^q(1-u)L_4M_4,\\&g_1(L_4,A_4)+e\beta^q(1-u)L_4M_4-\sigma^qL_4Z_4=(\mu_L^q+\alpha_L^q+ud^q)L_4,\\&(\mu_A^q+ud^q)A_4=\alpha_L^qL_4,\qquad \sigma^q\rho L_4=\eta.\end{aligned} \tag{3.164}$$

Utilizing these identities leads to the following result:

$$\begin{aligned}{}_a^cD_t^qU_4(t) \;\leq\; & g_0(M_4)\left(\frac{L}{L_4}+\frac{g_0(M)}{g_0(M_4)}-\frac{M_4}{M}\frac{g_0(M)}{g_0(M_4)}-\frac{LM}{L_4M_4}\frac{g_0(M)}{g_0(M_4)}\right)\\&+g_1(L_4,A_4)\left(1+\frac{g_1(L,A)}{g_1(L_4,A_4)}-\frac{L}{L_4}-\frac{L_4}{L}\frac{g_1(L,A)}{g_1(L_4,A_4)}\right)\end{aligned}$$

$$+e\beta^q(1-u)L_4M_4\left(1+\frac{LM}{L_4M_4}-\frac{L}{L_4}-\frac{L_4g_1(L,A)}{Lg(L_4,A_4)}\right)$$
$$+L_4\left(1+\frac{L}{L_4}-\frac{A}{A_4}-\frac{LA_4}{L_4A}\right). \tag{3.165}$$

It follows that if $M(t)=M_4$, $L(t)=L_4$, $A(t)=A_4$ and $Z(t)=Z_4$, then ${}_a^cD_t^qU_4(t)=0$. However, if

$$g_0(M_4)\left(\frac{L}{L_4}+\frac{g_0(M)}{g_0(M_4)}-\frac{M_4}{M}\frac{g_0(M)}{g_0(M_4)}-\frac{LM}{L_4M_4}\frac{g_0(M)}{g_0(M_4)}\right)$$
$$+g_1(L_4,A_4)\left(1+\frac{g_1(L,A)}{g_1(L_4,A_4)}-\frac{L}{L_4}-\frac{L_4}{L}\frac{g_1(L,A)}{g_1(L_4,A_4)}\right)$$
$$+e\beta^q(1-u)L_4M_4\left(1+\frac{LM}{L_4M_4}-\frac{L}{L_4}-\frac{L_4g_1(L,A)}{Lg(L_4,A_4)}\right)$$
$$+L_4\left(1+\frac{L}{L_4}-\frac{A}{A_4}-\frac{LA_4}{L_4A}\right)<0, \tag{3.166}$$

then ${}_a^cD_t^qU_4(t)<0$ and it follows that equilibrium point $\mathscr{E}_4$ is globally asymptotically stable, otherwise it is unstable. This completes the proof. $\square$

3.4.5 Implications of Time-Dependent Farming Awareness Campaigns

In order to understand effects of time dependent farming awareness campaigns versus non-time dependent, model (3.110) was modified into an optimal control problem-whose main goal was to determine optimal farming awareness levels that are capable of minimizing FAW population or lead to eradication at minimal costs. Hence the constant awareness campaign parameter u in model (3.110) was remodeled to be time dependent, that is, $0\le u(t)\le u_{\max}<1$, where $u_{\max}$ is the upper bound of the control $u(t)$, which reflect the practical limitation on the maximum rate of control that can be implemented in a given time period.

Based on the aforementioned goal, the appropriate objective functional was mathematically formulated as follows:

$$J[u(t)]=\int_0^T\left[L(t)+A(t)+\frac{W}{2}u^2(t)\right]dt, \tag{3.167}$$

subject to the system

$$\left.\begin{array}{rcl}
{}_a^cD_t^qM(t) & = & r^qM\left(1-\dfrac{M}{K_M^q}\right)-\beta^q(1-u(t))LM,\\
{}_a^cD_t^qL(t) & = & b_L^qA\left(1-\dfrac{L}{K_L^q}\right)+e\beta^q(1-u(t))LM-\sigma^qZL\\
 & & -(\mu_L^q+\alpha_L^q+u(t)d^q)L,\\
{}_a^cD_t^qA(t) & = & \alpha_L^qL-(\mu_A^q+u(t)d^q)A,\\
{}_a^cD_t^qZ(t) & = & \rho\sigma^qLZ-\eta^qZ.
\end{array}\right\} \tag{3.168}$$

In equation (3.167), W is known as the weight constant. The weight constant over the prescribed time frame, is a measure of the relative costs of the interventions over a finite time horizon. The optimal control problem hence becomes that, we seek an optimal function, $u^*(t)$, such that $J(u^*(t)) = \min\limits_{\Omega} J(u(t))$ subject to the state equations in system (3.168) with initial conditions (3.169):

$$M(0)\geq 0, \qquad L(0)\geq 0, \qquad A(0)\geq 0, \qquad Z(0)\geq 0. \tag{3.169}$$

The Pontryagin's maximum principle (Pontryagin, 2018; Shell, 1969) was to determine the necessary conditions that optimal controls must satisfy. Through Pontryagin's maximum principle system (3.168) was transformed into an equivalent problem, namely the problem of minimizing the Hamiltonian $H(t)$ given by:

$$\left.\begin{array}{rcl}
H(t) & = & L(t)+A(t)+\dfrac{W}{2}u^2(t)\\
 & & +\lambda_1\left[r^qM\left(1-\dfrac{M}{K_M^q}\right)-\beta^q(1-u(t))LM\right]\\
 & & +\lambda_2\left[b_L^qA\left(1-\dfrac{L}{K_L^q}\right)+e\beta^q(1-u(t))LM-\sigma^qZL\right.\\
 & & \left.-(\mu_L^q+\alpha_L^q+u(t)d^q)L\right]\\
 & & +\lambda_3\left[\alpha_L^qL-(\mu_A^q+u(t)d^q)A\right]+\lambda_4\left[\rho\sigma^qLZ-\eta^qZ\right].
\end{array}\right\} \tag{3.170}$$

From (3.170), $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$ and $\lambda_4(t)$ are the adjoint variables corresponding to the states $M(t), L(t), A(t)$ and $Z(t)$.

Given an optimal control $u^*(t)$ and the corresponding state solutions M, L, A and Z, there exist adjoint functions $\lambda_i(t)$, $i = 1,2,3,4$ satisfying (3.171):

$$\left.\begin{aligned}
{}_a^cD_t^q\lambda_1(T-t) &= \left[r^q - \frac{2r^qM(T-t)}{K_M^q} - \beta^q(1-u(T-t))L(T-t)\right]\lambda_1(T-t)\\
&\quad +e\beta^q(1-u(T-t))L(T-t)\lambda_2(T-t),\\
{}_a^cD_t^q\lambda_2(T-t) &= 1-\beta^q(1-u(T-t))M(T-t)\lambda_1(T-t)+\alpha_L^q\lambda_3(T-t)\\
&\quad +\sigma^q\rho Z(T-t)\lambda_4(T-t)+e\beta^q(1-u(T-t))M(T-t)\lambda_2(T-t)\\
&\quad -\left[(\alpha_L^q+\mu_L^q+u(T-t)d^q-\frac{b_L^qA(T-t)}{K_L}+\sigma^qZ(T-t)\right]\lambda_2(T-t),\\
{}_a^cD_t^q\lambda_3(T-t) &= 1-(\mu_A^q+u(T-t)d^q)\lambda_3(T-t)+b_L\left(1-\frac{L(T-t)}{K_L^q}\right)\lambda_2(T-t),\\
{}_a^cD_t^q\lambda_4(T-t) &= -\sigma^qL(T-t)\lambda_2(T-t)+(\sigma^q\rho L(T-t)-\eta^q)\lambda_4(T-T),
\end{aligned}\right\} \tag{3.171}$$

with transversality conditions $\lambda_i(T) = 0$ for $i = 1,2,3,4$. Furthermore, the optimal controls are characterized by the optimality conditions (3.172):

$$u(t) = \min\left\{\max\left\{0, \frac{(e\beta^qM+d^q)L\lambda_2+d^qA\lambda_3-\beta^qLM\lambda_1}{W}\right\}, u_{\max}\right\}. \tag{3.172}$$

3.5 Chapter Overview

In this chapter, three mathematical models meant to understand the effects of FAW infestation in a maize field on the final maize biomass. In the first model the implications of memory effects and control on FAW-maize interaction were assessed through a fractional-order mathematical framework. The second model was developed with a goal to understand the effects of seasonality on FAW-maize interaction resulting to changes on the final maize biomass. All development stages of the plant and pest which were considered to be strongly influenced by climate and weather changes were modeled by periodic functions. The third model was developed to assess the implications of natural FAW predators and mass farming media campaigns on minimizing maize destruction by FAW during an outbreak. The analytical and numerical results in all the proposed models in this chapter are in Chapter 4.

CHAPTER FOUR

RESULTS AND DISCUSSION

4.1 Introduction

In this chapter, mathematical models of maize-FAW interactions presented in the chapter three were simulated in matrix laboratory (MATLAB) programming language to support analytical findings and demonstrate the strength and impact of mitigation strategies.

4.2 Model Parametrization

In this section, a brief discussion on the baseline values for model parameters used is presented. FAW infestations in maize fields is a recent phenomena in Africa (Rukundo *et al.*, 2020), hence data remains scarce. Despite this challenge the present study made use of literature to draw baseline values for the model parameters. Below is the comprehensive discussion on baseline values for model parameters.

(i) Natural mortality rate of adult FAW μ_A : The life span of female adult FAW is (15-21) days. According to Westbrook *et al.* (2016), a female adult moth with a life span of 18 days can oviposit about 125 eggs per day. It follows that the natural mortality rate of the moth as defined by Hoenig *et al.* (1983) is:

$$\mu_A = \frac{1}{\text{expected lifetime}}.$$

Hence, in our simulations we set $\mu_A = 1/18$ per day.

(ii) Egg laying rate b and life span of adult moth μ_A^{-1}: During its entire life span of (15-21) days, the FAW adult female's total egg production per female averages about 1 500 with a maximum of over 2 000 (FAO, 2018*b*). The average daily egg laying rate can be expressed as follows:

$$\text{eggs laid per day} = \frac{\text{eggs laid in a lifetime}}{\text{expected lifetime}}.$$

Based on Westbrook *et al.* (2016), in our simulations $b = 125$ eggs per day.

(iii) Egg hatching rate α_E and gender ratio W: Mathematically the egg hatching rate is the inverse of average duration of the egg stage, that is,

$$\alpha_E = \frac{1}{\text{Average duration of the egg stage}}.$$

Despite being climate dependent, the duration of egg stage takes average period of (2-5) days (FAO, 2018*b*; Assefa & Ayalew, 2019). Westbrook *et al.* (2016) estimated a gender ratio of 50:50 males/females.

(iv) Average duration of the larval stage α_L^{-1} : The duration of the larval stage is influenced by climate changes. During summer periods the larval stage is about 14 days and 30 days during cool weather (FAO, 2018*b*; Assefa & Ayalew, 2019).

(v) Average duration of the pupal stage α_P^{-1} : Similar to the larval stage, the pupal stage also depends of climate. It is about 8-9 days during summer, but reaches 20-30 days during the winter (FAO, 2018*b*; Assefa & Ayalew, 2019).

Table 3: Model parameters and their baseline values

Symbol	Definition	Source
b	125 Eggs per moth per day	(FAO, 2018*b*; Westbrook *et al.*, 2016)
w	0.5	(FAO, 2018*b*; Westbrook *et al.*, 2016)
α_E^{-1}	3(3-5) Days	(FAO, 2018*b*; Assefa & Ayalew, 2019)
α_L^{-1}	14(14-30) Days	(FAO, 2018*b*; Assefa & Ayalew, 2019)
α_P^{-1}	9(8-30) Days	(FAO, 2018*b*; Assefa & Ayalew, 2019)
μ_A^{-1}	18(15-21) Days	(Westbrook *et al.*, 2016; Sisay *et al.*, 2019)
K_M	50 biomass plant^{-1}	(Al Basir *et al.*, 2019; Liu *et al.*, 2020).
K_E	10^8	(Al Basir *et al.*, 2018; Bokil *et al.*, 2019).
K_L	10^6	(Al Basir *et al.*, 2018; Bokil *et al.*, 2019).
μ_E, μ_L, μ_P	0.01 Day^{-1}	(Faithpraise *et al.*, 2015; Al Basir *et al.*, 2019)
β	5×10^{-8} Day^{-1}	(Faithpraise *et al.*, 2015; Bokil *et al.*, 2019)
r	0.05 Day^{-1}	(Al Basir *et al.*, 2018, 2019; Liu *et al.*, 2020)
e	0.2 Day^{-1}	(Faithpraise *et al.*, 2015; Bokil *et al.*, 2019)
u_E, u_L, u_P, u_A	0.3	(Al Basir *et al.*, 2018, 2019; Bokil *et al.*, 2019)

4.3 Simulation Results for FAW-Maize Interaction Model with Memory Effects and Control

4.3.1 Sensitivity Analysis of the Reproduction Number

The analytical results of the FAW-maize biomass interaction model showed that $\mathscr{R}_0$ is an important threshold parameter for the persistence and extinction of FAW during any outbreak. Since the parameters of the proposed model were mainly drawn from literature, there is need to investigate the influence of each parameter on the magnitude of $\mathscr{R}_0$ so as to understand the uncertainty regarding their values. In order to infer on the relationship between model parameters and individual parameters, sensitivity analysis of $\mathscr{R}_0$ was performed following the approach in (Arriola & Hyman, 2005).

Definition 4.7

The normalized sensitivity index of $\mathscr{R}_0$, which depends differentiably on a parameter say κ, is defined by

$$\Theta_{\kappa}^{\mathscr{R}_0} = \frac{\partial \mathscr{R}_0}{\partial \kappa} \times \frac{\kappa}{\mathscr{R}_0}. \tag{4.1}$$

Model parameters whose sensitivity indices are positive will increase the size of $\mathscr{R}_0$ whenever they are increased while those with negative indices decreases $\mathscr{R}_0$ whenever they are increased. It follows from (4.1) that the normalized sensitivity of $\mathscr{R}_0$ with regard to the model parameters that define it was obtained (4.2):

$$\left.\begin{array}{rcl}
\Theta_{b}^{\mathscr{R}_0} & = & 1, \qquad \Theta_{W}^{\mathscr{R}_0} = 1, \qquad \Theta_{\alpha_i}^{\mathscr{R}_0} = \dfrac{\mu_i + u_i}{\mu_i + \alpha_i + u_i} > 0, \\[2ex]
\Theta_{\mu_i}^{\mathscr{R}_0} & = & -\dfrac{\mu_i}{\mu_i + \alpha_i + u_i} < 0, \qquad \Theta_{\mu_A}^{\mathscr{R}_0} = -\dfrac{\mu_A}{\mu_A + u_A} < 0, \\[2ex]
\Theta_{u_i}^{\mathscr{R}_0} & = & -\dfrac{u_i}{\mu_i + \alpha_i + u_i} < 0, \qquad \Theta_{u_j}^{\mathscr{R}_0} = -\dfrac{u_A}{\mu_A + u_A} < 0, \\[2ex]
\text{for} & & i = E, P, L \ \text{ and } \ j = E, L, P, A.
\end{array}\right\} \tag{4.2}$$

Results in (4.2) demonstrated that model parameters b, W and α_i increase the size of $\mathscr{R}_0$ whenever they are increased whereas model parameters μ_j and u_j decrease the size of $\mathscr{R}_0$ whenever they are increased. Precisely an increase in either b or W by 10% may result in an increase in the magnitude of $\mathscr{R}_0$ by 10%. However, an increase by 10% of α_i will increase the size of $\mathscr{R}_0$ by a value less than 10%.

In addition, it was also noted that, the model parameter u_j has a negative effect on $\mathscr{R}_0$ implying that intervention strategies have an impact on the extinction and persistence of FAW in the environment. Using the baseline values in Table 3, the numerical outcomes for mathematical expression in (4.2) were obtained as presented in Table 4 and the graphic illustration is in Fig. 12.

Table 4: Sensitivity index of $\mathscr{R}_0$

Parameter	b	W	u_A	α_L	u_L	α_P	u_P	μ_A	α_E	u_E	μ_E	μ_L	μ_P
Sensitivity index	+1	+1	-0.64	+0.61	-0.55	+0.50	-0.45	-0.36	+0.25	-0.02	-0.05	-0.23	-0.06

Numerical results in Table 4 showed that pest control intervention strategies will have more effects on minimizing the FAW population in the field if such strategies target the adult FAW population.

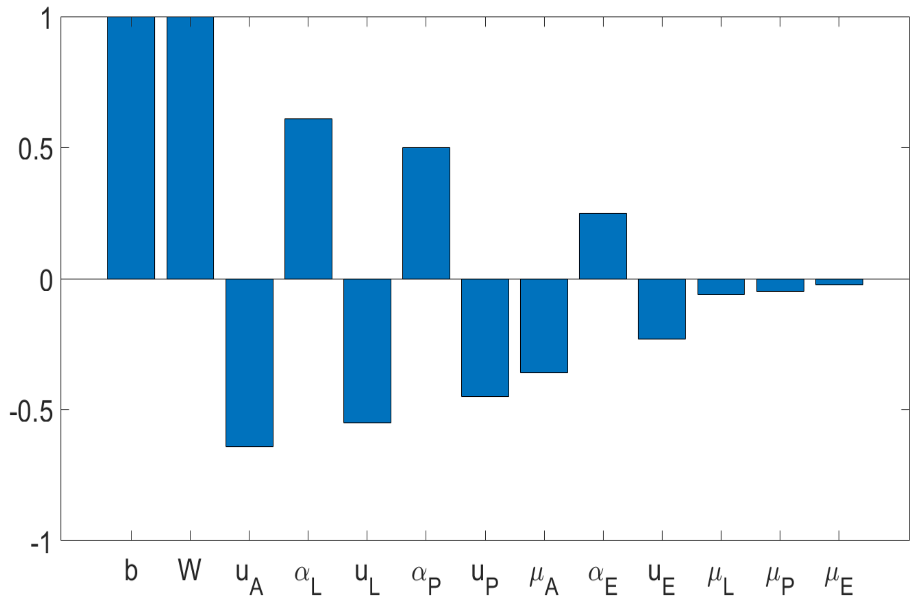
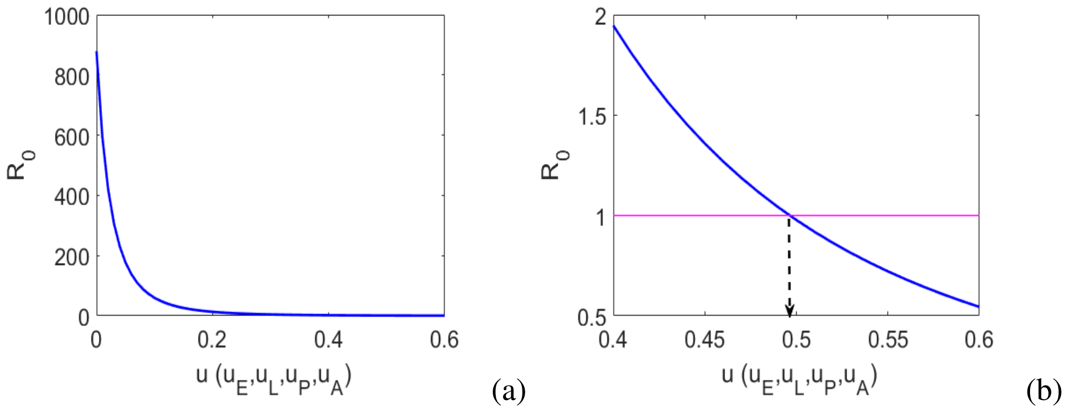


Figure 12: **Sensitivity analysis of $\mathscr{R}_0$ with respect to model parameters. Baseline values used are in Table 3**

The simulation results in Fig. 13 demonstrates the effects of varying the intervention strategies on extinction and persistence of pests in the field. The results showed that any value of $u > 0.5$ will lead to the extinction of the pest and persistence of the pests occurs for $u < 0.5$.



(a) (b)

Figure 13: **Effects of varying intervention strategies on the magnitude of $\mathscr{R}_0$**

4.3.2 Population Level Effects

In this section, numerical solutions determined from the simulations of model (3.6) in MATLAB programming language are presented. The algorithms utilized to simulated fractional-order model are comprehensively presented in (Diethelm, 2010). On simulating system (3.6) the following assumed initial population levels were considered: $E(0) = 1000$, $L(0) = P(0) = 0$, $A(0) = 500$ and $M(0) = 15$.

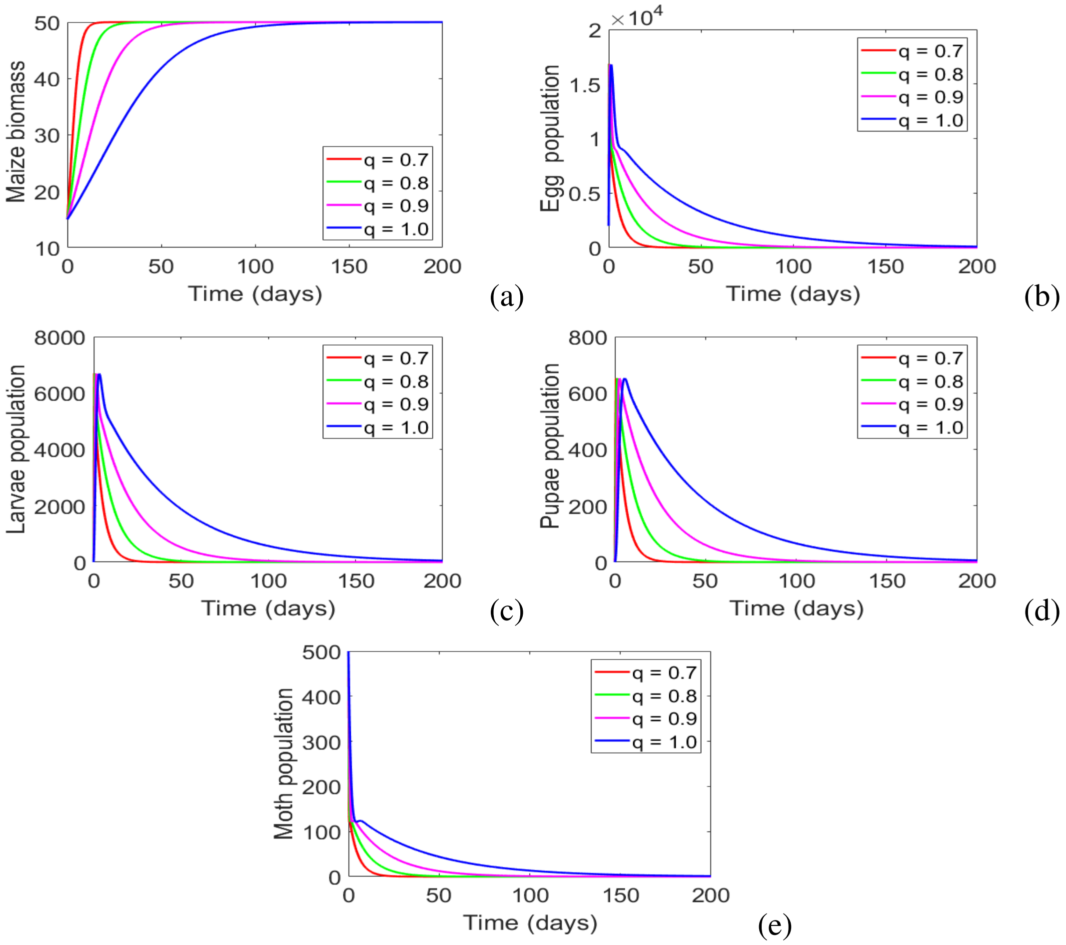


Figure 14: Numerical results of system (3.6) demonstrating the convergence of solutions to the pest-free equilibrium for $\mathscr{R}_0 \leq 1$

Numerical results in Fig. 14 demonstrated the dynamics of the pest and maize biomass whenever the reproduction number, $\mathscr{R}_0$, is less than unity that is $\mathscr{R}_0 = 0.8630$. As one can observe, if the moth cannot produce more than one off-spring then within a period of 150 days all the FAW populations (eggs, larvae, pupae and moth) will become extinct while the maize biomass will increase with time till it reaches the expected maximum biomass per plant (50 biomass plant^{-1}). Moreover, it is evident that convergence of solutions to their respective limiting points in time depends on the fractional-order, q, as q approaches unity the time taken by solutions to converge to the limiting point increases.

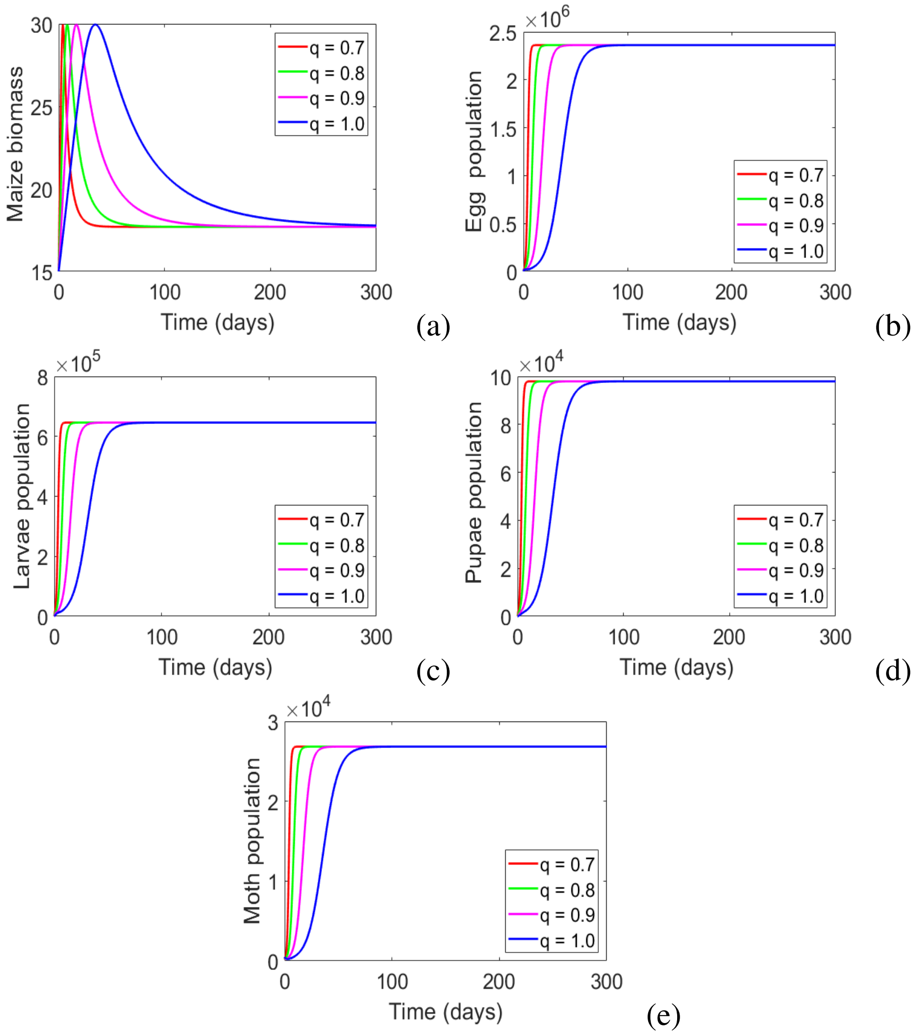


Figure 15: **Numerical results of system (3.6) demonstrating the convergence of solutions to the pest persistence equilibrium point for $\mathscr{R}_0 > 1$**

From the simulation results shown in Fig. 15, one can observe that whenever each female moth reproduce more than one off-spring, that is, $\mathscr{R}_0 > 1$ ($\mathscr{R}_0 = 2.8931$), then the pest population will persist in the field till the final harvesting time, $t = 150$ day. In addition, the final maize biomass per plant will be less than the expected 50 biomass plant^{-1}. Precisely, maize biomass will increase from the start and reaches a maximum of 50 biomass plant^{-1} , after approximately 100 days and after that it decreases gradually till it stabilizes at approximated 18 biomass plant^{-1}.

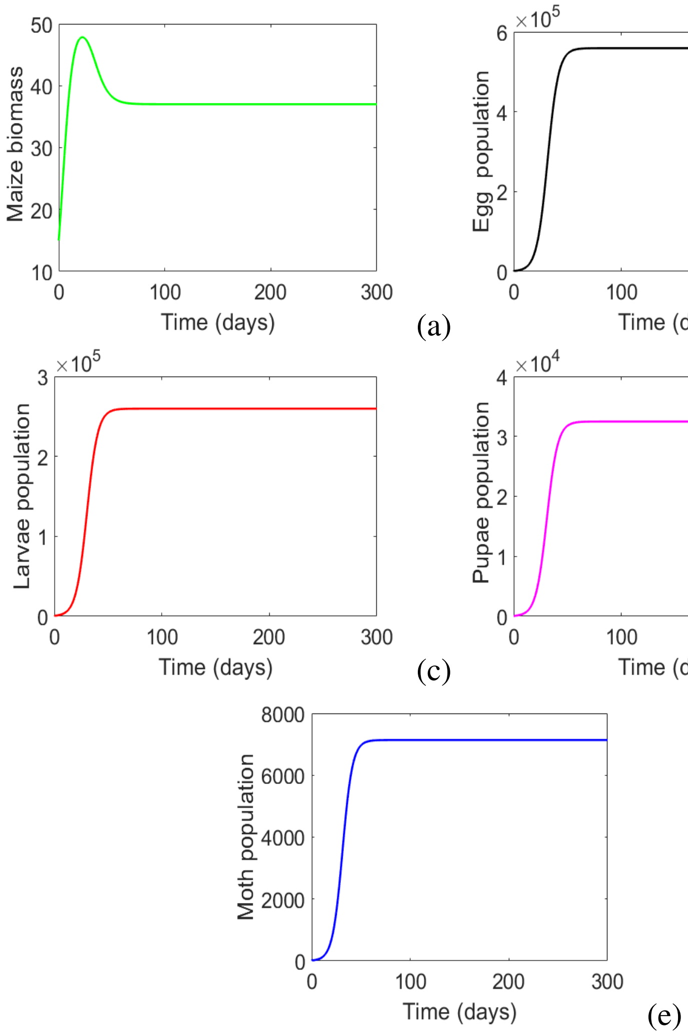


Figure 16: **Model solutions showing the effects of a FAW outbreak with a small initial pest life cycle population with fixed** $q = 0.8$ **and** $\mathscr{R}_0 = 1.3583$

Figure 16 shows the solutions of model system (3.6) for an experiment set up with small population sizes for the pest, that is, $E(0) = 100$, $L(0) = P(0) = 0$ and $A(0) = 50$, together with a control rate of $u_E = u_L = u_P = u_A = 0.45$ day $^{-1}$ leading to $\mathscr{R}_0 = 1.3583$. Furthermore, q has been fixed to 0.8. As one can observe, the pest population will increase rapidly within the first 100 days and stabilize thereafter. The maize biomass will also increase during the first 50 days and attains a maximum approximately close to the expected value, 50 biomass plant^{-1}, there after the biomass decreases gradually for approximately 50 days before it becomes stable at approximately 40 biomass plant^{-1}. Overall, the egg population will dominate all the pest populations.

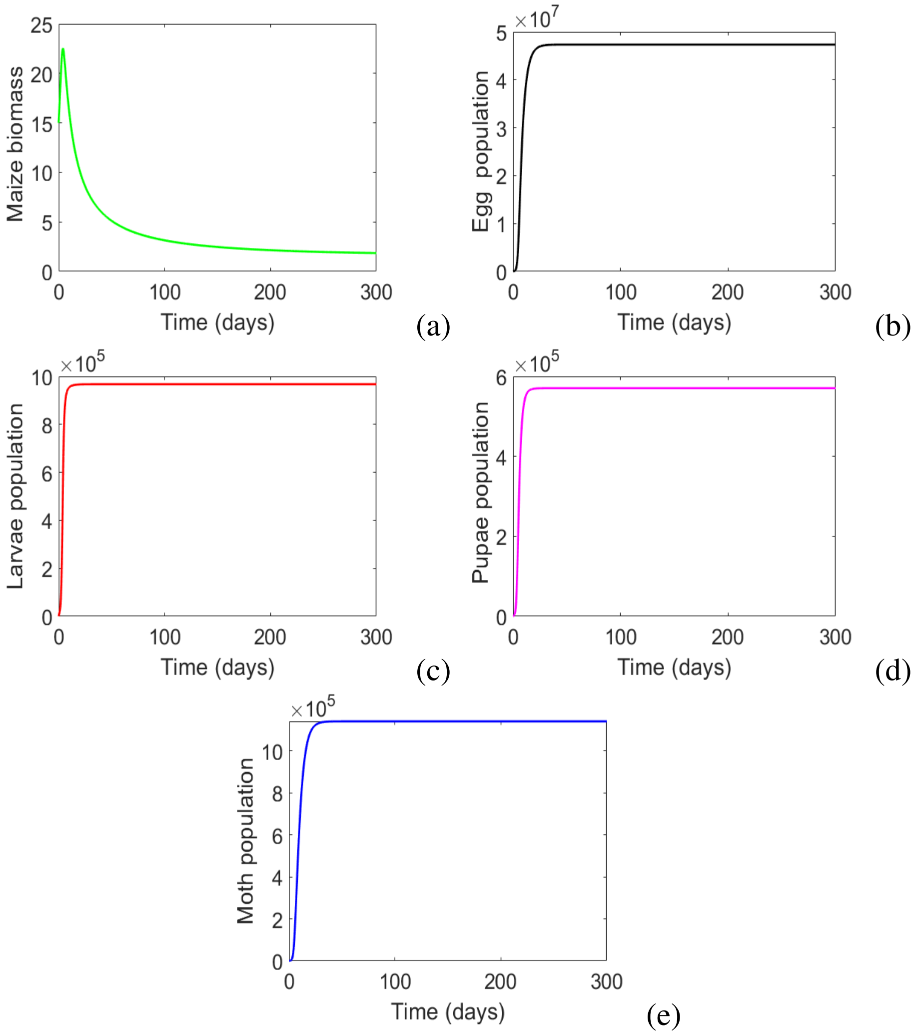


Figure 17: **Model solutions showing the effects of a FAW outbreak with a large initial pest life cycle population with fixed $q = 0.8$ and $\mathscr{R}_0 = 58$**

Numerical results in Fig. 17 depicts the effects of a FAW outbreak with a large initial pest life cycle population, $E(0) = 2000$, $L(0) = P(0) = 0$ and $A(0) = 15$, combined with less effective control measures, that is, $u_E = u_L = 0.45$, and $u_E = u_L = 0.45$, $u_P = u_A = 0$. More often pesticides which are known to effectively control FAW are expensive such that farmers in some areas rely on traditional methods of controlling the pest such as hand picking of the larvae, picking and destroying of egg masses, spraying lime, salt, oil and soap solution.

Prior studies suggests that traditional methods are less effective and are likely to eliminate the egg and larvae population. Hence, simulation in Fig. 17 were done to explore the effects of a FAW outbreak with a large initial pest life cycle population coupled with less effective control measures. As one can observe, an outbreak with a large pest population coupled with less effective control measure may result in the pest population increasing rapidly such that in less than 100 days they will reach their respective maximum. After an initial increase, the maize biomass would gradually decrease to a level below its initial biomass. The results highlight the importance of effective control measure on increasing maize biomass whenever there is a FAW outbreak.

4.4 Numerical Results for FAW-Maize Interaction Model with Seasonality

In this section, simulation results for model (3.64) are presented. Since model (3.64) is non-autonomous it follows that all model parameters which were modeled as periodic functions have a baseline value also known as the time averages of their respective periodic functions. In this section, baseline values in Table 3 were considered as time averages for their respective parameters. For example, $r = r_0$, $\beta = \beta_0$ and so on. In all the simulations the amplitude of oscillations was set to 0.8 and initial population levels were assumed as follows: $E(0) = 500$, $L(0) = 0$, $P(0) = 0$, $A(0) = 500$, and $M(0) = 10$. Additional parameters that are not in Table 3 are defined in Table 5:

Table 5: Model parameters and their baseline values

Parameter definition	Symbol	Baseline value	Source
Average density dependent mortality rate	θ_0	0.008 Day^{-1}	(Siekmann, 2009).
Half saturation constant	a_0	0.8	(Siekmann, 2009).

Since model (3.98) is an optimal control problem, simulations were performed by closely following the forward-backward sweep method (Lenhart & Workman, 2007). Precisely, the Simulation results presented in this Section are meant to explore the following scenarios:

(i) Effects of implementing traditional control measures alone,

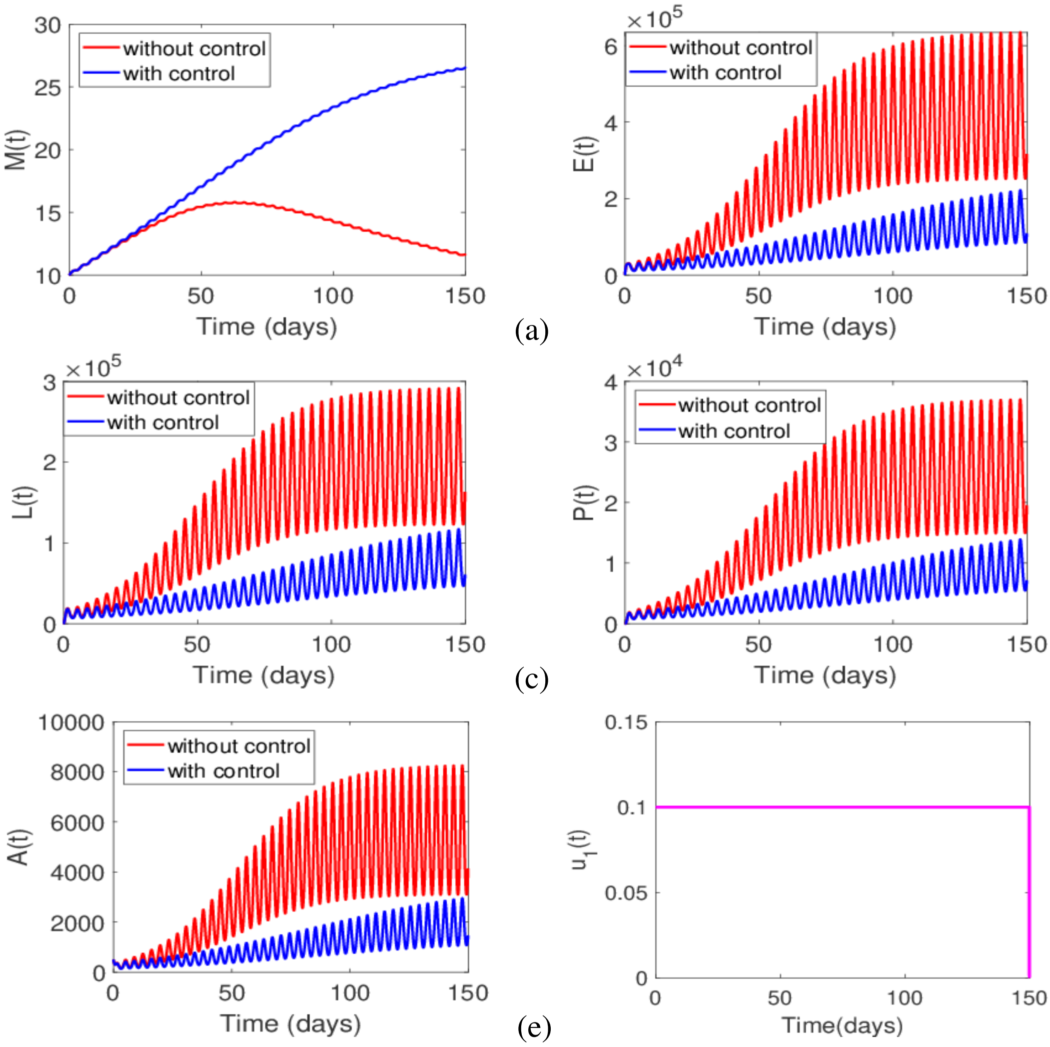
(ii) Effects of implementing time-dependent use of chemical insecticides alone, and

(iii)) Effects of combining time-dependent traditional methods with use of chemical insecticides.

In addition, since the impact of larvae on maize is so apparent, it was assumed that $C_1 \leq C_2$, that is, the minimization of the larvae is more important than that of FAW eggs. Furthermore, traditional methods of controlling FAW are known to be less costly compared to chemical insecticides and as such, it was assumed that $W_1 < W_2$.

4.4.1 Effects of Implementing Traditional Control Measures Alone

In resource limited settings, majority of the farmers cannot purchase pesticides to control FAW whenever there is an outbreak and more often they rely on traditional methods like handpicking and destroying of egg masses and larvae. Here, the goal is to understand the effects of time dependent implementation of such methods on the dynamics of FAW and maize interaction. To investigate this scenario model (3.98) was simulated with $u_1 \neq 0$ $(0 \leq u_1 \leq 0.1)$ and $u_2 \neq 0$ and the results obtained are in Fig. 18.



(a) (b) (c) (d) (e) (f)

Figure 18: Solution of model (3.98) with and without optimal control under scenario 1: $u_1 \neq 0$ **and** $u_2 = 0$**. We set** $0 \leq u_1(t) \leq 0.1$

As one can observe, the dynamics of the maize biomass and FAW populations, with and without control will be associated with oscillations which reflects seasonal variations. Further, the results also shows that without control, the maize biomass may not exceed 15 per plant, however, with timely control the biomass may exceed 25 per plant by the final time horizon ($t = 150$). Moreover, although, traditional methods will be capable of reducing FAW population and increasing maize biomass, they will not be able to completely eliminate the pest. Figure 18(f) portrays the optimal control profiles for $u_1(t)$. In addition, one can also note that the control profile for the control $u_1(t)$ starts from the maximum ($u_1 = 0.1$) and stays at that level for the entire duration. From the pattern of the optimal control profile we can conclude that a desirable outcome can be achieved only if the traditional methods are implemented throughout the entire time horizon.

4.4.2 Effects of Implementing Time-Dependent use of Chemical Insecticides Alone

Despite being expensive, chemical insecticides are known to be more efficient compared to traditional methods on controlling FAW. To explore the impact of chemical control measures on FAW dynamics model (3.98) was simulated with $u_1 = 0$ and $0 < u_2(t) < 0.1$ and the results are depicted in Fig. 19.

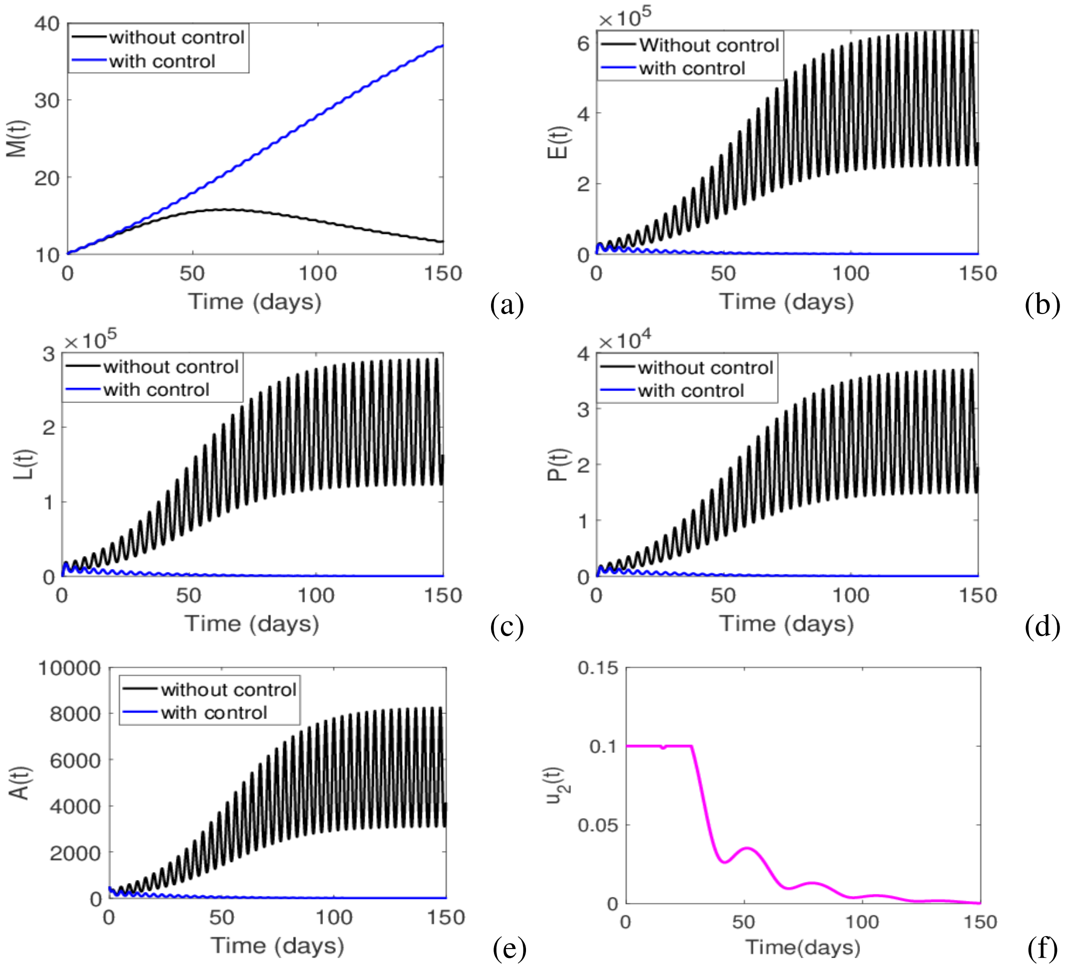


Figure 19: Solution of model (3.98) with and without optimal control under scenario 2: $0 < u_1 \neq 0$ **and** $0 < u_2 \leq 0.1$

When chemical insecticides are used, one can note that the population of FAW may become extinct in a period of 50 days. Moreover, the maize biomass per plant may exceed 35 per plant by the final time ($t = 150$ days). Comparing Fig. 18 and Fig. 19, one can conclude that use of chemical insecticides should be encouraged since the final biomass will be higher compared to when farmers rely on traditional methods only. The control profile for control $u_2(t)$ starts from the maximum initially, but only for a very short time ($t < 50$), followed by a decrease to some lower level till the final time horizon. This may attribute to the decrease in FAW populations. Hence, one can conclude that for chemical insecticides, intensity use needs to be maintained at maximum for a period of approximately 50 days, thereafter the intensity may be reduced till the final time.

4.4.3 Effects of Combining Time-Dependent Traditional Methods with use of Chemical Insecticides

To understand the implications of combining traditional methods with chemical insecticide use, model (3.98) was simulated with $0 < u_1(t) \leq 0.1$ and $0 < u_2(t) \leq 0.1$ over period of 150 days and the solution results are depicted in Fig. 20.

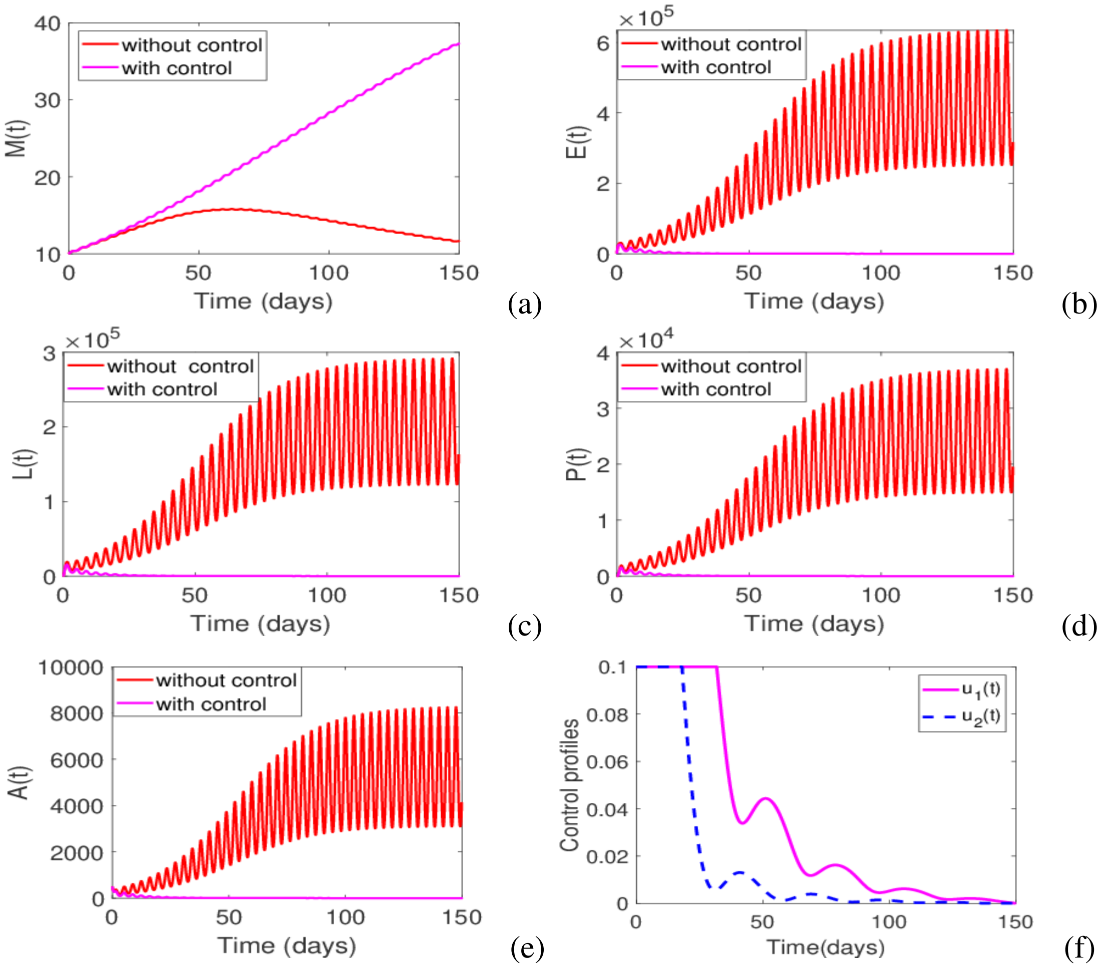


Figure 20: Solution of model (3.98) with and without optimal control under scenario 3: $0 < u_1 \leq 0.1$ **and** $0 < u_2 \leq 0.1$

One can note that when traditional methods are combined with chemical insecticides use then the time taken to eliminate the FAW from the field is less than the time that will be taken if only chemical insecticides are in use (Fig. 20). Although the time required to eliminate the FAW populations will decrease, the final maize biomass may not be significantly different to that obtained when only chemical insecticides are in use (Fig. 19).

In Figure 20 (f), one can observe that the control profiles for $u_1(t)$ and $u_2(t)$ starts at their respective maximum initially, but only for a very short time, followed by a decrease to some lower level till the final time. It is worth noting that the control profile for $u_1(t)$ remain at its maximum for a slightly longer period compared to that of $u_2(t)$ and this can be attributed to less cost associated with traditional methods relative to chemical insecticides use. As such we can conclude that when traditional methods are combined with chemical insecticides use, chemical control efforts may be ceased after approximately 50 days and the traditional methods can be implemented for additional 50 days or more but at low intensity.

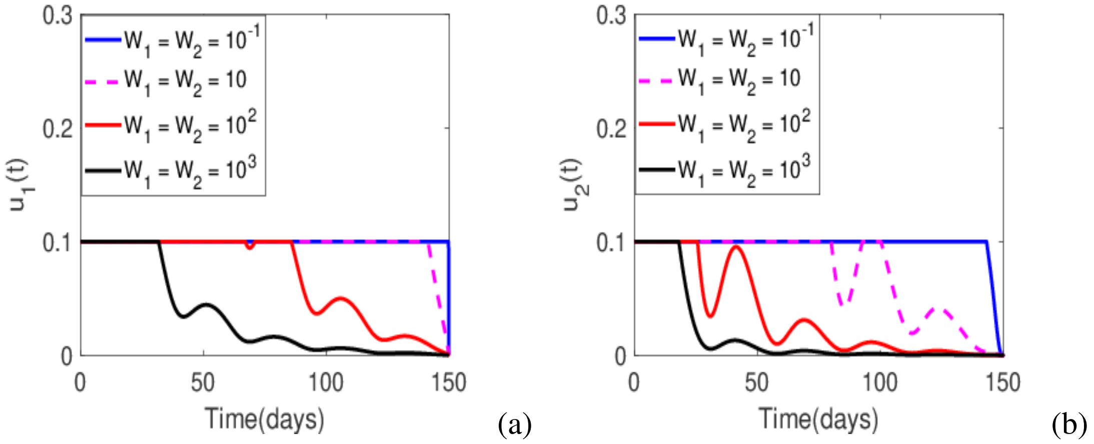


Figure 21: Simulation results for model system (3.98) illustrating the effects of varying the weights, W_1 and W_2 with $0 \leq u_1 \leq 0.1, 0 \leq u_2 \leq 0.1$

To assess the effects of costs on implementing the control efforts $u_1(t)$ and $u_2(t)$, the weight constants W_1 and W_2 were varied and the results are illustrated in Fig. 21. From the results one can note that if the costs are low, for example $W_1 = W_2 = 0.1$ then the associated control profile starts at their respective maximum and stays there till the final time horizon. However, as the costs increases the respective control profile starts at their respective maxima and stays there for a reduced duration compared to when the costs are low. In particular, as the costs increases the control profile for $u_2(t)$ stays at its maxima for a relatively short duration compared to that of $u_1(t)$. In a nutshell, one can deduce that depending on the cost parameters associated with the control, the optimal profiles of $u_1(t)$ and $u_2(t)$ stays at their respective maxima for a length duration, before eventually settling at their minimum levels.

4.5 Numerical Results for FAW-Maize Interaction Model with Farming Awareness and Larvae Predation

In this section, simulation results for model (3.168) are presented. Additional model parameters to those in Table 3 are given in Table 6. Since model (3.168) is an optimal control problem the simulation algorithm was adopted from (Lenhart & Workman, 2007) and initial population levels were assumed as follows: $M(0) = 15$, $L(0) = 500$, $A(0) = 100$, and $Z(0) = 50$.

Table 6: Model parameters and their baseline values

Symbol	Definition	Baseline value	Source
b_L	Growth rate of larva	1/14 day^{-1}	(Pearce *et al.*, 2006)
σ	Consumption rate of larva by predators	$5 \times 10^{-5}\text{Day}^{-1}$	(Pearce *et al.*, 2006).
ρ	Conversion rate of prey to predator	0.1 Day^{-1}	(Siekmann, 2009).
d	Mortality of FAW due to intervention strategies	0.01 Day^{-1}	Estimate.
η^{-1}	Average life span of predator	100 Days	(Pearce *et al.*, 2006).

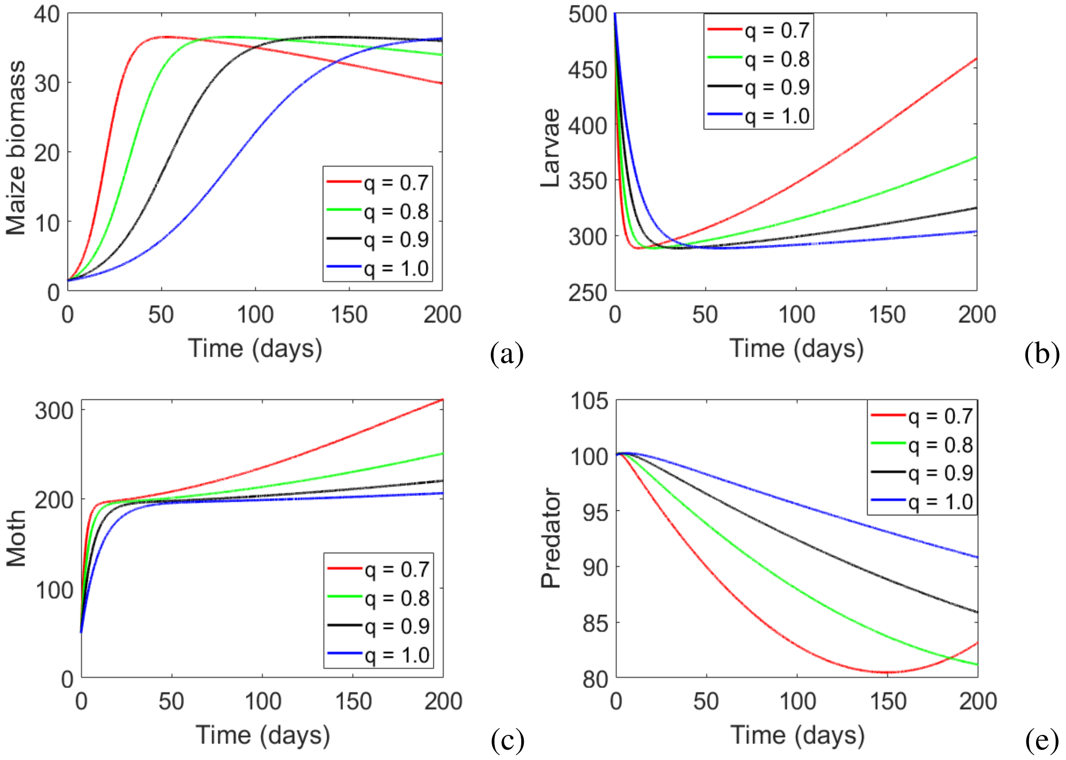


Figure 22: **Simulation results of model (3.168) with constant farming awareness $u = 0.1$ and different fractional order values**

Simulation results in Fig. 22 shows the impact of none time dependent farming awareness campaigns on minimizing or eradicating FAW in the maize field. From the simulation results one can observe that at this level of farming awareness, the maize biomass increases from the start and converges to 35 biomass per plant which is less than the expected 50 biomass per plant. This suggest that farming awareness will minimizing the effects of FAW on maize biomass to some extent but will not be highly effective for the farmer to achieve the expected biomass per plant. However, in Fig. 23 one can observe that if $u = 0.7$, then the level of maize biomass converges to the expected level even at different fractional order values. Thus, as the awareness level increases to levels close to 100% ($u = 1$), the FAW population decreases significantly and the final biomass level obtained reaches expected levels.

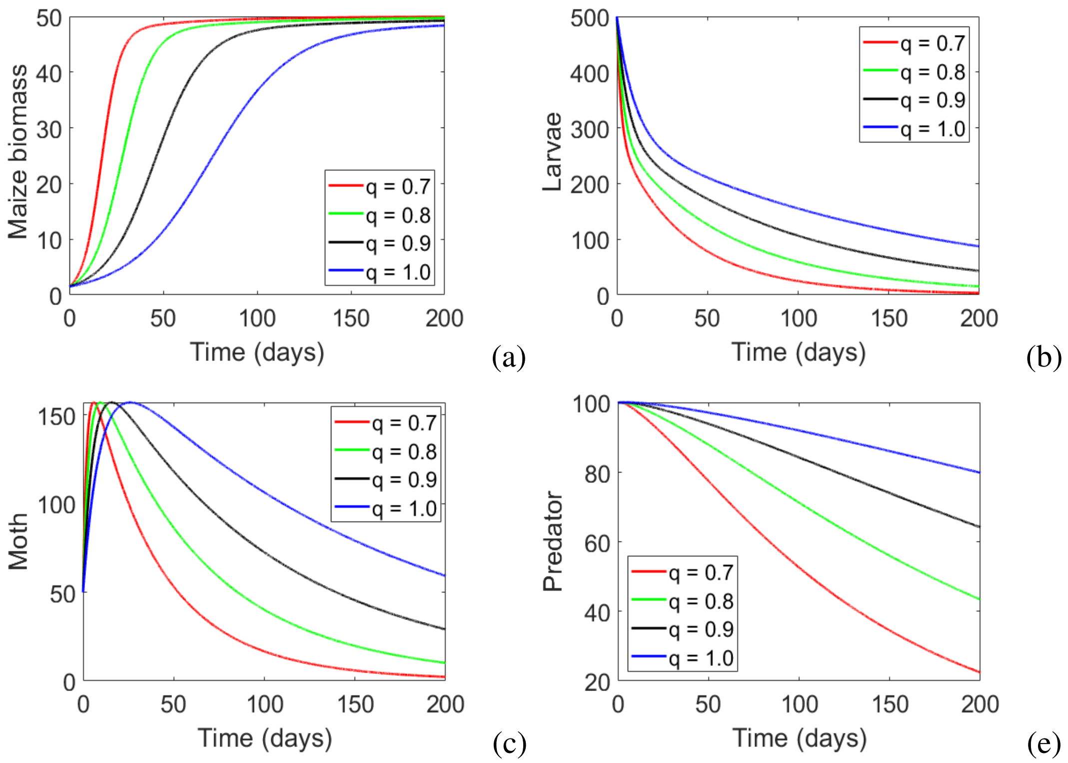
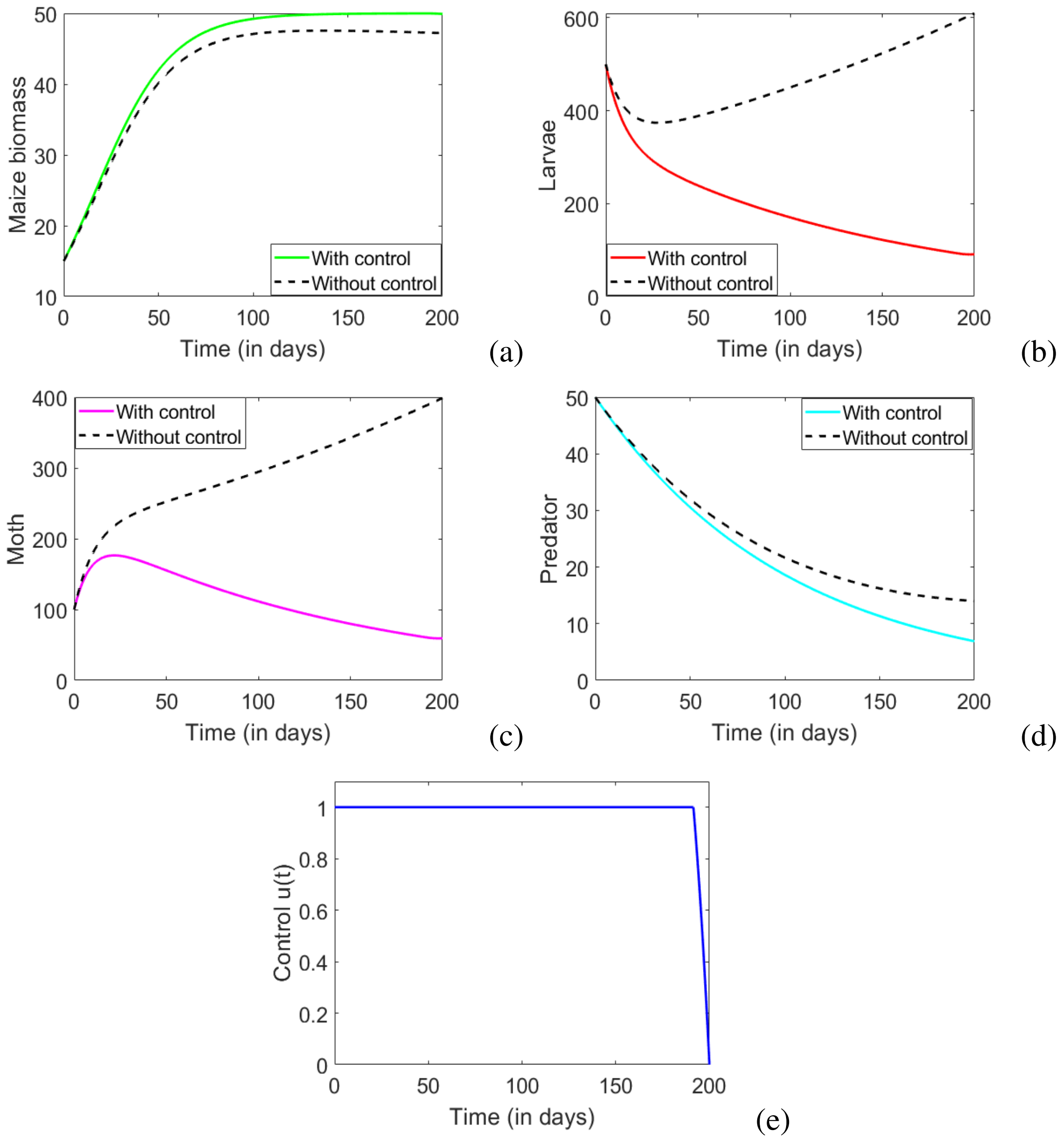


Figure 23: **Simulation results of model (3.168) with constant farming awareness $u = 0.7$ and different fractional order values**

In what follows, effects of time dependent farming awareness campaigns was determined. Without loss of generality the fractional-order q, and the control $u(t)$ were set to $u(t) = 0.03$ and $q = 0.9$, respectively. Associated numerical illustrations are in Fig. 24.



(a) (b) (c) (d) (e)

Figure 24: Simulation results of model (3.168) with time dependent constant farming awareness $0 \leq u(t) \leq 1$**,** $q = 0.9$ **and** $W = 10$

From the results Fig. 24, one can note that in the presence of time dependent farming awareness the FAW population (larvae and moth) decreases remarkably compared to when there is no time dependent farming awareness. Further one can also note that a significant decrease of the FAW larvae in the presence of optimal farming awareness will also lead to a slightly decrease in the predator population over time. The results further highlights that in the presence of optimal farming awareness the final maize biomass will be within the expected level. However, in the absence of optimal farming awareness the final biomass level will be always less than the expected final biomass. In addition, one can also observe that the optimal control profile Fig. 24 (e) starts at $u_{\text{max}} = 1$ and remain there for the greater part of time horizon ($0 \leq t \leq 195$ days) till it drops close to the final time period. This suggests that for one to attain the outcomes in Fig. 24, optimal farming awareness efforts need to be maintained at their maximum intensity for the greater part of the time horizon and can be ceased gradually till the final time.

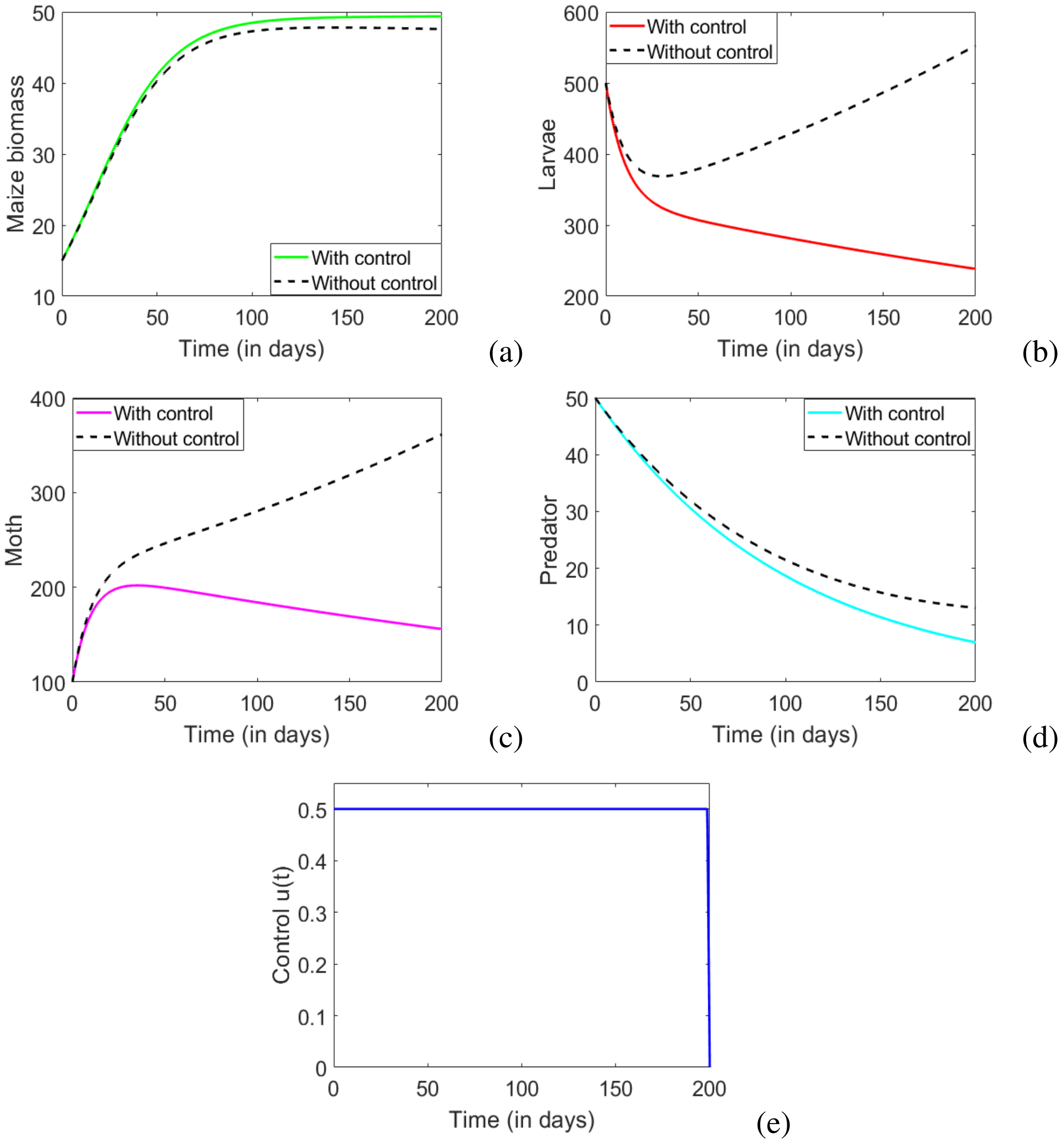


Figure 25: **Simulation results of model (3.168) at low maximum intensity $u_{\max} = 0.5$, with $q = 0.9$ and $W = 100$**

Simulation results in Fig. 25 shows the impact of the upper bound of the control variable $u_{\max}$ on model solutions. Here, the upper-bound of the control was set to $u_{\max} = 0.5$. From the results one can note that in this scenario the optimal efforts will need to be maintained at their maximum intensity throughout the entire time horizon in order for the final maize biomass to be within the expected level.

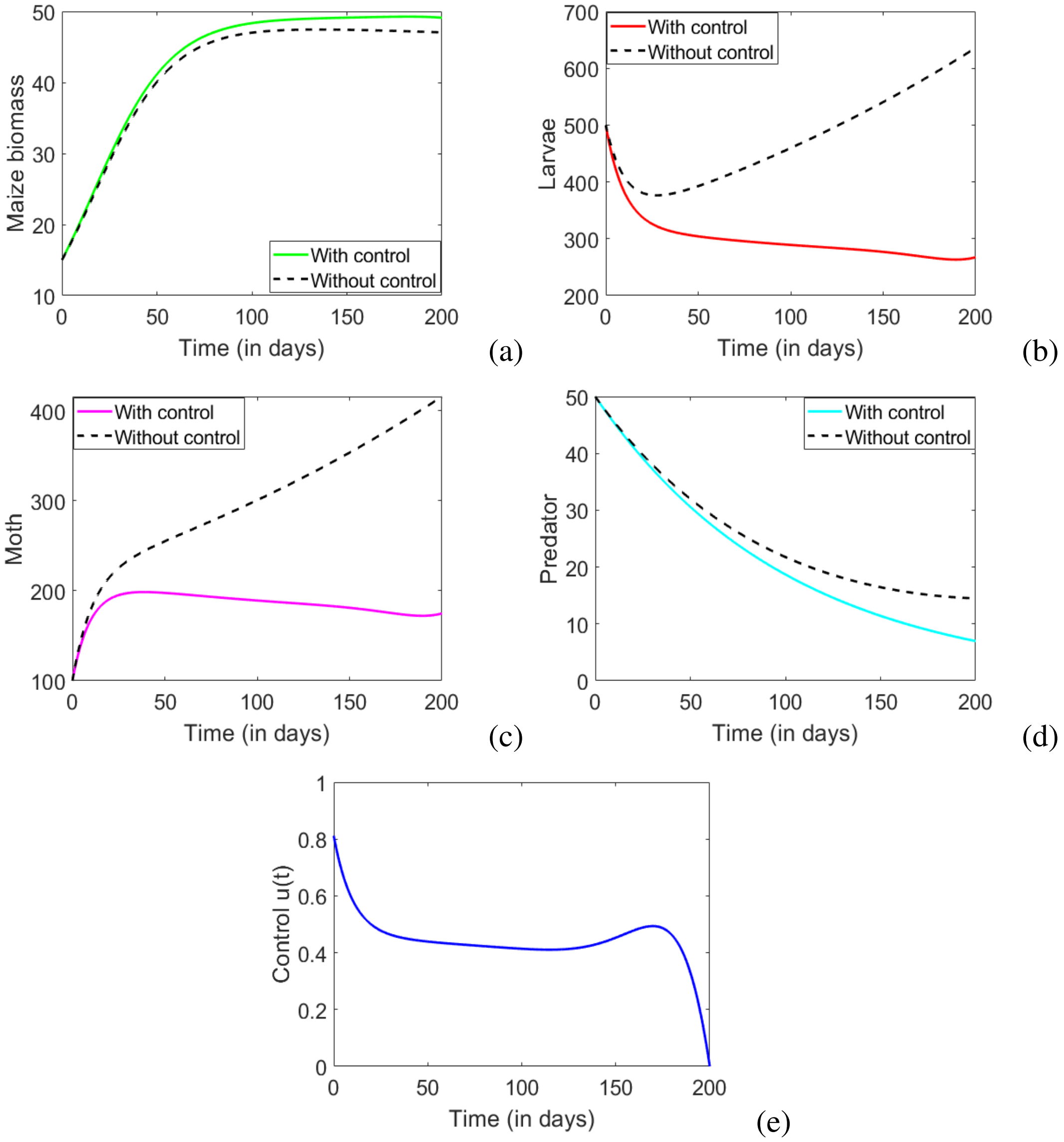


Figure 26: Simulation results of model (3.168) at high cost of implementation, $W = 1000$, $0 \leq u(t) \leq 1$, **and** $q = 0.9$

Simulation results in Fig. 26 show the impact of the costs on the implementation of optimal farming awareness. Here, the weight constant was set to $W = 1000$. The results shows that when the costs of implementing farming awareness are high, the control profile for $u(t)$ does not start at its maximum, $u_{\text{max}} = 1$, but begins on $u(t) = 0.8$ and this is followed by a gradual decrease before it stabilizes at $u(t) = 0.4$ after approximately 40 days from the start. The control profile stays at $u(t) = 0.4$ till the 150th day after which it increases slightly to $u(t) = 0.5$ and immediately drops gradually to its minimum until the final time horizon. Although, the pattern of the control profile is complex, one can deduce that optimum results can be attained if intensity of the control $u(t)$ is maintained between 0.4 and 0.5 ($0.4 \leq u(t) \leq 0.5$) for the greater part of the time horizon.

4.6 Chapter Overview

In this chapter, numerical solutions to the proposed models were presented and their biological implications were comprehensively discussed. Results from the first model demonstrated that, the reproduction number, $\mathscr{R}_0$ is a threshold parameter that account for the average number of off-spring generated by an adult female FAW during its entire life span. Further, it was revealed that, there was a highly positively correlation between b, the number of eggs laid per day per female moth and w, the proportion of adult female moth in the environment such that if either of the aforementioned parameters increase by say 10%, will increase $\mathscr{R}_0$ by the same size. Although model parameters accounting for FAW control at each stage were observed to have significant impact on reducing the size of the reproduction number, they were observed not to be as highly correlated as those that have positive influence.

In the second model the influence of seasonality on FAW-maize interaction and in-turn on maize biomass was noted to have a significant impact. The results showed that as the FAW populations fluctuates seasonally the maize biomass was also fluctuating periodically. In addition, it was also noted that the time-dependent spraying of insecticides could lead to the eradication on FAW in the maize field. In the third model, time-dependent farming awareness campaigns were shown to have more impact on reducing FAW population in the maize field compared to non-time dependent farming awareness campaigns. For non-time dependent farming awareness to be more effective, it was noted that they should be greater that 70% efficacy all the time. However, for time-dependent farming awareness campaigns around 10% efficacy it was found to bring the final maize biomass extremely close to the expected levels.

CHAPTER FIVE

CONCLUSION AND RECOMMENDATIONS

5.1 Conclusions

Invasive plants, insects and diseases are major threat to achieving the sustainable development goals (SDGs). Since 2016, the invasive fall armyworm (FAW), *Spodoptera frugiperda*, has been one of the most rapidly spreading and highly devastating maize pests across African and Asian countries. Since the pest is new to the African continent, little is known about its impact on the final maize biomass under different conditions. The present study aims to make use of mathematical models to investigate the impact of FAW infestations in maize fields and on the final maize biomass. Three mathematical models which were proposed and rigorously analyzed, anchored on the following objectives: (a) evaluate the impacts of memory effects and control on FAW-maize interaction and on the final maize biomass, (b) evaluate the effects of seasonality on FAW-maize interaction and on the final maize biomass and (c) evaluate the impact of farming awareness and larvae predation on FAW-maize interaction and on the final maize biomass.

Recently, several researchers have shown that fractional-order models can more accurately describe many real world phenomena compare to integer order models. It was further noted that, fractional-order models are non-local operators and contain all characteristics concerned with memory of the dynamical system. Owing to the advantages of fractional-order models, the first mathematical framework proposed in this study is a new dynamical system for FAW-maize interaction based on Caputo fractional-order. The model included all the relevant biological information and control. In particular, all the development stages namely, the egg stage, larvae, pupae and adults were included in the formulated model.

Mathematical analysis of the proposed model revealed that, four equilibrium points which revealed the existence of a threshold parameter defined by $\mathscr{R}_0$ were computed and analyzed. It was observed that, $\mathscr{R}_0$, the average number of newborns produced by one individual female moth during its life span was an integral component for stability of the aforementioned model equilibria. Further, it was noted that, if one female moth is not capable of producing more than one off-spring then within a period of 150 days all the FAW populations (eggs, larvae, pupae and moth) will become extinct while the maize biomass will increase with time till it reaches the expected maximum biomass per plant (50 biomass plant^{-1}) otherwise it persists. Prior studies on pest biology to the area where the pest is endemic revealed that, with control the pest extinct in a period of approximately 150 days while without control extinct in a period of approximately 300 days (Assefa & Ayalew, 2019; Chapman *et al.*, 2000). In a nutshell, the numerical results re-affirm that, once insecticides' use leads to elimination of the pest during the maize life span, then due to residual effects of the insecticides' and the pest's life cycle they will be no any other pest in the field unnecessary another one migrates from another field and the insecticide is no longer efficient enough.

Moreover, It was noted that a threshold parameter, the reproduction number was qualitatively and quantitatively used to investigate the local and global stability of the model's steady state. For the two steady states the model has (that is the pest-free equilibrium and the pest persistence equilibrium points), it was observed that they were both locally and globally stable. In particular, the pest-free equilibrium point is both locally and globally stable whenever the reproduction number is less than unity. However, when the reproduction number is greater than unity there exists a pest persistence equilibrium point which is also both locally and globally stable.

Further analysis of the model revealed that, convergence of solutions to their respective limiting points in time depends on the fractional-order, q, that is as q approaches unity the time taken by solutions to converge to the limiting point increases. For maize biomass in particular and when $\mathscr{R}_0 < 1$, the increase of the final maize biomass at the end of the season depends on fractional-order, q, as q decreases, the final maize biomass increases approaching the maximum carrying capacity in a period less than 100 days. However, as fractional order, q decreases and when $\mathscr{R}_0 > 1$, the final maize biomass per field increases but less than the expected 50 biomass plant^{-1} . It was also revealed that model parameters, egg laying rate and proportion of female moth in the environment have a strong positive influence on increasing the size of the reproduction number. Precisely, an increasing in the proportion of female moth in the environment by say 20% will increase the size of the reproduction number by the same size. Intervention strategies (controls) aimed at reducing FAW adult populations were observed to have a strong impact in reducing the size of the reproduction number than any other model parameter. In particular, an increase in reduction of adult population by 10% will be associated with a reduction of 6.4% on the reproduction number.

The influence of seasonality on the development of FAW is well documented. In particular, different development stages of insects are favored by different ranges of weather conditions, hence, variations of weather conditions influence the development rates, duration of life-cycles, and, ultimately, the survival of insects. Under unfavourable weather condition for the development and reproduction, the pest is forced to migrate to other suitable locations for survival. To explore the effects of seasonality of FAW-maize interaction, a non-autonomous mathematical model was proposed and analyzed. All plant and pest development stages that are influenced by changes in weather patterns were modeled as periodic functions.

The dynamical properties of the proposed model which were comprehensively investigated revealed that, the model solution was non-negative, unique, permanent and bounded admitting global asymptotic and continuous periodic function. Meanwhile, the proposed model was extended into an optimal control problem. The goal of the optimal control problem was to determine the level of using time dependent traditional methods and pesticides capable of minimizing or eradicating FAW in the field at minimal cost of implementation.

With sole use of chemical insecticides, it was noted that the population of FAW may become extinct in a period of 50 days and the final maize biomass per plant will be extremely close to the anticipated levels at the end of the season. With traditional methods in use, it was observed that the final maize biomass will be approximately 50% of the expected. However, when pesticides use and traditional methods are combined, it was observed that the FAW population will become extinction in approximately 20 days which is shorter than when pesticides alone are being used.

Further analysis of the model revealed that if traditional methods only are in use they will have to be implemented at their maximum intensity through out the entire season, however, when pesticides are in use they will have to be implemented at their maximum strength for approximately 30 days and their after their intensity can be reduced to 25% of the initial implementation level, and after 100 days they may be ceased completely. In a scenario when pesticides and traditional methods are being implemented, the final biomass is relatively higher than when a single control strategy is in use. Moreover, both methods will not need to be implemented at their maximum intensity through out the season. Precisely, after 20 days both methods may be implemented at levels slightly below 50% of their initial implementation level, and after 50 days pesticides use may be ceased while traditional methods will continue to be used for an additional 60 days after which they can be ceased.

Since 2016, when FAW outbreak was observed in Africa, African governments adopted emergency actions around chemical insecticides, however, due to financial challenges associated with governments of African countries and the cost associated with massive spraying programme of chemical insecticides and use of genetically modified crops (Bt maize), effective management of pest in the continent remains a challenge, there need to understand the role of farming awareness campaigns and natural enemies of FAW larvae on FAW dynamics.

Awareness campaigns, in particular through various media outlets such as radio, newspapers, TV and so on, does not only make farmers aware of FAW outbreak but also improve trust on integrated pest management (IPM) each nation will be advocating for (Abraha *et al.*, 2021). In recent times, attention to health news has been observed play an integral on disease management (Cui *et al.*, 2008). There is no doubt that correct and relevant knowledge about crop and its pests is very much essential to farmers (Abraha *et al.*, 2021). To explore the role of farming media campaigns a new fractional-order model was developed and analyzed. The proposed model included FAW larvae predators and media campaigns.

Dynamical analysis of the proposed model revealed that it has five equilibrium points which may all be globally asymptotically stable if certain conditions outlined in the Appendix 4 are satisfied. Simulation results of the model with constant awareness campaigns u, showed that $u = 0.7$ may lead to achievement of the expected maize biomass at the final time $t = 164$ for fractional-order values $q = 0.7, 0.8, 0.9$. However for $q = 1.0$ the final maize biomass at this level of awareness will be slightly less than the expected. For time dependent farming awareness it was observed that the expected maize biomass can be attained if the costs of implementing the strategy are low. In addition, it was observed that if the intensity of implementing are low then the efforts can be carried out at their maximum intensity throughout the time horizon. Although, the model was not exhaustive it illustrated the value of optimal control theory as a tool to suggest effective management strategies during FAW outbreaks.

5.2 Recommendations

Cereal crops play a vital role in the daily diets in Africa and account for up to 46% of the daily calorie consumption. Maize followed by sorghum are major staple food crops grown in diverse agro-ecological zones and farming systems in sub-Saharan Africa (Niassy *et al.*, 2021). In this study, mathematical models were proposed and analyzed to assess the effects FAW infestations in a maize field on the final maize biomass. Based on outcomes from this study the following recommendations are made:

(i) Coupling of time dependent traditional methods and insecticides use be considered and mathematical models be utilized to evaluate the success after a defined time horizon. In addition, insecticide be applied at intensity lower than that of traditional methods all the time. From the study it was noted that traditional methods of control FAW alone need to be at extremely higher efficacy (80%) to sufficiently eliminate FAW in the field. Since achieving a higher efficacy is practically challenging there is need to combine these methods with pesticides use which has shown that it can effectively reduce FAW population even when it is maintained at intensity levels around 10%.

(ii) Mathematical models be utilized together with other mitigation strategies to understand and evaluate effective ways of managing FAW during an outbreak.

(iii) Policy makers need to enhancing farming awareness campaigns in order to strength existing knowledge about the pest and the associated control strategies. In particular, from the study it was observed that reducing the population of adult FAW by 64% will reduce the generation of new offspring by the same magnitude.

5.2.1 Possible Future Works

This study is not exhaustive, in future Mathematical models proposed could be extended to incorporate the following aspects:

(i) Continuous replanting of maize crops. In the mathematical frameworks proposed, the dynamics were explored without factoring continuous replanting of maize.

(ii) Effects of multiple predators. FAW have several natural enemies. These predators affects different populations on FAW. Some of these predators can be reared on an alternative host, and use to control FAW. In this study only FAW larvae predator were considered.

(iii) Effects of knowledge heterogeneity among farmers. Farmers in a community do not posses uniform knowledge and attitude. One farmer may be aware of the pest and the associated mitigation strategy while another will not be. Hence the distribution and persistence of pests can be strongly affected by this heterogeneity.

(iv) Validation of the mathematical models presented in this study. The three mathematical models in this study have not been validated, therefore in future when data are available, I recommend to validate these models for effective exploring the impact of FAW-maize interaction dynamics.

5.2.2 Limitation of the Study

This study was limited to various constraints which need to be acknowledged. The following were the limitations of this study:

(i) The proposed model were calibrated based on data extracted from literature, however, model validation was not done due to unavailability of observed data for changes in FAW population and the resultant maize biomass over time due to roll-out of intervention strategies.

(ii) Unavailability of enough funds for travelling to meet my supervisors who were living outside the country for face to face consultation.

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RESEARCH OUTPUTS

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RESEARCH

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A mathematical model for fall armyworm management on maize biomass

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Abstract

Fall armyworm (*Spodoptera frugiperda*), a highly destructive and fast spreading agricultural pest native to North and South America, poses a real threat to global food security. In this paper, to explore the dynamics and implications of fall armyworm outbreak in a field of maize biomass, we propose a new dynamical system for maize biomass and fall armyworm interaction via Caputo fractional-order operator, which is not only a nonlocal operator but also contains all characteristics concerned with memory of the dynamical system. We define the basic reproduction number, which represents the average number of newborns produced by one individual female moth during its life span. We establish that the basic reproduction number is a threshold quantity, which determines persistence and extinction of the pest. Finally, we simulate the Caputo system using the Adam–Bashforth–Moulton method to illustrate the main results.

MSC: 92B05; 93A30; 93C15

Keywords: Fall armyworm; Fractional calculus; Mathematical modeling; Pest control strategies

1 Introduction

Fall armyworm (FAW), *Spodoptera frugiperda*, a highly destructive and fast spreading agricultural pest native to North and South America, poses a real threat to global food security. *Spodoptera frugiperda* remains an important pest of members of family *Poaceace* including major food crops such as corn, sorghum, rice wheat, maize, and diverse pasture [1]. According to FAO [2], food security is defined as a "situation that exists when all people, at all times, have physical, social, and economic access to sufficient, safe, and nutritious food that meets their dietary needs and food preferences for an active and healthy life" [2]. In recent years, the FAW has spread globally and emerged in countries where it had rarely or never before been presented, posing a real threat to global food security [3, 4]. Prior studies suggest that FAW pest is native to and widely distributed in the tropical and subtropical regions of America [1], and its invasion into Africa was reported for the first time in January 2016 [1]. Since then, it has become an epidemic pest in and beyond several African countries [1, 3–5]. FAW is regarded to be a major pest of maize biomass and other crops, such as rice, millet, sorghum, and cotton [5]. Due to the importance of maize biomass in many countries, there is a need to explore the implications of FAW outbreak in a field of maize biomass to increase the harvest of this crop.

Mathematical modeling has become a tool used to explore many real-world phenomena. Ordinary differential equations (ODEs) and partial differential equations (PDEs) with and without memory effects are some of the tools that have been commonly used to formulate equation(s) that mirror the real-world problem(s) [6–12]. In recent years a number of mathematical models were developed to explore plant–pest interaction [11–19]. Tang et al. [17] proposed impulsive differential equation models or hybrid dynamical system to model the introduction of a periodic IPM strategy, which includes periodic spraying of pesticide and release of natural enemies at critical time [14–17]. On the other hand, Tang et al. [17] developed an impulsive pest-natural enemy model, in which pulsing actions such as spraying pesticide and releasing natural enemies were considered with the assumption that the pesticide kills a pest instantly, whereas Chowdhury [18] formulated and extensively investigated continuous and discrete predator–prey models concerning IPM strategy. Discrete host parasitoid models were also proposed in circumstances where the timing of pesticide application leads to the death of parasitoid, and four different cases involving the timing of applications were investigated [11, 18].

The aforementioned studies and several other cited therein have certainly produced many useful results and improved the existing knowledge on plant–pest interaction. However, one of the limitations of these studies is that their models were based on integer-order ordinary differential equations. Recent studies suggest that models that use integer-order differential equations do not adequately capture memory effects and hereditary properties, which are inherent in many real-world problems [20]. As such, in recent years, fractional calculus has become an intriguing field. Several researchers have shown that models that utilize fractional calculus are more likely to replicate real-world problems compared those that use integer-order differential equations since fractional-order differential equations are able to capture memory effects [20, 21].

Therefore the present work aims to utilize fractional calculus to explore the implications of FAW infestation in a field planted with initial number of maize biomass at time $t = 0$ and obtaining maximum harvest of the biomass at the end of the season. The proposed model incorporates all the relevant biological information. In particular, the FAW population has been subdivided into egg population, larvae population, pupae population, and the adult population, also known as moth. Although the FAW has six larval instar stages, we have considered this as a single group to reduce complexity of the model. The proposed model also incorporates the use of nonbiological control methods such as use of pesticides and commonly known traditional methods such as hand picking of caterpillars. The role of biological control on FAW dynamics was comprehensively explored in [12], and hence we did not consider this aspect. At larval stage cannibalism is known to occur in FAW dynamics, and we also incorporate this aspect. We support qualitative and quantitative analytical results obtained in this study by numerical illustration.

We organize the paper as follows. In Sect. 2, we give some necessary definitions and some known properties of fractional calculus. In Sect. 3, we propose a fractional-order model for fall armyworm and maize biomass interaction. We investigate the local and global stability of the model equilibria. To support analytical findings, we carry out numerical simulation and present their results in Sect. 4. In the last section, we present the conclusions of this paper.

2 Preliminaries on the Caputo fractional calculus

We begin by introducing the definition of Caputo fractional derivative and state related theorems (see [22–24]), which we will utilize to derive important results in this work.

Definition 2.1 ([22–24]) Suppose that $q, a, t \in \mathbb{R}$, $q > 0, t > a$. The Caputo fractional derivative is given by

$$ {}_{a}^{c}D_{t}^{q} f(t) = \frac{1}{\Gamma(n-q)} \int_{a}^{t} \frac{f^{n}(\xi)}{(t-\xi)^{q+1-n}}\, d\xi, \quad n-1 < q, n \in \mathbb{N}, $$

where Γ is the gamma function.

Definition 2.2 ([22, 24]) The Riemann–Liouville fractional integral of arbitrary real order $q > 0$ of a function $f(t)$ is defined by the integral

$$ J^{q} f(t) = \frac{1}{\Gamma(q)} \int_{0}^{t} (t-\xi)^{q-1} f(\xi)\, d\xi, $$

and $J^{0} f(t) = f(t)$.

Definition 2.3 ([25]) Let $q > 0$, $n-1 < q < n$, $n \in \mathbb{N}$. Suppose that $f(t), f'(t), \ldots, f^{(n-1)}(t)$ are continuous on $[t_0, \infty)$ and have the exponential order and that ${}_{t_0}^{c}D_{t}^{q} f(t)$ is piecewise continuous on $[t_0, \infty)$. Then

$$ \mathcal{L}\left\{ {}_{t_0}^{c}D_{t}^{q} f(t) \right\} = s^{q} \mathcal{F}(s) - \sum_{k=0}^{n-1} s^{q-k-1} f^{(k)}(t_0), $$

where $\mathcal{F}(s) = \mathcal{L}\{f(t)\}$.

Lemma 2.1 ([26]) *Let $x(\cdot)$ be a continuous and differentiable function with $x(t) \in \mathbb{R}_{+}$. Then, for any time instant $t \geq t_0$, we have*

$$ {}_{t_0}^{c}D_{t}^{q} \left(x(t) - x^{*} - x^{*} \ln \frac{x(t)}{x^{*}} \right) \leq \left(1 - \frac{x^{*}}{x(t)} \right) {}_{t_0}^{c}D_{t}^{q} x(t), \quad x^{*} \in \mathbb{R}^{+}, \forall q \in (0,1). $$

3 Model formulation and analysis

3.1 Model formulation

A fractional-order model we introduce consists of two populations, maize biomass and the FAW population, where one of the populations is a stage-structured giving a total of five populations. Meanwhile, the FAW population is divided into four classes, which represent the FAW life cycle and are the egg stage $E(t)$, Larvae $L(t)$, pupal $P(t)$, and the adult stage (Moth) $A(t)$. Although the FAW typically has six larval instars, to reduce complexity of the model in a biological sensible way, all larval instars are represented by class $L(t)$. The life cycle of FAW starts when eggs are laid in masses on maize biomass, mostly underside of these biomass. The following equation describes the FAW egg population dynamics:

$$ {}_{a}^{c}D_{t}^{q} E(t) = b^{q} \left(1 - \frac{E}{K_{E}^{q}} \right) WA - \left(\alpha_{E}^{q} + u_{E}^{q} + \mu_{E}^{q} \right) E, $$

where b represents the egg laying rate for an adult female FAW, that is, an average number of eggs each female adult FAW will lay per day, K_E represents the egg carrying capacity, that is, the availability of space to lay eggs, W is the proportion of adult FAW that are females, α_E is the egg hatching rate, and u_E accounts for the effects of intervention strategies a farmer will implement once they observe eggs laid on the maize biomass, μ_E is the egg mortality rate. FAW larvae generally emerge simultaneously three to five days following oviposition. The following equation summarizes the population dynamics of the larvae stage:

$$ {}_{a}^{c}D_t^q L(t) = \alpha_E^q \left(1 - \frac{L}{K_L^q}\right) E + \theta^q LM - \left(\alpha_L^q + u_L^q + \mu_L^q\right) L, $$

where the transition rate from the egg stage to larvae is α_E. Prior studies [3, 27] suggest that whenever the food is limited, the older larvae of FAW exhibit a cannibalistic behavior on the smaller larvae. Hence to account for this aspect, we assume that the death rate due to lack of food is proportional to the smaller larvae $\alpha_E E$ and to the coefficient L/K_L that represent the availability of food for each larvae. Therefore K_L models the availability of food and space for the larvae population, μ_L represents the natural mortality rate of the larvae, and $1/\alpha_L$ is the average duration of the larvae stage, which is estimated to the of 14–30 days [3, 28, 29]. In particular, it is estimated that this duration is shorter, around 14 days during warm summer months and longer, and around 30 days during cooler weather [28, 29]. The parameter u_L models the role of intervention strategies implemented by the farmer, which may be use of pesticides or handpicking of the larvae. The term $\theta^q LM$ represents the interaction of larvae and maize biomass, which results in conversion of maize biomass into larvae biomass. Hence we can write $\theta^q = e^q \beta^q$, where the parameter e represents the efficiency with which caterpillar (FAW larvae) convert consumed maize biomass.

Pupation of the FAW normally occurs in the soil at a depth of 2–8 cm [29]. Here the larva constructs a loose cocoon, oval in shape and 20–30 mm in length, through tying together particles of soil with silk [3]. In areas where the soil is too hard, larvae web together leaf debris and other material to form a cocoon on the soil surface [3]. The following equation represents the dynamics of pupae stage:

$$ {}_{a}^{c}D_t^q P(t) = \alpha_L^q L - \left(\mu_P^q + \alpha_p^q + u_P^q\right) P, $$

where μ_P is the natural mortality rate, $1/\alpha_P$ represents the duration of the pupae stage, which is approximately 8–9 days during the summer; however, during winter it may reach 20–30 days [3, 28]. It is worth noting that the pupal stage of FAW does not enter a diapause period to withstand protracted periods winter or summer seasons in the absence of the host plant biomass [3, 30]. The effects of intervention strategies on reducing the population of the pupae is modeled by u_P. Adult FAW are 20–25-mm long with a wingspan of approximately 30–40 mm. Female and male adult FAW have different color pattern on their forewing. Adult female FAW are responsible for laying eggs on the surface of maize biomass, and this process usually starts after a preoviposition period of 3–4 days and continues until they become 3-week old. The following equation summarizes the population dynamics of the adult FAW:

$$ {}_{a}^{c}D_t^q A(t) = \alpha_P^q P - \left(\mu_A^q + u_A^q\right) A, $$

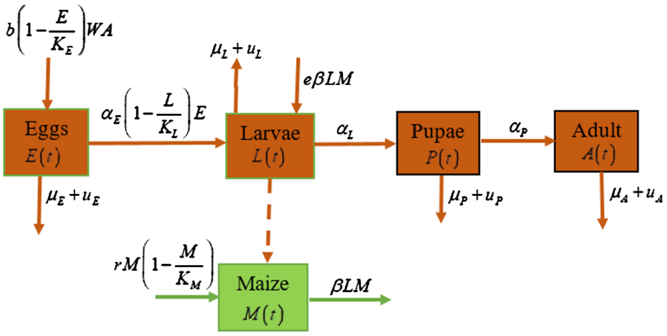

Figure 1 Model flow diagram illustrating the dynamics of fall armyworm in a field of maize biomass. The fall armyworm life cycle is divided into four classes: the egg stage *E*(*t*), Larvae *L*(*t*), pupal *P*(*t*), and the adult stage (Moth) *A*(*t*). The compartment *M*(*t*) represents maize plant population. The dotted line demonstrates that fall armyworm larvae are responsible for attacking the maize plant

where α_P accounts for the proportion of FAW pupa population that successfully progresses to the adult stage, u_A denotes the effects of intervention strategies, and $1/\mu_A$ is the life span, which is estimated to average about 10 days, with a range of about 7–21 days [3]. It is worth noting that the duration of the life cycle for FAW lasts for about 30 days at 28°C and may take longer, 60–90 days, when the weather is cooler [28]. In addition, under favorable conditions, the FAW larvae have a potential to feed and breed on maize biomass year-round [28].

Plant biomass (plant seeds) planted at time $t = 0$ emerges in a period of 0 to 7 days. We assume that planting of maize seed per hectare at the beginning of the season is done in a day. In this regard, we let $M(t)$ represent the population density of maize biomass per hectare Therefore the population dynamics of the maize biomass is governed by the equation

$$ {}_a^cD_t^q M(t) = r^q M\left(1 - \frac{M}{K_M^q}\right) - \beta^q LM, $$

where r is the growth rate of maize biomass, K_M is the maximum biomass of maize plants, and β is rate at which larvae (FAW larva) attacks the biomass of the maize plants.

Our assumptions on the dynamics of fall armyworm in a maize biomass population density are demonstrated in Fig. 1, and equations are presented in the system

$$ \left.\begin{aligned} {}_a^cD_t^q M(t) &= r^q M\left(1 - \tfrac{M}{K_M^q}\right) - \beta^q LM, \\ {}_a^cD_t^q E(t) &= b^q\left(1 - \tfrac{E}{K_E^q}\right)WA - \left(\alpha_E^q + u_E^q + \mu_E^q\right)E, \\ {}_a^cD_t^q L(t) &= \alpha_E^q\left(1 - \tfrac{L}{K_L^q}\right)E + \theta^q LM - \left(\alpha_L^q + u_L^q + \mu_L^q\right)L, \\ {}_a^cD_t^q P(t) &= \alpha_L^q L - \left(\mu_P^q + \alpha_p^q + u_P^q\right)P, \\ {}_a^cD_t^q A(t) &= \alpha_P^q P - \left(\mu_A^q + u_A^q\right)A. \end{aligned}\right\} \qquad (1) $$

3.2 Positivity and boundedness of solutions to model (1)

Theorem 3.1 *There exists a unique solution for the fractional-order model* (1) *in* $(0, \infty)$. *Moreover, the solution is nonnegative for all* $t > 0$ *and remains in* $\mathbb{R}_+^5$.

Proof We begin by demonstrating that $\mathbb{R}_+^5 = \{(M,E,L,P,A) \in \mathbb{R}_+^5 : M \geq 0, E \geq 0, L \geq 0, P \geq 0, A \geq 0\}$ is positively invariant. For that, we have to demonstrate that on each hyperplane bounding the nonnegative orthant the vector field points to $\mathbb{R}_+^5$. Let us consider the following cases.

Case 1. Assume that the exists $t_* > t_0$ such that $M(t_*) = 0$ and $M(t) < 0$ for $t \in (t_*, t_1]$, where t_1 is sufficiently close to t_*. If $M(t_*) = 0$, then

$${}_{t_0}^{c}D^q M(t_*) = 0.$$

Therefore ${}_{t_0}^{c}D^q M(t) \geq 0$ for all $t \in [t_*, t_1]$.

Case 2. Assume that the exists $t_* > t_0$ such that $E(t_*) = 0$ and $E(t) < 0$ for $t \in (t_*, t_1]$, where t_1 is sufficiently close to t_*. If $E(t_*) = 0$, then

$${}_{t_0}^{c}D^q E(t_*) = b^q WA > 0.$$

Therefore ${}_{t_0}^{c}D^q E(t) > 0$ for all $t \in [t_*, t_1]$.

Case 3. Assume that the exists $t_* > t_0$ such that $L(t_*) = 0$ and $L(t) < 0$ for $t \in (t_*, t_1]$, where t_1 is sufficiently close to t_*. If $L(t_*) = 0$, then

$${}_{t_0}^{c}D^q L(t_*) = \alpha_E^q E > 0.$$

From the last two equations of system (1) we can easily verify that

$$\left.\begin{aligned} {}_{t_0}^{c}D^q P(t) &= \alpha_L^q L > 0, \\ {}_{t_0}^{c}D^q A(t) &= \alpha_P^q P > 0. \end{aligned}\right\}$$

From the discussion above we observe that each hyperplane bounding the nonnegative orthant, the vector field points to $\mathbb{R}_+^5$, that is, all solutions of system (1), remains nonnegative for all $t \geq 0$. ☐

Theorem 3.2 *Let $\mathcal{X}(t) = (E(t), L(t), P(t), A(t))$ be a unique solution of the model* (1) *for $t \geq 0$. Then the solution $\mathcal{X}(t)$ is bounded above, that is, $\mathcal{X}(t) \in \Omega$, where Ω denotes the feasible region given by*

$$\Omega = \left\{ (E(t), L(t), P(t), A(t)) \,\middle|\, \begin{array}{l} 0 \leq E(t) \leq K_E \\ 0 \leq L(t) \leq C_L \\ 0 \leq P(t) \leq C_P \\ 0 \leq A(t) \leq C_A \end{array} \right\}.$$

Proof Here we will now demonstrate that the solutions of system (1) are bounded for all $t \geq 0$. For biological relevance, the least possible lower bound for each of the variables in system (1) is zero. Based on this, our discussion will be on determining the upper bound for these variables. Moreover, we can easily establish that the following conditions should hold: $0 \leq M(t) \leq K_M$ and $0 \leq E(t) \leq K_E$. For instance, we have

$${}_{a}^{c}D_t^q M(t) = r^q M\left(1 - \frac{M}{K_M^q}\right) - \beta^q LM$$

$$\leq r^q M\left(1-\frac{M}{K_M^q}\right).$$

Therefore it follows that $\limsup_{t\to\infty} M(t) \leq K_M$. Based on these bounds, we have

$$\begin{aligned}
{}_a^cD_t^q L(t) &= \alpha_E^q\left(1-\frac{L}{K_L^q}\right)E + \theta^q LM - \left(\alpha_L^q + u_L^q + \mu_L^q\right)L \\
&\leq \alpha_E^q K_E^q - \left[\left(\alpha_L^q + u_L^q + \mu_L^q + \frac{\alpha_E^q K_E^q}{K_L^q}\right) - \theta^q K_M^q\right]L.
\end{aligned}$$

Applying the Laplace transform leads to

$$s^q\mathcal{L}\big[L(t)\big] - s^{q-1}L(0) \leq \frac{\alpha_E^q K_E^q}{s} - \left[\left(\alpha_L^q + u_L^q + \mu_L^q + \frac{\alpha_E^q K_E^q}{K_L^q}\right) - \theta^q K_M^q\right]\mathcal{L}\big[L(t)\big].$$

Grouping like terms, we get

$$\begin{aligned}
\mathcal{L}\big(L(t)\big) &\leq \frac{\alpha_E^q K_E^q s^{-1}}{s^q + [(\alpha_L^q + u_L^q + \mu_L^q + \frac{\alpha_E^q K_E^q}{K_L^q}) - \theta^q K_M^q]} \\
&\quad + \frac{s^{q-1}L(0)}{s^q + [(\alpha_L^q + u_L^q + \mu_L^q + \frac{\alpha_E^q K_E^q}{K_L^q}) - \theta^q K_M^q]} \\
&= \frac{\alpha_E^q K_E^q s^{q-(1+q)}}{s^q + [(\alpha_L^q + u_L^q + \mu_L^q + \frac{\alpha_E^q K_E^q}{K_L^q}) - \theta^q K_M^q]} \\
&\quad + \frac{s^{q-1}L(0}{s^q + [(\alpha_L^q + u_L^q + \mu_L^q + \frac{\alpha_E^q K_E^q}{K_L^q}) - \theta^q K_M^q]}.
\end{aligned}$$

Applying the inverse Laplace transform leads to

$$\begin{aligned}
L(t) &\leq \mathcal{L}^{-1}\left\{\frac{\alpha_E^q K_E^q s^{q-(1+q)}}{s^q + [(\alpha_L^q + u_L^q + \mu_L^q + \frac{\alpha_E^q K_E^q}{K_L^q}) - \theta^q K_M^q]}\right\} \\
&\quad + \mathcal{L}^{-1}\left\{\frac{s^{q-1}L(0}{s^q + [(\alpha_L^q + u_L^q + \mu_L^q + \frac{\alpha_E^q K_E^q}{K_L^q}) - \theta^q K_M^q]}\right\} \\
&\leq \alpha_E^q K_E^q t^q E_{q,q+1}\left(-\left[\left(\alpha_L^q + u_L^q + \mu_L^q + \frac{\alpha_E^q K_E^q}{K_L^q}\right) - \theta^q K_M^q\right]t^q\right) \\
&\quad + L(0)E_{q,1}\left(-\left[\left(\alpha_L^q + u_L^q + \mu_L^q + \frac{\alpha_E^q K_E^q}{K_L^q}\right) - \theta^q K_M^q\right]t^q\right) \\
&\leq \max\left\{\frac{\alpha_E^q K_E^q}{[(\alpha_L^q + u_L^q + \mu_L^q + \frac{\alpha_E^q K_E^q}{K_L^q}) - \theta^q K_M^q]}, L(0)\right\} \\
&\quad \times \left(\left[\left(\alpha_L^q + u_L^q + \mu_L^q + \frac{\alpha_E^q K_E^q}{K_L^q}\right) - \theta^q K_M^q\right]t^q E_{q,q+1}\right. \\
&\quad \times \left(-\left[\left(\alpha_L^q + u_L^q + \mu_L^q + \frac{\alpha_E^q K_E^q}{K_L^q}\right) - \theta^q K_M^q\right]t^q\right)
\end{aligned}$$

$$
\begin{aligned}
&+ E_{q,1}\left(-\left[\left(\alpha_L^q + u_L^q + \mu_L^q + \frac{\alpha_E^q K_E^q}{K_L^q}\right) - \theta^q K_M^q\right] t^q\right)\right) \\
&= \frac{C}{\Gamma(1)} := C_L,
\end{aligned}
$$

where $C_L = \max\{[(\alpha_L^q + u_L^q + \mu_L^q + \frac{\alpha_E^q K_E^q}{K_L^q}) - \theta^q K_M^q], L(0)\}$. Thus $L(t)$ is bounded from above. From the equation for pupa population we have

$$
{}_a^cD_t^q P(t) = \alpha_L^q L - \left(\mu_P^q + \alpha_p^q + u_P^q\right)P \le \alpha_L^q C_L - \left(\mu_P^q + \alpha_p^q + u_P^q\right)P.
$$

Applying the Laplace transform leads to

$$
s^q \mathcal{L}\bigl[P(t)\bigr] - s^{q-1}P(0) \le \frac{\alpha_L^q C_L}{s} - \left(\mu_P^q + \alpha_p^q + u_P^q\right)\mathcal{L}\bigl[P(t)\bigr].
$$

Combining the like terms, we get

$$
\begin{aligned}
\mathcal{L}\bigl(P(t)\bigr) &\le \alpha_L^q C_L \frac{s^{-1}}{s^q + (\mu_P^q + \alpha_p^q + u_P^q)} + P(0)\frac{s^{q-1}}{s^q + (\mu_P^q + \alpha_p^q + u_P^q)} \\
&= \alpha_L^q C_L \frac{s^{q-(1+q)}}{s^q + (\mu_P^q + \alpha_p^q + u_P^q)} + P(0)\frac{s^{q-1}}{s^q + (\mu_P^q + \alpha_p^q + u_P^q)}.
\end{aligned}
$$

Applying the inverse Laplace transform leads to

$$
\begin{aligned}
P(t) &\le \mathcal{L}^{-1}\left\{\alpha_L^q C_L \frac{s^{q-(1+q)}}{s^q + (\mu_P^q + \alpha_p^q + u_P^q)}\right\} + P(0)\mathcal{L}^{-1}\left\{\frac{s^{q-1}}{s^q + (\mu_P^q + \alpha_p^q + u_P^q)}\right\} \\
&\le \alpha_L^q C_L t^q E_{q,q+1}\left(-\left(\mu_P^q + \alpha_p^q + u_P^q\right)t^q\right) + P(0)E_{q,1}\left(-\left(\mu_P^q + \alpha_p^q + u_P^q\right)t^q\right) \\
&\le \frac{\alpha_L^q C_L}{(\alpha_P^q + \mu_p^q + u_P^q)}\left(\mu_P^q + \alpha_p^q + u_P^q\right)t^q E_{q,q+1}\left(-\left(\alpha_P^q + \mu_p^q + u_P^q\right)t^q\right) \\
&\quad + P(0)E_{q,1}\left(-\left(\alpha_P^q + \mu_p^q + u_P^q\right)t^q\right) \\
&\le \max\left\{\frac{\alpha_L^q C_L}{(\alpha_P^q + \mu_p^q + u_P^q)}, P(0)\right\}\left(\left(\alpha_P^q + \mu_p^q + u_P^q\right)t^q E_{q,q+1}\left(-\left(\alpha_P^q + \mu_p^q + u_P^q\right)t^q\right)\right. \\
&\quad \left. + E_{\alpha,1}\left(-\left(\alpha_P^q + \mu_p^q + u_P^q\right)t^q\right)\right) \\
&= \frac{C}{\Gamma(1)} := C_P,
\end{aligned}
$$

where $C_P = \max\{\frac{\alpha_L^q C_L}{(\alpha_P^q + \mu_P^q)}, P(0)\}$. Thus $P(t)$ is bounded from above. From the last equation of system (1) we have

$$
{}_a^cD_t^q A(t) = \alpha_P^q P - \left(\mu_A^q + u_A^q\right)A \le \alpha_P^q C_P - \left(\mu_A^q + u_A^q\right)A.
$$

By applying the Laplace transform it follows that

$$
s^q \mathcal{L}\bigl[A(t)\bigr] - s^{q-1}A(0) \le \frac{\alpha_L^q C_P}{s} - \left(\mu_A^q + u_A^q\right)\mathcal{L}\bigl[A(t)\bigr].
$$

Grouping similar terms, we have

$$\begin{aligned}\mathcal{L}\big(A(t)\big) &\leq \alpha_P^q C_P \frac{s^{-1}}{s^q + (\mu_A^q + u_A^q)} + A(0)\frac{s^{q-1}}{s^q + (\mu_A^q + u_A^q)} \\ &= \alpha_L^q K_L^q \frac{s^{q-(1+q)}}{s^q + (\mu_P^q + \alpha_P^q)} + P(0)\frac{s^{q-1}}{s^q + (\mu_P^q + \alpha_P^q)}.\end{aligned}$$

Utilizing inverse Laplace transform, we get

$$\begin{aligned}A(t) &\leq \mathcal{L}^{-1}\left\{\alpha_P^q C_P \frac{s^{q-(1+q)}}{s^q + (\mu_A^q + u_A^q)}\right\} + A(0)\mathcal{L}^{-1}\left\{\frac{s^{q-1}}{s^q + (\mu_A^q + u_A^q)}\right\} \\ &\leq \alpha_P^q C_P t^q E_{q,q+1}\big(-\big(\mu_A^q + u_A^q\big)t^q\big) + A(0)E_{q,1}\big(-\big(\mu_A^q + u_A^q\big)t^q\big) \\ &\leq \frac{\alpha_P^q C_P}{(\alpha_A^q + u_A^q)}\big(\mu_A^q + u_A^q\big)t^q E_{q,q+1}\big(-\big(\mu_A^q + u_A^q\big)t^q\big) + A(0)E_{q,1}\big(-\big(\mu_A^q + u_A^q\big)t^q\big) \\ &\leq \max\left\{\frac{\alpha_P^q C_P}{(\mu_A^q + u_A^q)}, A(0)\right\}\big(\big(\mu_A^q + u_A^q\big)t^q \\ &\quad \times E_{q,q+1}\big(-\big(\mu_A^q + u_A^q\big)t^q\big) + E_{q,1}\big(-\big(\mu_A^q + u_A^q\big)t^q\big)\big) \\ &= \frac{C}{\Gamma(1)} = C_A,\end{aligned}$$

where $C_P = \max\{\frac{\alpha_P^q C_P}{(\mu_A^q + u_A^q)}, A(0)\}$. Thus $A(t)$ is bounded from above. This completes the proof. ☐

3.3 Model equilibria

By direct calculations we can observe that system (1) has four equilibrium points:

(i) Trivial equilibrium

$$\mathcal{E}^1 = \big\{E^1 = 0, L^1 = 0, P^1 = 0, A^1 = 0, M^1 = 0\big\}.$$

(ii) First axial equilibrium point

$$\mathcal{E}^2 = \big\{M^2 = K_M^q, E^2 = 0, L^2 = 0, P^2 = 0, A^2 = 0\big\}.$$

(iii) Second axial equilibrium point $\mathcal{E}^3 = \{M^3 E^3, L^3, P^3, A^3\}$, where

$$\mathcal{E}^3 : \left\{\begin{array}{l} E^3 = \frac{K_E^q K_L^q m_1 m_2 m_3 m_4}{\alpha_E^q (b^q W K_L^q \alpha_L^q \alpha_P^q + K_E^q m_1 m_3 m_4)}(\frac{b^q W \alpha_E^q \alpha_L^q \alpha_P^q}{m_1 m_2 m_3 m_4} - 1), \\ L^3 = \frac{K_E^q K_L^q m_1 m_2 m_3 m_4}{b^q W (K_E \alpha_E + K_L^q m_2)\alpha_L^q \alpha_P^q}(\frac{b^q W \alpha_E^q \alpha_P^q \alpha_L^q}{m_1 m_2 m_3 m_4} - 1), \\ P^3 = \frac{K_E^q K_L^q m_1 m_2 m_3 m_4}{b^q W (K_E \alpha_E + K_L^q m_2)\alpha_P^q m_3}(\frac{b^q W \alpha_E^q \alpha_L^q \alpha_P^q}{m_1 m_2 m_3 m_4} - 1), \\ A^3 = \frac{K_E^q K_L^q m_1 m_2 m_3 m_4}{b^q W (K_E \alpha_E + K_L^q m_2) m_3 m_4}(\frac{b^q W \alpha_E^q \alpha_L^q \alpha_P^q}{m_1 m_2 m_3 m_4} - 1), \\ M^3 = 0, \end{array}\right\} \qquad (2)$$

with

$$m_1 = \big(\mu_E^q + \alpha_E^q + u_E^q\big), \qquad m_2 = \big(\mu_L^q + \alpha_L^q + u_L^q\big),$$

$$m_3 = \left(\mu_P^q + \alpha_p^q + u_P^q\right), \qquad m_4 = \left(\mu_A^q + u_A^q\right).$$

We can observe that this equilibrium point makes biological sense whenever

$$\frac{b^q W \alpha_E^q \alpha_L^q \alpha_P^q}{m_1 m_2 m_3 m_4} > 1.$$

Let

$$\begin{aligned}\mathcal{R}_0 &= b^q W\left(\frac{\alpha_E^q}{\mu_E^q + \alpha_E^q + u_E^q}\right)\left(\frac{\alpha_L^q}{\mu_L^q + \alpha_L^q + u_L^q}\right)\left(\frac{\alpha_P^q}{\mu_P^q + \alpha_P^q + u_P^q}\right)\left(\frac{1}{\mu_A^q + u_A^q}\right)\\ &= \frac{b^q W \alpha_E^q \alpha_L^q \alpha_P^q}{m_1 m_2 m_3 m_4}.\end{aligned}$$

Biologically, $\mathcal{R}_0$ is a threshold quantity that accounts for the persistence of the FAW population, and thus when $\mathcal{R}_0 > 1$, the population of FAW persists and will be an attack on maize plants leaves, and finally the population of maize plants is extinct. Hence we can precisely define $\mathcal{R}_0$ as the average number of off-spring generated by an adult female FAW during its entire life span. Precisely, we can note that a proportion W of moth will each lay b eggs per day for an average duration of $\frac{1}{\mu_A^q + u_A^q}$; laid egg has the probability $\frac{\alpha_E^q}{\mu_E^q + \alpha_E^q + u_E^q}$ of surviving to become larva. Caterpillars that emerge following oviposition have the probability $\frac{\alpha_L^q}{\mu_L^q + \alpha_L^q + u_L^q}$ of surviving to become pupa, which also has the probability $\frac{\alpha_P^q}{\mu_P^q + \alpha_P^q + u_P^q}$ of surviving to become moth.

(iv) Interior equilibrium point

$$\mathcal{E}^4 : \left\{\begin{array}{l} E^4 = \frac{-b^q K_E^q W \alpha_L^q \alpha_P^q h_2 + b^q K_E^q W \alpha_P^q \sqrt{h_2^2 - 4h_1 h_3}}{-b^q W \alpha_L^q \alpha_P^q h_2 + b^q W \alpha_P^q \sqrt{h_2^2 - 4h_1 h_3} - 2K_E^q h_1 m_1 m_4}, \\ L^4 = \frac{-h_2 + \sqrt{h_2^2 - 4h_1 h_3}}{2h_1}, \\ P^4 = \frac{-\alpha_L^q h_2 + \alpha_L^q \sqrt{h_2^2 - 4h_1 h_3}}{2h_1 m_3}, \\ A^4 = \frac{-\alpha_P^q \alpha_L^q h_2 + \alpha_P^q \alpha_L^q \sqrt{h_2^2 - 4h_1 h_3}}{2h_1 m_3 m_4}, \\ M^* = \frac{2h_1 r^q K_M^q - \beta_q K_M^q h_2 + \beta^q K_M^q \sqrt{h_2^2 - 4h_1 h_3}}{2h_1 r}, \end{array}\right\}, \tag{3}$$

where

$$\left.\begin{aligned} h_1 &= b^q K_L^q \theta^q K_M^q W \alpha_P^q \alpha_L^q, \\ h_2 &= -(b^q \theta^q r^q W K_L^q K_M^q \alpha_L^q \alpha_P^q + \theta^q \beta^q e^q K_E^q K_L^q K_M^q m_1 m_2 m_3 \\ &\quad + b^q r^q W K_E^q K_L^q \alpha_E^q \alpha_L^q \alpha_P^q - b^q r^q W K_L^q \alpha_L^q \alpha_P^q m_2), \\ h_3 &= -(\theta^q K_E K_L^q K_M^q m_1 m_2 m_3 + r^q K_E^q K_L^q m_1 m_2^2 m_3), \end{aligned}\right\}$$

Based on (3), h_1, h_2, and h_3, $\Delta = (h_2^2 - 4h_1 h_3) > 0$ implies that the equilibrium point $\mathcal{E}^4$ has a unique feasible equilibrium.

3.4 Local stability of equilibrium points

In this section, we study the local stability behavior of the four equilibrium points computed earlier by using the Jacobian matrix

$$J = \begin{bmatrix} r^q - \frac{2Mr^q}{K_M} - \beta^q L & 0 & -\beta^q M & 0 & 0 \\ 0 & -m_1 - \frac{b^q WA}{K_E} & 0 & 0 & b^q W(1 - \frac{E}{K_E}) \\ \theta^q L & (1 - \frac{L}{K_L^q})\alpha_E^q & \theta^q M - m_2 - \frac{\alpha_E^q E}{K_L} & 0 & 0 \\ 0 & 0 & \alpha_L^q & -m_3 & 0 \\ 0 & 0 & 0 & \alpha_P^q & -m_4 \end{bmatrix}. \tag{4}$$

(a) Trivial equilibrium point. Evaluating the Jacobian matrix (4) about $\mathcal{E}^1$ leads to

$$J(\mathcal{E}^1) = \begin{bmatrix} r^q & 0 & 0 & 0 & 0 \\ 0 & -m_1 & 0 & 0 & b^q W \\ 0 & \alpha_E^q & -m_2 & 0 & 0 \\ 0 & 0 & \alpha_L^q & -m_3 & 0 \\ 0 & 0 & 0 & \alpha_P^q & -m_4 \end{bmatrix}. \tag{5}$$

The trivial equilibrium point is locally stable if all eigenvalues λ_i $(i = 1, 2, 3, 4)$ of $J(\mathcal{E}^1)$ satisfy the condition $|\arg(\lambda_i)| > \frac{q\pi}{2}$ [31]. We can observe that one of the eigenvalues of (5) is $r^q > 0$. The other equilibrium points are obtained from the characteristic equation

$$\lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda + c_4 = 0 \tag{6}$$

with

$$\left.\begin{aligned} c_1 &= m_1 + m_2 + m_4, \\ c_2 &= m_1 m_2 + (m_1 + m_2)(m_3 + m_4) + m_3 m_4, \\ c_3 &= m_1 m_2 (m_3 + m_4) + m_3 m_4 (m_1 + m_2), \\ c_4 &= m_1 m_2 m_3 m_4 - b^q W \alpha_E^q \alpha_L^q \alpha_P^q \\ &= m_1 m_2 m_3 m_4 (1 - \mathcal{R}_0). \end{aligned}\right\}$$

The Routh–Hurwitz criteria for local asymptotic stability of the equilibrium point $\mathcal{E}^1$ are

$$\left.\begin{aligned} &\mathcal{H}_1 = c_1 > 0, \qquad c_3 > 0, \qquad c_4 > 0, \\ &\mathcal{H}_2 = c_1 c_2 c_3 - c_3^2 - c_1^2 c_4 > 0. \end{aligned}\right\} \tag{7}$$

As we can observe, all the coefficients of the characteristic polynomial (6) are positive whenever $\mathcal{R}_0 < 1$, implying that condition $\mathcal{H}_1$ holds for $\mathcal{R}_0 < 1$. Since we have established that the trivial equilibrium point $\mathcal{E}^1$ has another eigenvalue r^q, which is always positive, we will not investigate the positivity of condition $\mathcal{H}_2$, and hence we conclude that $\mathcal{E}^1$ is an unstable equilibrium point.

(b) First axial equilibrium point $\mathcal{E}^2$. Evaluating the Jacobian matrix (4) about $\mathcal{E}^2$ leads to

$$J(\mathcal{E}^2) = \begin{bmatrix} -r^q & 0 & -\beta^q K_M^q & 0 & 0 \\ 0 & -m_1 & 0 & 0 & b^q W \\ 0 & \alpha_E^q & \theta^q K_M^q - m_2 & 0 & 0 \\ 0 & 0 & \alpha_L^q & -m_3 & 0 \\ 0 & 0 & 0 & \alpha_P^q & -m_4 \end{bmatrix}. \tag{8}$$

From (8) we can observe that one of the eigenvalues is $-r^q < 0$, and the other eigenvalues are roots of the characteristic equation

$$\lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4 = 0 \tag{9}$$

with

$$\left.\begin{aligned} b_1 &= m_1 + m_2 + m_4 - \theta^q K_M^q, \\ b_2 &= (m_1 + m_2)(m_3 + m_4) + m_1 m_2 + m_3 m_4 - \theta^q K_M^q (m_1 + m_3 + m_4), \\ b_3 &= m_1\bigl(m_3 m_4 + m_2(m_3 + m_4)\bigr) + m_2 m_3 m_4 - \theta^q K_M^q \bigl(m_1(m_3 + m_4) + m_3 m_4\bigr), \\ b_4 &= m_1 m_2 m_3 m_4 \bigl((1 - \mathcal{R}_0) - \theta^q K_M^q\bigr). \end{aligned}\right\}$$

The Routh–Hurwitz criteria for local asymptotic stability of the equilibrium point $\mathcal{E}^2$ are

$$\left.\begin{aligned} \widehat{\mathcal{H}}_1 &= b_1 > 0, \quad b_3 > 0, \quad b_4 > 0, \\ \widehat{\mathcal{H}}_2 &= b_1 b_2 b_3 - b_3^2 - b_1^2 b_4 > 0. \end{aligned}\right\} \tag{10}$$

If conditions specified in (10) hold, then the equilibrium point $\mathcal{E}^2$ is locally asymptotically stable.

(c) Second axial equilibrium point $\mathcal{E}^3$. Evaluating the Jacobian matrix (4) about $\mathcal{E}^3$, we get

$$J(\mathcal{E}^3) = \begin{bmatrix} r^q - \beta^q L^3 & 0 & 0 & 0 & 0 \\ 0 & -\widehat{n}_1 & 0 & 0 & \widehat{n}_2 \\ \widehat{n}_3 & \widehat{n}_4 & -\widehat{n}_5 & 0 & 0 \\ 0 & 0 & \alpha_L^q & -m_3 & 0 \\ 0 & 0 & 0 & \alpha_P^q & -m_4 \end{bmatrix} \tag{11}$$

with

$$\widehat{n}_1 = -m_1 - \frac{b^q W A^3}{K_E}, \qquad \widehat{n}_2 = b^q W\left(1 - \frac{E^3}{K_E}\right), \qquad \widehat{n}_3 = \theta^q L,$$
$$\widehat{n}_4 = \left(1 - \frac{L^3}{K_L^q}\right), \qquad \widehat{n}_5 = m_2 + \frac{\alpha_E^q E}{K_L}.$$

From (11) we can observe that $-r^q(\frac{\beta^q L^3}{r^q} - 1)$ is an eigenvalue, and other eigenvalues can be determined from the characteristic polynomial

$$\lambda^4 + d_1\lambda^3 + d_2\lambda^2 + d_3\lambda + d_4 = 0$$

with

$$d_1 = m_1 + m_3 + m_4 + \widehat{n_5},$$
$$d_2 = \widehat{n_1}\widehat{n_5} + m_3m_4 + (m_3 + m_4)(\widehat{n_1} + \widehat{n_5}),$$
$$d_3 = \widehat{n_1}\widehat{n_5}(m_3 + m_4) + m_3m_4(\widehat{n_1} + \widehat{n_5}),$$
$$d_4 = m_3m_4\widehat{n_1}\widehat{n_5} - \widehat{n_2}\widehat{n_4}\alpha_L^q\alpha_P^q.$$

Ahmed et al. [31] presented some Routh–Hurwitz stability conditions for fractional-order systems. One well-known Routh–Hurwitz condition is that an equilibrium point is locally stable if all eigenvalues of the community matrix satisfy the condition $|\arg(\lambda_i)| > \frac{q\pi}{2}$. The Routh–Hurwitz criteria for the local asymptotic stability of the equilibrium point $\mathcal{E}^3$ are

$$\left.\begin{array}{l} \xi_1 = d_1 > 0, \qquad d_3 > 0, \qquad d_4 > 0, \\ \xi_2 = d_1d_2d_3 - d_3^2 - d_1^2d_4 > 0. \end{array}\right\} \tag{12}$$

Since the existence of the equilibrium point $\mathcal{E}^3$ is based on $\mathcal{R}_0 > 1$, (2), we conclude that the equilibrium point $\mathcal{E}^3$ is locally asymptotically stable provided that conditions (12) hold and (i) $r^q < \beta^q L^3$ and (ii) $\mathcal{R}_0 > 1$.

(d) Interior equilibrium point $\mathcal{E}^4$. Evaluating the Jacobian matrix (4) about $\mathcal{E}^4$, we get

$$J(\mathcal{E}^4) = \begin{bmatrix} n_1 & 0 & -n_2 & 0 & 0 \\ 0 & -n_3 & 0 & 0 & n_4 \\ n_5 & n_6 & n_7 & 0 & 0 \\ 0 & 0 & \alpha_L^q & -m_3 & 0 \\ 0 & 0 & 0 & \alpha_P^q & -m_4 \end{bmatrix} \tag{13}$$

with

$$n_1 = r^q - \frac{2Mr^q}{K_M} - \beta^q L, \qquad n_2 = -\beta^q M, \qquad n_3 = -m_1 - \frac{b^q WA}{K_E},$$
$$n_4 = b^q W\left(1 - \frac{E}{K_E}\right), \qquad n_5 = \theta^q L, \qquad n_6 = \left(1 - \frac{L}{K_L^q}\right)\alpha_E^q,$$
$$n_7 = \theta^q M - m_2 - \frac{\alpha_E^q E}{K_L}.$$

The characteristic equation of (13) is

$$\lambda^5 + z_1\lambda^4 + z_2\lambda^3 + z_3\lambda^2 + z_4\lambda + z_5 = 0,$$

where

$$z_1 = m_3 + m_4 + n_3 - n_1 - n_7,$$
$$z_2 = n_2n_5 - n_1n_3 + m_3(m_4 + n_3 - n_1 - n_7) + n_1n_7 - n_3n_7 - m_4(n_1 - n_3 + n_7),$$
$$z_3 = n_3(n_1n_7 + n_2n_5) + m_4\bigl(n_2n_5 - n_3n_7 + n_1(n_7 - n_3)\bigr)$$

$$
\begin{aligned}
&\quad - m_3\big(n_1(n_3 - n_7) + n_3n_7 + m_4(n_1 - n_3 + n_7) - n_2n_5\big),\\
z_4 &= n_3m_4(n_2n_5 + n_1n_7) + m_3\big(n_3(n_2n_5 + n_1n_7) + m_4\big(n_2n_5 - n_3n_7 + n_1(n_7 - n_3)\big)\big)\\
&\quad - \alpha_L^q\alpha_P^q n_4n_6,\\
z_5 &= \alpha_L^q\alpha_P^q n_1n_4n_6 + n_3m_3m_4(n_2n_5 + n_1n_7).
\end{aligned}
$$

The Routh–Hurwitz criteria necessary and sufficient for local asymptotic stability of the equilibrium point $\mathcal{E}^4$ are that the Hurwitz determinants H_i are all positive [32]. For a fifth-degree polynomial, these criteria are

$$
\left.
\begin{aligned}
&H_1 = z_1 > 0,\\
&H_2 = z_1z_2 - z_3 > 0,\\
&H_3 = z_1z_2z_3 + z_1z_5 - z_1^2z_4 - z_3^2 > 0,\\
&H_4 = (Z_3z_4 - z_2z_5)(z_1z_2 - z_3) - (z_1z_4 - z_5)^2 > 0,\\
&H_5 = c_5H_4 > 0.
\end{aligned}
\right\}
\tag{14}
$$

Thus we have the following result.

Theorem 3.3 *The interior equilibrium point* $\mathcal{E}^4$ *is locally asymptotically stable if conditions in* (14) *hold*; *otherwise, it is unstable.*

3.5 Global stability of equilibrium points

In this section, we study the global stability of the equilibrium points $\mathcal{E}^1$, $\mathcal{E}^2$, $\mathcal{E}^3$, and $\mathcal{E}^4$ determined earlier.

(a) Trivial equilibrium point $\mathcal{E}^1$. Let us consider the Lyapunov function

$$
\begin{aligned}
\mathcal{U}_1(M,E,L,P,A) = M(t) &+ \left(\frac{m_4}{b^qW}\right)E(t) + \left(\frac{m_1m_4}{b^qW\alpha_E^q}\right)L(t)\\
&+ \left(\frac{m_1m_2m_4}{b^qW\alpha_E^q\alpha_L^q}\right)P(t) + \left(\frac{m_1m_2m_3m_4}{b^qW\alpha_E^q\alpha_L^q\alpha_P^q}\right)A(t).
\end{aligned}
$$

As we can observe, the Lyapunov functional $\mathcal{U}_1(M,E,L,P,A)$ is defined, continuous, and positive definite for all $M(t)$, $E(t)$, $L(t)$, $P(t)$, and $A(t)$. It is evident that $\mathcal{U}_1$ vanishes at $\mathcal{E}^1$. The fractional derivative of $\mathcal{U}(t)$ along the solutions of system (1) leads to

$$
\begin{aligned}
{}^c_{t_0}D_t^q\mathcal{U}_1(t) &= {}^c_{t_0}D_t^qM(t) + \left(\frac{m_4}{b^qW}\right){}^c_{t_0}D_t^qE(t) + \left(\frac{m_1m_4}{b^qW\alpha_E^q}\right){}^c_{t_0}D_t^qL(t)\\
&\quad + \left(\frac{m_1m_2m_4}{b^qW\alpha_E^q\alpha_L^q}\right){}^c_{t_0}D_t^qP(t) + \left(\frac{m_1m_2m_3m_4}{b^qW\alpha_E^q\alpha_L^q\alpha_P^q}\right){}^c_{t_0}D_t^\alpha A(t)\\
&= r^qM(t)\left(1 - \frac{M(t)}{K_M^q}\right) - \beta^qL(t)M(t)\\
&\quad + \left(\frac{m_4}{b^qW}\right)\left(b^q\left(1 - \frac{E(t)}{K_E^q}\right)WA(t) - m_1E(t)\right)\\
&\quad + \left(\frac{m_1m_4}{b^qW\alpha_E^q}\right)\left(\alpha_E^q\left(1 - \frac{L(t)}{K_L^q}\right)E(t) + \theta^qL(t)M(t) - m_2L(t)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{m_1m_2m_4}{b^qW\alpha_E^q\alpha_L^q}\right)\left(\alpha_L^qL(t)-m_3P(t)\right)\\
&+\left(\frac{m_1m_2m_3m_4}{b^qW\alpha_E^q\alpha_L^q\alpha_P^q}\right)\left(\alpha_P^qP(t)-m_4AV\right)\\
&=-m_4\frac{E(t)A(t)}{K_E}-m_1m_2\frac{E(t)L(t)}{K_L}-\frac{m_1m_2m_3m_4^2}{b^qW\alpha_E^q\alpha_L^q\alpha_P^q}\\
&\quad\times\left(1-\frac{b^qW\alpha_E^q\alpha_L^q\alpha_P^q}{m_1m_2m_3m_4}\right)A(t)-\theta^q\left(\frac{b^qW\alpha_E^q\beta^q}{m_1m_4\theta^q}-1\right)L(t)M(t)\\
&\quad+r^qM(t)\left(1-\frac{M(t)}{K_M^q}\right)\\
&=-m_4\frac{E(t)A(t)}{K_E}-m_1m_2\frac{E(t)L(t)}{K_L}-\frac{m_4}{\mathcal{R}_0}(1-\mathcal{R}_0)A(t)\\
&\quad-\theta^q\left(\frac{b^qW\alpha_E^q\beta^q}{m_1m_4\theta^q}-1\right)L(t)M(t)+r^qM(t)\left(1-\frac{M(t)}{K_M^q}\right).
\end{aligned}
$$

Note that ${}_{t_0}^{c}D_t^q\mathcal{U}_1(t)=0$ if $M(t)=K_M^q$, $\mathcal{R}_0=1$, and $m_1m_4\theta^q\leq b^qW\alpha_E^q\beta^q$. Thus ${}_{t_0}^{c}D_t^q\mathcal{U}_1(t)$ is negative definite if $M(t)=K_M^q$, $\mathcal{R}_0\leq 1$, and $m_1m_4\theta^q\leq b^qW\alpha_E^q\beta^q$. Therefore we have the following theorem.

Theorem 3.4 *The trivial equilibrium point $\mathcal{E}^1$ is globally asymptotically stable if $M(t)=K_M^q$, $\mathcal{R}_0\leq 1$, and $m_1m_4\theta^q\leq b^qW\alpha_E^q\beta^q$; otherwise, it is unstable.*

(b) First axial equilibrium point $\mathcal{E}^2$. Define the function

$$
\begin{aligned}
\mathcal{U}_2(M,E,L,P,A)&=M(t)-M^*-M^*\ln\frac{M(t)}{M^*}+\left(\frac{m_4}{b^qW}\right)E(t)\\
&\quad+\left(\frac{m_1m_4}{b^qW\alpha_E^q}\right)L(t)+\left(\frac{m_1m_2m_4}{b^qW\alpha_E^q\alpha_L^q}\right)P(t)\\
&\quad+\left(\frac{m_1m_2m_3m_4}{b^qW\alpha_E^q\alpha_L^q\alpha_P^q}\right)A(t).
\end{aligned}
$$

Evidently, the function $\mathcal{U}_2(M,E,L,P,A)$ is defined, continuous, and positive definite for all $M(t)$, $E(t)$, $L(t)$, $P(t)$, and $A(t)$. Furthermore, $\mathcal{U}_2$ vanishes at $\mathcal{E}^2$. Hence the fractional derivative of $\mathcal{U}_2(t)$ along the solutions of the system satisfies

$$
\begin{aligned}
{}_{t_0}^{c}D_t^q\mathcal{U}_2(t)&\leq\left(1-\frac{M^*}{M(t)}\right){}_{t_0}^{c}D_t^qM(t)+\left(\frac{m_4}{b^qW}\right){}_{t_0}^{c}D_t^qE(t)+\left(\frac{m_1m_4}{b^qW\alpha_E^q}\right){}_{t_0}^{c}D_t^qL(t)\\
&\quad+\left(\frac{m_1m_2m_4}{b^qW\alpha_E^q\alpha_L^q}\right){}_{t_0}^{c}D_t^qP(t)+\left(\frac{m_1m_2m_3m_4}{b^qW\alpha_E^q\alpha_L^q\alpha_P^q}\right){}_{t_0}^{c}D_t^{\alpha}A(t)\\
&=-r^qM^*\left(1-\frac{M(t)}{K_M^q}\right)\left(1-\frac{M(t)}{M^*}\right)-m_4\frac{E(t)A(t)}{K_E}-m_1m_2\frac{E(t)L(t)}{K_L}\\
&\quad-\frac{m_1m_2m_3m_4^2}{b^qW\alpha_E^q\alpha_L^q\alpha_P^q}\left(1-\frac{b^qW\alpha_E^q\alpha_L^q\alpha_P^q}{m_1m_2m_3m_4}\right)A(t)\\
&\quad-\frac{\beta^qb^qW\alpha_E^q+m_1m_4\theta^q}{b^qW\alpha_E^q}\left(1-\frac{\beta^qb^qW\alpha_E^q}{m_1m_4\theta^q+\beta^qb^qW\alpha_E^q}\frac{M^*}{M(t)}\right)
\end{aligned}
$$

$$\begin{aligned}
&= -r^q M^*\left(1-\frac{M(t)}{K_M^q}\right)\left(1-\frac{M(t)}{M^*}\right) - m_4\frac{E(t)A(t)}{K_E} - m_1 m_2\frac{E(t)L(t)}{K_L} \\
&\quad - \frac{\beta^q b^q W\alpha_E^q + m_1 m_4\theta^q}{b^q W\alpha_E^q}\left(1-\frac{\beta^q b^q W\alpha_E^q M^*}{(m_1 m_4\theta^q + \beta^q b^q W\alpha_E^q)M(t)}\right) \\
&\quad - \frac{m_1 m_2 m_3 m_4^2}{b^q W\alpha_E^q\alpha_L^q\alpha_P^q}(1-\mathcal{R}_0)A(t).
\end{aligned}$$

Therefore ${}_{t_0}^{c}D_t^q\mathcal{U}_2(t)$ is negative definite if the following conditions hold: (i) $\mathcal{R}_0 \le 1$, (ii) $M < M^*$, (iii) $\beta^q b^q W\alpha_E^q M^* \le (m_1 m_4\theta^q + \beta^q b^q W\alpha_E^q)M(t)$. Therefore we have the following theorem.

Theorem 3.5 *The trivial equilibrium point* $\mathcal{E}^2$ *is globally asymptotically stable if the following conditions hold*: (i) $\mathcal{R}_0 \le 1$, (ii) $M < M^*$, *and* (iii) $\beta^q b^q W\alpha_E^q M^* \le (m_1 m_4\theta^q + \beta^q b^q W\alpha_E^q)M(t)$; *otherwise, it is unstable.*

(c) Global stability of equilibrium points $\mathcal{E}^3$ and $\mathcal{E}^4$. We will use the following Lyapunov function to investigate the global stability of the equilibrium points $\mathcal{E}^3$ and $\mathcal{E}^4$:

$$\begin{aligned}
\mathcal{U}_3(t) &= a_0\left[M(t) - M^* - M^*\ln\left(\frac{M(t)}{M^*}\right)\right] + a_1\left[E(t) - E^* - E^*\ln\left(\frac{E(t)}{E^*}\right)\right] \\
&\quad + a_2\left[L(t) - L^* - L^*\ln\left(\frac{L(t)}{L^*}\right)\right] + a_3\left[P(t) - P^* - P^*\ln\left(\frac{P(t)}{P^*}\right)\right] \\
&\quad + a_4\left[A(t) - A^* - A^*\ln\left(\frac{A(t)}{A^*}\right)\right],
\end{aligned}$$

where a_1, a_2, a_3, and a_4 are positive constants to be determined. Let $g_0(M) = r^q(1-\frac{M}{K_M^q})$, $g_1(E,A) = b^q(1-\frac{E}{K_E^q})WA$, and $g_2(E,L) = \alpha_E^q(1-\frac{L}{K_L^q})E$. Recall that at this equilibrium, we have the following identities:

$$\left.\begin{array}{l}
g_0(M) = \beta^q L^* M^*, \qquad g_1(E^*,A^*) = m_1 E^*, \qquad g_2(E^*,L^*) + \theta^q L^* M^* = m_2 L^*, \\
\alpha_L^q L^* - m_3 P^*, \qquad \alpha_P^q P^* = m_4 A^*.
\end{array}\right\}$$

Setting

$$\left.\begin{array}{ll}
a_1 = g_2(E^*,L^*), & a_3 = \frac{g_1(E^*,A^*)g_2(E^*,L^*)}{\alpha_L^q L^*}, \\
a_2 = g_1(E^*,A^*), & a_4 = \frac{g_1(E^*,A^*)g_2(E^*,L^*)}{\alpha_P^q P^*},
\end{array}\right\}$$

it follows from Lemma 2.1 that

$$\begin{aligned}
&{}_{t_0}^{c}D_t^\alpha\mathcal{U}_2(t) \\
&\quad\le \left(1-\frac{M^*}{M(t)}\right){}_{t_0}^{c}D_t^q M(t) + g_2\left(E^*,L^*\right)\left(1-\frac{E^*}{E(t)}\right){}_{t_0}^{c}D_t^q E(t) \\
&\qquad + g_1\left(E^*,A^*\right)\left(1-\frac{L^*}{L(t)}\right){}_{t_0}^{c}D_t^q L(t)\left(\frac{g_1(E^*,A^*)g_2(E^*,L^*)}{\alpha_L^q L^*}\right)\left(1-\frac{P^*}{P(t)}\right){}_{t_0}^{c}D_t^q P(t) \\
&\qquad + \left(\frac{g_1(E^*,A^*)g_2(E^*,L^*)}{\alpha_P^q P^*}\right)\left(1-\frac{A^*}{A(t)}\right){}_{t_0}^{c}D_t^q A(t)
\end{aligned}$$

$$
\begin{aligned}
&= g_0\bigl(M^*\bigr)\left(1-\frac{M^*}{M}\right)\left(\frac{g(M)}{g(M^*)}-\frac{LM}{L^*M^*}\right)+\theta^q g_1\bigl(E^*,A^*\bigr)\left(1-\frac{L}{L^*}\right)\left(1-\frac{M}{M^*}\right)\\
&\quad+g_2\bigl(E^*,L^*\bigr)\left(1-\frac{E^*}{E}\right)\left(g_1(E,A)-g_1\bigl(E^*,A^*\bigr)\frac{E}{E^*}\right)\\
&\quad+g_1\bigl(E^*,A^*\bigr)\left(1-\frac{L^*}{L}\right)\left(g_2(E,L)-g_2\bigl(E^*,L^*\bigr)\frac{L}{L^*}\right)\\
&\quad+\left(\frac{g_1(E^*,A^*)g_2(E^*,L^*)}{\alpha_L^q L^*}\right)\left(1-\frac{P^*}{P}\right)\left(\alpha_L^q L-\alpha_L^q L^*\frac{P}{P^*}\right)\\
&\quad+\left(\frac{g_1(E^*,A^*)g_2(E^*,L^*)}{\alpha_P^q P^*}\right)\left(1-\frac{A^*}{A}\right)\left(\alpha_P^q P-\alpha_P^q P^*\frac{A}{A^*}\right)\\
&= g_0\bigl(M^*\bigr)\left(1-\frac{M^*}{M}\right)\left(\frac{g(M)}{g(M^*)}-\frac{LM}{L^*M^*}\right)+\theta^q g_1\bigl(E^*,A^*\bigr)\left(1-\frac{L}{L^*}\right)\left(1-\frac{M}{M^*}\right)\\
&\quad+g_1\bigl(E^*,A^*\bigr)g_2\bigl(E^*,L^*\bigr)\left(1-\frac{E}{E^*}-\frac{E^*g_1(E,A)}{Eg_1(E^*,A^*)}+\frac{g_1(E,A)}{g_1(E^*,A^*)}\right)\\
&\quad+g_1\bigl(E^*,A^*\bigr)g_2\bigl(E^*,L^*\bigr)\left(3-\frac{A}{A^*}-\frac{A^*P}{AP^*}-\frac{P^*L}{PL^*}-\frac{L^*g_2(E,L)}{Lg_2(E^*,L^*)}+\frac{g_2(E,L)}{g_2(E^*,L^*)}\right).
\end{aligned}
$$

Let $\Phi(x) = 1 - x + \ln x$ for $x > 0$. It follows that $\Phi(x) \leq 0$ with the equality if and only if $x = 1$. Using this relation, we have

$$
\begin{aligned}
&1-\frac{E}{E^*}-\frac{E^*g_1(E,A)}{Eg_1(E^*,A^*)}+\frac{g_1(E,A)}{g_1(E^*,A^*)}\\
&\quad=\Phi\left(\frac{E^*g_1(E,A)}{Eg_1(E^*,A^*)}\right)-\frac{E}{E^*}\\
&\qquad+\frac{g_1(E,A)}{g_1(E^*,A^*)}-\ln\left(\frac{E^*g_1(E,A)}{Eg_1(E^*,A^*)}\right)\\
&\quad\leq\frac{g_1(E,A)}{g_1(E^*,A^*)}-\ln\left(\frac{g_1(E,A)}{g_1(E^*,A^*)}\right)-\frac{E}{E^*}+\ln\left(\frac{E}{E^*}\right).
\end{aligned}
$$

Similarly, we can write

$$
\begin{aligned}
&3-\frac{A}{A^*}-\frac{A^*P}{AP^*}-\frac{P^*L}{PL^*}-\frac{L^*g_2(E,L)}{Lg_2(E^*,L^*)}+\frac{g_2(E,L)}{g_2(E^*,L^*)}\\
&\quad=\Phi\left(\frac{A^*P}{AP^*}\right)+\Phi\left(\frac{A^*P}{AP^*}\right)+\Phi\left(\frac{L^*g_2(E,L)}{Lg_2(E^*,L^*)}\right)-\frac{A}{A^*}\\
&\qquad+\frac{g_2(E,L)}{g_2(E^*,L^*)}-\ln\left(\frac{A^*g_2(E,L)}{Ag_2(E^*,L^*)}\right)\\
&\quad\leq\frac{g_2(E,L)}{g_2(E^*,L^*)}-\ln\left(\frac{g_2(E,L)}{g_2(E^*,L^*)}\right)-\frac{A}{A^*}+\ln\left(\frac{A}{A^*}\right).
\end{aligned}
$$

Therefore ${}_{t_0}^{c}D_t^q\mathcal{U}_3(t)$ is negative definite if the following conditions hold:

(i) $(1-\frac{M^*}{M})(\frac{g(M)}{g(M^*)}-\frac{LM}{L^*M^*})\leq 0$,

(ii) $(1-\frac{L}{L^*})(1-\frac{M}{M^*})\leq 0$.

Therefore we have the following theorem.

Theorem 3.6 *The equilibrium points* $\mathcal{E}^3$ *and* $\mathcal{E}^4$ *are globally asymptotically stable if the following conditions hold*:

(i) $(1 - \frac{M^*}{M})(\frac{g(M)}{g(M^*)} - \frac{LM}{L^*M^*}) \leq 0$

(ii) $(1 - \frac{L}{L^*})(1 - \frac{M}{M^*}) \leq 0$;

otherwise, they are unstable.

4 Numerical results

4.1 Model parameterization

In this section, we present the baseline values for the model parameters. Majority of the parameter values are taken from previously published studies, and a few not available in literature were estimated within plausible and reasonable ranges so as to draw reasonably realistic scenarios.

(i) Natural mortality rate of adult FAW μ_A: The life span of female adult FAW is 15–21 days. It follows that the natural mortality rate of the moth is

$$\mu_A = \frac{1}{\text{expected lifetime}}.$$

(ii) Egg laying rate b and life span of adult moth μ_A^{-1}: During its entire life span of 15–21 days and adult female FAW's total egg production per female averages about 1500 with a maximum of over 2000 [28]. The average daily egg laying rate can be expressed as follows:

$$\text{eggs laid per day} = \frac{\text{eggs laid in a lifetime}}{\text{expected lifetime}}.$$

Westbrook et al. [33] estimated that a female adult moth with a life span of 18 days can oviposit about 125 eggs. Hence in our simulations, we set $b = 125$ eggs per day and $\mu_A = 1/18$ per day.

(iii) Egg hatching rate α_E and gender ratio W: Mathematically, the egg hatching rate is the inverse of average duration of the egg stage, that is,

$$\alpha_E = \frac{1}{\text{Average duration of the egg stage}}.$$

Depending on the climate, the duration of egg stage takes an average period of 2–3 days [3, 28]. Westbrook et al. [33] estimated a gender ratio of 50:50 males/females.

(iv) Average duration of the larval stage α_L^{-1}: The duration of the larval stage is influenced by climate changes. During summer periods, the larval stage is about 14 days and 30 days during cool weather [3, 28, 29].

(v) Average duration of the pupal stage α_P^{-1}: Similarly to the larval stage, the pupal stage also depends on the climate. It is about 8–9 days during summer but reaches 20–30 days during the winter [3, 28].

4.2 Sensitivity analysis of the reproduction number

Analytical results of the model have shown that the basic reproduction number is an important threshold parameter for the persistence and extinction of FAW during any outbreak. Since the parameters of the proposed model have either been drawn from literature

Table 1 Sensitivity index of the basic reproduction number

Parameter	b	W	u_A	α_L	u_L	α_P	u_P	μ_A	α_E
Sensitivity index	+1	+1	−0.64	+0.61	−0.55	+0.50	−0.45	−0.36	+0.25
Parameter	u_E	μ_L	μ_P	μ_E					
Sensitivity index	−0.23	−0.06	−0.05	−0.02					

or estimated, there is need to investigate the influence of each parameter on the magnitude of the basic reproduction number $\mathcal{R}_0$ so as to understand the uncertainty regarding their values. To infer on the relationship between the model parameters and individual parameters, we conduct sensitivity analysis as follows.

Definition 4.1 (See [34]) The normalized sensitivity index of $\mathcal{R}_0$, which depends differentiably on a parameter, say κ, is defined by

$$\Theta_{\kappa}^{\mathcal{R}_0} = \frac{\partial \mathcal{R}_0}{\partial \kappa} \times \frac{\kappa}{\mathcal{R}_0}. \tag{15}$$

The model parameters whose sensitivity index is positive will increase the size of $\mathcal{R}_0$ whenever they are increased, whereas those with negative index decrease $\mathcal{R}_0$ whenever they are increased. It follows from (15) that the normalized sensitivity of $\mathcal{R}_0$ with regard to the model parameters that define it is given by

$$\left.\begin{array}{lll} \Theta_{b}^{\mathcal{R}_0} = 1, & \Theta_{W}^{\mathcal{R}_0} = 1, & \Theta_{\alpha_i}^{\mathcal{R}_0} = \frac{\mu_i+u_i}{\mu_i+\alpha_i+u_i} > 0, \\ \Theta_{\mu_i}^{\mathcal{R}_0} = -\frac{\mu_i}{\mu_i+\alpha_i+u_i} < 0, & \Theta_{\mu_A}^{\mathcal{R}_0} = -\frac{\mu_A}{\mu_A+u_A} < 0, & \\ \Theta_{u_i}^{\mathcal{R}_0} = -\frac{u_i}{\mu_i+\alpha_i+u_i} < 0, & \Theta_{u_j}^{\mathcal{R}_0} = -\frac{u_A}{\mu_A+u_A} < 0 & \text{for } i = E,P,L. \end{array}\right\} \tag{16}$$

As we can observe from (16), the model parameters b, W, and α_j ($j = E,L,P,A$) increase the size of $\mathcal{R}_0$ whenever they are increased, whereas the model parameters μ_j and u_j decrease the size of $\mathcal{R}_0$ whenever they are increased. It is worth noting that an increase in either b or W by 10% may result in an increase in the magnitude of $\mathcal{R}_0$ by 10%. However, an increase by 10% of α_j increases the size of $\mathcal{R}_0$ by a value less than 10%. In addition, note that u_j has a negative effect on $\mathcal{R}_0$, implying that intervention strategy has an impact on extinction and persistence of FAW in the environment. Without loss of generality, we set $u_E = u_L = u_P = u_A = 0.3$ and computed the sensitivity index for each model parameter that defines $\mathcal{R}_0$. The results are presented in Table 1 and Fig. 2.

Numerical results in Table 1 suggests that pest control intervention strategies more effects on minimizing the FAW population in the field if such strategies target the adult FAW population. Simulation results in Fig. 3 demonstrate the effects of varying the intervention strategies on extinction and persistence of pests in the field. For simplicity, we set $u = u_E = u_L = u_P = u_A$, whereas the other parameter values are taken from Table 2. From the results we can note that any value of $u > 0.5$ leads to the extinction of the pest, and persistence of the pests occurs for $u < 0.5$.

4.3 Population level effects

In this section, we conducted additional simulations to numerically illustrate the dynamical behavior of system (1) and to validate the analytical results such as the stability of the equilibria. We used the fractional Adam–Bashforth–Moulton method [35] to conduct the

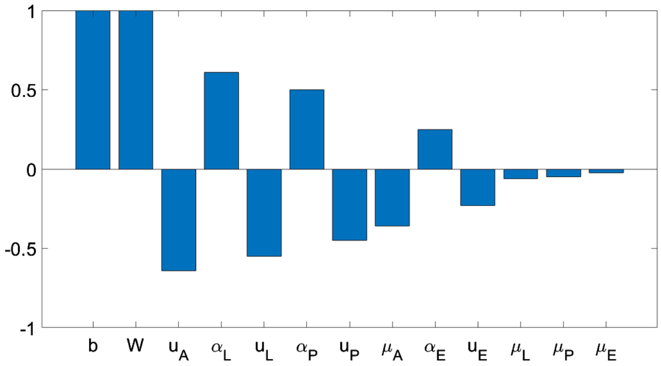

Figure 2 Sensitivity analysis of the basic reproduction number $\mathcal{R}_0$ with respect to model parameters. Baseline values used given in Table 2. The exact numerical indices are also presented in Table 1, whereas control parameters are set to $u_E = u_L = u_P = u_A = 0.3$. Overall, we can note that model parameters b and W are highly correlated with $\mathcal{R}_0$. In particular, an increase in either b or W by 10% may result in an increase in the magnitude of $\mathcal{R}_0$ by 10%

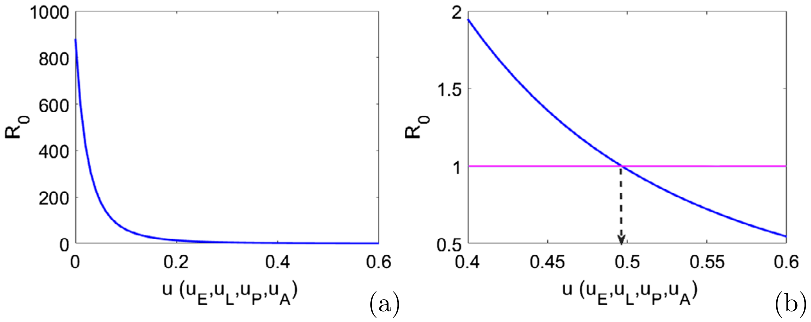

Figure 3 Effects of varying intervention strategies on the magnitude of $\mathcal{R}_0$. Here we set $u = u_E, u_L = u_P = u_A$. (**a**) Dynamics of $\mathcal{R}_0$ on a wide ranges of values for u. (**b**) Zoomed graph, that is, an illustration of the dynamics of $\mathcal{R}_0$ for a refined range of values for u. From the results we can note that any value of $u > 0.5$ leads to the extinction of the pest, and persistence of the pests occurs for $u < 0.5$

simulations, that is, for a differential equation

$$\frac{d^\alpha x(t)}{dt^\alpha} = f\bigl(t, x(t)\bigr),$$

the fractional variant of the one step Adam–Moulton method is given by

$$x_{n+1} = \sum_{i=0}^{[\alpha]-1} \frac{t_{n+1}^i}{i!} x_0^i + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n} a_{i,n+1} f(t_i, x_i) + \frac{h^\alpha}{\Gamma(\alpha+2)} f\bigl(t_{n+1}, x_{n+1}^p\bigr),$$

where $t_i = ih$ with some fixed h, and

$$a_{i,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, & i = 0, \\ (n-i+2)^{\alpha+1} + (n-i)^{\alpha+1} - 2(n-i+1)^{\alpha+1}, & 1 \le i \le n. \end{cases}$$

Table 2 Model parameters and their baseline values

Symbol	Definition	Baseline value	Source
b	Number of eggs laid per day per female moth	125 eggs per moth per day	[28]
W	Proportion of female adult moth	0.5	[33]
α_E^{-1}	Average duration of egg stage	3 (3–5) days	[28]
α_L^{-1}	Development time of the larva	14 (14–30) days	[28]
α_P^{-1}	Development time of pupae	9 (8–30) days	[28]
μ_A^{-1}	Moth life span	18 (15–21) days	[28]
K_M	Maximum biomass of maize plants	50 kg plant^{-1}	Estimate.
K_E, K_L	Egg environmental carrying capacity	10^8	Estimate.
K_L	Egg environmental carrying capacity	10^6	Estimate.
μ_E, μ_L, μ_P	Natural mortality rate of immature stages	0.01 day^{-1}	Estimate.
β	Plant attack rate by caterpillars	5×10^{-8} day^{-1}	Estimate.
r	Growth rate of maize plants	0.05 day^{-1}	Estimate.
e	Leaf impact factor	0.2 day^{-1}	Estimate.
u_E, u_L, u_P, u_A	Implications of parasite control	varied	

To determine the error in this method, by assuming that $t_i = ih = \frac{i\tau}{N}$ with some $N \in \mathbb{N}$, we have (see [35])

$$\max_{0\leq i\leq N} \left|x(t_i) - x_i\right| = \begin{cases} \mathcal{O}(h^2), & \alpha \leq 1, \\ \mathcal{O}(h^{1+\alpha}), & \alpha < 1. \end{cases}$$

Simulating system (1), we assumed the following initial population levels: $E(0) = 1000$, $L(0) = P(0) = 0$, $A(0) = 500$, and $M(0) = 15$.

Numerical results in Fig. 4 illustrate the dynamics of the pest and maize biomass whenever the reproduction ratio $\mathcal{R}_0$ is less than unity. As we can note, if the moth cannot produce more than one off-spring, then within a period of 200 days, all the FAW populations (eggs, larvae, pupae, and moth) will become extinct, whereas the maize biomass will increase with time till it reaches the expected maximum biomass per plant (50 kg plant^{-1}). We can also observe that the convergence of solutions to their respective limiting points in time depends on the fractional order q: as q approaches unity, the time taken by solutions to converge to the limiting point increases. From the simulation results shown in Fig. 5 we can observe that whenever each female moth reproduces more than one off-spring, that is, $\mathcal{R}_0 > 1$, then the pest population will persist in the field till the final harvesting time $t = 300$ day. In addition, the final maize biomass per plant will be less than the expected 50 k,plant^{-1}. Precisely, maize biomass increases from the start and reaches a maximum of 50 kg plant^{-1} after approximately 100 days, and after that, it decreases gradually till it stabilizes at approximated 18 kg plant^{-1}. Figure 6 shows the solutions of model system (1) for an experiment set up with small population sizes for the pest, that is, $E(0) = 100$, $L(0) = P(0) = 0$, and $A(0) = 50$, together with a control rate of $u_E = u_L = u_P = u_A = 0.45$ day $^{-1}$, leading to $\mathcal{R}_0 = 1.3583$. Furthermore, q was fixed to 0.8. As we can observe, the pest population increases rapidly within the first 100 days, and then it stabilizes. The maize biomass also increases during the first 50 days and attains a maximum approximately close to the expected value 50 kg plant^{-1}, and then the biomass decreases gradually for approximately 50 days before it becomes stable at approximately 50 kg plant^{-1}. Overall, the egg population will dominate all the pest populations.

Numerical results in Fig. 7 depicts the effects of a FAW outbreak with a large initial pest life cycle population, $E(0) = 2000$, $L(0) = P(0) = 0$, and $A(0) = 15$, combined with less effective control measures, that is, $u_E = u_L = 0.45$, and $u_P = u_A = 0$. More often, pesticides,

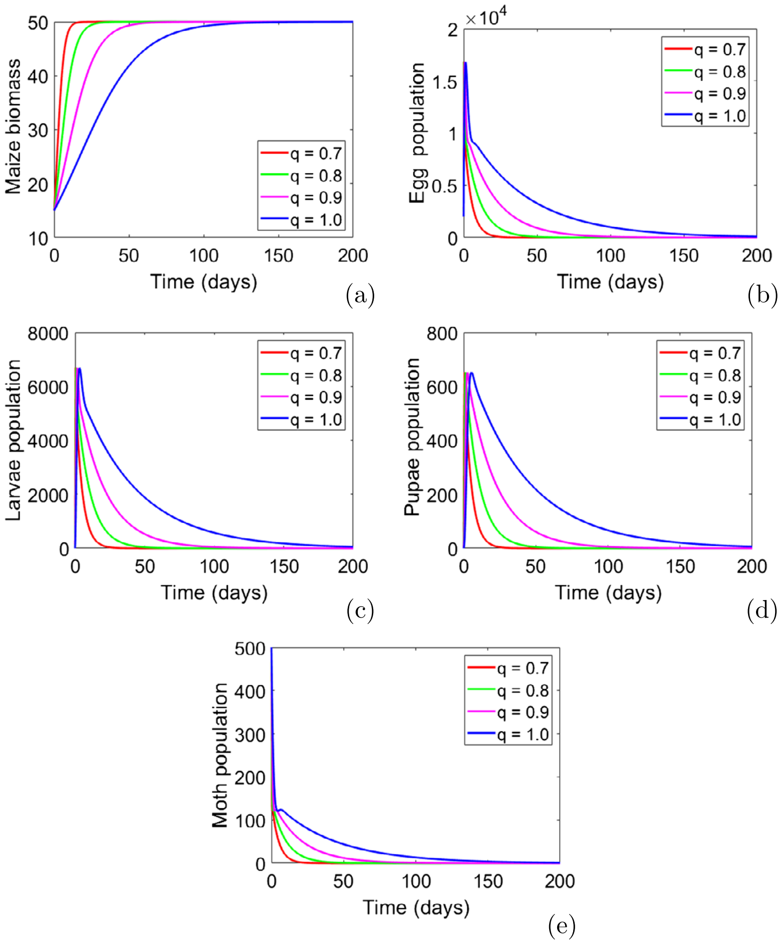

Figure 4 Numerical results of system (1) demonstrating the convergence of solutions to the pest-free equilibrium for $\mathcal{R}_0 \leq 1$. On construction of the simulations, we considered initial population levels discussed earlier, while baseline values for the model parameters are as in Table 2. In addition, we set $u_E = u_L = u_P = u_A = 0.52$ to obtain $\mathcal{R}_0 = 0.8630$. We can note that whenever $\mathcal{R}_0 < 1$, the pest population becomes extinct while maize biomass increases with time, reaching its expected maximum of 50 kg plant^{-1}

which are known to effectively control FAW, are expensive such that farmers in some areas rely on traditional methods of controlling the pest such as hand picking of caterpillars, picking and destroying egg masses, spraying lime, salt, oil, and soap solution. Prior studies suggests that traditional methods are less effective and are likely to eliminate the egg and larvae population. Hence, in Fig. 7, we explore the effects of a FAW outbreak with a large initial pest life cycle population coupled with less effective control measures. As we can observe, an outbreak with a large pest population coupled with less effective control measure may result in the pest population increasing rapidly so that in less than 100 days, they will reach their respective maximum. After an initial increase, the maize biomass would gradually decrease to a level below its initial biomass. The results highlight the importance of effective control measure on increasing maize biomass whenever there is a FAW outbreak.

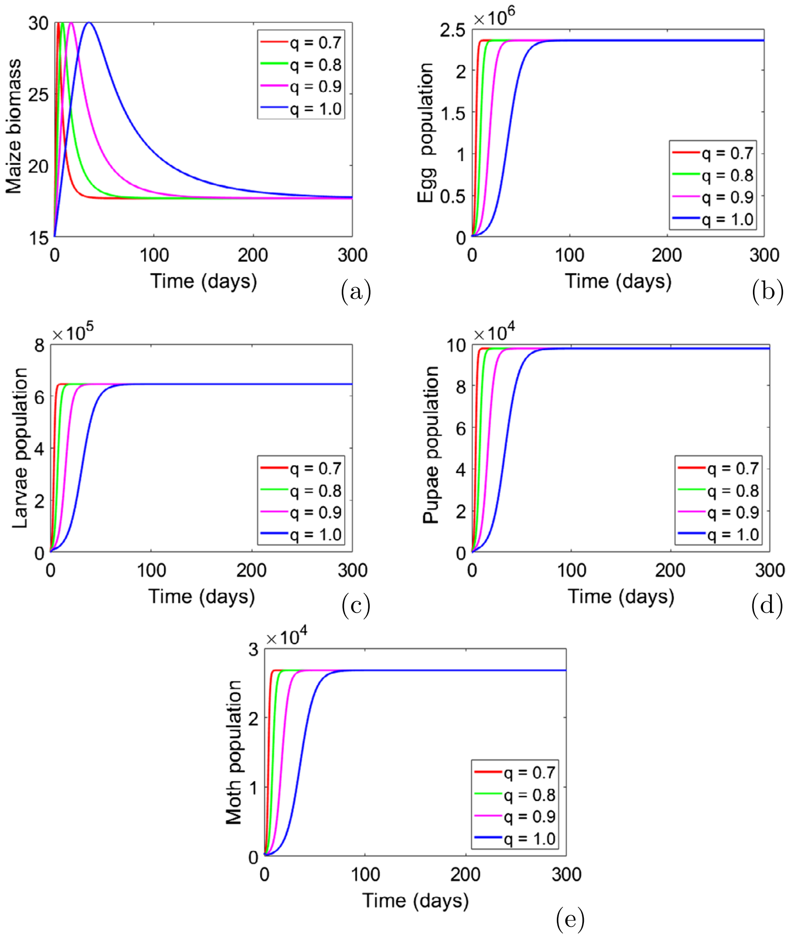

Figure 5 Numerical results of system (1) demonstrating the convergence of solutions to the pest persistence equilibrium point for $\mathcal{R}_0 > 1$. Baseline values for the model parameters are as in Table 2. In addition, we set $u_E = u_L = u_P = u_A = 0.35$ to obtain $\mathcal{R}_0 = 2.8931$. As we can observe, the solutions suggest that the pest will persist in the field and the final maize biomass will be less than the expected 50 kg plant^{-1}

5 Concluding remarks

In this study, we presented a Caputo fractional-order model for fall armyworm (*Spodoptera frugiperda*) infestation in a maize field. Fall armyworm (*Spodoptera frugiperda*) commonly known as FAW remain a major pest of maize. The pest has already been considered as one of the greatest threat to food security in Africa despite the fact that it was first detected in 2016. The pest is highly destructive and fast spreading. Moth are capable of flying up to 100 km in one night. Based on its destructiveness, there is need to gain understanding the dynamics of this pest and the maize plant whenever there is an outbreak. Through the use of mathematical models, it is possible to predict many real-world problems in various fields such as agriculture, economics, biology, engineering, and so on. In particular, mathematical models are capable of providing solutions to phenomena that are difficult to measure in the field. Here fractional derivatives model have been utilized to model the dynamics of FAW infestation in a maize field based on the fact that fractional calculus is naturally related to many adaptive systems with memory and hereditary properties, which widely exist in several fields such as biology, agriculture, medicine, physics, chemistry, and engineering [21].

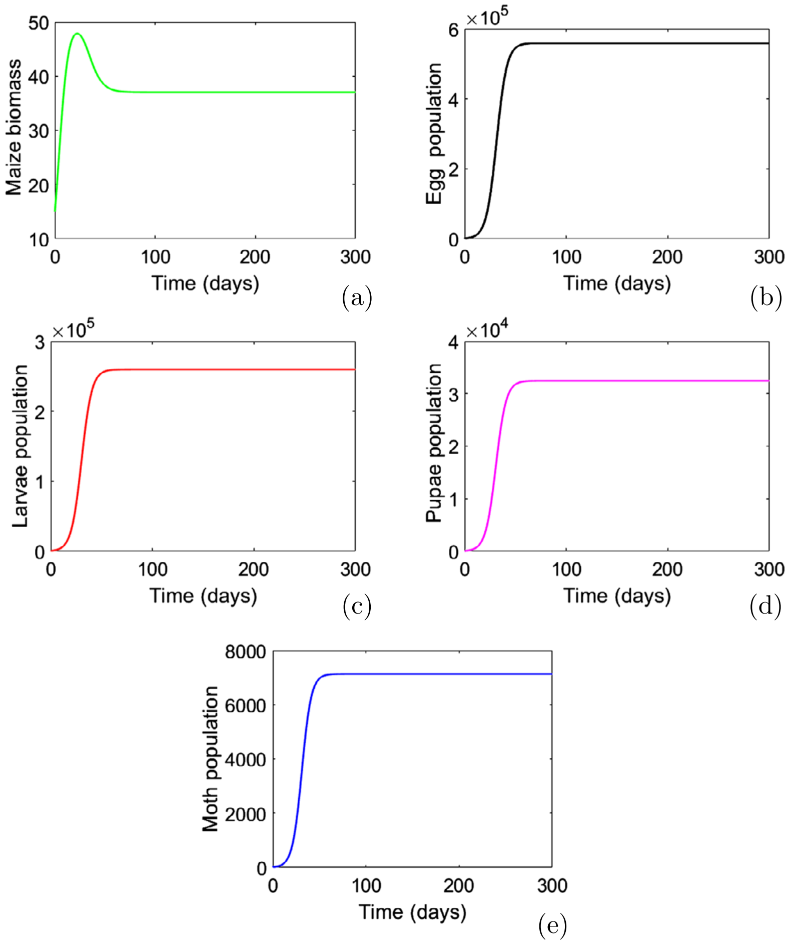

Figure 6 Model solutions showing the effects of a FAW outbreak with a small initial pest life cycle population. We considered $E(0) = 100$, $L(0) = P(0) = 0$, and $A(0) = 50$. Furthermore, we fixed $q = 0.8$ and $u_E = u_L = u_P = u_A = 0.45$ to obtain $\mathcal{R}_0 = 1.3583$

Mathematical analysis of the proposed model reveals that there exists a threshold parameter, the basic reproduction number, which governs the persistence and extinction of FAW in the field. Biologically, this basic reproduction number represents the average number of newborns produced by one individual female moth during its life span. We have noted that if one female moth is not capable of producing more than one off-spring, then the pest population becomes extinct; otherwise, it persists. The basic reproduction number was qualitatively and quantitatively used to investigate the local and global stability of the model steady state. For two steady states, the model has the pest-free equilibrium and the pest persistence equilibrium. We have observed that both are locally and globally stable. In particular, the pest-free equilibrium point is both locally and globally stable whenever the basic reproduction number is less than unity. However, when the basic reproduction number is greater than unity, there exists a pest persistence equilibrium point, which is also both locally and globally stable.

We have also noted that the model parameters, the egg laying rate and proportion of female moth in the environment, have a strong positive influence on increasing the size of the basic reproduction number. Precisely, increasing in the proportion of female moth

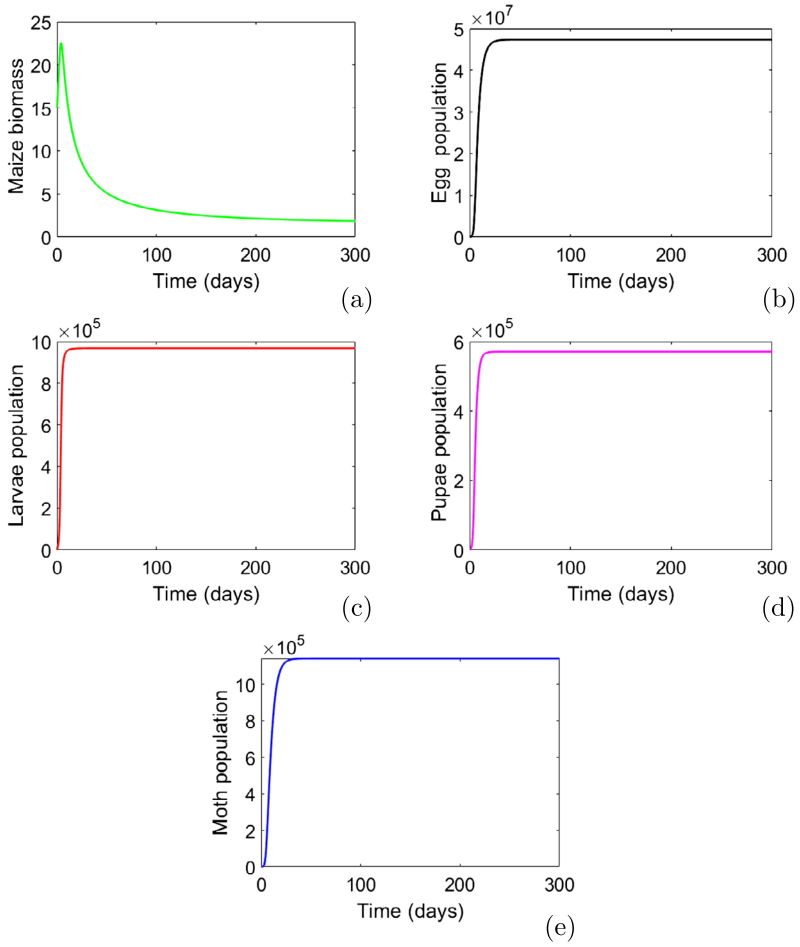

Figure 7 Model solutions showing the effects of a FAW outbreak with a large initial pest life cycle population, $E(0) = 2000$, $L(0) = P(0) = 0$, and $A(0) = 15$, coupled with less effective control measures, that is, $u_E = u_L = 0.45$ and $u_P = u_A = 0$, leading to $\mathcal{R}_0 = 58$. We set $q = 0.8$

in the environment by a certain percent increases the size of the basic reproduction by a similar percentage. FAW intervention strategies aimed at reducing moth population were observed to have a stronger impact on reducing the size of the basic reproduction number than any other model parameter. Numerical illustrations are included to support analytical results and to explore optimal intervention levels essential to minimize persistence of the pest population. We also used numerical simulations to illustrate the impact of initial pest population level during an outbreak on maize growth in a field.

The proposed model is not exhaustive. In the future work, we will explore the effects of temperature and seasonal variation on the dynamics of FAW and its implications on maize growth. In the current study, we found that there is a need for better metadata in plant population studies to help explain calibration and validation of proposed models. Although we did not manage to validate the proposed model with data, due to its unavailability, the proposed model and results will certainly improve the existing knowledge on FAW dynamics and its implications in maize crops.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Formal analysis and Methodology, SD; Supervision and writing review, LL, MK, DK, and SM. All the authors read and approved the final manuscript.

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A fractional-order fall armyworm-maize biomass model with naturally beneficial insects and optimal farming awareness

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ABSTRACT

Maize remains an important food crop in Africa. However, the production of this crop, and consequently the livelihood of the growers are threatened by the invasion and widespread infestation of the fall armyworm which causes substantial maize yield losses. In this paper, a fractional-order fall armyworm-maize biomass model with naturally beneficial insects and optimal farming awareness has been formulated. Comprehensive analysis of the model has shown that it contains five equilibrium points which are all locally and globally asymptotically stable if the conditions outlined in Lemma 2.1 and 2.2 are met. We also carried out numerical simulations to support the analytical results and to illustrate different dynamical regimes that can be observed in the model. We have found that time-dependent farming awareness can significantly reduce fall armyworm population if the cost of implementation is relatively low.

1. Introduction

In the last half century, invasive species have caused unprecedented challenges to agricultural systems globally. In sub-Saharan Africa (SSA), agriculture is considered the primary source of livelihoods for most households [1,2]. However, its contribution to food security and poverty reduction is hampered by several, often interacting, biotic and abiotic factors. For instance, the recent invasion of fall armyworm (FAW-*Spodoptera frugiperda* JE Smith) in SSA has become a major threat to food security in the region [2,3]. The first outbreak of FAW in Africa occurred in West Africa in 2016, and to date the pest has spread to 44 countries in the continent [2]. The FAW can cause damage to more than 80 crop species, including economically important crops such as maize, rice, sorghum, wheat, sugarcane and cotton just to mention a few.

Current estimates from 12 African countries suggest an annual loss of (4.1–17.6) million tons of maize due to FAW infestations [2]. In particular, farm-level estimates from Ghana and Zambia suggest yield losses of (22–67) per cent [3], 47% in Kenya [4] and 9.4% in Zimbabwe [5] due to FAW infestations. In maize, FAW attacks all cropping stages from

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seedling emergence through to ear development. They defoliate and destroy young plants whereby, whorl damage can result in yield losses, and ear feeding can result in the reduction of grain quality and yields [6]. The Management of the FAW involves the integration of several approaches, including the use of insecticides, host plant resistance, and biological control. However, all these approaches depend on several characteristics of the involved agro-ecosystems [7]. In South America where the pest has been a challenge for quite sometime, the common management strategy has been the use of insecticide sprays and genetically modified crops like Bt maize [6].

Due to financial challenges associated with most of African governments, alongside the cost associated with massive spraying programs of chemical insecticides and the use of genetically modified crops like Bt maize, the effective management of this pest in the continent remains a challenge [6]. In addition, excessive use of chemical insecticides is associated with negative environmental effects and can lead to the development of pest resistance [8]. At the backdrop of this, integrated pest management (IPM) has been gaining more attention among researchers and its application is also increasing the crop yields [9,10]. This approach seeks to minimize the reliance on pesticides use by emphasizing the application of biological control agents.

Mass media can affect the spread and attack poised by FAW during an outbreak. Furthermore, awareness campaigns, particularly through various media outlets such as radio, newspapers, TV and so on, do not only make farmers aware of FAW outbreaks but also improve trust on IPM a nation will be advocating for. In recent times, attention to health news has been observed to play an integral role in disease management [11]. There is no doubt that correct and relevant knowledge about maize crop and its pests is essential to farmers [10].

The main goal of this paper is to develop a mathematical model to assess the effects of media campaigns during a FAW outbreak. Mathematical models of plant-pest interactions have provided insights into effective methods for effective pest management as well as way of increasing plant productivity (e.g. [10,12–22]). In some of the studies (e.g. [10,12–14,16,23]), mathematical models were used to investigate the effects of biological control on the dynamics of plant pest interactions, while in other studies (e.g. [17–22]), pest management models based solely on chemical controls were proposed and analyzed. For instance, Liu and Teng [18] utilized a mathematical model to assess the impact of spraying pesticides at a fixed time on the pest reproductive cycles. Among several outcomes, their study showed that there exists an optimal time for pest control if the pesticides were to be applied just before each birth pulse of the cycle. Wei [24] proposed pest control models that incorporated birth pulse and were based on the assumption that pesticides killed adult pests or larvae or both. Making use of numerical simulation, the study demonstrated that with the different elimination rates for larvae and adults, the corresponding optimal times for pesticide applications were also different.

These studies and several others (e.g. [10,12–14,16–22,25]) have certainly produced many useful results and improved the existing knowledge on plant-pest interaction dynamics. Despite of all these efforts, mathematical models for FAW management during an outbreak are very few and of the few that exists there are some limitations; (i) majority of these few were general and not pest-specific, which implies that their results were also general. Practically, pests are not general, rather, they follow different biological development cycles, hence more informative plant-pest interaction models need to be pest-specific and closely follow the life cycle of the pest involved (ii) the presented models utilized integer-order differential equations (IDEs) which according to Caputo [26], do not replicate real-world problems nor capture memory effects as compared to fractional calculus.

Furthermore, unlike IDEs, models based on fractional calculus have been found to be more accurate with regard to describing rules and development processes of several phenomena in natural science [27] and this has been attributed to the fact that fractional order models possess memory effects and hereditary properties. Hence, there has been growing interest among researchers to use fractional calculus in modeling real-world problems, and some remarkable achievements have been made [27]. Cognizant of this, a fractional order pest-plant based model has been proposed in the present study with the aim to study the effects of educational campaigns and FAW larval predation on persistence and extinction of the pest in a maize field. The model incorporates the maize biomass and two essential development stages of the FAW, that is, the larval and the moth (adult). In addition, since FAW larvae are prey to several parasitoids, predators and pathogens like birds, rodents, beetles, earwigs [28], the proposed model incorporates the predator population.

The rest of the paper is organized as follows: In Section 2, a fractional-order FAW model is proposed and analyzed. In particular, the model's steady states have been computed and their stability has been investigate as well. In Section 3, we perform an optimal control study to determine the effects of farming on minimizing the effects of FAW on maize biomass, through both mathematical analysis and numerical simulation. Finally, we conclude the paper with some discussion in Section 4.

2. Model formulation and analytical results

2.1. Model formulation

We developed a mathematical model for FAW outbreak in a maize field focusing on investigating the effects of farming awareness and biological control (FAW predators). The proposed model is based on fractional calculus of Caputo type [26]. The FAW population is subdivided into two classes; the larvae $L(t)$ and the adult which also known as the moth $A(t)$. The FAW predator population is modeled by $Z(t)$. Meanwhile, the dynamics of maize biomass are represented by $M(t)$. The proposed model is governed by the following assumptions:

(i) We assume logistic growth for the density of maize biomass, with net growth rate r and carrying capacity K_M. Let β be the consumption rate by FAW larvae and e be the efficiency of biomass conversion. Awareness is assumed to reduce the attack rate of the maize crop by FAW larvae by a factor $1-u$, with $0 \leq u \leq 1$. Thus $u=0$ implies that awareness has no impact on reducing the attack rate of the maize plant by FAW whereas $u=1$ implies that farming awareness is 100% efficient in protecting the maize crop from FAW attack during an outbreak.

(ii) The dynamics of the FAW larvae are assumed to follow a logistic growth model, with net growth rate b_L and the carrying capacity K_L. The larvae are assumed to progress to the adult stage after approximately $1/\alpha_L$ days. The FAW larvae and adults suffer natural mortality at rates μ_L and μ_A, respectively. Apart from natural mortality, both populations diminish due to mortality attributed to the mitigation strategies carried out by farmers as a result of awareness, at the rate ud, where d is the mortality rate of the FAW larvae and adult. Note that if awareness does not have an impact ($u=0$) on FAW populations, then these populations suffer natural mortality only.

(iii) Even though biological control may not replace conventional insecticides, a number of parasitoids, predators and pathogens like birds, rodents, beetles and earwigs readily attack the larvae [28]. To account for the effect of larval predation, let σ be the attack rate of the larvae by predators and ρ be the efficiency of conversion. The average life span of predators is assumed to be $1/\eta$ days.

Based on the above assumptions, the proposed model is summarized by the following system of nonlinear ordinary differential equations (Fig. 1 shows the transition diagram):

$$\left.\begin{array}{rcl} {}_a^cD_t^qM(t) &=& r^qM\left(1-\dfrac{M}{K_M^q}\right)-\beta^q(1-u)LM,\\ {}_a^cD_t^qL(t) &=& b_L^qA\left(1-\dfrac{L}{K_L^q}\right)+e\beta^q(1-u)LM-\sigma^qZL-(\mu_L+\alpha_L+ud)L,\\ {}_a^cD_t^qA(t) &=& \alpha_L^qL-(\mu_A^q+ud)A,\\ {}_a^cD_t^qZ(t) &=& \rho\sigma^qLZ-\eta^qZ. \end{array}\right\} \tag{1}$$

with initial conditions as:

$$M(0)\geq 0,\qquad L(0)\geq 0,\qquad A(0)\geq 0,\qquad Z(0)\geq 0. \tag{2}$$

Here, ${}_a^cD_t^q$ represents the Caputo fractional derivative of order q ($0<q<1$). The Caputo fractional derivative of order q is defined [29]:

$${}_a^cD_t^qf(t)=\frac{1}{\Gamma(n-q)}\int_0^t\frac{f^n(\xi)}{(t-\xi)^{q+1-n}}d\xi,\quad n-1<q<n\in\mathbb{N},$$

where Γ represents the gamma function and the Riemann Liouville fractional integral of arbitrary real order $q>0$ of a function $f(t)$ is defined by the following integral:

$$J^qf(t)=\frac{1}{\Gamma(q)}\int_0^t(t-\xi)^{q-1}f(\xi)d\xi,$$

$$J^0f(t)=f(t).$$

Remark 2.1. Note that, in order to avoid flaws regarding the time dimension, we introduced q in the model parameters (right-hand side) of system (2) so that the dimensions of these parameters became $(time)^{-q}$ which is in agreement with the left-hand side of the model.

2.2. Positivity and boundedness of solutions

In this section, we study the positivity and boundedness of solutions of the proposed fractional order model (2) to establish if it is mathematically and biological poised. It follows from (2) that:

Theorem 2.1. *Model (2) is positively invariant and bounded in $\mathbb{R}_+^4$.*

Proof. This begin by demonstrating that $\mathbb{R}_+^4=\{(M,L,A,Z)\in\mathbb{R}_+^4 : M(0)\geq 0, L(0)\geq 0, A(0)\geq 0, Z(0)\geq 0\}$ is positively invariant. For that, we demonstrated that on each hyper-plane bounding the non-negative orthant, the vector field points to $\mathbb{R}_+^4$. Therefore, for $M(0)\geq 0$, $L(0)\geq 0$, $A(0)\geq 0$, $Z(0)\geq 0$, we have

$$\left.\begin{array}{rcl} {}_a^cD_t^qM(t)\mid_{M=0} &=& 0,\\ {}_a^cD_t^qL(t)\mid_{L=0} &=& b_L^qA\geq 0,\\ {}_a^cD_t^qA(t)\mid_{A=0} &=& \alpha_L^qL\geq 0,\\ {}_a^cD_t^qZ(t)\mid_{Z=0} &=& 0. \end{array}\right\} \tag{3}$$

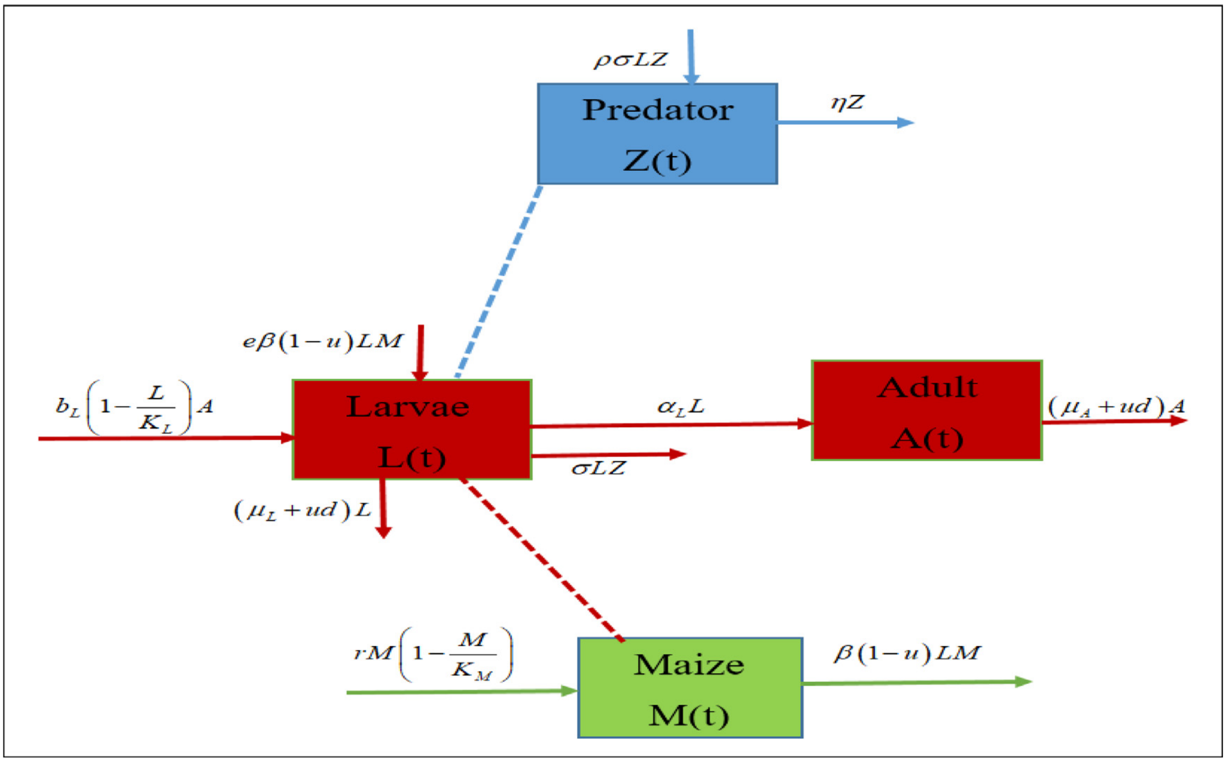


Fig. 1. Model flow diagram illustrating the dynamics of the FAW in a maize field. The fAW life cycle is divided into two classes; the larvae $L(t)$ and adult $A(t)$ population. The FAW predator and maize biomass population are represented by compartment $Z(t)$ and $M(t)$ respectively. Continuous lines indicate either inflow or outflow transition between compartments. Red and blue discontinuous arrows connecting compartment $L(t)$ with compartments $Z(t)$ and $M(t)$ show the interaction that occurs between the predators $Z(t)$ and FAW larvae $L(t)$ as well as with maize biomass $M(t)$. Note that the predator has an effect on larvae which in turn have an effect on maize biomass. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Based on the results in (3), it follows that model (2) is positively invariant in $\mathbb{R}^4_+$. Further, from the first equation of model (2), we demonstrate that $M(t) \le K_M$, $\forall t \ge 0$. If there exists t_0 such that $M(t_0) > K_M$, then due to the continuity of $M(t)$:

$$\exists B_\epsilon(t_0) : \forall t \in B_\epsilon(t_0) : M(t) > K_M, \tag{4}$$

so:

$$rM\left(1-\frac{M}{K_M}\right) < 0. \tag{5}$$

Thus ${}^c_{t_0}D^qM(t) < 0$. From the continuity of $M(t)$ and $\frac{dM}{dt} = \lim_{q\to 1^-} {}^c_{t_0}D^qM(t) < 0$, hence we conclude that $M(t)$ is a decreasing function for all $t \ge 0$ and it follows that $0 \le M(t) \le M(0) \le K_M$, $\forall t \ge 0$, and this is a contradiction to (4). Thus $M(t) \le K_M$, for all $t \ge 0$. Using a similar approach it can easily be verified that $0 \le L(t) \le K_L$. Now, from the third equation of system (2) we have:

$$\begin{aligned} {}^c_aD^q_tA(t) &= \alpha^q_L L - (\mu^q_A + ud^q)A \\ &\le \alpha^q K^q_L - (\mu^q_A + ud^q)A. \end{aligned} \tag{6}$$

Applying the Laplace transform one gets:

$$s^q\mathcal{L}[A(t)] - s^{q-1}A(0) \le \frac{\alpha^q_L K^q_L}{s} - (\mu^q_a + ud^q)\mathcal{L}[A(t)]. \tag{7}$$

After combining like terms one gets:

$$\begin{aligned} \mathcal{L}[A(t)] &\le \alpha^q_L K^q_L \frac{s^{-1}}{s^q + (\mu^q_a + ud^q)} + A(0)\frac{s^{q-1}}{s^q + (\mu^q_A + ud)} \\ &= \alpha^q_L K^q_L \frac{s^{q-(1+q)}}{s^q + (\mu^q_a + ud^q)} + A(0)\frac{s^{q-1}}{s^q + (\mu^q_a + ud^q)}. \end{aligned} \tag{8}$$

Applying the inverse Laplace transform leads to:

$$\begin{aligned} A(t) &\le \mathcal{L}^{-1}\left\{\alpha_L^q K_L^q \frac{s^{q-(1+q)}}{s^q + (\mu_a^q + ud^q)}\right\} + A(0)\mathcal{L}^{-1}\left\{\frac{s^{q-1}}{s^q + (\mu_a^q + ud^q)}\right\} \\ &\le \alpha_L^q K_L^q t^q E_{q,q+1}(-(\mu_a^q + ud^q)t^q) + A(0)E_{q,1}(-(\mu_a^q + ud^q)t^q) \\ &\le \frac{\alpha_L^q K_L^q}{(\mu_A^q + ud)}(\mu_A^q + ud)t^q E_{q,q+1}(-(\mu_a^q + ud^q)t^q) + A(0)E_{q,1}(-(\mu_a^q + ud^q)t^q) \\ &\le \max\left\{\frac{\alpha^q K_L^q}{(\mu_a^q + ud^q)}, A(0)\right\}((\mu_a^q + ud^q)t^q E_{q,q+1}(-(\mu_a^q + ud^q)t^q) + E_{q,1}(-(\mu_a^q + ud^q)t^q)) \\ &= \frac{C}{\Gamma(1)} = C_A, \end{aligned} \tag{9}$$

where E_q is the Mittag-Leffler function and $C_A = \max\left\{\frac{\alpha_L^q K_L^q}{(\mu_a^q + ud^q)}, P(0)\right\}$. Thus, $A(t)$ is bounded from above. From the last equation of system (2) we have:

$$\begin{aligned} {}_a^c D_t^q Z(t) &= \rho\sigma^q LZ - \eta^q Z \\ &\le -(\eta^q - \rho\sigma^q K_L^q)Z. \end{aligned} \tag{10}$$

Applying the Laplace transform in the previous inequality, we get:

$$s^q \mathcal{L}[Z(t)] - s^{q-1}Z(0) \le -(\eta^q - \rho\sigma^q K_L^q)\mathcal{L}[Z(t)], \tag{11}$$

which can be written as:

$$\mathcal{L}[Z(t)] \le Z(0)\frac{s^{q-1}}{s^q + (\eta^q - \rho\sigma^q K_L^q)}. \tag{12}$$

Applying the inverse Laplace transforms leads to

$$Z(t) \le Z(0)E_q[-(\eta^q - \rho\sigma^q K_L^q)t^q]. \tag{13}$$

Hence, we conclude that $Z(t)$ is bounded. □

2.3. Equilibrium points and their existence

The fractional-order model (2) has the following six equilibrium points:

(a) The trivial equilibrium point $\mathcal{E}_0 : (M_0, L_0, A_0, Z_0) = (0, 0, 0, 0)$ always exists.

(b) The pest-extinction equilibrium point $\mathcal{E}_1 : (M_1, L_1, A_1, Z_1) = (K_M, 0, 0, 0)$ always exists.

(c) The plant-extinction equilibrium point $\mathcal{E}_2 : (M_2, L_2, A_2, Z_2)$ where:

$$\left.\begin{aligned} M_2 &= 0, \qquad L_2 = \frac{\eta^q}{\rho}, \qquad A_2 = \frac{\eta^q \alpha^q}{(\mu_A^q + ud^q)\rho}, \\ Z_2 &= \frac{b^q\eta^q + \rho K_L^q(\mu_L^q + \alpha^q + ud^q)}{\rho\sigma^q K_L^q}\left(\frac{b^q \rho K_L^q}{b^q\eta^q + \rho K_L^q(\mu_L^q + ud^q + \alpha^q)} - 1\right). \end{aligned}\right\} \tag{14}$$

Thus, the equilibrium point $\mathcal{E}_2$ makes biological sense if $\frac{b^q \rho K_L^q}{b^q\eta^q + \rho K_L^q(\mu_L^q + ud^q + \alpha_L^q)} > 1$.

(d) The plant and predator-extinction equilibrium point $\mathcal{E}_3 : (M_3, L_3, A_3, Z_3)$ where:

$$\left.\begin{aligned} M_3 &= 0, \qquad L_3 = \frac{\eta^q K_L^q}{b^q}\left(\frac{b^q}{(\mu_L^q + \alpha_L^q + ud^q)} - 1\right), \\ A_3 &= \frac{\alpha_L^q K_K^q(\mu_L^q + \alpha_L^q + ud^q)}{b^q(mu_A^q + ud^q)}\left(\frac{b^q}{(\mu_L^q + \alpha_L^q + ud^q)} - 1\right), \\ Z_3 &= 0. \end{aligned}\right\} \tag{15}$$

Therefore the equilibrium point $\mathcal{E}_3$ exists and is biologically meaningful if $b^q > (\mu_L^q + \alpha_L^q + ud^q)$.

(e) The predator-extinction equilibrium point is $\mathcal{E}_4 : (M_4, L_4, A_4, Z_4)$ where:

$$\left.\begin{aligned}
M_4 &= \frac{K_M^q\left[r^qb^q + \beta^qK_L^q(\mu_L^q+\alpha_L^q+ud^q)\left(1-\dfrac{b^q}{(\mu_L^q+\alpha_L^q+ud^q)}\right)\right]}{r^qb^q+e(\beta^q(1-u))^2K_M^qK_L^q},\\
L_4 &= \frac{r^qK_L^q}{r^qb^q+e(\beta^q(1-u))^2K_M^qK_L^q}\left(\frac{b^q+e\beta^q(1-u)K_M^q}{(\mu_L^q+\alpha^q+ud^q)}-1\right),\\
A_4 &= \frac{r^q\alpha^qk_L^q}{(\mu_A^q+ud^q)(\mu_L^q+\alpha_L+ud^q)\tilde{n}}\left(\frac{b^q+e\beta^q(1-u)K_M^q}{(\mu_L^q+\alpha_L^q+ud^q)}-1\right),\\
Z_4 &= 0.
\end{aligned}\right\} \tag{16}$$

where $\tilde{n} = (r^qb^q + e(\beta^q(1-u))^2K_L^qK_M^q$. It follows that the equilibrium point $\mathcal{E}_4$ exists and is biologically feasible if $b^q + e\beta^q(1-u)K_M^q > (\mu_L^q+\alpha_L^q+ud^q)$ with $b^q < (\mu_L^q+\alpha_L^q+ud^q)$.

(f) The coexistence equilibrium point $\mathcal{E}_5 : (M_5, L_5, A_5, Z_5)$ where:

$$\left.\begin{aligned}
M_5 &= \frac{K_M^q\beta^q\eta^q}{r^q\rho}\left(\frac{r^q\rho}{\beta^q\eta^q}-1\right) \qquad L_5 = \frac{\eta^q}{\eta^q}, \qquad A_5 = \frac{\eta^q\alpha_L^q}{(\mu_A^q+ud^q)\rho},\\
Z_5 &= \frac{K_M^qe(\beta^q(1-u))^2\eta^q}{r^q\rho\sigma^q}\left(\frac{r^q\rho^q}{\beta^q\eta^q}-1\right)\\
&\quad+\frac{\sigma^q(K_L^qr^q\rho(\mu_L^q+\alpha_L^q+ud^q))+b^q}{r^q\rho}\left(\frac{b^qK_L^qr^q\rho}{b^qr^q\eta^q(\mu_L^q+\alpha_L^q+ud^q)K_L^qr^q\rho}-1\right).
\end{aligned}\right\} \tag{17}$$

Therefore, the equilibrium point $\mathcal{E}_5$ exists and is biologically meaningful if $b^qK_L^qr^q\rho > b^qr^q\eta^q(\mu_L^q+\alpha_L^q+ud^q)K_L^qr^q\rho$ with $r^q\rho > \beta^q\eta^q$.

2.4. Local stability analysis of the equilibrium points

The local stability analysis of for the fractional order model (2) around the above equilibrium points is obtained by computing the Jacobian matrix corresponding to equilibrium points. The Jacobian matrix of system (2) is as follows:

$$J(M,L,A,Z) = \begin{bmatrix} r^q-\beta^qL-\dfrac{2r^qM}{K_M^q} & -\beta^qM & 0 & 0\\ e\beta^q(1-u)L & n & 0 & -\sigma^qL\\ 0 & \alpha_L^q & -(\mu_A^q+ud^q) & 0\\ 0 & \rho Z & 0 & -\eta^q+\rho L\end{bmatrix}. \tag{18}$$

with $n = b^q + e\beta^q(1-u)M - \sigma^qZ - (\mu_L^q+\alpha_L^q+ud^q) - \frac{2b^qL}{K_L^q}$. The local stability of the equilibrium points of model (2) is now investigated making use of the Jacobian matrix (18) and Lemmas 2.1 and 2.2.

Lemma 2.1 ([30]). *Consider the following fractional order system:*

$$\left.\begin{aligned} {}_{t_0}^{c}D^qx(t) &= f(t,x),\\ x(0) &= x_0\end{aligned}\right\} \tag{19}$$

where $f(t,x) : \mathbb{R}^+\times\mathbb{R}^n \rightarrow \mathbb{R}^n$. *The equilibrium points* (14) *are locally asymptotically stable if all eigenvalues* λ_i *of the Jacobian matrix* $\frac{\partial f(t,x)}{\partial x}$ *evaluated at the equilibrium points satisfy the following condition:*

$$|\arg(\lambda_i) > \frac{q\pi}{2}.$$

Lemma 2.2 ([31], Routh–Hurwitz Criteria). *Given the polynomial:*

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + a_3\lambda^{n-3} + a_4\lambda^{n-4} + \cdots + a_{n-1}\lambda + a_n,$$

where the coefficients a_i *are real constants,* $i = 1,\ldots,n$, *define the* n *Hurwitz matrices using the coefficients* a_i *of the characteristic polynomial*

$$H_1 = \begin{bmatrix}a_1\end{bmatrix}, \quad H_2 = \begin{bmatrix}a_1 & 1\\ a_3 & a_2\end{bmatrix}, \quad H_3 = \begin{bmatrix}a_1 & 1 & 0\\ a_3 & a_2 & a_1\\ a_5 & a_4 & a_3\end{bmatrix},$$

and

$$H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix},$$

where $a_j = 0$ *if* $j > n$. *All of the roots of the polynomial* $P(\lambda)$ *are negative or have negative real part if and only if the determinants of all Hurwitz matrices are positive:*

$$det(H_j) > 0, \qquad j = 1, 2, \dots, n.$$

Routh–Hurwitz criteria for $n = 2, 3,$ *and* 4 *are as follows:*

$$\begin{aligned} &(C1)\ n = 2: && a_1 > 0, \quad and \quad a_2 > 0, \\ &(C2)\ n = 3: && a_1 > 0, \quad a_3 > 0, \quad and \quad a_1a_2 > a_3 \\ &(C3)\ n = 4: && a_1 > 0, \quad a_3 > 0, \quad a_4 > 0, \quad and \quad a_1a_2a_3 > a_3^2 + a_1^2a_4. \end{aligned}$$

Theorem 2.2.

(i) *The trivial equilibrium point* $\mathcal{E}_0$ *is locally asymptotically unstable.*
(ii) *If* $b^q < (\mu_L^q + \alpha_L^q + ud^q)$, *then the pest-extinction equilibrium point* $\mathcal{E}_1$ *is locally asymptotically stable.*
(iii) *If* $r^q\rho < \beta^q\eta^q$ *and condition* $(C1)$ *of* Lemma 2.2 *holds, then the equilibrium point* $\mathcal{E}_2$ *is locally asymptotically stable, otherwise it is unstable.*
(iv) *If* $b^q + e\beta^q(1-u)K_M^q < (\mu_L^q + \alpha^q + ud^q)$ *and condition* $(C1)$ *of* Lemma 2.2 *holds, then the equilibrium point* $\mathcal{E}_4$ *is locally asymptotically stable, otherwise it is unstable.*
(v) *If condition* $(C2)$ *of* Lemma 2.2 *holds, then the equilibrium point* $\mathcal{E}_5$ *is locally asymptotically stable, otherwise it is unstable.*

Proof.

(i) The Jacobian matrix of system (2) evaluated at $\mathcal{E}_0$ is

$$J(\mathcal{E}_0) = \begin{bmatrix} r^q & 0 & 0 & 0 & \\ 0 & b^q - (\mu_L^q + \alpha_L^q + ud^q) & 0 & 0 & \\ 0 & \alpha_L^q & -(\mu^q + ud^q) & 0 & \\ 0 & 0 & 0 & 0 & -\eta^q \end{bmatrix}.$$

The eigenvalues of matrix $J(\mathcal{E}_0)$ are $\lambda_1 = r^q > 0$, $\lambda_2 = b^q - (\mu_L^q + \alpha_L^q + ud^q)$, $\lambda_3 = -(\mu_A + ud^q)$ and $\lambda_4 = -\eta^q$. Since $\lambda_1 > 0$ it follows that the trivial equilibrium point $\mathcal{E}_0$ is locally asymptotically unstable.

(ii) The Jacobian matrix of system (2) evaluated at $\mathcal{E}_1$ is

$$J(\mathcal{E}_1) \begin{bmatrix} -r^q & 0 & 0 & 0 & \\ 0 & b^q - (\mu_L^q + \alpha_L^q + ud^q) & 0 & 0 & \\ 0 & \alpha_L^q & -(\mu^q + ud^q) & 0 & \\ 0 & 0 & 0 & 0 & -\eta^q \end{bmatrix}.$$

The eigenvalues of matrix $J(\mathcal{E}_0)$ are $\lambda_1 = -r^q$, $\lambda_2 = b^q - (\mu_L^q + \alpha_L^q + ud^q)$, $\lambda_3 = -(\mu_A + ud^q)$ and $\lambda_4 = -\eta^q$. Following Lemma 2.1, it can be observed that the equilibrium point $\mathcal{E}_1$ is locally asymptotically stable if $b^q < (\mu_L^q + \alpha_L^q + ud^q)$

(iii) The Jacobian matrix of system (2) evaluated at $\mathcal{E}_2$ is:

$$J(\mathcal{E}_2) = \begin{bmatrix} r^q - \beta^q L_2 & 0 & 0 & 0 \\ e\beta^q(1-u)L_2 & \widetilde{m} & 0 & -\sigma^q L_2 \\ 0 & \alpha_L^q & -(\mu_A^q + ud^q) & 0 \\ 0 & \rho Z_2 & 0 & -\eta^q + \rho L_2 \end{bmatrix}. \tag{20}$$

with $\widetilde{m} = b^q - \sigma^q Z_2 - (\mu_L^q + \alpha_L^q + ud^q) - \frac{2b^q L_2}{K_L^q}$. The eigenvalues of matrix (20) are $\lambda_1 = r^q - \frac{\beta^q\eta^q}{\rho}$, $\lambda_2 = -(\mu_A^q + ud^q)$ and the remaining eigenvalues can be obtained from the reduced matrix

$$\widetilde{J}(\mathcal{E}_2) = \begin{bmatrix} b^q - \sigma^q Z_2 - (\mu_L^q + \alpha_L^q + ud^q) - \dfrac{2b^q L_2}{K_L^q} & -\sigma^q L_2 \\ \rho Z_2 & -\eta^q + \rho L_2 \end{bmatrix}, \tag{21}$$

whose characteristic equation is as follows

$$\lambda^2 + a_1\lambda + a_2 = 0, \tag{22}$$

with

$$\begin{aligned}
a_1 &= \eta^q + \sigma^q Z_2 + (\mu_L^q + \alpha_L^q + ud^q) - b^q, \\
&= \eta^q + \sigma^q \frac{b^q\eta^q + \rho K_L^q(\mu_L^q + \alpha^q + ud^q)}{\rho\sigma^q K_L^q}\left(\frac{b^q\rho K_L^q}{b^q\eta^q + \rho K_L^q(\mu_L^q + ud^q + \alpha^q)} - 1\right) \\
&\quad + b^q\left(\frac{(\mu_L^q + \alpha_L^q + ud^q)}{b^q} - 1\right) \\
a_2 &= \sigma^q\eta^q Z_2 + (\eta^q - \rho L_2)\left((\mu_L^q + \alpha_L^q + ud^q) + \frac{2bL_2}{K_L} - b^q\right) \\
&= \sigma^q\eta^q \frac{b^q\eta^q + \rho K_L^q(\mu_L^q + \alpha^q + ud^q)}{\rho\sigma^q K_L^q}\left(\frac{b^q\rho K_L^q}{b^q\eta^q + \rho K_L^q(\mu_L^q + ud^q + \alpha^q)} - 1\right).
\end{aligned} \tag{23}$$

Therefore, if $r^q\rho < \beta^q\eta^q$ and condition $(C1)$ of Lemma 2.2 holds, then the equilibrium point $\mathcal{E}_2$ is locally asymptotically stable, otherwise it is unstable.

(iv) The Jacobian matrix of system (2) evaluated at $\mathcal{E}_3$ is

$$J(\mathcal{E}_3) = \begin{bmatrix} r^q - \beta^q L_3 & 0 & 0 & 0 \\ e\beta^q L_3 & b^q - (\mu_L^q + \alpha_L^q + ud^q) - \dfrac{2b^q L_3}{K_L^q} & 0 & -\sigma^q L_3 \\ 0 & \alpha_L^q & -(\mu_A^q + ud^q) & 0 \\ 0 & \rho Z_3 & 0 & -\eta^q + \rho L_3 \end{bmatrix}. \tag{24}$$

One can observe that, $\lambda_1 = r^q - \frac{\beta^q\eta^q}{\rho}$, $\lambda_2 = -(\mu_A^q + ud^q)$ are some of the eigenvalues of the Jacobian matrix (24), hence matrix (24) reduces to

$$\widetilde{J}(\mathcal{E}_3) = \begin{bmatrix} b^q - (\mu_L^q + \alpha_L^q + ud^q) - \dfrac{2b^q L_3}{K_L^q} & -\sigma^q L_3 \\ \rho Z_3 & -\eta^q + \rho L_3 \end{bmatrix}. \tag{25}$$

From (25) we have the characteristic equation

$$\lambda^2 + \bar{a}_1\lambda + \bar{a}_2 = 0, \tag{26}$$

with

$$\begin{aligned}
\bar{a}_1 &= b^q\left(\frac{(\eta^q + \mu_L^q + \alpha_L^q + ud^q)}{b^q} - 1\right) + 2\eta^q\left(\frac{b^q}{(\mu_L^q + \alpha^q + ud^q)} - 1\right), \\
\bar{a}_2 &= \eta^q(\mu_L^q + \alpha^q + ud^q)\left(1 - \frac{b^q}{(\mu_L^q + \alpha^q + ud^q)}\right)\left(1 - \frac{\rho K_L^q}{b^q}\left(\frac{b^q}{(\mu_L^q + \alpha^q + ud^q)} - 1\right)\right) \\
&\quad + 2(\eta^q)^2\left(1 - \rho\frac{\eta^q K_L^q}{b^q}\left(\frac{b^q}{(\mu_L^q + \alpha^q + ud^q)} - 1\right)\right)\left(\frac{b^q}{(\mu_L^q + \alpha^q + ud^q)} - 1\right).
\end{aligned} \tag{27}$$

Thus, if $r^q\rho < \beta^q\eta^q$ and condition $(C1)$ of Lemma 2.2 holds, then the equilibrium point $\mathcal{E}_3$ is locally asymptotically stable, otherwise it is unstable.

(iv) The Jacobian matrix of system (2) evaluated at $\mathcal{E}_4$ is

$$J(\mathcal{E}_4) = \begin{bmatrix} r^q - \beta^q L_4 - \dfrac{2r^q M_4}{K_M^q} & -\beta^q M_4 & 0 & 0 \\ e\beta^q(1-u)L_4 & \bar{n} & 0 & -\sigma^q L_4 \\ 0 & \alpha_L^q & -(\mu_A^q + ud^q) & 0 \\ 0 & 0 & 0 & -\eta^q + \rho L_4 \end{bmatrix}, \tag{28}$$

with $\bar{n} = b^q + e\beta^q(1-u)M_4 - (\mu_L^q + \alpha_L^q + ud^q) - \frac{2b^qL_4}{K_L^q}$. The eigenvalues of $J(\mathcal{E}_4)$ are;

$$\begin{aligned}
\lambda_1 &= -(\mu_A^q + ud^q)\\
\lambda_2 &= -\eta^q + \rho L_4\\
&= -\eta^q - \frac{\rho r^q K_L^q}{r^qb^q + e(\beta^q(1-u))^2K_M^qK_L^q}\left(1 - \frac{b^q + e\beta^q(1-u)K_M^q}{(\mu_L^q + \alpha^q + ud^q)}\right).
\end{aligned} \tag{29}$$

Hence, matrix (28) reduces to

$$\widetilde{J}(\mathcal{E}_4) = \begin{bmatrix} w_1 & -\beta^qM_4\\ e\beta^q(1-u)L_4 & w_2\end{bmatrix}, \tag{30}$$

with

$$\begin{aligned}
w_1 &= r^q - \beta^qL_4 - \frac{2r^qM_4}{K_M^q}\\
&= r^q - \frac{\beta^q r^q K_L^q}{r^qb^q + e(\beta^q(1-u))^2K_M^qK_L^q}\left(\frac{b^q + e\beta^q(1-u)K_M^q}{(\mu_L^q + \alpha^q + ud^q)} - 1\right)\\
&\quad - \frac{2r^q\left[r^qb^q + \beta^qK_L^q(\mu_L^q + \alpha_L^q + ud^q)\left(1 - \frac{b^q}{(\mu_L^q + \alpha_L^q + ud^q)}\right)\right]}{r^qb^q + e(\beta^q(1-u))^2K_M^qK_L^q},\\
w_2 &= b^q + e\beta^q(1-u)M_4 - (\mu_L^q + \alpha_L^q + ud^q) - \frac{2b^qL_4}{K_L^q}\\
&= b^q + e\beta^q(1-u)\frac{K_M^q\left[r^qb^q + \beta^qK_L^q(\mu_L^q + \alpha_L^q + ud^q)\left(1 - \frac{b^q}{(\mu_L^q + \alpha_L^q + ud^q)}\right)\right]}{r^qb^q + e(\beta^q(1-u))^2K_M^qK_L^q}\\
&\quad - (\mu_L^q + \alpha_L^q + ud^q) - \frac{2b^qr^q}{r^qb^q + e(\beta^q(1-u))^2K_M^qK_L^q}\left(\frac{b^q + e\beta^q(1-u)K_M^q}{(\mu_L^q + \alpha^q + ud^q)} - 1\right).
\end{aligned}$$

From (30), the corresponding characteristic equation is

$$\lambda^2 + \tilde{a}_1\lambda + \tilde{a}_2 = 0, \tag{31}$$

with

$$\begin{aligned}
\tilde{a}_1 &= -(w_1 + w_2),\\
\tilde{a}_2 &= w_1w_2 + \frac{e\beta^{2q}(1-u)^2r^qK_L^q}{r^qb^q + e(\beta^q(1-u))^2K_M^qK_L^q}\left(\frac{b^q + e\beta^q(1-u)K_M^q}{(\mu_L^q + \alpha_L^q + ud^q)} - 1\right).
\end{aligned}$$

Therefore, if $b^q + e\beta^q(1-u)K_M^q < (\mu_L^q + \alpha_L^q + ud^q)$ and condition $(C1)$ of Lemma 2.2 holds, then the equilibrium point $\mathcal{E}_4$ is locally asymptotically stable, otherwise it is unstable.

(vi) Since all the variables are non-zero at the coexistence equilibrium point, it follows that matrix J (18) is the Jacobian matrix of system (2) at this equilibrium point. From (18) one can observe, that $\lambda_1 = -(\mu_A^q + ud^q)$ and the remainder can be obtained from the following reduced matrix:

$$\bar{J}(\mathcal{E}_5) = \begin{bmatrix} \bar{w}_1 & -\beta^qM_5 & 0\\ e\beta^qL_5 & \bar{w}_2 & -\sigma^qL_5\\ 0 & \rho Z_5 & \bar{w}_3\end{bmatrix}. \tag{32}$$

where

$$\begin{aligned}
\bar{w}_1 &= r^q - \beta^qL_5 - \frac{2r^qM_5}{K_M^q},\\
\bar{w}_2 &= b^q + e\beta^q(1-u)M_5 - \sigma^qZ_5 - (\mu_L^q + \alpha_L^q + ud^q) - \frac{2b^qL_5}{K_L^q},\\
\bar{w}_3 &= -\eta^q + \rho L_5.
\end{aligned} \tag{33}$$

The corresponding characteristic equation at $\mathcal{E}_5$ becomes

$$\lambda^3 + a_1^*\lambda^2 + a_2^*\lambda + a_3^* = 0,$$

with

$$\begin{aligned}
a_1^* &= -(\bar{w}_1 + \bar{w}_2 + \bar{w}_3),\\
a_2^* &= \bar{w}_1(\bar{w}_2 + \bar{w}_2) + \bar{w}_2\bar{w}_3 + \sigma\rho L_5 Z_5 + e(\beta^q)^2 L_5 M_5,\\
a_3^* &= -\bar{w}_1(\sigma^q \rho L_5 Z_5 + \bar{w}_2\bar{w}_3) - e(\beta^q)^2 L_5 M_5 \bar{w}_3.
\end{aligned}$$

Since $\lambda_1 < 0$, it follows that condition $(C2)$ of Lemma 2.2 holds, then the equilibrium point $\mathcal{E}_5$ is locally asymptotically stable, otherwise it is unstable. This completes the proof. □

2.5. Global stability analysis of the equilibrium points

In this section, Lyapunov functions will be constructed in order to investigate the global stability of the equilibrium points of the model. To simplify the analysis, let $g_0(M) = r^q M(1 - M/K_M)$ and $g_1(L, A) = b_L(1 - L/K_L)A$.

Theorem 2.3. *The trivial equilibrium point $\mathcal{E}_0$ is globally asymptotically stable whenever:*

$$eg_0(M) + g_1(L, A) \leq \frac{(\mu_L^q + \alpha_L^q + ud^q)(\mu_A^q + ud^q)A}{\alpha_L^q} + \frac{\sigma^q \eta^q}{\rho} Z.$$

Proof. Let us consider the following Lyapunov function:

$$U_0(t) = eM(t) + L(t) + \frac{(\mu_L^q + \alpha_L^q + ud^q)}{\alpha_L^q} A(t) + \frac{1}{\rho} Z(t). \tag{34}$$

The fractional derivative of (34) along the solutions of system (2) leads to:

$$\begin{aligned}
{}_a^cD_t^q U_0(t) &\leq {}_a^cD_t^q[eM(t)] +{}_a^c D_t^q L(t) +{}_a^c D_t^q \left[\frac{(\mu_L^q + \alpha_L^q + ud^q)}{\alpha_L^q} A(t)\right] +{}_a^c D_t^q \left[\frac{1}{\rho} Z(t)\right]\\
&= e[g_0(M) - \beta^q(1-u)LM] + g_1(L, A) + e\beta^q(1-u)LM - \sigma^q ZL - (\mu_L^q + \alpha_L^q + ud^q)L\\
&\quad + \frac{(\mu_L^q + \alpha_L^q + ud^q)}{\alpha_L^q}\left[\alpha_L^q L - (\mu_A^q + ud^q)A\right] + \frac{1}{\rho}\left[\rho\sigma^q LZ - \eta^q Z\right]\\
&= eg_0(M) + g_1(L, A) - \frac{(\mu_L^q + \alpha_L^q + ud^q)(\mu_A^q + ud^q)A}{\alpha_L^q} - \frac{\eta^q}{\rho} Z.
\end{aligned} \tag{35}$$

It follows that if $M(t) = M_0$, $L(t) = L_0$, $A(t) = A_0$ and $Z(t) = Z_0$, then ${}_a^cD_t^q U_1(t) = 0$. However, if:

$$eg_0(M) + g_1(L, A) \leq \frac{(\mu_L^q + \alpha_L^q + ud^q)(\mu_A^q + ud^q)A}{\alpha_L^q} + \frac{\sigma^q \eta^q}{\rho} Z < 0,$$

then ${}_a^cD_t^q U_1(t) < 0$ and the trivial equilibrium point $\mathcal{E}_0$ is globally asymptotically stable, otherwise it is unstable. This completes the proof. □

Theorem 2.4. *The equilibrium point $\mathcal{E}_1$ is globally asymptotically stable whenever:*

$$eg_0(M)\left(1 - \frac{M^*}{M} + \beta^q(1-u)\frac{LM^*}{g_0(M)}\right) + g_1(L, A) - \frac{(\mu_L^q + \alpha_L^q + ud^q)(\mu_A^q + ud^q)A}{\alpha_L^q} - \frac{\sigma^q \eta^q}{\rho} Z \leq 0.$$

Proof.

$$U_1(t) = e\left[M(t) - M_1 - M_1 \ln\left(\frac{M(t)}{M_1}\right)\right] + L(t) + \frac{(\mu_L^q + \alpha_L^q + ud^q)}{\alpha_L^q} A(t) + \frac{1}{\rho^q} Z(t). \tag{36}$$

The fractional derivative of (36) along the solutions of system (2) leads to:

$$\begin{aligned}
{}_a^cD_t^q U_1(t) &\leq e\left(1 - \frac{M^*}{M(t)}\right) {}_a^cD_t^q M(t) +{}_a^c D_t^q L(t) +{}_a^c D_t^q \left[\frac{(\mu_L + \alpha_L + ud)}{\alpha_L} A(t)\right] +{}_a^c D_t^q \left[\frac{1}{\rho} Z(t)\right]\\
&= e\left(1 - \frac{M_1}{M(t)}\right)(g_0(M) - \beta^q(1-u)LM) + g_1(L, A) + e\beta^q(1-u)LM - \sigma^q ZL
\end{aligned}$$

$$
\begin{aligned}
&-(\mu_L+\alpha_L+ud)L+\frac{(\mu_L^q+\alpha_L^q+ud^q)}{\alpha_L^q}\left[\alpha_L^q L-(\mu_A^q+ud^q)A\right]+\frac{1}{\rho}\left[\rho\sigma^q LZ-\eta^q Z\right]\\
&=eg_0(M)\left(1-\frac{M_1}{M}+\beta(1-u)\frac{LM_1}{g_0(M)}\right)+g_1(L,A)-\frac{(\mu_L+\alpha_L+ud)(\mu_A^q+ud^q)A}{\alpha_L^q}\\
&-\frac{\sigma^q\eta^q}{\rho}Z.
\end{aligned}
$$

It follows that if $M(t)=M_1$, $L(t)=L_1$, $A(t)=A_1$ and $Z(t)=Z_1$, then ${}_a^cD_t^qU_1(t)=0$. However, if:

$$
eg_0(M)\left(1-\frac{M_1}{M}+\beta^q(1-u)\frac{LM_1}{g_0(M)}\right)+g_1(L,A)-\frac{(\mu_L^q+\alpha_L^q+ud^q)(\mu_A^q+ud)A}{\alpha_L^q}-\frac{\sigma^q\eta^q}{\rho}Z<0,
$$

then ${}_a^cD_t^qU_1(t)<0$ and the trivial equilibrium point $\mathcal{E}_1$ is globally asymptotically stable, otherwise it is unstable. This completes the proof. □

Theorem 2.5. *The equilibrium point $\mathcal{E}_2$ is globally asymptotically stable whenever:*

$$
\begin{aligned}
&g_1(L_2,A_2)\left(1-\frac{L}{L_2}-\frac{L_2g_1(L,A)}{L_2g(L_2,A_2)}+\frac{g_1(L,A)}{g_1(L_2,A_2)}\right)+L_2\left(1+\frac{L}{L_2}-\frac{A}{A_2}-\frac{LA_2}{L_2A}\right)\\
&+eg_0(M)-e\beta^q(1-u)L_2M\le 0.
\end{aligned}
$$

Proof. Consider the Lyapunov functional:

$$
\begin{aligned}
U_1(t)=\;&eM(t)+\left[L(t)-L_2-L_2\ln\left(\frac{L(t)}{L_2}\right)\right]+\frac{1}{\alpha_L^q}\left[A(t)-A_2-A_2\ln\left(\frac{A(t)}{A_2}\right)\right]\\
&+\frac{1}{\rho}\left[Z(t)-Z_2-Z_2\ln\left(\frac{Z(t)}{Z_2}\right)\right].
\end{aligned}
\tag{37}
$$

The fractional derivative of (37) along the solutions of system (2) leads to:

$$
\begin{aligned}
{}_a^cD_t^qU_2(t)\le\;&e\,{}_a^cD_t^qM(t)+\left(1-\frac{L^3}{L(t)}\right){}_a^cD_t^qL(t)+\frac{1}{\alpha_L^q}\left(1-\frac{M^3}{M(t)}\right){}_a^cD_t^qA(t)\\
&+\frac{1}{\rho}\left(1-\frac{Z^3}{Z(t)}\right){}_a^cD_t^qZ(t).
\end{aligned}
\tag{38}
$$

At the equilibrium point $\mathcal{E}_2$ we have the following identities:

$$
(\mu_L^q+\alpha_L^q+ud^q)L_2=g_1(L_2,A_2)-\sigma^qZ_2L_2,\qquad(\mu_A^q+ud^q)A_2=\alpha_L^qL_2,\qquad\eta^q=\sigma^q\rho L_2.
$$

Making use of these identities leads to

$$
\begin{aligned}
{}_a^cD_t^qU_2(t)\le\;&g_1(L_2,A_2)\left(1-\frac{L}{L_2}-\frac{L_2g_1(L,A)}{L_2g_1(L_2,A_2)}+\frac{g_1(L,A)}{g_1(L_2,A_2)}\right)+L_2\left(1+\frac{L}{L_2}-\frac{A}{A_2}-\frac{LA_2}{L_2A}\right)\\
&+eg_0(M)-e\beta^q(1-u)L_2M.
\end{aligned}
\tag{39}
$$

We can note that, at the equilibrium point $\mathcal{E}_3$ one can easily verify that ${}_a^cD_t^qU_2(t)=0$ and ${}_a^cD_t^qU_2(t)<0$ if and only if:

$$
\begin{aligned}
&g_1(L_2,A_2)\left(1-\frac{L}{L_2}-\frac{L_2g_1(L,A)}{L_2g_1(L_2,A_2)}+\frac{g_1(L,A)}{g_1(L_2,A_2)}\right)+L_2\left(1+\frac{L}{L_2}-\frac{A}{A_2}-\frac{LA_2}{L_2A}\right)\\
&+eg_0(M)-e\beta^q(1-u)L_2M<0.
\end{aligned}
$$

Hence, if the above condition holds then $\mathcal{E}_2$ is globally asymptotically stable. This completes the proof. □

Theorem 2.6. *The equilibrium point $\mathcal{E}_3$ is globally asymptotically stable whenever:*

$$\begin{aligned}&g_1(L_3,A_3)\left(1-\frac{L}{L_3}-\frac{L_3g_1(L,A)}{L_3g_1(L_3,A_3)}+\frac{g_1(L,A)}{g_1(L_3,A_3)}\right)+L_3\left(1+\frac{L}{L_3}-\frac{A}{A_3}-\frac{LA_3}{L_3A}\right)\\&+eg_0(M)-e\beta^q(1-u)L_3M-\eta^qZ\left(1-\frac{\sigma^q}{\eta^q}L_3\right)\le 0.\end{aligned}$$

Proof. Consider the Lyapunov functional:

$$\begin{aligned}U_3(t)=eM(t)+&\left[L(t)-L_3-L_3\ln\left(\frac{L(t)}{L_3}\right)\right]+\frac{1}{\alpha_L^q}\left[A(t)-A_3-A_3\ln\left(\frac{A(t)}{A_3}\right)\right]\\&+\frac{1}{\rho}Z(t).\end{aligned}\tag{40}$$

At the equilibrium point $\mathcal{E}_3$ we have the identities:

$$(\mu_L^q+\alpha_L^q+ud^q)L_3=g_1(L_3,A_3),\qquad(\mu_A^q+ud^q)A_3=\alpha_L^qL_3.$$

Utilizing these identities leads to the following result:

$$\begin{aligned}{}^{c}_{a}D^{q}_{t}U_3(t)\le&g_1(L_3,A_3)\left(1-\frac{L}{L_3}-\frac{L_3g_1(L,A)}{L_3g_1(L_3,A_3)}+\frac{g_1(L,A)}{g_1(L_3,A_3)}\right)+L_3\left(1+\frac{L}{L_3}-\frac{A}{A_3}-\frac{LA_3}{L_3A}\right)\\&+eg_0(M)-e\beta^q(1-u)L_3M-\eta^q\left(1-\frac{\sigma^q}{\eta^q}L_3\right).\end{aligned}$$

It follows that if $M(t)=M_3$, $L(t)=L_3$, $A(t)=A_3$ and $Z(t)=Z_3$, then ${}^{c}_{a}D^{q}_{t}U_3(t)=0$. However, if:

$$\begin{aligned}&g_1(L_3,A_3)\left(1-\frac{L}{L_3}-\frac{L_3g_1(L,A)}{L_3g_1(L_3,A_3)}+\frac{g_1(L,A)}{g_1(L_3,A_3)}\right)+L_3\left(1+\frac{L}{L_3}-\frac{A}{A_3}-\frac{LA_3}{L_3A}\right)\\&+eg_0(M)-e\beta^q(1-u)L_3M-\eta^q\left(1-\frac{\sigma^q}{\eta^q}L_3\right)<0,\end{aligned}$$

then ${}^{c}_{a}D^{q}_{t}U_3(t)<0$ and it follows that equilibrium point $\mathcal{E}_3$ is globally asymptotically stable, otherwise it is unstable. This completes the proof. □

Theorem 2.7. *The equilibrium point $\mathcal{E}_4$ is globally asymptotically stable whenever:*

$$\begin{aligned}&g_0(M_4)\left(\frac{L}{L_4}+\frac{g_0(M)}{g_0(M_4)}-\frac{M_4}{M}\frac{g_0(M)}{g_0(M_4)}-\frac{LM}{L_4M_4}\frac{g_0(M)}{g_0(M_4)}\right)+L_4\left(1+\frac{L}{L_4}-\frac{A}{A_4}-\frac{LA_4}{L_4A}\right)\\&+g_1(L_4,A_4)\left(1+\frac{g_1(L,A)}{g_1(L_4,A_4)}-\frac{L}{L_4}-\frac{L_4}{L}\frac{g_1(L,A)}{g_1(L_4,A_4)}\right)\\&+e\beta^q(1-u)L_4M_4\left(1+\frac{LM}{L_4M_4}-\frac{L}{L_4}-\frac{L_4g_1(L,A)}{Lg(L_4,A_4)}\right)\le 0.\end{aligned}\tag{41}$$

Proof. Consider the Lyapunov functional:

$$\begin{aligned}U_4(t)=+&\left[M(t)-M_4-M_4\ln\left(\frac{M(t)}{M_4}\right)\right]+\left[L(t)-L_4-L_4\ln\left(\frac{L(t)}{L_4}\right)\right]\\&+\frac{1}{\alpha_L^q}\left[A(t)-A_4-A_4\ln\left(\frac{A(t)}{A_4}\right)\right]+\frac{1}{\rho}\left[Z(t)-Z_4-Z_4\ln\left(\frac{Z(t)}{Z_4}\right)\right].\end{aligned}\tag{42}$$

At the equilibrium point $\mathcal{E}_4$ we have the following identities:

$$\begin{aligned}&g_0(M_4)=\beta^q(1-u)L_4M_4,\\&g_1(L_4,A_4)+e\beta^q(1-u)L_4M_4-\sigma^qL_4Z_4=(\mu_L^q+\alpha_L^q+ud^q)L_4,\end{aligned}$$

$$(\mu_A^q + ud^q)A_4 = \alpha_L^q L_4, \qquad \sigma^q \rho L_4 = \eta.$$

Utilizing these identities leads to the following result:

$$\begin{aligned}
{}_a^cD_t^q U_4(t) \le\; & g_0(M_4)\left(\frac{L}{L_4} + \frac{g_0(M)}{g_0(M_4)} - \frac{M_4}{M}\frac{g_0(M)}{g_0(M_4)} - \frac{LM}{L_4M_4}\frac{g_0(M)}{g_0(M_4)}\right)\\
& + g_1(L_4, A_4)\left(1 + \frac{g_1(L,A)}{g_1(L_4,A_4)} - \frac{L}{L_4} - \frac{L_4}{L}\frac{g_1(L,A)}{g_1(L_4,A_4)}\right)\\
& + e\beta^q(1-u)L_4M_4\left(1 + \frac{LM}{L_4M_4} - \frac{L}{L_4} - \frac{L_4 g_1(L,A)}{Lg(L_4,A_4)}\right)\\
& + L_4\left(1 + \frac{L}{L_4} - \frac{A}{A_4} - \frac{LA_4}{L_4A}\right).
\end{aligned}$$

It follows that if $M(t) = M_4$, $L(t) = L_4$, $A(t) = A_4$ and $Z(t) = Z_4$, then ${}_a^cD_t^q U_4(t) = 0$. However, if:

$$\begin{aligned}
& g_0(M_4)\left(\frac{L}{L_4} + \frac{g_0(M)}{g_0(M_4)} - \frac{M_4}{M}\frac{g_0(M)}{g_0(M_4)} - \frac{LM}{L_4M_4}\frac{g_0(M)}{g_0(M_4)}\right)\\
& + g_1(L_4, A_4)\left(1 + \frac{g_1(L,A)}{g_1(L_4,A_4)} - \frac{L}{L_4} - \frac{L_4}{L}\frac{g_1(L,A)}{g_1(L_4,A_4)}\right)\\
& + e\beta^q(1-u)L_4M_4\left(1 + \frac{LM}{L_4M_4} - \frac{L}{L_4} - \frac{L_4 g_1(L,A)}{Lg(L_4,A_4)}\right)\\
& + L_4\left(1 + \frac{L}{L_4} - \frac{A}{A_4} - \frac{LA_4}{L_4A}\right) < 0,
\end{aligned}$$

then ${}_a^cD_t^q U_4(t) < 0$ and it follows that equilibrium point $\mathcal{E}_4$ is globally asymptotically stable, otherwise it is unstable. This completes the proof. □

3. Optimal control problem

In this section, we investigate the role of time-dependent intervention strategies on minimizing the growth of the FAW population during an outbreak. Precisely, we investigate the effects of time dependent awareness campaigns as an intervention to control the growth of FAW population. Hence the constant awareness campaign parameter u in model (2) is now considered to be time-dependent, that is, $0 \le u(t) \le u_{\max} < 1$, where $u_{\max}$ is the upper bound of the control $u(t)$, which reflects practical limitation on the maximum rate of control that can be implemented in a given period. In what follows, we introduce an objective functional J which will be utilized to formulate the optimization problem of interest. In particular, the overall objective here is to minimize the number of FAW larvae and moths over a finite time interval $[0, T]$ at minimal costs. Mathematically, this can be captured as follows:

$$J[u(t)] = \min_{\Omega} \int_0^T \left[L(t) + A(t) + \frac{W}{2}u^2(t)\right] dt, \tag{43}$$

subject to the system:

$$\left.\begin{aligned}
{}_a^cD_t^q M(t) &= r^q M\left(1 - \frac{M}{K_M^q}\right) - \beta^q(1-u(t))LM,\\
{}_a^cD_t^q L(t) &= b_L^q A\left(1 - \frac{L}{K_L^q}\right) + e\beta^q(1-u(t))LM - \sigma^q ZL - (\mu_L^q + \alpha_L^q + u(t)d^q)L,\\
{}_a^cD_t^q A(t) &= \alpha_L^q L - (\mu_A^q + u(t)d^q)A,\\
{}_a^cD_t^q Z(t) &= \rho\sigma^q LZ - \eta^q Z.
\end{aligned}\right\} \tag{44}$$

In Eq. (43), W is known as the weight constant. The weight constant over the prescribed time frame is a measure of the relative costs of the interventions over a finite time horizon. The optimal control problem hence becomes that, we seek an optimal function, $u^*(t)$, such that $J(u^*(t)) = \min_{\Omega} J(u(t))$ subject to the state equations in system (44) with initial conditions (2).

3.1. Optimality system

We use Pontryagin's maximum principle [32,33] to determine the necessary conditions that optimal controls must satisfy. Through Pontryagin's maximum principle, system (44) is converted into an equivalent problem, namely the problem of minimizing the Hamiltonian $H(t)$ given by:

$$\left.\begin{aligned} H(t) &= L(t)+A(t)+\frac{W}{2}u^2(t) \\ &\quad +\lambda_1\left[r^qM\left(1-\frac{M}{K_M^q}\right)-\beta^q(1-u(t))LM\right] \\ &\quad +\lambda_2\left[b_L^qA\left(1-\frac{L}{K_L^q}\right)+e\beta^q(1-u(t))LM-\sigma^qZL-(\mu_L^q+\alpha_L^q+u(t)d^q)L\right] \\ &\quad +\lambda_3\left[\alpha_L^qL-(\mu_A^q+u(t)d^q)A\right]+\lambda_4\left[\rho\sigma^qLZ-\eta^qZ\right], \end{aligned}\right\} \tag{45}$$

where $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$ are $\lambda_4(t)$ are the adjoint variables corresponding to the states $M(t)$, $L(t)$, $A(t)$ and $Z(t)$.

Given an optimal control $u^*(t)$ and the corresponding state solutions M, L, A and Z, there exist adjoint functions $\lambda_i(t)$, $i = 1, 2, 3, 4$ satisfying:

$$\left.\begin{aligned} {}_a^cD_t^q\lambda_1(T-t) &= \left[r^q-\frac{2r^qM(T-t)}{K_M^q}-\beta^q(1-u(T-t))L(T-t)\right]\lambda_1(T-t) \\ &\quad +e\beta^q(1-u(T-t))L(T-t)\lambda_2(T-t), \\ {}_a^cD_t^q\lambda_2(T-t) &= 1-\beta^q(1-u(T-t))M(T-t)\lambda_1(T-t)+\alpha_L^q\lambda_3(T-t) \\ &\quad +\sigma^q\rho Z(T-t)\lambda_4(T-t)+e\beta^q(1-u(T-t))M(T-t)\lambda_2(T-t) \\ &\quad -\left[\alpha_L^q+\mu_L^q+u(T-t)d^q-\frac{b_L^qA(T-t)}{K_L}+\sigma^qZ(T-t)\right]\lambda_2(T-t), \\ {}_a^cD_t^q\lambda_3(T-t) &= 1-(\mu_A^q+u(T-t)d^q)\lambda_3(T-t)+b_L\left(1-\frac{L(T-t)}{K_L^q}\right)\lambda_2(T-t), \\ {}_a^cD_t^q\lambda_4(T-t) &= -\sigma^qL(T-t)\lambda_2(T-t)+(\sigma^q\rho L(T-t)-\eta^q)\lambda_4(T-T), \end{aligned}\right\} \tag{46}$$

with transversality conditions $\lambda_i(T) = 0$ for $i = 1, 2, 3, 4$. Furthermore, the optimal controls are characterized by the optimality conditions:

$$u(t) = \min\left\{\max\left\{0, \frac{(e\beta^qM+d^q)L\lambda_2+d^qA\lambda_3-\beta^qLM\lambda_1}{W}\right\}, u_{\max}\right\}. \tag{47}$$

4. Numerical results and discussions

In this section, we present some numerical results to support the analytical results presented in Sections 2 (2.2, 2.3, 2.4. 2.5) and 3. For the numerical simulations, we use a forward–backward sweep iterative scheme [33]. The initial population levels were assumed as follows: $M(0) = 15$, $L(0) = 500$, $A(0) = 100$, and $Z(0) = 50$. All simulation of the model (2) was done using the baseline values for model parameters presented in Table 1 obtained from different literature.

Before investigating the effects of time-dependent farming awareness on minimizing or eradicating FAW in the maize field, we first simulate the model system (2) with constant awareness campaigns u. From the simulation in Fig. 2, we can observe that at this level of farming awareness ($u = 0.1$), the maize biomass will increase from the start and converge to 35 biomass per plant which is less than the expected 50 biomass per plant. This suggests that while farming awareness may minimize the effects of FAW on maize biomass, to some extent it cannot be highly effective towards achieving the expected biomass per plant. However, in Fig. 3 we can observe that if $u = 0.7$, then the level of maize biomass converges to the expected level even at different fractional order values. Thus, as the awareness level increases to levels close to 100% ($u = 1$), the FAW population decreases significantly and the final maize biomass reaches expected levels.

Next, we investigate the effects of time-dependent awareness campaigns $u(t)$ on minimizing the damage on maize biomass by FAW. Without loss of generality, we set $q = 0.9$ and $u(t) = 0.03$ per day with an upper bound of $u_{\max} = 1$. The simulation results are presented in Fig. 4.

From the results in Fig. 4, one can note that in the presence of time-dependent farming awareness, the FAW population (larvae and moth) decreases remarkably compared to when there is no time dependent farming awareness. We also note

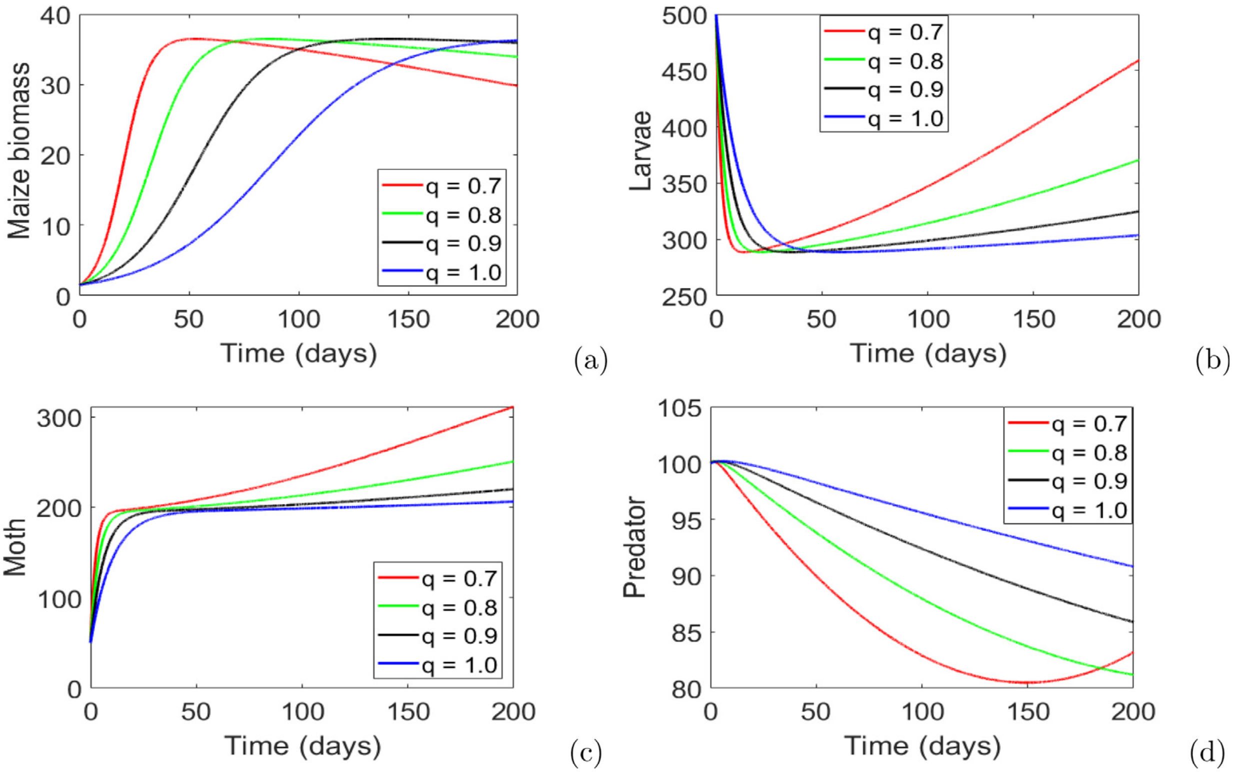


Fig. 2. Simulation results of model (2) with constant farming awareness $u = 0.1$ and different fractional order values.

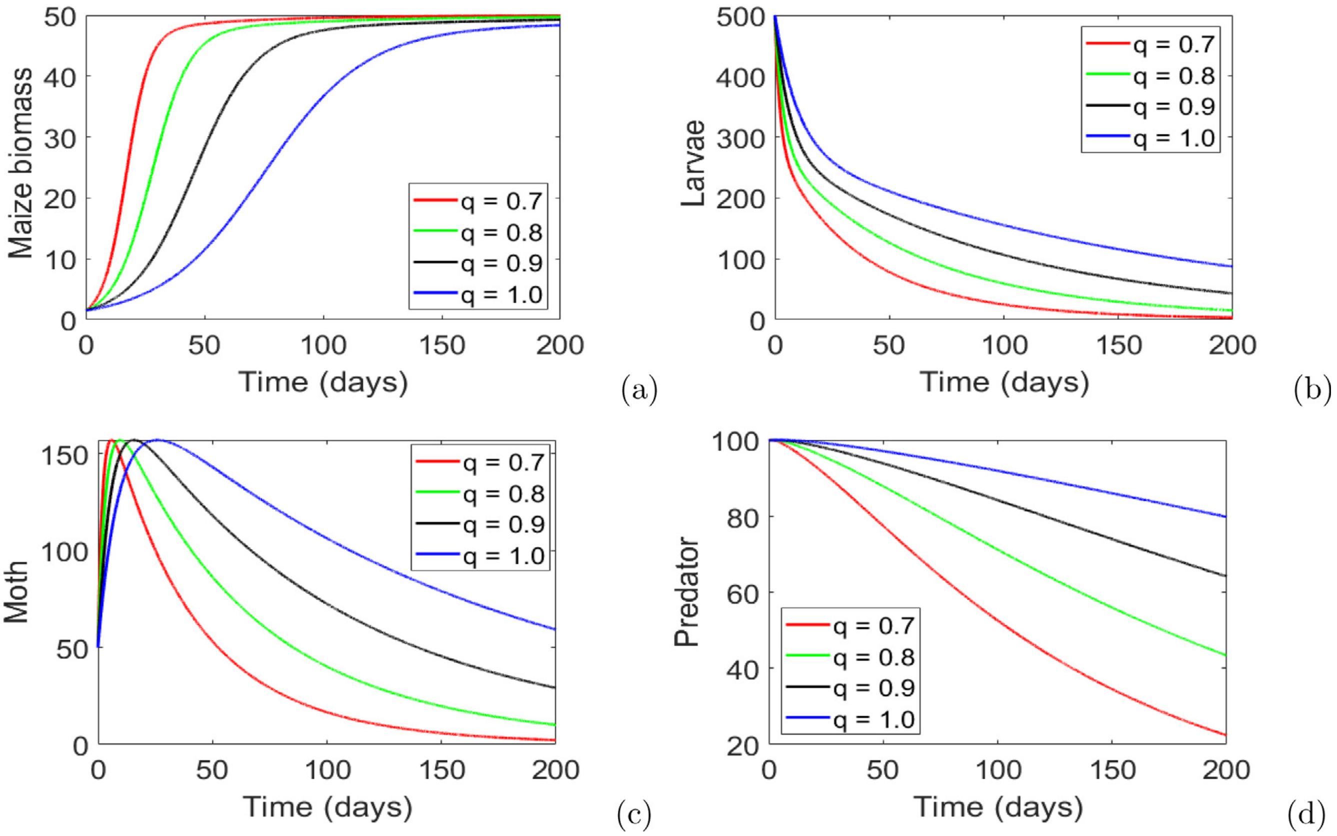


Fig. 3. Simulation results of model (2) with constant farming awareness $u = 0.7$ and different fractional order values.

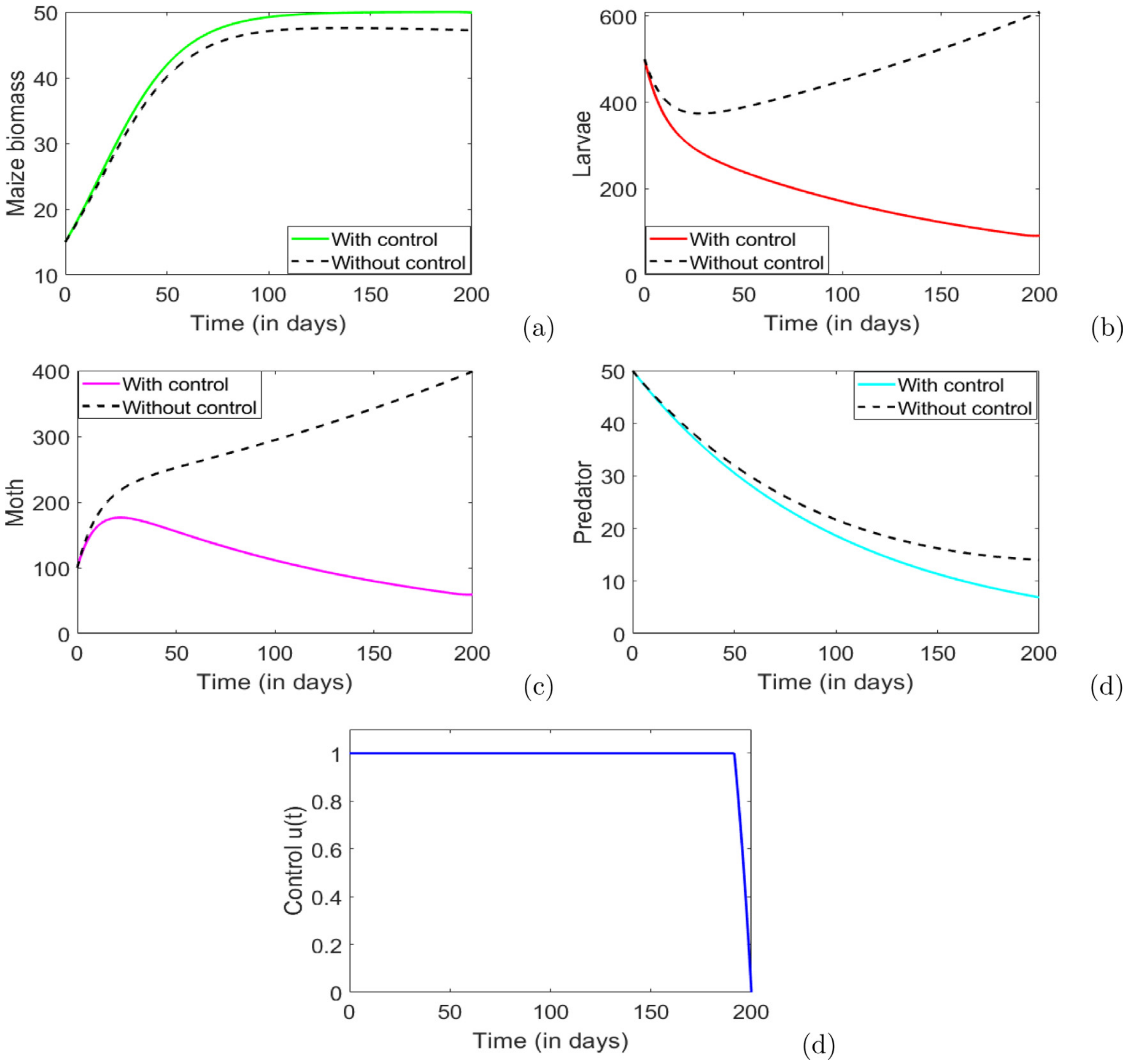


Fig. 4. Simulation results of model (2) with time-dependent constant farming awareness $0 \le u(t) \le 1$, $q = 0.9$ and $W = 10$.

Table 1
Model parameters and their baseline values.

Symbol	Definition	Baseline value	Source
b_L	Growth rate of larva	1/14 day^{-1}	[34]
α_L^{-1}	Average development time of the larva	30 Days	[34]
μ_A^{-1}	Average moth life span	21 Days	[34]
K_M	Maximum biomass of maize plants	50 plant^{-1}	[35].
K_L	Egg environmental carrying capacity	10^6	[35].
μ_L	Natural mortality rate of larva	0.01 Day^{-1}	[35].
r	Growth rate of maize plants	0.05 Day^{-1}	[35].
e	Efficiency of biomass conversion	0.2	[35].
β	Plant attack rate by larvae	5×10^{-5} Day^{-1}	[36].
σ	Consumption rate of larva by predators	5×10^{-5} Day^{-1}	[37].
ρ	Conversion rate of prey to predator	0.1 Day^{-1}	[38].
d	Mortality of FAW due to intervention strategies	0.01 Day^{-1}	[37].
η^{-1}	Average life span of predator	100 Days	[39].

that a significant decrease of the FAW larvae in the presence of optimal farming awareness will also lead to a slight decrease of the predator population over time. The results also show that in the presence of optimal farming awareness, the final maize biomass will be within the expected level. However, in the absence of optimal farming awareness the final

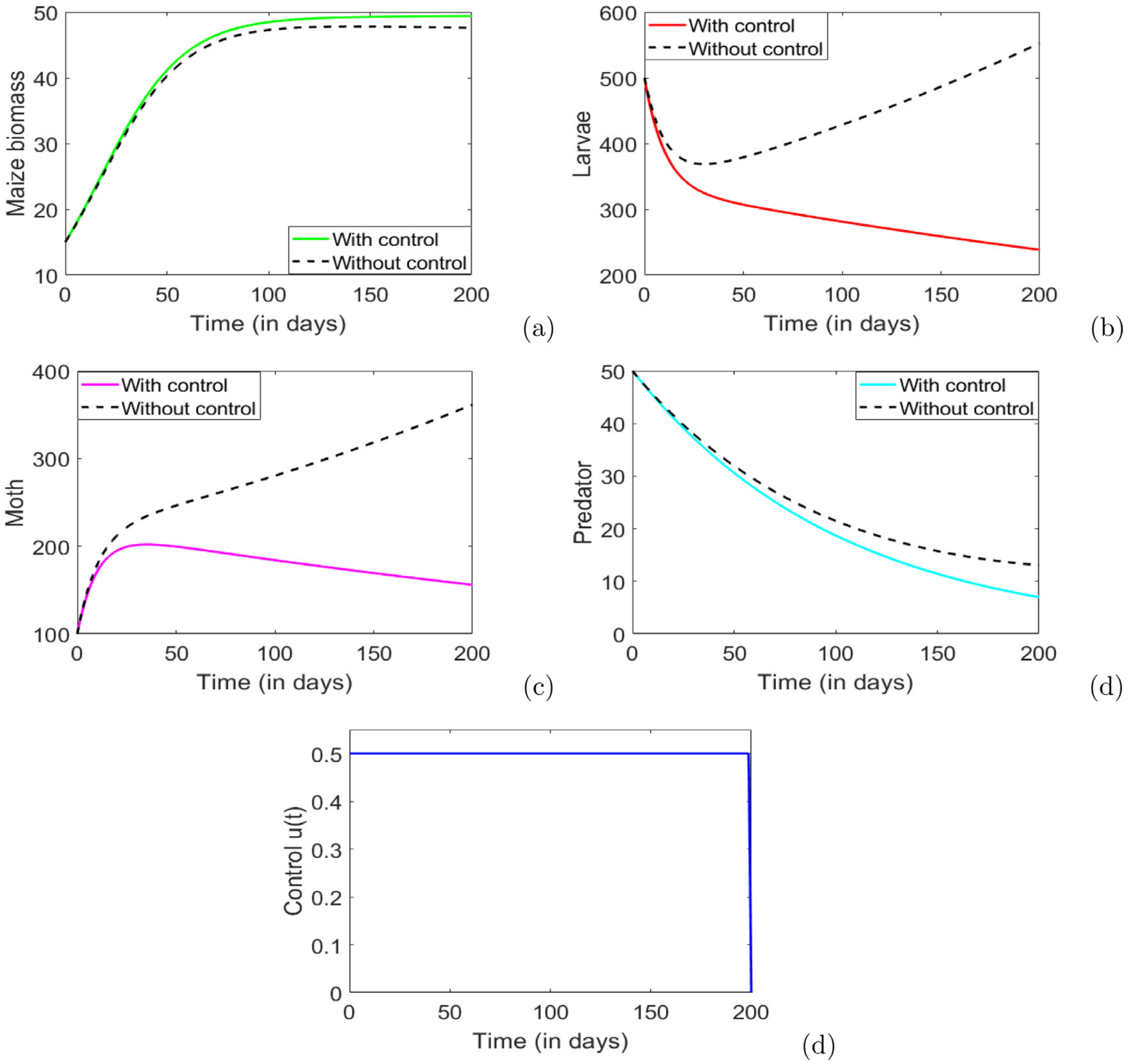
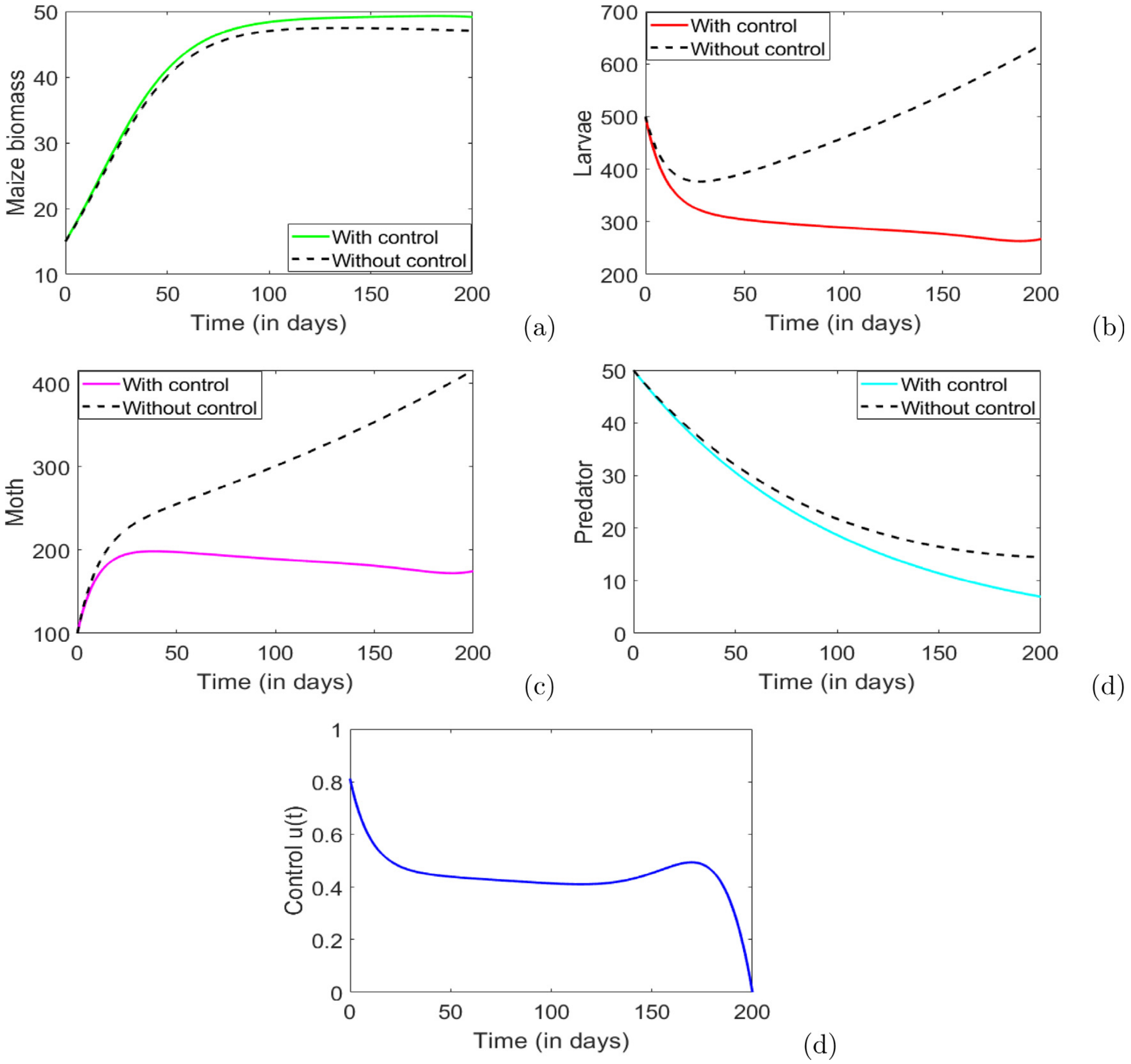


Fig. 5. Simulation results of model (2) at low maximum intensity $u_{\max} = 0.5$, with $q = 0.9$ and $W = 100$.

biomass level will always be less than the expected final biomass. In addition, one can observe that the optimal control profile (Fig. 4(d)) starts at $u_{\max} = 1$ and remains there for the greater part of time horizon ($0 \leq t \leq 195$ days) till it drops close to the final period. This suggests that for one to attain the outcomes in Fig. 4, optimal farming awareness efforts need to be maintained at their maximum intensity for the greater part of the time horizon and thereafter ceased gradually till the final time.

The simulation results in Fig. 5 show the impact of the upper bound of the control variable $u_{\max}$ on model solutions. Here, we set $u_{\max} = 0.5$. We can note that in this scenario, the optimal efforts will need to be maintained at their maximum intensity throughout the entire time horizon in order for the final maize biomass to be within the expected level.

The simulation results in Fig. 6 show that the impact of the costs on the implementation of optimal farming awareness. Here we set $W = 1000$. We note that when the costs of implementing farming awareness are high, the control profile for $u(t)$ does not start at its maximum, $u_{\max} = 1$, but begins on $u(t) = 0.8$, followed by a gradual decrease before it stabilizes at $u(t) = 0.4$ after approximately 40 days from the start. The control profile stays at $u(t) = 0.4$ till the 150th day after which it increases slightly to $u(t) = 0.5$ and immediately drops gradually to its minimum until the final time horizon. Although the pattern of the control profile is complex, one can deduce that optimum results can be attained if the intensity of the control $u(t)$ is maintained between 0.4 and 0.5 ($0.4 \leq u(t) \leq 0.5$) for a greater part of the time horizon.



(a) (b) (c) (d) (d)

Fig. 6. Simulation results of model (2) at high cost of implementation, $W = 1000$, $0 \leq u(t) \leq 1$, and $q = 0.9$.

5. Concluding remarks

We have formulated a fractional-order model that incorporates naturally beneficial insects and optimal farming awareness. Dynamical analysis of the proposed model revealed that it has six equilibrium points which are all locally and globally asymptotically stable if the conditions outlined in Lemmas 2.1 and 2.2 are met. The simulation results for the model with constant awareness campaigns u, showed that $u = 0.7$ may lead to the achievement of the expected maize biomass at the end of the season (that is $t = 160$ days) for fractional-order values $q = 0.7, 0.8, 0.9$. However for $q = 1.0$, the final maize biomass at this level of awareness will be slightly less than the expected. For time-dependent farming awareness, we observed that the expected maize biomass can be attained if the costs of implementing the strategy are low. In addition, we observed that if the intensity of implementing is low, then the efforts can be carried out at their maximum intensity throughout the time horizon but when costs are high, the control profile for $u(t)$ does not start at its maximum, $u_{\max} = 1$, rather at $u(t) = 0.8$ followed by a gradual decrease before it stabilizes at $u(t) = 0.4$ after approximately 40 days from the start. Although this study is not exhaustive, it has illustrated the value of optimal control theory as tool to suggest effective management strategies during FAW outbreaks. In future, we will explore the effects of temperature and seasonal variation, migration of the moth and include the parameter of continuous replanting of maize crops on the dynamics of FAW and its implications on maize biomass.

CRediT authorship contribution statement

Salamida Daudi: Model formulation, Analysis. **Livingstone Luboobi:** Supervision, Writing – review. **Moatlhodi Kgosimore:** Supervision, Writing – review. **Dmitry Kuznetsov:** Supervision, Writing – review.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability statement

The data used to support the findings of this study are included within the article.

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Research article

Dynamics for a non-autonomous fall armyworm-maize interaction model with a saturation functional response

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Abstract: In this study, we present a non-autonomous model with a Holling type II functional response, to study the complex dynamics for fall armyworm-maize biomass interacting in a periodic environment. Understanding how seasonal variations affect fall armyworm-maize dynamics is critical since maize is one of the most important cereals globally. Firstly, we study the dynamical behaviours of the basic model; that is, we investigate positive invariance, boundedness, permanence, global stability and non-persistence. We then extended the model to incorporate time dependent controls. We investigate the impact of reducing fall armyworm egg and larvae population, at minimal cost, through traditional methods and use of chemical insecticides. We noted that seasonal variations play a significant role on the patterns for all fall armyworm populations (egg, larvae, pupae and moth). We also noted that in all scenarios, the optimal control can greatly reduce the sizes of fall armyworm populations and in some scenarios, total elimination may be attained. The modeling approach presented here provides a framework for designing effective control strategies to manage the fall armyworm during outbreaks.

Keywords: fall armyworm; maize; seasonal variations; mathematical model; optimal control

1. Introduction

Maize (*Zea mays*) is one of the most important cereals globally and is also referred as the "Queen of Cereals" due to its high yield potential [1]. Demand for maize is increasing, not only because of its higher nutritional benefits but also its ability to feed the growing global population and contribution to food security [2]. According to food and agriculture organization (FAO) [3], food security is a "situation that exists when all people, at all times, have physical, social, and economic access to sufficient, safe, and nutritious food that meets their dietary needs and food preferences for an active and healthy life".

In many African countries, agriculture remains an important contributor to food security, despite its inability to provide sufficient output to meet the needs of most of their populations. One of the main threats to food security in these countries is the recent invasion by fall armyworm, (FAW-*Spodoptera frugiperda*), a major pest of maize [2, 3], native to tropical and subtropical parts of America [2–4], where it has more than 350 different crop and non-crop host plants [4]. The FAW was first reported in West and Central parts of Africa in 2016 but it rapidly spread to other parts of the continent with high devastating effect on maize production [2, 5, 6].

According to Goergen et al [8], the infestation of African countries with the FAW has huge consequences for their economies, agricultural yield and access to overseas markets [8]. It is estimated that about $US13 billion per annum in crop losses throughout sub-Saharan Africa are due to FAW infestation, thus, threatening the livelihoods with a majority of poor farmers [7, 9]. For instance, a recent research on the impact of FAW on maize biomass in Ghana and Zambia revealed that the national mean loss of maize crops was 45% (range 22–67%) and 40% (range 25–50%) respectively [2].

Like any other insect pests, weather conditions in a season have an effect on maize biomass and FAW dynamics. Prior studies on pest biology have shown that the distribution and abundance of pests is largely influenced by relationship between their developmental rates and temperature [10, 11]. In particular, different development stages of insects are favored by different temperature ranges, hence, temperature variations influence the development rates, duration of life cycles, and, ultimately, the survival of insects [11]. Furthermore, an increase in the ambient temperature to the near thermal optimum for insects causes increase in their metabolism, and, consequently, their activities [11]. Since temperature fluctuates in the natural environment, it follows that the development rates of insects vary seasonally. For FAW in particular, prior studies suggest that, populations in a given area directly depend on the time in a season, host plants availability, and weather conditions [5]. Under unfavourable weather condition and scarcity of food for the development and reproduction, FAW is forced to migrate to other suitable locations for survival [5, 12].

As the evidence for climate impacts on FAW has increased, it is imperative that the mathematical models designed to explore the relationship between FAW and maize crops accommodate the effects of seasonal variations. The main goal of this study is to develop and analyze a non-autonomous FAW-maize interaction model. Despite a considerable number of studies on plant-pest interactions (see, for example, [15–23]), there are few studies that have been devoted to explore FAW and maize interaction.

One of the notable recent mathematical models for FAW and maize interaction was proposed by Faithpraise and coworkers [16] who evaluated the effects of biocontrol on managing FAW infestations in cereal crops among several other outcomes. The findings from their study revealed that, biocontrol could significantly control FAW infestations in cereal crops. Although this study improved the existing knowledge on FAW, one of its limitations was that development rates for the pest were all assumed to be constant yet in reality these depends on time. Thus, the present study is motivated by this existing research gap. Our results are new and, to our knowledge, very little work has been done so far on modelling and analysing the effects of seasonal variation in a FAW-maize interaction model with a saturation functional response.

2. Basic non-autonomous model

2.1. Model derivations

Biologically, maize seed planted at the beginning of the season at time $t = 0$ germinates in $0 - 7$ days [13]. Depending on the variety of maize seed planted, harvest of this crop which occurs at the end of the season (90–164) days is influenced by the weather variations within a season. Since maize growth is affected by weather condition fluctuation, the growth rates of their parts such as leaves, cobs, kernel, and stems which in fact called biomass according to Chowdhury and Battude [14,22] are weather dependent [13]. Motivated by recent mathematical models for plant-pest interactions (see., for example [15–23], in this study we develop a non-autonomous model for FAW infestations in a field of maize biomass under assumption that (i) once the FAW moths migrate into the field, no migration out of the field occurs before the harvest (ii) weather condition fluctuation in a season have an impact on maize biomass and FAW dynamics. The developed model subdivides the FAW population of life cycle into subclasses as: eggs population $E(t)$, larvae population $L(t)$, pupae population $P(t)$ and adult population which are also known as moth $A(t)$. On the other hand, we let the variable $M(t)$ denote the population density of maize biomass which grow logistically in the absence of the larvae with carrying capacity K_M and a net seasonal growth rate $r(t)$. We assume that larvae with a mortality rate $\mu_L(t)$ is the only threat to maize biomass throughout its growth period and the adult moth takes over in the reproduction process. The life cycle of FAW starts when eggs are laid in masses on maize biomass, mostly underside of these biomass [36]. We also assume that production of FAW egg is a logistic growth $b(t)\left[1 - \dfrac{E(t)}{K_E(t)}\right] wA(t)$ with $b(t)$ representing the average number of eggs laid by a proportion w of moth, $A(t)$ which are females. Since the growth of maize plants depends on seasonal variations, it suffices to assume that the egg carrying capacity $K_E(t)$, egg hatching rate $\alpha_E(t)$ and egg mortality rate $\mu_E(t)$ are season-dependent. Furthermore, FAW larvae generally emerge simultaneously three to five days following oviposition. Although, the FAW has six larval instar stages, we have considered this as single group called larvae in order to reduce complexity of the model. Since the population size of maize biomass is finite and independent of weather fluctuation and because the rate at which FAW larvae consumes food decreases, a Holling type II functional response also known as the saturating functional response is included in the equation capturing the dynamics of maize density and larvae population with half saturation constant $a(t)$. In particular, when FAW larvae feed on maize biomass, the FAW larvae with an average duration of $\frac{1}{\alpha_L(t)}$ in the larval stage convert maize biomass into larvae's biomass at the rate $e(t)$. Finally, pupation of the FAW normally occurs in the soil, at a depth of 2–8 cm [33]. We assume that, duration of the pupal stage with natural mortality rate $\mu_P(t)$ is denoted by $\frac{1}{\alpha_{P}(t)}$ which after 8 days in the soil escapes as adult moth and start the cycle again. The model explanations above can be represented schematically in Figure 1:

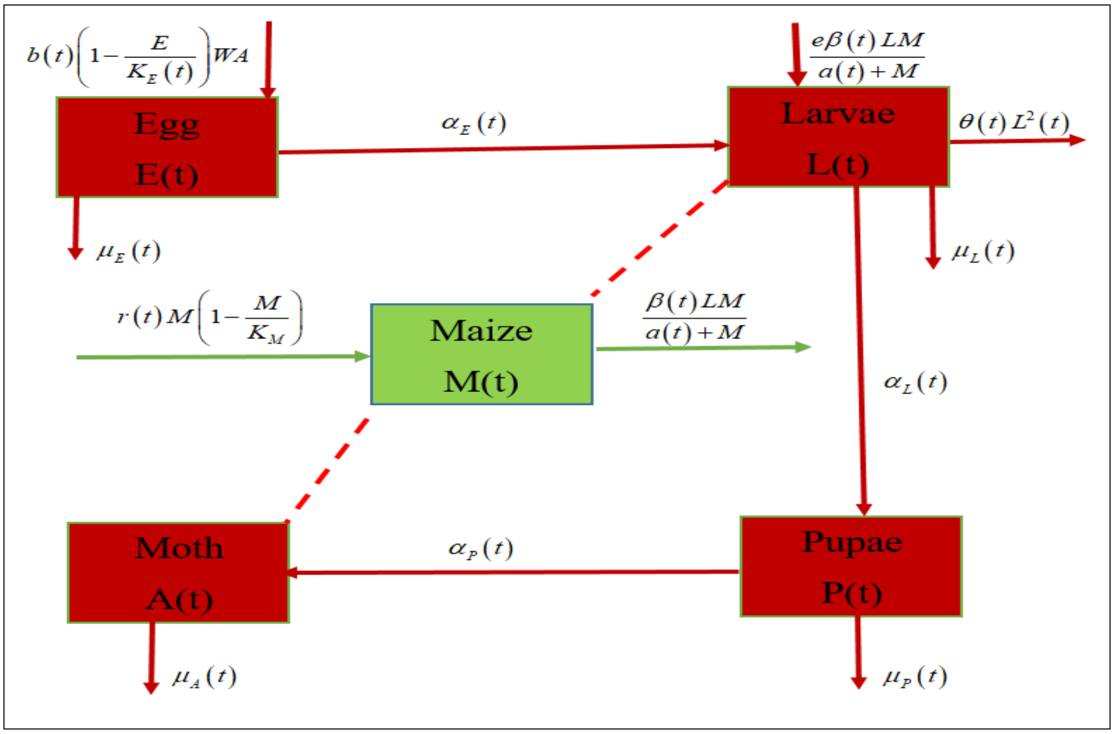

Figure 1. Schematic representation of the model (1). The green square box represents Maize biomass and the other dark red square boxes represent the four epidemiological stages of the FAW, that is., egg $E(t)$, larvae $L(t)$, pupae $P(t)$ and moth also known as the adult stage of FAW $A(t)$. Continuous arrows indicate either inflow or outflow transition between compartments. Red dotted lines connecting compartments $M(t)$ with compartments $A(t)$ and $L(t)$ shows the interaction that occurs between the plant and adult FAW A(t) as well as the larvae $L(t)$. Note that the moth interacts with the maize plant when it lays eggs on maize biomass, while the larvae feed on maize biomass.

The proposed model is summarized by the following system of nonlinear ordinary differential Equations in (1)

$$\left.\begin{aligned}
\frac{dM(t)}{dt} &= r(t)\left[1-\frac{M(t)}{K_M}\right]M(t)-\frac{\beta(t)M(t)}{a(t)+M(t)}L(t),\\
\frac{dE(t)}{dt} &= b(t)wA(t)\left[1-\frac{E(t)}{K_E(t)}\right]-[\mu_E(t)+\alpha_E(t)]E(t),\\
\frac{dL(t)}{dt} &= \alpha_E(t)E(t)+\frac{e\beta(t)M(t)}{a(t)+M(t)}L(t)-[\alpha_L(t)+\mu_L(t)]L(t)-\theta(t)L^2(t),\\
\frac{dP(t)}{dt} &= \alpha_L(t)L(t)-[\alpha_p(t)+\mu_P(t)]P(t),\\
\frac{dA(t)}{dt} &= \alpha_P(t)P(t)-\mu_A(t)A(t).
\end{aligned}\right\} \tag{1}$$

where $\mu_A(t)$ represents natural mortality rate of the moth (adult moth) and $\theta(t)$ is the density-dependent death rate of larvae population. Thus, $-\theta(t)L^2(t)$ reflects of predation, intra-specific and interspecific competition that is known to exist on FAW larave population. Precisely, prior studies suggest that when food is limited, the older FAW larvae exhibit a cannibalistic behavior on the smaller larvae [24, 25].

All model parameters that are functions of time depend on seasonal variations. For biological significance, we assume that all these parameters are continuous and bounded functions defined on $\mathbb{R}_+$. We now provide a comprehensive definition to model parameters that are meant to capture seasonal fluctuations, that is;

$$\left.\begin{array}{rcll}
r(t) & = & r_0[1+r_1\cos(2\pi t\omega^{-1})], & b(t)=b_0[1+b_1\cos(2\pi t\omega^{-1})], \\
\beta(t) & = & \beta_0[1+\beta_1\cos(2\pi t\omega^{-1})], & K_E(t)=K_{E_0}1+K_{E_1}\cos(2\pi t\omega^{-1}), \\
a(t) & = & a_0[1+a_1\cos(2\pi t\omega^{-1}), & \alpha_E(t)=\alpha_{E_0}[1+\alpha_{E_1}\cos(2\pi t\omega^{-1}), \\
\mu_E(t) & = & \mu_{E_0}[1+\mu_{E_1}\cos(2\pi t\omega^{-1}), & \alpha_L(t)=\alpha_{L_0}[1+\alpha_{L_1}\cos(2\pi t\omega^{-1}), \\
\mu_L(t) & = & \mu_{L_0}[1+\mu_{L_1}\cos(2\pi t\omega^{-1}), & \theta(t)=\theta_0[1+\theta_1\cos(2\pi t\omega^{-1}), \\
\alpha_P(t) & = & \alpha_{P_0}[1+\alpha_{P_1}\cos(2\pi t\omega^{-1}), & \mu_A(t)=\mu_{A_0}[1+\mu_{A_1}\cos(2\pi t\omega^{-1}), \\
\mu_P(t) & = & \mu_{P_0}[1+\mu_{P_1}\cos(2\pi t\omega^{-1}), & e(t)=e_0[1+e_1\cos(2\pi t\omega^{-1})],
\end{array}\right\} \tag{2}$$

where $\omega > 0$ represent the period. Further, r_0, β_0, K_{E_0}, a_0, b_0, α_{E_0}, μ_{E_0}, e_0, α_{L_0}, μ_{L_0}, θ_0, α_{P_0} and μ_{A_0} are the baseline values or the time averages of $r(t)$, $K_E(t)$, $b(t)$, $\alpha_E(t)$, $\mu_E(t)$, $e(t)$, $\beta(t)$, $\alpha_L(t)$, $\mu_L(t)$, $\theta(t)$, $\alpha_P(t)$, $\mu_A(t)$, $a(t)$, respectively, and r_1, K_{M_1}, β_1, K_{E_1}, a_1, b_1, α_{E_1}, μ_{E_1}, e_1, α_{L_1}, μ_{L_1}, θ_1, α_{P_1} and μ_{A_1} denote the magnitude of seasonal fluctuations. Note that $0 < r_1 < 1$, $0 < \beta_1 < 1$, $0 < K_{E_1} < 1$, $0 < a_1 < 1$, $0 < b_1 < 1$, $0 < \alpha_{E_1} < 1$, $0 < \mu_{E_1} < 1$, $0 < e_1 < 1$, $0 < \alpha_{L_1} < 1$, $0 < \mu_{L_1} < 1$, $0 < \theta_1 < 1$, $0 < \alpha_{P_1} < 1$ and $0 < \mu_{A_1} < 1$. From Eq (2) we can observe that all model parameters that account for seasonal fluctuations are periodic (with period $\omega > 0$ days), continuous and bounded below and above. Since $\left|\cos(2\pi t\omega^{-1})\right| \leq 1$ and $0 < r_1 < 1$, it follows that:

$$r_0(1-r_1) \leq r(t) \leq r_0(1+r_1). \tag{3}$$

Therefore, we conclude that $r(t)$ is bounded below and above. By following the same approach one can easily verify that all the other periodic model parameters are bounded below and above. Next, we investigate the dynamics of system (1), in particular we will focus on the positive invariance, nonpersistence, permanence, global attractivity of the bounded positive solutions and the boundary solution.

2.2. Positivity, boundedness and permanence of model solutions

In this subsection, we will prove for positivity, boundedness, global asymptotic stability of the bounded positive solution and permanence of system (1).

Definition 1. *The set of solution for the system (1) is said to be ultimately bounded if* $\exists \mathcal{M} > 0$, *such that for each solution* $(M(t), E(t), L(t), P(t), A(t))$ *of (1), there* $\exists T > 0$, *such that* $\|(M(t), E(t), L(t), P(t), A(t))\| \leq \mathcal{M}\ \forall t > T$, *with* $\mathcal{M}$ *independent of a particular solution and* T *may depend on the solution.*

Definition 2. *Model system (1) is said to be permanent if there exists* δ_+ *and* Δ_+ *with* $0 < \delta_+ < \Delta_+$ *such that:*

$$\min\left\{\liminf_{t\to+\infty} M(t),\quad \liminf_{t\to+\infty} E(t),\quad \liminf_{t\to+\infty} L(t),\quad \liminf_{t\to+\infty} P(t),\quad \liminf_{t\to+\infty} A(t)\right\} \geq \delta \textit{ and,}$$

$$\max\left\{\limsup_{t\to+\infty} M(t),\quad \limsup_{t\to+\infty} E(t),\quad \limsup_{t\to+\infty} L(t),\quad \limsup_{t\to+\infty} P(t),\quad \limsup_{t\to+\infty} A(t)\right\} \le \Delta,$$

for all solutions of (1) with initial values being positive hold. Model system (1) is said to be non-persistent if there exists a positive solution $(M(t), E(t), L(t), P(t), A(t))$ *of system (1) satisfying:*

$$\min\left\{\liminf_{t\to+\infty} M(t),\quad \liminf_{t\to+\infty} E(t),\quad \liminf_{t\to+\infty} L(t),\quad \liminf_{t\to+\infty} P(t),\quad \liminf_{t\to+\infty} A(t)\right\} = 0.$$

Given $u \in C(\mathbb{R})$, we define $\Phi(u(s)) = \dfrac{u(s)}{a(s)+u(s)}$ and observe that, $\Phi(u)$ is a monotonic increasing function. Defining;

$$\left.\begin{array}{l}
\mathcal{M}_1 = \sup_{t\in\mathbb{R}}\{K_M\}, \qquad m_1 = \inf_{t\in\mathbb{R}}\left\{\dfrac{K_M}{r(t)}[r(t)-\beta(t)\mathcal{M}_3]\right\},\\[2ex]
\mathcal{M}_2 = \sup_{t\in\mathbb{R}}\left\{\dfrac{b(t)w\mathcal{M}_5}{b(t)w\mathcal{M}_5+\alpha_E(t)+\mu_E(t)}\right\}, \quad m_2 = \sup\limits_{t\in\mathbb{R}}\left\{\dfrac{b(t)wm_5}{b(t)wm_5+\alpha_E(t)+\mu_E(t)}\right\},\\[2ex]
\mathcal{M}_3 = \sup_{t\in\mathbb{R}}\left\{\alpha_E(t)\mathcal{M}_2 + \dfrac{e\beta(t)\Phi(\mathcal{M}_1)(t)-(\mu_L(t)+\alpha_L(t))}{\theta(t)}\right\}, \quad \mathcal{M}_4 = \sup\limits_{t\in\mathbb{R}}\left\{\dfrac{\alpha_L(t)\mathcal{M}_3}{\mu_P(t)+\alpha_P(t)}\right\},\\
\mathcal{M}_5 = \sup_{t\in\mathbb{R}}\left\{\dfrac{\alpha_P(t)\mathcal{M}_4}{\mu_A(t)}\right\}, \quad m_3 = \inf\limits_{t\in\mathbb{R}}\left\{\alpha_E(t)m_2 + \dfrac{e\beta(t)\Phi(m_1)(t)-(\mu_L(t)+\alpha_L(t))}{\theta(t)}\right\},\\[2ex]
m_4 = \inf_{t\in\mathbb{R}}\left\{\dfrac{\alpha_L(t)m_3}{\mu_P(t)}\right\}, \qquad m_5 = \inf\limits_{t\in\mathbb{R}}\left\{\dfrac{\alpha_P(t)m_4}{\mu_A(t)}\right\}.
\end{array}\right\} \tag{4}$$

Then, we define condition (H1) using Eq (4) as follows:

$$(H1):\left\{\begin{array}{l}
\inf_{t\in\mathbb{R}}\{[r(t)-\beta(t)\mathcal{M}_3]\} > 0,\\
\inf_{t\in\mathbb{R}}\left\{\alpha_E(t)\mathcal{M}_2 + \dfrac{e\beta(t)\Phi(\mathcal{M}_1)(t)-(\mu_L(t)+\alpha_L(t))}{\theta(t)}\right\} > 0,\\
\inf_{t\in\mathbb{R}}\left\{\alpha_E(t)m_2 + \dfrac{e\beta(t)\Phi(m_1)(t)-(\mu_L(t)+\alpha_L(t))}{\theta(t)}\right\} > 0.
\end{array}\right\} \tag{5}$$

Using condition (H1) in Eq (5), we define the set:

$$\Omega = \left\{ \Big(M(t), E(t), L(t), P(t), A(t)\Big) \,\middle|\, \begin{array}{l} m_1 \le M(t) \le \mathcal{M}_1,\\ m_2 \le E(t) \le \mathcal{M}_2,\\ m_3 \le L(t) \le \mathcal{M}_3,\\ m_4 \le P(t) \le \mathcal{M}_4,\\ m_5 \le A(t) \le \mathcal{M}_5 \end{array} \right\}.$$

Based on definitions 1 and 2, we have Theorem 1 and its proof is in Appendix A.

Theorem 1.

(i) *The solution set* Ω *of the system (1) is said to be positively invariant if condition (H1) holds and it is permanent if the solution set* Ω_δ *of the system (1) defined by:*

$$\Omega_\delta = \left\{ \Big(M(t), E(t), L(t), P(t), A(t)\Big) \middle| \begin{array}{l} m_1^\delta \le M(t) \le \mathcal{M}_1^\delta, \\ m_2^\delta \le E(t) \le \mathcal{M}_2^\delta, \\ m_3^\delta \le L(t) \le \mathcal{M}_3^\delta, \\ m_4^\delta \le P(t) \le \mathcal{M}_4^\delta, \\ m_5^\delta \le A(t) \le \mathcal{M}_5^\delta, \end{array} \right\}.$$

is ultimately a bounded region of (1), where $\delta > 0$ *sufficiently small so that* $m_i^\delta > 0 (i = 1, ..., 5)$ *and condition (H1) holds when* $\mathcal{M}_i$ *and* m_i *are replaced by* $\mathcal{M}_i^\delta$ *and* m_i^δ*, respectively. Further, we define* $\mathcal{M}_i^\delta = \mathcal{M}_i + \delta$, $m_i^\delta = m_i - \delta$.

(ii) *System (1) has at-least one* $\omega-$ *periodic solution* $(M^*(t), E^*(t), L^*(t), P^*(t), A^*(t)) \in \Omega$ *if condition (H1) holds.*

Biological implications of Theorem 1: Theorem 1 implies that model system (1) is biologically well-poised, that is., the population of species under consideration are non-negative and bounded.

3. Dynamical behaviors of the non-autonomous model with optimal control

3.1. Model formulation

In this section, we extend the basic system (1) to incorporate time dependent intervention strategies with the main goal of reducing FAW egg and larvae populations. Considering the extent of damage FAW can cause in a short period of time, it is imperative that once this pest has been identified in a maize field, necessary control approaches should be implemented timeously. Prior studies suggest that the effective management of FAW depends on the integration of several control strategies which include biological control, host-plant resistance, and use of chemical insecticides [6]. Here, we reformulate system (1) to incorporate new parameter $u_1(t)$ and $u_2(t)$. Control $u_1(t)$ models the efforts of traditional control methods like handpicking and destruction of FAW egg masses and larvae on FAW dynamics. Control $u_2(t)$ accounts for the efforts of chemical pesticide use on FAW dynamics. Without loss in generality, herein we will use the term traditional methods to denote handpicking and destruction of FAW egg masses and larvae. Utilizing similar variable and parameter names as in Eq (1), the new system of nonlinear differential equations incorporating time dependent controls is given by:

$$\left.\begin{aligned}
\frac{dM(t)}{dt} &= r(t)\left[1-\frac{M(t)}{K_M}\right]M(t)-\frac{M(t)\beta(t)}{M(t)+a(t)}L(t),\\
\frac{dE(t)}{dt} &= b(t)wA(t)\left[1-\frac{E(t)}{K_E(t)}\right]-[\alpha_E(t)+\mu_E(t)+u_1(t)]E(t),\\
\frac{dL(t)}{dt} &= \alpha_E(t)E(t)+\frac{e\beta(t)M(t)}{a(t)+M(t)}L(t)-[\alpha_L(t)+\mu_L(t)+u_1(t)+u_2(t)]L(t)-\theta(t)L^2(t),\\
\frac{dP(t)}{dt} &= \alpha_L(t)L(t)-[\alpha_p(t)+\mu_P(t)]P(t),\\
\frac{dA(t)}{dt} &= \alpha_P(t)P(t)-\mu_A(t)A(t).
\end{aligned}\right\} \tag{6}$$

As we can observe, control efforts $u_1(t)$ aims to reduce the egg and larvae population while the use of chemical insecticide $u_2(t)$ mainly reduces the density of larvae population only. For Eq (6) to be biologically meaningful we set:

$$M(0)\geq 0, \qquad E(0)\geq 0, \qquad L(0)\geq 0, \qquad P(0)\geq 0, \qquad A(0)\geq 0.$$

The control set for the controls is defined as:

$$\Gamma=\{(u_1(t),u_2(t)))\,\big|\,0\leq u_1(t)\leq u_{1\max}, 0\leq u_2(t)\leq u_{2\max}\}, \tag{7}$$

where $u_{1\max}$ and $u_{2\max}$ represents the upper bounds for the efforts of traditional methods and use of chemical insecticide respectively. If $u_i = 0$, $(i = 1, 2)$, it implies absence of time dependent control measures.

A control strategy is said to be successful control strategy if it can reduce the egg and larvae population thereby reducing or eradicating FAW population in the field. As such, our aim here is to identify a pair of characterized control strategy (u_1^*, u_2^*) that reduces the population of FAW egg and larvae at minimal cost. To obtain a pair of characterized control strategy (u_1^*, u_2^*), we propose an objective functional with quadratic in control which according to Lahrouz and Gaff [34, 35], (i) controls will not disappear after differentiation and this implies that the solution is unique (ii) all control strategies (u_1^*, u_2^*) are bounded implying that efforts for implementing these controls are also bounded and characterized. Thus, the following objective functional is proposed:

$$J(u_1(t),u_2(t))=\int_0^T\left[C_1E(t)+C_2L(t)+\frac{W_1}{2}u_1^2(t)+\frac{W_2}{2}u_2^2(t)\right]dt\,. \tag{8}$$

subject to the constraints (6) and where C_1, C_2, W_1 and W_2 are balancing coefficients (non-negative) converting the integrals into monetary quantity over a finite period of time, T days.

The optimal control problem, thus, becomes that, we seek optimal functions, $(u_1^*(t), u_2^*(t))$, so that:

$$J(u_1^*(t),u_2^*(t))=\min_U J(u_1(t),u_2(t)) \tag{9}$$

subject to the state equations in system (6) with initial conditions. In order to study the existence of an optimal control pair (u_1^*, u_2^*), we will make use of Fleming and Rishel's work [30]. Theorem 2 (i) and (ii) state the existence of the optimal controls and their characterization.

Theorem 2.

(i) *There* $\exists$ *an optimal control pair* (u_1^*, u_2^*) *to the problem (6).*

(ii) *Given an optimal control* $u = (u_1, u_2) \in U$ *and corresponding state solutions* M, E, L, A *and* P, *there exists adjoint functions* λ_i, $i = 1, ..., 5$ *satisfying:*

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= -\left[r(t) - \frac{2r(t)M(t)}{K_M} - \frac{\beta(t)L(t)}{a(t)+M(t)} + \frac{\beta(t)L(t)M(t)}{(a(t)+M(t))^2}\right]\lambda_1(t) \\
&\quad -\left[\frac{e\beta(t)L(t)M(t)}{a(t)+M(t)} - \frac{e\beta(t)L(t)M(t)}{(a(t)+M(t))^2}\right]\lambda_3(t), \\
\frac{d\lambda_2}{dt} &= -C_1 + \left[\frac{b(t)wA(t)}{K_E(t)} + \alpha_E(t) + \mu_E(t) + u_1(t)\right]\lambda_2(t) - \alpha_E(t)\lambda_3(t), \\
\frac{d\lambda_3}{dt} &= -C_2 + \frac{\beta(t)M(t)}{a(t)+M(t)}\lambda_1(t) - \left[\frac{e\beta(t)M(t)}{a(t)+M(t)} - 2\theta(t)L(t) - \alpha_L(t) - \mu_L(t)\right]\lambda_3(t) \\
&\quad +[u_1(t) + u_2(t)]\lambda_3(t) - \alpha_L(t)\lambda_4(t), \\
\frac{d\lambda_4}{dt} &= (\alpha_P(t) + \mu_P(t))\lambda_4(t) - \alpha_P(t)\lambda_5(t), \\
\frac{d\lambda_5}{dt} &= -b(t)w\left(1 - \frac{E(t)}{K_E(t)}\right)\lambda_2(t) + \mu_A(t)\lambda_5(t),
\end{aligned} \tag{10}$$

with transversality condition $\lambda_i(T) = 0$, *for* $i = 1, ..., 5$. *Moreover, these optimal controls are characterized by:*

$$\left.\begin{aligned}
u_1 &= \min\left[u_{1\max}, \max\left(\frac{E(t)\lambda_2(t) + L(t)\lambda_3(t)}{W_1}, 0\right)\right], \\
u_2 &= \min\left[u_{2\max}, \max\left(\frac{L(t)\lambda_3(t}{W_2}, 0\right)\right]
\end{aligned}\right\} \tag{11}$$

Theorem 2(i) There exists an optimal control pair (u_1^*, u_2^*) to the problem (6).

Proof. Suppose that $\mathbf{f}(t, \mathbf{x}, \mathbf{u})$ be the right hand side of the (6) whereby $\mathbf{x} = (M, E, L, P, A)$ and $\mathbf{u} = (u_1(t), u_2(t))$ represent the vector of state variables and control functions respectively. We list the requirements for the existence of optimal control as presented in Fleming and Rishel (1975) [30]:

1) The function $\mathbf{f}$ is of class C^1 and there exists a constant C such that $|\mathbf{f}(t, 0, 0)| \le C, |\mathbf{f}_{\mathbf{x}}(t, \mathbf{x}, \mathbf{u})| \le C(1 + |\mathbf{u}|), |\mathbf{f}_{\mathbf{u}}(t, \mathbf{x}, \mathbf{u})| \le C$;
2) the admissible set of all solutions to system (6) with corresponding control in Ω is non empty;
3) $\mathbf{f}(t, \mathbf{x}, \mathbf{u}) = \mathbf{a}(t, \mathbf{x}) + \mathbf{b}(t, \mathbf{x})\mathbf{u}$;
4) the control set $U = [0, u_{1\max}] \times [0, u_{2\max}]$ is compact, closed, and convex;
5) the integrand of the objective functional is convex in U.

To verify these conditions, we write:

$$\mathbf{f}(t,\mathbf{x},\mathbf{u}) = \begin{bmatrix} r(t)M(t)\left[1-\dfrac{M(t)}{K_M}\right]-\dfrac{\beta(t)M(t)}{a(t)+M(t)}L(t), \\ b(t)wA(t)\left[1-\dfrac{E(t)}{K_E(t)}\right]-[\alpha_E(t)+\mu_E(t)+u_1(t)]E(t), \\ \alpha_E(t)E(t)+\dfrac{e\beta(t)M(t)}{a(t)+M(t)}L(t)-[\alpha_L(t)+\mu_L(t)+u_1(t)+u_2(t)]L(t)-\theta(t)L^2(t), \\ \alpha_L(t)L(t)-[\alpha_p(t)+\mu_P(t)]P(t), \\ \alpha_P(t)P(t)-\mu_A(t)A(t). \end{bmatrix}. \tag{12}$$

From Eq (12), it is evident that $\mathbf{f}(t,\mathbf{x},\mathbf{u})$ is of class C^1 and $|\mathbf{f}(t,\mathbf{0},\mathbf{0})| = 0$. Furthermore, one can easily compute $|\mathbf{f_x}(t,\mathbf{x},\mathbf{u})|$ and $|\mathbf{f_u}(t,\mathbf{x},\mathbf{u})|$, and demonstrate that:

$$|\mathbf{f}(t,\mathbf{0},\mathbf{0})| \le C, \quad |\mathbf{f_x}(t,\mathbf{x},\mathbf{u})| \le C(1+|\mathbf{u}|) \quad \text{and} \quad |\mathbf{f_u}(t,\mathbf{x},\mathbf{u})| \le C.$$

Due to the condition 1, the existence of the unique solution for condition 2 for bounded control is satisfied. On the other hand, the quantity $\mathbf{f}(t,\mathbf{x},\mathbf{u})$ is expressed as a linear function of the control variables which satisfy the condition 3. □

Theorem 2(ii) Given an optimal control $u = (u_1,u_2) \in U$ and corresponding state solutions M, E, L, P and P, there exists adjoint functions λ_i, $i = 1,2,3,4,5$ satisfying Eqs (10) and (11).

Proof. To characterize our optimal control problem we use Pontryagin's Maximum Principle [31], to formulate the following Hamiltonian function:

$$\begin{aligned} H(t) &= C_1E(t)+C_2L(t)+\frac{W_1}{2}u_1^2(t)+\frac{W_2}{2}u_2^2(t) \\ &+\lambda_1(t)\left[r(t)M(t)\left[1-\frac{M(t)}{K_M}\right]-\frac{\beta(t)M(t)}{a(t)+M(t)}L(t)\right] \\ &+\lambda_2(t)\left[b(t)wA(t)\left[1-\frac{E(t)}{K_E(t)}\right]-[\alpha_E(t)+\mu_E(t)+u_1(t)]E(t)\right] \\ &+\lambda_3(t)\left[\alpha_E(t)E(t)+\frac{e\beta(t)M(t)}{a(t)+M(t)}L(t)-[\alpha_L(t)+\mu_L(t)+u_1(t)+u_2(t)]L(t)-\theta(t)L^2(t)\right] \\ &+\lambda_4(t)\left[\alpha_L(t)L(t)-[\alpha_p(t)+\mu_P(t)]P(t)\right] \\ &+\lambda_5(t)\left[\alpha_P(t)P(t)-\mu_A(t)A(t)\right]. \end{aligned}$$

Next we determine the adjoint equations as follows; $\dfrac{\partial \lambda_i}{dt} = -\dfrac{\partial H}{\partial \mathbf{x}}$, where $\mathbf{x} = (M(t),E(t),L(t),P(t),A(t))$, with transversality condition $\lambda_i(T) = 0$ for $i = 1,2,3,4,5$, and obtained the results in equation (10). Now, we minimize the Hamiltonian with respect to the controls. Note that we have required the convexity for minimization, $\dfrac{\partial^2 H}{\partial u_i^2} = W_i > 0$, $i = 1,2$. On the interior of the control set, we have:

$$\left.\frac{\partial H}{\partial u_1} = 0 \Rightarrow u_1 = \frac{E(t)\lambda_2(t)+L(t)\lambda_3(t)}{W_1}, \qquad \text{and,} \qquad \frac{\partial H}{\partial u_2} = 0 \Rightarrow u_2 = \frac{L(t)\lambda_3(t}{W_2}. \right\}$$

Using the standard arguments and the bounds for the controls, one gets the characterization of this optimal pair (11). □

3.2. Numerical results

In this section, we support analytical results of this study by simulating model (6) by making use of the following assumed initial conditions:$E(0) = 500$, $L(0) = 0$, $P(0) = 0$, $A(0) = 500$, and $M(0) = 10$, as well as parameter values in Table 1.

Table 1. Model parameters and their baseline values.

Parameter definition	Symbol	Baseline value	Source
Average eggs laid per female moth per day	b_0	125	[32]
Proportion of adult female moths	w	0.5	[12]
Average duration of egg stage	α_{E0}^{-1}	3(3–5)Days	[32]
Development time of the larva	α_{L}^{-1}	14 (14–30)Days	[32]
Development time of pupae	α_{P0}^{-1}	9(8-30)Days	[32]
Moth life span	μ_{A0}^{-1}	18(15–21) Days	[32]
Maximum biomass of maize plants	K_M	50 kg plant^{-1}	[36].
Egg environmental carrying capacity	K_{E0}	10^8	[36].
Averaged natural death rate of immature stages	$\mu_{E0}, \mu_{L0}, \mu_{P0}$	0.01 Day^{-1}	[36].
Rate of plant attack by larvae	β_0	5×10^{-8}Day^{-1}	[36].
Growth rate of maize plants	r_0	0.05 Day^{-1}	[36].
Efficiency of biomass conversion	e	0.2	[36].
Average density dependent mortality rate	θ_0	0.008 Day^{-1}	[36].
Half saturation constant	a_0	0.8	Estimate.

Without loss of generality, we will fix all parameters that model the amplitude of seasonal dependent parameters (r_1, K_{M_1}, β_1, K_{E_1}, a_1, b_1, α_{E_1}, μ_{E_1}, e_1, α_{L_1}, μ_{L_1}, θ_1, α_{P_1} and μ_{A_1}) to 0.8. Furthermore, we set $\omega = 7$, that is., a small period was considered since the life span of FAW and maize in the field is very short.

It is worth noting that numerical simulations presented here were obtained by solving system (6) using the forward-backward sweep method [37] and the parameter values in Table 1. In addition, we considered the following assumed initial conditions; $E(0) = 500$, $L(0) = 0$, $P(0) = 0$, $A(0) = 500$, and $M(0) = 10$. The initial step of the forward-backward sweep method is to assign an initial guess for the controls and then solve the system forward in time, followed by solving the adjoint state backward in time. Then these optimal controls are updated for optimality using the Hamiltonian of the optimal system. "The controls are then updated by using a convex combination of the previous controls and the value from the characterizations of the controls. This process is repeated and iterations are ceased if the values of the unknowns at the previous iterations are very close to the ones at the present iterations" [37]. For a detailed discussion we refer the reader to [37]. The numerical results presented in this Section are based on the following scenarios:

(i) *Effects of implementing traditional control measures alone*,

(ii) *Effects of implementing time-dependent use of chemical insecticides alone*, and

(iii) *Effects of combining time-dependent traditional methods with use of chemical insecticides.*

In addition, since the impact of larvae on maize is so apparent, we will assume that $C_1 \leq C_2$, that is, the minimization of caterpillars is more important than that of FAW eggs. Furthermore, traditional methods of controlling FAW are known to be less costly compared to chemical insecticides and as such, we will assume that $W_1 < W_2$.

3.2.1. Effects of implementing traditional control measures alone

In resource limited settings, majority of the farmers cannot purchase pesticides to control FAW whenever there is an outbreak and more often they rely on traditional methods like handpicking and destroying of egg masses and larvae. Here, we seek to understand the effects of time dependent implementation of such methods on the dynamics of FAW and maize interaction. To investigate this scenario, we simulated model (6) with $u_1 \neq 0$ $(0 \leq u_1 \leq 0.1)$ and $u_2 \neq 0$ and we obtained the results illustrated in Figure 2. As we can observe, the dynamics of the maize biomass and FAW populations,

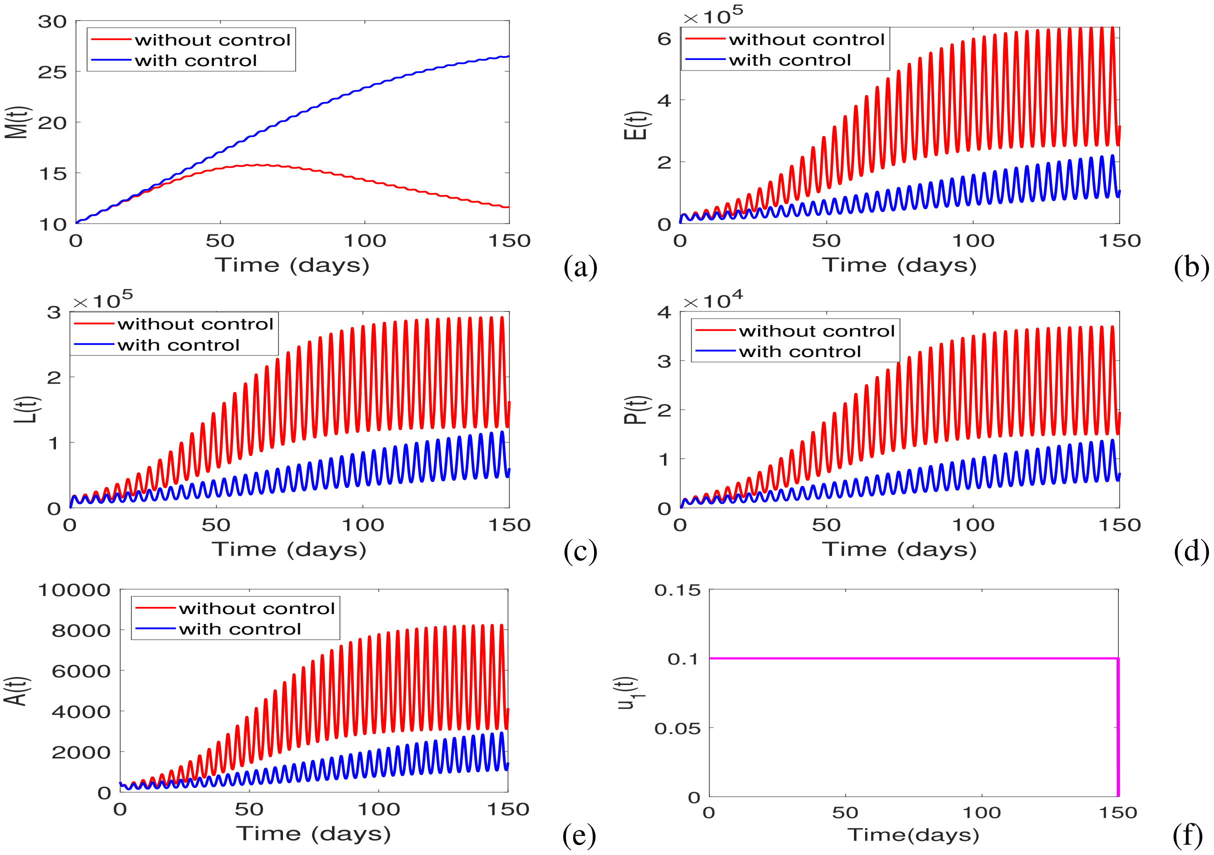

Figure 2. Solution of model (6) with and without optimal control under scenario 1: $u_1 \neq 0$ and $u_2 = 0$. We set $0 \leq u_1(t) \leq 0.1$. In (a)–(e), the red solid trend-lines represents the dynamics of the respective populations in the absence of control and the blue solid curves depict the dynamics in the presence of control. Fig. (f) illustrates the optimal control profile. As we can observe, with control $u_1(t)$ being implemented, the respective populations of the FAW decrease compared to when there are no controls.

with and without control, will be associated with oscillations which reflect seasonal variations. Further, we can also observe that without control, the maize biomass may not exceed 15 kg per plant, however, with timely control, the biomass may exceed 25 kg per plant by the final time horizon ($t = 150$). Moreover, although traditional methods will be capable of reducing FAW population and increasing maize biomass, they will not be able to completely eliminate the pest. Figure 2(f) portrays the optimal control profiles for $u_1(t)$. We clearly observe that $u_1(t)$ starts from the maximum ($u_1 = 0.1$) and stays at that level for the entire duration. From the pattern of the optimal control profile we can conclude that a desirable outcome can be achieved only if the traditional methods are implemented throughout the entire time horizon.

3.2.2. Effects of implementing time-dependent use of chemical insecticides alone

Despite being expensive, chemical insecticides are known to be more efficient compared to traditional methods on controlling FAW. To explore the impact of chemical control measures on FAW dynamics we simulated model (6) with $u_1 = 0$ and $0 < u_2(t) < 0.1$ and the results are depicted in Figure 3. When chemical insecticides are used, we can note that the population of FAW may become extinct in a period of 50 days. Moreover, the maize biomass per plant may exceed 35 kg per plant by the final time ($t = 150$ days). Comparing the results portrayed in Figures 2 and 3, we conclude that the use of chemical insecticides should be encouraged since the final biomass will be higher

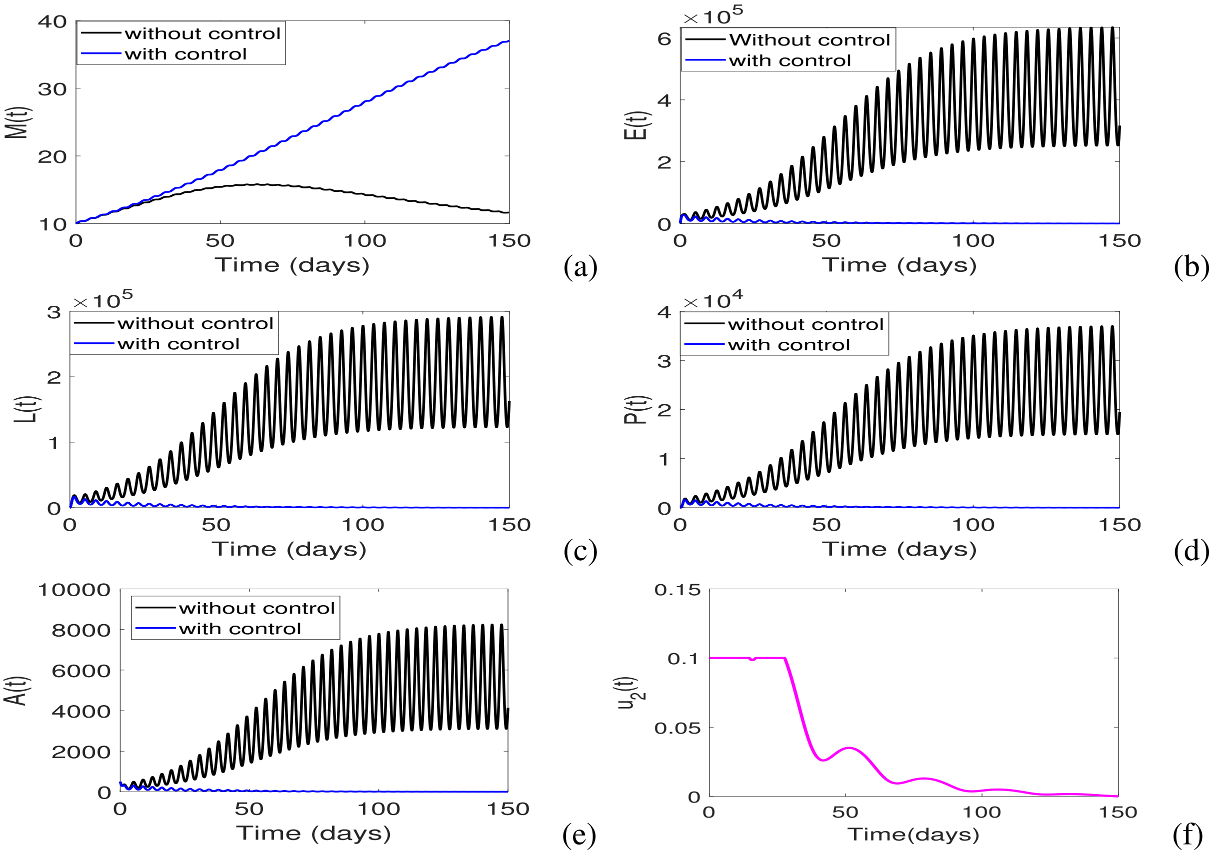

Figure 3. Solution of model (6) with and without optimal control under scenario 2: $0 < u_1 \neq 0$ and $0 < u_2 \leq 0.1$.

compared to when farmers rely on traditional methods only. The control profile for control $u_2(t)$ starts from the maximum initially, but only for a very short time ($t < 50$), followed by a decrease to some lower level till the final time horizon. This may attribute to the decrease in FAW populations. Hence, we conclude that for chemical insecticides, intensity use needs to be maintained at maximum for a period of approximately 50 days, thereafter the intensity may be reduced till the final time.

3.2.3. Effects of combining time dependent traditional methods with use of chemical insecticides

To understand the impact of combining traditional methods with use of chemical insecticides, we simulated model (6) with $0 < u_1(t) \leq 0.1$ and $0 < u_2(t) \leq 0.1$ over period of 150 days and the solution results are depicted in Figure 4. We can note that when traditional methods are combined with chemical insecticides use then the time taken to eliminate the FAW from the field is less than the time that will be taken if chemical insecticides were in use (Figure 4). Although the time required to eliminate the FAW populations will decrease, the final maize biomass may not be significantly different from that obtained when only chemical insecticides were in use (Figure 3). In Figure 4(f), we can observe that the control profiles for $u_1(t)$ and $u_2(t)$ starts at their respective maximum initially, but only for a very short time, followed by a decrease to some lower level till the final time. It is worth noting that the control profile for $u_1(t)$ remain at its maximum for a slightly longer period compared to that of $u_2(t)$ and this can be attributed to less cost associated with traditional methods relative to chemical insecticides

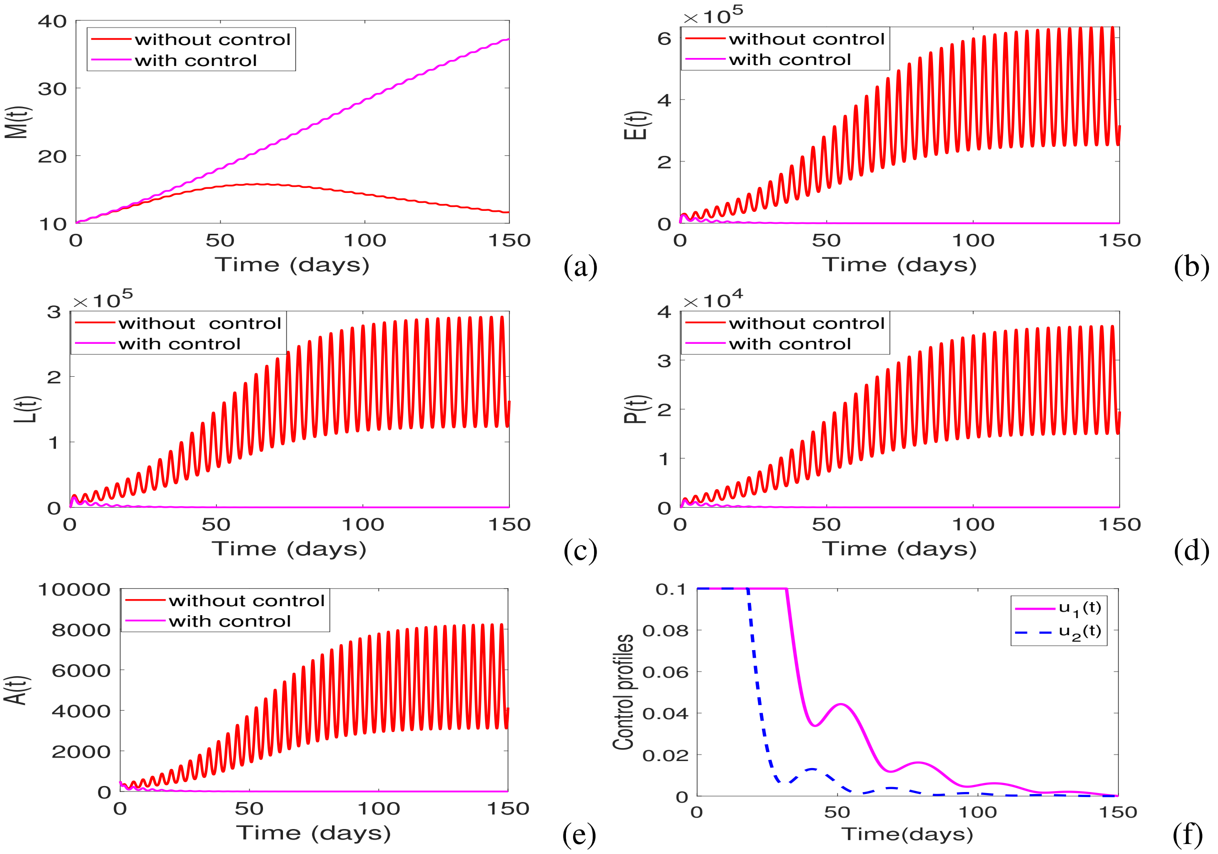

Figure 4. Solution of model (6) with and without optimal control under scenario 3: $0 < u_1 \leq 0.1$ and $0 < u_2 \leq 0.1$.

use. As such we can conclude that when traditional methods are combined with chemical insecticides use, chemical control efforts may be ceased after approximately 50 days and the traditional methods can be implemented for additional 50 days or more but at low intensity. To assess the effects of costs on implementing the control efforts $u_1(t)$ and $u_2(t)$, we varied the weight constants W_1 and W_2 and the results are illustrated in Figure 5. From the results we can note that if the costs are low, for example, $W_1 = W_2 = 0.1$ then the associated control profile starts at their respective maximum and stays there till the final time horizon. However, as the cost increases the respective control profile starts at their respective maxima and stays there for a reduced duration compared to when the costs are low. In particular, as the cost increases the control profile for $u_2(t)$ stays at its maxima for a relatively short duration compared to that of $u_1(t)$. In a nutshell, we can deduce that depending on the cost parameters associated with the control, the optimal profiles of $u_1(t)$ and $u_2(t)$ stay at their respective maxima for a longer duration, before eventually settling at their minimum levels.

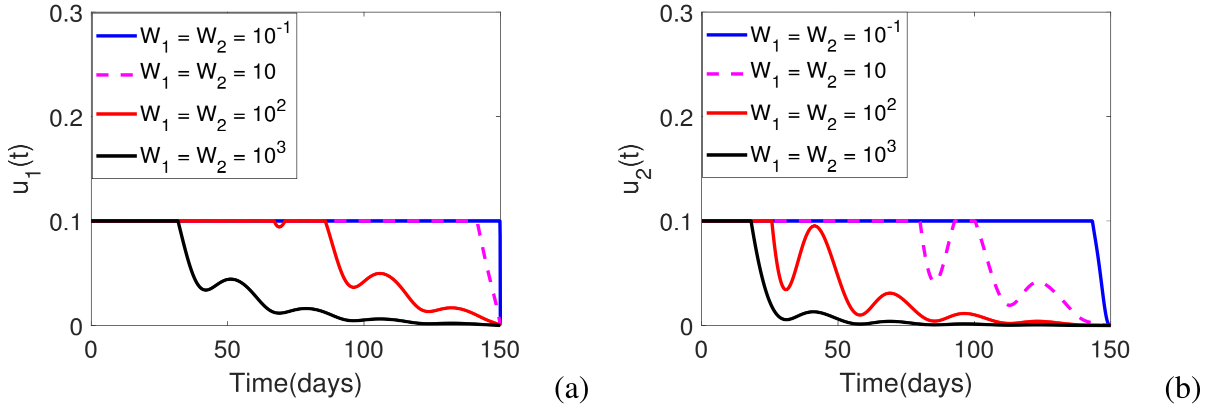

Figure 5. Simulation results for model system (1) illustrating the effects of varying the weights, W_1 and W_2. We set $0 \leq u_1 \leq 0.1$, $0 \leq u_2 \leq 0.1$, C_1 and C_2 are fixed to 1 and 2, respectively, and the rest of the model parameters are as in Table 1.

4. Conclusions

We have formulated a mathematical model to investigate the effects of seasonal variations on the dynamics of maize biomass and FAW interaction. After a comprehensive analysis of the dynamical behavior of the proposed framework, we extended it to incorporate time-dependent control strategies, namely traditional methods (like handpicking and destruction of egg masses and larvae) and the use of chemical insecticides. Our optimal control is aimed at minimizing the numbers of the eggs and larvae population at minimal costs. Our results show that, in all the scenarios, the optimal control can greatly reduce the FAW population and in some instances, complete elimination of the pest may be attained. Future research could expand our analysis to include climate-sensitive aspects of FAW such as temperature and predict changes in population dynamics at various temperature ranges.

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Conflict of interest

The authors declare that they have no competing interests.

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Appendix

A. Proof of Theorem 1 and the discussion on global asymptotic stability of the boundary solution

In this section, we will provide the proof of Theorem 1 and we will discuss the global asymptotic stability of the boundary solutions of the model. We will begin our discussion by considering Theorem 1 (i):

To prove Theorem 1 (i), we will make use of the lemma as given by Bai et al [26], which was used to demonstrate the permanence of a non-autonomous prey-predator model with a generalist predator.

Lemma 1. *Suppose $p(t)$ and $q(t)$ are bounded and continuous functions in $\mathbb{R}$ with $\inf_{t\in\mathbb{R}} p(t) > 0$ and $\inf_{t\in\mathbb{R}} q(t) > 0$. If there exist a positive function $u(t)$ which satisfies:*

$$u'(t) \leq p(t)u(t) - q(t)u^2(t), \qquad t \in [t_0, +\infty),$$

then $\limsup_{t\to+\infty} u(t) \leq \sup_{t\in\mathbb{R}} \frac{p(t)}{q(t)}$. Moreover, $u(t) \leq \sup_{t\in\mathbb{R}} \frac{p(t)}{q(t)}$ for all $t \in [t_0, +\infty)$ if $0 < u(t_0) \leq \sup_{t\in\mathbb{R}} \frac{p(t)}{q(t)}$. On the other hand, if $u(t)$ satisfies:

$$u'(t) \geq p(t)u(t) - q(t)u^2(t), \qquad t \in [t_0, +\infty),$$

then $\liminf_{t\to+\infty} u(t) \geq \inf_{t\in\mathbb{R}} \frac{p(t)}{q(t)}$. Moreover, $u(t) \geq \inf_{t\in\mathbb{R}} \frac{p(t)}{q(t)}$ for all $t \in [t_0, +\infty)$ if $0 < u(t_0) \geq \inf_{t\in\mathbb{R}} \frac{p(t)}{q(t)}$.

We now demonstrate the proof for Theorem 1 as follows; Considering system (1), we have the following expressions:

$$\left.\begin{array}{rcl} M(t) & = & M(t_0)\exp\left\{\displaystyle\int_{t_0}^{t}\left[r(s)M(s)\left[1-\frac{M(s)}{K_M}\right]-\frac{\beta(s)M(s)}{a(s)+M(s)}L(s)\right]ds\right\}, \\ E(t) & = & E(t_0)\exp\left\{\displaystyle\int_{t_0}^{t}\left[b(s)wA(s)\left[1-\frac{E(s)}{K_E(s)}\right]-[\alpha_E(s)+\mu_E(s)]E(s)\right]ds\right\}, \\ L(t) & = & L(t_0)\exp\left\{\displaystyle\int_{t_0}^{t}\left[\alpha_E(s)E(s)+\frac{e\beta(s)M(s)}{a(s)+M(s)}L(s)-[\alpha_L(s)+\mu_L(s)]L(s)\right.\right. \\ & & \left.\left.-\theta(s)L^2(s)\right]ds\right\}, \\ P(t) & = & P(t_0)\exp\left\{\displaystyle\int_{t_0}^{t}\left[\alpha_L(s)L(s)-[\alpha_p(s)+\mu_P(s)]P(s)\right]ds\right\}, \\ A(t) & = & A(t_0)\exp\left\{\displaystyle\int_{t_0}^{t}\left[\alpha_P(s)P(s)-\mu_A(s)A(s)\right]ds\right\}. \end{array}\right\} \tag{13}$$

From Eq (13), we can observe that all the solutions of model (1) are non-negative. We now demonstrate that, the solution set Ω of the system (1) is positively invariant. Let ($M(t)$, $E(t)$, $L(t)$, $P(t)$, $A(t)$) be a unique solution of system (1) with ($M(t_0)$, $E(t_0)$, $L(t_0)$, $P(t_0)$, $A(t_0)$) $\in \Omega$. From the first equation of (1) and the positivity solutions of (1), we have:

$$M'(t) \;\leq\; r(t)M(t)\left[1-\frac{M(t)}{K_M}\right], \qquad t \geq t_0,$$

and by Lemma 1 and $0 < M(t_0) \leq \mathcal{M}_1$, $M(t) \leq \mathcal{M}_1$, $t \geq t_0$. Considering the second equation of the system (1), we have:

$$\begin{array}{rcl} E(t) & \leq & b(t)wA(t) - [b(t)wA(t) + \alpha_E(t) + \mu_E(t)]E(t) \\ & \leq & b(t)w\mathcal{M}_5 - [b(t)w\mathcal{M}_5 + \alpha_E(t) + \mu_E(t)]E(t),, \qquad t \geq t_0, \end{array}$$

by Lemma 1 and $0 < E(t_0) \leq \mathcal{M}_2$, $E(t) \leq \mathcal{M}_2$, $t \geq t_0$. From the third equation of model (1), we have:

$$L'(t) \;\leq\; \alpha_E(t) + e\beta(t)\Phi(\mathcal{M}_1)(t)L(t) - [\alpha_L(t) + \mu_L(t)]L(t) - \theta(t)L^2(t), \qquad t \geq t_0,$$

by Lemma 1 and $0 < L(t_0) \leq \mathcal{M}_3$, $L(t) \leq \mathcal{M}_3$, $t \geq t_0$. From the fourth equation of system (1), we have:

$$P'(t) \leq \ [\mu_P(t) + \alpha_P(t)]\left[\frac{\alpha_L(t)\mathcal{M}_3}{[\mu_P(t) + \alpha_P(t)]} - P(t)\right], \qquad t \geq t_0,$$

by Lemma 1 and $0 < P(t_0) \leq \mathcal{M}_4$, $P(t) \leq \mathcal{M}_4$, $t \geq t_0$. From the last equation of system (1), we have:

$$A'(t) \leq \ \mu_A(t)\left[\frac{\alpha_P(t)\mathcal{M}_4}{\mu_A(t)} - A(t)\right], \qquad t \geq t_0.$$

Again from the first equation of system (1), we have:

$$\begin{aligned} M'(t) &\geq r(t)M(t) - \frac{r(t)}{K_M}M^2(t) - \beta(t)L(t)M(t), \\ &\geq (r(t) - \beta(t)\mathcal{M}_3) - \frac{r(t)}{K_M}M^2(t), \qquad t \geq t_0, \end{aligned}$$

and by Lemma 1 and $M(t_0) \geq m_1 > 0$, we get $M(t) \geq m_1$, $t \geq t_0$. From the second equation of system (1), we have:

$$\begin{aligned} E(t) &\geq b(t)wA(t) - [b(t)wA(t) + \alpha_E(t) + \mu_E(t)]E(t) \\ &\geq b(t)wm_5 - [b(t)wm_5 + \alpha_E(t) + \mu_E(t)]E(t), \qquad t \geq t_0. \end{aligned}$$

By Lemma 1 and $E(t_0) \geq m_2 > 0$, it follows that $E(t) \geq m_2$, holds for $t \geq t_0$. From the third equation of system (1), we have:

$$L'(t) \ \geq \ \alpha_E(t) + e\beta(t)\Phi(m_1)(t)L(t) - [\alpha_L(t) + \mu_L(t)]L(t) - \theta(t)L^2(t), \qquad t \geq t_0,$$

It follows from Lemma 1 and $L(t_0) \geq m_3 > 0$ that$L(t) \geq m_3$, holds for $t \geq t_0$. From the fourth equation of system (1), we have:

$$P'(t) \geq \ [\mu_P(t) + \alpha_P(t)]\left[\frac{\alpha_L(t)m_3}{[\mu_P(t) + \alpha_P(t)]} - P(t)\right], \qquad t \geq t_0,$$

By Lemma 1 and $P(t_0) \geq m_4 > 0$ we have thus $P(t) \geq m_3$, holds fo $t \geq t_0$. Furthermore, from the last equation of system (1), we have:

$$A'(t) \geq \ \mu_A(t)\left[\frac{\alpha_P(t)m_4}{\mu_A(t)} - A(t)\right], \qquad t \geq t_0,$$

which implies that by Lemma 1 and $A(t_0) \geq m_5$ that $A(t) \geq m_5$ holds $\forall t \geq t_0$. Hence, the solution set Ω of the system (1) is positive invariant.

Suppose if the condition (H1) holds, now we prove that the model system (1) is permanent. We let $(M(t), E(t), L(t), P(t), A(t))$ be a unique solution of system (1) with positive initial value $(M(t_0), E(t_0), L(t_0), P(t_0), A(t_0))$. Choose $\delta > 0$ which is sufficiently small so that $m_i^\delta(i = 1, 2, 3, 4, 5)$, and each inequality of (H1) holds when $\mathcal{M}_i$ and m_i are replaced by $\mathcal{M}_i^\delta > 0$ and $m_i^\delta > 0$, respectively. By Lemma 1, it follows that $\limsup_{t\to+\infty} M(t) \leq \mathcal{M}_1$, which follows that there exists $T_0 > t_0$ such that for $t > T_0$, $M(t) \leq \mathcal{M}_1^\delta$. Then from the first equation of system (1), we have:

$$M'(t) \leq r(t)M(t) - \frac{r(t)}{K_M}M^2(t), \qquad t > T_0,$$

which yields by Lemma 1 that:

$$\limsup_{t\to+\infty} M(t) \leq \sup_{t\in\mathbb{R}}\{K_M\}.$$

Hence, by the arbitrariness of δ, we obtain $\limsup_{t\to+\infty} M(t) \leq \mathcal{M}_1$. Then, there exists $T_1 > T_0$ such that for $t > T_1$, $M(t) \leq \mathcal{M}_1^\delta$, and:

$$E(t) \quad \leq \quad b(t)w\mathcal{M}_5^\delta - [b(t)w\mathcal{M}_5^\delta + \alpha_E(t) + \mu_E(t)]E(t), \qquad t > T_1.$$

It follows from Lemma 1 and $\inf_{t\in\mathbb{R}}\{b(t)w\mathcal{M}_5^\delta - [b(t)w\mathcal{M}_5^\delta + \alpha_E(t) + \mu_E(t)]E(t)\} > 0$ that

$$\limsup_{t\to+\infty} E(t) \leq \sup_{t\in\mathbb{R}}\left\{\frac{b(t)w\mathcal{M}_5^\delta}{[b(t)w\mathcal{M}_5^\delta + \alpha_E(t) + \mu_E(t)]E(t)}\right\}.$$

Hence, by the arbitrariness of δ, we have $\limsup_{t\to+\infty} E(t) \leq \mathcal{M}_2$, and there exists $T_2 > T_1$ such that for $t > T_2$, $E(t) \leq \mathcal{M}_2^\delta$, and

$$L'(t) \quad \leq \quad \alpha_E(t) + e\beta(t)\Phi(\mathcal{M}_1^\delta)(t)L(t) - [\alpha_L(t) + \mu_L(t)]L(t) - \theta(t)L^2(t), \qquad t > T_2,$$

It follows from Lemma 1 and $\inf_{t\in\mathbb{R}}\{e\beta(t)\Phi(\mathcal{M}_1^\delta)(t)L(t) - [\alpha_L(t) + \mu_L(t)]\} > 0$ that:

$$\limsup_{t\to+\infty} L(t) \leq \sup_{t\in\mathbb{R}}\left\{\alpha_E(t)\mathcal{M}_2^\delta + \frac{e\beta(t)\Phi(\mathcal{M}_1^\delta)(t) - (\mu_L(t) + \alpha_L(t))}{\theta(t)}\right\}.$$

Hence, by the arbitrariness of δ, we have $\limsup_{t\to+\infty} L(t) \leq \mathcal{M}_3$, and there exists $T_3 > T_2$ such that for $t > T_3$, $L(t) \leq \mathcal{M}_3^\delta$, and:

$$P'(t) \leq \alpha_L(t)\mathcal{M}_3^\delta - (\mu_P(t) + \alpha_P(t))P(t), \qquad t > T_3,$$

which yields by Lemma 1 that:

$$\limsup_{t\to+\infty} P(t) \leq \sup_{t\in\mathbb{R}}\left\{\frac{\alpha_L(t)\mathcal{M}_3^\delta}{[\mu_P(t) + \alpha_P(t)]}\right\}.$$

Hence, by the arbitrariness of δ, we obtain $\limsup_{t\to+\infty} P(t) \leq \mathcal{M}_4$. Then, there exists $T_4 > T_3$ such that for $t > T_4$, $P(t) \leq \mathcal{M}_4^\delta$, and:

$$A'(t) \leq \alpha_P(t)\mathcal{M}_4^\delta - \mu_A(t)P(t), \qquad t > T_4,$$

which yields by Lemma 1 that:

$$\limsup_{t\to+\infty} A(t) \leq \sup_{t\in\mathbb{R}}\left\{\frac{\alpha_P(t)\mathcal{M}_4^\delta}{\mu_A(t)}\right\}.$$

Hence, by the arbitrariness of δ, we obtain $\limsup_{t\to+\infty} A(t) \leq \mathcal{M}_5$. Then, there exists $T_5 > T_4$ such that for $t > T_4$, $A(t) \leq \mathcal{M}_5^\delta$. This completes the proof of Theorem 1(i).

In what follows, we will investigate the global asymptotic stability of the boundary solution. We will assume that the maize biomass is the only food source for the FAW in this case; such that in the absence of the maize plant biomass, the FAW population becomes extinct. Hence the only boundary solution of system (1) is $(M(t), 0, 0, 0, 0)$. Substituting this boundary solution into system (1) one gets:

$$\frac{dM(t)}{dt} = r(t)M(t)\left(1 - \frac{M(t)}{K_M}\right). \tag{14}$$

Equation (14) is well known non-autonomous logistic equation. As illustrated in equation (3), $r(t)$ is continuous $\omega-$periodic, bounded below and above by positive constants. According to Fan and Wang [27], we have the following results:

Lemma 2. *(Reference [27]): If $r(t)$ is a continuous $\omega-$ periodic function, and bounded below and above by strictly positive reals for all $t \in \mathbb{R}$, the logistic Eq (14) has exactly a solution $M^g(t)$ bounded below and above by positive constants. Precisely, this solution is given by:*

$$M^g(t) = \left[\exp\left(\int_0^{\omega} r(s)ds\right) - 1\right] \cdot \left[\int_t^{t+\omega} \frac{r(s)}{K_M} \cdot \exp\left(-\int_s^t r(\tau)d\tau)ds\right)\right]^{-1}. \tag{15}$$

In addition, $M^g(t)$ is globally asymptotically stable for $M(t)$ with positive initial value $M(t_0) = M_0 > 0$ in the sense $\lim_{t\to+\infty} |M(t) - M^g(t)| = 0$.

By Lemma 2, we obtain the following result:

Lemma 3. *System (1) admits a unique positive $\omega-$periodic solution $M^g(t), 0, 0, 0, 0)$ which is globally asymptotically stable for $M(t)$ with positive initial value $M(t_0) = M_0 > 0$ in the sense* $\lim_{t\to+\infty} |M(t) - M^g(t)| = 0$.

For a continuous and periodic function $g(t)$ with periodic ω, we denote:

$$\mathcal{A}(g) := \frac{1}{\omega}\int_0^{\omega} g(t)dt. \tag{16}$$

Lemma 4. *(Reference [28]): If $r(t)$ is a continuous $\omega-$periodic function, then the null solution of (14) is globally asymptotically stable provided that one of the following two conditions is met:*

(1) $\mathcal{A}(r) < 0$;

(2) $\mathcal{A}(r) = 0$, *and* $\mathcal{A}(r/K_M) < 0$.

Note: If $\mathcal{A}(r) > 0$ and $\mathcal{A}(r/K_M) > 0$, then (14) has a unique positive $\omega-$periodic solution $M^g(t)$ which is globally asymptotically stable (see Tineo [29]). Thus, when r/K_M is non-negative with $\mathcal{A}(r/K_M) > 0$, the null solution of (14) is globally stable if and only if $\mathcal{A}(r) \leq 0$.

Finally, we we provide the proof of Theorem 1(ii):

Define a Poincare mapping $\mathcal{F} : \mathbb{R}^5 \to \mathbb{R}^5$ as follows:

$$\begin{aligned} \mathcal{F}(\xi) &= (M(t_0+\omega, t_0, \xi), E(t_0+\omega, t_0, \xi), L(t_0+\omega, t_0, \xi), P(t_0+\omega, t_0, \xi), A(t_0+\omega, t_0, \xi)) \\ \xi &= (M_0, E_0, L_0, P_0, A_0) \in \mathbb{R}^5, \end{aligned}$$

where $(M(t_0+\omega, t_0, \xi), E(t_0+\omega, t_0, \xi), L(t_0+\omega, t_0, \xi), P(t_0+\omega, t_0, \xi), A(t_0+\omega, t_0, \xi))$ represents the solution of (1) through (t_0, ξ), $\xi = (M_0, E_0, L_0, P_0, A_0) \in \mathbb{R}^5$. By the positive invariant property of Ω, $\mathcal{F}(\Omega) \cap \Omega$. The continuity of $\mathcal{F}$ can be guaranteed by the continuity of solution of Eq (1) with respect to initial value. Note that Ω is closed, bounded, convex set in $\mathbb{R}^5$. Therefore, it follows from Brouwer's theorem of fixed point that the operator $\mathcal{F}$ has at least one fixed point $\xi^* = (M^*(t), E^*(t), L^*(t), P^*(t), A^*(t))$ in Ω, which is a positive $\omega-$ periodic solution of system (1). The proof is complete.

AIMS Press

Dynamics for a non-autonomous fall armyworm-maize biomass interaction model with a saturation functional response

Introduction

Maize (*Zea mays*) is ranked the third most important cereal grain after wheat and rice globally and is also referred the "Queen of Cereals" due to its high genetic yield potential [2]. One of the main threats to food security in these countries is the invasion by fall armyworm (FAW), *Spodoptera frugiperda*, a major pest of maize [1, 2], native to tropical and subtropical parts of America [1, 2].

Basic non autonomous Mathematical model

Model derivations

Motivated by recent mathematical models for plant-pest interactions (see., for example [1], in this work we propose a non-autonomous model for FAW infestations in a maize field. The proposed model subdivides the FAW population into subclasses as: eggs population $E(t)$, larvae population $L(t)$, pupae population $P(t)$ and adult population which are also known as moth $A(t)$ Meanwhile, let the variable $M(t)$ represents maize biomass population. Due to finite sizes of fields of maize biomass, we assume a logistic growth for the biomass of maize plant, with a net seasonal growth rate $r(t)$ and seasonal carrying capacity K_M. The proposed model is summarized by the following system of non-linear ordinary differential equations:

$$\left.\begin{aligned}
\frac{dM(t)}{dt} &= r(t)M(t)\left[1-\frac{M(t)}{K_M}\right]-\frac{\beta(t)M(t)}{a(t)+M(t)}L(t),\\
\frac{dE(t)}{dt} &= b(t)wA(t)\left[1-\frac{E(t)}{K_E(t)}\right]-[\alpha_E(t)+\mu_E(t)]E(t),\\
\frac{dL(t)}{dt} &= \alpha_E(t)E(t)+\frac{e\beta(t)M(t)}{a(t)+M(t)}L(t)-[\alpha_L(t)+\mu_L(t)]L(t)\\
&\quad -\theta(t)L^2(t),\\
\frac{dP(t)}{dt} &= \alpha_L(t)L(t)-[\alpha_p(t)+\mu_P(t)]P(t),\\
\frac{dA(t)}{dt} &= \alpha_P(t)P(t)-\mu_A(t)A(t).
\end{aligned}\right\}$$

All model parameters that are functions of time depend on seasonal variations. For biological significance, we assume that all these parameters are continuous and bounded functions defined on $\mathbb{R}$.

Mathematical model with optimal control

Model derivations

In this section, we re-formulate the problem as an optimal control problem with a goal to determine the impact of time-dependent intervention approaches on reducing or eradicating FAW population in maize field. Here, we re-formulate the model without time dependent control to incorporate new parameters $u_1(t)$ and $u_2(t)$. Control $u_1(t)$ models the efforts of traditional control methods like handpicking and destruction of FAW egg masses and larvae on FAW dynamics. Control $u_2(t)$ accounts for the efforts of chemical pesticide use on FAW dynamics.

$$\left.\begin{aligned}
\frac{dM(t)}{dt} &= r(t)M(t)\left[1-\frac{M(t)}{K_M}\right]-\frac{\beta(t)M(t)}{a(t)+M(t)}L(t),\\
\frac{dE(t)}{dt} &= b(t)wA(t)\left[1-\frac{E(t)}{K_E(t)}\right]-[\alpha_E(t)+\mu_E(t)+u_1(t)]E(t),\\
\frac{dL(t)}{dt} &= \alpha_E(t)E(t)+\frac{e\beta(t)M(t)}{a(t)+M(t)}L(t)-[\alpha_L(t)+\mu_L(t)+u_1(t)+u_2(t)]L(t)\\
&\quad -\theta(t)L^2(t),\\
\frac{dP(t)}{dt} &= \alpha_L(t)L(t)-[\alpha_p(t)+\mu_P(t)]P(t),\\
\frac{dA(t)}{dt} &= \alpha_P(t)P(t)-\mu_A(t)A(t).
\end{aligned}\right\}$$

Numerical results

Effects of implementing traditional control measures alone.

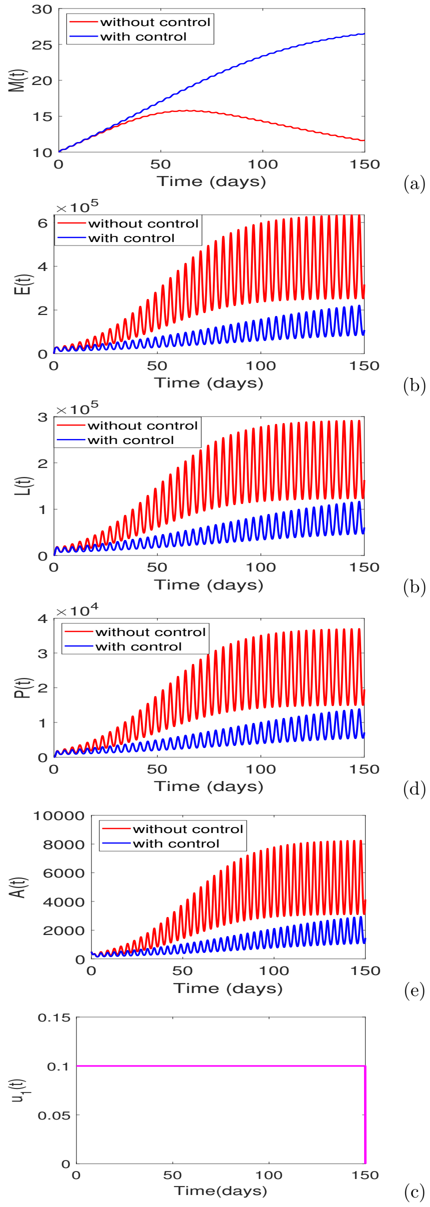

Figure 1: Solution of model with and without optimal control under scenario 1: $u_1 \neq 0$ and $u_2 = 0$. We set $0 \leq u_1(t) \leq 0.1$.

Effects of combining time dependent traditional methods with use of chemical insecticides

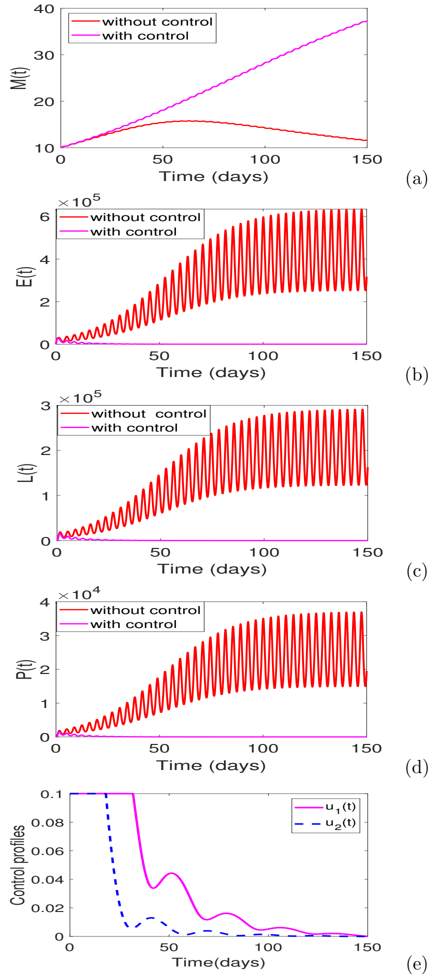

Figure 2: Solution of model with and without optimal control under scenario 3: $0 < u_1 \leq 0.1$ and $0 < u_2 \leq 0.1$.

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