# Physics student ideas on quantum state and its formal representations

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**Summary.** — Developing a quantum way of thinking is a core and challenging task for physics students. The concept of quantum state, whose physical meaning is connected to the formal structure of the theory, plays an important role in the construction of a quantum perspective and in student difficulties elicited by research. A questionnaire and interview protocol were devised to explore student understanding of the state concept in connection to the properties of its formal representations and to quantum behavior. Results of a calibration of research instruments performed on 6 physics students from different universities are here presented.

PACS `01.40.Fk` – Research in physics education.
PACS `03.65.-w` – Quantum mechanics.
PACS `03.65.Ca` – Formalism.

## 1. – Introduction

Conceptual change from a classical to a quantum perspective is a crucial and complex process for physics students. It is crucial in the education of future researchers, professionals and teachers, which requires getting a firm grasp of quantum physics because of its fundamental scientific content, its widespread technological applications and its cultural significance. But it is complex, as the emergence of quantum behavior entails the re-interpretation of basic concepts (such as the state of a system) and the introduction of new concepts (such as incompatibility of observables and entanglement), whose construction is non-intuitive and very different from the classical one students are used to. In addition, the physical meaning of theoretical entities is deeply linked to the formal structure of the theory, which is in turn a new and highly mathematical language. Therefore, purely verbal or iconic representation of processes alone is not sufficient to build solid mental models of quantum concepts and their interconnections. For this purpose, it is essential to care about the founding role of mathematics and the need to continuously operate a transition from mathematics to physical meaning and vice versa.

The concept of quantum state is a paradigmatic example of these multiple layers of complexity. While the classical state of a system provides the value of each physical quantity (i.e. each measurable property of the system), a quantum state provides the sets of probability distributions for the measurement of each physical quantity on the system. A fundamental difference between the classical and the quantum notion of state is that the possible measurement outcomes on a quantum system generally don't coincide with its properties immediately before measurement, but they are acquired in the process. In order to make this distinction, it's necessary to develop an understanding of new features of quantum behavior such as the active nature of quantum measurement, its stochastic character, indeterminism and incompatibility of observables. Most of these features are in turn encoded in the likewise new formal constructs of Hilbert space: vectors to represent the state of a system (instead of a point in phase space as in classical Hamiltonian mechanics) and operators to represent physical quantities (instead of real valued functions in phase space). Connecting the quantum state and its formal representations to measurement entails taking into account two further elements: the concept of eigenstates of the measured observable as possible output states, over whose basis the state vector must be developed; the Born rule, i.e. the interpretation of the coefficient of each eigenstate as entity encoding the corresponding transition probability. This connection enables the assignment of a physical meaning to vector superposition in quantum mechanics (QM) and its use to make predictions on measurement. In conclusion, employing the concept of quantum state in order to interpret and structure physical situations involves understanding how conceptual elements and their interconnections are embedded in the formal entities representing them.

Educational research at university level confirms the importance of quantum state in leaning difficulties. This role is testified by the widespread "misconceptions", i.e. student typical mistakes, elicited by research: for instance, students struggle to interpret the representational role of $\psi$, e.g. using its modulus as the expression of particle energy [1], and resort to naïve mechanisms to describe time evolution of state in the absence of measurement [2]. Analogous significance is ascribed by research to the relation between mathematical structures and physical meaning, as shown by difficulties to make qualitative inferences from quantitative tools [3], and in general to making meaning of mathematics in QM [4]. Although a valuable empirical basis has been established in the last ten years by the physics education research community, many aspects of student ideas on quantum state and on physical information contained in its formal description still await exploration. For instance, there is scant research on the role of phase relations in QM, as well as a need to explore student understanding of the notion of eigenstate as a junction between quantum state and measurement of observables as well as between quantum state and its time evolution. More in general, it is important to understand how student use the formal representations of quantum state to interpret/structure physical situations: students need to mathematize, i.e. transpose patterns of experimental data obtained in lab space into elements of quantum state formal expressions; to derive, for instance time evolution of the state in Hilbert space, given the state vector at a point in time; to interpret resulting expressions as observations in lab space. This can be framed as recognizing the structural role of mathematics in a physical theory, and developing the corresponding structural skills [5]. At the same time, there is a need to understand how student use the concept of state to identify and interpret basic features of quantum behavior, especially in connection to measurement.

For these purposes, an empirical investigation was designed on university physics students. As this research is part of a project aimed at the construction of teaching/learning proposals devised to overcome student difficulties, the Model of Educational Reconstruction (MER) was adopted as a theoretical framework [6]. Consequently, the investigation was grounded on a clarification of science content, performed by means of a review of specific literature on QM at university level, and in particular on students learning problems [7]. Data gathering instruments consist of a multi-perspective written questionnaire and an interview protocol on each questionnaire item, developed in two stages by case study method. In this paper, results of the second stage are reported.

## 2. – Research questions

The transition from a classical to a quantum perspective involves the acquisition of non-intuitive conceptual constructions and the ability of employing mathematical knowledge for structuring physical situations and recognizing the mutual influence of conceptual and formal aspects. Therefore, student understanding of quantum physics needs to be investigated at different levels: a cultural one, concerning student ways of looking at emerging features of new quantum behavior; a conceptual one, concerning the recognition of essential features of the quantum state concept; a structural one, concerning the way in which they transpose physical information into elements of state formal representation and interpret them in order to address specific problems. As a consequence, following general research questions were stated:

**RQ1**: how do students identify and interpret quantum behavior in general and with respect to measurement?

**RQ2**: what kind of relations do student evidence on the concept of eigenstate in connection to measurement and time evolution?

**RQ3**: how do students transpose physical information into elements of state formal representations and interpret them in the application context of specific problems?

## 3. – Instruments and methods

Research instruments were developed by means of a preliminary study to collect student perspectives, their ways of looking and their points of view. For this purpose, an open questionnaire was developed, allowing students to choose approach, relevant aspects and allowing researchers to examine how these aspects are discussed. The investigation was conducted by case study method on three 3rd year physics students of University of Perugia after the first half of their first QM course.

On the basis of content analysis and case study results [8], a comprehensive grid was elaborated, describing aspects to be included and contexts to be explored. The grid is organized in six sections: quantum behavior and domain of applicability of the theory and its formalism; physical information encoded in formal representations of state - at a point in time; physical information encoded in formal representations of state - time evolution; a time problem: understanding models for the analysis of 1-dimensional quantum scattering; interpreting and sketching $\Re\{\psi\}$ and $|\psi|^2$ graphs; formal transposition of patterns of experimental data.

The grid was used to structure a new questionnaire made up of 21 items and organized on three levels: cultural, qualitative-conceptual and formal (state vector, wave function, $\Re\{\psi\}$ graphs, $|\psi|^2$ graphs). For each aspect, at least two items were designed in order to cross-examine it from different perspectives. An interview protocol was then devised to deepen student reasoning on questionnaire items and crucial elements. The protocol is structured in two parts: first with rogersian method [9], then by asking a stimulus question on student's written answer and following the dynamical evolution of student reasoning path by means of reinforcing stimuli.

A new case study was planned to finalize data gathering instruments and get preliminary results. Main outcomes are here presented, and concern data collected in three Italian universities, by administering the questionnaire to six volunteer 3rd year physics students from Perugia (1), Calabria (4); Roma-La Sapienza (1). All participants had followed QM course and 4/6 passed the exam two months earlier in the same year. Students were given two-hour time to complete the questionnaire. Two of them were interviewed on each item, a third one on a selection of items. Aspects of the results of this investigation have been described in other articles [7, 10].

## 4. – Data and findings

In the present section, a wide selection of questionnaire items are discussed, reporting main results. Hereafter, answers given by students are labeled with the symbols S1-S6.

**4**˙1. *Item A1: Quantum behavior*. –

| What elements characterize/identify the quantum behavior of a system? |
|---|

Students answer this question by focusing either on formal entities characterizing quantum systems (2/6): "the Hamiltonian determines the features of the system"(S1), "a system is characterized by a total state (ket) given by a combination of substates" (S4); or on quantum behavior properly said (3/6), e.g. "lack of determinism" (S3). One student mentions both aspects but doesn't discriminates between them, first describing the formal structure of state representation: "a quantum state is a vector in an infinite-dimensional Hilbert space" and then the process of time evolution in the absence of measurement: "it evolves in time according to Schrödinger equation" (S6). Most students cope with item A1 by looking through the properties of formal elements: a) states as vectors (S4, S6); b) observables as operators (S1, S6: "observables are represented by operators on the state"); c) non-commutation of observables as formal representation of indeterminacy (S2: "commutation relations can reveal indetermination principles between observables"). When discussing quantum behavior, students mention primarily "quantization" of physical quantities, interpreted as discreteness of the set of measurable values. They ascribe this property to observables in general (S3: "possible results of any measurement are quantized and discrete", S2: "quantized observables") or to energy in particular, regardless of the physical context (S4: "some observables, such as energy, are quantized"). Student answers to item A1 give a very poor picture of quantum behavior, focusing mostly on an aspect already evident in the old quantum theory (discreteness), and neglecting notable features such as entanglement and indistinguishability of particle. Instead, they ascribe importance to the identification of the new formal structures used to describe quantum systems, often identified with physical behavior itself (3/6).

**4**˙2. *Items A2-A3: Measurement*. –

| A2)How does the role of measurement change in the transition from classical physics to QM?<br>A3)Predictability and objective nature of measured properties: how do these concepts change in the transition from classical physics to QM? |
|---|

A richer image of the peculiarities of quantum physics emerges from the global questions on the measurement process. In answers to A2, students highlight following features of quantum measurement: perturbation of the system (4/6), e.g. "measurement alters the systems and determines the value of physical quantities" (S2), and end of deterministic view (3/6), e.g. "we can't measure two physical quantities at the same time

(Heisenberg principle). Collapse of determinism" (S3).

In answers to A3, all of them identify the probabilistic nature of quantum predictions, but only S2 explicitly connects it with the fact that a definite value is gained only at the time of measurement ("through measurement a physical quantity acquires a defined value", S2). Three other students display a conceptual approach to the issue of measurement, basing their reasoning on indetermination principle (S1, S3, S5). See for instance S5's answer, who also connects uncertainty in measurement with the formal expression of incompatibility: "In quantum mechanics events acquire a probabilistic nature [...] the concept of intedeterminacy is introduced by means of the famous principle, according to which there is uncertainty on measurement of non-commuting observables".

The richer conceptual content of these answers is consistent with the attention given by textbooks (and presumably lecturers) to the fundamental topic of quantum measurement, which is addressed also from a qualitative and cultural point of view [11, 12, 13, 14]. Nevertheless, a prevailing attitude of procedural kind is still observed in two cases. S4 describes the quantum state as a tool to make predictions on measurement results: "just by looking at the ket of the system ($|\alpha >= \sum_i c_i|\alpha_i >$) it is possible to determine on which states the system can be found after measurement ($|\alpha_k >$), with probability given by corresponding coefficient $c_k$ ($\rho(a_k) = |c_k|^2$)". Besides, both S4 and S6 describe the measurement process through the application of an unspecified operator to the state vector: "the concept of measurement is linked to an operator on an Hilbert space which, if applied to the state, modifies its properties: the state collapses on an eigenstate of the operator" (S6).

Statements like this are tricky indeed, because while state collapse can be described *ex post* by the application of a projector to the state, often students identify the results of a measurement with the application of the operator corresponding to the measured observable, a well known and widespread issue [3]. In particular, the link between the operator nature of observables and the concept of eigenstate appears to be a critical aspect. This is shown by other students' statements, mixing up the properties of involved entities, e.g. : "measurement collapses the observable on an eigenstate of the system" (S1), a claim repeated in the interview: "when I observe a physical quantity, it behaves in a different way because it's perturbed, falling in an eigenstate of the system under consideration".

**4**˙3. *Item A6: Incompatibility in measurement.* –

> While analyzing concepts involved in quantum measurement, some students made the following statements:
> 1. *If we measure the position of a particle in the ground state of an infinite well potential, its position will change, but not its energy, because a position measurement doesnt alter the energy*
> 2. *A particle in a stationary state doesnt change its energy after a position measurement because the system remains isolated*
> 3. *A position measurement on a system in the ground state of an hydrogen atom gives a definite value because its Bohr radius is well defined*
>
> Briefly discuss aspects you agree/disagree with in each student statement.

All three statements concern position measurements on an energy eigenstate, and can be found false by recognizing incompatibility between energy and position and its consequences for measurement. Among tested students, only S5 explicitly looks at incompatibility to decide about a statement ("I don't agree with the first statement because energy and position are incompatible"), while S2's assessment of the first statement implicitly implies it ("False. Measurement can alter energy"). In general, only S2 uses consistent claims to answer all sub-questions: "1. False. Measurement can alter energy; 2. False. The act of measuring implies interaction between the system and something else; 3. False. Average [position] value for ground state corresponds to Bohr radius" (actually, the most probable value). Statement 3 is consistently handled by 5/6 of the students, who recognize that a position measurement on an energy eigenstate can give different results, thus discriminating between quantum model and old quantum physics' one. Statements 1 and 2, on the contrary, are found true by 4/6 of the students, either handling energy and position as compatible observables: "The first two statements are true: the act of measurement doesn't always alter other properties of the system" (S3), or using an hybrid classical-quantum reasoning: "a stationary state doesn't vary in time. Consequently, energy is conserved" (S4), or without giving any reason (e.g. S5's answer to Statement 2).

As evidenced by S5's case, identifying energy and position as incompatible observables does not imply recognizing that a position measurement on a stationary state will alter its energy. In the analysis of items on stationarity we'll see that lack of recognition of coincidence between energy eigenstates and stationary states is consistent with results like this.

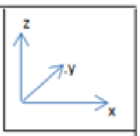
Student S6 is a different matter: as in his answer to A3, he identifies the action of a suitable operator on the state of a system with a physical process changing system's properties: "During position measurement, the energy of the system is not altered, but it can be modified by applying a $\hat{a}^{\pm} = \hat{x} \pm i\hat{p}$ to the state vector". In the non-relativistic QM course, creation and annihilation operators are introduced as part of a formal procedure in order to identify energy eigenstates of a quantum harmonic oscillator, and don't represent a process in which the energy of a generic system described by a generic state vector is raised or lowered. It is clear, therefore, that difficulties concerning the physical role of operator's action on the state may not be limited to Hermitian ones, representing observables, but can also involve operators of different nature. A further example of this conceptual pattern, concerning the time evolution operator, shall be shown in section **4**˙5.

**4**˙4. *Items B4-B5: Physical information encoded in phase relations*. –

For items' text, refer to Fig. 1. Among five students answering B4.1 in the written test, three of them reconstruct the phenomenology they expect to observe (e.g. identifying the number of spots visible on screen), two express relations on $\alpha$ and $\beta$: S2 writes that "as the spots are identical $\rightarrow |\alpha|^2 = |\beta|^2$", S3 translates information on equal brightness of the spots in the assumption that $\alpha = \beta = 1/\sqrt{2}$, with further specifications such as "equal probability of z-component of spin being $|\uparrow_z>$ or $|\downarrow_z>$" (S3). In interviews, S1 and S6 express the same statement as S3. With the exception of S2, tested students don't provide consistent answers on the way in which information obtained by experimental outcomes is transposed in state formalism.

Item B4.2 explores how student deal with predictions on a subsequent observation on the same state, how they use basis change equations (provided in item's text) to perform such prediction, and if they recognize the role of phase in this process. Students S3 and S4 start from a state vector whose coefficients contain no phase difference, appropriately using given basis change equations (see items in figure) and making predictions consistent with their initial assumption. Students S1 and S6 answer this item by using only basis change equations, without any reference to state vector, and conclude that only x spin-up spot will appear because $|\uparrow_z> -|\downarrow_z>= 0 = |\downarrow_x>$. They interpret transformation

**B4.1** A beam of silver atoms (spin-½), identically prepared in spin state α | ↑$_z$> + β| ↓$_z$>, propagates along the y direction (orthogonal to x and z). The beam is sent through a Stern Gerlach device with a vertical magnetic field gradient in the z direction. On the screen we observe two identical spots.




Transpose this experimental outcome in quantum state formalism and describe your reasoning

**B 4.2** Afterwards the device is replaced by a similar one, whose magnetic gradient is in the x direction, and the beam is sent through it. Can we expect to see two spots, one markedly brighter than the other? Explain your reasoning

To answer the question it may be useful to know that:

$$|\uparrow>_z = \frac{|\uparrow>_x + |\downarrow>_x}{\sqrt{2}}$$

$$|\downarrow>_z = \frac{|\uparrow>_x - |\downarrow>_x}{\sqrt{2}}$$

**B5** Given a system's probability distribution in position space, is it possible to deduce its probability distribution in momentum space? Explain your answer

Fig. 1. – Items B4-B5: Physical information encoded in phase relations

equations as containing information on the system under consideration.

Regarding item B5, all students answer that Fourier transform allows to obtain system's probability distribution in momentum space, starting from probability distribution in position space, as in following example: "given the fact that we are dealing with conjugate variables, we only need to apply a Fourier transform to the position distribution" (S2).

When explicitly asked in interview, S3 and S6 claim that coefficients $\alpha$, $\beta$ and their square modulus have the same physical meaning, as well as $\psi(x)$ and $|\psi(x)|^2$. Only S1 recognizes that $\psi(x)$ contains more information than $|\psi(x)|^2$, by referring to interference context, even if this difference is not explicitly connected by the student to the physical meaning of complex phase function. Interference is a familiar context to students in which they can effectively recognize the role of complex nature of description of quantum state, and in particular of phases. At the same time, phase relations are closely tied to information completeness of quantum state, and their significance emerges in general situations, such as predictions on measurements of observables not commuting with the one on whose basis the state is expanded. For this reason, its important to emphasize in different contexts how and why physical information is encoded in phase relations.

4·5. *Items B6, B7, B8, B12, B13: Stationarity and time evolution.* – For a summary of topics addressed in these items and of representations used, refer to Fig. 2. Students identify energy (E) eigenstates as stationary states both in the familiar situation of infinite potential well (5/6) and in the qualitative-conceptual item B8 (3 on 4 answering students). At the same time, also positions (x) eigenstates and eigenstates of other observables are included among stationary states, possibly because of the failure to identify incompatibility of E and x: "energy eigenstates are stationary ... also eigenstates of other operators can be stationary" (S5) and "In some cases, position eigenstates are energy eigenstates too" (S5). One student, anyway, ascribes stationarity to x eigenstates even after recognizing that x and E are incompatible: "energy eigenstates aren't position eigenstates $[\hat{H}, \hat{x}] \neq 0$" and at the same time "eigenstates of $\hat{H}$ are stationary ...

| CODE | REPRESENTATION | TOPIC | DESCRIPTION |
|---|---|---|---|
| **B6.1-B6.4** | Formal-analitic Formal-graphical | A time problem. models for the analysis of 1d quantum scattering. | Temporal relations in plane wave model and wave packet model. |
| **B7,B8** | Qualitative Conceptual | Coincidence between energy eigenstates and stationary states. | Students are asked to identify possible relations between given sets of states. |
| **B12, B13** | Formal-grahpical $\|\psi(x)\|^2$ | Time evolution for stationary and non-stationary states. | Given $\|\psi(x)\|^2$ graph of a stationary state (B12) and-non stationary state (B13), items ask to sketch $\|\psi(x)\|^2$ graphs at two different instants. |

Fig. 2. – Summary of items B6, B7, B8, B12, B13: Stationarity and time evolution

position eigenstates can be stationary" (S6). In another case, linear combinations of E eigenstates are interpreted as E eigenstates, without discriminating between degenerate and non-degenerate eigenvalues (S2). Thus, they included them among stationary states, showing issues with the concept of eigenstate and the physical meaning of superposition.

The discrimination between E eigenstates and other eigenstates is resolved when students are able to exploit commutation relations commonly examined in textbooks, such as $[\hat{H}, \hat{p}] = \hat{V}(x)$. This formal representation of the relation between observables is used by most in the familiar situation of comparison between $\hat{H}$ and $\hat{p}$ operators (4/6), only by a minority in case of $\hat{H}$ and $\hat{x}$ (2/6).

Students relate E eigenstates and their linear combinations respectively to time invariance and time evolution in the familiar situation of infinite potential well, while uncertainties emerge in the crucial context of scattering, especially when dealing with wave-packet mode: e.g. "incident, reflected, transmitted wave packets are to be considered related to the same instant or all instants because a frequency distribution of electrons gives us a complete spectrum" (S5).

Also in answers on time evolution, S6 uses operator action on a state vector as a conceptual reference, sometimes productively, e.g. identifying incident, reflected and transmitted components of a plane wave as formal elements associated to the same instant ("this wave function describes the state of the system and not its time evolution, for which the action of an operator is needed"), sometimes not - as when asked if stationary eigenstates composing a wave function can be associated to different instants ("it depends on the effect of time evolution operator on each of these states").

## 5. – Discussion and Conclusion

Within the framework of MER [6], we are conducting a research on university student understanding of quantum mechanics. On the basis of literature analysis, content analysis, and the results of a first case study, the following research instruments were elaborated: a 21 item questionnaire including multi-representation, an interview protocol devised to follow student reasoning path in depth. A calibration stage was designed to finalize data-gathering instruments and get preliminary results. The study was conducted on six 3rd year students coming from different Italian universities, after taking QM course and passing the exam (4/6). Written questionnaire was administered to all of them. Two students were interviewed on each item and one on a selection of them. Main findings are the following:

In describing quantum behavior, student highlight discreteness of measurable values, also mentioning lack of determinism and time evolution according to Schrödinger's equation, but neglecting other notable features such as entanglement or indistinguishability of particles. Alternatively, they focus on formal entities characterizing quantum systems. Most students look through the properties of formal elements: states as vectors, observables as operators and non-commutation of observables as formal representation of indeterminacy. A richer conceptual content is evidenced in addressing the widely covered topic of quantum measurement, where students focus as before on the end of deterministic view, but also on perturbation of systems and probabilistic nature of predictions. Anyway, they display uncertainties on the concept of eigenstate, at times interpreted as a proper state of a system in general rather than of a specific observable. Another critical aspect on measurement is the connection between measurement process and its formal representation which, if identified with the application to the state vector of the operator corresponding to the measured observable, can hide deep issues with the structural role of Hermitian operators (RQ1).

As already seen, the concept of eigenstate can be tricky for students. This emerges both in measurement and in the discussion of stationarity. On a global level, while students discriminate between energy and momentum eigenstates, they often treat position eigenstates and energy eigenstates as they were the same, neglecting incompatibility. As tested students identify the stationarity of energy eigenstates, it follows that also position eigenstates are sometimes considered such. The only student recognizing coincidence between stationary states and energy eigenstates, includes among them their linear combinations, evidencing again a need to discuss how the concept of eigenstate is related to the operator structure of observables. When asked to decide on the result of a position measurement on different energy eigenstates, students find more or less difficulties depending on the physical context of the item. As before, some answers are consistent with a lack of recognition of incompatibility of energy and position. The issue is resolved when students are able to exploit commutation relations. This happens frequently in the familiar situation of comparison between $\hat{H}$ and $\hat{p}$ operators, but only rarely when $\hat{H}$ and $\hat{x}$ are involved (RQ2).

Mathematization was tested in the context of spin, where all students but one don't provide consistent answers on the way in which information obtained by experimental outcomes is transposed in state formalism, either neglecting phase relations or trying to reconstruct the phenomenology on a qualitative level (even if a qualitative description of the physical situation was already provided in item's text). This result is peculiar, as students shouldn't be unfamiliar with this kind of assignment. Indeed, more abstract and technically difficult versions of the task, requiring empirical reconstruction of the state vector, are not unusual in Italian courses' examinations. Consistently with general findings of physics education research, this suggests that building student knowledge of QM on formalism and problem solving in situations which are totally disconnected from a phenomenological background trains students to master formal tools and techniques, but they need significant phenomenological contexts as a basis for building the conceptual foundation of QM formal entities. Vectors in basis change equations are interpreted by some students as containing information on the system under consideration, thus displaying difficulties with incompatibility and the operator structure of observables. Basis change is indeed a procedure embedding a structural meaning in QM, as concerning the relation between different observables and the derivation of predictions on measurement of observables non-commuting with that on whose basis the state has been developed. Precisely in this respect, phase relations reveal their physical content, conveying information on measurement of those non-commuting observbles. Tested students did not consider the role of phase relations, neither facing vector formalism in Stern-Gerlach context, nor continuous analytical formalism in an abstract context. As a consequence, state vector coefficients and wave function on the one hand and their square modulus on the other hand were interpreted as different ways to convey same physical information. Implications for teaching include discussing in different physical situations how and why physical information is encoded in phase relations (RQ3).

It is important to observe that, while tested students show a richer knowledge structure on quantum measurement as a general topic, they struggle to interpret physical situations in the application context of specific problems, both when concepts embedding deep formal relations are involved (i.e. eigenstate), or in tasks requiring mathematization/derivation/interpretation of formal entities. A need emerges to support students in developing an awareness of the role of the formal entities related to quantum state in structuring physical situations, and the ability to operate a transition from mathematics to physical meaning and vice versa. A unifying element of most issues is their correlation with the structural role of non-commutative algebra of operators and with the related role of incompatibility in measurement and time evolution. Students need to build an understanding of incompatibility both in global terms and situated ones, according to the different roles it plays in QM processes, and to discuss how this concept is encoded in the structure of Hilbert space constructs. Research instruments shall be modified in order to explore student understanding of incompatibility and related formal aspects, at the basis of quantum theory and of deep learning difficulties.

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