Grabbing Olives on Linear Pizzas and Pissaladières

Jean-Claude Bermond, Michel Cosnard, Frédéric Havet

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Hungry Coatis[1] ✉
Université Côte d'Azur, Coati, I3S, CNRS, INRIA, Sophia Antipolis, France

Abstract

In this paper we revisit the problem entitled SHARING A PIZZA stated by P. Winkler by considering a new puzzle called SHARING A PISSALADIERE. The game is played by two polite coatis Alice and Bob who share a pissaladière (a $p \times q$ grid) which is divided into rectangular slices. Alice starts in a corner and then the coatis alternate removing a remaining slice adjacent to at most two other slices. On some slices there are precious olives of Nice and the aim of each coati is to grab the maximum number of olives. We first study the particular case of $1 \times n$ grid (i.e. a path) where the game is a graph grabbing game known as SHARING A LINEAR PIZZA. In that case each player can take only an end vertex of the remaining path. These problems are particular cases of a new class of games called d-degenerate games played on a graph with non negative weights assigned to the vertices with the rule that coatis alternatively take a vertex of degree at most d.

Our main results are the following. We give optimal strategies for paths (linear pizzas) with no two adjacent weighty vertices. We also give a recurrence formula to compute the gains which depend only on the parity of n and of the respective parities of weighty vertices with a complexity in $\mathcal{O}(h^2)$ where h denotes the number of parity changes in the weighty vertices. When the weights are only $\{0, 1\}$ we reduce the computation of the average number of olives collected by each player to a word counting problem. We solve SHARING A PISSALADIÈRE with $\{0, 1\}$ weights, when there is one olive or 2 olives. In that case Alice (resp. Bob) grabs almost all the olives if the number of vertices of the grid $n = p \times q$ is odd (resp. even). We prove that for a $2 \times q$ grid with a fixed number k of olives Bob grabs at least $\left\lceil \frac{k-1}{3} \right\rceil$ olives and almost always grabs all the k olives.

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1 Introduction

One of our motivations for this article came from the problem entitled SHARING A PIZZA stated by P. Winkler in the new edition of his famous book *Mathematical Puzzles* [14]. We copy here the statement.

Alice and Bob are preparing to share a circular pizza, divided by radial cuts into some arbitrary number of slices of various sizes. They will be using the "polite pizza protocol": Alice picks any slice to start; thereafter, starting with Bob, they alternate taking slices but always from one side or the other of the gap. Thus after the first slice, there are just two choices at each turn until the last slice is taken (by Bob if the number of slices is even, otherwise by Alice). Is it possible for the pizza to have been cut in such a way that Bob has the advantage?in other words, so that with best play, Bob gets more than half the pizza?

In the solution page 286, P. Winkler wrote: *If the number of slices is even (as with most pizzas), one can apply the solution of the puzzle Coins in a Row (see problem in Chapter 7 [14] but also the first problem of the book in 2004 [12]). Alice can always get at least half the pizza. Indeed Alice can just number the slices 1, 2, etc. starting clockwise (say) from the*

[1] J.-C. Bermond, F. Havet, and M. Cosnard

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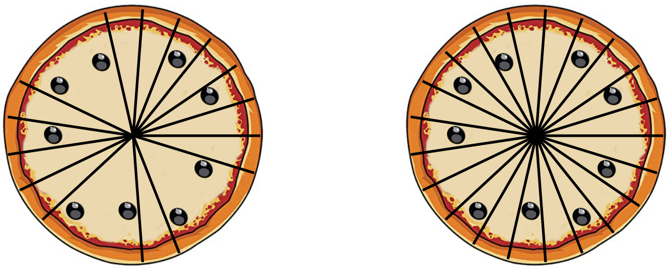

Figure 1 Two examples of pizzas where Bob can always get 5 of the 9 olives.

cut, and play so as to take all the even-numbered slices or all the odd, whichever is better for her. The argument fails if the number of slices is odd. But the odd case sounds even better for Alice since then she ends up with more slices. How can we get a handle on the odd case?

P. Winkler indicates that this puzzle was already devised by D. E. Brown in 1996, and has attracted a lot of attention, in part because of his conjecture [13] that Alice could always get at least 4/9 of the pizza. Indeed he gave the surprising example of a 15-slice circular pizza with slice-sizes 0, 1, and 2 of which nothing can stop Bob from acquiring 5/9. The conjecture has been proved independently by two groups [2, 6].

At this stage, P. Winkler rose an interesting question *Can a slice be of size zero ?* and answered: *Mathematically, no problem; gastronomically, think of a slice of size ϵ.* In fact, in the figure of his book, he put pepperoni to indicate the slice sizes and he wrote *No matter how she plays, Alice can never get more than 4 of the 9 pepperoni chunks against a smart, hungry Bob.* We give here again the example with olives (instead of pepperoni). In both pizzas of Figure 1, there are 9 olives (placed in the same positions). The pizza on the left has 15 slices, three of them having two olives. The pizza on the right has 21 slices with at most one olive on each of them. It is obtained from the left pizza by splitting the slices with two olives into three parts, two with one olive and the middle one with none. One can check that on both pizzas Bob can always grab 5 olives.

Another interesting question is : " What is the right shape for a pizza ? " Many people might answer round (circular). But according to Wikipedia[2] the traditional Sicilian pizza is typically rectangular, unlike the Neapolitan pizza which is circular. The roman "Pizza al taglio" (pizza by the slice) is baked in large rectangular trays, and generally divided in rectangular slices. Note also that the largest pizza[3] ever baked is a linear pizza of length 1,853.88 meters.

Of higher importance to the authors is "pissaladière". This marvelous dish is originally a specialty of the French city of Nice (under the name "pissaladeria") and of Liguria (under the name "piscialandrea"). The classical pissaladière is presented as a rectangular grid where the slices are small rectangles (see Figure 2). Its traditional topping consists of caramelised (almost pureed) onions, anchovies and delicious small black olives of Nice, of which coatis are fond.

The original aim of our research was to tackle a new puzzle similar to Winkler's one that we call Sharing a Pissaladière. The game is now played on a pissaladière, which is a rectangle, divided into rectangular slices by horizontal and vertical lines (knife cuts).

[2] `https://en.wikipedia.org/wiki/Sicilian_pizza`
[3] `https://www.this-is-italy.com/world-biggest-pizza/`

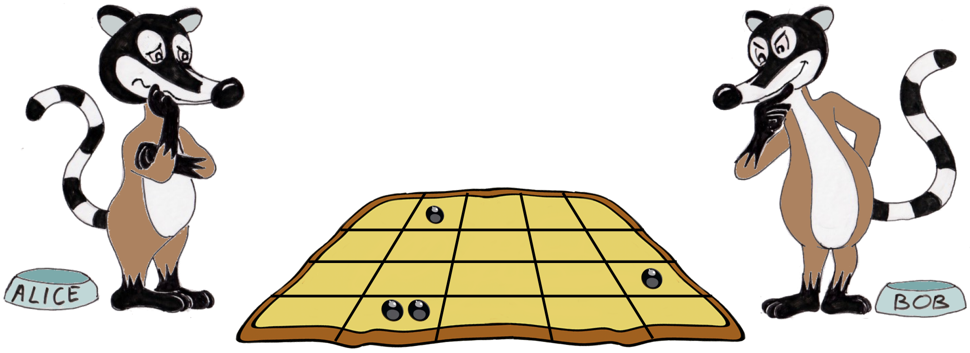

Figure 2 Two coatis trying to (politely) grab as many olives as possible on a pissaladière.

Olives are a precious ingredient and so there are only a small number k of olives; therefore many slices have no olives and some slices have one or more olives. The game is played by two coatis named Alice and Bob (see Figure 2). They both want to eat as many olives as possible, but they are civilized. The politeness imposes them to alternately pick a slice which is adjacent to at most two other slices. Hence, in the first move Alice can only pick one of the four corners of the pizza, and in the second move Bob can only pick one of the three remaining corners or one of the two slices adjacent to the corner removed by Alice. On Figure 2 you can see that the malicious Bob is content (and Alice disappointed). Indeed, with the disposal of the olives, Bob will grab all the 4 olives.

One particular interesting case is the $1 \times n$ grid or path of order n. In this case, the coati politeness imposes a stronger rule than the above, because otherwise any slice can be picked and the coati would rush to slices with olives. With the stronger rule, a coati can only pick one of the two slices at the ends of the (remaining) path. We will refer to this puzzle as Sharing a Linear Pizza. Note that this problem appears in Sharing a Pizza after the first move. Indeed, when Alice has picked the first slice, it remains a linear pizza. Therefore to solve an instance of Sharing a Pizza, it suffices to solve n instances of Sharing a Linear Pizza. This problem is equivalent to the puzzle Coins in a Row described in Winkler's books [12, 14] as follows: *On a table is a row of 50 coins, of various denominations. Alix picks a coin from one of the ends and puts it in her pocket; then Bert chooses a coin from one of the (remaining) ends, and the alternation continues until Bert pockets the last coin. Prove that Alix can play so as to guarantee at least as much money as Bert.* The main difference with Sharing a Linear Pizza is that Alice is now Alix and Bob is now Bert. We note that P. Winkler mentioned the following in his book : *In fact, for Alix to play optimally, she needs to analyze all the possible situations that may later arise. This can be done by a technique called "dynamic programming".* More precisely, for an instance on the path of order n, $O(n^2)$ situations have to be considered, corresponding to the $O(n^2)$ subpaths, so the optimal strategies may be found in $O(n^2)$ time.

Various questions can be asked for these puzzles.

(Q1) What is the minimum number of olives each coati can grab in function of the size of the pizza or pissaladière? This is the question asked by Winkler for the circular pizza.

(Q2) For a given setting of k olives, can we give optimal strategies for the two coatis ? In particular can we compute the number of olives each coati will grab (when they both play optimally) in a time polynomial in k (the number of olives) and independent of the size of the pizza ?

(Q3) What is the average number of olives grabbed by each coati if the k olives are uniformly distributed ?

(Q4) What is the probability that a given coati (Alice or Bob) wins, that is grab at least half the olives ?

2 Statement of the problem and results

2.1 Graph-grabbing games

The three puzzles described above are particular cases of scoring games played on graphs (for scoring games, see for example the survey [7]). In particular, Sharing a Pizza and Sharing a Linear Pizza belongs to the class of *graph-grabbing* or *graph-sharing* games [3, 8, 9, 10]. In such games, two players Alice and Bob share a **connected graph with non-negative weights assigned to the vertices**. In this mathematical setting, the weights are real numbers, but in practical life the weights represent the number of olives on a slice and so are integers. The information is complete in the sense that both players know exactly the weights; in particular they know which slices have a zero weight (no olive). They alternately take the vertices one by one and collect their weights. The first turn is Alice's. There are some differences according to the rule taken to grab a vertex. In [3, 9] the authors consider two rules: the first one consists in removing a non cut vertex (i.e. the subgraph induced by the remaining vertices is connected during the whole game). The authors called this rule (R) (for "Remaining part connected"). The second rule, called (T) (for "Taken part connected"), imposes that the subgraph induced by the taken vertices is connected during the whole game. Hence Sharing a Pizza is a graph-grabbing game on a cycle with any of the two rules (or both) because for a cycle (R) and (T) are identical; Sharing a Linear Pizza is a graph-grabbing game on a path with rule (R). More generally, playing a graph-grabbing game on a tree with rule (R) imposes each player to take a leaf. For such games, Seacrest and Seacrest [11] showed that if a tree has an even number of vertices, Alice has a winning strategy (i.e. a strategy to obtain at least half of the total weight), thus answering a conjecture of Micek and Walczak [8]. They also conjectured that the same statement holds for every weighted connected bipartite graph with even order. Egawa et al. [4] proved that the conjecture holds for some particular bipartite graphs called $K_{m,n}$-trees. As noted in [8], any odd tree on at least three vertices can be weighted so that Alice gets nothing (it suffices to put the whole weight at one non-leaf vertex). Eoh and Choi [5] gave general classes of graphs for which Alice has a winning strategy with rule (R) when the weights are 0 or 1. Cibulka et al. [3] showed, when playing with any of rules (T) and (R) or both, then for every $\epsilon > 0$ and for every $k \geq 1$, there is a k-connected graph G for which Bob has a strategy to obtain $(1 - \epsilon)$ of the total weight of the vertices. This contrasts with the original pizza game played on a cycle. They also show that the problem of deciding whether Alice has a winning strategy is PSPACE-complete if condition (R) or both conditions (T) and (R) are required. Micek and Walczak [9] showed the importance of parity in graph-grabbing games. For example, they proved that, with rule (T) on a tree with an odd number of vertices, Alice has a strategy to get at least 1/4 of the weight.

2.2 d-degenerate games

Note that playing with rules (R) or (T) on a pissaladière (i.e. a (rectangular) grid) with at least two rows and two columns is everything but polite. Most of the time, the set of vertices that a player can remove is not very restricted; on her first move, Alice can always remove the vertex she wants. Although this has been successfully practiced by barbarian soldiers like Attila or Alexandre the Great, civilized coatis need more restricted rules to share a pissaladière. Therefore we introduce the d-degenerate rule (D_d) yielding the ***d*-degenerate game**.

(D_d) : At each move, a player can only pick a ***d*-removable** vertex which is a vertex of degree at most d in the remaining graph.

Of course, with such a rule, all the vertices can be removed only if at each step there is a removable vertex. This is the case if and only if the graph is d-degenerate. Recall that a graph is ***d*-degenerate** if each of its subgraphs has a vertex of degree at most d. A graph G is d-degenerate if and only if it has a ***d*-degenerate ordering** that is an ordering $(v_1, \dots, v_n)$ of its vertices such that v_i has degree at most d in $G\langle\{v_i, \dots, v_n\}\rangle$ for all $i \in [n]$.

Note that we could play the d-degenerate game on a graph which is not d-degenerate. Then the two coatis share a fraction of the graph until there is no d-removable vertex. But we shall not consider it in this paper, and shall only play the d-degenerate game on d-degenerate graphs.

The 1-degenerate connected graphs are the trees. In a tree, a 1-removable vertex is a leaf. Therefore, on trees, the 1-degenerate game is equivalent to the graph-grabbing game with rule (R). In particular, the 1-degenerate game on path is exactly Sharing a Linear Pizza. Grids are a nice examples of 2-degenerate graphs and the 2-degenerate game on grids is exactly Sharing a Pissaladière.

2.3 Our results

When we started our study, our aim was to obtain results on Sharing a Pissaladière, which corresponds to the 2-degenerate game on a grid. We were interested in the case of integer-valued weight function, which corresponds to the case where an olive cannot be cut and is on a unique slice, and especially $\{0, 1\}$-weight function, which corresponds to the case where there is at most one olive per slice. But we realized that questions (Q2) to (Q4) were not answered in the literature for Sharing a Linear Pizza. Consequently, we first study this problem.

Consider the path with n vertices $P_n = (1, 2, \dots, n)$ together with a weight function $w : V(P_n) \to \mathbb{R}^+$. A **weighty** vertex is a vertex whose weight is nonzero. In that case question (Q1) is easily answered when n is even as noted by Winkler. Alice can use the even-odd strategy in which she removes either all the even vertices or all the odd ones, thus grabbing at least half of the weight (olives). However, the even-odd strategy is not necessarily optimal. For example, consider a path with $n = 12$ vertices and weights $w(2) = 1, w(5) = 2, w(7) = 3, w(10) = 4$ and all the other weights equal to zero. Playing the even-odd strategy Alice collects 5 olives, while we can show that she can get 6 olives (see Example in subsection3.1).

Therefore our aim in Section 3 is to describe optimal strategies on paths. We first consider **0-ordinary** weight function in which the end-vertices have weight zero and there are no two adjacent weighty vertices. In Subsection 3.1, we describe some strategies called Σ_0 and show that they are optimal (Theorem 4). It allows us to give a recursive formula to compute the gain of both players and to show that the weights collected by each player depend only on the parity of n and on the respective parities of the weighty vertices and not on the precise value of n nor of the values of the weights of the vertices. So, for 0-ordinary weight functions, one can compute the optimal gain in time $\mathcal{O}(k^2)$ where k is the number of weighty vertices. We then show that in fact, the complexity is in $\mathcal{O}(h^2)$ where h denotes the number of parity changes in the weighty vertices (Proposition 6). In Subsection 3.2, we extend strategy Σ_0 to weight functions where the end-vertices may be weighty but two weighty vertices are still not adjacent. Then in Subsection 3.3, we revisit Sharing a Pizza in the case where there is no adjacent weighty vertices: we show that Alice has an optimal strategy whose first move consist in grabbing a weighty vertex, and derive that the optimal gain can be computed in $\mathcal{O}(kh^2)$.

If we have consecutive weighty vertices, the situation is more complex as there are examples where a player has no interest in taking a weighty end-vertex. Consider the path on four vertices with $0 < w(1) < w(2)$ and $w(3) = w(4) = 0$. If at the first move Alice removes the weighty end-vertex 1, then she will gain only $w(1)$. But if she removes vertex 4, then whatever Bob plays she will grab vertex 2 and gains the weight $w(2)$.

In section 4, we show that for a d-degenerate game with $\{0, 1\}$ weight function, there is an optimal play where a coati grabs immediately an olive if possible (Lemma 8). This lemma applied for linear pizza enables us to delete two consecutive vertices of weight 1 giving one olive to each player (Proposition 9). Hence, for linear pizzas with a $\{0, 1\}$ weight function we completely answer question (Q2). Furthermore, in that case we show how to relate the computation of the gain of each player to counting on binary words. That allows us to give a way to compute asymptotically the average number of olives grabbed by each player and to answer questions (Q3) and (Q4) when the number k of olives is small. See Table 1.

In Section 5 we deal with our original problem Sharing a Pissaladière. We first consider the simple case where there is a single olive on the pissaladière: we prove that Alice collects the olive if and only if the pissaladière has an odd number of slices or if the olive is on a corner of the pissaladière. This allows us to completely answer questions (Q1) to (Q4) in this case (Corollary 13.) Then, we consider the game with k olives on a pissaladière with two rows of slices. We show in Corollary 15 that Alice might collect no olive while, in contrast, Bob can always collect $\lceil \frac{k-1}{3} \rceil$ olives (best possible). We also show in Proposition 16, that almost always Bob collects all the olives (and Alice none). Finally, we consider the case where there are two olives on different slices of the pissaladière. This case is more complex, and we do not answer (Q2). However we prove that if the number of slices is odd (resp. even), then Alice (resp. Bob) almost always grab the two olives (Theorem 18).

3 Sharing a linear pizza

In this section, we are given the path $P_n = (1, 2, \ldots, n)$ of order n with a non-negative weight function w on the vertices. The weight of vertex i is denoted by $w(i)$. If the $w(i)$ are integers, we can consider vertices as slices of a linear pizza, the weight representing the number of olives on each slice.

We denote by $O = O(w)$ the set of k vertices $\{i_j \mid 1 \leq j \leq k\}$ which have a nonzero weight $w(i_j)$ called **weighty vertices** (slices with olives in the integral model). The other vertices have weight zero (no olive). The set O is **ordinary** if it contains no two adjacent vertices, that is if $|O \cap \{i, i+1\}| \leq 1$ for all $i \in [1, n-1]$. A set which is not ordinary is called **rare**. It is s**-ordinary** ($s = 0, 1, 2$) if it is ordinary and $|O \cap \{1, n\}| = s$. So s is the number of weighty end-vertices. The weight function w is **ordinary** (resp. **rare**, s**-ordinary**) if the set O is ordinary (resp. rare, s-ordinary).

Ordinary weight functions are useful because we can design for them optimal strategies to determine the final weight collected by each player and compute recursively this number in polynomial time (quadratic in k and not depending of n). Indeed after removing a weighty vertex, the remaining set of weighty vertices is still ordinary. Ordinary sets are important especially in the case of $\{0, 1\}$-weight functions (or more generally integer-valued weight functions) because the proportion of ordinary sets over all possible sets O of k olives tends to 1 when n tends to $+\infty$. **In this section, we consider only ordinary weight functions.**

3.1 Optimal strategy for 0-ordinary weight functions

Strategies Σ_0. At a given stage an end-vertex of the remaining path can be of three types: (1) "weighty" ; (2) "critical" if it has weight zero and is adjacent to a weighty vertex; (3) "safe" if it has weight zero and is not adjacent to a weighty vertex. In strategy Σ_0, at any move if a player X can remove a weighty vertex, then X will do so and will gain the weight of this vertex. Otherwise, if the end-vertices have weight zero, two cases can appear. Either one end-vertex is safe and the player will remove it. Otherwise the two end-vertices are critical; the player is stacked and obliged to let in the next move the other player grab a weighty vertex; but the player can choose to remove the critical vertex which ensures the maximum gain at the end of the game. Then we apply recursively the strategy on the path P_n with a new weight function where the weight of the removed vertex is now 0 and so with a smaller set of weighty vertices.

At the beginning of the game, if the weight function is 0-ordinary, the end-vertices have weight 0 (that is $i_1 > 1$ and $i_k < n$). The set of safe vertices is $S = \{1 \leq i \leq i_1 - 2\} \cup \{i_k + 2 \leq i \leq n\}$. So, during the first $|S|$ moves, the strategy Σ_0 consists for each player to remove a safe vertex (the order in which the safe vertices are chosen has no importance). Then, at move $|S| + 1$, one player, denoted by $\overline{X}$, is obliged to take one of the two critical vertices ($i_1 - 1$ or $i_k + 1$). Then at move $|S| + 2$ the other player, denoted by X, will take the neighbor x of this vertex which is either i_1 or i_k. X is either Alice or Bob according the parity of $|S|$ which itself depends only on the parity of n and of the respective parities of i_1 and i_k. More precisely

- $X = A$ when $|S|$ is odd, that is when either $\{n$ is even and i_1 and i_k have the same parity$\}$, or $\{n$ is odd and i_1 and i_k have different parity $\}$;
- $X = B$ when $|S|$ is even, that is when either $\{n$ is odd and i_1 and i_k have the same parity$\}$, or $\{n$ is even and i_1 and i_k have different parity $\}$.

Then we apply strategy Σ_0 on P_n with the weight of x reduced to zero. More precisely, let us denote for every $1 \leq \alpha \leq \beta \leq n$, by $w[\alpha, \beta]$ the weight function defined by $w[\alpha, \beta](i) = w(i)$ if $\alpha \leq i \leq \beta$, and $w[\alpha, \beta](i) = 0$ otherwise. We apply strategy Σ_0 on P_n with the new weight function $w[i_2, i_k]$ if $x = i_1$ and $w[i_1, i_{k-1}]$ if $x = i_k$. Note that the first $|S| + 2$ moves already done corresponds to successive removal of safe vertices in the strategy Σ_0 on P_n with this new weight function.

Example. Let us apply strategy Σ_0 on the example of the introduction: a path with $n = 12$ vertices and 4 weighty vertices $w(2) = 1, w(5) = 2, w(7) = 3, w(10) = 4$. At the first move vertex 12 is safe (and 1 is critical) and so Alice will remove 12. Then at the second move there are two critical vertices 1 and 11. Bob can choose which vertex he will remove and one can verify that the best choice is to remove 1. Then Alice grabs the weighty vertex 2. Then Bob removes the safe vertex 3. Now at move 5 Alice is stacked as there are two critical vertices 4 and 11. But she has the choice. If she does the greedy choice of removing the vertex 4 with the largest weight one can verify that she will grab at the end only 5 olives. But, if at move 5 Alice chooses the apparently bad vertex 11, Bob will be happy to grab at move 6 the 4 olives of vertex 10. At move 7 Alice removes the safe vertex 9. Then, whatever Bob does, Alice will grab at moves 9 and 11 the weighty vertices $5, 7$. In summary, Alice will get the olives of vertices $2, 5, 7$ that is 6 olives as announced in the introduction.

The strategy Σ_0 is partly greedy in the sense that the player X who takes the first vertex x in O is uniquely determined. However, the other player $\overline{X}$ can choose the vertex which maximizes his/her final gain or equivalently minimizes the gain of X. But as seen in the example this choice is not greedy.

The total weight collected by a player X with strategy Σ_0 on P_n with weight function w is denoted by $\sigma_0^X(n;w)$. The following lemma follows immediately from the definition. It gives a recursive formula to compute the gain of the players and shows that the weight collected by each player depends only on the parity of n and on the respective parities of the i_j and not on the precise value of n nor of the values of the i_j.

▶ **Lemma 1.** *Let P_n be a path with n vertices and w a 0-ordinary weight function with set $O = \{i_1, i_2, \ldots, i_k\}$ of weighty vertices. Assume that the two players play according to strategy Σ_0. Let $X = A$ if either $\{n$ is even and i_1 and i_k have the same parity$\}$, or $\{n$ is odd and i_1 and i_k have different parity $\}$; otherwise $X = B$. Then*

$$\sigma_0^X(n;w) = \min\left\{w(i_1) + \sigma_0^X(n; w[i_2, i_k])\ ,\ w(i_k) + \sigma_0^X(n; w[i_1, i_{k-1}])\right\},$$
$$\sigma_0^{\overline{X}}(n;w) = W - \sigma_0^X(n;w) = \max\left\{\sigma_0^{\overline{X}}(n; w[i_2, i_k])\ ,\ \sigma_0^{\overline{X}}(n; w[i_1, i_{k-1}])\right\}.$$

Our aim is now to prove that the strategies Σ_0 are optimal. To do so, we first prove that the strategies Σ_0 have the following property which is easy to prove for optimal strategies.

▶ **Lemma 2.** *Let P_n be a path with n vertices and w a 0-ordinary weight function. Then, for any player X: $\sigma_0^X(n;w) \geq \max\left\{\sigma_0^X(n; w[i_2, i_k])\ , \sigma_0^X(n; w[i_1, i_{k-1}])\right\}$.*

Proof. We proceed by induction on k, the result being trivial when $k = 1$. Suppose that the lemma is true until $k-1$. Consider a game on P_n with the set of weighty vertices O of size k. By Lemma 1, two cases can appear for the player X.

(i) Either $\sigma_0^X(n;w) = \max\left\{\sigma_0^X(n; w[i_2, i_k])\ ,\ \sigma_0^X(n; w[i_1, i_{k-1}])\right\}$, or

(ii) $\sigma_0^X(n;w) = \min\left\{w(i_1) + \sigma_0^X(n; w[i_2, i_k])\ ,\ w(i_k) + \sigma_0^X(n; w[i_1, i_{k-1}])\right\}$.

In case (i), there is nothing to prove. Assume now that we are in case (ii). W.l.o.g. we may also assume that $w(i_1) + \sigma_0^X(n; w[i_2, i_k]) \leq w(i_k) + \sigma_0^X(n; w[i_1, i_{k-1}])$. Then $\sigma_0^X(n;w) = w(i_1) + \sigma_0^X(n; w[i_2, i_k])$. So it remains to prove the following claim.

▷ Claim 3. $\sigma_0^X(n; w[i_1, i_{k-1}]) \leq w(i_1) + \sigma_0^X(n; w[i_2, i_k])$.

Proof. By Lemma 1, $\sigma_0^X(n; w[i_1, i_{k-1}])$ is either $\max\{\sigma_0^X(n; w[i_2, i_{k-1}])\ ,\ \sigma_0^X(n; w[i_1, i_{k-2}])\ \}$, or $\min\left\{w(i_1) + \sigma_0^X(n; w[i_2, i_{k-1}])\ ,\ w_{k-1} + \sigma_0^X(n; w[i_1, i_{k-2}])\right\}$.
Assume $\sigma_0^X(n; w[i_1, i_{k-1}]) = \min\left\{w(i_1) + \sigma_0^X(n; w[i_2, i_{k-1}])\ ,\ w_{k-1} + \sigma_0^X(n; w[i_1, i_{k-2}])\right\}$. By the induction hypothesis, $\sigma_0^X(n; w[i_2, i_{k-1}]) \leq \sigma_0^X(n; w[i_2, i_k])$. So, $\sigma_0^X(n; w[i_1, i_{k-1}]) \leq w(i_1) + \sigma_0^X(n; w[i_2, i_{k-1}]) \leq w(i_1) + \sigma_0^X(n; w[i_2, i_k])$ proving the claim.

Assume $\sigma_0^X(n; w[i_1, i_{k-1}]) = \sigma_0^X(n; w[i_2, i_{k-1}])$. By the induction hypothesis, we have $\sigma_0^X(n; w[i_2, i_{k-1}]) \leq \sigma_0^X(n; w[i_2, i_k])$, so $\sigma_0^X(n; w[i_1, i_{k-1}]) \leq \sigma_0^X(n; w[i_2, i_k]) < w(i_1) + \sigma_0^X(n; w[i_2, i_k])$ proving the claim.

Henceforth, it remains to deal with the case where $\sigma_0^X(n; w[i_1, i_{k-1}]) = \sigma_0^X(n; w[i_1, i_{k-2}])$.

By Lemma 1, $\sigma_0^X(n; w[i_1, i_{k-2}])$ is either $\min\{w(i_1) + \sigma_0^X(n; w[i_2, i_{k-2}])\ ,\ w(i_{k-2}) + \sigma_0^X(n; w[i_1, i_{k-3}])\}$, or $\max\{\sigma_0^X(n; w[i_2, i_{k-2}])\ ,\ \sigma_0^X(n; w[i_1, i_{k-3}])\}$. As above, if it is either $\min\{w(i_1) + \sigma_0^X(n; w[i_2, i_{k-2}])\ ,\ w(i_{k-2}) + \sigma_0^X(n; w[i_1, i_{k-3}])\}$ or $\sigma_0^X(n; w[i_2, i_{k-2}])$, we get the result using the induction hypothesis. Henceforth, we have to deal with the case where $\sigma_0^X(n; w[i_1, i_{k-1}]) = \sigma_0^X(n; w[i_1, i_{k-2}]) = \sigma_0^X(n; w[i_1, i_{k-3}])$.

And so on, by induction on $h = 1, \ldots, k-2$, either we get the result for some h, or we have $\sigma_0^X(n; w[i_1, i_{k-1}]) = \sigma_0^X(n; w[i_1, i_{k-1-h}])$ for every h. In particular, for $h = k-1$, we get $\sigma_0^X(n; w[i_1, i_{k-1}]) = \sigma_0^X(n; w[i_1, i_1]) \leq w(i_1)$. So the claim holds. ◁

The claim finishes the proof of the lemma. ◀

▶ **Theorem 4.** *Strategies Σ_0 are optimal on P_n with 0-ordinary weight functions.*

Proof. The proof is by induction on the number k of weighty vertices. Suppose that the strategies Σ_0 are optimal for 0-ordinary weight functions having up to $k-1$ weighty vertices. Let us consider the game on P_n with a 0-ordinary weight function w with a set O of k weighty vertices.

By Lemma 1, with strategy Σ_0, the gain of player X is $\sigma_0^X(n;w) = \min\{w(i_1) + \sigma_0^X(n;w[i_2,i_k])\ ,\ w(i_k) + \sigma_0^X(n;w[i_1,i_{k-1}])\}$, and the gain of player $\overline{X}$ is $\sigma_0^{\overline{X}}(n;w) = \max\{\sigma_0^{\overline{X}}(n;w[i_2,i_k])\ ,\ \sigma_0^{\overline{X}}(n;w[i_1,i_{k-1}])\}$.

Consider an optimal strategy played by the two players. Let c be the first critical vertex removed by a player, and let x be its weighty neighbour. Either $c = i_1 - 1$ and $x = i_1$, or $c = i_k + 1$ and $x = i_k$.

Suppose for a contradiction that c is removed by X. In the following move, $\overline{X}$ can remove x. Let $w' = w[i_2,i_k]$ if $x = i_1$ and $w' = w[i_1,i_{k-1}]$ if $x = i_k$. The vertices removed so far (including c and x) are safe for w'. Therefore it can be seen as the beginning of a strategy Σ_0 for w' which is optimal for w' by the induction hypothesis. Thus the gain of X using this strategy for w is at most its optimal gain for w' that is $\sigma_0^X(n;w[i_1,i_{k-1}])$ if $x = i_k$ and $\sigma_0^X(n;w[i_2,i_k])$ if $x = i_1$. In both cases, by Lemma 2, this gain is smaller than $\sigma_0^X(n;w)$, a contradiction.

Therefore c is removed by $\overline{X}$. In the following move X can remove x. Let $w' = w[i_2,i_k]$ if $x = i_1$ and $w' = w[i_1,i_{k-1}]$ if $x = i_k$. The vertices removed so far (including c and x) are safe for w'. Therefore it can be seen as the beginning of a strategy Σ_0 for w' which is optimal for w' by the induction hypothesis. Thus the gain of $\overline{X}$ using this strategy for w is at most its optimal gain for w' which is $\sigma_0^{\overline{X}}(n;w[i_1,i_{k-1}])$ if $x = i_k$ and $\sigma_0^{\overline{X}}(n;w[i_2,i_k])$ if $x = i_1$. In both cases, it is no greater than $\sigma_0^{\overline{X}}(n;w)$, and it is strictly smaller if removing c does not maximize $\overline{X}$'s gain. So $\overline{X}$ has no interest of playing differently from strategy Σ_0 and can remove safe vertices until it is forced to pick a critical vertex. Moreover, when he does so, he must remove a critical vertex that maximizes its gain (as in strategy Σ_0). On his side, X must also remove safe vertices as long as $\overline{X}$ does because it cannot remove a critical vertex by the above paragraph. Therefore the strategy Σ_0 is optimal. ◀

Complexity. Since strategy Σ_0 is optimal, computing the gain of each player when they play optimally is computing $\sigma_0^A(n;w)$ and $\sigma_0^B(n;w)$. Using Lemma 1, one can do it in $\mathcal{O}(k^2)$ time. But we can do it faster. In fact, analyzing strategy σ_0, we can prove that the players collect weighty vertices by blocks and not one by one. A **(parity) block** of olives in a 0-ordinary set of weighty vertices $O = \{i_1, \ldots, i_k\}$ is a maximum set of successive weighty vertices having the same parity. The following lemma shows that that X collects either the rightmost block of olives or the leftmost one.

▶ **Lemma 5.** *Let w be a 0-ordinary weight function on P_n with set of weighty vertices $O = \{i_1, \ldots i_k\}$. Let $\{i_1, \ldots, i_\alpha\}$ be its leftmost parity block of O and let $\{i_\beta, \ldots, i_k\}$ be its rightmost parity block.*

Let $X = A$ if either $\{n$ is even and i_1 and i_k have the same parity$\}$, or $\{n$ is odd and i_1 and i_k have different parity $\}$; otherwise $X = B$. Then

$$\begin{aligned} \sigma_0^{\overline{X}}(n;w) &= \max\left\{\sigma_0^{\overline{X}}(n;w[i_{\alpha+1},i_k])\ ,\ \sigma_0^{\overline{X}}(n;w[i_1,i_{\beta+-1}])\right\}, \quad \textit{and so} \\ \sigma_0^X(n;w) &= \min\left\{\sum_{j=1}^{\alpha} w(i_j) + \sigma_0^X(n;w[i_{\alpha+1},i_k])\ ,\ \sum_{j=\beta}^{k} w(i_j) + \sigma_0^X(n;w[i_1,i_{\beta-1}])\right\}. \end{aligned}$$

Proof. By Lemma 1, we have $\sigma_0^{\overline{X}}(n;w) = \max\left\{\sigma_0^{\overline{X}}(n;w[i_2,i_k])\ ,\ \sigma_0^{\overline{X}}(n;w[i_1,i_{k-1}])\right\}$. $\sigma_0^{\overline{X}}(n;w[i_2,i_k]) \geq \sigma_0^{\overline{X}}(n;w[i_{\alpha+1},i_k])$ and $\sigma_0^{\overline{X}}(n;w[i_1,i_{k-1}]) \geq \sigma_0^{\overline{X}}(n;w[i_1,i_{\beta-1}])$ by induction. Thus, $\sigma_0^{\overline{X}}(n;w) \geq \max\left\{\sigma_0^{\overline{X}}(n;w[i_{\alpha+1},i_k])\ ,\ \sigma_0^{\overline{X}}(n;w[i_1,i_{\beta-1}])\right\}$.

Let us now prove the opposite inequality by induction on k, the result holding when $k=2$. By the induction hypothesis $\sigma_0^{\overline{X}}(n;w[i_2,i_k]) = \max\left\{\sigma_0^{\overline{X}}(n;w[i_{\alpha+1},i_k])\ ,\ \sigma_0^{\overline{X}}(n;w[i_2,i_{\beta-1}])\right\}$ and by Lemma 2, $\sigma_0^{\overline{X}}(n;w[i_2,i_{\beta-1}]) \leq \sigma_0^{\overline{X}}(n;w[i_1,i_{\beta-1}])$.
Similarly by induction: $\sigma_0^{\overline{X}}(n;w[i_1,i_{k-1}]) = \max\left\{\sigma_0^{\overline{X}}(n;w[i_{\alpha+1},i_{k-1}])\ ,\ \sigma_0^{\overline{X}}(n;w[i_1,i_{\beta-1}])\right\}$ and by Lemma 2, $\sigma_0^{\overline{X}}(n;w[i_{\alpha+1},i_{k-1}]) \leq \sigma_0^{\overline{X}}(n;w[i_{\alpha+1},i_k])$. Therefore in both cases, $\sigma_0^{\overline{X}}(n;w[i_1,i_{k-1}]) \leq \max\left\{\sigma_0^{\overline{X}}(n;w[i_{\alpha+1},i_k])\ ,\ \sigma_0^{\overline{X}}(n;w[i_1,i_{\beta-1}])\right\}$. ◀

Using Lemma 5, we get the following proposition.

▶ **Proposition 6.** *Let P_n be a path with n vertices and w a 0-ordinary weight function. Then $\sigma_0^X(n;w)$ and $\sigma_0^{\overline{X}}(n;w)$ can be computed in $O(h^2)$ arithmetic operations where h is the number of parity blocks of the set of weighty vertices (or equivalently the number of parity changes of the weighty vertices plus 1).*

3.2 Optimal strategies for 1- and 2-ordinary weight functions

Based on strategies Σ_0 for 0-ordinary weight functions, we now define the strategies Σ_1 and Σ_2 for 1-ordinary and 2-ordinary, respectively, weight functions.

Strategies Σ_1. If the players have to play on the path P_n with a 1-ordinary weight function w, then the players do the following: At the first move A removes the weighty end-vertex; then the players play according to strategy Σ_0 on P_n with 0-ordinary weight function $w' = w[1,n-1] = w[i_1,i_{k-1}]$ if the removed vertex was n, or $w' = w[2,n] = w[i_2,i_k]$ if the removed vertex was 1.

Strategies Σ_2. If the players have to play on the path P_n with a 2-ordinary weight function w, then the players do the following: At the first move, A removes the end-vertex with the largest weight and at the second move B removes the other end-vertex; Then the players play according to strategy Σ_0 on P_n with 0-ordinary weight function $w' = w[2,n-1] = w[i_2,i_{k-1}]$.

It can be proved that Strategies Σ_1 (resp. Σ_2) are optimal on P_n with 1-ordinary (resp. 2-ordinary) weight functions. See [1].

3.3 Back on Circular pizza

Here, we are given the cycle $C_n = (1,2,\ldots,n,1)$ of order n with a non-negative weight function w on the vertices. As for paths, the weight function is **ordinary** if there are no adjacent weighty vertices. We show the following proposition (see[1] for its proof).

▶ **Proposition 7.** *Let C_n be a cycle with n vertices and w a 0-ordinary weight function with k weighty vertices.*

Alice has an optimal strategy whose first move consists in removing a weighty vertex. So the weight collected by Alice and Bob if they both play optimally can be computed in $O(kh^2)$ arithmetic operations where h is the number of parity blocks.

4 d-degenerate games with a $\{0,1\}$-weight function

A $\{0,1\}$-weight function w is completely determined by the set $O = \{v \mid w(v) = 1\}$. This set is then called the **olive set** and its elements the **olives**. Let G be a graph G on n vertices and an olive set $O \subseteq V(G)$. Coatis Alice and Bob are denoted by their initials A and B respectively. An **optimal play** is a sequence of moves made alternately by Alice and Bob when they play optimally. For $X \in \{A, B\}$, we denote by $\mathrm{col}_d^X(G, O)$ the number of olives that X collects in an optimal play. Note that $\mathrm{col}_d^A(G, O) + \mathrm{col}_d^B(G, O) = |O|$. For any integer $k \in [n]$, we say that a coati **secures** s olives among k on G if he/she collects at least s olives for any set of k olives.

Question (Q1) consists in determining $\mathrm{sec}_d^A(G, k)$ (resp. $\mathrm{sec}_d^B(G, k)$) the maximum number of olives Alice (resp. Bob) secures.

Question (Q3) asks for the average number of olives $\overline{\mathrm{col}}_d^X(G, k)$ coati X collects in the d-degenerate game over all possible k olive sets of G.

Finally, Question (Q4) asks for $\mathrm{win}_d^A(G, k)$ (resp. $\mathrm{win}_d^B(G, k)$) which is the probability that Alice (resp. Bob) wins, that is collects at least half the olives.

The following lemma shows that there is an optimal play where a coati grabs immediately an olive if possible. We denote by $R_d(G, O)$ the set of **removable olives** in the d-degenerate game on G, that are the vertices of O that have degree at most d in G.

▶ **Lemma 8.** *Let G be a d-degenerate graph and let O be a set of olives. If $R_d(G, O) = r > 0$, then for any order $o_1, o_2, \dots, o_r$ of the r vertices of $R_d(G, O)$, there is an optimal play starting with $o_1, o_2, \dots, o_r$*

Proof. We prove the result by induction on the number of vertices of G, the result holding trivially if G has one vertex. We note that to prove the lemma it suffices to prove that there is an optimal play starting with o_1. Indeed, by induction hypothesis there exist an optimal play in $(G, O) - \{o_1\}$ starting with $o_2, \dots, o_r$ as these vertices are vertices of $O - o_1$ removable.

Let σ be an optimal play for (G, O) and suppose that σ starts with the vertex v_1. If $v_1 = o_1$ there is nothing to prove. We distinguish two cases.

- $v_1 \in O$. So, let $v_1 = o_j$, with $2 \le j \le r$. By the induction hypothesis applied to $(G, O) - \{o_j\}$, there is an optimal play in $(G, O) - \{o_j\}$ starting with o_1. So we have an optimal play in $G, O)$ $o_j, o_1, v_3, \dots v_n$. But the play $\sigma' = (o_1, o_j, v_3, \dots v_n)$ is also optimal as A and B get the same number of olives (the only difference is that now A grabs o_1 instead of o_j). Note that the proof will not be valid if the weighty vertices have different weights.
- $v_1 \notin O$. By the induction hypothesis, we have an optimal play in $(G, O) - \{v_1\}$ starting with $o_1, o_2, \dots, o_r$ and so in (G, O) an optimal play $(v_1, o_1, \dots, o_r, v_{r+1}, v_{r+2}, \dots, v_n)$. The gain of A is therefore $\mathrm{col}^A((G, O) = \lfloor r/2 \rfloor + G(A)$ where $G(A)$ is the gain of A in the optimal play $(v_{r+1}, v_{r+2}, \dots, v_n)$ in $(G, O) - \{v_1, o_1, \dots, o_r\}$ with A playing first (resp. second) if r is odd (resp. even).
  Now consider the play where A plays first o_1. By the induction hypothesis there is an optimal play in $(G, O) - \{o_1\}$ starting with $o_2, \dots, o_r$.
  If $r \ge 2$, A can remove in move 3 the vertex v_1 and then A will gain $G(A)$ in the play in $(G, O) - \{o_1, v_1, o_2, o_3 \dots, o_r\}$ (with B starting). So we have a play $\sigma' = (o_1, o_2, v_1, o_3, \dots, o_r, v_{r+1}, v_{r+2}, \dots, v_n)$ where A gains $\lfloor r/2 \rfloor + G(A)$ which the optimum and so σ' is an optimal play satisfying the lemma.
  If $r = 1$ the gain of A is now 1 plus the gain $H(A)$ in an optimal play on $(G, O) - \{o_1\}$ with B playing first. To compute the gain $H(A)$ we will us the trick of computing the

gain if we add an olive in v_1. Let $H'(A)$ be the gain of A if we add an olive in v_1 that is in the game $(G, O \cup \{v_1\}) \setminus \{o_1\}$ with B playing first. As Alice can play on this game, the same strategy as if she was playing on $(G, O \setminus \{o_1\}$, we have $H(A) \geq H'(A) - 1$. By the induction hypothesis, in this game, there is an optimal play where B starts with v_1 and then consisting of the optimal play $(v_2, \dots, v_n)$ and so the gain of A is exactly $H'(A) = G(A)$. Therefore $H(A) \geq G(A) - 1$. In summary there is a play σ' starting with o_1, where A gains at least $G(A)$; that is exactly the value of an optimal play and so σ' is also an optimal play. ◀

4.1 Sharing a linear pizza with a $\{0, 1\}$-weight function

The olive set $O = \{v \mid w(v) = 1\}$ is **rare** or **α-ordinary** for some $\alpha \in \{0, 1, 2\}$ if it corresponds to a rare or α-ordinary weight function. Lemma 8 implies the following proposition (see proof in [1]).

▶ **Proposition 9.** *Suppose that O contains two adjacent olives $\{i_j, i_j + 1\}$. Let $P'_{n-2} = P_n \setminus \{i_j, i_j + 1\}$ and $O' = O \setminus \{i_j, i_j + 1\}$. Then $\mathrm{col}_1^A(P_n, O) = 1 + \mathrm{col}_1^A(P'_{n-2}, O')$ and $\mathrm{col}_1^B(P_n, O) = 1 + \mathrm{col}_1^B(P'_{n-2}, O')$.*

This proposition enables us to deal with adjacent olives and so to answer Question (Q2) for linear pizza with a $\{0, 1\}$-weight function. If there are h pairs of adjacent olives, we give h olives to each player and delete the corresponding vertices. We are left with a path of size $n - 2h$ and with an ordinary set of olives. Then we apply strategy Σ to compute the remaining gains of A and B.

Now we can give answers to questions (Q3) and (Q4). First we show that that among the $\binom{n}{k}$ sets of k olives on P_n, the 0-ordinary sets prevail. Therefore, to obtain asymptotic formulas on $\overline{\mathrm{col}}_1^A(P_n, k)$, (and so $\overline{\mathrm{col}}_1^B(P_n, k) = k - \overline{\mathrm{col}}_1^A(P_n, k)$), $\mathrm{win}_1^A(P_n, k)$, $\mathrm{win}_1^B(P_n, k)$, for k fixed, we can only consider 0-ordinary sets. Recall that when n is even $\mathrm{win}_1^A(P_n, k) = 1$ by Winkler's result. In that case we associate to a 0-ordinary set $O = \{i_1, \dots, i_k\}$ of k olives a **parity word** $W_O = \gamma_1\gamma_2 \cdots \gamma_k$ where $\gamma_j = 1$ if i_j is odd, and $\gamma_j = 0$ otherwise. We then show that the problem is reduced to computations of words. All the details are given in [1]. Table 1 summarizes the obtained values when $k \leq 6$ and suggests Conjecture 10.

Table 1 Asymptotic value of $\overline{\mathrm{col}}_1^A(P_n, k)$ and $\mathrm{win}_1^A(P_n, k)$ for $1 \leq k \leq 6$.

	k	1	2	3	4	5	6
$\overline{\mathrm{col}}_1^A(P_n, k)$	n even	1	$3/2 + \mathcal{O}(\frac{1}{n})$	$9/4 + \mathcal{O}(\frac{1}{n})$	$23/8 + \mathcal{O}(\frac{1}{n})$	$7/2 + \mathcal{O}(\frac{1}{n})$	$131/32 + \mathcal{O}(\frac{1}{n})$
	n odd	$O(\frac{1}{n})$	$1/2 + \mathcal{O}(\frac{1}{n})$	$3/4 + \mathcal{O}(\frac{1}{n})$	$9/8 + \mathcal{O}(\frac{1}{n})$	$3/2 + \mathcal{O}(\frac{1}{n})$	$61/32 + \mathcal{O}(\frac{1}{n})$
$\mathrm{win}_1^A(P_n, k)$	n even	1	1	1	1	1	1
	n odd	$2/n$	$1/2 + \mathcal{O}(\frac{1}{n})$	$\mathcal{O}(\frac{1}{n})$	$1/4 + \mathcal{O}(\frac{1}{n})$	$\mathcal{O}(\frac{1}{n})$	$3/16 + \mathcal{O}(\frac{1}{n})$

▶ **Conjecture 10.** *Let $C_k \lim_{n\to\infty, n \text{ even}} \overline{\mathrm{col}}_1^A(P_n, k)$ and $D_k = \lim_{n\to\infty, n \text{ odd}} \mathrm{win}_1^A(P_n, k)$. Then $\dfrac{C_k}{k} \to \dfrac{1}{2}$ when $k \to \infty$ and $D_{2p} \to 0$ when $p \to \infty$.*

5 Sharing a pissaladière

In this section, we study SHARING A PISSALADIÈRE, the 2-degenerate game on rectangular grids $G_{p \times q}$ with p rows and q columns. We always consider $p \leq q$ and $p > 1$. Indeed if $p = 1$ the grid is reduced to a path considered in the preceding sections with the stronger rule of picking only vertices of degree 1. Vertices are denoted (a, b) where a is the index of the row and b the index of the column. A vertex of $G_{p\times q}$ is a **corner vertex** if it is in $\{(1, 1), (p, 1), (1, q), (p, q)\}$. Otherwise it is a **central vertex**.

At a given stage in the game, a vertex is **critical** if it is removable and its removal leaves an olive removable. This happens when the olive has three remaining neighbours, in which case its removable neighbours are critical. A vertex is **safe** if it is removable, not an olive and not critical.

The strategy for a coati is to take a safe vertex and when possible to force the other coati to take a critical vertex.

5.1 One olive

We first consider the case where the pissaladière contains only one olive. Here again parity is important. Alice will collect the olive if pq is odd or if it is in a corner and Bob will collect a central olive if pq is even. The following lemma is more general since it deals with grid subgraphs and will be useful for the case of 2 olives.

▶ **Lemma 11.** *Let $O = \{o\}$ be a set of one olive in a subgraph H of the grid (not necessarily connected) on $n(H)$ vertices. Then $\mathrm{col}_2^A(H, O) = 1$ if and only if $n(H)$ is odd or $d_H(o) \leq 2$.*

Proof. If $d_H(o) \leq 2$, then Alice collects the olive at her first move. Henceforth, we may assume $d_H(o) \geq 3$.

Assume $n(H)$ is odd. Alice's strategy is the following. She removes a safe vertex until Bob removes a critical vertex. Then the olive becomes removable and Alice grabs it. Let us prove that Alice can always take a safe vertex. Assume that it is Alice's turn to play and no critical vertex has been removed. Since $n(H)$ is odd, the remaining graph has an odd number of vertices and contains at least o and three of its neighbours and hence it has at least 5 vertices. Therefore, at least one of the vertices is neither critical nor o. By symmetry, we may assume that there is such a vertex $x = (a', b')$ in the northwest part from $o = (a, b)$ that is a vertex with $a' \leq a$ and $b' \leq b$. Moreover, we may consider x such that there is no vertex in the northwest part from it. Then x is removable, and thus safe concluding the proof. If $n(H)$ is even, Bob can play the same strategy as Alice's above, and so collects the olive. ◀

A particular case of the previous lemma is when the subgraph is the grid itself.

▶ **Proposition 12.** *Let o be the single olive in $G_{p\times q}$. Then $\mathrm{col}_2^A(G_{p\times q}, o) = 1$ if and only if pq is odd or o is a corner.*

With Proposition 12, we can answer questions (Q1) to (Q4) when there is a single olive on the pissaladière. Note that for any $X \in \{A, B\}$, $\mathrm{win}_2^X(G_{p\times q}, 1) = \overline{\mathrm{col}}_2^X(G_{p\times q}, 1)$

▶ **Corollary 13.** *(i) if pq is odd then $\mathrm{sec}_2^A(G_{p\times q}, 1) = 1$, and $\mathrm{sec}_2^A(G_{p\times q}, 1) = 0$, else (ii) $\mathrm{sec}_2^B(G_{p\times q}, 1) = 0$.*

(iii) If pq is odd then $\overline{\mathrm{col}}_2^A(G_{p\times q}, 1) = 1$ and $\overline{\mathrm{col}}_2^B(G_{p\times q}, 1) = 0$, else (iii) $\overline{\mathrm{col}}_2^A(G_{p\times q}, 1) = \frac{4}{pq}$ and $\overline{\mathrm{col}}_2^B(G_{p\times q}, 1) = 1 - \frac{4}{pq}$.

5.2 Pissaladière of height 2

In this subsection, we consider Sharing a Pissaladière when the pissaladière has two rows. The two coatis plays the 2-degenerate game on the grid $G_{2\times q}$. We shall prove that $\mathrm{sec}_2^A(G_{2\times q}, k) = 0$ and $\mathrm{sec}_2^B(G_{2\times q}, k) = \lceil \frac{k-1}{3} \rceil$ (Corollary 15) and $\overline{\mathrm{col}}_2^A(G_{2\times q}, k)$ tends to 0 when $q \to \infty$ and k is fixed (Proposition 16).

Let O be a set of olives in $G_{2\times q}$. An **olive component** of O in G is a component of $G_{2\times q}\langle O\rangle$. The set of olive components of size ≥ 2 will be denoted $\mathcal{C}'(O)$. An olive in a component of size 1 is said to be **lonesome**; said otherwise an olive is lonesome if none of its neighbours is in O. Two lonesome olives are **related** if they have two common neighbours. The **lonesome graph** is the graph whose vertices are the lonesome olives and in which there is an edge between two lonesome olives if and only if they are related. A **wave** is a connected component in the lonesome graph. We denote by $\mathcal{W}(O)$ the set of waves. If there is a lonesome corner olive in O which is in an odd wave, then we set $\epsilon(O) = 1$, otherwise we set $\epsilon(O) = 0$.

▶ **Lemma 14.** *Let O be a set of olives in $G_{2\times q}$. Then*

$$\mathrm{col}_2^B(G_{2\times q}, O) \geq \sum_{W\in\mathcal{W}(O)} \left\lceil \frac{|W|}{2} \right\rceil - \epsilon(O) + \sum_{C\in\mathcal{C}'(O)} \left\lfloor \frac{|C|}{2} \right\rfloor .$$

Proof. Let v be a vertex. We denote by $L(v)$ (resp. $R(v)$) the set of vertices that are left (resp. right) to v and not adjacent to v:

$$L(a,b) = \{(a',b') \mid b' < b\} \setminus \{(a, b-1)\} \quad \text{and} \quad R(a,b) = \{(a',b') \mid b' > b\} \setminus \{(a, b+1)\}.$$

Note that if v is a central vertex then $L(v)$ and $R(v)$ are both odd.

Bob's strategy is the following.

1. If an olive is removable, then Bob collects it.
2. If there is no removable olive, then Bob takes a removable vertex not adjacent to a remaining lonesome olive (as we will see such a vertex always exists).

Let us show that when there is no removable olive, Bob has always the possibility to take a removable vertex not adjacent to a lonesome olive and so 2. can be applied. That is clear if there is no lonesome olive. If a lonesome olive remains, then it is a central one. Let o_ℓ be the leftmost (that is the one with smallest second coordinate) remaining lonesome olive and o_r the rightmost (that is the one with largest second coordinate) remaining lonesome olive. Observe that all removable vertices are in $L(o_\ell) \cup R(o_r)$ which cardinal is even.Then Bob removes a vertex in this set. Doing so Alice will not be able to collect a lonesome olive in her next move. The only way for Alice to collect a lonesome central olive is to remove a vertex adjacent to two related lonesome olives, to let Bob collect one of them by 1., and to collect the other. However Alice may collect a lonesome olive in a corner on her first move. Therefore Bob collects at least $\sum_{W\in\mathcal{W}(O)} \lceil |W|/2 \rceil - \epsilon(O)$ lonesome olives.

Moreover, for any olive component C of size greater than 1, Bob collects at least $\lfloor |C|/2 \rfloor$ olives in C. Indeed each time Alice collects an olive in the component if there is a remaining olive Bob also collects another one. ◀

▶ **Corollary 15.** *Let k be a positive integer.*
(i) If $q \geq 2k+1$, then $\mathrm{sec}_2^A(G_{2\times q}, k) = 0$.
(ii) If $q \geq k+1$, then $\mathrm{sec}_2^B(G_{2\times q}, k) = \lceil \frac{k-1}{3} \rceil$.

Proof. (i) Let $O = \{(1,b) \mid 2 \leq b \leq 2k+1 \text{ and } b \text{ even}\}$. Then O is made of k lonesome central olives which are not related to each other. Hence, by Lemma 14, $\mathrm{col}_2^B(G_{2\times q}, O) \geq |\mathcal{C}^1(O)| = |\mathcal{C}(O)| = k$.

(ii) Using Lemma 14, one can show that $\mathrm{sec}_2^B(G_{2\times q}, k) \geq \lceil \frac{k-1}{3} \rceil$. Let us now prove $\mathrm{sec}_2^B(G_{2\times q}, k) \leq \lceil \frac{k-1}{3} \rceil$. Since sec_2^B is non-decreasing it suffices to prove this statement when $k \equiv 1 \mod 3$, say $k = 3h+1$ for some integer h.

Let $O_i = \{(1, 3i), (2, 3i), (2, 3i+1)\}$ for all $i \in [h]$, and let $O = \{(1,1)\} \cup \bigcup_{i=1}^{h} O_i$.
Set $L_i = \{(a,b) \mid b < 3i-1\}$ and $R_i = \{(a,b) \mid b > 3i+2\} \cup \{(1, 3i+2\}$.

Let us describe an optimal strategy for Alice. She first removes the olive $(1,1)$. Then at each move she does the following. If it is possible to collect an O_i, then she collects the first olive. By lemma 8 Bob takes the second olive, then Alice takes the third.

Otherwise let ℓ (resp. r) be the smallest (resp. largest) integer i such that O_i has not been removed. Then Alice removes a vertex of $L_\ell \cup R_r$. This is always possible because all removed vertices are in this set and this set has an odd number of vertices. Doing so Bob will not be able to collect an olive at his next move. (In particular, it will not collect the first olive of any O_i).

Applying this strategy Alice collects the first and third olives of all O_i. Therefore Alice collects two olives per O_i. Since she also collects the olive $(1,1)$ in her first move, she collects $2h+1$ olives. Consequently Bob collects $h = \frac{k-1}{3}$ olives. ◀

▶ **Proposition 16.** *Let be fixed positive integer k.*
$\overline{\mathrm{col}}_2^A(G_{2\times q}, k) = \mathcal{O}(1/q)$ *and* $\overline{\mathrm{col}}_2^B(G_{2\times q}, k) = k + \mathcal{O}(1/q)$*;*
$\mathrm{win}_2^A(G_{2\times q}, k) = \mathcal{O}(1/q)$*, and* $\mathrm{win}_2^B(G_{2\times q}, k) = 1 + \mathcal{O}(1/q)$*.*

Proof. We distinguish two kinds of olive sets: *ordinary sets* in which the k olives are central lonesome and not related, and the ones that are not ordinary that we call *rare* sets. For an ordinary set O, by Lemma 14, $\mathrm{col}_2^B(G_{2\times q}, O) = k$ and so $\mathrm{col}_2^A(G_{2\times q}, O) = 0$.

A usual combinatorial computation gives that the number of rare sets is $\mathcal{O}(q^{k-1})$. The result follows. ◀

5.3 Two olives on a pissaladière

In this subsection, the coatis are given a pissaladière $G_{p\times q}$ with two olives thrown uniformly at random on two different slices. Our aim is to estimate the number of olives that Alice (or Bob) will collect when playing the 2-degenerate game. Unsurprisingly, the parity of the number $n = p \times q$ of slices plays an important role. Therefore, we define **Content** (C) to be Alice (resp. Bob) and **Defeated** (D) to be Bob (resp. Alice) if n is odd (resp. even).

The following lemma shows that Content always collects at least one olive.

▶ **Lemma 17.** *For any set of two olives O, Content collects at least one olive.*

Proof. Let O be a set of two olives.

If O contains two corners, then Content can remove one of them on its first move, so $\mathrm{col}_2^C(G_{p\times q}, O) \geq 1$.

Assume now that one of the two olives, say o_1, is not a corner. Playing its optimal strategy for $\{o_1\}$, Content collects at least $\mathrm{col}_2^C(G_{p\times q}, \{o_1\})$ olives. Thus $\mathrm{col}_2^C(G_{p\times q}, O) \geq \mathrm{col}_2^C(G_{p\times q}, \{o_1\}) = 1$, by Lemma 11. ◀

We shall prove that Content almost always grabs the two olives.

▶ **Theorem 18.** $\overline{\mathrm{col}}_2^C(G_{p\times q}, 2) = 2 + \mathcal{O}(1/n)$.

For the proof we will use the following notation. A vertex $v = (a,b)$ has (at most) four neighbours: its **north neighbour** $\mathbb{N}(v) = (a-1, b)$, its **south neighbour** $\mathbb{S}(v) = (a+1, b)$, its **west** neighbour $\mathbb{W}(v) = (a, b-1)$ and its **east neighbour** $\mathbb{E}(v) = (a, b+1)$.

The (open) **neighbourhood** of a vertex v, denoted by $N_G(v)$, in a graph G is the set of vertices adjacent to it. The **closed neighbourhood** of a vertex v is $N_G[v] = N_G(v) \cup \{v\}$. When the graph G is clear from the context, we often drop the superscript and use $N(v)$ and $N[v]$ instead of $N_G(v)$ and $N_G[v]$ respectively.

Let $S = V \setminus (N[o_1] \cup N[o_2])$. Let S^* be the set of vertices of S that can be removed without removing any vertex of $N[o_1] \cup N[o_2]$.

For a matching (set of independent edges) M, we denote by $V(M)$ the set the vertices incident to edges of this matching, and for every vertex v in $V(M)$ we denote by $M(v)$ the vertex matched to v by M (that is such that $\{v, M(v)\} \in M$).

At a given stage in the game, a vertex is **critical** if it is removable and its removal leaves an olive removable. This happens when the olive has three remaining neighbours, in which case its removable neighbours are critical. A vertex is **safe** if it is removable, not a olive and not critical. A non-olive vertex which is not removable is called **locked**. Observe that a locked vertex may become removable and thus either critical or safe. A removable vertex which is is one of the four neighbours of an olive goes from safe to critical when another neighbour of the olive is removed. It might go from critical to safe when its neighbouring olive is removed. Note there always exists a safe vertex in S^* unless all vertices of S^* have been removed.

Let $O = \{o_1, o_2\}$ be a set of two olives in $G_{p\times q}$. We set $o_1 = (a_1, b_1)$ and $o_2 = (a_2, b_2)$ and **we always assume that o_1 is smaller than o_2 in the lexicographical order, that is either $a_1 < a_2$, or $a_1 = a_2$ and $b_1 < b_2$.** In other words, either o_1 is north of o_2 or o_1 and o_2 are in the same latitude (row) and o_1 is west of o_2.

For each $i \in [2]$, we abbreviate $\mathbb{N}(o_i)$, $\mathbb{S}(o_i)$, $\mathbb{W}(o_i)$, $\mathbb{E}(o_i)$ in, respectively, $\mathbb{N}_i, \mathbb{S}_i, \mathbb{W}_i, \mathbb{E}_i$. Furthermore, for every $\mathbb{X}, \mathbb{Y} \in \{\mathbb{N}, \mathbb{S}, \mathbb{W}, \mathbb{E}\}$, we also abbreviate $\mathbb{X}(\mathbb{Y}_i)$ into $\mathbb{XY}_i$, and so on.

We now split the sets of two olives in two categories: rare and ordinary.

A set of of two olives is **rare** if it is of one of the following types:

- Type A: At least one olive is a corner.
- Type B: The two olives are at distance at most 4, that is $|a_2 - a_1| + |b_2 - b_1| \le 4$.
- Type C : The two olives are on a same row along the two opposite sides, that is $o_1 = (a, 1)$ and $o_2 = (a, q)$ with $1 < a < p$. Or the two olives are on a same column along the two opposite sides, that is $o_1 = (1, b)$ and $o_2 = (p, b)$ with $1 < b < q$.

The sets that are not rare are called **ordinary**.

We shall now prove that Content collects the two olives if the set is ordinary. As the number of ordinary sets of two olives is obviously $\binom{n}{2} + \mathcal{O}(n)$, this directly implies Theorem 18. Note that for many rare sets, Content also collects the two olives. Due to lack of space and because it is sufficient to prove Theorem 18, we only prove it here for ordinary sets. For more details, see [1].

▶ **Theorem 19.** *Let O be an ordinary set of two olives in $G_{p\times q}$. Then Content collects the two olives.*

The strategy for Content is to take a safe vertex and when possible to force Defeated to take a critical vertex. The proof of this theorem is in two parts; in one we prove that Content grabs the first olive, and in the other we prove that Content grabs the second olive. As it is the easiest, we begin with this later part.

▶ **Lemma 20.** *Let O be a set of two olives at distance at least 3 in $G_{p\times q}$. Then Content collects the second olive.*

Proof. Let X be the coati that collected the first olive and $\overline{X}$ its opponent. By Lemma 8, we may assume that X collected this olive as soon as possible, and so right after $\overline{X}$ removed a critical vertex adjacent to it. Hence, after the removal of the first olive, we are left with a subgraph H of the grid in which the second olive is not removable (for otherwise it would have been possible to remove it earlier since the two olives have no common neighbours). If X is Defeated (resp. Content), then H has an odd (resp. even) number of vertices, and so by Lemma 11 with Content (resp. Defeated) in the role of Alice, Content collects the second olive. ◀

▶ **Lemma 21.** *Let O be an ordinary set of two olives in $G_{p\times q}$. Then Content collects the first olive.*

Proof. Set $V = V(G_{p\times q})$ and $O = (o_1, o_2)$ with $o_1 = (a_1, b_1)$ and $o_2 = (a_2, b_2)$. Recall that we may assume, as stated above, that $a_1 < a_2$, or $a_1 = a_2$ and $b_1 < b_2$.

Since O is not of Type A, it contains no corner. Therefore a coati collects the first olive if and only its opponent is the first to remove a critical vertex. Therefore the strategy of both coatis is to force its opponent to be the first to remove a critical vertex. Therefore we may assume that **both coatis remove a safe vertex until it is impossible. Let R be the set of vertices removed until a coati is blocked, that is there is no more safe vertex to remove. We shall prove that $|R| \equiv pq \mod 2$, which implies that Content collects the first olive.** Recall that as long as there remains at least one vertex of S^* in the graph, one of them is safe and the coatis are not blocked. So $S^* \subseteq R$.

In the figures, we put the two olives and their neighbours in general position. Note that if o_1 (resp. o_2) has degree 3 – that is on the border –, then one of $\mathbb{N}_1, \mathbb{W}_1, \mathbb{S}_1$ (resp. $\mathbb{N}_2, \mathbb{E}_2, \mathbb{S}_2$) does not exist and the three neighbours of o_1 (resp. o_2) are critical. Recall that there is no olive in a corner (otherwise it will be a rare set of Type 1). We indicate with a $*$ the vertices which are in S^*. We put nothing for vertices locked at the beginning and recall that they might at some stage become removable and thus either critical and safe.

By symmetry, we can suppose furthermore that $b_2 \geq b_1$ and that $a_2 - a_1 \leq b_2 - b_1$. We distinguish various cases. We consider them in decreasing order of the possible values of $a_2 - a_1$ and for each value of $a_2 - a_1$ in decreasing order of the values of $b_2 - b_1$.

▷ Claim 22. If $a_2 - a_1 \geq 3$ (and so $b_2 - b_1 \geq 3$), then Content collects the first olive.

Proof. One can check that $S^* = S$. (See Table 2.)

Table 2 Two olives with $a_2 - a_1 = 3$ and $b_2 - b_1 = 3$. Vertices in S^* are marked with $*$.

$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
$*$	$*$	$\mathbb{N}_1$	$*$	$*$	$*$	$*$	$*$
$*$	$\mathbb{W}_1$	o_1	$\mathbb{E}_1$	$*$	$*$	$*$	$*$
$*$	$*$	$\mathbb{S}_1$	$*$	$*$	$*$	$*$	$*$
$*$	$*$	$*$	$*$	$*$	$\mathbb{N}_2$	$*$	$*$
$*$	$*$	$*$	$*$	$\mathbb{W}_2$	o_2	$\mathbb{E}_2$	$*$
$*$	$*$	$*$	$*$	$*$	$\mathbb{S}_2$	$*$	$*$
$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$

If one olive has four neighbours R contains also a neighbour of this olive. Thus $|R| = pq - 8$, and so Content collects the first olive. ◁

▷ Claim 23. If $a_2 - a_1 = 2$ and $b_2 - b_1 \geq 4$, then Content collects the first olive.

Proof. The proof is identical to Claim 22 as again $S^* = S$. ◁

▷ Claim 24. If $a_2 - a_1 = 2$ and $b_2 - b_1 = 3$, then Content collects the first olive.

Proof. One can check that $S^* = S \setminus \{\mathbb{SE}_1, \mathbb{NW}_2\}$. (See Table 3). We distinguish two cases:

Assume first that either $\{o_1$ is of degree 3 or R contains $\mathbb{W}_1$ or $\mathbb{N}_1\}$ and either $\{o_2$ is of degree 3 or R contains $\mathbb{E}_2$ or $\mathbb{S}_2\}$. Then $\mathbb{SE}_1$ and $\mathbb{NW}_2$ remain locked, that is do not belong to R. Thus $|R| = pq - 10$, and so Content collects the first olive.

Otherwise R contains a vertex in $\{\mathbb{S}_1, \mathbb{E}_1, \mathbb{N}_2, \mathbb{W}_2\}$. Then R contains necessarily $\mathbb{SE}_1$ and $\mathbb{NW}_2$. Thus $|R| = pq - 8$, and so Content collects the first olive. ◁

Table 3 Two olives with $a_2 - a_1 = 2$ and $b_2 - b_1 = 3$. Vertices in S^* are marked with $*$.

*	*	*	*	*	*	*	*
*	*	$\mathbb{N}_1$	*	*	*	*	*
*	$\mathbb{W}_1$	o_1	$\mathbb{E}_1$	*	*	*	*
*	*	$\mathbb{S}_1$			$\mathbb{N}_2$	*	*
*	*	*	*	$\mathbb{W}_2$	o_2	$\mathbb{E}_2$	*
*	*	*	*	*	$\mathbb{S}_2$	*	*
*	*	*	*	*	*	*	*

▷ Claim 25. If $a_2 - a_1 = 1$ and $b_2 - b_1 \geq 4$, then Content collects the first olive.

Proof. The proof is similar to Claim 24. Here $S^* = S \setminus L$, where $L = \{(a_1, b) \mid b_1 + 1 < b < b_2\} \cup \{(a_1 + 1, b) \mid \ b_1 < b < b_2 - 1\}$. If we set $d = b_2 - b_1$ $(d \geq 4)$, then $|L| = 2(d-2)$. We have the same two cases as in the preceding claim. In the first case $|R| = pq - 2d - 6$, and in the second case $|R| = pq - 8$. In both cases, Content collects the first olive. ◁

▷ Claim 26. If $a_1 = a_2$ and $b_2 - b_1 \geq 5$, then Content collects the first olive.

Proof. Set $d = b_2 - b_1$. We have $d \geq 3$. Moreover, since O is not of Type C, $d \leq q - 2$.

If $a_1 > 1$, let $L_{\mathbb{N}} = \{(a_1 - 1, b) \mid b_1 < b < b_2\}$, in which case $|L_{\mathbb{N}}| = d - 1$ and if $a_1 = 1$, $L_{\mathbb{N}} = \emptyset$. If $a_1 < q$, let $L_{\mathbb{S}} = \{(a_1 + 1, b) \mid \ b_1 < b < b_2\}$, in which case $|L_{\mathbb{S}}| = d - 1$ and if $a_1 = q$, $L_{\mathbb{S}} = \emptyset$. Let $L_{\text{eq}} = \{(a_1, b) \mid \ b_1 + 1 < b < b_2 - 1\}$; then $|L_{\text{eq}}| = d - 3$. Let $L = L_{\mathbb{N}} \cup L_{\text{eq}} \cup L_{\mathbb{S}}$. One can check that $S^* = S \setminus L$. (See Table 4).

Table 4 Two olives with $a_2 = a_1$ and $b_2 - b_1 = 5$. Vertices in S^* are marked with $*$.

*	*	*	*	*	*	*	*	*	*
*	*	$\mathbb{N}_1$					$\mathbb{N}_2$	*	*
*	$\mathbb{W}_1$	o_1	$\mathbb{E}_1$			$\mathbb{W}_2$	o_2	$\mathbb{E}_2$	*
*	*	$\mathbb{S}_1$					$\mathbb{S}_2$	*	*
*	*	*	*	*	*	*	*	*	*

If $a_1 = 1$ or $a_1 = q$, then $b_1 \neq 1$ and $b_q \neq q$ because O is not of Type 1. The two olives have only three neighbours, so $R = S^*$ and $|R| = pq - 2d - 4 \equiv pq \mod 2$. Thus Content collects the first olive. In what follows **we suppose that** $1 < a_1 < q$ and so $|L| = 3d - 5$.

Observe the following facts:

- If $\mathbb{N}_1$ or $\mathbb{N}_2$ is in R, then $L_{\mathbb{N}} \subset R$. Otherwise $L_{\mathbb{N}} \cap R = \emptyset$.
- If $\mathbb{S}_1$ or $\mathbb{S}_2$ is in R, then $L_{\mathbb{S}} \subset R$. Otherwise $L_{\mathbb{S}} \cap R = \emptyset$.
- $R \cap \{\mathbb{E}_1, \mathbb{W}_2\} = \emptyset$.
- If $\{\mathbb{N}_1, \mathbb{S}_2\} \subset R$ or $\{\mathbb{S}_1, \mathbb{N}_2\} \subset R$, then $L_{\text{eq}} \subseteq R$. Otherwise $L_{\text{eq}} \cap R = \emptyset$.

If R contains at least one vertex in $\{\mathbb{N}_1, \mathbb{N}_2, \mathbb{S}_1, \mathbb{S}_2\}$, then $|R| \equiv pq \mod 2$ so Content collects the first olive. Indeed suppose for example that R contains N_1, then either it contains S_2 and $|R| = pq - 8$, otherwise $|R| = pq - 2d - 4$ (using the above facts). If $R \cap \{\mathbb{N}_1, \mathbb{N}_2, \mathbb{S}_1, \mathbb{S}_2\} = \emptyset$, then all the vertices of $L_{\mathbb{N}} \cup L_{\text{eq}} \cup L_{\mathbb{S}}$ remain locked. Thus $|R| = pq - 3d - 3$.

If d is odd then $|R| \equiv pq \mod 2$ in both cases, so Content collects the first olive.

If d is even, Content has to be careful. He must removes safe vertices so that $R \cap \{\mathbb{N}_1, \mathbb{N}_2, \mathbb{S}_1, \mathbb{S}_2\} \neq \emptyset$. Since O is not of Type 5, then either $b_1 > 1$ or $b_2 < q$. By symmetry, we may assume $b_1 > 1$. we now describe a strategy for Content such that R does not contain $\mathbb{W}_1$.

Let M be the matching $\{(i, b_1 - 1), (i, b_1)\}$. Observe that M matches $\mathbb{W}_1$ with o_1, $\mathbb{NW}_1$ with $\mathbb{N}_1$, and $\mathbb{SW}_1$ with $\mathbb{S}_1$. Content applies the following procedure until either s/he collects an olive, or a vertex in $\{\mathbb{N}_1, \mathbb{S}_1, \mathbb{N}_2, \mathbb{S}_2\}$ is removed.

(R) If Defeated just removed a vertex in $V(M)$, then Content removes the vertex to which it is matched by M. Otherwise, it removes a safe vertex not in $V(M)$.

Let us prove that this strategy is valid, i.e. that rule can be applied. Assume Defeated removed a vertex v in $V(M)$, then $M(v)$ has not been removed earlier. Furthermore, if v is , then one of its neighbours w in $\{\mathbb{N}(v), \mathbb{S}(v)\}$ has been removed before and so $M(w)$ has been removed and therefore $M(v)$ becomes removable and Content can remove it. Note also that $R \setminus V(M) = V(G) \setminus \{V(M) \cup L \cup \{\mathbb{E}_1, \mathbb{W}_2, \mathbb{N}_2, o_2, \mathbb{S}_2\}\}$ ($\mathbb{N}_2$ and $\mathbb{S}_2$ cannot have been removed otherwise the procedure stopped). Therefore, $|R \setminus V(M)| = pq - 2p - 3d - 10 \equiv pq \mod 2$ (as d is even) and so when Content plays it remains at least one removable vertex in $R \setminus V(M)$ and the second part of Rule (R) can be applied.

Finally, suppose for a contradiction that $\mathbb{W}_1$ is removed during the procedure. Before being removed one of the two vertices $\mathbb{NW}_1$ or $\mathbb{SW}_1$ has been removed (because o_1 is not removed since the procedure did not stop). But those vertices are matched with $\mathbb{N}_1$ and $\mathbb{S}_1$ respectively. Thus, by Rule (R), one of $\mathbb{N}_1$ and $\mathbb{S}_1$ has been removed before and the procedure stopped, a contradiction.

Thus, applying the above procedure, one vertex in $\{\mathbb{N}_1, \mathbb{N}_2, \mathbb{S}_1, \mathbb{S}_2\}$ is removed. Hence, as noted before $|R| \equiv pq \mod 2$ and Content collects the first olive. ◁

Since O is not of type B, the only possibles cases are the ones considered in the above claims. ◀

6 Conclusion

In this paper, we studied some particular cases of graph-grabbing games. We made an extensive analysis of Sharing a Linear Pizza, including optimal strategies and complexity results. However there are still many open problems : study the case of adjacent weighty vertices for general weights and prove the conjecture that for $\{0, 1\}$ weights each player grabs asymptotically half of the olives.

We also introduced d-generate games for which we obtained preliminary results and performed a more extensive study for the grids case that we called Pissaladière. We proved that for two olives A (resp. B) almost always grabs the two olives if the size is odd (resp. even). We conjecture that the same result holds for k olives.

▶ **Conjecture 27.** *Let O be an ordinary set of k olives in $G_{p\times q}$. If $n = pq$ is odd (resp. even), then A (resp. B) grabs the k olives. If n is odd $\overline{\text{col}}_2^A(G_{p\times q}, k) = k + \mathcal{O}(1/n)$. If n is even $\overline{\text{col}}_2^B(G_{p\times q}, 2) = k + \mathcal{O}(1/n)$*

Finally, many problems remain open for other d-generate games.

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