Improved Bounds for Gossiping in Mesh Bus Networks

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Abstract

Improved bounds for the minimum gossiping time in mesh bus networks of arbitrary dimension for 1-port model are given. More precisely, the gossiping protocol consists of steps during which messages are sent via buses and at the end of the protocol, all the nodes should know all the information. Furthermore, during one step a bus can carry at most one message, and each node can either send or receive (not both) on at most one bus The minimum gossiping time of a bus network G is the minimum number of steps required to perform a gossip under this model. Here we determine almost exactly the minimum gossip time for 2-dimensional mesh bus networks and give tight bounds for d-dimensional mesh bus networks.

1 Introduction

A *bus network* is a communication network in which the nodes (representing nodes of a physical network) are connected by a set of shared buses (representing the communication media). These networks have been considered in the literature and are usually modelled by hypergraphs (see [1] [3] [12]). They have also gained interest recently in optical networks (see [2] [11] [13] [14] [15]).

Here we are interested in structured communication problems, and in particular, gossiping. *Gossiping* in a network refers to the process of disseminating the information from all the nodes in a given bus network to all the other nodes. We will also consider *broadcasting*, where a given node has to send its information to all the other nodes, and *accumulation*, where one node wants to know (accumulate) all the information from the other nodes. The time required to perform a protocol will depend on the model used. Here we first suppose that we are in the "store and forward" model (also called "packet switched"), where a node can only send the content of a message after it has received the whole message; so an intermediate node has to "store" the message before being able to "forward" it.

We also assume that the network functions in some synchronous way; more exactly, the protocol consists of steps, and, during one step, messages are transmitted from certain nodes to certain other nodes on buses. We suppose furthermore that, during one step, a bus can carry at most one message.

Finally, we consider the "1-port hypothesis", where during one step, a node can either send or receive (but not both at the same step) on at most one bus. We will denote $g(G)$ (resp. $b(G)$, $a(G)$) the minimum gossiping time (resp. broadcasting time, accumulation time) of a bus network G under the 1-port model, that is the minimum number of steps required to perform a gossip (resp. broadcast, accumulation) protocol in G under the 1-port model. Various results have been obtained on these parameters. For example, in [7] the following result was obtained.

$$max\{\lceil log_2 n\rceil, \lceil n/m\rceil\} \leq g(G) \leq min\{n-1, (l-1)D\Delta\} + 2\lfloor n/l\rfloor,$$

where G is any bus network of order n, maximum degree Δ, and diameter D, having m buses and maximum bus length l. (The *length* of a bus is the number of nodes on the bus.)

If we restrict the structure of the bus networks in question, then the bounds can be improved substantially.

A bus network in which any subset of vertices of size $l \leq k$, where $k \geq 2$ is a constant, is connected by a shared bus of length l, is called a *complete bus network*. It was proved in [5] that if G is a complete bus network, then $g(G) \leq clog_2 n + log_k n + O(1)$.

Here we are interested in mesh bus networks. These networks have been extensively studied and shown to be efficient for many algorithms, in particular routing, sorting (see [9]) and gossiping (see [4] and [6]).

A *mesh bus network* of dimension d, denoted $M(n_1, n_2, ..., n_d)$, is a hypergraph where the vertex set is $\{(x_1, x_2, ..., x_d) : x_i \in \{1, 2, ..., n_i\}\}$ and an edge contains the nodes which agree on all coordinates but one. Let M_n^d be a mesh bus network of dimension d, where $n_i = n, 1 \leq i \leq n_d$.

In [8], it is proved that $b(M_n^d) = d$,

$\lfloor n/2 \rfloor + \lceil log_2 n \rceil \leq g(M_n^2) \leq \lfloor n/2 \rfloor + 2\lceil log_2 n \rceil + 1$, and

$\lfloor n/d \rfloor + \lceil (d-1) log_2 n \rceil \leq g(M_n^d) \leq \lceil n/d \rceil + 2(d-1)\lceil log_2 n \rceil + 3d - 3$.

In this paper we present the following improved bounds for gossiping in mesh bus networks (this question was asked in [7]). First we gave results on accumulation time and then we use them to obtain the improved bounds for gossiping. In the 2-dimensional case, we obtain some exact answers or very close bounds.

Theorem 1.1

If n is even, $a(M_n^2) = n/2 + \lceil log_2 n \rceil$
If n is odd, $a(M_n^2) = (n-3)/2 + \lceil log_2 3n \rceil$

Theorem 1.2

If n is even, $n/2 + \lceil log_2 n \rceil + 1 \leq g(M_n^2) \leq n/2 + \lceil log_2 n \rceil + 2$.
If n is odd, $(n-1)/2 + \lceil log_2(3n) \rceil \leq g(M_n^2) \leq (n-1)/2 + \lceil log_2(3n) \rceil + 1$.

Note that the upper bounds of Theorem 1.2 follow from Theorem 1.1 and the fact that we can realize a gossiping protocol by first performing an accumulation protocol, and then performing, in two steps, a broadcasting protocol from the node at which the information has been accumulated. We will call such a vertex which knows all the information of the other nodes an expert node or an expert. The lower bounds will be obtained in Section 2.

In some cases we can improve the lower bounds and obtain exact answers.

Theorem 1.3
If $n = 2^k$, then $g(M_n^2) = n/2 + log_2 n + 2$.
If $3n = 2^k - 1$, then $g(M_n^2) = (n+1)/2 + \lceil log_2(3n) \rceil$.

For d-dimensional mesh networks, we have the following results.

Theorem 1.4 *For $d \geq 3$, $\lfloor n/d \rfloor + \lceil log_2(d+r)n^{d-1} \rceil \leq a(M_n^d) \leq \lfloor n/d \rfloor + (d-1)\lceil log_2 n \rceil + \lceil log_2(d+r) \rceil + d - 2$, where $n = qd + r, 0 \leq r < d$.*

In the d-dimensional case, the expert can finish broadcasting in d steps. Thus we obtain the following upper bounds for the gossiping time.

Theorem 1.5 *For $d \geq 3$, $g(M_n^d) \leq a(M_n^d) + d$.*

2 Lower bounds

In this section we give lower bounds for the minimum accumulation and gossiping times in the d-dimensional mesh bus network M_n^d, under the 1-port hypothesis. We begin with some general lemmas concerning accumulation and gossiping times in arbitrary bus networks under the 1-port hypothesis.

Lemma 2.1 *Under the 1-port hypothesis, the minimum gossiping time for any bus network is at least one more than the minimum accumulation time for that network.*

Proof. Let G be a bus network, and let $a(G)$ denote the minimum accumulation time in G. Then in any gossiping algorithm for G, no vertex knows all the information before step $a(G)$. Therefore the gossiping time for G is at least $a(G)$. However, at least one vertex must send a message during step $a(G)$; this vertex cannot receive during step $a(G)$, and in particular cannot receive its last message during this step. Since it cannot have received its last message any earlier (by definition of $a(G)$), it must receive its last message during or after step $a(G) + 1$. The lemma follows. ■

Lemma 2.2 *Suppose that in an accumulation algorithm for a network G under the 1-port model, there is a set A of vertices none of which has sent any message in the first s steps. Then at least $log_2|A|$ more steps are needed to complete the accumulation algorithm. Furthermore, if $|A| = 2^p$ or $2^p - 1$, then at step $s + p$, there are at most $L - 1$ experts, where L is the maximum length of buses in G.*

Proof. For the first part, it suffices to prove that at step $s + p$ an arbitrary vertex v can have received information from at most $2^p - 1$ elements of A, where $p < \lceil log_2|A| \rceil$. Let u be the vertex which sends to v at step $s + p$. Recall that v can receive from at most one vertex at each step.

If $p = 1$, the result follows from the fact that even if u belongs to A, u does not know that information from any other element of A; so the message it sends to v contains at most one piece of information from A.

Let $p > 1$. By induction, u has received the information from at most $2^{p-1} - 1$ elements of A before step $s + p$. So at step $s + p$, u can send to v at most 2^{p-1} (the information it received plus its own if it belongs to A) pieces of information from A. Hence v has received at most $2^{p-1} - 1$ (before step $s + p$) plus 2^{p-1} (at step $s + p$) pieces of information from A, that is all together $2^p - 1$.

It is not difficult to see that if $|A| = 2^p$, then only one vertex of A can be expert after step $s + p$ as this expert can not have sent its information to any other vertices. It is also clear that none of the vertices outside A can become experts either.

Now suppose $|A| = 2^p - 1$. Observe first that any vertex which has received $2^{i-1} - 1$ pieces of information from A during steps $s+1, \ldots, s+i-1$, where $1 \leq i \leq p$, cannot have sent any messages during steps $s + 1, \ldots, s + i - 1$. Moreover, any vertex which knows 2^{i-1} pieces of information from A at the end of step $s + i - 1$ must itself belong to A.

Consider a vertex v which becomes an expert at the end of step $s + p$. If v has never sent out its information, then it is the unique expert. Otherwise we have two cases to consider.

If $v \in A$ and v sent out its information, then it must have sent its information at step $s + 1$ (otherwise, the total number of pieces of information it could have received from A by the end of step $s + p$ would be at most $(2^p - 1) - 2$, and it would not be an expert). Furthermore, v should have received 2^{p-1} pieces of information of A from some vertex u at step $s+p$.

If $v \notin A$, then it should have also received 2^{p-1} pieces of information of A from some vertex u at step $s+p$.

By the observation above, in both cases, $u \in A$ and u has not sent its information in the first $s+p-1$ steps.

Now suppose that two vertices v and v' are experts at the end of step $s+p$. Let u and u' be the two vertices from which v and v' received the information at step $s+p$. Then it is necessary that $u = u'$ as otherwise v would not learn the information of u' and so it would not be an expert. Hence the experts can be informed at the last step by a unique vertex u of A and they are all on the same bus containing u. Therefore the total number of experts is at most $L-1$. ∎

We now give a lower bound for the accumulation time in M_n^d.

Lemma 2.3 *Let* $n = qd + r, 0 \leq r < d$.
Then $a(M_n^d) \geq q - 1 + \lceil log_2(d+r)n^{d-1} \rceil$.

Proof. Since M_n^d has dn^{d-1} buses (n^{d-1} in each dimension), at most dn^{d-1} messages can be sent during any single step (recall that by the hypothesis, a bus can carry at most one message per step). After $q-1$ steps, where $q = \lfloor n/d \rfloor$, at most $(q-1)dn^{d-1}$ vertices have sent their messages. Let A be the set of vertices which have not sent any messages before step q. Then we have $|A| \geq n^d - (q-1)dn^{d-1} = (d+r)n^{d-1}$. So by Lemma 2.2 we need at least $\lceil log_2|A| \rceil$ steps to complete the accumulation algorithm and hence, $a(M_n^d) \geq q - 1 + \lceil log_2(d+r)n^{d-1} \rceil$. ∎

Using Lemma 2.1, we deduce the following corollary.

Corollary 2.4 $g(M_n^d) \geq q + \lceil log_2(d+r)n^{d-1} \rceil$.

In the special case $d = 2$, Lemma 2.3 and Corollary 2.4 can be stated explicitly as follows.

Lemma 2.5 *If* n *is even,* $a(M_n^2) \geq n/2 + \lceil log_2 n \rceil$
and $g(M_n^2) \geq n/2 + \lceil log_2 n \rceil + 1$.

If n is odd, $a(M_n^2) \geq (n-3)/2 + \lceil log_2(3n) \rceil$
and $g(M_n^2) \geq (n-1)/2 + \lceil log_2(3n) \rceil$.

In the case $d = 2$ and $n = 2^k$ or $3n = 2^k - 1$, the bounds in Lemma 2.5 for gossiping can be further improved.

Lemma 2.6 *If* $n = 2^k$, $g(M_n^2) \geq n/2 + \lceil log_2 n \rceil + 2$.
If $3n = 2^k - 1$, $g(M_n^2) \geq (n-1)/2 + \lceil log_2 3n \rceil + 1$.

Proof. Let $n = 2^k$. From the proof of Lemma 2.3, there will be a set of vertices of size at least $2n$ which have not sent out any information during the first $n/2 - 1$ steps. From the proof of Lemma 2.2, there will be at most one expert after step $n/2 + \lceil log_2 n \rceil$ $(= n/2 - 1 + \lceil log_2(2n) \rceil)$ and Lemma 2.5 guarantees that there is no expert before that step. Clearly, at least two more steps are required to complete the gossip, as the unique expert needs two steps to broadcast its messages to the others.

Let $3n = 2^k - 1$. Here n is odd and from the proof of Lemma 2.3, there will be a set of vertices of size at least $3n$ which have not sent out any information during the first $(n-3)/2$ steps. From Lemma 2.2 and 2.5, there will be at most $n - 1$ experts after step $(n-3)/2 + \lceil log_2(3n) \rceil$. Again two more steps will be required to complete gossiping. ■

3 Upper bounds for $d = 2$

In this section we first give the proof of Theorem 1.1, determining $a(M_n^2)$, the exact values of the accumulation time of M_n^2. From this we deduce an upper bound for the gossiping time in M_n^2. It suffices to consider a gossiping protocol obtained by first applying an accumulation protocol in time $a(M_n^2)$, followed by a broadcast, which can be done in two steps.

Theorem 3.1 $g(M_n^2) \leq a(M_n^2) + 2 = \begin{cases} n/2 + \lceil log_2 n \rceil + 2, & n \textit{ even}, \\ (n-3)/2 + \lceil log_2(3n) \rceil + 2, & n \textit{ odd}. \end{cases}$

When $n = 2^k$ or $3n = 2^k - 1$, by Lemma 2.6 and Theorem 3.1 we have Theorem 1.3, which determines the exact gossiping time in M_n^2 for the given n.

Before we present the proof of Theorem 3.1, we will give some more definitions and notations.

In what follows we will label the vertices of M_n^2 with ordered pairs (x, y), using "matrix notation", where x is the row index and y is the column index. Recall that M_n^2 contains $2n$ buses, with n in each dimension. We refer to those buses in the first dimension (that is, those buses of the form $\{(\alpha, y) : 1 \leq \alpha \leq n\}$, where y is fixed) as *vertical buses*, and those in the second dimension as *horizontal buses.*

Our protocol will follow from the way we obtained the lower bound. The first stage consists of $\lfloor n/2 \rfloor - 1$ steps during which the information from M_n^2 will be collected inside an intermediate set A_n which must satisfy the requirement:

- In each step of the first stage, it is feasible to send $2n$ different messages from M_n^2 to A_n.

Observe that if our protocol is to be optimal, then the set A_n must also satisfy:

- $|A_n| = 2n$ if n is even, and $|A_n| = 3n$ if n is odd.

In the second stage, the information now concentrated in A_n will be accumulated at a fixed vertex of A_n, which in fact will always be the vertex $(1, 1)$. This leads to a third requirement:

- During the second stage, the information lying inside A_n can be accumulated at $(1, 1)$ in $\lceil log_2(|A_n|) \rceil$ steps (and of course we need $(1, 1) \in A_n$).

To satisfy the above property, at each step of an accumulation algorithm for A_n, the number of vertices which have not yet sent their information to $(1, 1)$ must be halved. In other words, at each step, one half of these vertices must send their information to the other half. It is worth bearing this in mind when the sets A_n are defined.

Now observe that if, in the first stage, we wish to send $2n$ messages during a single step, then we must send n messages from M_n^2 to A_n along the horizontal buses and n more along the vertical buses. Consequently, the set A_n must contain at least one vertex from every horizontal bus and one vertex from every vertical bus. In addition, the vertices from which the $2n$ messages originate must be suitably placed in M_n^2. A *transversal* of M_n^2, that is, a set of n vertices intersecting each bus in a single vertex, is a particularly useful configuration in this respect, since the elements of a transversal can all send their messages simultaneously, either horizontally or vertically. It is with this in mind that we make the following definitions.

Definition 3.2 Let S be any set of vertices in M_n^2 of the form
$S = \{(a+\alpha, b+\beta) : 1 \leq \alpha, \beta \leq s\}$,
for some choice of a, b and s. We say a subset T of S is a *transversal* of S if T contains a unique vertex of each bus which intersects S. It is not difficult to verify that every such set S can be partitioned into transversals.

We use $T(S,i)$, $1 \leq i \leq s$, to denote a partition of the elements of S into transversals. We will always assume that $T(S,1) = \{(a+\alpha, b+\alpha) : 1 \leq \alpha \leq s\}$ (the diagonal elements of S), but otherwise the choice of the transversals is arbitrary.

Notation 3.3 We say $f_h^i(X) = Y$ (or $f_h^i : X \to Y$) if at step i of the algorithm, the vertices in X send their information to the vertices in Y along horizontal buses. Clearly this presupposes that X and Y satisfy $|X| = |Y|$, and that X and Y intersect the same set of $|X|$ horizontal buses. The notation $f_v^i(X) = Y$ (or $f_v^i : X \to Y$) is analogous, the messages being sent along vertical buses in this case.

Finally, we make the following definitions in order to facilitate the description of certain subsets of vertices of M_n^2.

Definition 3.4

1. We use $T_{a,b}$ to denote the following translation:

   $T_{a,b}((x,y)) = (x+a, y+b)$, and

   $T_{a,b}(X) = \{(x+a, y+b) : (x,y) \in X\}$ for a set X of vertices.

2. We use D_s to denote the first s elements of the main diagonal in M_n^2; that is,

   $D_s = \{(\alpha, \alpha) : 1 \leq \alpha \leq s\}$.

   Note that for the set S of Definition 3.2, $T(S, 1) = T_{a,b}(D_s)$.

Proof of Theorem 3.1

We are now ready to present the proof of Theorem 3.1. We will divide the proof into three cases, according to whether n is a power of 2, n is even, or n is odd. In each case we will define a subset A_n of M_n^2, with $|A_n| = 2n$ if n is even and $|A_n| = 3n$ if n is odd, and show that the information in M_n^2 can be sent to A_n in $\lfloor n/2 \rfloor - 1$ steps, and that the information in A_n can be accumulated at $(1, 1)$ in $\lceil log_2(|A_n|) \rceil$ steps.

Case 1: $n = 2^k, k \geq 1$.

We let $A_n = H_n \cup V_n$, where

$$\begin{aligned} H_2 &= \{(1,1),(2,2)\}, \\ H_{2^{i+1}} &= H_{2^i} \cup T_{0,2^i}(D_{2^i}); \\ V_2 &= \{(1,2),(2,1)\}, \\ V_{2^{i+1}} &= V_{2^i} \cup T_{2^i,0}(D_{2^i}). \end{aligned}$$

Note that $|A_n| = 2n$, and that V_n contains exactly one vertex from each horizontal bus while H_n contains exactly one vertex from each vertical bus.

Lemma 3.5 *If* $n = 2^k$*, then the information in* A_n *can be accumulated at* $(1, 1)$ *in* $\lceil log_2 |A_n| \rceil = k + 1$ *steps.*

Proof. The proof is by induction on k; and the result is clear for $k = 1$. Observe that for each $k > 1$, $H_{2^k} \backslash H_{2^{k-1}}$ contains exactly one vertex from each horizontal bus intersecting $H_{2^{k-1}}$, and similarly $V_{2^k} \backslash V_{2^{k-1}}$ contains exactly one vertex from each vertical bus intersecting $V_{2^{k-1}}$.

Therefore for $k > 1$, the information contained in $A_{2^k} \backslash A_{2^{k-1}}$ can be sent to $A_{2^{k-1}}$ in one step as follows. Observe that $H_{2^k} \backslash H_{2^{k-1}} = T_{0,2^{k-1}}(D_{2^{k-1}})$ and $V_{2^k} \backslash V_{2^{k-1}} = T_{2^{k-1},0}(D_{2^{k-1}})$, and let

$f_h^1 : T_{0,2^{k-1}}(D_{2^{k-1}}) \to V_{2^{k-1}}$, and

$f_v^1 : T_{2^{k-1},0}(D_{2^{k-1}}) \to H_{2^{k-1}}$. ■

Lemma 3.6 *If $n = 2^k$, the information contained in M_n^2 can be accumulated in A_n in $n/2 - 1 = 2^{k-1} - 1$ steps.*

Proof. The proof is again by induction on k, and is clear for $k = 1$; so we let $k \geq 2$.

We first partition M_n^2 into four subsets, X, P, Q and R, where $X = M_{2^{k-1}}^2$, $P = T_{0,2^{k-1}}(X)$, $Q = T_{2^{k-1},0}(X)$ and $R = T_{2^{k-1},2^{k-1}}(X)$. Note that $A_{2^k} = A_{2^{k-1}} \cup T(P,1) \cup T(Q,1)$.

By induction, all information contained in X can be collected inside $A_{2^{k-1}}$ in $2^{k-2} - 1$ steps. We will add to each of these steps the following transmissions, using only the rows and columns of $M_n^2 \backslash M_{2^{k-1}}^2$:

$f_h^i(T(R, 2i-1)) = T(Q, 1)$; and
$f_v^i(T(R, 2i)) = T(P, 1), 1 \leq i \leq 2^{k-2} - 1$.

To define the remaining steps, we let $j = i - 2^{k-2} + 1$.

For $i = 2^{k-2}, 2^{k-2}+1, \ldots, 2^{k-1}-2$,

$f_h^i(T(P, i+1)) = V_{2^{k-1}}$,
$f_h^i(T(Q, j+1)) = T(Q, 1)$);

$f_v^i(T(Q, i+1)) = H_{2^{k-1}}$,
$f_v^i(T(P, j+1)) = T(P, 1)$.

For $i = 2^{k-1} - 1$,

$f_h^{2^{k-1}-1}(T(P, 2^{k-1}) = V_{2^{k-1}}$,
$f_h^{2^{k-1}-1}(T(R, 2^{k-1}-1)) = T(Q, 1)$;

$f_v^{2^{k-1}-1}(T(Q, 2^{k-1})) = H_{2^{k-1}}$,
$f_v^{2^{k-1}-1}(T(R, 2^{k-1})) = T(P, 1)$.

It is straightforward to verify that all information from M_n^2 has now been collected in A_n. ■

Case 2: n even, $n \neq 2^k$ for any k.

Let $n = 2^k + r$, where $0 < r < 2^k$ and r is even. Let

$$\begin{aligned} H_n &= H_{2^k} \cup T_{0,2^k}(D_r), \\ V_n &= V_{2^k} \cup T_{2^k,0}(D_r) \\ \text{and } A_n &= V_n \cup H_n. \end{aligned}$$

Note that $|A_n| = 2n$. For the purpose of the proofs, we define $H_1 = \{(1,1)\}$ and $V_1 = \{(1,2)\}$.

Lemma 3.7 *If n is even, the information in A_n can be accumulated at $(1,1)$ in $\lceil log_2(2n) \rceil = 1 + \lceil log_2 n \rceil$ steps.*

Proof. In one step, we can send all information in $A_n \backslash A_{2^k}$ into A_{2^k}, via

$f_h^1(T_{0,2^k}(D_r)) = V_r$, and
$f_v^1(T_{2^k,0}(D_r)) = H_r$.

In the remaining $\lceil log_2 n \rceil = 1 + \lceil log_2(2^k) \rceil$ steps, we can accumulate all information now in A_{2^k} at $(1,1)$, as in Lemma 3.5. ■

Lemma 3.8 *If n is even, the information in M_n^2 can be accumulated in A_n in $n/2 - 1$ steps.*

Proof. Recall that $n = 2^k + r$, where $0 < r < 2^k$ and r is even. Let $s = 2^k - r$. Partition M_n^2 into the following subsets:

$$\begin{aligned} X &= M_{2^k}^2, \\ P &= T_{0,2^k}(M_r^2), \\ Q &= T_{2^k,0}(M_r^2), \\ R &= T_{2^k,2^k}(M_r^2), \\ U &= T_{r,2^k}(M(s,r)), \\ W &= T_{2^k,r}(M(r,s)), \end{aligned}$$

where $M(s,r)$ and $M(r,s)$ are s by r and r by s arrays respectively. We further partition each of U and W into four "identical" quadrants; each quadrant in U is an $s/2$ by $r/2$ array, and each quadrant in W is an $r/2$ by $s/2$ array. More precisely, we let $U = U_{11} \cup U_{12} \cup U_{21} \cup U_{22}$, where

$U_{11} = T_{r,2^k}(M(s/2, r/2))$, $U_{12} = T_{0,r/2}(U_{11})$, $U_{21} = T_{s/2,0}(U_{11})$ and $U_{22} = T_{s/2,r/2}(U_{11})$. Similarly, we let $W_{11} = T_{2^k,r}(M(r/2, s/2))$, $W_{12} = T_{0,s/2}(W_{11})$, $W_{21} = T_{r/2,0}(W_{11})$ and $W_{22} = T_{r/2,s/2}(W_{11})$.

Finally, we let U_h^i be the union of the ith row of U_{11} and the ith row of U_{22}, $1 \le i \le s/2$, and U_v^j be the union of the jth column of U_{12} and the jth column of U_{21}, $1 \le j \le r/2$. We make the corresponding definitions for W.

Note that $A_n = A_{2^k} \cup T(P, 1) \cup T(Q, 1)$.

From Lemma 3.6, the information contained in X can be collected inside A_{2^k} in $2^{k-1} - 1$ steps. During these $2^{k-1} - 1$ steps we also perform the following transmissions, which use only the rows and columns of $M_n^2 \backslash X$.

For $1 \le i \le r/2$,

$f_h^i(T(R, 2i)) = T(Q, 1)$, and
$f_v^i(T(R, 2i - 1)) = T(P, 1)$.

For $r/2 + 1 \le i \le 2^{k-1} - 1$,

$f_h^i(W_v^{i-r/2}) = T(Q, 1)$, and
$f_v^i(U_h^{i-r/2}) = T(P, 1)$.

Thus after the first $2^{k-1} - 1$ steps, all information from R has been sent to A_n, as has that from every U_h^i but one and every W_v^i but one. There remain $r/2$ steps in which all rows and columns of M_n^2 can participate; we describe these steps below.

For $i = 2^{k-1}$,

$f_h^i(T(P, r/2 + 1)) = V_r$,
$f_h^i(U_v^1) = V_{2^k} \backslash V_r$,
$f_h^i(W_v^{2^{k-1}-r/2}) = T(Q, 1)$;

and

$f_v^i(T(Q, r/2 + 1)) = H_r$,
$f_v^i(W_h^1) = H_{2^k} \backslash H_r$,
$f_v^i(U_h^{2^{k-1}-r/2}) = T(P, 1)$.

Now let $j = i - 2^{k-1} + 1$. For $2^{k-1} + 1 \le i \le 2^{k-1} + r/2 - 1$ (and, hence, $2 \le j \le r/2$),

$f_h^i(T(P, j)) = V_r$,

$f_h^i(U_v^j) = V_{2^k} \backslash V_r,$
$f_h^i(T(Q, j + r/2)) = T(Q, 1);$

and

$f_v^i(T(Q, j)) = H_r,$
$f_v^i(W_h^j) = H_{2^k} \backslash H_r,$
$f_v^i(T(P, j + r/2)) = T(P, 1).$

It is straightforward to check that after these steps, all the remaining information in M_n^2 has been transferred to A_n. ■

Case 3: n odd and $n \geq 5$.

Let $n = 2m + 1$. In this case, we will set $|A_n| = 3n = 6m + 3$. Let $p = \lfloor log_2 m \rfloor$.

Note that $p + 3 \leq \lceil log_2(3n) \rceil \leq p + 4$, and that $\lceil log_2(3n) \rceil = p + 4$ if and only if $6m + 3 \geq 2^{p+3} + 1$, or equivalently, $m \geq 2^{p+2}/3 - 1/3$.

Let $S_p = \{m : 2^p < m < 2^{p+2}/3 - 1/3\}$. We will distinguish two cases, according to whether $m \in S_p$ for some p.

Case 3.1: $m \notin S_p$ for any p.

In this case, there is some p for which either

• $m = 2^p$, so that $\lceil log_2(3n) \rceil = \lceil log_2(6m + 3) \rceil = p + 3$ as $p \geq 1$, and $\lceil log_2(2(n - 1)) \rceil = \lceil log_2(4m) \rceil = p + 2$,

or

• $2^{p+2}/3 - 1/3 \leq m < 2^{p+1}$, so that $\lceil log_2(3n) \rceil = \lceil log_2(6m + 3) \rceil = p + 4$, and $\lceil log_2(2(n - 1)) \rceil = \lceil log_2(4m) \rceil = p + 3$.

Thus if $m \notin S_p$ for any p, then $\lceil log_2(3n) \rceil = \lceil log_2(2(n - 1)) \rceil + 1$. Our strategy in this case will consist of defining A_n so that all the information in $A_n \backslash A_{n-1}$ can be sent to A_{n-1} in one step; the accumulation within A_n (to $(1, 1)$) will then require $1 + \lceil log_2(2(n - 1)) \rceil = \lceil log_2(3n) \rceil$ steps.

Case 3.2: $m \in S_p$ for some p.

In this case, we will define A_n so that the information from $A_n \backslash A_{2^{p+1}}$ can be sent to $A_{2^{p+1}}$ in one step. The accumulation within A_n will then require $1 + log_2(2^{p+1}) + 1 = p + 3 = \lceil log_2(3n) \rceil$ steps.

We now define A_n. Recall that $n = 2m + 1$ and $p = \lfloor log_2 m \rfloor$, and let $2m = 2^{p+1} + r$.

We let $A_n = A_{2m} \cup E_m \cup F_m$, where

$$E_m = \begin{cases} \{(i,n) : 1 \leq i \leq m+2\}, & m \notin S_p \\ \{(r+i,n) : 1 \leq i \leq m+2\}, & m \in S_p; \end{cases}$$

and

$$F_m = \begin{cases} \{(n,i) : 1 \leq i \leq m+1\}, & m \notin S_p \\ \{(n,r+i) : 1 \leq i \leq m+1\}, & m \in S_p. \end{cases}$$

Note that $|A_n| = 3n$, as desired.

Lemma 3.9 *If n is odd, the information in A_n can be accumulated at $(1,1)$ in $\lceil log_2(3n) \rceil$ steps.*

Proof. From the above discussion, it suffices to show that if $m \notin S_p$, then all information in $A_n \backslash A_{n-1}$ can be sent to A_{n-1} in one step, and if $m \in S_p$, then all information in $A_n \backslash A_{2^{p+1}}$ can be sent to $A_{2^{p+1}}$ in one step.

If $m \notin S_p$, we let $f_h^1(E_m) = V_{m+2}$, and $f_v^1(F_m) = H_{m+1}$.

If $m \in S_p$, we let $f_h^1(E_m \cup T_{0,2^{p+1}}(D_r)) = V_{r+m+2}$, and $f_v^1(F_m \cup T_{2^{p+1},0}(D_r)) = H_{r+m+1}$.

With regard to this last step, note that for $m \in S_p$,

$r + m + 2 = 3m - 2^{p+1} + 2 < 2^{p+2} - 1 - 2^{p+1} + 2 = 2^{p+1} + 1$, so that $V_{r+m+2}, H_{r+m+1} \subseteq A_{2^{p+1}}$. ■

Lemma 3.10 *If n is odd, the information in M_n^2 can be accumulated in A_n in $(n-3)/2$ steps.*

Proof. Note that $(n-3)/2 = m - 1$. From Lemma 3.8, the information in M_{n-1}^2 can be accumulated in A_{n-1} in $(n-3)/2$ steps. We add the following transmissions to these steps.

For $m \notin S_p$:

$f_h^i((n, m+1+i)) = (n,1)$, and
$f_v^i((m+2+i, n)) = (1,n), \ 1 \leq i \leq m-1.$

For $m \in S_p$:

For $1 \leq i \leq r$,

$f_h^i(n,i) = (n, r+1)$
$f_v^i(i,n) = (r+1, n).$

For $r+1 \leq i \leq m-1$,

$f_h^i(n, m+1+i) = (n, r+1)$
$f_v^i(m+2+i, n) = (r+1, n)$.

In both cases it is straightforward to check that during these extra transmissions, all information in $M_n^2 \backslash M_{n-1}^2$ has been sent to A_n. ■

Finally we deal directly with $n = 3$ which is not covered above. We will show that $a(M_3^2) \leq 4$. The accumulation protocol is defined by the following steps.

$f_h^1(\{(3,3),(1,3),(2,3)\}) = \{(3,1),(1,1),(2,1)\}$ and $f_v^1(\{(3,2)\} = \{(1,2)\}$
$f_v^2(\{(2,1)\}) = \{(3,1)\}$
$f_v^3(\{(3,1),(2,2)\}) = \{(1,1),(1,2)\}$
$f_h^4(\{(1,2)\}) = \{(1,1)\}$

After these steps, the vertex $(1,1)$ clearly has all the information. ■

It is clear that the upper bounds in Theorem 1.1 follow immediately from the above lemmas. Combining with the Lemmas 2.3 and 2.5, we have Theorems 1.1 and 1.2.

4 Upper bounds for $d > 2$

In this section we establish an upper bound for the accumulation time, and therefore the gossiping time, in M_n^d. The accumulation algorithm is constructed along the same lines as those of Section 3. We will define an intermediary set A_n^d, of cardinality $2n^{d-1}$, which will satisfy the following.

Lemma 4.1 The information in M_n^d can be sent to A_n^d in at most $\lfloor n/d \rfloor + \lceil log_2(d+r) \rceil - 1$ steps, where $n = qd + r$, $0 \leq r < d$.

Lemma 4.2 The information in A_n^d can be sent to $(1, 1, \ldots, 1)$ in at most $(d-1)\lceil log_2 n \rceil + (d-1)$ steps.

This immediately yields the following result.

Theorem 4.3 The accumulation time in M_n^d is at most $\lfloor n/d \rfloor + (d-1)\lceil log_2 n \rceil + \lceil log_2(d+r) \rceil + (d-2)$.

Since we can broadcast the information from $(1,1,\ldots,1)$ to all of M_n^d in d additional steps, we have the following bound.

Theorem 4.4 The gossiping time in the d-dimensional mesh bus network satisfies $g(M_n^d) \le \lfloor n/d \rfloor + (d-1)\lceil log_2 n \rceil + \lceil log_2(d+r) \rceil + 2d - 2$.

Construction of the sets A_n^d and proof of Lemma 4.2

For $d = 2$, recall that A_n was defined in such a way that $|A_n| = 2n$ for even n, and $|A_n| = 3n$ for odd n. Now as we do not want to consider the congruence, we need a set which looks like A_n, but with slight modification. Let A_n^2 be the $2n$-element set defined as follows:
Let $n = 2^k + r, 0 \le r < 2^k$. Then $A_n^2 = A_{2^k} \cup T_{0,2^k}(D_r) \cup T_{2^k,0}(D_r)$, where A_{2^k} was defined in Case 1 of the proof of Theorem 3.1.

Note that $|A_n^2| = 2n$, and that for even n, $A_n^2 = A_n$ as defined in Section 3. As in the proof of Lemma 3.7 we can send the information in $A_n^2 \backslash A_{2^k}^2$ to $A_{2^k}^2$ in one step, and the information in $A_{2^k}^2$ can be accumulated at $(1,1)$ in $k+1$ steps. Thus we have the following lemma.

Lemma 4.5 *The information in* A_n^2 *can be accumulated at* $(1,1)$ *in* $\lceil log_2 n \rceil + 1$ *steps.*

Let $a(A_n^d)$ be the (accumulation) time needed to send all information from A_n^d to $(1,1,\ldots,1)$.

We can view M_n^d as the union of n copies of M_n^{d-1} (and of course the buses between them), where the ith copy contains those vertices whose last co-ordinate is equal to i. If we let A_n^d be the union of n copies of A_n^{d-1}, one in each copy of M_n^{d-1}, then the information in A_n^d can be collected at the vertices $\{(1,1,\ldots,1,i) : 1 \le i \le n\}$ in $a(A_n^{d-1})$ steps, by applying the accumulation algorithm for A_n^{d-1} in parallel. However, the vertices $\{(1,1,\ldots,1,i) : 1 \le i \le n\}$ comprise one bus in M_n^d; the accumulation at $(1,1,\ldots,1,1)$ will not, therefore, be optimal.

Consequently, we will select, for each $i, 1 \le i \le n$, a set $A_n^{d-1}(i)$ in the ith copy of M_n^d, having the same accumulation properties as A_n^{d-1}, except that the information is accumulated at a vertex $(x_i, y_i, 1, \ldots, 1, i)$ (or (x_i, y_i, i) if $d = 3$), which in general is different from $(1,1,\ldots,1,i)$. If the n vertices $\{(x_i, y_i, 1, \ldots, 1, i) : 1 \le i \le n\}$ are distinct, then in the same step each can send its information to the corresponding vertex $(x_i, y_i, 1, \ldots, 1, 1)$. If, in addition, the set $X = \{(x_i, y_i) : 1 \leq i \leq n\}$ of n vertices has the optimal accumulation time of $\lceil log_2 n \rceil$ in M_n^2, then the information in $\{(x_i, y_i, 1, \ldots, 1, i) : 1 \leq i \leq n\}$ can be accumulated at $(1, \ldots, 1)$ in $\lceil log_2 n \rceil$ more steps.

With this in mind, we make the following definition.

Definition 4.6 *Let $n = 2^k + r$, where $0 \leq r < 2^k$. We let*
$X = A^2_{2^{k-1}} \cup T_{0,2^{k-1}}(D_{\lfloor r/2 \rfloor}) \cup T_{2^{k-1},0}(D_{\lceil r/2 \rceil})$.

Note that when n is even, X is just $A_{n/2}$ as defined in Section 3. Clearly $|X| = n$; and using a proof analogous to that of Lemma 3.7, we can show that the information in X can be sent to $(1, 1)$ in $\lceil log_2 n \rceil$ steps.

We now let $A_n^{d-1}(i)$ be obtained from A_n^{d-1} by the permutation of the buses in the first two dimensions which sends vertex $(1, 1, \ldots, 1)$ to vertex $(x_i, y_i, 1, \ldots, 1)$, where $\{(x_i, y_i) : 1 \leq i \leq n\} = X$; and we let $A_n^d = \cup_{i=1}^n A_n^{d-1}(i)$.

We now have the following:

- By induction, the information in $A_n^{d-1}(i)$ can be accumulated at vertex $(x_i, y_i, 1, \ldots, 1)$ in $a(A_n^{d-1})$ steps, and these n accumulation protocols can be applied in parallel;
- The information from the vertices $\{(x_i, y_i, 1, \ldots, 1, i) : 1 \leq i \leq n\}$ can be sent to the vertices $\{(x_i, y_i, 1, \ldots, 1) : 1 \leq i \leq n\}$ in one step, and this last set constitutes a copy of X lying in copy 1 of M_n^{d-1};
- The information in this copy of X can be accumulated at $(1, 1, \ldots, 1)$ in $\lceil log_2 n \rceil$ steps.

It follows that $a(A_n^d) \leq a(A_n^{d-1}) + 1 + \lceil log_2 n \rceil$ steps, and using $a(A_n^2) = \lceil log_2 n \rceil + 1$ we obtain Lemma 4.2.

Proof of Lemma 4.1

We begin by partitioning M_n^d into n sets, $M_1, \ldots, M_n$, where M_i is defined by

$$M_i = \{(x_1, \ldots, x_d) : \sum_{j=1}^{n} x_j \equiv i \pmod{n}\}.$$

Note that given a fixed dimension, each set M_i contains exactly one vertex from each of the buses in that dimension. It follows that any M_i can send all of its information to any M_j along the buses of any given dimension; moreover, up to d such transmissions can take place simultaneously.

Let $n = qd + r$, where $0 \leq r \leq d$. During the first $q - 1$ steps of the algorithm, we send the information from the sets $M_{d+1}, \dots, M_{qd}$ to the vertices of $M_1 \cup \dots \cup M_d$. To do this, at each step we send the information from d M_l's to $M_1 \cup \dots \cup M_d$. More precisely, for $j = 1, \dots, q-1$, we define the following steps.

$f_i^j(M_{jd+i}) = M_i, 1 \leq i \leq d$. (Here f_i^j generalises f_h^j and f_v^j of Section 3, and describes the transmissions using buses in dimension i at step j.)

After these $q - 1$ steps, the information from M_n^d is contained in $M_1 \cup \dots \cup M_d \cup M_{qd+1} \cup \dots \cup M_{qd+r}$.

During the next $\lceil log_2(d + r) \rceil - 1$ steps, this information will be sent to the vertices of $M_1 \cup M_2$. The steps are defined so that if at some step j, the sets $M_1, \dots, M_c$ receive the messages sent, then at step $j + 1$ the sets $M_{\lceil c/2 \rceil+1}, \dots, M_c$ will send their information to the sets $M_1, \dots, M_{\lfloor c/2 \rfloor}$.

It can be verified that after step $q - 1 + \lceil log_2(d + r) \rceil - 1$, the information from M_n^d is held by the vertices of $M_1 \cup M_2$.

Finally, we transfer the information from $M_1 \cup M_2$ to A_n^d. Observe that because of the recursive definition, A_n^d consists of n^{d-2} 'permuted' copies of A_n^2, one in each of the n^{d-2} copies of M_n^2 whose vertices all agree in the last $d - 2$ coordinates.

Now each of M_1 and M_2 contains exactly n vertices from each copy of M_n^2, and these n vertices form a (2-dimensional) transversal of M_n^2. Thus if S denotes a fixed copy of M_n^2, we can send the information in $S \cap M_1$ to $S \cap A_n^d$ along the horizontal buses (of dimension 1), and the information in $S \cap M_2$ to $S \cap A_n^d$ along the vertical buses (of dimension 2), in one step. Moreover, horizontal buses induced by two different such sets S are vertex-disjoint, and the same holds for the vertical buses; consequently, we can in fact send the information from all of $M_1 \cup M_2$ to A_n^d in a single step.

This last step completes the accumulation of information from M_n^d at the vertices of A_n^d; the total number of steps used is $q - 1 + \lceil log_2(d + r) \rceil = \lfloor n/d \rfloor + \lceil log_2(d + r) \rceil - 1$

This completes the proof of Lemma 4.1. ■

Theorem 4.4 follows immediately. Combining this with Lemma 2.6, we have the upper bound in Theorem 1.4.

5 Conclusion

In the case $d = 2$, we believe that the upper bounds for gossiping time in Theorem 1.1 are the exact answers. Therefore it would be interesting to find an argument to improve the lower bounds. In the case $d > 2$, there should be room to improve both bounds, but probably some new techniques are required. For accumulation time, we conjecture that the lower bounds in Theorem 1.4 are the exact answers. Other than for $d = 2$, we can also verify the conjecture for $d = 3$ and $n = 3(2^k)$, but the proof is omitted here. One can also consider the same problem for other bus networks or for different hypothesis (see [7] [10]).

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